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APPENDIX C

DERIVATION OF $\mathbf{P}_{\mathrm{NL}}$

In order to shorten the descriptions, let us put

$$\begin{aligned} a &= E_1 e^{-j\omega_1 t} \\ b &= E_2 e^{-j\omega_2 t} \\ c &= E_3 e^{-j\omega_3 t} \\ d &= E_4 e^{-j\omega_4 t} \end{aligned} \tag{C.1}$$

Equation (8.53) becomes

$$\mathbf{P}_{\mathrm{NL}} = \hat{\mathbf{x}}\epsilon_0 \frac{\chi_{xxxx}}{8}(a + a^* + b + b^* + c + c^* + d + d^*)^3 \tag{C.2}$$

Putting

$$q = a + b + c + d \tag{C.3}$$

$$P_{\mathrm{NL}} = \epsilon_0 \frac{\chi_{xxxx}}{8}(q + q^*)^3 \tag{C.4}$$

Note that

$$(q + q^*)^3 = q^3 + 3q^2 q^* + \text{c.c.} \tag{C.5}$$

Frequencies associated with q^3 are too high and are out of the range of interest. The q^3 terms will be discarded.

$$P_{\mathrm{NL}} = \tfrac{3}{8}\epsilon_0 \chi_{xxxx}(q^2 q^* + \text{c.c.}) \tag{C.6}$$

Discarding q^3 makes a substantial reduction in the number of calculations. The last step is inserting Eq. (C.3) into Eq. (C.6) and performing the multiplication. The result is

$$q^2 q^* + \text{c.c.} = a(|a|^2 + 2|b|^2 + 2|c|^2 + 2|d|^2) + b(2|a|^2 + |b|^2 + 2|c|^2 + 2|d|^2)$$

$$
\begin{aligned}
&+ c(2|a|^2 + 2|b|^2 + |c|^2 + 2|d|^2) + d(2|a|^2 + 2|b|^2 + 2|c|^2 + |d|^2) \\
&+ 2a^*(bc + cd + db) + 2b^*(ac + cd + da) + 2c^*(ab + bd + da) \\
&+ 2d^*(ab + bc + ca) + a^*(b^2 + c^2 + d^2) + b^*(a^2 + c^2 + d^2) \\
&+ c^*(a^2 + b^2 + d^2) + d^*(a^2 + b^2 + c^2) + \text{c.c.} \qquad \text{(C.7)}
\end{aligned}
$$

These terms generate a variety of beat frequencies. From Eq. (C.1), terms such as a^*bc, a^*cd, and a^*db create frequency components of $\omega_2 + \omega_3 - \omega_1, \omega_3 + \omega_4 - \omega_1$, and $\omega_2 + \omega_4 - \omega_1$, respectively. Moreover, for the set of equations that are commensurate with each other,

$$\omega_4 = \omega_1 + \omega_2 - \omega_3 \qquad \text{(C.8)}$$

and these frequency components become $2\omega_2 - \omega_4, \omega_2$, and $2\omega_2 - \omega_3$, respectively. Note, in particular, that b^*cd, a^*cd, d^*ab, and c^*ab become $\omega_1, \omega_2, \omega_3$, and ω_4, respectively.

Rewriting Eq. (C.7) using Eqs. (C.1) and (C.8) reduces Eq. (C.6) to

$$
\begin{aligned}
\mathbf{P}_{\mathrm{NL}} = \tfrac{1}{2}\hat{\mathbf{x}}[&P_{\mathrm{NL}}(\omega_1)e^{j\omega_1 t} + P_{\mathrm{NL}}(\omega_2)e^{j\omega_2 t} + P_{\mathrm{NL}}(\omega_3)e^{j\omega_3 t} + P_{\mathrm{NL}}(\omega_4)e^{j\omega_4 t} \\
&+ P_{\mathrm{NL}}(2\omega_1 - \omega_2) + P_{\mathrm{NL}}(2\omega_1 - \omega_3) + P_{\mathrm{NL}}(2\omega_1 - \omega_4) \\
&+ P_{\mathrm{NL}}(2\omega_2 - \omega_1) + P_{\mathrm{NL}}(2\omega_1 - \omega_3) + P_{\mathrm{NL}}(2\omega_2 - \omega_4) \\
&+ P_{\mathrm{NL}}(2\omega_3 - \omega_1) + P_{\mathrm{NL}}(2\omega_3 - \omega_2) + P_{\mathrm{NL}}(2\omega_3 - \omega_4) \\
&+ P_{\mathrm{NL}}(2\omega_4 - \omega_1) + P_{\mathrm{NL}}(2\omega_4 - \omega_2) + P_{\mathrm{NL}}(2\omega_4 - \omega_3) + \text{c.c.}]
\end{aligned}
$$

where

$$
\begin{aligned}
P_{\mathrm{NL}}(\omega_1) &= \chi_{\mathrm{eff}}[(|E_1|^2 + 2|E_2|^2 + 2|E_3|^2 + 2|E_4|^2)E_1 + 2E_3E_4E_2^*] \\
P_{\mathrm{NL}}(\omega_2) &= \chi_{\mathrm{eff}}[(2|E_1|^2 + |E_2|^2 + 2|E_3|^2 + 2|E_4|^2)E_2 + 2E_3E_4E_1^*] \\
P_{\mathrm{NL}}(\omega_3) &= \chi_{\mathrm{eff}}[(2|E_1|^2 + 2|E_2|^2 + |E_3|^2 + 2|E_4|^2)E_3 + 2E_1E_2E_4^*] \\
P_{\mathrm{NL}}(\omega_4) &= \chi_{\mathrm{eff}}[(2|E_1|^2 + 2|E_2|^2 + 2|E_3|^2 + |E_4|^2)E_4 + 2E_1E_2E_3^*]
\end{aligned}
$$

$$
\begin{aligned}
P_{\mathrm{NL}}(2\omega_1 - \omega_2) &= \chi_{\mathrm{eff}}E_1^2E_2^* \\
P_{\mathrm{NL}}(2\omega_1 - \omega_3) &= \chi_{\mathrm{eff}}(E_1^2E_3^* + 2E_1E_4E_2^*) \\
P_{\mathrm{NL}}(2\omega_1 - \omega_4) &= \chi_{\mathrm{eff}}(E_1^2E_4^* + 2E_1E_3E_2^*) \\
P_{\mathrm{NL}}(2\omega_2 - \omega_1) &= \chi_{\mathrm{eff}}E_2^2E_1^* \\
P_{\mathrm{NL}}(2\omega_2 - \omega_3) &= \chi_{\mathrm{eff}}(E_2^2E_3^* + 2E_2E_4E_1^*) \\
P_{\mathrm{NL}}(2\omega_2 - \omega_4) &= \chi_{\mathrm{eff}}(E_2^2E_4^* + 2E_2E_3E_1^*) \\
P_{\mathrm{NL}}(2\omega_3 - \omega_1) &= \chi_{\mathrm{eff}}(E_3^2E_1^* + 2E_2E_3E_4^*) \\
P_{\mathrm{NL}}(2\omega_3 - \omega_2) &= \chi_{\mathrm{eff}}(E_3^2E_2^* + 2E_1E_3E_4^*) \\
P_{\mathrm{NL}}(2\omega_3 - \omega_4) &= \chi_{\mathrm{eff}}E_3^2E_4^*
\end{aligned}
$$

$$P_{\mathrm{NL}}(2\omega_4 - \omega_1) = \chi_{\mathrm{eff}}(E_4^2 E_1^* + 2E_2 E_4 E_3^*)$$
$$P_{\mathrm{NL}}(2\omega_4 - \omega_2) = \chi_{\mathrm{eff}}(E_4^2 E_2^* + 2E_1 E_4 E_3^*)$$
$$P_{\mathrm{NL}}(2\omega_4 - \omega_3) = \chi_{\mathrm{eff}} E_4^2 E_3^*$$
$$\chi_{\mathrm{eff}} = \frac{3\epsilon_0}{4}\chi_{xxxx}$$