4 Mueller Matrices

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Abstract. The measurement of Mueller matrices provides a useful way of increasing the information obtainable from scattering experiments. Nevertheless, the analysis of this information is not a simple matter because of the complicated relationships between the elements of these matrices. This chapter deals with the properties of Mueller matrices and with the analysis of physical parameters measurable by polarimetric techniques. First, the case of scattering systems that do not reduce the degree of polarization of totally-polarized incident light is studied. The "pure Mueller matrices" corresponding to these kinds of system are studied and classified by means of their polar decomposition. The general structure of Mueller matrices is then studied and is applied to the "parallel decomposition" of the scattering system. A comparison of Mueller matrices with "Stokes matrices", the Transmittance Condition, a Purity Criterion, the Degree of Purity and Polarizance parameters are also dealt with.

1 Introduction

Polarimetric techniques are useful in order to increase the information obtained in scattering measurements. The effects of a scattering system in the polarization of light beams that interact with it can be expressed mathematically by means of the so-called Mueller matrix (or "scattering matrix"). The polarization state of a light beam can be represented by means of the Stokes vector, constituted by the four real Stokes parameters. The Mueller matrix is a real 4x4 matrix that gives the linear transformation of the Stokes vector of the incident beam into the Stokes vector of the emerging beam. So, the Mueller matrix depends on the particular interaction conditions as well as the spectral characteristics of the incident light.

Thus, the 16 elements of the Mueller matrix contain all the information obtainable by means of polarimetric measurements. These elements are not totally independent in the sense that they are restricted by several bilinear and quadratic inequalities. In certain important cases some of these inequalities becomes equalities and the effective number of free parameters is reduced [1].

In consequence, a high degree of knowledge of the relationships between the elements of a Mueller matrix is important for analyzing the information contained in such matrices obtained from experimental measurements. Mueller matrix measurements allow us to identify and analyze properties such as diattenuation, birefringence and depolarization. These properties are related

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with the nature and distribution of the scatterers that constitute the whole scattering system [2,3].

From a theoretical point of view, the interest comes from the necessity of establishing models that fit with the differences in behavior exhibited by the scattering systems. The main problem basically consists of extracting and understanding the physical information contained in the elements of a Mueller matrix.

This chapter deals with the properties of Mueller matrices and with the analysis of physical parameters measurable by polarimetric techniques. In Sect. 2, the case of scattering systems that do not reduce the degree of polarization of totally-polarized incident light is studied. The "pure Mueller matrices" [4,5] corresponding to this kind of system are studied and classified in Sect. 3 by means of their polar decomposition. The general structure of Mueller matrices is studied in Sect. 4, and is then applied to the "parallel decomposition" of the scattering system (Sect. 5). Section 6 is devoted to clarifying the relationship between the set of physical Mueller matrices and the set of "Stokes matrices" mathematically defined as the matrices that transform any Stokes vector into another Stokes vector. The conditions derived from the restriction that the transmittance of the passive optical system is not higher than 1 are considered in Sect. 7. In Sect. 8 we deal with a "Purity Criterion" for Mueller matrices, as well as purity and polarizance parameters.

2 Basic Transformation of Polarization

First, let us consider the Jones formalism in order to represent the effects of a deterministic-nondepolarizing optical system. It should be noted that we use the term "deterministic-nondepolarizing" in the sense of totally polarized incident light always emerging totally polarized. This observation is important because the degree of polarization of a partially polarized light beam can decrease when it interacts with some kinds of deterministic optical systems[6] (for example a diattenuator). Furthermore, there exist deterministic optical systems that can depolarize totally polarized polychromatic light[7], resulting in an emerging light beam that can not be described by a stable Jones vector, but by the corresponding Stokes vector. Thus, we can say that any deterministic-nondepolarizing optical system can be represented by its corresponding Jones matrix. This circumstance occurs in several phenomena such as scattering by a single particle or by a collection of identical particles [2]. Let us consider the analytical signal representations $\varepsilon_1(t)$, $\varepsilon_2(t)$

$$\begin{aligned}\varepsilon_1(t) &= E_1(t) + i\tilde{E}_1(t),\\ \varepsilon_2(t) &= E_2(t) + i\tilde{E}_2(t).\end{aligned} \tag{1}$$

where $E_1(t)$, $E_2(t)$ are the real components of the electric field, and $\tilde{E}_1(t)$, $\tilde{E}_2(t)$ are its Hilbert transforms [8,9].

It is well known that when light incides on a linear passive optical system, the transformation of the field state can be written as

$$\varepsilon'(t) = T\varepsilon(t), \tag{2}$$

where

$$\varepsilon(t) \equiv \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix}, \; \varepsilon'(t) \equiv \begin{pmatrix} \varepsilon'_1(t) \\ \varepsilon'_2(t) \end{pmatrix}, \tag{3}$$

are the emerging and incident Jones vectors and T is the Jones matrix of the system.

In general both emerging and incident fields can fluctuate. Now we construct the polarization matrix [8], namely, the 2×2-covariance matrix of the two variables $\varepsilon_1(t), \varepsilon_2(t)$

$$\Phi = \langle \varepsilon \otimes \varepsilon^\dagger \rangle = \begin{pmatrix} \varepsilon_1\varepsilon_1^* & \varepsilon_1\varepsilon_2^* \\ \varepsilon_2\varepsilon_1^* & \varepsilon_2\varepsilon_2^* \end{pmatrix}, \tag{4}$$

where the angular brackets denote time averaging, $\otimes$ denotes direct (Kronecker) product, $\varepsilon^\dagger$ denotes the complex conjugate transpose of ε, and ε_i^* denotes the complex conjugate of ε_i.

The relation between Φ and the corresponding Stokes parameters is given by [10,8]

$$\Phi = \frac{1}{2}\sum_{k=0}^{3} s_k\sigma_k = \frac{1}{2}\begin{pmatrix} s_0 + s_1 & s_0 + is_3 \\ s_0 - is_3 & s_0 - s_1 \end{pmatrix}, \tag{5}$$

where $\sigma_k (k = 1, 2, 3)$ are the Pauli matrices and σ_0 is the identity matrix

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{6}$$

The intensity and the degree of polarization are invariant under transformation of the reference axis, and thus have relevant physical meaning. They can be expressed as

$$I = \mathrm{Tr}\Phi, \tag{7}$$

$$P = \left[1 - \frac{4\det\Phi}{(\mathrm{Tr}\Phi)^2}\right]. \tag{8}$$

The polarization matrix of the emerging beam is

$$\Phi' = \langle \varepsilon' \otimes \varepsilon'^{\dagger} \rangle = \langle (T\varepsilon) \otimes (T\varepsilon)^{\dagger} \rangle = \langle T\varepsilon \otimes \varepsilon^{\dagger} T^{\dagger} \rangle = T \langle \varepsilon \otimes \varepsilon^{\dagger} \rangle T^{\dagger} = T \Phi T^{\dagger}. \tag{9}$$

In order to make subsequent straightforward calculations, we introduce the following alternative notation for the elements of Φ (which we will also use, where convenient, for other 2x2 matrices)

$$\varphi_0 \equiv \varphi_{11}, \quad \varphi_1 \equiv \varphi_{12}\ , \quad \varphi_2 \equiv \varphi_{21}, \quad \varphi_3 \equiv \varphi_{22}. \tag{10}$$

The elements of the polarization matrix can also be written as a column-vector φ [11] defined by

$$\varphi \equiv \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \varepsilon_1^* \\ \varepsilon_1 \varepsilon_2^* \\ \varepsilon_2 \varepsilon_1^* \\ \varepsilon_2 \varepsilon_2^* \end{pmatrix} = \langle \varepsilon \otimes \varepsilon^* \rangle. \tag{11}$$

The polarization vector φ and the corresponding Stokes vector $\mathbf{s}$ are related by means of the following expression

$$\mathbf{s} = A\varphi, \quad A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix}. \tag{12}$$

The polarization vector φ' of the emerging beam is given by

$$\varphi' = \langle \varepsilon' \otimes \varepsilon'^* \rangle = \langle (T\varepsilon) \otimes (T\varepsilon)^* \rangle = \langle (T\varepsilon) \otimes (T^* \varepsilon^*) \rangle, \tag{13}$$

which, applying the properties of the Kronecker product, can be written as

$$\varphi' = \langle (T \otimes T^*) (\varepsilon \otimes \varepsilon^*) \rangle = (T \otimes T^*) \langle (\varepsilon \otimes \varepsilon^*) \rangle = (T \otimes T^*) \varphi, \tag{14}$$

or as

$$\varphi' = V\varphi,\ V \equiv (T \otimes T^*)\,, \tag{15}$$

where the deterministic-nondepolarizing optical system is represented by the "complex Mueller-Jones matrix" V [12]. The corresponding Mueller-Jones matrix N of the optical system is related to V and T by[12][13]

$$N = AVA^{-1} = A\,(T \otimes T^*)\,A^{-1}, \tag{16}$$

or, in components

$$n_{kl} = \frac{1}{2} \mathrm{Tr}\left(\sigma_k T \sigma_l T^{\dagger} \right), \quad (k, l = 0, 1, 2, 3)\,. \tag{17}$$

In the next section we will deal with optical systems whose effect on polarized light can not be represented by means of Jones matrices, but can be represented by Mueller matrices. Therefore we will distinguish between Mueller-Jones matrices ("pure Mueller matrices"), which correspond to deterministic-nondepolarizing optical systems, i. e., characterizable by Jones matrices, and Mueller matrices in general. For reasons of clarity, hereafter we will use the following notation for the different types of matrices: V, complex pure Mueller matrices; L, complex Mueller matrices; N, pure Mueller matrices, and M, Mueller matrices. The justification for using "pure" to refer to certain sort of matrices will be clear from subsequent sections.

It is now pertinent to consider the reversibility properties of the matrices that represent optical systems, i. e. the operation of interchanging the incident and emerging light beams. For a given optical system characterized by a Jones matrix T, the Jones matrix representing the same system when the incident and emerging beams are interchanged is T^T [14] (absence of Faraday effects assumed). For the complex Mueller-Jones matrix $V(T)$ it is easy to prove that $V(T^T) = V^T(T)$, and that the corresponding reverse Mueller-Jones matrix is given by[15]

$$N^R = N(T^T) = QN^TQ, \tag{18}$$

where Q is the diagonal matrix $D(1,1,1,-1)$ that performs the sign change of the element s_3 in the reverse interaction.

3 Polar Decomposition of Pure Matrices

Let us now consider the polar decomposition of Jones matrices and pure Mueller matrices [16–18]. Any Jones matrix can be written as $J = J_P J_R$, or $J = J'_R J'_P$, where J_P, J'_P are Hermitian matrices and J_R, J'_R are unitary matrices.

In the Jones formalism, Hermitian matrices are associated with diattenuators (partial polarizers). These matrices can be written as

$$J_P(\alpha,\delta,k_1,k_2) = \begin{pmatrix} k_1 c_\alpha^2 + k_2 s_\alpha^2 & (k_1-k_2)\, c_\alpha s_\alpha e^{-i\delta} \\ (k_1-k_2)\, c_\alpha s_\alpha e^{i\delta} & k_1 s_\alpha^2 + k_2 c_\alpha^2 \end{pmatrix}$$
$$(c_\alpha \equiv \cos\alpha,\ s_\alpha \equiv \sin\alpha)\,. \tag{19}$$

k_1, k_2 ($k_1 > k_2$) are the principal coefficients of amplitude transmission for the two orthogonal polarization eigenstates of J_P. Their azimuths (namely χ and $\chi + \frac{\pi}{2}$) and their ellipticities (ω and -ω) are given by

$$\begin{aligned} \tan 2\chi &= \tan 2\alpha \cos\delta, \\ \sin 2\omega &= \sin 2\alpha \sin\delta. \end{aligned} \tag{20}$$

This kind of Jones matrix satisfies $\det J_P = k_1 k_2$.

Interesting specific cases are: $\delta = 0$, linear diattenuator (linear partial polarizer); $k_2 = 0$, total polarizer.

The unitary Jones matrices are associated with pure retarders and can be written as

$$J_R(\beta, \gamma, \Delta) = \begin{pmatrix} c_\beta^2 e^{i\frac{\Delta}{2}} + s_\beta^2 e^{-i\frac{\Delta}{2}} & \left(e^{i\frac{\Delta}{2}} - e^{-i\frac{\Delta}{2}}\right) e^{-i\gamma} s_\beta c_\beta \\ \left(e^{i\frac{\Delta}{2}} - e^{-i\frac{\Delta}{2}}\right) e^{i\gamma} s_\beta c_\beta & s_\beta^2 e^{i\frac{\Delta}{2}} + c_\beta^2 e^{-i\frac{\Delta}{2}} \end{pmatrix}$$
$$(c_\beta \equiv \cos\beta, \; s_\beta \equiv \sin\beta). \quad (21)$$

Δ stands for the retardation caused by the retarder between the two elliptical polarized eigenstates of J_R. The respective azimuths ψ, $\psi + \frac{\pi}{2}$, and ellipticities $(\nu, -\nu)$ of these eigenstates are given by

$$\tan 2\psi = \tan 2\beta \cos\gamma,$$
$$\sin 2\nu = \sin 2\beta \sin\gamma. \quad (22)$$

Interesting specific cases are: $\nu = 0$, linear retarder; $\nu = \pm\frac{\pi}{2}$, $\beta = \frac{\pi}{4}$, circular retarder.

The polar decomposition of a Jones matrix shows that the polarization effects of any "pure" scatterer are equivalent to that given by a system composed of a serial combination of an elliptical diattenuator and an elliptical retarder in either of the two possible relative positions. These results can be applied to the Mueller representation, so that a pure Mueller matrix N can be written as $N = N_P N_R$, or $N = N_R^{'} N_P^{'}$, where N_P, $N_P^{'}$ are real symmetric matrices and N_R, $N_R^{'}$ are orthogonal matrices.

Although the equivalent system is not unique, the physical parameters $(\alpha, \delta, k_1, k_2, \beta, \gamma, \Delta)$, $(\alpha', \delta', k_1', k_2', \beta', \gamma', \Delta')$ corresponding to the components of the respective "equivalent systems" provide all information obtainable by polarimetric techniques, and are useful in order to study properties of the scattering system.

The four parameters of the "equivalent diattenuator" are obtained through the following expressions [17]

$$k_1 = k_1' = (n_{00} + q) \;,$$
$$k_2 = k_2' = (n_{00} - q) \;, \quad (23)$$

$$\sin 2\alpha = \frac{\left(n_{02}^2 + n_{03}^2\right)^{\frac{1}{2}}}{q}, \quad \cos 2\alpha = \frac{n_{01}}{q} \;, \quad (24)$$

$$\sin 2\alpha' = \frac{\left(n_{20}^2 + n_{30}^2\right)^{\frac{1}{2}}}{q}, \quad \cos 2\alpha' = \frac{n_{10}}{q}, \quad (25)$$

$$\tan\left(\frac{\delta}{2}\right) = \frac{n_{30}}{n_{20}}\ ; \quad \tan\left(\frac{\delta'}{2}\right) = \frac{n_{03}}{n_{02}}\ , \tag{26}$$

where

$$q^2 \equiv n_{01}^2 + n_{02}^2 + n_{03}^2 = n_{10}^2 + n_{20}^2 + n_{30}^2. \tag{27}$$

To obtain the parameters of the "equivalent retarder" it is necessary to distinguish two cases depending on the value of $\det N$.

a) $\det N \neq 0$

Since $\det N_R = 1$, then $\det N_P = \det N_P' = \det N \neq 0$. It can thus be proved that the inverse matrix of N_P is

$$[N_P\,(k_1, k_2, \alpha, \delta)]^{-1} = N_P\left(\frac{1}{k_1}, \frac{1}{k_2}, \alpha, \delta\right). \tag{28}$$

Although this is not a physical Mueller matrix (because $\frac{1}{k_1} \geq 1$, $\frac{1}{k_2} \geq 1$), the Mueller matrix of the equivalent retarders can be obtained as

$$N_R = N_P\left(\frac{1}{k_1}, \frac{1}{k_2}, \alpha, \delta\right) N; \quad N_R' = N N_P'\left(\frac{1}{k_1'}, \frac{1}{k_2'}, \alpha', \delta'\right)\ . \tag{29}$$

Once N_R and N_R' (whose elements are denoted by r_{ij}, r_{ij}' respectively) are calculated, the parameters (Δ, ν, ψ) , $(\Delta', \nu', \psi')$of the equivalent retarders are given by [17]

$$\cos^2\left(\frac{\Delta}{2}\right) = \frac{1}{4}\mathrm{Tr}N_R, \quad \cos^2\left(\frac{\Delta'}{2}\right) = \frac{1}{4}\mathrm{Tr}N_R', \tag{30}$$

$$\sin 2\nu = \frac{r_{12} - r_{21}}{2\sin\left(\frac{\Delta}{2}\right)}, \quad \sin 2\nu' = \frac{r_{12}' - r_{21}'}{2\sin\left(\frac{\Delta'}{2}\right)}, \tag{31}$$

$$\sin 2\psi = \frac{r_{31} - r_{13}}{2\cos 2\nu \sin\Delta}, \quad \sin 2\psi' = \frac{r_{31}' - r_{13}'}{2\cos 2\nu' \sin\Delta'}, \tag{32}$$

which leads to

$$\cos 2\beta = \cos 2\nu \cos 2\psi, \quad \cos 2\beta' = \cos 2\nu' \cos 2\psi', \tag{33}$$

$$\tan\gamma = \frac{\tan 2\nu}{\sin 2\psi}, \quad \tan\gamma' = \frac{\tan 2\nu'}{\sin 2\psi'}. \tag{34}$$

b) $\det N = 0$

In this case $k_2 = 0$ and N_R, N_R' are not unique. An arbitrary value can be chosen for γ, for example $\gamma = 0$. Thus, the Mueller matrix N can be expressed in terms of equivalent systems composed of a total polarizer and a linear retarder $N = N_P(\alpha, \delta, k_1, 0)\, N_R(\beta, 0, \Delta)$; $N = N'_R(\beta', 0, \Delta')\, N'_P(\alpha', \delta, k_1, 0)$. Now β, β' are given by

$$\tan 2\beta = \tan 2\beta' = \frac{n_{10} - n_{01}}{n_{02} - n_{20}}, \tag{35}$$

and Δ (Δ') have two possible solutions Δ_1, Δ_2 (Δ'_1, Δ'_2)

$$\tan \Delta_1 = \tan \Delta'_1 = \frac{bn_{30} - an_{03}}{b^2 - n_{03}^2}, \tag{36}$$

$$\tan \Delta_2 = \tan(\pi - \Delta'_2) = \frac{bn_{30} + an_{03}}{b^2 - n_{03}^2}, \tag{37}$$

where

$$a \equiv n_{01} \sin \beta - n_{02} \cos \beta, \qquad b \equiv n_{10} \sin \beta - n_{20} \cos \beta. \tag{38}$$

In the case of $a = n_{03}$, these results must be replaced by $\Delta = \Delta' = \frac{\pi}{2}$.

The obtainment of the physical parameters of the equivalent system corresponding to experimentally measured pure Mueller scattering matrices, provides an appropriate way of inspecting the polarimetric properties of the sample, avoiding potentially erroneous analysis due to the complicated implicit relationships between the elements of the Mueller matrix. Thus, when the experimental conditions of scattering ensure the coherent superposition of the emerging light beams, the Mueller matrix of the system is a pure Mueller matrix and this procedure can be applied. Moreover, several equivalence theorems, such as those summarized below, could be useful in the analysis of pure Mueller matrices[14][19].

- "Any serial combination of retarders is optically equivalent to a retarder (in general elliptical)".
- "Any serial combination of retarders is optically equivalent to a serial combination of two linear retarders (the solution is not unique)".
- "Any serial combination of retarders is optically equivalent to a serial combination of a linear retarder and a rotator".
- "Any elliptical diattenuator is optically equivalent to a linear diattenuator placed between two identical linear retarders with orthogonal birefringence axis".

4 Mueller Matrices for Incoherent Scattering

Depending on the characteristics of the scattering system as well as the coherence of the light, for certain directions scattered light could be a superposition of several incoherent wavelets. In this case the optical system can not be represented by means of a Jones matrix nor a pure Mueller matrix, but can be considered as an ensemble [20–22] composed of deterministic-nondepolarizing (pure) elements, each one with a well-defined Jones matrix, in such a manner that the incident light beam is shared among these different elements (or particles). Let I_i be the intensity of the portion of light that incides on the "i" particle. The ratio between I_i and the intensity I of the whole beam is denoted by a respective coefficient p_i so that

$$p_i = \frac{I_i}{I}, \ \sum_i p_i = 1. \tag{39}$$

Now we denote by $T^{(i)}, V^{(i)}$ and $N^{(i)}$ the matrices representing the "i" element in the different formalisms (i. e.: Jones matrix, complex Mueller matrix and Mueller matrix). Thus, for a given scattering direction, the Jones vector of the emerging wavelet is

$$\varepsilon_i'(t) = T^{(i)} \left(\sqrt{p_i}\varepsilon(t)\right), \tag{40}$$

and the corresponding polarization vector is given by

$$\begin{aligned} \varphi_i' &= \langle \varepsilon_i' \otimes \varepsilon_i'^{*} \rangle = \langle \left(T^{(i)} \sqrt{p_i}\varepsilon(t)\right) \otimes \left(T^{(i)} \sqrt{p_i}\varepsilon(t)\right)^{*} \rangle \\ &= p_i \langle \left(T^{(i)} \otimes T^{(i)*}\right) (\varepsilon \otimes \varepsilon^{*}) \rangle = p_i V^{(i)} \varphi. \end{aligned} \tag{41}$$

The polarization state of the complete emerging beam (for the scattering direction in question) is given by the incoherent superposition of the beams emerging from each element. Then, the polarization vector of the emerging beam is

$$\varphi' = L\varphi; \ L \equiv \left(\sum_i p_i V^{(i)} \right), \tag{42}$$

showing that the corresponding Mueller matrix M is given by

$$M = A \left(\sum_i p_i V^{(i)} \right) A^{-1} = \sum_i p_i \left(A V^{(i)} A^{-1} \right) = \sum_i p_i N^{(i)}. \tag{43}$$

We have obtained this result by considering the optical system as composed of a set of parallel pure elements, but it is worth pointing out that the same result could be obtained by considering the system as an ensemble[23] so that each realization "i", characterized by a well-defined Jones matrix $T^{(i)}$, occurs with a probability p_i.We can therefore consider the optical system as composed of such an ensemble, and will refer to ensemble averages by means of brackets $\langle x \rangle \equiv \sum_i p_i x^{(i)}$ so that $\langle N \rangle \equiv \sum_i p_i N^{(i)}$, or, in components $\langle n_{kl} \rangle \equiv \sum_i p_i n_{kl}^{(i)}$, $(k, l = 0, 1, 2, 3)$.

5 Parallel Decomposition of Mueller Matrices

By analyzing the expressions of the elements of the matrix L obtained in the previous section

$$L = \begin{pmatrix} \langle t_0 t_0^* \rangle & \langle t_0 t_1^* \rangle & \langle t_1 t_0^* \rangle & \langle t_1 t_1^* \rangle \\ \langle t_0 t_2^* \rangle & \langle t_0 t_3^* \rangle & \langle t_1 t_2^* \rangle & \langle t_1 t_3^* \rangle \\ \langle t_2 t_0^* \rangle & \langle t_2 t_1^* \rangle & \langle t_3 t_0^* \rangle & \langle t_3 t_1^* \rangle \\ \langle t_2 t_2^* \rangle & \langle t_2 t_3^* \rangle & \langle t_3 t_2^* \rangle & \langle t_3 t_3^* \rangle \end{pmatrix}, \tag{44}$$

we can clearly see that its elements can be suitably reordered to construct a new matrix H whose elements are defined by

$$h_{kl} \equiv \frac{1}{2} \langle t_k t_l^* \rangle \, ; \;\; k, l = 0, 1, 2, 3. \tag{45}$$

This structure corresponds to a positive-semidefinite Hermitian matrix [24], consequently characterized by four non-negative eigenvalues. The explicit relation between M and H is given by the following expressions

$$H = \frac{1}{4} \begin{pmatrix} \begin{matrix} m_{00} + m_{01} \\ +m_{10} + m_{11} \end{matrix} & \begin{matrix} m_{02} + m_{12} \\ +i\,(m_{03} + m_{13}) \end{matrix} & \begin{matrix} m_{20} + m_{21} \\ -i\,(m_{30} + m_{31}) \end{matrix} & \begin{matrix} m_{22} + m_{33} \\ +i\,(m_{23} - m_{32}) \end{matrix} \\[2ex] \begin{matrix} m_{02} + m_{12} \\ -i\,(m_{03} + m_{13}) \end{matrix} & \begin{matrix} m_{00} - m_{01} \\ +m_{10} - m_{11} \end{matrix} & \begin{matrix} m_{22} - m_{33} \\ -i\,(m_{23} + m_{32}) \end{matrix} & \begin{matrix} m_{20} - m_{21} \\ -i\,(m_{30} - m_{31}) \end{matrix} \\[2ex] \begin{matrix} m_{20} + m_{21} \\ +i\,(m_{30} + m_{31}) \end{matrix} & \begin{matrix} m_{22} - m_{33} \\ +i\,(m_{23} + m_{32}) \end{matrix} & \begin{matrix} m_{00} + m_{01} \\ -m_{10} - m_{11} \end{matrix} & \begin{matrix} m_{02} - m_{12} \\ +i\,(m_{03} - m_{13}) \end{matrix} \\[2ex] \begin{matrix} m_{22} + m_{33} \\ -i\,(m_{23} - m_{32}) \end{matrix} & \begin{matrix} m_{20} - m_{21} \\ +i\,(m_{30} - m_{31}) \end{matrix} & \begin{matrix} m_{02} - m_{12} \\ -i\,(m_{03} - m_{13}) \end{matrix} & \begin{matrix} m_{00} - m_{01} \\ -m_{10} + m_{11} \end{matrix} \end{pmatrix}, \tag{46}$$

$$M = \begin{pmatrix} \begin{matrix} h_{00} + h_{11} \\ +h_{22} + h_{33} \end{matrix} & \begin{matrix} h_{00} - h_{11} \\ +h_{22} - h_{33} \end{matrix} & \begin{matrix} h_{01} + h_{10} \\ +h_{23} + h_{32} \end{matrix} & \begin{matrix} -i\,(h_{01} - h_{10}) \\ -i\,(h_{23} - h_{32}) \end{matrix} \\[2ex] \begin{matrix} h_{00} + h_{11} \\ -h_{22} - h_{33} \end{matrix} & \begin{matrix} h_{00} - h_{11} \\ -h_{22} + h_{33} \end{matrix} & \begin{matrix} h_{01} + h_{10} \\ -h_{23} - h_{32} \end{matrix} & \begin{matrix} -i\,(h_{01} - h_{10}) \\ +i\,(h_{23} - h_{32}) \end{matrix} \\[2ex] \begin{matrix} h_{02} + h_{20} \\ +h_{13} + h_{31} \end{matrix} & \begin{matrix} h_{02} + h_{20} \\ -h_{13} - h_{31} \end{matrix} & \begin{matrix} h_{03} + h_{30} \\ +h_{12} + h_{21} \end{matrix} & \begin{matrix} -i\,(h_{03} - h_{30}) \\ +i\,(h_{12} - h_{21}) \end{matrix} \\[2ex] \begin{matrix} i\,(h_{02} - h_{20}) \\ +i\,(h_{13} - h_{31}) \end{matrix} & \begin{matrix} i\,(h_{02} - h_{20}) \\ -i\,(h_{13} - h_{31}) \end{matrix} & \begin{matrix} i\,(h_{03} - h_{30}) \\ +i\,(h_{12} - h_{21}) \end{matrix} & \begin{matrix} h_{03} + h_{30} \\ -h_{12} - h_{21} \end{matrix} \end{pmatrix}. \tag{47}$$

Let us note that H determines the Mueller matrix uniquely and vice-versa, and that the relation can also be written as

$$H = \frac{1}{4} \sum_{k,l=0}^{3} m_{kl} E_{kl}, \tag{48}$$

where E_{kl} $(k, l = 0, 1, 2, 3)$ denotes the set of 16 Hermitian and unitary Dirac matrices

$$E_{kl} \equiv \sigma_k \otimes \sigma_l, \tag{49}$$

which constitutes a complete set for expanding a Hermitian operator into a linear combination with real coefficients. These coefficients are 16 directly measurable magnitudes, namely the 16 elements of the Mueller matrix M associated with H. The Dirac matrices (as well as the Pauli matrices) are trace orthogonal, in the sense that the trace of the product of any two of these operators is equal to zero, and constitute a complete set of base matrices. This expansion provides a fundamental significance for the elements of the Mueller matrix, in addition to their phenomenological significance. The relation between the matrix H and its corresponding Mueller matrix is, in fact, analogous to that between the polarization matrix and its corresponding Stokes parameters. It is possible to classify the Mueller matrices according to the rank of H, i. e. according to the number of non-zero eigenvalues. For example, H corresponds to a pure Mueller matrix when only one eigenvalue is non-zero. Since H is Hermitian, it can be diagonalized through a similarity transformation, given by a unitary matrix U, such as

$$H = U \Lambda U^{-1}, \tag{50}$$

where Λ is the eigenvalue diagonal matrix $D(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, with $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$.

Then, we can write H as

$$\begin{aligned} H = {} & \lambda_0 U D(1,0,0,0) U^{-1} + \lambda_1 U D(0,1,0,0) U^{-1} \\ & + \lambda_2 U D(0,0,1,0) U^{-1} + \lambda_3 U D(0,0,0,1) U^{-1}, \end{aligned} \tag{51}$$

which is a linear combination, with non-negative coefficients, of four matrices, each with only one eigenvalue equal to 1. By carrying this result to the Mueller formalism, we get

$$M = \lambda_0 N_0 + \lambda_1 N_1 + \lambda_2 N_2 + \lambda_3 N_3. \tag{52}$$

Thus, we see that any Mueller matrix can be decomposed into one to four pure Mueller matrices weighted by the eigenvalues of H. This decomposition provides a physical interpretation analogous to the decomposition of a polarization matrix into a superposition of two orthogonal-polarized, incoherent, linearly independent beams whose intensities are the two eigenvalues of the polarization matrix[25].

We see that the system can be considered a parallel combination of one to four pure elements and that an appropriate procedure of analysis of an experimentally obtained Mueller matrix is the following

- From M, calculate H,
- Calculate the eigenvalues of H and the eigenvectors matrix U,
- Calculate the pure components N_i of the "parallel decomposition" of M,
- Apply the polar decomposition theorem for each component N_i and obtain the parameters of the equivalent systems,
- Interpret the characteristics of the scattering system in terms of these results.

It should be noted that the equivalent parameters of the four parallel components are non independent. In fact, the total 16 free parameters are shared among the four parallel components in such a manner that N_0, N_1, N_2 and N_3, depend on $7, 5, 3$ and 1 independent parameters respectively.

A peculiar characteristic of the above parallel decomposition is that the weights are the eigenvalues of H. Nevertheless there are infinite possible decompositions where the weights are different from λ_i.

6 Mueller Matrices and Stokes Matrices

In this section we refer to "Stokes matrices" for the set of matrices B characterized by the fact that, for any Stokes vector $\mathbf{s}$, the transformed vector $B\mathbf{s}$ is also a Stokes vector. The implicit or explicit identification between the set of Stokes matrices with the set of Mueller matrices is the starting point of several works dealing with the properties of Mueller matrices. The general characterization of Mueller matrices through the non-negativity of the eigenvalues of H enables us to analyze whether both sets are in fact identical. First of all, we can observe that, obviously, any Mueller matrix is a Stokes matrix. Nevertheless the converse is not true, i. e., there exist Stokes matrices that are not Mueller matrices. As a counterexample, let us consider the Stokes matrix $D(1,0,1,1)$ whose corresponding H matrix, calculated with Eq. 46, has negative determinant. We find that some properties arise from the peculiar structure of Mueller matrices and not exclusively from the fact that they transform Stokes vectors into Stokes vectors.

7 Transmittance Condition

As we have seen, the non-negativity of the eigenvalues of H matrices leads to restrictive inequalities between the elements of Mueller matrices. Here we will consider the restrictions derived from considering passive systems (such as scattering systems) whose maximum intensity transmittance t_{max} is less or equal to 1.

For a given Mueller matrix M, and an incident light beam with a Stokes vector $\mathbf{s}$, the intensity transmittance "t" is given by the ratio of the exiting intensity s'_0 to the incident intensity s_0

$$t = \frac{s'_0}{s_0} = \frac{m_{00}s_0 + m_{01}s_1 + m_{02}s_2 + m_{03}s_3}{s_0}. \tag{53}$$

The intensity transmittance averaged over all incident polarization states is equal to m_{00}, which is also the transmittance for unpolarized light ($s_1 = s_2 = s_3 = 0$). The maximum and minimum of t

$$t_+ = m_{00} + \left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{\frac{1}{2}}, \tag{54}$$

$$t_- = m_{00} - \left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{\frac{1}{2}}, \tag{55}$$

are reached for the following respective incident Stokes vectors [26]

$$\mathbf{s}_+ = \left(\left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{\frac{1}{2}}, m_{01}, m_{02}, m_{03}\right)^T, \tag{56}$$

$$\mathbf{s}_- = \left(\left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{\frac{1}{2}}, -m_{01}, -m_{02}, -m_{03}\right)^T. \tag{57}$$

Thus, the "Transmittance Condition" can be expressed as

$$t_+ = m_{00} + \left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{\frac{1}{2}} \leq 1. \tag{58}$$

On the other hand, the condition $t_- \geq 0$ does not lead to any additional restriction because any Mueller matrix satisfy the following inequality

$$\left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{\frac{1}{2}} \leq m_{00}. \tag{59}$$

8 Purity Criterion and Purity Index

As we have seen, incoherent superposition of the wavelets scattered in a given direction results in depolarization, and an equivalent system constituted by pure components requires parallel combination. Once a Mueller matrix is measured, it is very useful to apply a simple criterion in order to check (within the experimental limits of accuracy) if the matrix corresponds to a pure system or to a parallel incoherent combination. This "Purity Criterion" can be stated as the following theorem: "A necessary and sufficient condition for a Mueller matrix M to be a pure Mueller matrix is that M satisfies the equality $\mathrm{Tr}\left(M^T M\right) = 4m_{00}^2$ " [27].

In general, any Mueller matrix satisfies $\mathrm{Tr}\left(M^T M\right) \leq 4m_{00}^2$, and the equality is reached for pure Mueller matrices [20,27,28].

The terms of this condition can be expressed in terms of the Euclidean norm of M [27],

$$\|M\|_e \equiv \left(\sum_{i,j=0}^{3} m_{ij}^2\right)^{\frac{1}{2}} = \left[\mathrm{Tr}\left(M^T M\right)\right]^{\frac{1}{2}}, \tag{60}$$

and in terms of m_{00}, which coincides with the norm defined by the maximum of the elements of M, (it can be proved that $m_{00} \geq 0$, $m_{00} \geq m_{ij}$),

$$\|M\|_m \equiv m_{00}. \tag{61}$$

Then, the general condition results in the "Condition of the Norms"

$$\|M\|_e \leq 2\,\|M\|_m\,. \tag{62}$$

These results have similar well-known conditions for Stokes vectors. In fact, the elements of Stokes vectors verify the condition $s_1^2 + s_2^2 + s_3^2 \leq s_0^2$, which can be expressed as

$$\|\mathbf{s}\|_e \leq \sqrt{2}\,\|\mathbf{s}\|_m\,. \tag{63}$$

As with the degree of polarization of light beams, it is possible to define a non-dimensional parameter for measuring the "Degree of Purity" $P(M)$ of an optical system represented by a Mueller matrix M[29]

$$P(M) \equiv \left[\frac{\mathrm{Tr}\left(M^T M\right) - m_{00}^2}{3m_{00}^2}\right]^{\frac{1}{2}} = \left[\frac{1}{3}\left(\frac{4\mathrm{Tr}\left(H^2\right)}{\left(\mathrm{Tr}H\right)^2} - 1\right)\right]^{\frac{1}{2}}. \tag{64}$$

The value of $P(M)$ is restricted by the following limits

$$0 \leq P(M) \leq 1. \tag{65}$$

$P(M) = 0$ corresponds to a total depolarizer whose Mueller matrix elements are zero except m_{00}.

$P(M) = 1$ corresponds to a pure Mueller matrix.

Other kinds of parameters are related to the diattenuation effects. The "Polarizance" K, defined as the degree of polarization of the emerging light when unpolarized light is incident

$$K \equiv \frac{\left(m_{01}^2 + m_{02}^2 + m_{03}^2\right)^{\frac{1}{2}}}{m_{00}}, \tag{66}$$

provides a measure of the polarizing power of the optical system. The “Reverse Polarizance”[29] is defined in the same way

$$K_r \equiv \frac{\left(m_{10}^2 + m_{20}^2 + m_{30}^2\right)^{\frac{1}{2}}}{m_{00}}, \tag{67}$$

referring to the same system when the incident and emerging light beams are interchanged.

The limits for the values of K and K_r are

$$0 \leq K \leq 1;\ 0 \leq K_r \leq 1. \tag{68}$$

In the case of pure Mueller matrices $K_d = K_r$, but in general their values can be different.

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