Elements of Photonics, Volume I: In Free Space and Special Media. Keigo Iizuka
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APPENDIX B

WHY THE ANALYTIC SIGNAL METHOD IS NOT APPLICABLE TO THE NONLINEAR SYSTEM

Let us review the common approach to solving an *RL* network by the method of the analytic signal (phasor). A rule that electrical engineering students use is to replace $V \cos \omega t$ by $Ve^{j\omega t}$, but not by $\frac{1}{2}(Ve^{j\omega t} + \text{c.c.})$, which is the mathematically exact equivalent, and to "*take the real part*" of the final solution instead of both real and imaginary parts of the solution.

Method I This is how an electrical engineering student solves the problem of the *RL* circuit shown in Fig. B.1. The differential equation for the analytic signal $Ve^{j\omega t}$ is

$$L\frac{di_a}{dt} + Ri_a = Ve^{j\omega t} \tag{B.1}$$

The method of undetermined coefficients is used. Let an assumed solution be

$$i_a(t) = Ie^{j\omega t} \tag{B.2}$$

Inserting Eq. (B.2) into (B.1) gives

$$(j\omega L + R)I = V \tag{B.3}$$

$$I = \frac{V}{j\omega L + R} \tag{B.4}$$

$$= \frac{V}{\sqrt{(\omega L)^2 + R^2}} e^{-j\phi} \tag{B.5}$$

where

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

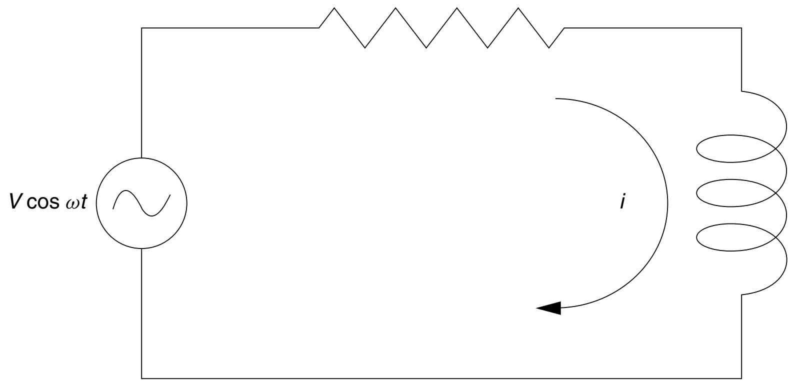

Figure B.1 *RL* circuit driven by $V\cos\omega t$.

From Eqs. (B.2) and (B.5),

$$i_a(t) = \frac{V}{\sqrt{(\omega L)^2 + R^2}} e^{j(\omega t - \phi)} \tag{B.6}$$

The final step is to take the real part of Eq. (B.6). The answer is

$$i(t) = \text{Re } i_a(t) = \frac{V}{\sqrt{(\omega L)^2 + R^2}} \cos(\omega t - \phi) \tag{B.7}$$

Method II The same problem will be solved by driving with

$$v(t) = V\cos\omega t \tag{B.8}$$

Equation (B.8) is rewritten as

$$v(t) = \frac{V}{2}(e^{j\omega t} + e^{-j\omega t}) \tag{B.9}$$

Now, since the differential equation (B.1) is linear, the law of superposition holds true. The solution will be obtained by adding i_1 when driven by $(V/2)e^{j\omega t}$ and i_2 when driven by $(V/2)e^{-j\omega t}$.

The value of i_1 is immediately obtained by replacing V by $V/2$ in Eq. (B.6) as

$$i_1(t) = \frac{V}{2}\frac{1}{\sqrt{(\omega L)^2 + R^2}} e^{j(\omega t - \phi)} \tag{B.10}$$

and

$$i_1(t) = \tfrac{1}{2} i_a \tag{B.11}$$

Next, the circuit is driven by $(V/2)e^{-j\omega t}$. An assumed solution

$$i_2(t) = Ie^{-j\omega t} \tag{B.12}$$

is put into Eq. (B.1):

$$I = \frac{V}{2}\left(\frac{1}{-j\omega L + R}\right) \tag{B.13}$$

Inserting Eq. (B.13) into (B.12) gives

$$i_2(t) = \frac{V}{2}\frac{1}{\sqrt{(\omega L)^2 + R^2}}e^{-j(\omega t - \phi)} \tag{B.14}$$

and

$$i_2(t) = \tfrac{1}{2}i_a^* \tag{B.15}$$

Using the law of superposition, the solution of Eq. (B.9) is

$$i(t) = i_1(t) + i_2(t) = \frac{V}{\sqrt{(\omega L)^2 + R^2}}\cos(\omega t - \phi) \tag{B.16}$$

Thus, it has been proved that Methods I and II provide the same answer.

Let's examine the key points that made the two answers the same. The law of superposition gave

$$i(t) = i_1(t) + i_2(t) \tag{B.17}$$

From Eqs. (B.11) and (B.15),

$$i(t) = \tfrac{1}{2}[i_a(t) + i_a^*(t)] \tag{B.18}$$

Since Re $z = \frac{1}{2}(z + z^*)$ the operation of Eq. (B.18) is identical with the operation of "taking the real part of $i_a(t)$." Thus,

$$i(t) = \text{Re } i_a(t) \tag{B.19}$$

The equality of the solutions by the two methods cannot be realized if the law of superposition expressed by Eq. (B.17) is not true. The law of superposition, however, is realized only when the differential equation is linear, as explained below. This leads to the conclusion that the method of analytic signal cannot be used for solving a nonlinear differential equation, and expressions such as Eq. (B.8) or (B.9) have to be used to express the driving voltage.

Finally, we give a proof that the law of superposition is realized only when the differential equation is linear. The proof is made using the same differential equation. When the circuit is driven by v_1, the solution i_1 has to satisfy

$$L\frac{di_1}{dt} + Ri_1 = v_1 \tag{B.20}$$

whereas when driven by v_2, the solution i_2 has to satisfy

$$L\frac{di_2}{dt} + Ri_2 = v_2 \tag{B.21}$$

Now, let the solution be i when the circuit is excited by $v_1 + v_2$:

$$L\frac{di}{dt} + Ri = v_1 + v_2 \tag{B.22}$$

Inserting Eqs. (B.20) and (B.21) into the right-hand side of Eq. (B.22) gives

$$L\frac{di}{dt} + Ri = L\frac{d}{dt}(i_1 + i_2) + R(i_1 + i_2) \tag{B.23}$$

Comparing both sides of Eq. (B.23), we see that

$$i = i_1 + i_2 \tag{B.24}$$

is the solution when the circuit is driven by $v_1 + v_2$.

In other words, when i_1 is the response of stimulus v_1 and i_2 is the response of stimulus v_2, the response of the two stimuli together, $v_1 + v_2$, is the addition of the two responses, $i_1 + i_2$. This is the law of superposition, which holds true only when the differential equation is linear; meaning that L and R, which are the coefficients of di/dt and i in Eq. (B.1), are not a function of i.

Let us now examine the case of the nonlinear differential equation. As an example of a nonlinear differential equation, let us consider the case when the value of the resistance R is changed due to the generated heat. Such a system may be represented approximately by the differential equation

$$L\frac{di}{dt} + Ri^3 = v \tag{B.25}$$

When the circuit is driven by v_1, the current i_1 has to satisfy

$$L\frac{di_1}{dt} + Ri_1^3 = v_1 \tag{B.26}$$

Similarly, when it is driven by v_2, i_2 has to satisfy

$$L\frac{di_2}{dt} + Ri_2^3 = v_2 \tag{B.27}$$

Next, when v_1 and v_2 are applied simultaneously,

$$L\frac{di}{dt} + Ri^3 = v_1 + v_2 \tag{B.28}$$

Inserting Eqs. (B.25) and (B.26) into the right-hand side of Eq. (B.28) gives

$$L\frac{di}{dt} + Ri^3 = \frac{d}{dt}(i_1 + i_2) + R(i_1 + i_2)^3 - 3R(i_1^2 i_2 + i_1 i_2^2) \tag{B.29}$$

Comparing both sides of Eq. (B.29), we see that

$$i = i_1 + i_2$$

is no longer the solution of Eq. (B.28). The law of superposition no longer holds true with a nonlinear differential equation.