Elements of Photonics, Volume II: For Fiber and Integrated Optics. Keigo Iizuka

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APPENDIX B

MODE DENSITY

It will be proved that the density $m(\nu)$ of the modes per unit frequency per unit volume of a blackbody radiator is

$$m(\nu) = \frac{8\pi n_1^3}{c^3}\nu^2$$

The general expression for a sinusoidally oscillating wave is

$$\Psi(x, y, z, t) = \psi(x, y, z)e^{-j2\pi\nu t} \tag{B.1}$$

Equation (B.1) has to satisfy the wave equation:

$$\nabla^2\Psi - \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{B.2}$$

Using the method of separation of variables, Ψ will be found. Assume the solution is expressed by

$$\psi(x, y, z) = X(x)Y(y)Z(z) \tag{B.3}$$

Inserting Eq. (B.3) into (B.2) and then dividing the equation by Ψ gives

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} - \left(\frac{2\pi\nu}{v}\right)^2 = 0 \tag{B.4}$$

Each term of Eq. (B.4) is a function of x, y, and z, respectively. For Eq. (B.4) to be satisfied at any location in space, each individual term has to be constant.

$$\frac{X''(x)}{X(x)} = k_x, \qquad \frac{Y''(y)}{Y(y)} = k_y, \qquad \frac{Z''(z)}{Z(z)} = k_z \tag{B.5}$$

with

$$k_x^2 + k_y^2 + k_z^2 = \left(\frac{2\pi\nu}{v}\right)^2 \tag{B.6}$$

The general solution for $X(x)$ is

$$X(x) = A \sin k_x X + B \cos k_x X \tag{B.7}$$

$X(x)$ has to satisfy the *boundary conditions*.

Two ways of setting up the boundary conditions will be presented: the periodic boundary condition that was not mentioned in the text and the standing-wave boundary condition that has already been mentioned in the text.

Can we really take the size of a crystal as the boundary of a gigantic size potential well in the y and z directions? Strictly speaking, if indeed they are the boundaries of a big potential well, the properties of the crystal would change if the crystal is broken in half. The surface shape or surface roughness of the crystal would change the shape of the potential well. Would the properties of the material then depend on the shape?

The periodic boundary was proposed as an alternative boundary. The periodic boundary is applicable to a crystal of infinite size. The assumption is made that there is a certain distance L at which the wavefunction repeats itself in a gigantic size crystal. The L is taken definitely much larger than 100 Å but much smaller than the physical dimensions of the crystal. In most cases, L drops out from the final answer by expressing the quantity concerned per volume or per area. In the present case as well, the same value of the density of states can be obtained whether the physical size or periodic boundary is used as a boundary as explained later. With the periodic boundary condition,

$$X(0) = X(L_x) \tag{B.8}$$

and

$$k_x L_x = 2\pi n_x \tag{B.9}$$

where n_x is an integer including zero for both forward and backward waves.

$$n_x = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{B.10}$$

Similarly, for k_y and k_z

$$k_y = 2\frac{n_y}{L_y}\pi \tag{B.11}$$

$$k_z = 2\frac{n_z}{L_z}\pi \tag{B.12}$$

where

$$\begin{aligned} n_y &= 0, \pm 1, \pm 2, \pm 3, \ldots \\ n_z &= 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned} \tag{B.13}$$

The k_x, k_y, k_z set of coordinates shown in Fig. B.1a is useful for counting the number of modes. The values of k_x, k_y, and k_z are discrete as specified by the integers, and the values form lattice points. Every lattice point represents one mode. The wavenumber k_i with a particular frequency ν_i is $k_i = 2\pi\nu_i/v$ and is represented by the length connecting the origin to the lattice point. Modes whose resonance frequency is less than ν are inside a sphere of radius $k = 2\pi\nu/v$.

The total number of modes whose resonance frequency is less than ν is now calculated. The total number M of lattice points inside the sphere with radius $2\pi\nu/v$ is the volume of the sphere divided by the density of the lattice points $(2\pi/L)^3$ with the assumption $L_x = L_y = L_z = L$.

$$\begin{aligned} M &= \frac{4}{3}\pi\left(\frac{2\pi\nu}{v}\right)^3 \bigg/ \left(\frac{2\pi}{L}\right)^3 \\ &= \frac{4}{3}\pi\left(\frac{L\nu}{v}\right)^3 \end{aligned} \tag{B.14}$$

There are two orthogonal directions of polarization and the total number M_t of modes taking polarization into account is

$$M_t = 2M \tag{B.15}$$

Thus, the increase $m(\nu)\,d\nu$ due to an increase in frequency from ν to $\nu + d\nu$ per unit volume is

$$m(\nu)d\nu = \frac{1}{L^3}\frac{dM_t}{d\nu}d\nu \tag{B.16}$$

From Eqs. (B.14) to (B.16), the mode density is

$$m(\nu) = \frac{8\pi\nu^2}{v^3} \tag{B.17}$$

where $v = c/n$ and where n is the refractive index of the blackbody radiator.

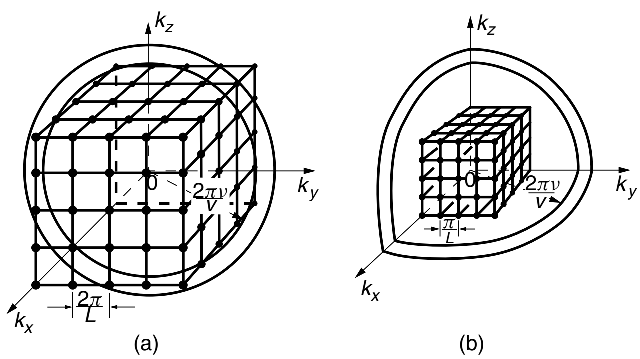

Figure B.1 Lattice points representing the modes whose frequencies are lower than ν. (a) When the periodic boundary condition is used. (b) When the standing-wave boundary condition is used.

Lastly, it will be shown that the standing-wave boundary condition provides the same result as the periodic boundary condition does. The condition for a standing wave to exist in a medium bordered by $x = 0$ and $x = L$ is that Eq. (B.7) satisfy

$$\begin{aligned} &B = 0 \\ &k_x = \frac{n_x}{L}\pi \quad \text{with } n_x = 1, 2, 3, 4, \ldots \end{aligned} \tag{B.18}$$

where n_x is called the mode number. Because

$$\sin(-k_x x) = -\sin k_x x$$

the difference between the standing waves with k_x and $-k_x$ is a difference in the sign of the amplitude. After all, the sine function oscillates between 1 and -1 with time. Standing waves whose amplitudes differ only in sign are considered as the same modes.

Consequently, only positive numbers are used to designate the mode number.

The spacing between adjacent modes when using the standing-wave boundary condition is

$$\frac{\pi}{L}$$

whereas the spacing between modes when using the periodic boundary condition is

$$2\frac{\pi}{L}$$

In one dimension, the mode density for the standing-wave boundary condition is twice as large as for the periodic boundary condition, and in three dimensions, the density is eight times as large.

Next, the total number M of modes that corresponds to Eq. (B.14) will be calculated for the case of the standing-wave boundary condition.

This time, only the positive numbers of k_x, k_y, k_z are allowed, as shown in Fig. B.1b. The total number M of modes whose resonance frequency is less than ν is contained only inside the positive octant of the sphere, but the density of the lattice points is 2^3 times larger than the case of the periodic boundary. The result is

$$M = \frac{1}{8}\frac{4}{3}\pi \left(\frac{2\pi\nu}{v}\right)^3 \Big/ \left(\frac{\pi}{L}\right)^3 \tag{B.19}$$

which is identical to Eq. (B.14).