Ideal MHD Equilibria

keppens@rijnh.nl

- steady state ($\partial t = 0$) smoothly varying solutions to MHD equations

  $\Rightarrow$ solutions without discontinuities

  $\Rightarrow$ conservative or non-conservative formulation equivalent

- stationary MHD equations governed by

$$\left\{\begin{array}{lcl}\nabla \cdot (\rho \mathbf{v}) & = & 0 \\ \rho\,(\mathbf{v} \cdot \nabla)\,\mathbf{v} & = & -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g} \\ (\mathbf{v} \cdot \nabla)\,p & = & -\gamma p \nabla \cdot \mathbf{v} \\ \nabla \times (\mathbf{v} \times \mathbf{B}) & = & 0 \\ & \text{and} & \quad \nabla \cdot \mathbf{B} = 0 \end{array}\right.$$

  $\Rightarrow$ given $\mathbf{g}$ and parameter γ, solve for ρ, $\mathbf{v}$, p, $\mathbf{B}$

- dramatic simplication when considering **STATIC** equilibria

  $\Rightarrow \mathbf{v} = 0$ leaves only

$$\begin{cases} -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g} = 0 \\ \text{and } \nabla \cdot \mathbf{B} = 0 \end{cases}$$

  $\Rightarrow$ governing equations for magnetostatic equilibria

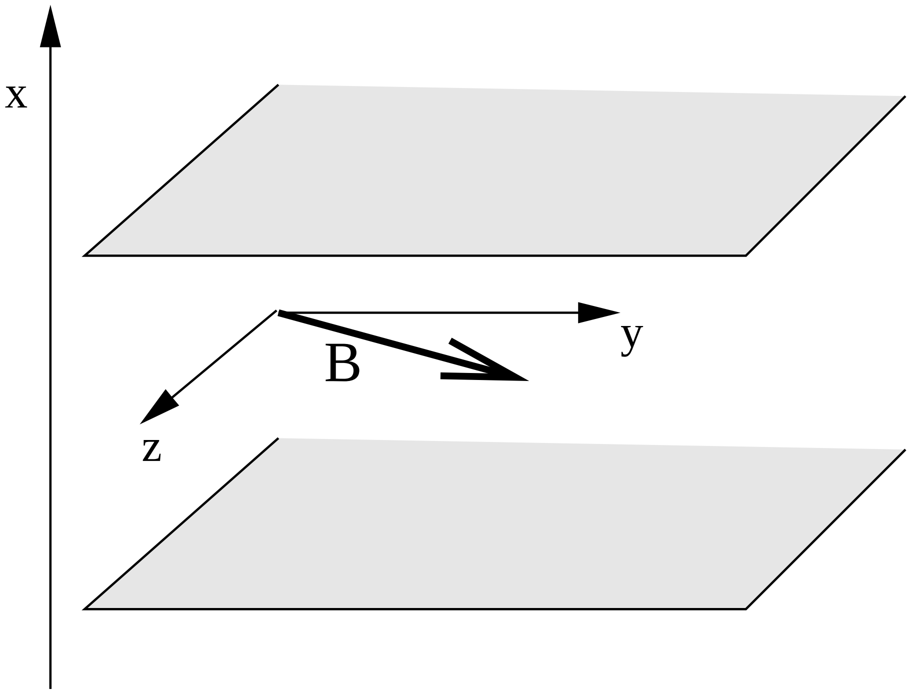
- without external gravity $\mathbf{g} = 0$

$$\begin{cases} \underbrace{\nabla p}_{\text{pressure gradient}} = \underbrace{(\nabla \times \mathbf{B}) \times \mathbf{B}}_{\text{Lorentz force}} \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$$

- magnetostatic case without gravity: choose dimensionality and geometry
- 1D cartesian geometry $p(x)$, $B_x(x)$, $B_y(x)$, $B_z(x)$

  $\Rightarrow$ constant (or zero) B_x from $\nabla \cdot \mathbf{B} = 0$

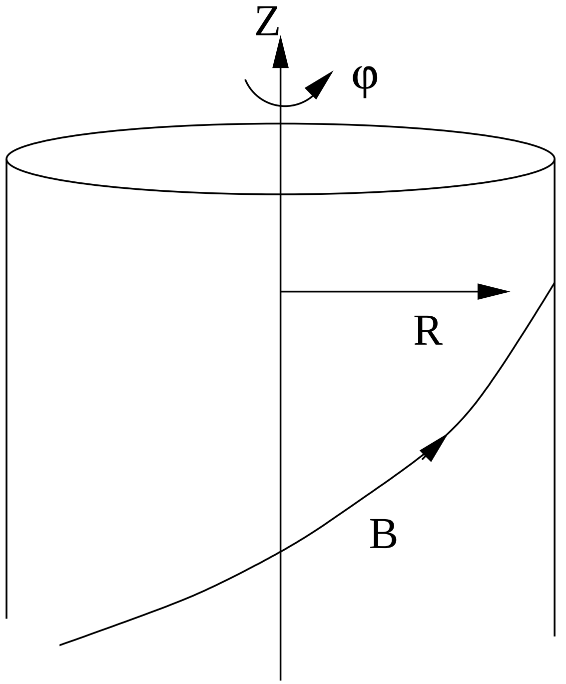
$$\frac{d}{dx}\underbrace{\left(p + \frac{B_y^2 + B_z^2}{2}\right)}_{\text{total pressure}} = 0$$



- 1D cylindrical geometry $p(R)$, $B_R(R)$, $B_\varphi(R)$, $B_Z(R)$

  $\Rightarrow$ (constant RB_R hence) zero B_R from $\nabla \cdot \mathbf{B} = 0$

$$\frac{d}{dR}\underbrace{\left(p + \frac{B_\varphi^2 + B_Z^2}{2}\right)}_{\text{total pressure}} = \underbrace{-\frac{B_\varphi^2}{R}}_{\text{radially inward tension}}$$



- ODE governs 1D static equilibria: 4 functions ρ, p, B_2, B_3

  $\Rightarrow$ 3 free profiles, namely ρ and 2 from p, B_2, B_3

- 1D cylindrical case: dimensions from

  ⇒ cylinder radius a, axial field strength B_0, axial density ρ_0

  ⇒ 2 essential dimensionless parameters

$$\beta_0 = \frac{2p_0}{B_0^2} \text{ and } \mu_0 = \frac{aB_\varphi}{RB_Z}|_{R=0} = \frac{aJ_{Z0}}{2B_0}$$

  ⇒ axial plasma beta β_0

  ⇒ inverse pitch $\mu = aB_\varphi / RB_Z$

- Latter uses axial current

$$J_{Z0} = J_Z(R=0) = \frac{1}{R}\frac{d}{dR}(RB_\varphi)\,|_{R=0}$$

$\Rightarrow$ introduce dimensionless quantities $\tilde{f}$ and unit profiles $\bar{f}$

$$\tilde{R} \equiv \frac{R}{a} \equiv \bar{R}$$

$$\tilde{\rho} \equiv \frac{\rho}{\rho_0} \equiv \bar{\rho}$$

$$\tilde{p} \equiv \frac{p}{B_0^2} = \frac{\beta_0}{2}\frac{p}{p_0} \equiv \frac{\beta_0}{2}\bar{p}$$

$$\tilde{B}_Z \equiv \frac{B_Z}{B_0} \equiv \bar{B}_Z$$

$$\tilde{B}_\varphi \equiv \frac{B_\varphi}{B_0} \equiv \mu_0 \bar{\mu} \bar{R} \bar{B}_Z$$

- 1D cylindrical equilibrium fully specified by β_0, μ_0 and

    $\Rightarrow$ two (three with $\bar{\rho}$) unit profiles $\bar{p}(\bar{R})$ and $\bar{\mu} = \mu/\mu_0$

    $\Rightarrow$ determine $\bar{B}_Z$ from equilibrium relation

$$\frac{d}{d\bar{R}}\left[\frac{\beta_0}{2}\bar{p} + \frac{\bar{B}_Z{}^2}{2}\left(\bar{R}^2\mu_0^2\bar{\mu}^2 + 1\right)\right] = -\bar{R}\mu_0^2\bar{\mu}^2\bar{B}_Z{}^2$$

    $\Rightarrow$ find $\bar{B}_\varphi \equiv \bar{R}\bar{\mu}\bar{B}_Z$ and $\tilde{B}_\varphi = \mu_0\bar{B}_\varphi$

- core problem, freedom in parameter values and unit profile variations
- for cylinder of finite length $L \equiv 2\pi R_0$: alternatively

    $\Rightarrow$ introduce $\epsilon = a/R_0$ and work with $q_0 = \epsilon/\mu_0$ instead

    $\Rightarrow$ q-profile from $q = RB_Z/R_0B_\varphi$

Grad-Shafranov equation

- consider axisymmetric ($\partial\varphi = 0$), 2D static equilibria

  $\Rightarrow$ assume right-handed coordinate system (R, Z, φ)

  $\Rightarrow$ to solve for 5 two-dimensional functions

$$\rho(R,Z),\;\; p(R,Z),\;\; B_R(R,Z),\;\; B_Z(R,Z),\;\; B_\varphi(R,Z)$$

  $\Rightarrow$ density profile arbitrary, not in equilibrium $\nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}$

- from $\nabla \cdot \mathbf{B} = 0$ or $\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_Z}{\partial Z} = 0$

  $\Rightarrow$ flux function $\psi(R,Z)$ with $\mathbf{B} = \frac{1}{R}\hat{e_\varphi} \times \nabla\psi + B_\varphi \hat{e_\varphi}$ such that

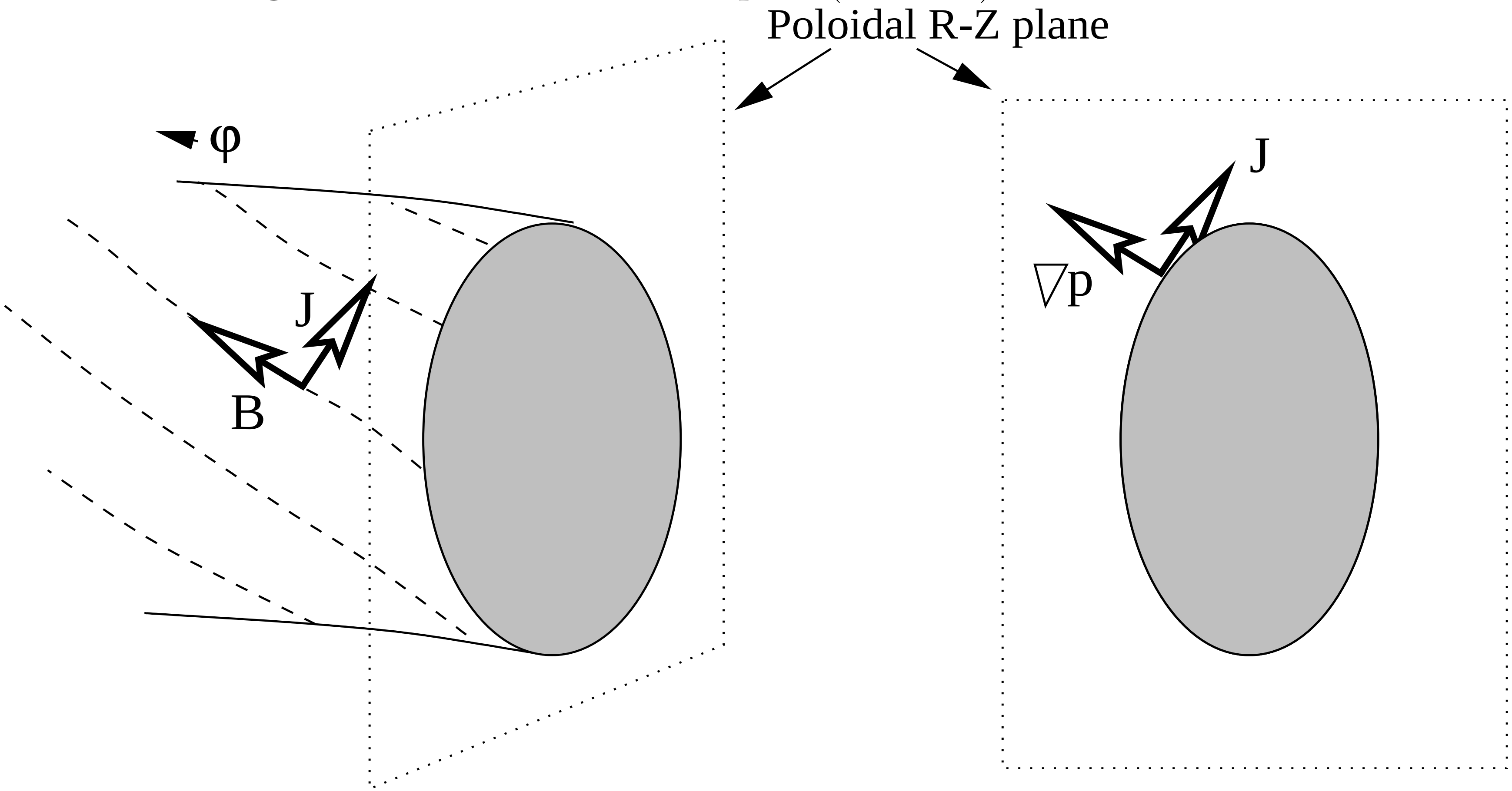
$$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial Z} \;\;\text{and}\;\; B_Z = \frac{1}{R}\frac{\partial\psi}{\partial R}$$

- note that $\mathbf{B} \cdot \nabla\psi = 0$

  $\Rightarrow$ constant ψ surfaces (or contours in (R, Z)-plane) are 'flux surfaces'

  $\Rightarrow$ field lines lie on these flux surfaces

  $\Rightarrow$ 3 orthogonal directions from $\nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}$



- since $\mathbf{J} = \nabla \times \mathbf{B}$ we have $\nabla \cdot \mathbf{J} = 0$

  $\Rightarrow$ analogously define current stream function $I(R, Z)$

$$\mathbf{J} = -\frac{1}{R}\hat{e_\varphi} \times \nabla I + J_\varphi \hat{e_\varphi}$$

  $\Rightarrow$ this yields

$$J_R = \frac{1}{R}\frac{\partial I}{\partial Z} \quad \text{and} \quad J_Z = -\frac{1}{R}\frac{\partial I}{\partial R}$$

  $\Rightarrow$ from $\mathbf{J} = \nabla \times \mathbf{B}$ we have as well

$$J_R = \frac{\partial B_\varphi}{\partial Z} \Rightarrow I = RB_\varphi$$

- φ component of $\nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}$

$$0 = J_R B_Z - J_Z B_R = \frac{1}{R^2}\left(\frac{\partial I}{\partial Z}\frac{\partial \psi}{\partial R} - \frac{\partial I}{\partial R}\frac{\partial \psi}{\partial Z}\right)$$

  $\Rightarrow$ since commutator vanishes, find $I(\psi)$

- R and Z component of force balance yield

$$\frac{\partial p}{\partial R} = J_Z B_\varphi - J_\varphi B_Z = \left(-\frac{II'}{R^2} - \frac{J_\varphi}{R}\right)\frac{\partial \psi}{\partial R}$$

$$\frac{\partial p}{\partial Z} = J_\varphi B_R - J_R B_\varphi = \left(-\frac{II'}{R^2} - \frac{J_\varphi}{R}\right)\frac{\partial \psi}{\partial Z}$$

$\Rightarrow$ again find $p(\psi)$ and conclude

$$p' = -\frac{II'}{R^2} - \frac{J_\varphi}{R}$$

- use $\mathbf{J} = \nabla \times \mathbf{B}$ to find

$$J_\varphi = \frac{\partial B_Z}{\partial R} - \frac{\partial B_R}{\partial Z}$$

$\Rightarrow$ in terms of ψ we get $RJ_\varphi = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2}$

- Grad-Shafranov equation thus reads

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2} = -II' - p'R^2$$

$\Rightarrow$ to solve for $\psi(R, Z)$ under given profiles $p(\psi)$ and $I(\psi)$

- GS: 2nd order PDE

  ⇒ character from $a\psi_{RR} + 2b\psi_{RZ} + c\psi_{ZZ} + \ldots$

  ⇒ find $b^2 - ac = -1 < 0$: elliptic

  ⇒ Boundary value problem well-posed

- for elliptic Laplace equation: solution reaches extremum on boundary

  ⇒ here we have (different from Laplace $\nabla \cdot \nabla\psi = 0$):

$$R^2 \nabla \cdot \frac{1}{R^2} \nabla\psi = -II' - p'R^2$$

  ⇒ normally have $\psi \in [0, \psi_1]$: zero at magnetic axis, fixed BV ψ_1

Grad-Shafranov for tokamak equilibria

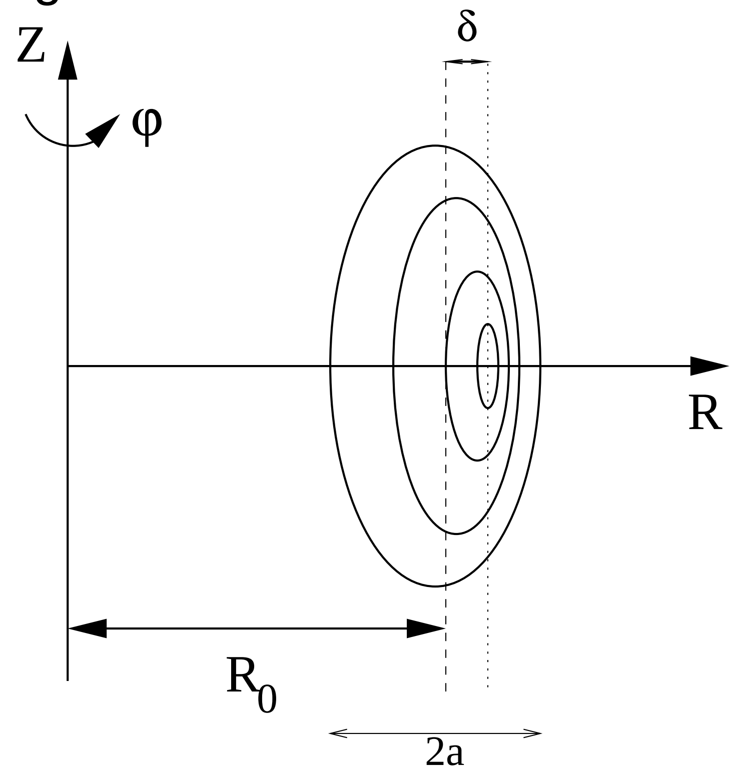
- tokamak $=$ toroidal vessel for achieving controlled nuclear fusion

  $\Rightarrow$ task to achieve steady state magnetically confined plasma

  $\Rightarrow$ within poloidal cross-section of vessel: ideal MHD equilibrium

  $\Rightarrow$ static plasma and axisymmetry: solve GS within given boundary

- overall configuration: 'donut' with nested flux surfaces



Scaling for Grad-Shafranov

- natural to set units by a (length) and B_0 (magnetic and pressure) from
  - $\Rightarrow$ $2a$ horizontal diameter of vessel
  - $\Rightarrow$ geometric center of vessel at $R_0 \rightarrow \epsilon \equiv a/R_0$ inverse aspect ratio
  - $\Rightarrow$ use strength of vacuum field at R_0 for B_0
- value of flux on outer boundary ψ_1
  - $\Rightarrow$ use to define a unit flux label $\bar{\psi} = \psi/\psi_1$
  - $\Rightarrow$ define dimensionless 'inverse flux' $\alpha \equiv a^2 B_0/\psi_1$
- change to dimensionless poloidal coordinates (x, y)

$$\left\{ \begin{array}{l} R \\ Z \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{lll} x = & \frac{R-R_0}{a} & = \tilde{R} - \frac{1}{\epsilon} \\ y = & \frac{Z}{a} & = \tilde{Z} \end{array} \right.$$

- LHS of GS changes to

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2} \quad\Rightarrow\quad \frac{\psi_1}{a^2}\left[\frac{\partial^2\bar{\psi}}{\partial x^2}-\frac{\epsilon}{1+\epsilon x}\frac{\partial\bar{\psi}}{\partial x}+\frac{\partial^2\bar{\psi}}{\partial y^2}\right]$$

- RHS of GS becomes

$$-I\frac{dI}{d\psi}-R^2\frac{dp}{d\psi} \quad\Rightarrow\quad -\frac{a^2B_0^2}{\psi_1}\left[\tilde{I}\frac{d\tilde{I}}{d\bar{\psi}}+\tilde{R}^2\frac{d\tilde{p}}{d\bar{\psi}}\right]$$

$\Rightarrow$ using $\tilde{I}\equiv RB_\varphi/aB_0$ and $\tilde{p}\equiv p/B_0^2$

- introduce dimensionless profiles

  $\Rightarrow$ scaled pressure $P = P(\bar{\psi}) \equiv \frac{\alpha^2}{\epsilon}\tilde{p}$

  $\Rightarrow Q = Q(\bar{\psi}) \equiv -\frac{\epsilon\alpha^2}{2a^2B_0^2}\left[I^2(\psi) - R_0^2B_0^2\right]$

  $\Rightarrow G = G(\bar{\psi}) \equiv -\frac{1}{\epsilon}\left[Q(\bar{\psi}) - P(\bar{\psi})\right]$

- under these definitions, GS becomes

$$\frac{\partial^2\bar{\psi}}{\partial x^2} - \frac{\epsilon}{1+\epsilon x}\frac{\partial\bar{\psi}}{\partial x} + \frac{\partial^2\bar{\psi}}{\partial y^2} = -\frac{dG}{d\bar{\psi}} - \frac{dP}{d\bar{\psi}}x(2+\epsilon x)$$

  $\Rightarrow$ two arbitrary profiles $G' \equiv dG/d\bar{\psi}$ and $P' \equiv dP/d\bar{\psi}$

- core problem now identified, separate freedom in G' and P'

  $\Rightarrow$ magnitude and shape: amplitude and unit profile

$$G' \equiv -A\Gamma(\bar{\psi}) \quad \text{with} \quad \Gamma(0) = 1$$

$$P' \equiv -\frac{1}{2}AB\Pi(\bar{\psi}) \quad \text{with} \quad \Pi(0) = 1$$

  $\Rightarrow$ roughly related to current profile and pressure gradient profile

  $\Rightarrow$ Final form of GS is then

$$\frac{\partial^2\bar{\psi}}{\partial x^2} - \frac{\epsilon}{1+\epsilon x}\frac{\partial\bar{\psi}}{\partial x} + \frac{\partial^2\bar{\psi}}{\partial y^2} = A\left[\Gamma(\bar{\psi}) + Bx(1+\frac{\epsilon x}{2})\Pi(\bar{\psi})\right]$$

- boundary conditions

  $\Rightarrow$ $\bar{\psi} = 1$ on boundary

  $\Rightarrow$ $\bar{\psi} = \bar{\psi}_x = \bar{\psi}_y = 0$ at magnetic axis $(\delta, 0)$

  $\Rightarrow$ 2nd order PDE with 4 BCs: overdetermined!!!

- A (and B if input δ) to be determined with solution ('eigenvalues')
- once solution $\bar{\psi}$, A, and B identified

$\Rightarrow$ reconstruct pressure from

$$\tilde{p} = \frac{\epsilon}{\alpha^2}\left[\int^{\bar{\psi}}\left(-\frac{1}{2}AB\Pi\right)d\bar{\psi} + P(0)\right]$$

$\Rightarrow$ magnetic field components from

$$\tilde{B}_R = -\frac{1}{\alpha}\frac{\epsilon}{1+\epsilon x}\frac{\partial\bar{\psi}}{\partial y}$$

$$\tilde{B}_Z = \frac{1}{\alpha}\frac{\epsilon}{1+\epsilon x}\frac{\partial\bar{\psi}}{\partial x}$$

$$\tilde{B}_\varphi = \frac{1}{1+\epsilon x}\left[1 - 2\frac{\epsilon}{\alpha^2}\int^{\bar{\psi}} A(\epsilon\Gamma - \frac{1}{2}B\Pi)d\bar{\psi}\right]^{1/2}$$

$\Rightarrow$ different realization of same core problem solution for different α!

The Soloviev solution

- analytic solution to Grad-Shafranov

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}=-II'-p'R^2$$

$\Rightarrow$ for linear profiles $I^2/2=I_0^2/2-E\psi$ and $p=p_0-F\psi$

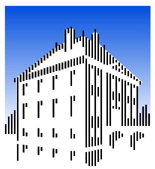
$\Rightarrow$ E, F free profile parameters

- polynomial solution derived by Soloviev

$$\psi=\left(C+DR^2\right)^2+\frac{1}{2}\left[E+\left(F-8D^2\right)R^2\right]Z^2$$

$\Rightarrow$ two additional parameters C, D

$\Rightarrow$ verify by insertion



- use Soloviev for creating tokamak equilibria
- instructive to change to $\bar{\psi}$ and (x, y) coordinates

  ⇒ Soloviev solution rewritten as

$$\bar{\psi} = \left[x - \frac{\epsilon}{2}(1 - x^2)\right]^2 + \left(1 - \frac{\epsilon^2}{4}\right)\left[1 + \tau\epsilon x(2 + \epsilon x)\right]\frac{y^2}{\sigma^2}$$

  ⇒ unit contour through 4 points $(x = \pm 1, y = 0)$ $(0, y = \pm\sigma)$

  ⇒ $\epsilon = a/R_0$ inverse aspect ratio wrt geometric center

  ⇒ τ measures triangularity (ellips-like for $\tau = 0$)

- $\bar{\psi} = 0$ at magnetic axis $(\delta, 0)$ with

$$\delta = \frac{1}{\epsilon}\left[\sqrt{1 + \epsilon^2} - 1\right]$$

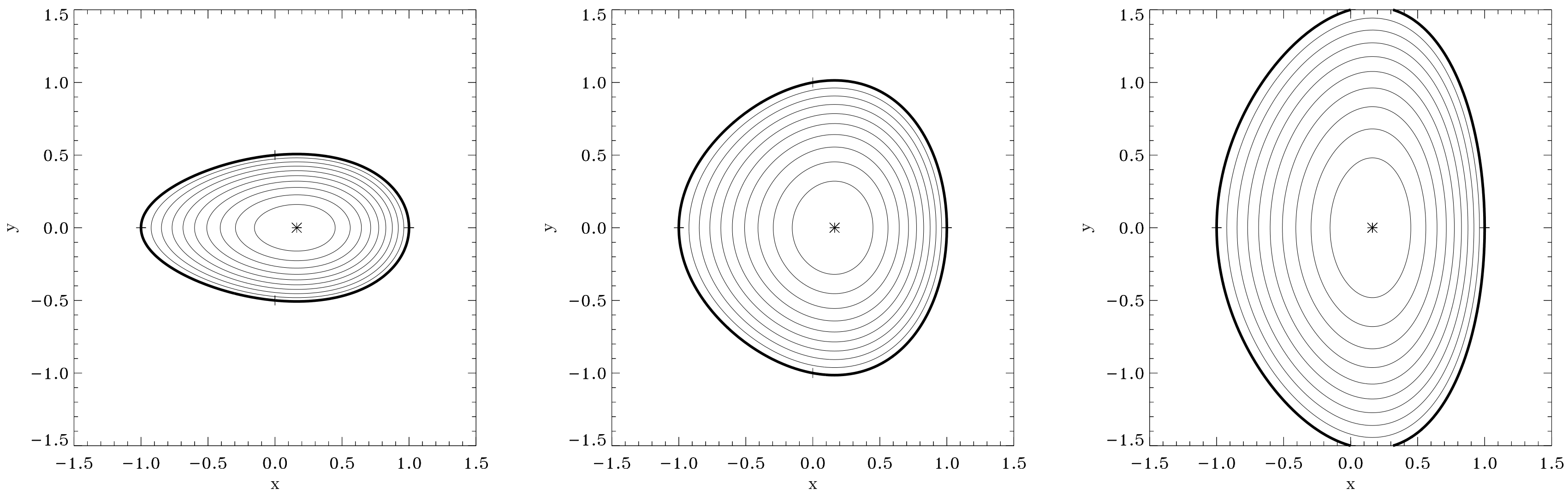
  ⇒ outward Shafranov shift of magnetic axis

  ⇒ geometric effect required for equilibrium in torus

- did not consider 'boundary' as given here

  $\Rightarrow$ determined as unit contour from solution

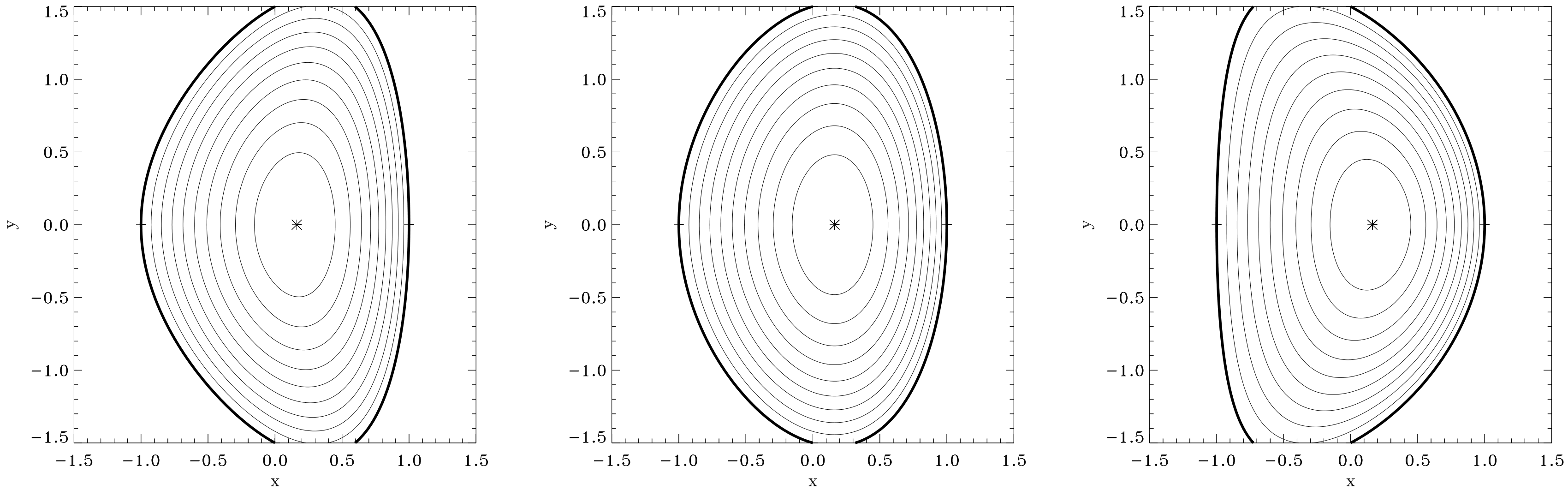
- cases $\epsilon = 0.333$, $\tau = 0$ and $\sigma = (0.5, 1, 1.5)$



$\Rightarrow$ 'elliptic' flux contours

- varying the triangularity parameter τ

$\Rightarrow$ case $\tau = -0.5$ versus $\tau = 0$ and $\tau = 1.5$



$\Rightarrow$ 'triangular' flux contours, sign of τ for direction

Force-Free equilibria

- So far: 1D and 2D static ideal MHD equilibria (no gravity, no flow)

  $\Rightarrow$ pressure gradient balancing Lorentz force

- in low beta plasma $\beta \ll 1$ only solve

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$$

  $\Rightarrow$ plus solenoidal constraint

- current $\mathbf{J} = \nabla \times \mathbf{B}$ must be $\parallel \mathbf{B}$

  $\Rightarrow$ force-free magnetic configuration, can be 1D, 2D, or 3D

- simplest case: vanishing current $\mathbf{J} = 0$ or 'potential fields'

  $\Rightarrow$ vacuum magnetic field solution

  $\Rightarrow$ since $\nabla \times \mathbf{B} = 0$ define ϕ from $\mathbf{B} = -\nabla\phi$

  $\Rightarrow$ since $\nabla \cdot \mathbf{B} = 0$ solve

$$\nabla^2 \phi = 0$$

  $\Rightarrow$ elliptic Laplace equation, BVP well-posed

- frequently used for reconstructing 'coronal' magnetic field

  $\Rightarrow$ normal $\mathbf{B}$ component from magnetograms of solar photosphere

  $\Rightarrow$ $B_n|_{\mathrm{photosphere}} = -\frac{\partial \phi}{\partial n}|_{\mathrm{photosphere}}$ given

- more general: field-aligned currents $\mathbf{J} = \alpha \mathbf{B}$

  $\Rightarrow$ from $\nabla \cdot (\nabla \times \mathbf{B}) = 0$ find

$$\mathbf{B} \cdot \nabla \alpha = 0$$

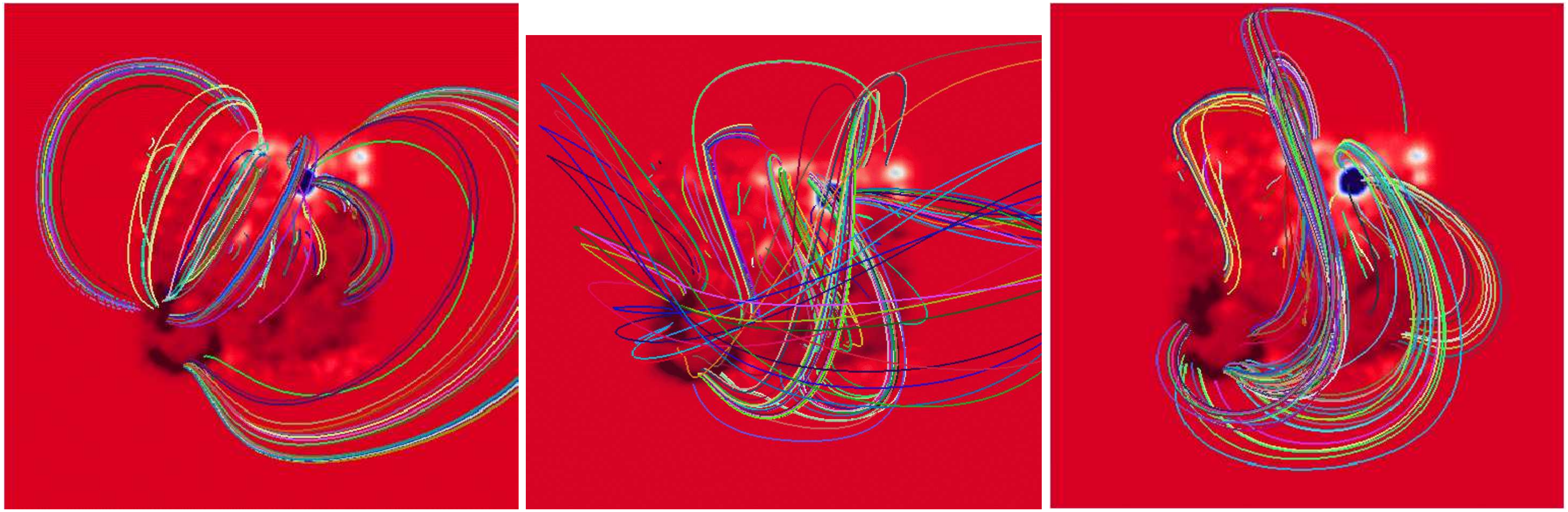
  $\Rightarrow$ α is constant along field line

- 1D examples are

  $\Rightarrow$ slab: $B_y(x) = \sin(\alpha x)$ and $B_z(x) = \cos(\alpha x)$

  $\Rightarrow$ cylinder: $B_Z(R) = J_0(\alpha R)$ and $B_\varphi(R) = J_1(\alpha R)$

- example taken from Régnier & Amari



$\Rightarrow$ three reconstructions: potential, constant α, varying α

$\Rightarrow$ best agreement with observations for nonlinear force-free case

Stationary equilibria

- static ($\mathbf{v} = 0$) case only left momentum balance

  $\Rightarrow$ stationary solutions $\mathbf{v} \neq 0$ involve all equations

- 1D case still fairly trivial

  $\Rightarrow$ slab: when $v_x = 0$ same as before

  $\Rightarrow$ 1D slab with arbitrary background flow $v_y(x)$ and $v_z(x)$ obeys

$$\frac{d}{dx}\underbrace{\left(p + \frac{B_y^2 + B_z^2}{2}\right)}_{\text{total pressure}} = -\rho g$$

  $\Rightarrow$ with gravity $\mathbf{g} = -g\hat{e}_x \Rightarrow$ density enters

  $\Rightarrow$ example solution: force-free (B_y, B_z) and $\rho = 1 - Dx$ on $x \in [0, 1]$

  $\Rightarrow$ density contrast D and pressure $p = p_0 - g(x - \frac{Dx^2}{2})$

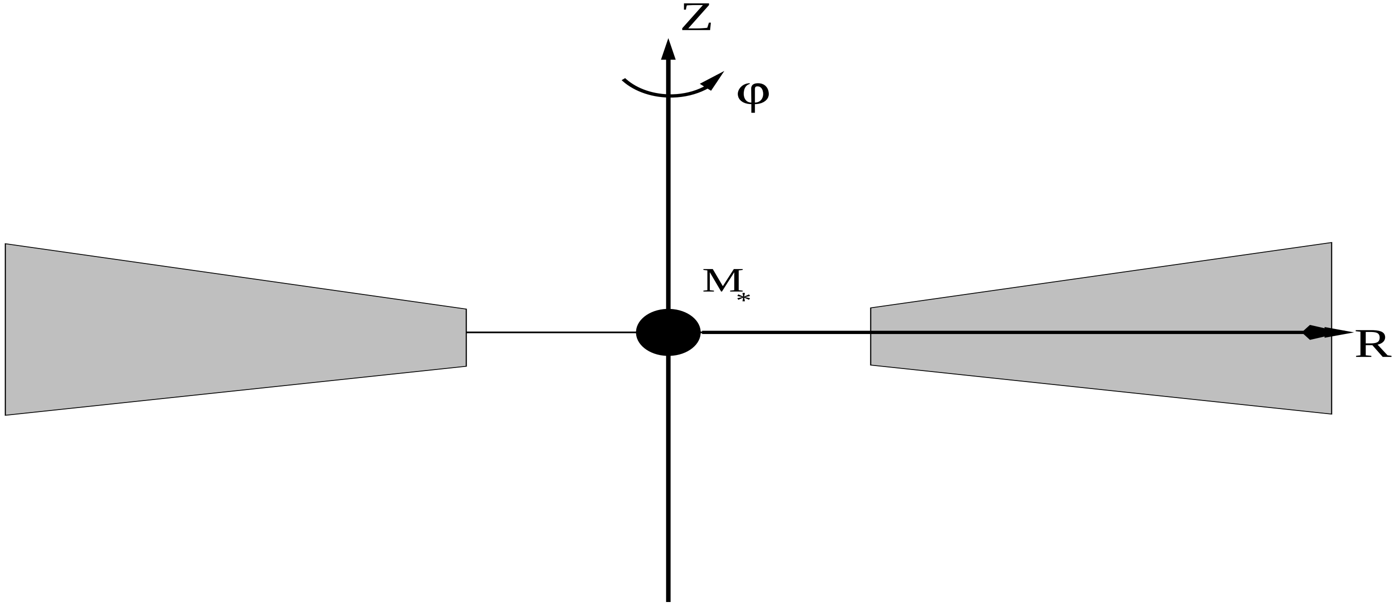
- 1D cylindrical equilibria

  ⇒ take $v_R = 0$ then again only force balance remains

  ⇒ assume gravitational 'line' source $\mathbf{g} = -\frac{GM_*}{R^2}\hat{e}_R$

$$\frac{d}{dR}\underbrace{\left(p + \frac{B_\varphi^2 + B_Z^2}{2}\right)}_{\text{total pressure}} = \underbrace{-\frac{B_\varphi^2}{R}}_{\text{radially inward tension}} + \underbrace{\frac{\rho v_\varphi^2}{R}}_{\text{centrifugal force}} \underbrace{-\rho\frac{GM_*}{R^2}}_{\text{gravity}}$$

  ⇒ suitable for accretion disks $R \in [R_{inner}, R_{outer}]$: midplane



  ⇒ for jets: neglect gravity and take $R \in [0, R_{jet}]$

- 2D (axisymmetric) stationary ideal MHD equilibria

  $\Rightarrow$ allow for both toroidal and poloidal flow in tokamak steady-state

- core problem is again 2nd order PDE for flux $\psi(R, Z)$

  $\Rightarrow$ $\mathbf{BUT}$ additional algebraic equation for $M^2(R,Z) \equiv \rho v_p^2/B_p^2$

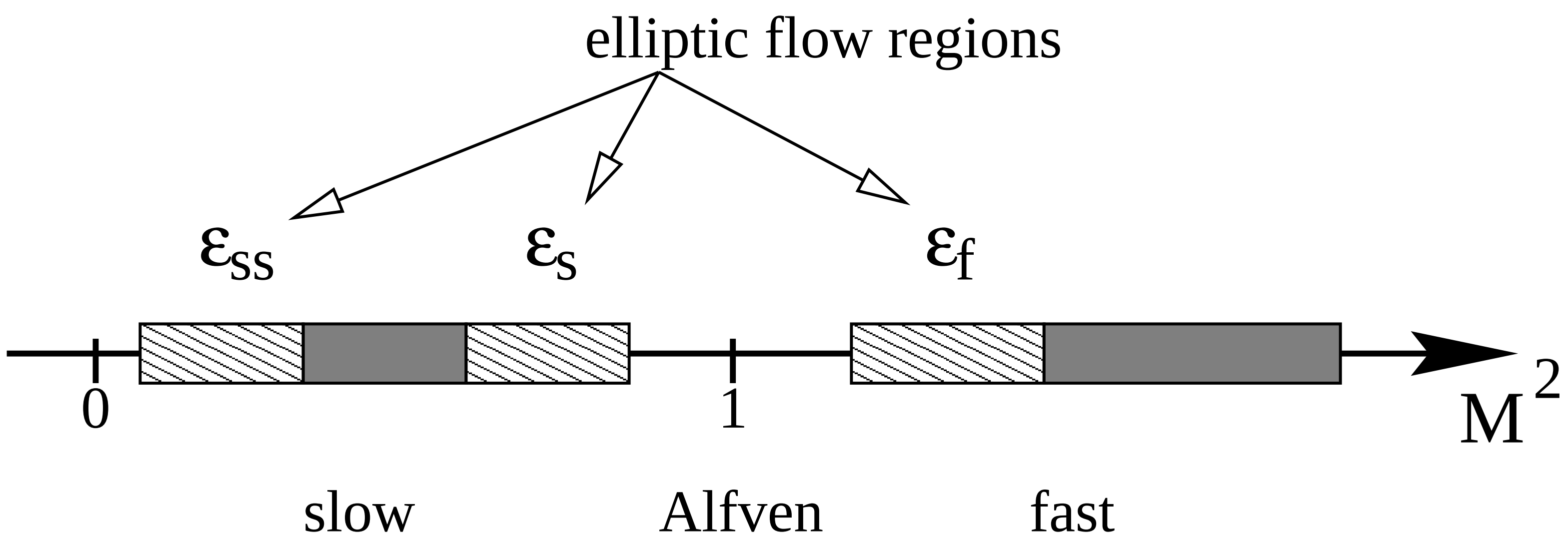
  $\Rightarrow$ latter expresses energy conservation along fieldline

- $\mathbf{much}$ more complex problem:

  $\Rightarrow$ character of governing PDE can change from location to location

  $\Rightarrow$ elliptic or hyperbolic nature depends on M^2

  $\Rightarrow$ turns out: 3 elliptic regimes: subslow, slow, fast



References

- J.P. Goedbloed, *Transactions of Fusion Technology* **37**, 2T, march 2000
- B.C. Low, Lecture series HAO 1994
- J.P. Goedbloed & S. Poedts, *Advanced Magnetohydrodynamics*, Lecture series 1997-1998 K.U. Leuven (CPA, Doctoraatsopleiding Wiskunde)
- J.P. Goedbloed & S. Poedts, *Principles of Magnetohydrodynamics*, Cambridge University Press, to appear 2003
- J.P. Goedbloed, *Magnetohydrodynamics of astrophysical plasmas*, Lectures at Utrecht University 2001
- J.P. Goedbloed & A.E. Lifschitz, *Phys. Plasmas* **4**, 1997, 3544
- J.P. Goedbloed, *Physica Scripta* **T98**, 2002, 43
- S. Régnier web site `solar.physics.montana.edu/regnier`
- T. Amari web site `www.cpht.polytechnique.fr/cpht/amari`