Elements of Photonics, Volume I: In Free Space and Special Media. Keigo Iizuka

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APPENDIX A

DERIVATION OF THE FRESNEL–KIRCHHOFF DIFFRACTION FORMULA FROM THE RAYLEIGH–SOMMERFELD DIFFRACTION FORMULA

The Rayleigh–Sommerfeld diffraction formula uses the Fourier transform of the input field, but Fresnel–Kirchhoff's integral equation uses the input field directly to find the diffraction field. This appendix shows how the latter is derived from the former formula (Kazuo Tanaka, private communication).

By combining Eq. (1.177) with (1.178), an expression for the diffraction pattern can be obtained directly from the input field as

$$E(x_i, y_i, z_i) = \iint \left(\iint E(x_0, y_0, 0) e^{-j2\pi f_x x_0 - j2\pi f_y y_0} dx_0\, dy_0 \right) \times e^{j2\pi\sqrt{f_s^2 - f_x^2 - f_y^2} z_i} e^{j2\pi f_x x_i + j2\pi f_y y_i}\, df_x\, df_y \tag{A.1}$$

Reversing the order of integration gives

$$E(x_i, y_i, z_i) = \iint dx_0\, dy_0 E(x_0, y_0, 0) \iint e^{j2\pi\sqrt{f_s^2 - f_x^2 - f_y^2} z_i} \times e^{j2\pi f_x (x_i - x_0) + j2\pi f_y (y_i - y_0)}\, df_x\, df_y \tag{A.2}$$

The integration Eq. (A.2) can be performed using Weyl's expansion theorem [1], which expresses a spherical wavefront in integral form as

$$\frac{e^{j2\pi f_s r}}{r} = j \iint \frac{e^{j2\pi f_x (x_i - x_0) + j2\pi f_y (y_i - y_0) + j2\pi f_z z}}{f_z}\, df_x\, df_y \tag{A.3}$$

where

$$r^2 = (x_i - x_0)^2 + (y_i - y_0)^2 + z_i^2 \tag{A.4}$$

The factor $1/f_z$ in Eq. (A.3) must be removed in order to use it in the integral of Eq. (A.2). This can be accomplished by differentiating both sides of Eq. (A.3) with respect to z. The result is

$$-\frac{1}{2\pi}\frac{\partial}{\partial z}\left(\frac{e^{j2\pi f_s r}}{r}\right) = \iint e^{2\pi f_x(x_i - x_0) + j2\pi f_y(y_i - y_0) + j2\pi f_z z}\, df_x\, df_y \tag{A.5}$$

Inserting Eq. (A.5) into (A.2) gives

$$E(x_i, y_i, z_i) = -\frac{1}{2\pi}\iint E(x_0, y_0, 0)\frac{\partial}{\partial z}\left(\frac{e^{j2\pi f_s r}}{r}\right) dx_0\, dy_0 \tag{A.6}$$

The derivative with respect to z in Eq. (A.6) is first performed as

$$\frac{\partial}{\partial z}\left(\frac{e^{j2\pi f_s r}}{r}\right) = \left(-\frac{1}{r^2}e^{j2\pi f_s r} + j2\pi f_s \frac{e^{j2\pi f_s r}}{r}\right)\frac{dr}{dz} \tag{A.7}$$

For a large value of r, the second term on the right-hand side of Eq. (A.7) dominates, and

$$\frac{\partial}{\partial z}\left(\frac{e^{j2\pi f_s r}}{r}\right) \doteqdot j2\pi f_s \frac{e^{j2\pi f_s r}}{r}\frac{z}{r} \tag{A.8}$$

where z/r was obtained from the derivative of Eq. (A.4).

From the para-axial approximation, namely, in the region where

$$z^2 \gg (x_i - x_0)^2 + (y_i - y_0)^2 \tag{A.9}$$

the ratio $z/r \cong 1$. Thus Eq. (A.8) becomes

$$\frac{\partial}{\partial z}\left(\frac{e^{j2\pi f_s r}}{r}\right) \doteqdot j2\pi f_s \frac{e^{j2\pi f_s r}}{r} \tag{A.10}$$

From this result, and by substituting $f_s = 1/\lambda$, Eq. (A.6) becomes

$$E(x_i, y_i, z_i) = \frac{1}{j\lambda}\iint E(x_0, y_0, 0)\frac{e^{j(2\pi/\lambda)r}}{r}\, dx_0\, dy_0 \tag{A.11}$$

This equations is known as the *Fresnel–Kirchhoff integral* of diffraction, which represents the diffraction pattern for a given input field.

The final equation, the Fresnel–Kirchhoff integral (Eq. (A.11)), is identical to Eq. (1.28), which was derived earlier without rigorous proof. The analysis here has proved that the constant K given by Eq. (1.29) is true.

REFERENCE

1. E. Wolf and M. Nieto-Vesperinas, "Analyticity of the angular spectrum amplitude of scattered fields and some of its consequences," *J. Opt. Soc. Am. A* **2**(6), 886–890 (1985).