Elements of Photonics, Volume I: In Free Space and Special Media. Keigo Iizuka

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PROPAGATION OF LIGHT IN ANISOTROPIC CRYSTALS

Electrooptic materials change their indices of refraction when an external field is applied. These materials have many applications and are used in integrated optic devices such as phase or amplitude modulators, optical switches, optical couplers, optical deflectors, and harmonic frequency generators. They are extremely important materials in integrated optics. Both electrooptic and acoustooptic media, however, are generally anisotropic and the optical properties in one direction are different from those in other directions.

The manner of light propagation in anisotropic media is more complicated than propagation in isotropic media. A good understanding of how light propagates in electrooptic materials is essential in order to avoid undesirable effects or to take advantage of special properties to build more sophisticated devices.

With the right approach, the analysis is not all that complicated. A good starting point is to allow for the fact that only two types of waves exist in the medium. The first type follows the usual laws of propagation and does not create new problems. However, the second wave does not necessarily propagate in the expected direction perpendicular to the wavefront, and herein lies the challenge. Added to this challenge is the fact that the index of refraction varies with the direction of propagation. Fortunately, the two types of waves are easily identifiable by the direction of polarization. Moreover, the two waves can be treated separately, one at a time, and the two results can be added to obtain the final result.

This chapter and the next are twin chapters. They can almost be considered as one chapter. Chapter 4 is devoted to the manner of propagation in a crystal for a given set of anisotropy parameters, while Chapter 5 looks at what kind of external control field is needed to change the anisotropy parameters so as to achieve the desired manner of light propagation.

Chapter 4 starts with a brief explanation of the meaning of polarization in a crystal and the qualitative difference between propagation in isotropic and anisotropic media. Then, differences are derived quantitatively using the wave equation. There are two popular methods of obtaining a graphical solution to the wave equation — the wavevector method and the indicatrix method. Each of these is explained. Using both graphical and analytical methods, the laws of refraction across the boundary between two anisotropic media are examined. As a practical application, in Chapter 6 various optical elements are described that manipulate the state of polarization of the light. These optical elements are used to construct devices such as polarizers, quarter-waveplates, and half-waveplates.

4.1 POLARIZATION IN CRYSTALS

Solids can broadly be classified as either crystalline or amorphous. Crystalline solids are characterized by a regular periodic sequence of the building block molecules. Amorphous solids such as glass or plastic have an irregular molecular arrangement. Optical, electrical, or mechanical properties of the crystalline solids are generally direction sensitive because of the anisotropy.

Figure 4.1a is a two-dimensional model of a crystalline solid. Coulomb's forces acting between the nucleus and the electrons, or among the electrons themselves, are stably balanced within the constraint of overall electrical neutrality and minimum energy. When an external electric field is applied to the crystal by placing the crystal between two capacitor plates, the entire electron pattern is translated toward the positive electrode, as indicated in Fig. 4.1b. The translation of the electron cloud creates a positive excess charge layer on the top surface of the crystal, and a negative charge layer at the bottom surface. These surface charges establish an additional electric field in the crystal. Such a phenomenon is called polarization of the crystal due to the external field [1].

The switch of the battery in Fig. 4.1b is first turned on so as to charge up the capacitor plates with free electrons. The switch remains open so that the surface density of the free electrons on the plate remains fixed. Then, the crystal is inserted. The fields before and after the insertion of the crystal are compared to determine the degree of polarization of the crystal.

As shown in Fig. 4.1c, the direction of the field established by the polarization is opposite to that of the field due to the original charges on the electrode, resulting in a

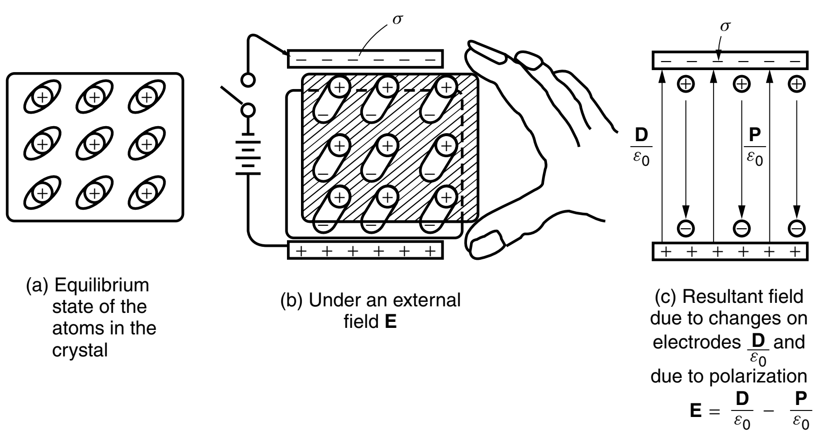


Figure 4.1 Change in the electric field **E** in the capacitor. First, the air-filled capacitor is charged by the battery and then the switch is left open during the experiment. The **E** field is measured before and after the insertion of the crystal.

reduction of the electric field between the capacitor plates. The field strength is reduced to ϵ_r^{-1} of the original field.

If the same crystal is rotated by 90° in the plane of the page, ϵ_r will be different for anisotropic crystalline structures. Chemists use polarization information to study molecular structure.

There are basically three different mechanisms for inducing polarization: (1) atomic polarization, (2) orientation polarization, and (3) space-charge polarization. These are illustrated in Fig. 4.2.

A typical example of atomic polarization is the polarization associated with a hydrogen atom shown in Fig. 4.2a. Without the external electric field, the nucleus is surrounded by a spherically symmetric electron cloud and the "center of gravity" of the negative charges coincides with that of the positive charges and no dipole moment exists. When, however, an external electric field is applied, both the nucleus and the electron cloud shift away from each other along the direction of the external electric field and the centers of gravity of the positive and negative charges no longer coincide. A dipole moment is created, resulting in the polarization.

Some molecules posses a permanent dipole moment, which is already present even before the external electric field is applied. A good example of this kind of molecule is the water molecule. The atoms of the water molecule are arranged in a triangular shape, as shown in Fig. 4.2b. The "center of gravity" of the positive charges of the three nuclei

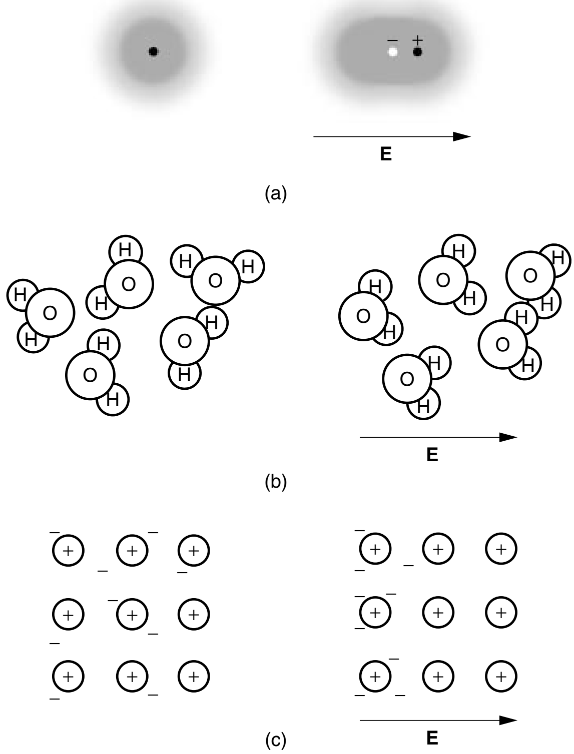


Figure 4.2 Mechanisms of polarization. (a) Atomic polarization. (b) Orientation polarization. (c) Space-charge polarization.

would be somewhere inside the triangle. The location of the center of the three electron clouds (two belonging to the hydrogen atoms and one belonging to the oxygen atom) will be at the same location as the center of gravity of the nuclei, provided the electron clouds are perfectly spherically symmetric. However, in reality, this symmetry is broken by the binding process. When a molecular bond is formed, electrons are displaced toward the stronger binding atom. In the water molecule, oxygen is the stronger binding atom, so that electrons are displaced toward the oxygen atom. In a water molecule, the centers of gravity of the positive and negative charges no longer coincide, and a dipole moment exists even without an external electric field. Ironically, this permanent dipole moment is 10^3–10^4 times larger than that of the atomic dipole moment of a hydrogen atom under an external electric field of 3×10^4 V/cm, which is the maximum external electric field that can be applied without causing electrical arcing [2].

Both atomic and orientation polarizations are produced by the displacement or reorientation of the bound charges. The space-charge polarization shown in Fig. 4.2(c), however, is produced by the buildup of traveling charge carriers within a specific volume or on the surface at an interface.

Note in Fig. 4.1 that the direction of the electric field $\mathbf{P}/\epsilon_0$ due to the polarization is opposite to that of $\mathbf{D}/\epsilon_0$. The resultant field $\mathbf{E}$ including the contribution from the original charge on the electrodes is

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} - \frac{\mathbf{P}}{\epsilon_0} \tag{4.1}$$

Hence,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{4.2}$$

In the linear range of polarization and in an isotropic medium, $\mathbf{P}$ is proportional to $\mathbf{E}$ and

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} \tag{4.3}$$

where χ is the electric susceptibility. Inserting Eq. (4.3) into (4.2) gives

$$\mathbf{D} = \epsilon_0 n^2 \mathbf{E} \tag{4.4}$$

where

$$\epsilon_r = n^2 = (1 + \chi) \tag*{(4.5)*}$$

4.2 SUSCEPTIBILITY OF AN ANISOTROPIC CRYSTAL

The susceptibility χ of an anisotropic crystal varies according to the direction. For an anisotropic crystal, χ in Eq. (4.3) has to be replaced by a general form of susceptibility tensor defined as

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \epsilon_0 \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{4.6}$$

* ϵ_r of water is known to be 81. Does this mean $n = 9$? It is important that ϵ_r and n are measured at the same frequency. $\epsilon_r = 81$, however, is the value measured at 114 MHz and $n = 9$ is not applicable at optical frequencies.

The values of the tensor elements depend on the choice of coordinates with respect to the crystal axis. In general, it is possible to choose the coordinates so that $\overleftrightarrow{\chi}$ becomes a diagonal matrix [2–4].

If the new coordinates are chosen in the directions parallel to the eigenvectors, the susceptibility tensor converts into a much more manageable form. This form is

$$\overleftrightarrow{\chi} = \begin{bmatrix} \chi'_{11} & 0 & 0 \\ 0 & \chi'_{22} & 0 \\ 0 & 0 & \chi'_{33} \end{bmatrix} \tag{4.7}$$

where χ'_{ii} are called the principal susceptibilities. The coordinate axes for which the susceptibility tensor is reduced to diagonal form are called the principal axes. The corresponding index of refraction tensor becomes

$$\mathbf{D} = \epsilon_0 \begin{bmatrix} n_\alpha^2 & 0 & 0 \\ 0 & n_\beta^2 & 0 \\ 0 & 0 & n_\gamma^2 \end{bmatrix} \mathbf{E} \tag{4.8}$$

where n_α, n_β, and n_γ are called the principal refractive indices. Crystals with two identical principal indices are called uniaxial crystals, and those with three different principal indices are called biaxial crystals.

Analysis with biaxial crystals is significantly more complicated than with uniaxial crystals. Since most of the optically transparent crystals that are used for electrooptic devices are uniaxial, this chapter will concentrate on uniaxial crystals.

With a uniaxial crystal it is always possible to find an axis of rotational symmetry and such an axis is called the *optic axis* of the crystal. Sometimes it is casually called the *crystal axis* or simply the *c axis*. Uniaxial crystals have only one optic axis (taken as the z axis in this chapter), while biaxial crystals have two optic axes.

With the uniaxial crystal,

$$\begin{aligned} n_\alpha &= n_\beta = n_o \\ n_\gamma &= n_e \end{aligned} \tag{4.9}$$

Equation (4.8) becomes

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \epsilon_0 \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{4.10}$$

Solving for E_x, E_y, and E_z leads to

$$\epsilon_0 \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \dfrac{1}{n_o^2} & 0 & 0 \\ 0 & \dfrac{1}{n_o^2} & 0 \\ 0 & 0 & \dfrac{1}{n_e^2} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} \tag{4.11}$$

If $n_e > n_o$, the birefringence is called *positive birefringence*, and if $n_e < n_o$, it is called *negative birefringence*.

4.3 THE WAVE EQUATION IN AN ANISOTROPIC MEDIUM

The propagation of a lightwave in an anisotropic medium will be analyzed [5,6]. Maxwell's equations for isotropic and anisotropic media are quite similar. The only difference is that the dielectric constant is a tensor $\overleftrightarrow{\epsilon}_r$ in the anisotropic case, whereas it is a scalar in the isotropic case. Maxwell's equations are

$$\nabla \times \mathbf{H} = -j\omega\mathbf{D} \tag{4.12}$$

$$\nabla \times \mathbf{E} = j\omega\mu\mathbf{H} \tag{4.13}$$

where

$$\mathbf{D} = \epsilon_0 \overleftrightarrow{\epsilon}_r \mathbf{E} \tag{4.14}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{4.15}$$

and where ρ is the free-charge density. The curl operation in Eq. (4.13) with the use of Eq. (4.12) immediately leads to

$$\nabla \times \nabla \times \mathbf{E} = \omega^2\mu\mathbf{D} \tag{4.16}$$

Equation (4.16) is essentially a wave equation and is one of the most basic expressions governing propagation in any kind of media. It will be used often in this chapter to deal with anisotropy. Sometimes, a simple recast of Eq. (4.16) provides an answer.

In this entire chapter, the analysis is restricted to plane waves. The plane wave implies that the sole temporal and spatial dependency in $\mathbf{E}$ and $\mathbf{H}$ is a factor expressed by

$$e^{-j\omega t + j\mathbf{k}\cdot\mathbf{r}} \tag{4.17}$$

where

$$\mathbf{k} = k_x\hat{\mathbf{i}} + k_y\hat{\mathbf{j}} + k_z\hat{\mathbf{k}} \tag{4.18}$$

is the propagation vector.

With the assumption of Eq. (4.17), $(\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$ in the curl operation becomes $(jk_x,\ jk_y,\ jk_z)$ and the curl operation is simply

$$\nabla \times \mathbf{E} = j\mathbf{k} \times \mathbf{E} \tag{4.19}$$

Then, Eqs. (4.12) and (4.13) become

$$\mathbf{k} \times \mathbf{H} = -\omega\mathbf{D} \tag{4.20}$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H} \tag{4.21}$$

Equation (4.20) specifies that $\mathbf{D}$ is perpendicular to $\mathbf{H}$, and Eq. (4.21) specifies that $\mathbf{E}$ is also perpendicular to $\mathbf{H}$. Thus, both $\mathbf{E}$ and $\mathbf{D}$ have to be perpendicular to $\mathbf{H}$. As shown in Fig. 4.3, both $\mathbf{E}$ and $\mathbf{D}$ are in the same plane but this does not necessarily

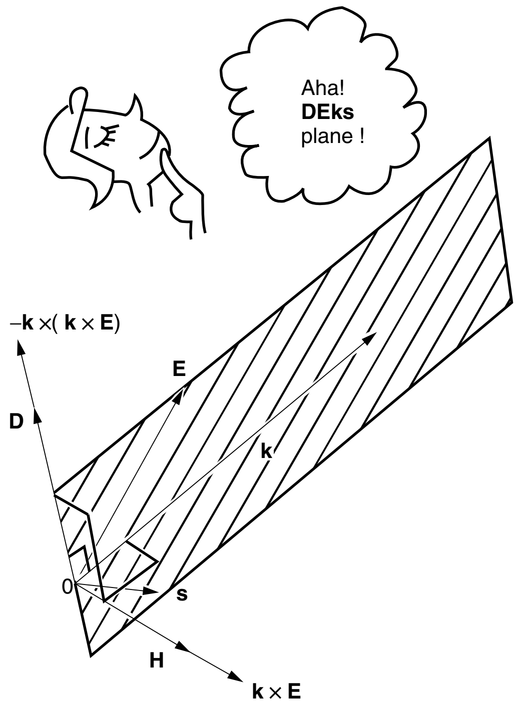


Figure 4.3 Orientations of **k**, **E**, **k** × **E**, **H**, −**k** × (**k** × **E**), and **D**.

mean that **E** and **D** are parallel. Equation (4.21) also specifies that **k** is perpendicular to **H**; thus, **E**, **D**, and **k** are all in the same plane. Equations (4.20) and (4.21) lead to

$$-\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \omega^2 \mu \mathbf{D} \tag{4.22}$$

In short, we can make the following observations: (1) **D**, **E**, and **k** are in the same plane, which is perpendicular to **H** and (2) **D** is perpendicular to **k**.

Multiplying both sides of Eq. (4.22) by ϵ_0 and using Eq. (4.11) gives

$$-\mathbf{k} \times \left[\mathbf{k} \times \left(\overset{\leftrightarrow}{\frac{1}{n^2}} \right) \mathbf{D} \right] = k_0^2 \mathbf{D} \tag{4.23}$$

Equation (4.23) is called the *generalized wave equation.*

4.4 SOLVING THE GENERALIZED WAVE EQUATION IN UNIAXIAL CRYSTALS

It will be shown that, in an anisotropic medium, only two waves are allowed to propagate. These waves are the *ordinary wave*, or *o-wave*, and the *extraordinary wave*, or *e-wave*. The existence of these two waves, and only these two waves, is the most important fact about propagation in anisotropic media.

4.4.1 Graphical Derivation of the Condition of Propagation in a Uniaxial Crystal

If the optic axis of the unaxial crystal is taken along the z direction, the medium is cylindrically symmetric with respect to the z axis. In this case, it is always possible to choose coordinates such that $k_y = 0$,

$$\mathbf{k} = \begin{bmatrix} k_x \\ 0 \\ k_z \end{bmatrix} \tag{4.24}$$

meaning that $\mathbf{k}$ is in the x–z plane.

$\mathbf{D}$, which is in the plane perpendicular to $\mathbf{k}$, is represented as

$$\mathbf{D} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} \tag{4.25}$$

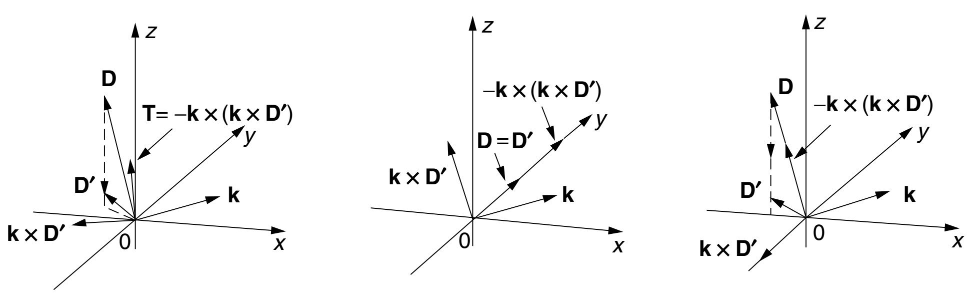
Only certain directions of the vector $\mathbf{D}$ within this plane will satisfy Eq. (4.23). A short proof follows. Let the vector $\mathbf{T}$ represent the left-hand side of Eq. (4.23).

$$\mathbf{T} = -\mathbf{k} \times \left[\mathbf{k} \times \left(\frac{\overleftrightarrow{1}}{n^2} \right) \mathbf{D} \right] \tag{4.26}$$

In order for $\mathbf{D}$ to satisfy the wave equation, $\mathbf{T}$ must point in the same direction as $\mathbf{D}$, which is the right-hand side of Eq. (4.23).

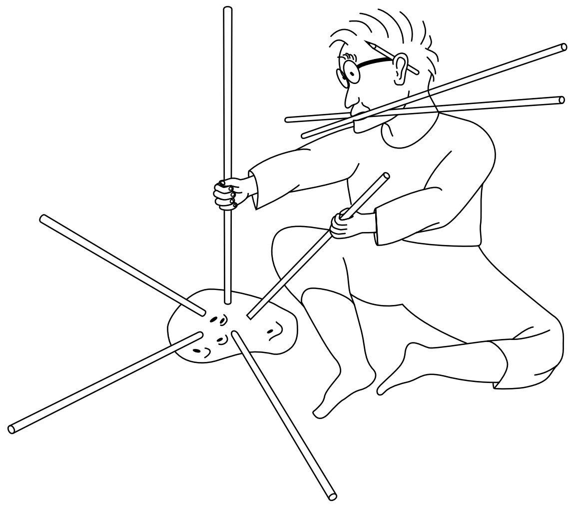
Referring to Fig. 4.4a, let us start with an arbitrary vector $\mathbf{D}$ except that it is in a plane perpendicular to $\mathbf{k}$ satisfying Eq. (4.20). The first operation of Eq. (4.26) is that

$$\mathbf{D}' = \left(\frac{\overleftrightarrow{1}}{n^2} \right) \mathbf{D} \tag{4.27}$$



(a) **D** in an arbitrary direction (b) o-wave −**k** × (**k** × **D**′) parallel to **D** (c) e-wave −**k** × (**k** × **D**′) parallel to **D**

Figure 4.4 Vector diagram for explaining that only o-waves and e-waves exist in an anisotropic medium.



By poking stick-like objects, such as chop sticks, knitting needles, or skewers, into a potato you can make your own study kit.

where $\mathbf{D}'$ is a quantity proportional to $\mathbf{E}$ from Eq. (4.11). With the hypothetical example of

$$\left(\frac{\overleftrightarrow{1}}{n^2}\right) = \frac{1}{n_o^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}$$

$\mathbf{D}'$ is expressed as

$$\mathbf{D}' = \frac{1}{n_o^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \frac{1}{n_o^2}\begin{bmatrix} D_x \\ D_y \\ \frac{1}{10}D_z \end{bmatrix} \tag{4.28}$$

Note that D_z in this example has been reduced to one-tenth of its original value.

Next, vector $\mathbf{k} \times \mathbf{D}'$ is in a plane perpendicular to $\mathbf{k}$. Finally, vector $\mathbf{T}$ in Eq. (4.26) is drawn. $\mathbf{T}$ is also in a plane perpendicular to $\mathbf{k}$. Thus, $\mathbf{D}$, $\mathbf{k} \times \mathbf{D}'$, and $\mathbf{T}$ are all in the same plane perpendicular to $\mathbf{k}$. In this plane, even though $\mathbf{T}$ is perpendicular to $\mathbf{k} \times \mathbf{D}'$, $\mathbf{D}$ is not necessarily perpendicular to $\mathbf{k} \times \mathbf{D}'$; thus, $\mathbf{T}$ and $\mathbf{D}$ are not necessarily in the same direction, and Eq. (4.23) is not satisfied in general. There are, however, two particular directions of $\mathbf{D}$ for which $\mathbf{T}$ and $\mathbf{D}$ point in the same direction to satisfy Eq. (4.23). The waves corresponding to these two particular directions of $\mathbf{D}$ are the ordinary wave (o-wave) and the extraordinary wave (e-wave). In the o-wave, $\mathbf{D}$ is polarized in the direction perpendicular to both $\mathbf{k}$ and the optic axis, namely, in the y direction and in the e-wave, $\mathbf{D}$ is polarized in the plane defined by $\mathbf{k}$ and the optic axis z, namely, in the x–z plane as shown in Figs. 4.4b and 4.4c, respectively.

We will now further explain how the o- and e-waves satisfy Eq. (4.23). First, with the o-wave, the operations of Eq. (4.26) are performed as shown in Fig. 4.4b. Since **D** is polarized in the y direction, Eq. (4.25) becomes

$$\mathbf{D} = \begin{bmatrix} 0 \\ D_y \\ 0 \end{bmatrix} \tag{4.29}$$

The operation of

$$\mathbf{D}' = \left(\frac{\overset{\leftrightarrow}{1}}{n^2} \right) \mathbf{D} \tag{4.30}$$

does not change the direction of **D**, and both **D** and **D**′ point in the same direction. Next, the vector $\mathbf{k} \times \mathbf{D}'$ is in the x–z plane, and both **k** and $\mathbf{k} \times \mathbf{D}'$ are in the same plane. Finally, the vector $\mathbf{T} = -\mathbf{k} \times (\mathbf{k} \times \mathbf{D}')$ points in the y direction, which is the direction of **D** and Eq. (4.23) is satisfied.

Next, with the e-wave, the operations of Eq. (4.26) are performed as shown in Fig. 4.4c. **D** is polarized in the x–z plane and is also perpendicular to **k**. Eq. (4.25) for the e-wave is

$$\mathbf{D} = \begin{bmatrix} D_x \\ 0 \\ D_z \end{bmatrix} \tag{4.31}$$

The first operation

$$\mathbf{D}' = \left(\frac{\overset{\leftrightarrow}{1}}{n^2} \right) \mathbf{D} \tag{4.32}$$

brings **D** to **D**′. Using the hypothetical example in Eq. (4.28), **D**′ becomes

$$\mathbf{D}' = \frac{1}{n_o^2} \begin{bmatrix} D_x \\ 0 \\ \frac{1}{10} D_z \end{bmatrix} \tag{4.33}$$

Even though the z component of **D**′ is reduced to one-tenth of D_z, **D**′ is still in the x–z plane. Next, the vector $\mathbf{k} \times \mathbf{D}'$ points in the $-y$ direction. What is important here is that $\mathbf{k} \times \mathbf{D}'$ always points in the $-y$ direction regardless of the value of D_z'.

The final cross product with **k** brings the vector

$$\mathbf{T} = -\mathbf{k} \times (\mathbf{k} \times \mathbf{D}')$$

into the x–z plane and also is perpendicular to **k**, which is the original direction of **D**. Thus, the directions of **T** and **D** match and the wave equation Eq. (4.23) is satisfied.

Thus, in conclusion, we have learned that there are two types of waves in an anisotropic crystal: o- and e-waves. The directions of polarization of **D** for both o- and e-waves are in the plane perpendicular to **k**, but that of the o-wave is the y direction and that of the e-wave is in the plane containing **k** and the optic axis. The directions of polarization of **D** for these waves are orthogonal to each other. No other directions of polarization are allowed. How then can light incident with an arbitrary direction of polarization be analyzed? The answer is to decompose the incident wave into two waves, each with the allowed direction of polarization. Propagation of each component wave is treated separately and summed at the exit of the crystal to obtain the expression of the transmitted light.

4.4.2 Analytical Representation of the Conditions of Propagation in a Uniaxial Crystal

In the previous section, a vector diagram argument was presented to show that only e-waves and o-waves propagate in a uniaxial crystal. Here, an analytical representation of each operation will be made. Let us start with an arbitrary **D** given by

$$\mathbf{D} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} \tag{4.34}$$

Since **D** and **k** are perpendicular to each other from Eq. (4.20),

$$\mathbf{k} \cdot \mathbf{D} = 0 \tag{4.35}$$

Inserting Eqs. (4.24) and (4.34) into Eq. (4.35) gives

$$D_z = -\frac{k_x}{k_z} D_x \tag{4.36}$$

Let $(\overleftrightarrow{1}/n^2)$ be

$$\left(\frac{\overleftrightarrow{1}}{n^2}\right) = \frac{1}{n_o^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{n_o^2}{n_e^2} \end{bmatrix} \tag{4.37}$$

Combining Eqs. (4.34), (4.36), and (4.37) gives

$$\left(\frac{\overleftrightarrow{1}}{n^2}\right) \mathbf{D} = \frac{1}{n_o^2} \begin{bmatrix} D_x \\ D_y \\ -\dfrac{n_o^2}{n_e^2} \dfrac{k_x}{k_z} D_x \end{bmatrix} \tag{4.38}$$

The value inside the square bracket of Eq. (4.26) is

$$\begin{aligned} \mathbf{k} \times \left[\left(\frac{\overleftrightarrow{1}}{n^2}\right) \mathbf{D}\right] &= \frac{1}{n_o^2} \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ k_x & 0 & k_z \\ D_x & D_y & -\dfrac{n_o^2}{n_e^2} \dfrac{k_x}{k_z} D_x \end{bmatrix} \\ &= \frac{1}{n_o^2} \begin{bmatrix} -k_z D_y \\ \left(k_z + \dfrac{n_o^2}{n_e^2} \dfrac{k_x^2}{k_z}\right) D_x \\ k_x D_y \end{bmatrix} \end{aligned} \tag{4.39}$$

Another cross product with $\mathbf{k}$ is performed to find $\mathbf{T}$ of Eq. (4.26):

$$\mathbf{T} = -\mathbf{k} \times \mathbf{k} \times \left[\left(\overleftrightarrow{\frac{1}{n^2}} \right) \mathbf{D} \right] = \frac{1}{n_o^2} \begin{bmatrix} \left(k_z^2 + \dfrac{n_o^2 k_x^2}{n_e^2} \right) D_x \\ k^2 D_y \\ -\dfrac{k_x}{k_z} \left(k_z^2 + \dfrac{n_o^2 k_x^2}{n_e^2} \right) D_x \end{bmatrix} \tag{4.40}$$

where

$$k^2 = k_x^2 + k_z^2 \tag{4.41}$$

The conditions for $\mathbf{T}$ to become parallel to $\mathbf{D}$ are found by setting the cross product of the two to zero:

$$\mathbf{D} \times \mathbf{T} = 0 \tag{4.42}$$

Inserting Eqs. (4.34), (4.36), and (4.40) into Eq. (4.42) gives

$$\mathbf{D} \times \mathbf{T} = \frac{1}{n_o^2} \begin{bmatrix} \dfrac{k_x}{k_z} \left[k^2 - \left(k_z^2 + k_x^2 \dfrac{n_o^2}{n_e^2} \right) \right] D_x D_y \\ 0 \\ \left[k^2 - \left(k_z^2 + k_x^2 \dfrac{n_o^2}{n_e^2} \right) \right] D_x D_y \end{bmatrix} \tag{4.43}$$

Thus, for $\mathbf{D}$ and $\mathbf{T}$ to be parallel requires that each component of Eq. (4.43) is zero. These conditions are categorized as follows.

Case 1: $n_o^2/n_e^2 = 1$

This condition means that the medium is isotropic. From Eq. (4.41), the value in the inner square bracket inside Eq. (4.43) becomes zero; thus D_x and D_y can be arbitrary. Note, however, that D_z has to satisfy Eq. (4.36) so that $\mathbf{D}$ remains perpendicular to $\mathbf{k}$.

Case 2: $n_o^2/n_e^2 \neq 1$

The medium is anisotropic. This situation can be split into three subcases.

(a) *o-Wave*

$$\begin{aligned} D_x &= 0 \\ D_y &\neq 0 \\ D_z &= -\frac{k_x}{k_z} D_x = 0 \end{aligned} \tag{4.44}$$

This condition means $\mathbf{D}$ is parallel to the y direction and is perpendicular to the plane of the optic axis (x–z plane) and $\mathbf{k}$. Thus, this case fits the description of the o-wave.

(b) *e-Wave*

$$\begin{aligned} D_x &\neq 0 \\ D_y &= 0 \\ D_z &= -\frac{k_x}{k_z} D_x \neq 0 \end{aligned} \tag{4.45}$$

This means that **D** is in the plane made by the optic axis of the crystal and **k** and yet is perpendicular to **k**. This is precisely the description of the e-wave.

(c) *Propagation along the z axis*. The factors in the inner square bracket of the first and third row in Eq. (4.43) can be rewritten using Eq. (4.41) as

$$k_x^2 \left(1 - \frac{n_o^2}{n_e^2}\right) = 0 \tag{4.46}$$

which means

$$k_x = 0$$

Since k_y is already zero, only k_z is nonzero, meaning that propagation is along the z axis. D_x and D_y can be arbitrary but D_z has to be zero because k_x in Eq. (4.36) is zero.

4.4.3 Wavenormal and Ray Direction

The direction of the energy flow of light, which is called the ray direction, is expressed by the Poynting vector

$$\mathbf{s} = \mathbf{E} \times \mathbf{H} \tag{4.47}$$

The ray direction **s** is perpendicular to both **E** and **H**, and **s** is also included in the **DEks** plane in Fig. 4.3. The vector **s** shows the direction of energy flow and is the direction that your eyes have to be positioned at, if you want to see the light. It is, however, **k** but not **s** that follows Snell's law.

Now, referring to Fig. 4.5, **H** is taken along the y axis, hence **s** is in the x–z plane. The relationship between the directions of **s** and **k** will be calculated. Let

$$\mathbf{s} = s_x \hat{\mathbf{i}} + s_z \hat{\mathbf{k}} \tag{4.48}$$

and

$$\tan\phi = \frac{s_x}{s_z} \tag{4.49}$$

where ϕ is the angle that **s** makes with the z axis (or the optic axis). The vector **s** is perpendicular to **E** from Eq. (4.47) and $\mathbf{E} \cdot \mathbf{s} = 0$; hence,

$$E_x s_x + E_z s_z = 0 \tag{4.50}$$

With Eq. (4.50), Eq. (4.49) becomes

$$\frac{E_z}{E_x} = -\tan\phi \tag{4.51}$$

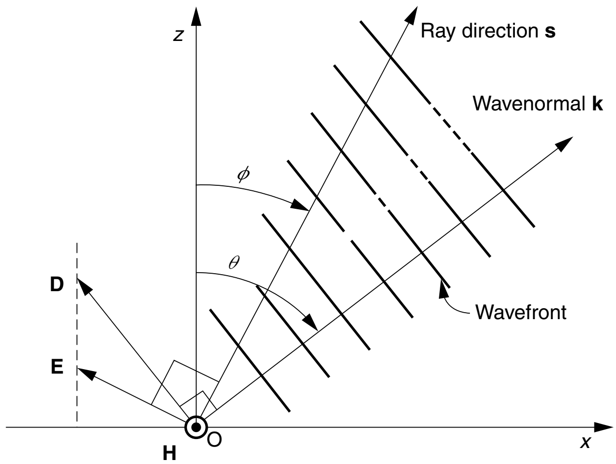


Figure 4.5 The wavefront, wave normal, and ray direction of the e-waves. (The y axis is into the page.)

Similarly,

$$\mathbf{k} = k_x\hat{\mathbf{i}} + k_z\hat{\mathbf{k}}$$

and

$$\tan\theta = \frac{k_x}{k_z} \tag{4.52}$$

Since $\mathbf{k}$ and $\mathbf{D}$ are perpendicular, $\mathbf{k} \cdot \mathbf{D} = 0$:

$$k_x D_x + k_z D_z = 0 \tag{4.53}$$

From Eqs. (4.52) and (4.53), we have

$$\frac{D_z}{D_x} = -\tan\theta \tag{4.54}$$

From Eq. (4.10), the e-wave components are related as

$$\frac{D_z}{D_x} = \left(\frac{n_e}{n_o}\right)^2 \frac{E_z}{E_x} \tag{4.55}$$

Equations (4.51), (4.54), and (4.55) give

$$\tan\theta = \left(\frac{n_e}{n_o}\right)^2 \tan\phi \tag{4.56}$$

The results of the e-wave calculations are shown in Fig. 4.5. Vector $\mathbf{s}$, which indicates the direction of the flow of light energy, does not coincide with $\mathbf{k}$, which indicates the direction of the normal to the wavefront. Also, the direction of polarization of $\mathbf{E}$ is not on the surface of the wavefront.

An analogy can be made with a sheet of cardboard paper blown into the air. The direction of the flight is not necessarily normal to the surface of the cardboard.

Another analogy is that of the skier. When a downhill skier climbs up a steep hill, the skis are set parallel to the contour line of the hill to prevent slips and the skier climbs off the fall line to decrease the effective slope. The parallel lines of the ski tracks resemble the wavefront, the fall line of the hill resembles the wavenormal **k**, and the movement of the skier resembles the ray path **s**.

Now, the double image seen through a calcite crystal (Iceland spar) can be explained. As shown in Fig. 4.6, a spotlight P is placed on the left surface of a calcite crystal

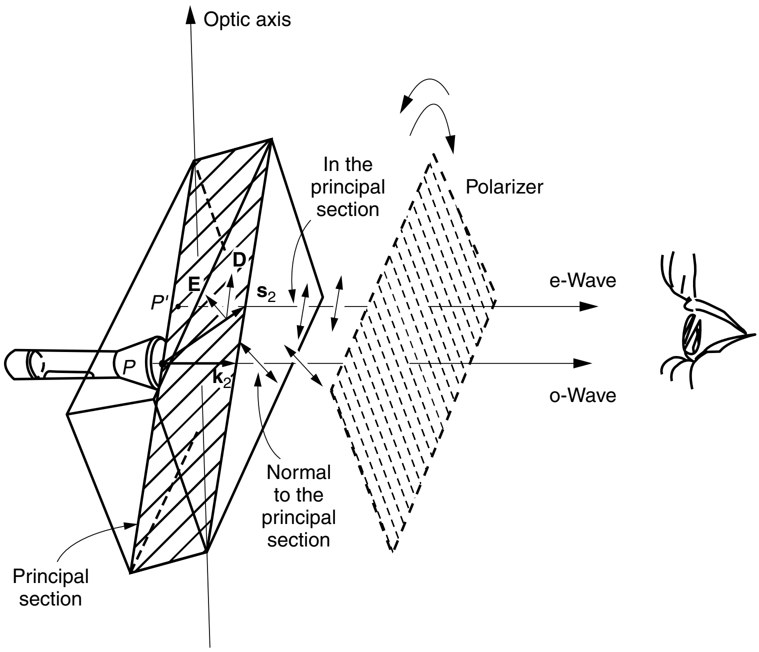
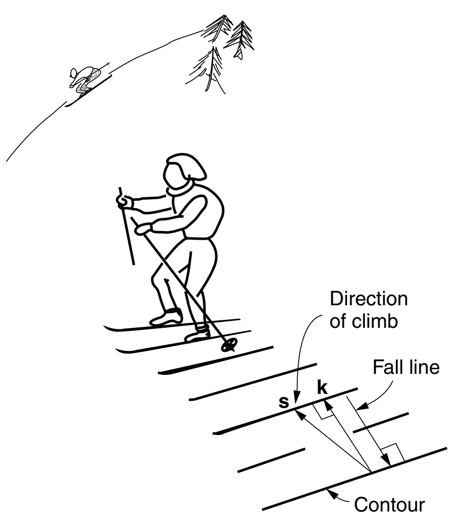


Figure 4.6 Birefringent image made by calcite crystal.



When a skier climbs a steep hill, the skiis are set parallel to the contour lines of the hill to avoid sliding, and the path of the climb is set off the fall line to decrease the effective slope. The direction of the skier is **s**.

and is observed from the right. The hatched plane, which contains the optic axis and also is perpendicular to the pair of cleaved surfaces is called a *principal section*. The incident light is polarized in an arbitrary direction. The energy of the o-wave polarized perpendicular to the principal section goes straight through following $\mathbf{k}_2$ while the energy of the e-wave polarized in the principal section follows the direction of $\mathbf{s}_2$. To the observer, one spot appears as if it were two spots. If the e- and o-wave images pass through a polarizer, then spots P and P' can be selectively seen by rotating the polarizer. The quantitative treatment of the same configuration will be described in Example 4.3.

Example 4.1 Draw the allowed directions of **D**, **E**, **s**, and **H** for o- and e-waves for the following three cases: (a) **k** is along the x axis, (b) **k** is at an arbitrary angle θ with respect to the z axis but in the x–z plane, and (c) **k** is along the z axis. The crystal axis is along the z axis.

Solution The answers are summarized in Fig. 4.7. The direction of **H** is always perpendicular to the **DEks** plane. □

Example 4.2 As shown in Fig. 4.8, the unit vector $\hat{\mathbf{s}}$ of the ray direction **s** in an anisotropic crystal is $(1/2, 0, \sqrt{3}/2)$. The optic axis of the crystal is along the z axis. The tensor refractive index is

$$|n^2| = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Find the directions (unit vectors) $\hat{\mathbf{e}}$, $\hat{\mathbf{d}}$ and $\hat{\boldsymbol{k}}$, respectively, for **E**, **D**, and **k**. Find those for both the o- and e-waves. (Note that in order to distinguish the unit vector for the z direction from the unit vector for the wavenormal, script $\hat{\boldsymbol{k}}$ is used for the unit vector for the wavenormal.)

Solution For the o-wave, the unit polarization vectors $\hat{\mathbf{e}}$ and $\hat{\mathbf{d}}$ are (0, 1, 0) or along the y axis, and $\hat{\boldsymbol{k}} = \hat{\mathbf{s}} = (1/2, 0, \sqrt{3}/2)$.

For the e-wave, the fact that **E** and **s** are perpendicular gives

$$\mathbf{s} \cdot \mathbf{E} = 0$$

$$\frac{1}{2}E_x + \frac{\sqrt{3}}{2}E_z = 0$$

$$\frac{E_x}{E_z} = -\sqrt{3}$$

$$\tan^{-1} -\sqrt{3} = -60^\circ$$

The unit vector $\hat{\mathbf{e}}$ is $(-\sqrt{3}\hat{\mathbf{i}}, 0\hat{\mathbf{j}}, \hat{\mathbf{k}})/\sqrt{1+3}$:

$$\hat{\mathbf{e}} = \frac{\mathbf{E}}{|\mathbf{E}|} = \frac{1}{2}(-\sqrt{3}\,\hat{\mathbf{i}}, 0\hat{\mathbf{j}}, \hat{\mathbf{k}})$$

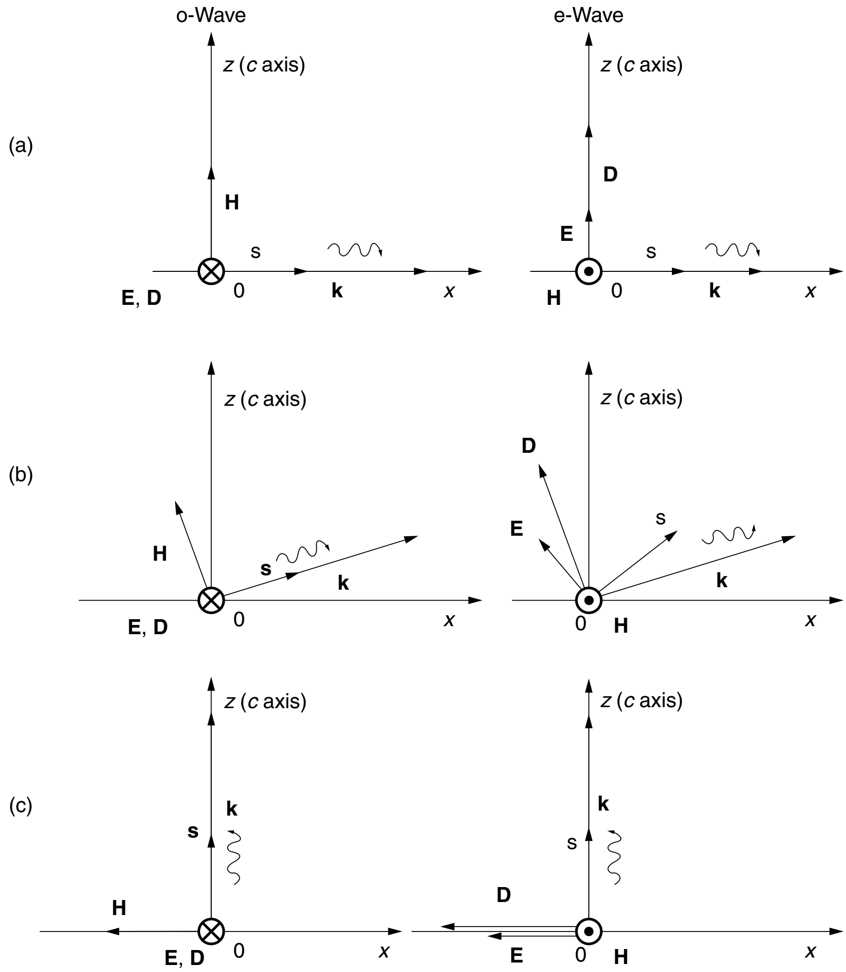


Figure 4.7 **D**, **E**, **H**, and **s** for various **k** of the o- and e-waves. The crystal axis is along the *z* axis. (The *y* axis is into the page.)

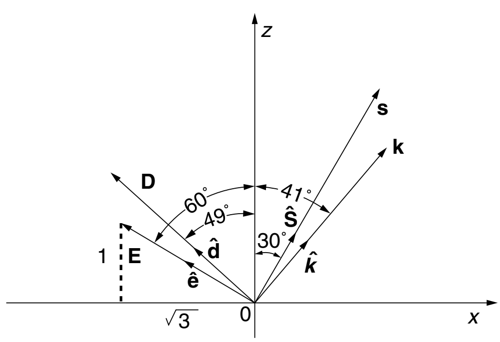


Figure 4.8 Given **s**, find the directions of **E** and **D** of the e-wave in an anisotropic crystal with $|\overset{\leftrightarrow}{n^2}| = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. (The *y* axis is into the page.)

The tensor operation is performed as follows:

$$\mathbf{D} = \epsilon_0(\overleftrightarrow{n}^2)\hat{\mathbf{e}}E$$

$$\mathbf{D} = \epsilon_0\frac{E}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -\sqrt{3} \\ 0 \\ 1 \end{bmatrix}$$

$$= \epsilon_0 E\begin{bmatrix} -\sqrt{3} \\ 0 \\ 3/2 \end{bmatrix}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{D}}{|\mathbf{D}|} = \frac{1}{\sqrt{4\times 3+9}}(-2\sqrt{3}\hat{\mathbf{i}}, 0\hat{\mathbf{j}}, 3\hat{\mathbf{k}})$$

$$= \frac{1}{\sqrt{21}}(-2\sqrt{3}\hat{\mathbf{i}}, 0\hat{\mathbf{j}}, 3\hat{\mathbf{k}})$$

The angle that $\hat{\mathbf{d}}$ makes with the optical axis is 49°.
Next, $\hat{\boldsymbol{k}}$ is found. $\mathbf{k}$ is perpendicular to $\mathbf{D}$.

$$-2\sqrt{3}k_x + 3k_z = 0$$

$$\theta = \tan^{-1}\left(\frac{k_x}{k_z}\right) = \tan^{-1}\left(\frac{3}{2\sqrt{3}}\right) = 41^\circ$$

$$\hat{\boldsymbol{k}} = \frac{\mathbf{k}}{|\mathbf{k}|} = \frac{1}{\sqrt{21}}(3\hat{\mathbf{i}}, 0, 2\sqrt{3}\hat{\mathbf{k}})$$ □

4.4.4 Derivation of the Effective Index of Refraction

The emphasis has been placed on the direction of the propagation constant $\mathbf{k}$, and the magnitude k has not yet been obtained. The value of k depends on the direction of $\mathbf{D}$. When $\mathbf{D}$ is pointed along the optic axis, the dielectric constant is $\epsilon_0 n_e^2$ and the propagation constant is $n_e k_0$; on the other hand, when $\mathbf{D}$ is in the direction perpendicular to the optic axis, the propagation constant is $n_o k_0$. The magnitude of the wavenormal $\mathbf{k}$ from an arbitrary direction of $\mathbf{D}$ will be obtained here. Using Eq. (4.26), Eq. (4.23) can be rewritten as

$$\mathbf{T} = k_0^2\mathbf{D} \tag{4.57}$$

Equation (4.40) is rewritten in diagonal matrix form as

$$\mathbf{T} = \begin{bmatrix} \frac{k_z^2}{n_o^2}+\frac{k_x^2}{n_e^2} & 0 & 0 \\ 0 & \frac{k^2}{n_o^2} & 0 \\ 0 & 0 & \frac{k_z^2}{n_o^2}+\frac{k_x^2}{n_e^2} \end{bmatrix}\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} \tag{4.58}$$

where Eq. (4.36) was used to convert D_x into D_z.

Inserting Eq. (4.58) into (4.57) gives

$$\begin{bmatrix} \dfrac{k_z^2}{n_o^2} + \dfrac{k_x^2}{n_e^2} - k_0^2 & 0 & 0 \\ 0 & \dfrac{k^2}{n_o^2} - k_0^2 & 0 \\ 0 & 0 & \dfrac{k_z^2}{n_o^2} + \dfrac{k_x^2}{n_e^2} - k_0^2 \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = 0 \tag{4.59}$$

Since the right-hand side is zero, the determinant has to vanish in order for D_x, D_y, and D_z to have nonzero solutions:

$$\left(\frac{k_z^2}{n_o^2} + \frac{k_x^2}{n_e^2} - k_0^2 \right)^2 \left(\frac{k^2}{n_o^2} - k_0^2 \right) = 0 \tag{4.60}$$

Equation (4.60) is called the characteristic equation of k, which can be separated into

$$\frac{k_z^2}{n_o^2} + \frac{k_x^2}{n_e^2} = k_0^2 \tag{4.61}$$

which is the equation of an ellipse in the k_x–k_z plane and

$$\frac{k^2}{n_o^2} = k_0^2 \tag{4.62}$$

which is the equation of a circle. Equation (4.59) is rewritten as

$$\left(\frac{k_z^2}{n_o^2} + \frac{k_x^2}{n_e^2} - k_0^2 \right) D_x = 0 \tag{4.63}$$

$$\left(\frac{k^2}{n_o^2} - k_0^2 \right) D_y = 0 \tag{4.64}$$

$$\left(\frac{k_z^2}{n_o^2} + \frac{k_x^2}{n_e^2} - k_0^2 \right) D_z = 0 \tag{4.65}$$

If Eq. (4.61) is the condition, Eqs. (4.63) and (4.65) mean that D_x and D_z can be nonzero, and this is precisely the condition of the e-wave. Similarly, if Eq. (4.62) is the condition, then D_y can be nonzero, and this is the condition for the o-wave.

With the e-wave, the magnitude k of the propagation constant depends on the direction of **k**. **k** is inclined at an angle θ with respect to the z axis, and

$$k_x = k \sin\theta, \qquad k_z = k \cos\theta \tag{4.66}$$

Inserting Eq. (4.66) into Eq. (4.61) gives the value of k for a given direction of propagation.

$$k = \pm n_{\text{eff}}(\theta) k_0 \tag{4.67}$$

where

$$n_{\text{eff}}(\theta) = \frac{1}{\sqrt{\left(\dfrac{\cos\theta}{n_o}\right)^2 + \left(\dfrac{\sin\theta}{n_e}\right)^2}} \tag{4.68}$$

For the e-wave, $n_{\text{eff}}(\theta)$ is the effective index of refraction when the angle of incidence is θ.

Similarly, for the o-wave from Eq. (4.62), the magnitude of the propagation constant is

$$k = \pm n_o k_0 \tag{4.69}$$

and k is independent of the angle of incidence.

4.5 GRAPHICAL METHODS

Methods that provide **D**, **E**, **k**, **H**, and **s** graphically, thereby alleviating some of the calculation, will be presented in this section. Two kinds of graphical methods will be explained: the wavevector and indicatrix methods. The wavevector method makes use of the **k**-space concept and emphasis is placed on propagation of the wavefront. The indicatrix method is based on the space of the indices of refraction and emphasis is on the optical properties of the medium.

4.5.1 Wavevector Method

The wavevector method [7] combines the ellipses calculated in Section 4.4.4 with a method for obtaining **k** and **s**.

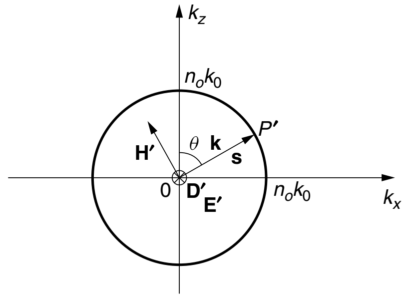
For simplicity, the uniaxial crystal case is considered. The characteristic equations (4.61) and (4.62) are represented in k_x, k_y, and k_z coordinates. Figure 4.9 shows the cross section in the $k_y = 0$ plane. Equation (4.61) is an ellipse with semiaxes, $n_e k_0$ and $n_o k_0$ and Eq. (4.62) is a circle with radius $n_o k_0$. With regard to the ellipse, note that the semiaxis $n_e k_0$ lies on the k_x axis, and the semiaxis $n_e k_0$ lies on the k_z axis. Let us now start to dig out as much information as possible from the graph in Fig. 4.9.

Let the wavenormal of the incident light be at an angle θ with respect to the k_z axis, as represented by $\overline{0P}'$ in Fig. 4.9a and $\overline{0P}$ in Fig. 4.9b. The quantities: **D**, **E**, **H**, **s**, **k**, n_{eff}, and the wavenormal or the wavefront are discussed in point form below.

1. There are two **D**'s that are orthogonal to each other, and they are both in the plane perpendicular to **k**. The electric displacement field for the o-wave, labeled as **D**$'$ in Fig. 4.9a, is in the k_y direction. The **D** field associated with the e-wave lies in the k_x–k_z plane in Fig. 4.9b.
2. The direction of **E** for the e-wave is tangent to the ellipse at point P. As this statement may not be immediately evident, a short proof is given.

   Taking the derivative of Eq. (4.61) with respect to k_x gives

$$\frac{2k_z}{n_o^2}\frac{dk_z}{dk_x} + \frac{2k_x}{n_e^2} = 0 \tag{4.70}$$



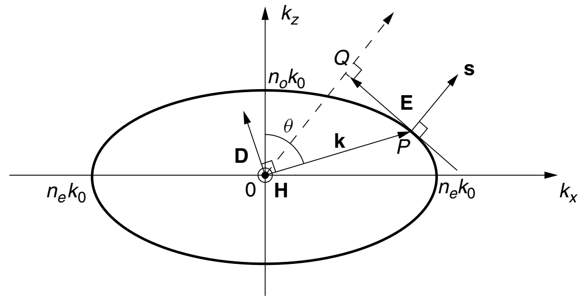


Figure 4.9 Wavevector diagram in a uniaxial crystal with $n_e > n_o$ (positive crystal). (a) o-wave. (b) e-wave.

and hence

$$\frac{dk_z}{dk_x} = -\left(\frac{n_o}{n_e}\right)^2 \frac{k_x}{k_z} \tag{4.71}$$

Using Eq. (4.54) with (4.52) gives

$$\frac{dk_z}{dk_x} = \frac{E_z}{E_x} \tag{4.72}$$

Thus, the tangent of the ellipse is the direction of the **E** field of the e-wave.

3. The direction of **H** is perpendicular to the plane made by **k** and **E** as indicated by Eq. (4.21). The **H** field for the e-wave is in the y direction. The $\mathbf{H}'$ field for the o-wave is in the x−z plane and perpendicular to **k**.
4. The Poynting vector **s** is obtained from $\mathbf{E} \times \mathbf{H}$. The direction of **s** for the e-wave is normal to **E** and thus normal to the ellipse at P. The Poynting vector $\mathbf{s}'$ of the o-wave is in the same direction as **k**.
5. Lastly, it should be mentioned that it is usually the ray direction **s** that is given in the laboratory. In many cases the direction of the wavenormal **k** is not explicitly given. For a given **s**, however, **k** can be found. Let the given direction of **s** be represented by the dotted line $\overline{0Q}$. Find the tangent to the ellipse that is normal to $\overline{0Q}$. The intersection of the tangent line with the ellipse determines the point P. The line $\overline{0P}$ is **k**.
6. $n_{\text{eff}}(\theta)k_0$ is represented by $\overline{0P}$.

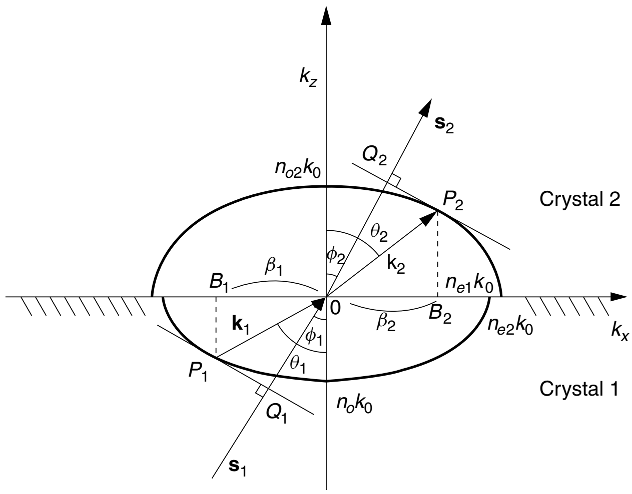


Figure 4.10 Finding the transmitted ray direction $\mathbf{s}_2$ for a given incident ray direction $\mathbf{s}_1$, using the wavevector diagram.

Next, a method of solving the refraction problem using the wavevector diagram will be described by a sequence of steps with brief descriptions for each step.

As indicated in Fig. 4.10, the optic axes of the two crystals are collinear, and the wavevector diagrams of the two crystals share a common origin. Only half of each wavevector diagram is shown in the figure. As described in the previous section, the light ray $\mathbf{s}_1$ is incident from the lower crystal 1 to crystal 2. The problem is to find the direction of the transmitted ray in crystal 2.

Step 1. In order to use Snell's law (phase matching) find the direction θ_1 of the wavenormal $\mathbf{k}_1$ from direction ϕ_1 of $\mathbf{s}_1$, as demonstrated earlier. The tangent line $\overline{P_1Q_1}$, which is perpendicular to $\mathbf{s}_1$, is drawn. The line drawn from the origin to P_1 is the direction of $\mathbf{k}_1$. The analytic expression relating the two angles is Eq. (4.56).

Step 2. The incident wave normal $\mathbf{k}_1$, is $\overline{0P_1}$. The analytic expression for k_1 is Eq. (4.67).

Step 3. Find the propagation constant β_1 along the interface, which is $\overline{0B_1}$, where B_1 is the normal from P_1 to the boundary surface.

$$\beta_1 = k_1 \sin\theta_1 \tag{4.73}$$

Step 4. In order to satisfy the condition of the phase matching across the boundary, find the point B_2 such that

$$\overline{0B}_1 = \overline{0B}_2 \tag{4.74}$$

or

$$\beta_1 = \beta_2 \tag{4.75}$$

Step 5. The angle θ_2 of the emergent wavenormal is found. Draw normal $\overline{P_2B_2}$ from the interface to intersect the wavevector diagram of crystal 2. $\overline{0P_2}$ is the direction of the emergent wavenormal.

The analytical expression for β_2 can be found from Eqs. (4.67) and (4.68):

$$\beta_1 = \beta_2 = \frac{k_0 \sin\theta_2}{\sqrt{\left(\dfrac{\cos\theta_2}{n_{o2}}\right)^2 + \left(\dfrac{\sin\theta_2}{n_{e2}}\right)^2}} \tag{4.76}$$

which is a quadratic equation in $\sin\theta_2$. The solution of Eq. (4.76) is

$$\sin\theta_2 = \frac{\left(\dfrac{\beta_1}{k_0 n_{o_2}}\right)}{\sqrt{1 + \left(\dfrac{\beta_1}{k_0}\right)^2 \left(\dfrac{1}{n_{o2}^2} - \dfrac{1}{n_{e2}^2}\right)}} \tag{4.77}$$

Step 6. Finally, $\mathbf{s}_2$, which is perpendicular to the tangent line $\overline{P_2Q_2}$, is drawn and the emergent ray direction ϕ_2 is found. The alternative is the use of Eq. (4.56).

Next, a description of total internal reflection is added. The condition of total internal reflection is that β_1 starts to exceed $n_{e_2}k_0$;

$$\beta_1 = n_{e_2}k_0 \tag{4.78}$$

With this condition the critical angle θ_c is obtained as

$$\sin\theta_c = \frac{n_{e2}}{n_{o1}} \frac{1}{\sqrt{1 + n_{e2}^2 \left(\dfrac{1}{n_{o1}^2} - \dfrac{1}{n_{e1}^2}\right)}} \tag{4.79}$$

4.5.2 Indicatrix Method

Whereas the wavevector method is closely linked to $\mathbf{k}$ space, the indicatrix method [8] utilizes the space formed by the indices of refraction of the crystal. The indicatrix method is an elegant way of handling crystal optics.

Using the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{4.80}$$

Eq. (4.22) is rewritten as

$$k^2\mathbf{E} - \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) = \omega^2 \mu \mathbf{D} \tag{4.81}$$

The scalar product of $\mathbf{D}$ with Eq. (4.81) eliminates the second term due to the fact that $\mathbf{D} \cdot \mathbf{k} = 0$, and Eq. (4.81) becomes

$$k^2\mathbf{E} \cdot \mathbf{D} = \omega^2 \mu \mathbf{D} \cdot \mathbf{D} \tag{4.82}$$

Equation (4.82) is rewritten as

$$\frac{k^2}{\omega^2 \mu \epsilon_0} \left(\frac{D_x}{\epsilon_{rx}}\hat{\mathbf{i}} + \frac{D_y}{\epsilon_{ry}}\hat{\mathbf{j}} + \frac{D_z}{\epsilon_{rz}}\hat{\mathbf{k}}\right) \cdot \mathbf{D} = D^2 \tag{4.83}$$

Equation (4.83) can be recast as

$$\left(\frac{x}{n_1}\right)^2 + \left(\frac{y}{n_2}\right)^2 + \left(\frac{z}{n_3}\right)^2 = 1 \tag{4.84}$$

where

$$\begin{aligned} x &= \frac{k}{k_0}\frac{D_x}{D} \qquad n_1^2 = \epsilon_{rx} \\ y &= \frac{k}{k_0}\frac{D_y}{D} \qquad n_2^2 = \epsilon_{ry} \\ z &= \frac{k}{k_0}\frac{D_z}{D} \qquad n_3^2 = \epsilon_{rz} \\ D &= |\mathbf{D}| \end{aligned} \tag{4.85}$$

Equation (4.84) is an ellipsoid in the x, y, z coordinates that are the normalized D_x, D_y, D_z coordinates. The principal axes of the ellipsoid, n_1, n_2, and n_3, are the principal refractive indices of the crystal. Such an ellipsoid is called the optical indicatrix, or simply indicatrix, or Fletcher's indicatrix after the scientist who first proposed the method in 1891 [9].

The procedure for obtaining **D**, **E**, **k**, **s**, and n_{eff} using the indicatrix will be explained in seven steps using the uniaxial case of $n_1 = n_2 = n_o$ and $n_3 = n_e$. Some mathematical verifications will follow.

Step 1. We assume that the direction of the wavenormal is known and is in the x–z plane. Draw line $\overline{0P}$ from the center 0 of the ellipsoid to the direction of the wavenormal, as shown in Fig. 4.11. Figure 4.12 is the cross section of the index ellipsoid in the x–z plane.

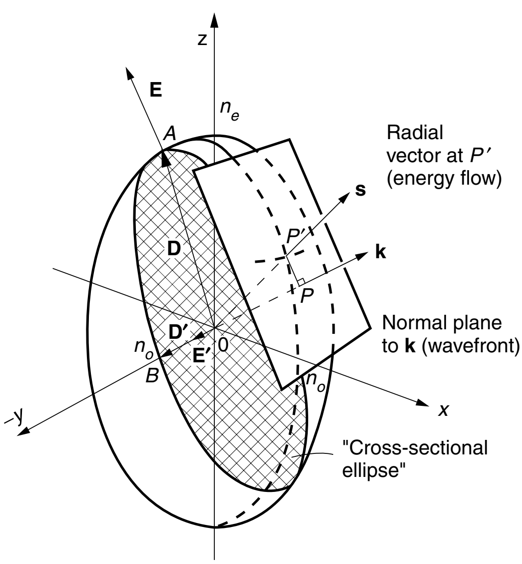


Figure 4.11 Method of the optical indicatrix.

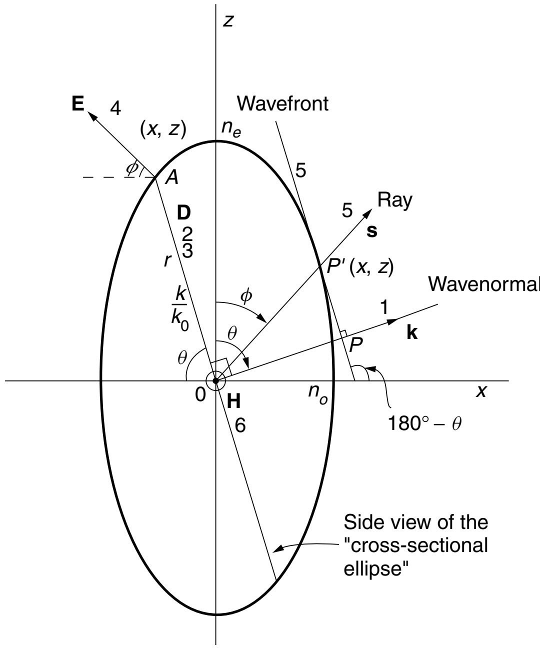


Figure 4.12 The *e*-wave fields in a uniaxial crystal obtained by the indicatrix. Numbers correspond to the appropriate steps in the text. (The *y* axis is into the page.)

Step 2. Find the plane that passes through the center of the ellipsoid in the direction perpendicular to $\overline{0P}$. The cross section of the ellipsoid cut by this plane is an ellipse, as indicated by the hatched area in the figure. This "cross-sectional ellipse" is of great importance for investigating propagation in the crystal.

Step 3. The conditions on the polarization direction of **D** in Section 4.4.2, described for the case when **k** is in the x–z plane, are summarized. Only two directions of polarization are allowed for a wave to propagate in an anisotropic medium. One polarization is in the y direction (o-wave) and the other is in the direction defined by the intersection of the plane perpendicular to **k** with the x–z plane (e-wave). With regard to the geometry of the indicatrix, these two allowed directions of polarization are in the directions of the major and minor axes **D** and **D**′ of the "cross-sectional ellipse." No other direction of polarization is allowed [10].

Step 4. The normals to the ellipse at A and B are the directions of **E** and **E**′.

Step 5. Find the plane that is tangent to the ellipsoid as well as perpendicular to $\overline{0P}$ or **k**. This tangent plane is the wavefront, and the **D**'s are parallel to this plane. The direction of energy flow is found from the contact point P' of the tangent plane to the ellipsoid. $\overline{0P'}$ is the direction of the ray vector **s**. (The proof is given later in this section.)

Step 6. The directions of **H** and **H**′ (not indicated in Fig. 4.11) are the directions perpendicular to the planes made by **k** and **E** and **k** and **E**′.

Step 7. When **k** is in a more general direction, one can always rotate the coordinates so that **k** is in the x–z plane because of the cylindrical symmetry of the indicatrix. However, by expanding the rule for using the indicatrix, there is no need to rotate the

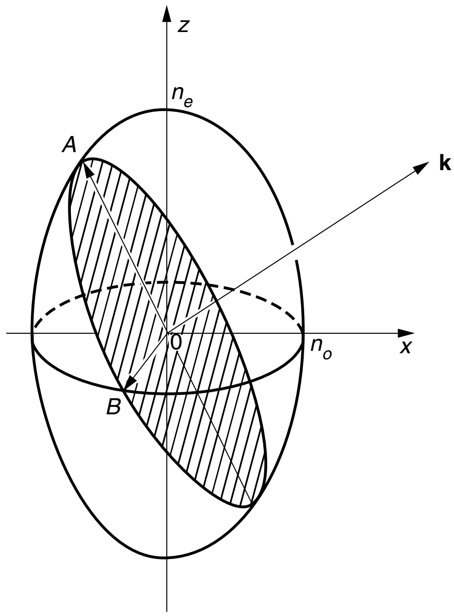


Figure 4.13 The "cross-sectional ellipse" in the indicatrix determines the allowed directions of polarization $\overline{OA}$ (e-wave) and $\overline{OB}$ (o-wave). They are in the directions of the principal axes.

coordinates. The general rule for the use of the indicatrix is as follows. As shown in Fig. 4.13, the intercept of the plane perpendicular to **k** with the ellipsoid generates the "cross-sectional ellipse" with principal semiaxes $\overline{0A}$ and $\overline{0B}$. The only allowed directions of the polarization are $\overline{0A}$ (e-wave) and $\overline{0B}$ (o-wave). The lengths of $\overline{0A}$ and $\overline{0B}$ are the effective refractive indices n_{eff} for the e- and o-waves, respectively.

The proofs of some of the steps will be given.

Let us examine now in more detail the relationship between the quantities shown in Fig. 4.12 and equations derived earlier for the e-wave in a uniaxial crystal. We will now prove that the normal to the ellipse at point A in Fig. 4.12 is the direction of E (as stated in Step 4 of the indicatrix method).

By taking the gradient of the left-hand side of Eq. (4.84) with $y = 0$ and $n_1 = n_o$ and $n_3 = n_e$, the normal vector **N** is obtained. Let (see boxed note)

$$F = \left(\frac{x}{n_o}\right)^2 + \left(\frac{z}{n_e}\right)^2 \tag{4.86}$$

The normal vector **N** is

$$\mathbf{N} = \nabla F \tag{4.87}$$

and

$$\mathbf{N} = \frac{2x}{n_o^2}\hat{\mathbf{i}} + \frac{2z}{n_e^2}\hat{\mathbf{k}} \tag{4.88}$$

Let's examine the meaning of the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = F \tag{4.89}$$

If, for example, $F = 1.1^2$, then this equation defines an ellipse that is inflated by 10% compared to the size of the $F = 1$ ellipse, a and b.

$$\frac{x^2}{(1.1a)^2} + \frac{y^2}{(1.1b)^2} = 1$$

The size of the ellipse is determined by the right-hand side F of Eq. (4.89). The component of the gradiant $(\partial F/\partial x)\mathbf{i}$ is the rate of expansion of the ellipse in the x direction.

By replacing x and z at A in Eq. (4.88) by those in Eq. (4.85), the normal vector is expressed as

$$\mathbf{N} = \frac{2k}{k_0 D}\left(\frac{D_x}{n_o^2}\hat{\mathbf{i}} + \frac{D_z}{n_e^2}\hat{\mathbf{k}}\right) \tag{4.90}$$

From Eq. (4.10), $\mathbf{N}$ is rewritten as

$$\mathbf{N} = \frac{2k}{k_0 D}[E_x\hat{\mathbf{i}} + E_z\hat{\mathbf{k}}] \tag{4.91}$$

and

$$\mathbf{N} = \frac{2k}{k_0 D}\mathbf{E} \tag{4.92}$$

Thus, it has been proved that $\mathbf{E}$ is in the direction of the normal.

Referring to Fig. 4.12, we will now prove that $\overline{0P}'$ is perpendicular to $\mathbf{E}$ and hence parallel to $\mathbf{s}$ (as stated in Step 5 of the indicatrix method). P' is the contact point on the ellipse of the tangent line that is perpendicular to $\mathbf{k}$. With the conditions $y = 0$, $n_1 = n_o$, and $n_3 = n_e$, and taking the derivative with respect to x of both sides of Eq. (4.84) at point P', one obtains

$$\frac{dz}{dx} = -\frac{x}{z}\left(\frac{n_e}{n_o}\right)^2 \tag{4.93}$$

The tangent that is perpendicular to $\mathbf{k}$ and hence parallel to $\mathbf{D}$ is

$$\frac{dz}{dx} = \frac{D_z}{D_x} \tag{4.94}$$

Equations (4.11) and (4.94) are put into Eq. (4.93) to obtain

$$\frac{z}{x} = -\frac{E_x}{E_z} \tag{4.95}$$

The line $\overline{0P}'$ connecting the origin and the contact point P' is

$$\overline{0P}' = x\hat{\mathbf{i}} + z\hat{\mathbf{k}} \tag{4.96}$$

From Eqs. (4.95) and (4.96),

$$\overline{0P}' \cdot \mathbf{E} = 0 \tag{4.97}$$

Thus, $\overline{0P}'$ is perpendicular to $\mathbf{E}$ and in the direction of $\mathbf{s}$ because of Eq. (4.47) with $\mathbf{H}$ in the $-y$ direction.

Finally, the value of k is examined. When vector $\mathbf{k}$ is at angle θ with respect to the optic axis, the coordinates (x, z) at point A in Fig. 4.12 are $(-r\cos\theta, r\sin\theta)$. Since this point is on the ellipse expressed by Eq. (4.84), we have

$$r^2\left[\left(\frac{\cos\theta}{n_o}\right)^2 + \left(\frac{\sin\theta}{n_e}\right)^2\right] = 1 \tag{4.98}$$

Next, the physical meaning of r is found. The length r in the x, z coordinates can be rewritten using Eq. (4.85) as

$$r = \sqrt{x^2 + z^2} = \frac{k}{k_0}\left(\sqrt{\frac{D_x^2 + D_z^2}{D}}\right) \tag{4.99}$$

Since the value inside the square root is unity, Eq. (4.99) with Eq. (4.98) becomes

$$r = \frac{k}{k_0} = n_{\text{eff}} \tag{4.100}$$

where

$$n_{\text{eff}} = \frac{1}{\sqrt{\left(\dfrac{\cos\theta}{n_o}\right)^2 + \left(\dfrac{\sin\theta}{n_e}\right)^2}} \tag{4.101}$$

which is the same expression as Eq. (4.68). Thus, it has been proved that the optical indicatrix results are identical with the wavevector results.

Both the indicatrix and the wavevector diagram are used for treating refraction at the boundary between anisotropic media. Crystals whose refractive indices change due to electrooptic or acoustooptic effects are characterized by the indicatrix, and the indicatrix method is generally a better choice for this situation. However, the wavevector approach is usually preferred when a knowledge of the phase velocity is required, such as problems involving wave dispersion or the interaction of lightwaves with other types of waves like acoustic waves. Either way, the difference between the two approaches is rather slim. The indicatrix approach to boundary problems will be presented in Section 4.6.

Example 4.3 A light ray is incident from air onto the principal section (Example 4.1) of a calcite crystal whose optic axis is titled by 48° with respect to the front surface, as shown in Fig. 4.14. If the thickness of the crystal is 2 cm, what is the distance between the two spots seen on the emergent surface of the crystal. The indices of refraction are $n_o = 1.658$ and $n_e = 1.486$.

Obtain the qualitative solutions by means of the wavevector method and indicatrix method, and then calculate the distance numerically.

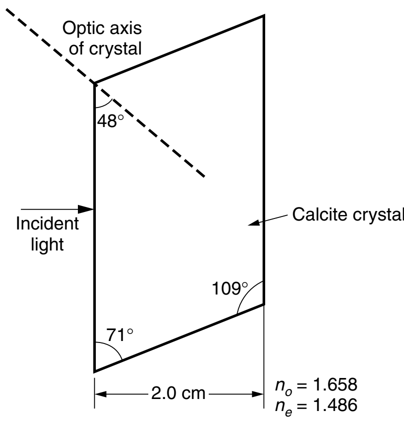


Figure 4.14 A light ray with two orthogonal polarizations is incident onto calcite. The parallelogram represents the principal section shown in Fig. 4.6.

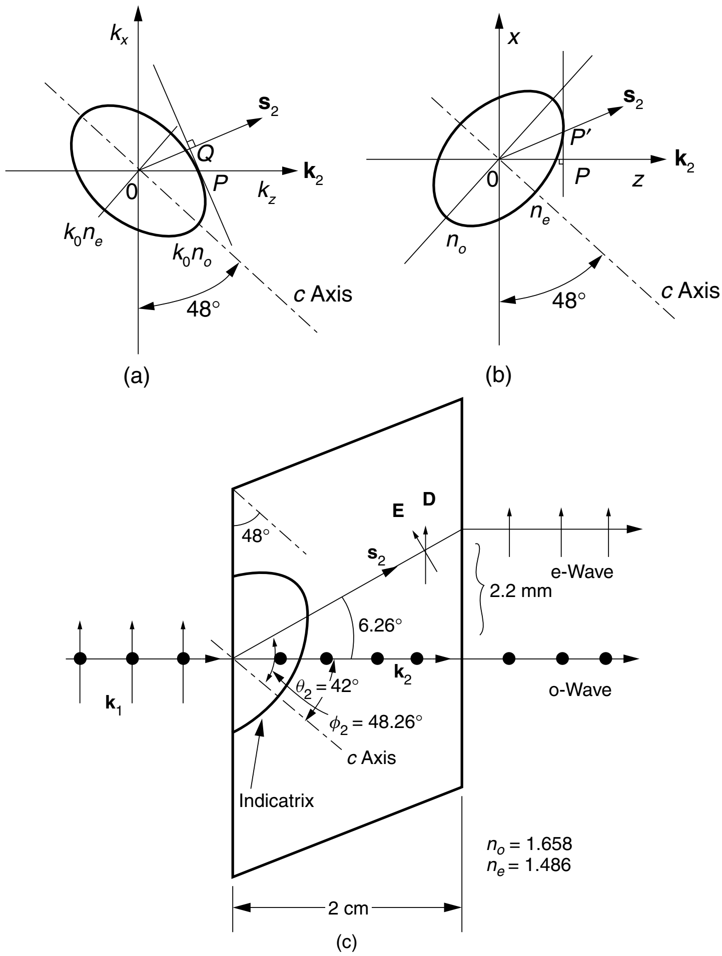


Figure 4.15 Directions of o- and e-waves transversing a calcite crystal. (a) By wavevector diagram. (b) By indicatrix method. (c) By numerical solution.

Solution Graphical solutions are shown in Figs. 4.15a and 4.15b and the analytical solution in Fig. 4.15c. The incident light ray is decomposed into two orthogonal directions of polarization. The o-wave obeys Snell's law, and because it enters the crystal at normal incidence, it propagates straight through the crystal.

The e-wave is slightly different. $\mathbf{k}_1$ and $\mathbf{k}_2$ obey Snell's law, and both are normal to the boundary. The angle of transmittance with respect to the crystal axis is therefore, from Fig. 4.15c, $\theta_2 = 42^\circ$. By Eq. (4.56) the transmitted angle ϕ_2 of the light ray is $\phi_2 = 48.26^\circ$. Thus, the e-wave is refracted at the boundary by 6.26° from the normal to the boundary. □

4.6 TREATMENT OF BOUNDARY PROBLEMS BETWEEN ANISOTROPIC MEDIA BY THE INDICATRIX METHOD

When the light encounters a boundary between two media with different optical properties, part of the energy is transmitted into the second medium, and the other part is reflected back into the original medium. In this section, the boundary between uniaxial anisotropic media is treated [5,11,12]. Section 4.6.1 treats the transmitted wave; Section 4.6.2, the reflected wave; and Section 4.6.3, total internal reflection. Before beginning, a few important general remarks made earlier will be repeated here.

On the boundary between the two media with different indices of refraction, the incident light changes its direction of propagation for the sake of phase matching. Recall that it is the direction of the *wavenormal* that obeys Snell's law. It is not the direction of the *energy flow* (ray direction) that obeys Snell's law. This is because Snell's law is the law that synchronizes the phasefronts of the incident, transmitted, and reflected light across the boundary, as explained in Section 2.2.

In dealing with refraction between uniaxial anisotropic media, the incident wave has to be first decomposed into two component waves: one wave is the o-wave and the other is the e-wave. The respective amplitudes of the two waves are found as the projection of the amplitude of the incident wave into these two directions of polarization. The two waves are treated separately and the results are added to reach the final answer.

Since the direction of the wavenormal of the o-wave is identical to the ray direction, the transmitted and reflected waves of the o-wave can be obtained in exactly the same manner as described in Section 2.3 and it will not be repeated here. The explanation in this section is devoted to the e-wave.

4.6.1 Refraction of the e-Wave at the Boundary of Anisotropic Media

For a given incident e-wave ray $\mathbf{s}_1$, the transmitted e-wave ray $\mathbf{s}_2$ will be found [13]. It is the direction of the light ray $\mathbf{s}$ that is usually given. In order to utilize Snell's law, the direction of the light ray $\mathbf{s}$ has to be converted into that of the wavenormal $\mathbf{k}$. After using Snell's law, the direction of the wavenormal is converted back to the ray direction.

It is assumed that the optic axes of the two crystals are collinear, as shown in Fig. 4.16. The indices of refraction of the bottom uniaxial crystal 1 are denoted as

$$\begin{bmatrix} n_{o1}^2 & 0 & 0 \\ 0 & n_{o1}^2 & 0 \\ 0 & 0 & n_{e1}^2 \end{bmatrix}$$

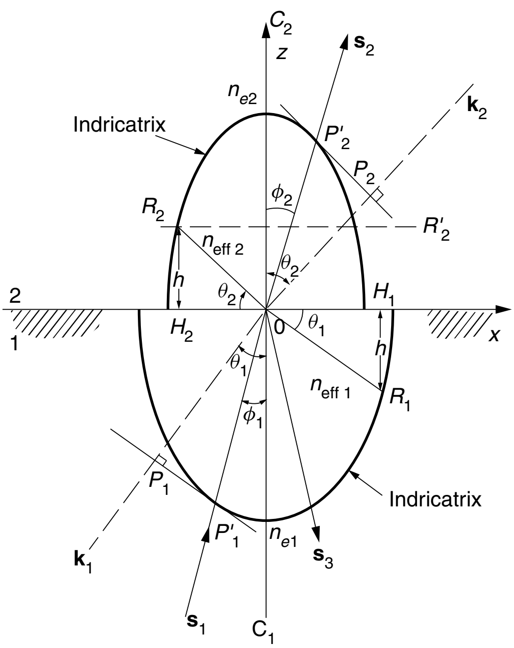


Figure 4.16 Refraction at the boundary of two uniaxial crystals. The optic axes of crystals 1 and 2 are collinear.

and those of the top crystal 2 by

$$\begin{bmatrix} n_{o2}^2 & 0 & 0 \\ 0 & n_{o2}^2 & 0 \\ 0 & 0 & n_{e2}^2 \end{bmatrix}$$

The indicatrixes of the two crystals share a common origin and only half of each indicatrix is drawn in the figure.

The light ray $\mathbf{s}_1$ is incident from the lower crystal 1 to the upper crystal 2. This ray direction is first converted into the wavenormal $\mathbf{k}_1$. The light ray $\mathbf{s}_1$ intersects the lower ellipse at point P'_1. The tangent to the ellipse at P'_1 is drawn. A line is drawn from the origin perpendicular to this tangent line. Let P_1 denote the intersection of the line from the origin with the tangent line. The wavefront of the incident wave is $\overline{P_1P'_1}$ and the wavenormal $\mathbf{k}_1$ is along $\overline{0P_1}$. The angle $\mathbf{k}_1$ makes with the z axis is θ_1.

Now that the angle θ_1 of the incident wavenormal is found, Snell's law can be applied. The effective index of refraction $n_{\text{eff}\,1}$ of crystal 1 is readily found by drawing line $\overline{0R_1}$ from the origin perpendicular to $\mathbf{k}_1$. Snell's law is

$$n_{\text{eff}\,1} \sin\theta_1 = n_{\text{eff}\,2} \sin\theta_2 \tag{4.102}$$

For the special orientation of the optic axis of the crystal perpendicular to the boundary, the angle that $\overline{0R_1}$ makes with the x axis is identical to the angle θ_1 of incidence. Thus, the value of the left-hand side of Eq. (4.102) is graphically represented by the height h of R_1 from the x axis.

Now let us consider the right-hand side of Eq. (4.102). A horizontal line is drawn at a height h above the x axis. The intersection R_2 that the horizontal line $\overline{R_2R_2'}$ makes with the indicatrix is determined. The length $\overline{0R_2}$ now graphically represents the effective index of refraction $n_{\text{eff}\,2}$ of the second medium. The direction perpendicular to $\overline{0R_2}$ is the direction of the wavenormal of the transmitted wave. Since the optic axis of the crystal is perpendicular to the boundary, the angle that $\overline{0R_2}$ makes with the x axis is identical to the transmitted angle θ_2 that $\mathbf{k}_2$ makes with the optic axis. Finally, the transmitted ray $\mathbf{s}_2$ is obtained by finding the line $\overline{P_2P_2'}$, which is tangent to the ellipse as well as perpendicular to $\mathbf{k}_2$. Thus, $\mathbf{s}_2$ was found from $\mathbf{s}_1$.

The above graphical method will be supplemented by a brief description of the analytical method. First, the incident ray direction ϕ_1 is converted into θ_1 of the incident wavenormal using Eq. (4.56).

The value of h of the incident light is

$$h = n_{\text{eff}\,1} \sin\theta_1 \tag{4.103}$$

where the effective refractive index $n_{\text{eff}\,1} = k_1/k_0$ is obtained from Eq. (4.68).

$$h = \frac{\sin\theta_1}{\left[\left(\dfrac{\cos\theta_1}{n_{o1}}\right)^2 + \left(\dfrac{\sin\theta_1}{n_{e1}}\right)^2\right]^{1/2}} \tag{4.104}$$

The angle θ_2 of the transmitted light is calculated from the fact that h for medium 1 remains the same as that for medium 2. The value of θ_2 has to be calculated for a given h value.

$$h = \frac{\sin\theta_2}{\left[\left(\dfrac{\cos\theta_2}{n_{o2}}\right)^2 + \left(\dfrac{\sin\theta_2}{n_{e2}}\right)^2\right]^{1/2}} \tag{4.105}$$

This quadratic equation in sin θ_2 can be solved in the same manner as Eq. (4.77) was obtained from Eq. (4.76). The result is

$$\sin\theta_2 = \frac{\left(\dfrac{h}{n_{o2}}\right)}{\sqrt{1 + h^2\left(\dfrac{1}{n_{o2}^2} - \dfrac{1}{n_{e2}^2}\right)}}. \tag{4.106}$$

Finally, the angle θ_2 of the wavenormal of the transmitted light is converted into angle ϕ_2 of the emergent light ray using Eq. (4.56).

Problem 4.3 deals with the case when the two optic axes of the crystal are at an angle.

4.6.2 Reflection of the e-Wave at the Boundary of Anisotropic Media

Regardless of the type of media, the direction of the reflected wave is such that the $\mathbf{k}$ component of the reflected light that is parallel to the boundary is phase matched with that of the incident light. When the optic axis of the crystal is normal to the boundary, as in Fig. 4.16, the angle of the reflected wavenormal is equal to the angle of the incident wavenormal. However, when the optic axes of the crystal are not normal to the boundary, such as shown in Fig. 4.17, the angle of reflection differs from the angle of incidence.

Referring to Fig. 4.17, light is incident from the lower crystal 1. Only the e-wave that is reflected back into crystal 1 is considered here. Let the incident wavenormal be $\mathbf{k}_1$. The angle of the reflected wavenormal has to be such that the wavelength along the boundary matches that of the incident wave. The phase matching conditions (Section 2.2) between the wavefronts of the incident and reflected light are

$$h_1 = n_{\mathrm{eff}\,1} \sin\theta_1' \tag{4.107}$$

$$h_3 = n_{\mathrm{eff}3} \sin\theta_3' \tag{4.108}$$

where θ_1' is the angle between the incident wavenormal to the boundary and the normal to the boundary, and θ_3' is the angle between the reflected wavenormal and the normal to the boundary. The phase matching condition is satisfied for

$$h_1 = h_3 = h \tag{4.109}$$

The graphical solution for θ_3' is as follows. Referring to Fig. 4.17, the line $\overline{0R_1}$ perpendicular to $\mathbf{k}_1$, drawn from the origin to the ellipse, is the effective index $n_{\mathrm{eff}\,1}$ of the incident wave. Line $\overline{R_1H_1}$ to the x axis represents h_1. The intersection of the horizontal line $\overline{R_1R_3}$ and the ellipse determines h_3, and $\overline{0R_3}$ represents $n_{\mathrm{eff}\,3}$. The direction normal to $\overline{0R_3}$ gives $\mathbf{k}_3$ of the reflected wave. Thus, the angle of reflection

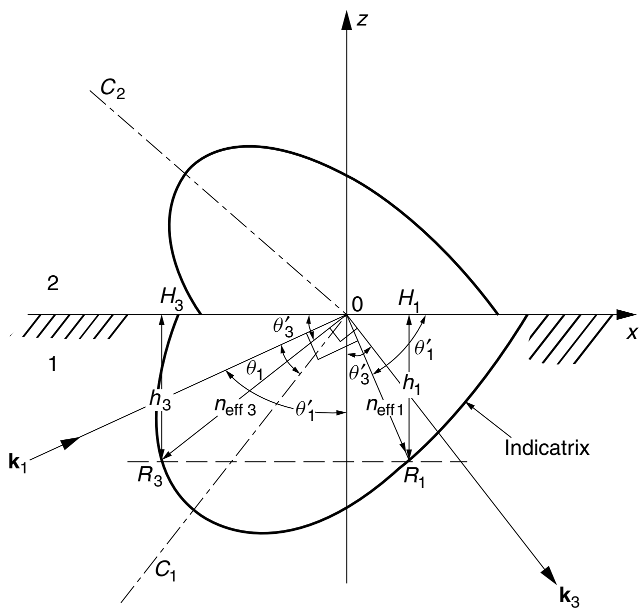


Figure 4.17 Boundary of two crystals where the optic axes are not perpendicular to the boundary. $\overline{C_1 0}$, $\overline{C_2 0}$, and $\mathbf{s}_1$ are all in the $y = 0$ plane.

was obtained from the angle of incidence. It should be noted that $\mathbf{k}_1$ and $\mathbf{k}_3$ are not symmetric with respect to the normal to the boundary when the optic axis of crystal 1 is not normal to the boundary. Transmission across such a boundary is left to Problem 4.3.

Note that the angle of reflection is independent of medium 2. As in the isotropic case, the *amount* of reflection is influenced by medium 2 but not the *angle* of reflection.

4.6.3 Total Internal Reflection of the e-Wave at the Boundary of Anisotropic Media

Total internal reflection takes place when the effective index of refraction of the second medium is not large enough to satisfy Snell's law given by Eq. (4.102). The case shown in Fig. 4.16 is taken as an example. In this case, the indicatrix of the top medium is smaller than that of the bottom medium, and when light is incident from the bottom medium to the top medium, the condition for total internal reflection can exist. The h value of the incident wave is

$$h = n_{\text{eff}\,1} \sin\theta \tag{4.110}$$

The h values of the top and bottom crystals have to be matched but the largest value that h can take in the top crystal 2 is n_{e2}; thus, total internal reflection takes place at an incident angle θ_c such that

$$n_{\text{eff}\,1} \sin\theta_c = n_{e2} \tag{4.111}$$

where θ_c is the critical angle.

Using Eq. (4.68), Eq. (4.111) becomes

$$n_{e2} = \frac{\sin\theta_c}{\sqrt{\left(\dfrac{\cos\theta_c}{n_{o1}}\right)^2 + \left(\dfrac{\sin\theta_c}{n_{e1}}\right)^2}}$$

Mere comparison of the form of this equation with that of Eq. (4.105) gives

$$\sin\theta_c = \frac{n_{e2}}{n_{o1}} \cdot \frac{1}{\sqrt{1 + n_{e2}^2\left(\dfrac{1}{n_{o1}^2} - \dfrac{1}{n_{e1}^2}\right)}} \tag{4.112}$$

where the subscripts 1 and 2 denote crystals 1 and 2. Equation (4.112) is identical to Eq. (4.79), which was obtained by the wavevector method in Section 4.5.1. Equation (4.112) reduces to the familiar expression for the critical angle of the isotropic case when $n_{o1} = n_{e1}$ and $n_{02} = n_{e2}$.

Example 4.4 Lithium niobate ($LiNbO_3$) is deposited over lithium tantalate ($LiTaO_3$) with the optic axes of both crystals normal to the interface. The indicatrixes are shown in Fig. 4.18. A light ray is incident on the boundary from the bottom layer, lithium

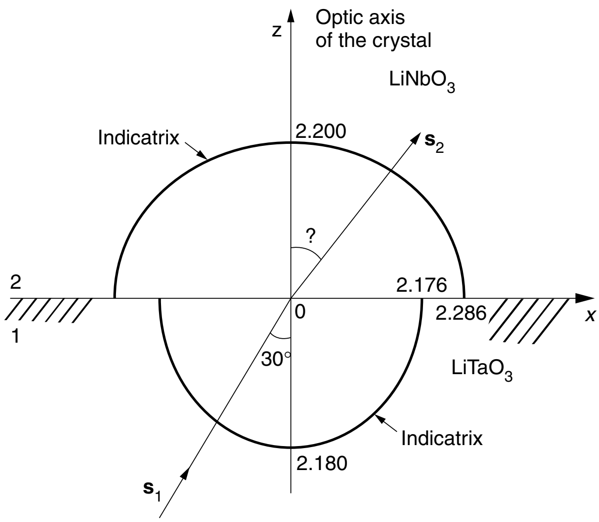


Figure 4.18 Refraction at the boundary between anisotropic media.

tantalate, at an incident angle $\theta_1 = 30°$. Calculate the emergent ray direction of the e-wave.

	Material	n_o	n_e
Medium 2	$LiNbO_3$	2.286	2.200
Medium 1	$LiTaO_3$	2.176	2.180

The wavelength of the incident light is $\lambda = 0.633$ μm.

Solution The emergent ray direction will be found. First, the ray direction is converted into that of the wavenormal using Eq. (4.56) with $\phi_1 = 30°$:

$$\theta_1 = 30.09°$$

The effective index of refraction k/k_0 at this incident angle is calculated using Eq. (4.68),

$$n_{\text{eff}\,1} = 2.177$$

and

$$h = n_{\text{eff}\,1} \sin\theta_1 \tag{4.113}$$

Equation (4.106) can be used for finding θ_2 by putting h = 1.092. The value of θ_2 is

$$\theta_2 = 28.82°$$

Finally, using Eq. (4.56), θ_2 is converted into the emergent ray direction:

$$\phi_2 = 30.72°$$

The results are summarized in Fig. 4.19. □

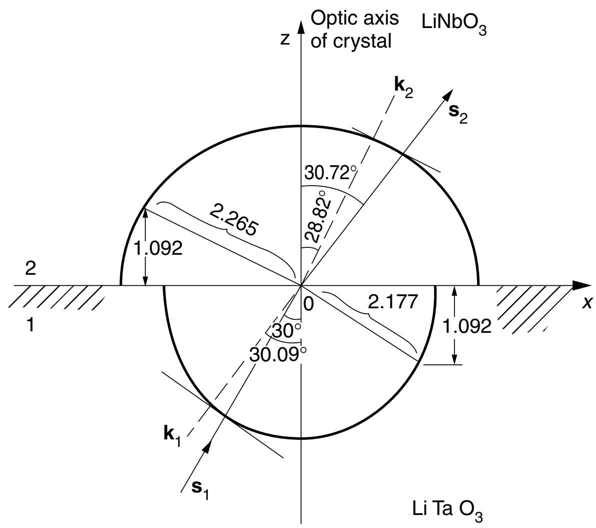


Figure 4.19 Solution to Example 4.4.

PROBLEMS

4.1 What condition enables Eqs. (4.61) and (4.62) to be satisfied simultaneously?

4.2 Obtain the characteristic matrix for the general case including the y components of the wave in an anisotropic medium with

$$\overleftrightarrow{\epsilon}_r = \begin{bmatrix} n_\alpha^2 & 0 & 0 \\ 0 & n_\beta^2 & 0 \\ 0 & 0 & n_\gamma^2 \end{bmatrix}$$

and let

$$\mathbf{k} = k_x\hat{\mathbf{i}} + k_y\hat{\mathbf{j}} + k_z\hat{\mathbf{k}}$$
$$\mathbf{E} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$$

4.3 Two crystals have a common boundary in the $z = 0$ plane. The optic axes of the crystals are not normal to the boundary and both are in the $y = 0$ plane with indicatrixes as shown in Fig. P4.3. The light ray $\mathbf{s}_1$ is incident from crystal 1 at the bottom into crystal 2 at the top. The incident light ray is in the $y = 0$ plane. Using the indicatrix, find the following quantities graphically:

(a) The vector $\mathbf{s}_2$ of the transmitted (refracted) ray.
(b) The vector $\mathbf{s}_3$ of the reflected ray when total internal reflection takes place.
(c) The angle of the reflected ray.

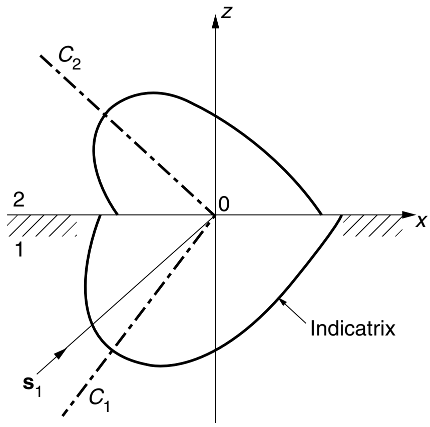


Figure P4.3 Boundary of two crystals.

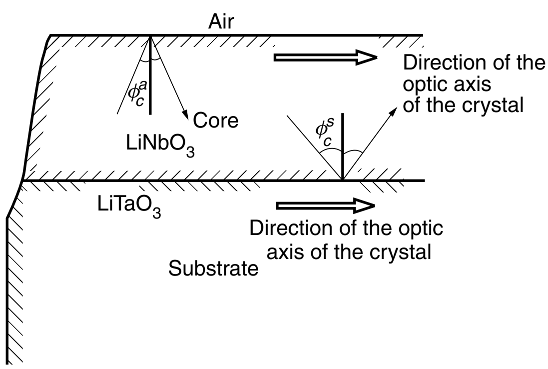


Figure P4.4 Critical angles of an anisotropic optical guide.

4.4 Find the critical angles of the rays of the o- and e-waves inside the core region of an $LiNbO_3$–$LiTaO_3$ optical guide. The optic axes of both the $LiNbO_3$ and $LiTaO_3$ crystals are oriented parallel to the boundaries (Y-cut) as in Fig. P4.4. The indices of refraction of the media are listed in Example 4.4.

4.5 **(a)** Figure P4.5 shows a contour whose radius r at the ray direction ϕ corresponds to the ray velocity (not phase velocity) in that direction. This ray velocity surface is called Huygens' wavelet of the e-wave ellipsoid. Derive the expression for Huygens' wavelet of the e-wave in terms of ϕ, u, v_e, and v_o, where $v_o = c/n_o$ and $v_e = c/n_e$.

(b) In the process of second harmonic generation (SHG), the energy of the light at the fundamental frequency is converted into that of the second higher harmonic frequency during the transmission in a nonlinear crystal. It is the ray velocities of the two waves rather than the phase velocities that have to be matched in order to optimize the efficiency of the energy conversion between the two waves. How would you use Huygens' wavelet to find the optimum ray direction in the SHG experiment [7]?

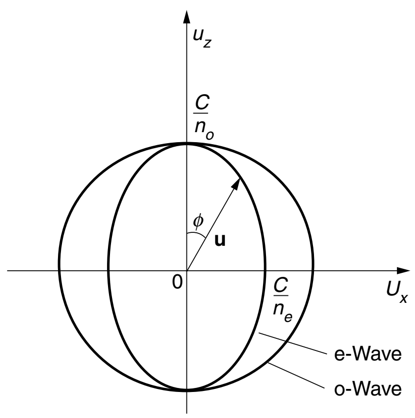


Figure P4.5 Huygens' wavelet. Note the differences of the coordinates among the wavevector diagram, the indicatrix, and Huygens' wavelet.

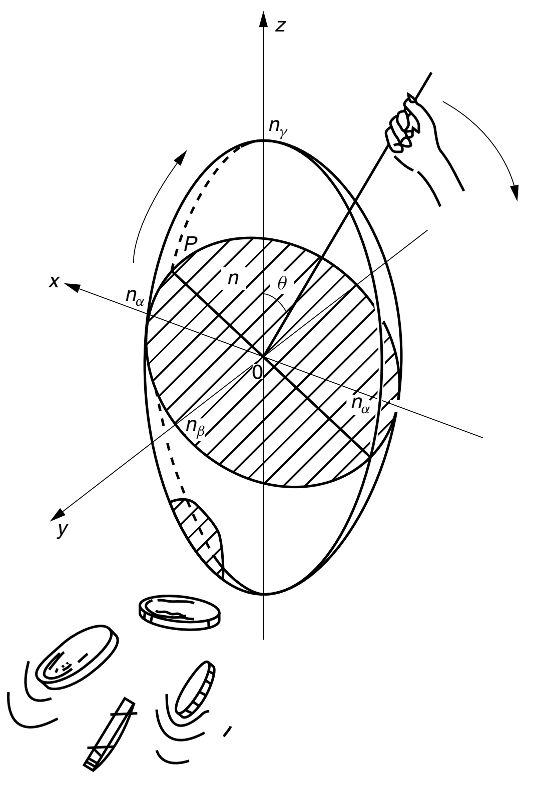


Figure P4.6 Search for the optic axes of a biaxial crystal.

4.6 Find the optic axes of the biaxial crystal shown in Fig. P4.6. The indices of refraction in the x, y, and z directions are n_α, n_β, and n_γ, respectively; assume that $n_\alpha < n_\beta < n_\gamma$. The direction of the optic axis is the direction of propagation such that the elliptic cross-section of the indicatrix made by the intersection with the plane perpendicular to the direction of propagation becomes a circle. In order to find the direction of the optic axes, the direction **k** of propagation is tilted from the z axis in the x–z plane until the elliptic cross section becomes a circle (Fig. P4.6). Prove that the optic axes are at an angle θ from the z axis, which is given by

$$\sin\theta = \pm\frac{n_\gamma}{n_\beta}\sqrt{\frac{n_\beta^2 - n_\alpha^2}{n_\gamma^2 - n_\alpha^2}}$$

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