Fundamentals of Photonics
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APPENDIX

A

FOURIER TRANSFORM

This appendix provides a brief review of the Fourier transform, and its properties, for functions of one and two variables.

A.1. One-Dimensional Fourier Transform

The harmonic function $F \exp(j2\pi\nu t)$ plays an important role in science and engineering. It has frequency ν and complex amplitude F. Its real part $|F|\cos(2\pi\nu t + \arg\{F\})$ is a cosine function with amplitude $|F|$ and phase $\arg\{F\}$. The variable t usually represents time; the frequency ν has units of cycles/s or Hz. The harmonic function is regarded as a building block from which other functions may be obtained by a simple superposition.

In accordance with the Fourier theorem, a complex-valued function $f(t)$, satisfying some rather unrestrictive conditions, may be decomposed as a superposition integral of harmonic functions of different frequencies and complex amplitudes,

$$\boxed{f(t) = \int_{-\infty}^{\infty} F(\nu) \exp(j2\pi\nu t)\, d\nu.} \tag{A.1-1}$$

Inverse Fourier Transform

The component with frequency ν has a complex amplitude $F(\nu)$ given by

$$\boxed{F(\nu) = \int_{-\infty}^{\infty} f(t) \exp(-j2\pi\nu t)\, dt.} \tag{A.1-2}$$

Fourier Transform

$F(\nu)$ is termed the **Fourier transform** of $f(t)$, and $f(t)$ is the **inverse Fourier transform** of $F(\nu)$. The functions $f(t)$ and $F(\nu)$ form a Fourier transform pair; if one is known, the other may be determined.

In this book we adopt the convention that $\exp(j2\pi\nu t)$ represents positive frequency, whereas $\exp(-j2\pi\nu t)$ is a harmonic function representing negative frequency. The opposite convention is used by some authors who define the Fourier transform in (A.1-2) with a positive sign in the exponent, and use a negative sign in the exponent of the inverse Fourier transform (A.1-1).

In communication theory, the functions $f(t)$ and $F(\nu)$ represent a signal, with $f(t)$ its time-domain representation and $F(\nu)$ its frequency-domain representation. The squared-absolute value $|f(t)|^2$ is called the **signal power**, and $|F(\nu)|^2$ is the energy spectral density. If $|F(\nu)|^2$ extends over a wide frequency range, the signal is said to have a wide bandwidth.

Properties of the Fourier Transform

Some important properties of the Fourier transform are provided below. These properties can be proved by direct application of the definitions (A.1-1) and (A.1-2) (see any of the books in the reading list).

- *Linearity.* The Fourier transform of the sum of two functions is the sum of their Fourier transforms.
- *Scaling.* If $f(t)$ has a Fourier transform $F(\nu)$, and τ is a real scaling factor, then $f(t/\tau)$ has a Fourier transform $|\tau|F(\tau\nu)$. This means that if $f(t)$ is scaled by a factor τ, its Fourier transform is scaled by a factor $1/\tau$. For example, if $\tau > 1$, then $f(t/\tau)$ is a stretched version of $f(t)$, whereas $F(\tau\nu)$ is a compressed version of $F(\nu)$. The Fourier transform of $f(-t)$ is $F(-\nu)$.
- *Time Translation.* If $f(t)$ has a Fourier transform $F(\nu)$, the Fourier transform of $f(t-\tau)$ is $\exp(-j2\pi\nu\tau)F(\nu)$. Thus delay by time τ is equivalent to multiplication of the Fourier transform by a phase factor $\exp(-j2\pi\nu\tau)$.
- *Frequency Translation.* If $F(\nu)$ is the Fourier transform of $f(t)$, the Fourier transform of $f(t)\exp(j2\pi\nu_0 t)$ is $F(\nu-\nu_0)$. Thus multiplication by a harmonic function of frequency ν_0 is equivalent to shifting the Fourier transform to a higher frequency ν_0.
- *Symmetry.* If $f(t)$ is real, then $F(\nu)$ has Hermitian symmetry [i.e., $F(-\nu) = F^*(\nu)$]. If $f(t)$ is real and symmetric, then $F(\nu)$ is also real and symmetric.
- *Convolution Theorem.* If the Fourier transforms of $f_1(t)$ and $f_2(t)$ are $F_1(\nu)$ and $F_2(\nu)$, respectively, the inverse Fourier transform of the product

$$F(\nu) = F_1(\nu)F_2(\nu) \tag{A.1-3}$$

is

$$\boxed{f(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)\,d\tau.} \tag{A.1-4}$$

Convolution

The operation defined in (A.1-4) is known as the convolution of $f_1(t)$ with $f_2(t)$. Convolution in the time domain is therefore equivalent to multiplication in the Fourier domain.

- *Correlation Theorem.* The correlation between two complex functions is defined as

$$\boxed{f(t) = \int_{-\infty}^{\infty} f_1^*(\tau)f_2(t+\tau)\,d\tau.} \tag{A.1-5}$$

Correlation

The Fourier transforms of $f_1(t)$, $f_2(t)$, and $f(t)$ are related by

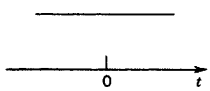
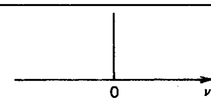
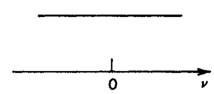
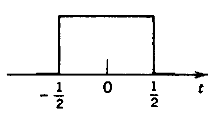
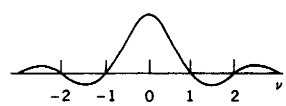
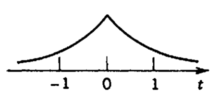
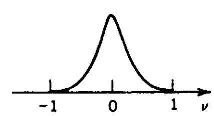
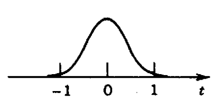
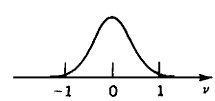
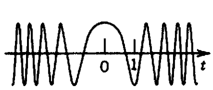
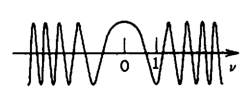
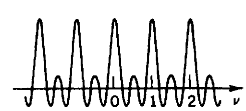
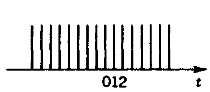
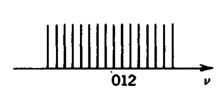
$$F(\nu) = F_1^*(\nu)F_2(\nu). \tag{A.1-6}$$

- *Parseval's Theorem.* The signal energy, which is the integral of the signal power $|f(t)|^2$, equals the integral of the energy spectral density $|F(\nu)|^2$, so that

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(\nu)|^2\, d\nu.} \tag{A.1-7}$$

Parseval's Theorem

TABLE A.1-1 Selected Functions and Their Fourier Transforms

Function		$f(t)$	$F(\nu)$	
Uniform		1	$\delta(\nu)$	
Impulse		$\delta(t)$	1	
Rectangular		$\mathrm{rect}(t)$	$\mathrm{sinc}(\nu)$	
Exponential[a]		$\exp(-\lvert t\rvert)$	$\dfrac{2}{1+(2\pi\nu)^2}$	
Gaussian		$\exp(-\pi t^2)$	$\exp(-\pi\nu^2)$	
Chirp[b]		$\exp(j\pi t^2)$	$e^{j\pi/4}\exp(-j\pi\nu^2)$	
Sum of $M=2S+1$ impulses		$\sum_{n=-S}^{S} \delta(t-n)$	$\dfrac{\sin(M\pi\nu)}{\sin(\pi\nu)}$	
Infinite sum of impulses		$\sum_{n=-\infty}^{\infty} \delta(t-n)$	$\sum_{n=-\infty}^{\infty} \delta(\nu-n)$	

[a]The double-sided exponential function is shown. The Fourier transform of the single-sided exponential, $f(t) = \exp(-t)$ with $t \geq 0$, is $F(\nu) = 1/[1 + j2\pi\nu]$. Its magnitude is $1/[1 + (2\pi\nu)^2]^{1/2}$.
[b]The functions $\cos(\pi t^2)$ and $\cos(\pi\nu^2)$ are shown. The function $\sin(\pi t^2)$ is shown in Fig. 4.3-6.

Examples

The Fourier transforms of some important functions used in this book are listed in Table A.1-1. By use of the properties of linearity, scaling, delay, and frequency translation, the Fourier transforms of other functions may be readily obtained. In this table:

- $\mathrm{rect}(t) \equiv 1$ for $|t| \le \frac{1}{2}$, and is 0 elsewhere, i.e., it is a pulse of unit height and unit width centered about $t = 0$.
- $\delta(t)$ is the impulse function (Dirac delta function), defined as $\delta(t) = \lim_{\alpha \to \infty} \alpha\, \mathrm{rect}(\alpha t)$. It is the limit of a rectangular pulse of unit area as its width approaches zero (so that its height approaches infinity).
- $\mathrm{sinc}(t) = \sin(\pi t)/(\pi t)$ is a symmetric function with a peak value of 1.0 at $t = 0$ and zeros at $t = \pm 1, \pm 2, \ldots$.

A.2. Time Duration and Spectral Width

It is often useful to have a measure of the width of a function. The width of a function of time $f(t)$ is its time duration and the width of its Fourier transform $F(\nu)$ is its spectral width (or bandwidth). Since there is no unique definition for the width, a plethora of definitions are in use. *All definitions, however, share the property that the spectral width is inversely proportional to the temporal width, in accordance with the scaling property of the Fourier transform.* The following definitions are used at different places in this book.

The Root-Mean-Square Width

The *root-mean-square (rms) width* σ_t of a nonnegative real function $f(t)$ is defined by

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} (t - \bar{t})^2 f(t)\, dt}{\int_{-\infty}^{\infty} f(t)\, dt}, \qquad \text{where } \bar{t} = \frac{\int_{-\infty}^{\infty} t f(t)\, dt}{\int_{-\infty}^{\infty} f(t)\, dt}. \tag{A.2-1}$$

If $f(t)$ represents a mass distribution (t representing position), then $\bar{t}$ represents the centroid and σ_t the radius of gyration. If $f(t)$ is a probability density function, these quantities represent the mean and standard deviation, respectively. As an example, the *Gaussian function* $f(t) = \exp(-t^2/2\sigma_t^2)$ has an rms width σ_t. Its Fourier transform is given by $F(\nu) = (1/\sqrt{2\pi}\,\sigma_\nu)\exp(-\nu^2/2\sigma_\nu^2)$, where

$$\sigma_\nu = \frac{1}{2\pi\sigma_t} \tag{A.2-2}$$

is the rms spectral width.

This definition is not appropriate for functions with negative or complex values. For such functions the rms width of the squared-absolute value $|f(t)|^2$ is used,

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} (t - \bar{t})^2 |f(t)|^2\, dt}{\int_{-\infty}^{\infty} |f(t)|^2\, dt}, \qquad \text{where } \bar{t} = \frac{\int_{-\infty}^{\infty} t |f(t)|^2\, dt}{\int_{-\infty}^{\infty} |f(t)|^2\, dt}.$$

We call this version of σ_t the *power-rms width*.

With the help of the Schwarz inequality, it can be shown that the product of the power rms widths of an arbitrary function $f(t)$ and its Fourier transform $F(\nu)$ must be greater than $1/4\pi$,

$$\boxed{\sigma_t \sigma_\nu \geq \frac{1}{4\pi},} \tag{A.2-3}$$

Duration–Bandwidth
Reciprocity Relation

where the spectral width σ_ν is defined by

$$\sigma_\nu^2 = \frac{\int_{-\infty}^{\infty} (\nu - \bar{\nu})^2 |F(\nu)|^2 \, d\nu}{\int_{-\infty}^{\infty} |F(\nu)|^2 \, d\nu}, \qquad \text{where } \bar{\nu} = \frac{\int_{-\infty}^{\infty} \nu |F(\nu)|^2 \, d\nu}{\int_{-\infty}^{\infty} |F(\nu)|^2 \, d\nu}.$$

Thus the time duration and the spectral width cannot simultaneously be made arbitrarily small.

The *Gaussian function* $f(t) = \exp(-t^2/4\sigma_t^2)$, for example, has a power-rms width σ_t. Its Fourier transform is also a Gaussian function, $F(\nu) = (1/2\sqrt{\pi}\,\sigma_\nu)\exp(-\nu^2/4\sigma_\nu^2)$, with power-rms width

$$\sigma_\nu = \frac{1}{4\pi\sigma_t}. \tag{A.2-4}$$

Since $\sigma_t\sigma_\nu = 1/4\pi$, the Gaussian function has the minimum permissible value of the duration–bandwidth product. In terms of the angular frequency $\omega = 2\pi\nu$,

$$\sigma_t \sigma_\omega \geq \tfrac{1}{2}. \tag{A.2-5}$$

If the variables t and ω, which usually describe time and angular frequency (rad/s), are replaced with the position variable x and the spatial angular frequency k (rad/m), respectively, then (A.2-5) translates to

$$\sigma_x \sigma_k \geq \tfrac{1}{2}. \tag{A.2-6}$$

In quantum mechanics, the position x of a particle is described by the wavefunction $\psi(x)$, and the wavenumber k is described by a function $\phi(k)$ which is the Fourier transform of $\psi(x)$. The uncertainties of x and k are the rms widths of the probability densities $|\psi(x)|^2$ and $|\phi(k)|^2$, respectively, so that σ_x and σ_k are interpreted as the uncertainties of position and wavenumber. Since the particle momentum is $p = \hbar k$ (where $\hbar = h/2\pi$ and h is Planck's constant), the position–momentum uncertainty product satisfies the inequality

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2},} \tag{A.2-7}$$

Heisenberg
Uncertainty Relation

which is known as the **Heisenberg uncertainty relation.**

The Power-Equivalent Width

The power-equivalent width of a signal $f(t)$ is the signal energy divided by the peak signal power. If $f(t)$ has its peak value at $t = 0$, for example, then the power-equivalent width is

$$\tau = \int_{-\infty}^{\infty} \frac{|f(t)|^2}{|f(0)|^2}\, dt. \tag{A.2-8}$$

The *double-sided exponential function* $f(t) = \exp(-|t|/\tau)$, for example, has a power-equivalent width τ, as does the Gaussian function $f(t) = \exp(-\pi t^2/2\tau^2)$. This definition is used in Sec. 10.1, where the coherence time of light is defined as the power-equivalent width of the complex degree of temporal coherence.

The power-equivalent spectral width is similarly defined by

$$\mathscr{B} = \int_{-\infty}^{\infty} \frac{|F(\nu)|^2}{|F(0)|^2}\, d\nu. \tag{A.2-9}$$

If $f(t)$ is real, so that $|F(\nu)|^2$ is symmetric, and if it has its peak value at $\nu = 0$, the power-equivalent spectral width is usually defined as the positive-frequency width,

$$B = \int_{0}^{\infty} \frac{|F(\nu)|^2}{|F(0)|^2}\, d\nu. \tag{A.2-10}$$

In the case $F(\nu) = \tau/(1 + j2\pi\nu\tau)$, for example,

$$B = \frac{1}{4\tau}. \tag{A.2-11}$$

This definition is used in Sec. 17.5A to describe the bandwidth of photodetector circuits susceptible to photon and circuit noise (see also Problem 17.5-5).

Using Parseval's theorem (A.1-7) and the relation $F(0) = \int_{-\infty}^{\infty} f(t)\, dt$, (A.2-10) may be written in the form

$$B = \frac{1}{2T}, \tag{A.2-12}$$

where

$$T = \frac{\left[\int_{-\infty}^{\infty} f(t)\, dt\right]^2}{\int_{-\infty}^{\infty} f^2(t)\, dt} \tag{A.2-13}$$

is yet another definition of the time duration [the square of the area under $f(t)$ divided by the area under $f^2(t)$]. In this case, the duration–bandwidth product $BT = \frac{1}{2}$.

The 1/e-, Half-Maximum, and 3-dB Widths

Another type of measure of the width of a function is its duration at a prescribed fraction of its maximum value ($1/\sqrt{2}$, $1/2$, $1/e$, or $1/e^2$, as examples). Either the half-width or the full width on both sides of the peak is used. Two commonly encountered measures are the full-width at half-maximum (FWHM) and the half-width at $1/\sqrt{2}$-maximum, called the 3-dB width. The following are three important examples:

- The *exponential function* $f(t) = \exp(-t/\tau)$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$, which describes the response of a number of electrical and optical systems, has a $1/e$-maximum width $\Delta t_{1/e} = \tau$. The magnitude of its Fourier transform $F(\nu) = \tau/(1 + j2\pi\nu\tau)$ has a 3-dB width (half-width at $1/\sqrt{2}$-maximum)

$$\Delta\nu_{3\text{-dB}} = \frac{1}{2\pi\tau}. \tag{A.2-14}$$

- The *double-sided exponential function* $f(t) = \exp(-|t|/\tau)$ has a half-width at $1/e$-maximum $\Delta t_{1/e} = \tau$. Its Fourier transform $F(\nu) = 2\tau/[1 + (2\pi\nu\tau)^2]$, known as the *Lorentzian distribution*, has a full-width at half-maximum

$$\Delta\nu_{\text{FWHM}} = \frac{1}{\pi\tau}, \tag{A.2-15}$$

  and is usually written in the form $F(\nu) = (\Delta\nu/2\pi)/[\nu^2 + (\Delta\nu/2)^2]$ where $\Delta\nu = \Delta\nu_{\text{FWHM}}$. The Lorentzian distribution describes the spectrum of certain light emissions (see Sec. 12.2D).
- The *Gaussian function* $f(t) = \exp(-t^2/2\tau^2)$ has a full-width at $1/e$-maximum $\Delta t_{1/e} = 2\sqrt{2}\,\tau$. Its Fourier transform $F(\nu) = \sqrt{2\pi}\,\tau \exp(-2\pi^2\tau^2\nu^2)$ has a full-width at $1/e$-maximum

$$\Delta\nu_{1/e} = \frac{\sqrt{2}}{\pi\tau} \tag{A.2-16}$$

  and a full-width at half-maximum

$$\Delta\nu_{\text{FWHM}} = \frac{(2\ln 2)^{1/2}}{\pi\tau}, \tag{A.2-17}$$

  so that

$$\Delta\nu_{\text{FWHM}} = (\ln 2)^{1/2}\,\Delta\nu_{1/e} = 0.833\,\Delta\nu_{1/e}. \tag{A.2-18}$$

  The Gaussian function is also used to describe the spectrum of certain light emissions (see Sec. 12.2D) as well as to describe the spatial distribution of light beams (see Sec. 3.1).

A.3. Two-Dimensional Fourier Transform

We now consider a function of two variables $f(x, y)$. If x and y represent the coordinates of a point in a two-dimensional space, then $f(x, y)$ represents a spatial pattern (e.g., the optical field in a given plane). The harmonic function $F \exp[-j2\pi(\nu_x x + \nu_y y)]$ is regarded as a building block from which other functions may be composed by superposition. The variables ν_x and ν_y represent spatial frequencies in the x and y directions, respectively. Since x and y have units of length (mm), ν_x and ν_y have units of cycles/mm, or lines/mm. Examples of two-dimensional harmonic functions are illustrated in Fig. A.3-1.

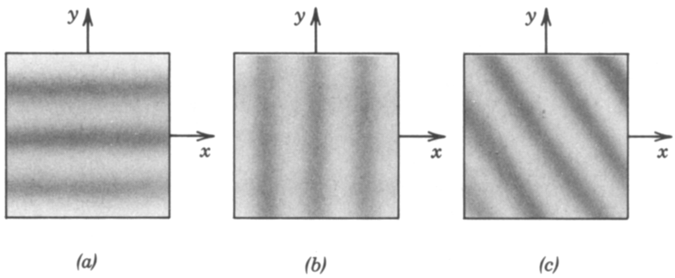


Figure A.3-1 The real part $|F|\cos[2\pi\nu_x x + 2\pi\nu_y y + \arg\{F\}]$ of a two-dimensional harmonic function: (a) $\nu_x = 0$; (b) $\nu_y = 0$; (c) arbitrary case. For this illustration we have assumed that $\arg\{F\} = 0$ so that dark and white points represent positive and negative values of the function, respectively.

The Fourier theorem may be generalized to functions of two variables. A function $f(x, y)$ may be decomposed as a superposition integral of harmonic functions of x and y,

$$\boxed{f(x, y) = \iint_{-\infty}^{\infty} F(\nu_x, \nu_y) \exp\left[-j2\pi(\nu_x x + \nu_y y)\right] d\nu_x\, d\nu_y} \tag{A.3-1}$$

Inverse Fourier Transform

where the coefficients $F(\nu_x, \nu_y)$ are determined by use of the two-dimensional Fourier transform

$$\boxed{F(\nu_x, \nu_y) = \iint_{-\infty}^{\infty} f(x, y) \exp\left[j2\pi(\nu_x x + \nu_y y)\right] dx\, dy.} \tag{A.3-2}$$

Fourier Transform

Our definitions of the two- and one-dimensional Fourier transforms, (A.3-2) and (A.1-2) respectively, differ in the sign of the exponent. The choice of this sign is, of course, arbitrary, as long as opposite signs are used in the Fourier and inverse Fourier transforms. In this book we have adopted the convention that $\exp(j2\pi\nu t)$ has positive temporal frequency ν, whereas $\exp[-j2\pi(\nu_x x + \nu_y y)]$ has positive spatial frequencies ν_x and ν_y. We have elected to use different signs in the spatial (two-dimensional) and temporal (one-dimensional) cases in order to simplify the notation used in Chap. 4 (Fourier optics), in which the traveling wave $\exp(+j2\pi\nu t)\exp[-j(k_x x + k_y y + k_z z)]$ has temporal and spatial dependences with opposite signs.

Properties

The two-dimensional Fourier transform has many properties that are obvious generalizations of those of the one-dimensional Fourier transform, and others that are unique to the two-dimensional case:

- *Convolution Theorem.* If $f(x, y)$ is the two-dimensional convolution of two functions $f_1(x, y)$ and $f_2(x, y)$ with Fourier transforms $F_1(\nu_x, \nu_y)$ and $F_2(\nu_x, \nu_y)$, respectively, so that

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x', y') f_2(x - x', y - y')\, dx'dy', \tag{A.3-3}$$

then the Fourier transform of $f(x, y)$ is

$$F(\nu_x, \nu_y) = F_1(\nu_x, \nu_y) F_2(\nu_x, \nu_y). \tag{A.3-4}$$

Thus, as in the one-dimensional case, convolution in the space domain is equivalent to multiplication in the Fourier domain.

- *Separable Functions.* If $f(x, y) = f_x(x) f_y(y)$ is the product of one function of x and another of y, then its two-dimensional Fourier transform is a product of one function of ν_x and another of ν_y. The two-dimensional Fourier transform of $f(x, y)$ is then related to the product of the one-dimensional Fourier transforms of $f_x(x)$ and $f_y(y)$ by $F(\nu_x, \nu_y) = F_x(-\nu_x) F_y(-\nu_y)$. For example, the Fourier transform of $\delta(x - x_0)\delta(y - y_0)$, which represents an impulse located at (x_0, y_0), is the harmonic function $\exp[j2\pi(\nu_x x_0 + \nu_y y_0)]$; and the Fourier transform of the Gaussian function $\exp[-\pi(x^2 + y^2)]$ is the Gaussian function $\exp[-\pi(\nu_x^2 + \nu_y^2)]$; and so on.
- *Circularly Symmetric Functions.* The Fourier transform of a circularly symmetric function is also circularly symmetric. For example, the Fourier transform of

$$f(x, y) = \begin{cases} 1, & (x^2 + y^2)^{1/2} \leq 1 \\ 0, & \text{otherwise}, \end{cases} \tag{A.3-5}$$

denoted by the symbol circ(x, y) and known as the *circ function*, is

$$F(\nu_x, \nu_y) = \frac{J_1(2\pi\nu_\rho)}{\nu_\rho}, \qquad \nu_\rho = (\nu_x^2 + \nu_y^2)^{1/2}, \tag{A.3-6}$$

where J_1 is the Bessel function of order 1. These functions are illustrated in Fig. A.3-2.

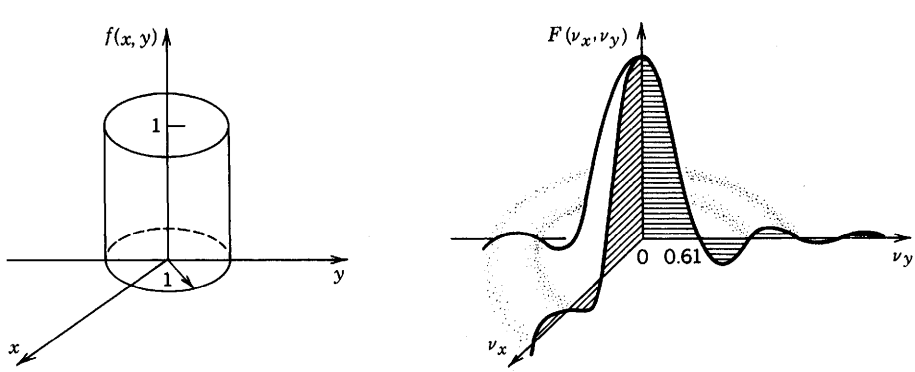


Figure A.3-2 The circ function and its two-dimensional Fourier transform.

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