Elements of Photonics, Volume I: In Free Space and Special Media. Keigo Iizuka

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7

HOW TO CONSTRUCT AND USE THE POINCARÉ SPHERE

In this chapter, the Argand diagram and the Poincaré sphere will be introduced as additional graphical methods for dealing with polarization.

The Poincaré sphere was proposed as early as 1892 by the French scientist Henrie Poincaré [1–6], but only in recent years has the Poincaré sphere been given the attention it really deserves [6]. The Poincaré sphere is a projection of the Argand diagram onto a spherical surface to make the diagram spherically symmetric [3]. Spherical symmetry eliminates the step of rotating and rerotating the coordinates, which is necessary when using the Argand diagram.

The Poincaré sphere can be used in:

1. Problems associated with any retardance or orientation of the fast axis.
2. Problems associated with a polarizer with any orientation.
3. Determining the Stokes parameters, which are the projections of a point on the sphere to the equatorial plane.

Although primarily used for polarized waves, the Poincaré sphere can be extended to partially polarized waves.

A special feature of the Poincaré sphere is the simplicity of manipulation. Regardless of whether the incident wave is linearly polarized or elliptically polarized, whether the desired quantities are the optical parameters of a lumped element or a distributed element like the twist rate of an optical fiber [7], or whether the axis of the optical element is horizontally oriented or tilted at an arbitrary angle, the procedure remains the same. Multiple retarders undergo the same kind of manipulation. The answer after each stage is provided as the manipulation continues.

Moreover, when parameters of optical components have to be selected to achieve a particular state of polarization, the Poincaré sphere becomes even more valuable.

This chapter uses many results of Chapter 6 and should be considered as an extension of Chapter 6.

7.1 COMPONENT FIELD RATIO IN THE COMPLEX PLANE

Let the complex number representation of the x and y component fields be

$$E_x = Ae^{j(-\omega t+\beta z+\phi_x)} \tag{7.1}$$

$$E_y = Be^{j(-\omega t+\beta z+\phi_y)} \tag{7.2}$$

A new quantity, which is the ratio of these complex fields, is defined in polar form as

$$\frac{E_y}{E_x} = \left(\frac{B}{A}\right) e^{j\Delta} \tag{7.3}$$

where Δ is the retardance. The retardance Δ is the phase of the y component with respect to that of the x component and is expressed as

$$\Delta = \phi_y - \phi_x \tag{7.4}$$

As mentioned in Chapter 6, Eq. (7.3) is called the component field ratio. The component field ratio is represented as a point P on the u–v complex plane, as shown in Fig. 7.1. The magnitude $0P$ represents the amplitude ratio $B/A (= \tan\alpha)$ and the phase angle represents the retardance Δ.

The real and imaginary parts of the component field ratio are

$$\left(\frac{B}{A}\right)\cos\Delta + j\left(\frac{B}{A}\right)\sin\Delta = u + jv \tag{7.5}$$

$$u = \tan\alpha\cos\Delta$$

$$v = \tan\alpha\sin\Delta \tag{7.6}$$

with

$$B/A = \tan\alpha \tag{7.7}$$

Hence, the ratio v/u is simply

$$\frac{v}{u} = \tan\Delta \tag{7.8}$$

Each point on the u–v complex plane of the component field ratio corresponds to a state of polarization because Eqs. (6.99) and (6.112) give the values of θ and β from given α and Δ values. This correspondence between states of polarization and points in the u–v complex plane is illustrated in Fig. 7.2. Some observations will be made concerning Figs. 7.1 and 7.2.

1. In the upper half-plane, $0 < \Delta < \pi$ and $\sin\Delta > 0$. From Eq. (6.126), all states in this region are left-handed. In the lower half-plane, $\pi < \Delta < 2\pi$ and $\sin\Delta < 0$, and here the states are all right-handed.

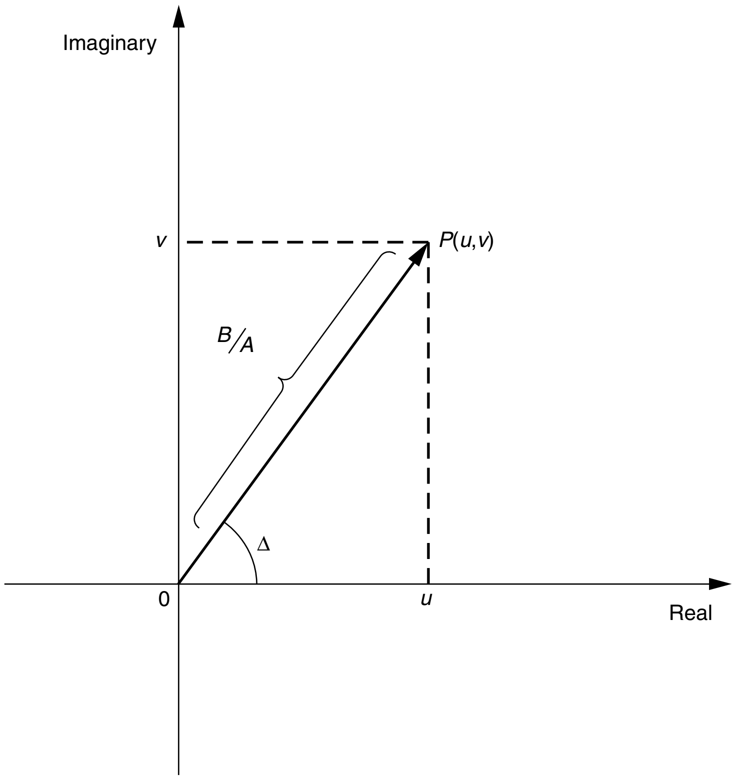


Figure 7.1 Component field ratio in the complex plane.

2. Along the positive u axis, $\Delta = 0$ and there is no phase difference between E_x and E_y. The states are all linearly polarized, as seen from Eq. (6.112). According to Eq. (6.99), the azimuth θ of the linear polarization is equal to α and increases with u from horizontally polarized at $u = 0$ to vertically polarized at $u = \infty$.
3. Since $B/A > 0$, the sign of u and v are determined solely by the value of the retardance Δ. Along the negative u axis, $\Delta = 180°$. The retardance $\Delta = 180°$ means that the direction of E_y is reversed from that of E_y in the corresponding positive u direction as shown in Fig. 7.3. The azimuth θ along the negative u axis is the mirror image of that along the positive u axis.
4. Along the positive v axis, $\Delta = 90°$ and with an increase in v, $\epsilon(= \tan\beta)$ increases within the range given by Eq. (6.108). According to Eq. (6.112), $\beta = \alpha$. In the region above the u axis, the states are all left-handed elliptically polarized waves from Eq. (6.126). Their major axis is horizontal in the region $v < 1$ and their minor axis is horizontal in the region $v > 1$.
5. Along the negative v axis, $\Delta = -90°$. With an increase in $|v|$, ϵ increases within the range given by Eq. (6.108). As with positive v, $\beta = \alpha$. Below the u axis all states of polarization are right-handed elliptically polarized waves with their major axis horizontal when $|v| < 1$ and with their minor axis horizontal when $|v| > 1$.

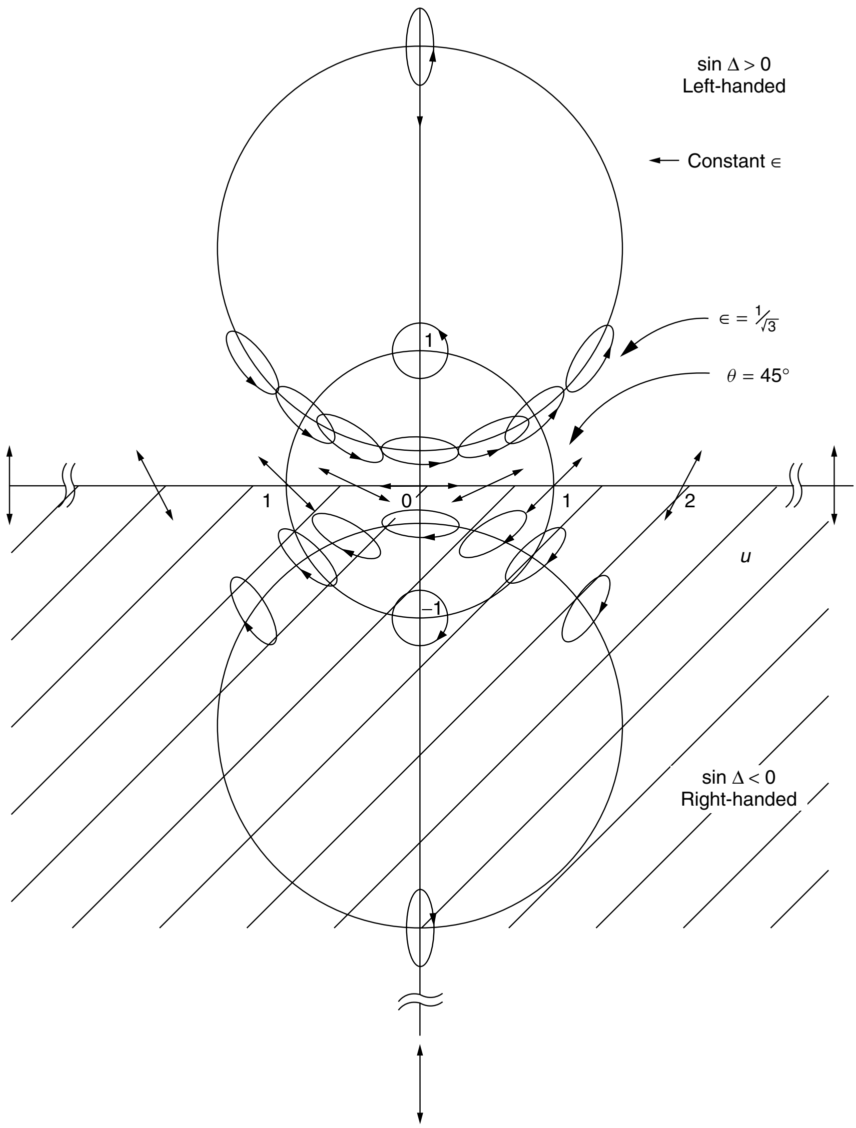


Figure 7.2 Ellipses on the complex plane of the complex amplitude ratio.

6. On the unit circle, $B/A = 1$ and α is either 45° or 135°. The results shown in Fig. 6.4 are arranged along the unit circle in Figs. 6.5 and 7.2. From Eq. (6.112), the relationship between β and Δ is

$$\beta = \tfrac{1}{2}|\Delta| \tag{7.9}$$

The intercepts of the unit circle with the positive and negative v axes are left-handed and right-handed circularly polarized waves, respectively.

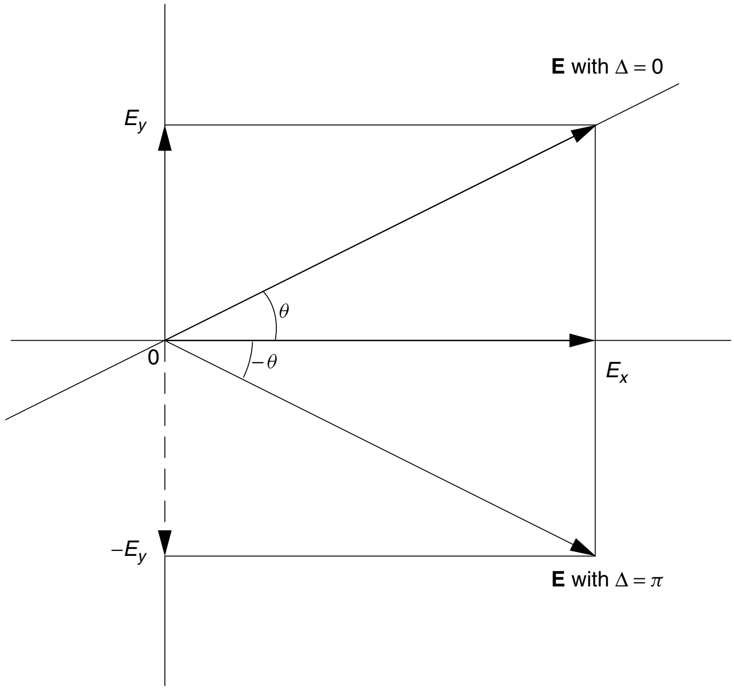


Figure 7.3 **E** with $\Delta = 0$ and **E** with $\Delta = \pi$.

7.2 CONSTANT AZIMUTH θ AND ELLIPTICITY ϵ LINES IN THE COMPONENT FIELD RATIO COMPLEX PLANE

In Fig. 7.2, the states of polarization were drawn in the component field ratio plane. The values of θ and ϵ were superposed on the values of α and Δ. In this section the points having the same θ values will be connected together, as well as the points having the same ϵ values. The two sets of θ and ϵ loci will be put together in one component field ratio complex plane. This plane is referred to as the Argand diagram.

7.2.1 Lines of Constant Azimuth θ

In this section, a set of curves of constant θ will be generated. The constant θ line can be generated if θ is expressed in terms of u and v. Equations (6.99) and (7.6) are used for this purpose.

$$\tan 2\theta = \frac{\left|2\dfrac{u}{\cos\Delta}\right|}{1-\left(\dfrac{u}{\cos\Delta}\right)^2}\cos\Delta \tag{7.10}$$

where the double-angle relationship of Eq. (6.96) was applied to the angle α.

Inverting both sides of Eq. (7.10) gives

$$2u \cot 2\theta = 1 - \left(\frac{u}{\cos \Delta}\right)^2 \tag{7.11}$$

Now, an alternate way of expressing Eq. (7.8) is

$$\frac{1}{\cos^2 \Delta} = 1 + \left(\frac{v}{u}\right)^2 \tag{7.12}$$

Inserting Eq. (7.12) into (7.11) gives

$$u^2 + v^2 + 2u \cot 2\theta - 1 = 0 \tag{7.13}$$

which can be further rewritten as

$$(u + \cot 2\theta)^2 + v^2 = \operatorname{cosec}^2 2\theta \tag{7.14}$$

Equation (7.14) is the expression of a circle with radius $\operatorname{cosec} 2\theta$ centered at $(-\cot 2\theta, 0)$. This is the contour of constant azimuth θ. To be more exact, it is the contour of constant $\tan 2\theta$.

The intersections of the circle with the u and v axes are investigated. From Eq. (7.13) with $u = 0$, the intersects P_1 and P_2 with the v axis are found:

$$v = \pm 1 \tag{7.15}$$

As a matter of fact, since the θ-dependent third term on the left-hand side of Eq. (7.13) vanishes with $u = 0$, the curves for all values of θ pass through the points P_1 and P_2.

From Eq. (7.14) with $v = 0$, the intersects with the u axis are found:

$$\begin{aligned} u &= -\cot 2\theta \pm \frac{1}{\sin 2\theta} \\ &= \frac{-\cos^2\theta + \sin^2\theta \pm 1}{2\cos\theta\sin\theta} \end{aligned} \tag{7.16}$$

The intersections are

$$u_1 = \tan\theta \tag{7.17}$$

$$u_2 = -\cot\theta = \tan(\theta + 90^\circ) \tag{7.18}$$

A series of circles for different values of θ are drawn in Fig. 7.4.

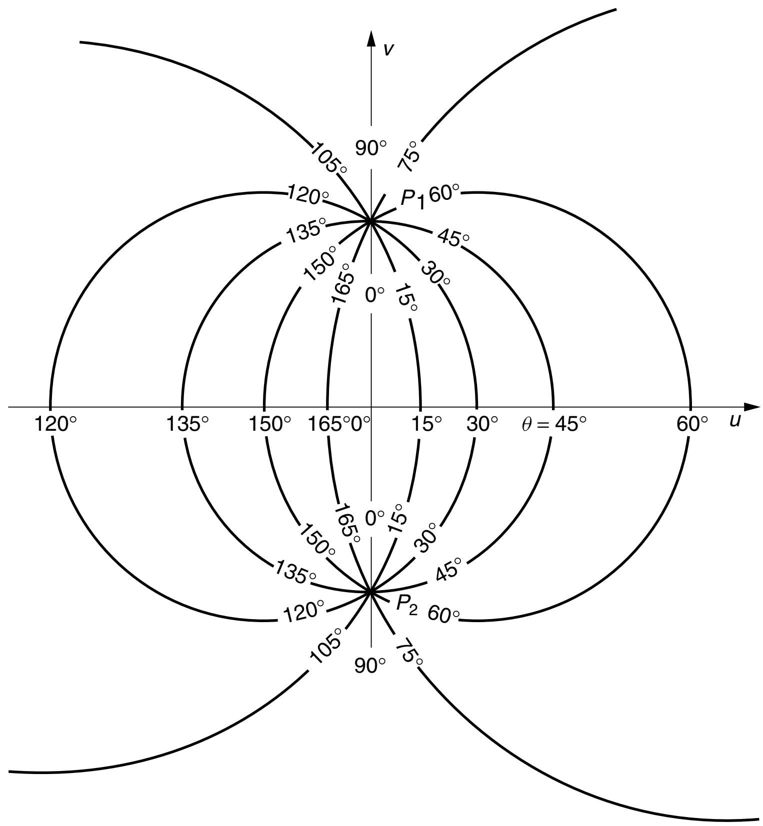


Figure 7.4 Constant θ curves.

A word of caution should be added. As mentioned at the beginning of this section, the above curves are not curves of constant θ but curves of constant $\tan 2\theta$. The multivalue problem of $\tan 2\theta$ has to be resolved. There is more than one value of θ that gives the same value of $\tan 2\theta$. For example, both $\theta = 30°$ and $\theta = 120°$ give the same value of $\tan 2\theta$. The value of $\tan 2\theta$ is the same for θ and $\theta \pm 90°n$, where n is an integer and the correct value of θ has to be selected. Equation (6.123) is useful for making this selection:

$$(a^2 - b^2)\cos 2\theta = A^2 - B^2 \tag{6.123}$$

If a is chosen as the length of the major axis, then $a^2 - b^2$ is always a positive number. Inside the unit circle in the u–v plane, $B/A < 1$ and $A^2 - B^2$ is also positive. Only the θ that satisfies $\cos 2\theta > 0$ is permitted inside the unit circle.

On the other hand, outside the unit circle, $A^2 - B^2 < 0$ and, hence, only the θ that satisfies $\cos 2\theta < 0$ is permitted outside the unit circle. The values of θ indicated in Fig. 7.4 are based on this selection.

7.2.2 Lines of Constant Ellipticity ϵ

Similar to the lines of constant azimuth θ, the lines of constant ellipticity ϵ can be obtained by expressing $\epsilon(=\tan\beta)$ in terms of u and v. First, the case of the left-handed rotation, or $|\sin\Delta| = \sin\Delta$, is treated. Equation (6.112) will be used to find the constant ϵ line. Using the trigonometric relationship of Eq. (6.105), and then expressing $\tan\alpha$ in terms of $v/(\sin\Delta)$ by means of Eq. (7.6), Eq. (6.112) is rewritten as

$$\sin 2\beta = \frac{2v}{1+\left(\dfrac{v}{\sin\Delta}\right)^2} \tag{7.19}$$

Now, $\sin^2\Delta$ is converted into $\tan^2\Delta$ so that Eq. (7.8) can be used. Equation (7.19) becomes

$$u^2 + v^2 - 2v\,\mathrm{cosec}\,2\beta + 1 = 0 \tag{7.20}$$

which can be rewritten further as

$$u^2 + (v - \mathrm{cosec}\,2\beta)^2 = \cot^2 2\beta \tag{7.21}$$

Equation (7.21) is the equation of a circle with radius $\cot 2\beta$ centered at (0, $\mathrm{cosec}\,2\beta$). The constant ϵ lines are plotted in Fig. 7.5.

Specific points on the circle will be investigated. The intersection with the u axis is examined first. Setting $v = 0$ in Eq. (7.20) gives $u^2 + 1 = 0$, and there is no intersection with the u axis. The intersections with the v axis are found from Eq. (7.20) with $u = 0$ as

$$v^2 - 2v\,\mathrm{cosec}\,2\beta + 1 = 0 \tag{7.22}$$

The two solutions of Eq. (7.22) are

$$\begin{aligned} v &= \mathrm{cosec}\,2\beta \pm \sqrt{\mathrm{cosec}^2 2\beta - 1} \\ &= \frac{1 \pm \cos 2\beta}{\sin 2\beta} \end{aligned} \tag{7.23}$$

The solutions corresponding to the positive and negative signs in Eq. (7.23) are, respectively,

$$v_1 = \cot\beta = \frac{1}{\epsilon} \tag{7.24}$$

$$v_2 = \tan\beta = \epsilon \tag{7.25}$$

Since $\epsilon < 1$, and hence $v_1 > v_2$, the lower intersection explicitly represents ϵ. From this information, the constant ϵ lines for the left-handed rotation are drawn. These circles all stay in the upper half of the u–v plane.

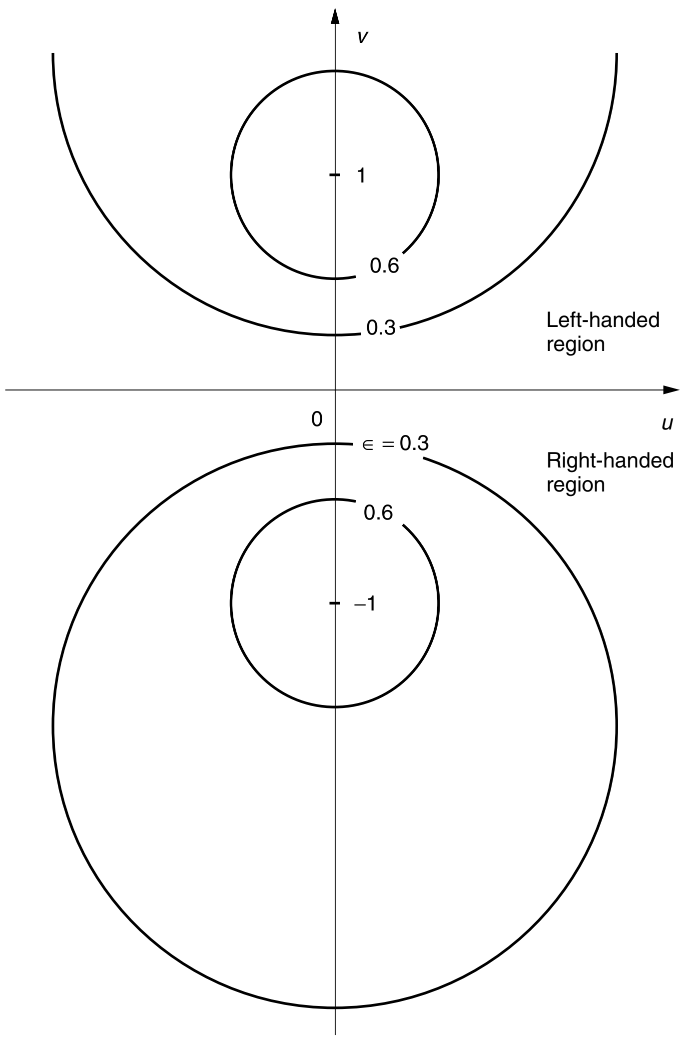


Figure 7.5 Constant ϵ circles.

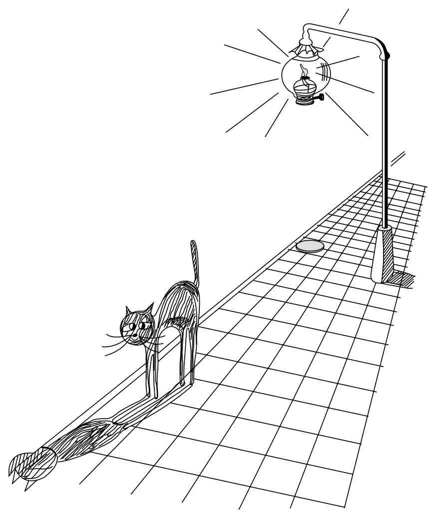
Exactly the same procedure is repeated for the right-handed rotation except that $|\sin\Delta| = -\sin\Delta$ is used in Eq. (6.112). The result is

$$u^2 + (v + \operatorname{cosec} 2\beta)^2 = \cot^2 2\beta \tag{7.26}$$

The circles represented by Eq. (7.26) fill the lower half-plane.

7.3 ARGAND DIAGRAM

The loci of the constant θ and those of the constant ϵ are combined together on the same plane in Fig. 7.6. This diagram is called an Argand diagram [3,6] because its shape resembles the spherical Argand gas lamps commonly used for street lighting in



Argand lamp.

days of old [1]. The Argand diagram graphically represents ϵ and θ on the α and Δ plane. It permits us to find any two parameters in terms of the other two.

Just like the circle diagram and the Jones matrix described in Chapter 6, the Argand diagram not only simplifies the procedure of the calculation but also provides the states of polarization after each stage of the optical system.

The Poincaré sphere is nothing but a projection of the Argand diagram onto a unit sphere. Knowledge about the Argand diagram is essential to a clear understanding of the operation of the Poincaré sphere.

7.3.1 Solution Using a Ready-Made Argand Diagram

There are two ways of using the Argand diagram. One way is to draw in the lines on the fully completed Argand diagram. The other way is to construct the constant θ and ϵ curves only as needed. The advantage of the former way is simplicity, and that of the latter is accuracy. Both ways will be explained using examples. For the first two examples, the incident light is linearly polarized. Following these, two examples are given where the incident light is elliptically polarized.

Example 7.1 Using the Argand diagram, answer the questions in Example 6.9, which concerned the emergent state of polarization from a retarder with $\Delta = 38^\circ$ and its fast axis in the x direction. The parameters of the incident linear polarization were $E_x = 2.0$ V/m and $E_y = 3.1$ V/m.

Solution With the given values of $B/A = 1.55$ and $\Delta = 38^\circ$, point P is picked as shown in Fig. 7.6. The answer is read from the diagram as $\theta = 60^\circ$ and $\epsilon = 0.3$, which matches the results that were given in Example 6.9. □

Example 7.2 Linearly polarized light with azimuth $\theta = 165^\circ$ is incident on a $\lambda/4$ plate. Find the state of polarization of the emergent wave when the fast axis of the $\lambda/4$ plate is (a) along the x axis and (b) along the y axis.

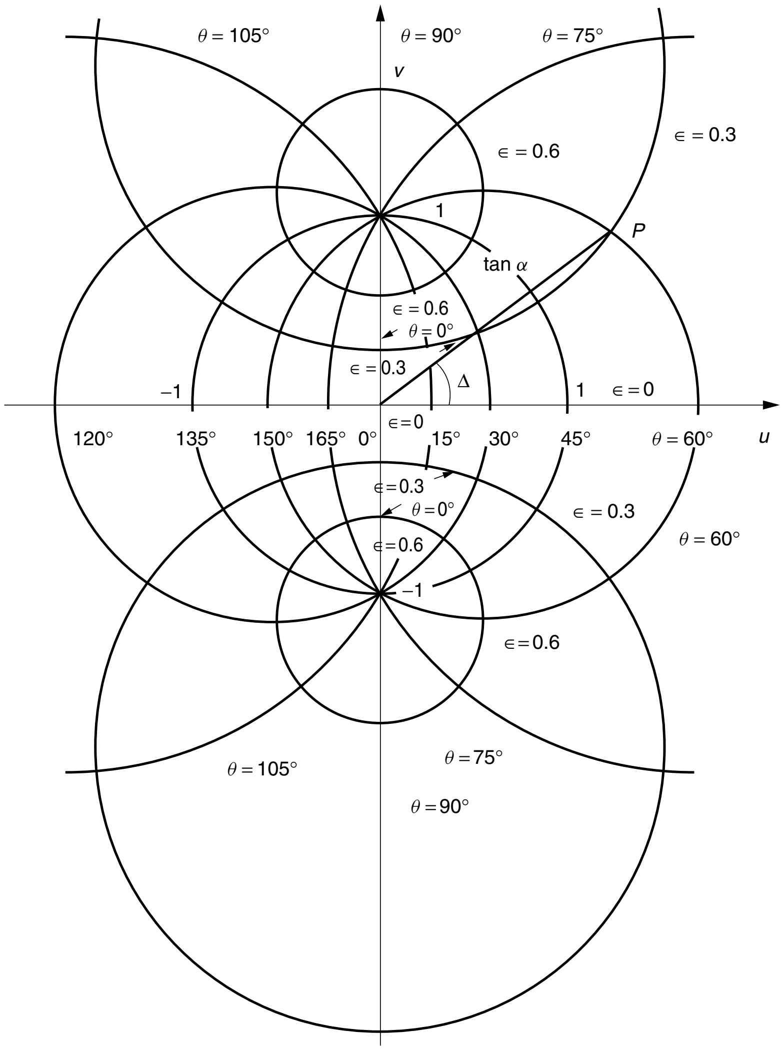


Figure 7.6 Argand diagram.

Solution

(a) The Argand diagram facilitates the calculation. Figure 7.7 shows the answer. Since the incident light is linearly polarized, $\epsilon = 0$. The point P_1, which represents the incident light, is $\theta = 165°$, $\epsilon = 0$; or $\theta = 165°$ on the u axis. From the measurement of the length $0P_1 = 0.27$, the incident wave is

$$\left(\frac{B}{A}\right) e^{j\Delta} = 0.27\ e^{j180°}$$

The introduction of the $\lambda/4$ plate delays the phase of the y component by $\Delta = 90°$ more than that of the x component and the component field ratio becomes $0.27\ e^{j270°}$. Point P_1 moves to point P_2. The answer is right-handed elliptical polarization with $\theta = 0°$ and $\epsilon = 0.27$.

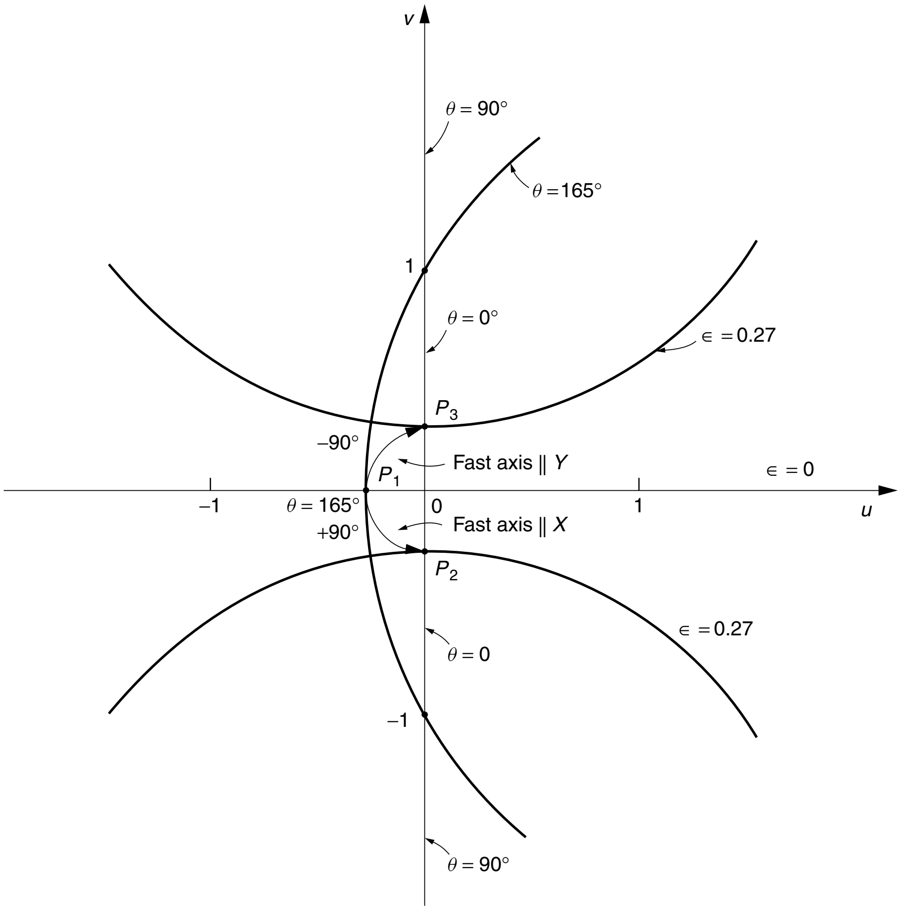


Figure 7.7 Solution of Example 7.2.

(b) This situation is similar to (a) but with $\Delta = -90°$. The point P_1 moves to point P_3 and the answer is the left-handed elliptical polarization with $\theta = 0°$ and $\epsilon = 0.27$. □

Example 7.3 Example 6.1 looked at how the state of polarization of a horizontally linearly polarized wave changes as the azimuth of the fast axis of the $\lambda/4$ plate is rotated. Repeat this exercise using the Argand diagram. Solve only for $\Theta = 22.5°$, for $\Theta = 67.5°$, and for $\Theta = 112.5°$.

Solution It should be remembered that Eqs. (6.1) and (6.2) are based on the condition that the x axis is parallel to the fast axis of the retarder. Just as was done analytically in Section 6.8.1, the incident field **E** has to be represented in the coordinates rotated by Θ so that the new x' axis aligns with the fast axis of the retarder. After this has been done, the retardation is accounted for and, finally, the coordinates are rotated back by $-\Theta$ to express the result in the original coordinates.

Let us start with the case $\Theta = 22.5°$. The incident light is $(\theta, \epsilon) = (0, 0)$ and is represented by P_0 in Fig. 7.8. The coordinates are rotated by $+22.5°$ so that the new x'

axis aligns with the fast axis. The azimuth of the incident light in the new coordinates decreases by the same amount and in the rotated coordinates, $\theta = -22.5^\circ$, which is equivalent to $\theta = 157.5^\circ$. Recall that $\theta = 157.4^\circ$ rather than $\theta = -22.5^\circ$ has to be used because of the restriction of $0 \leq \theta < 180^\circ$. This point is represented by P_1^1 at $(\theta, \epsilon) = (157.5^\circ, 0)$. The $\lambda/4$ plate rotates P_1^1 by 90° as indicated by the dotted line to P_2^1 at $(\theta, \epsilon) = (0, 0.42)$. The coordinates are then rotated back by -22.5° in order to express the emergent light in the original coordinates. The azimuth of the emergent light in the rotated back coordinates increases the same amount. P_2^1 moves along the $\epsilon = 0.42$ line to point P_3^1 at $(\theta, \epsilon) = (22.5^\circ, 0.42)$. The handedness is right-handed.

Next, the case with $\Theta = 67.5^\circ$ will be solved. The incident light is again at P_0. The coordinates are rotated by $+67.5^\circ$ to match the new x' axis to the fast axis. The azimuth of the incident light in the new coordinates decreases by the same amount and becomes $\theta = -67.5^\circ$, which is equivalent to $\theta = 112.5$, which meets the restriction $0 \leq \theta < 180^\circ$. The incident light is represented by P_1^3 at $(\theta, \epsilon) = (112.5^\circ, 0)$. The retardance of 90° brings the point P_1^3 as indicated by the dotted line to P_2^3 at $(\theta, \epsilon) = (90^\circ, 0.42)$.

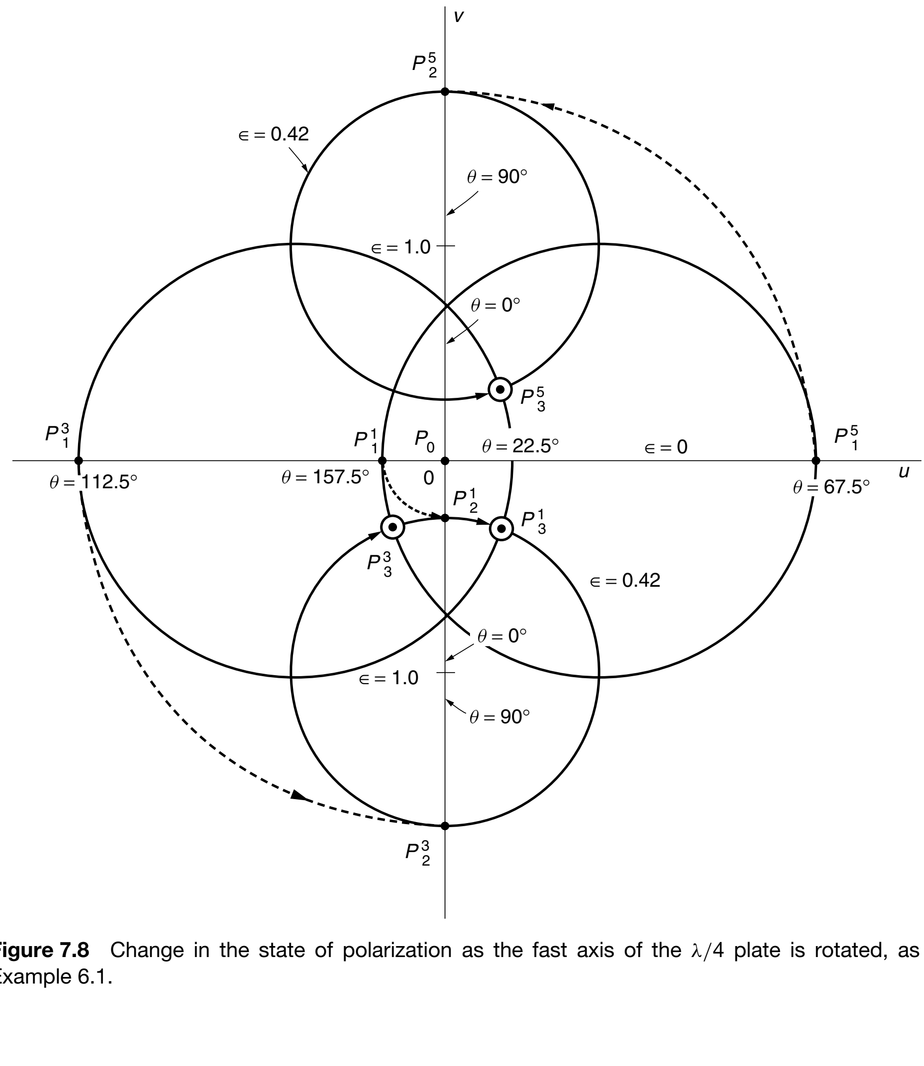


Figure 7.8 Change in the state of polarization as the fast axis of the $\lambda/4$ plate is rotated, as in Example 6.1.

Back-rotation of the coordinates moves P_2^3 along the constant ϵ line to P_3^3 at $(\theta, \epsilon) = (157.5^\circ, 0.42)$. The handedness is also right-handed.

Finally, the case the $\Theta = 112.5^\circ$ will be solved. The coordinates are rotated by $+112.5^\circ$ and as a result the azimuth of the incident light in the new coordinates is $\theta = -112.5^\circ$, which is equivalent to $\theta = 67.5^\circ$ to meet the restriction of $0 \leq \theta < 180^\circ$. The incident light is indicated by P_1^5 at $(\theta, \epsilon) = (67.5^\circ, 0)$. The retardance rotates P_1^5 as indicated by the dotted line to P_2^5, where $(\theta, \epsilon) = (90^\circ, 0.42)$. Back-rotation of the coordinates by -112.5° along $\epsilon = 0.42$ brings P_2^5 to P_3^5 at $\theta = 202.5^\circ$, which is equivalent to $\theta = 22.5^\circ$, which meets the restriction $0 \leq \theta < 180^\circ$. The final result is at P_3^5 with $(\theta, \epsilon) = (22.5, 0.42)$. The handedness is left-handed. □

The incident waves of the next two examples are elliptically polarized.

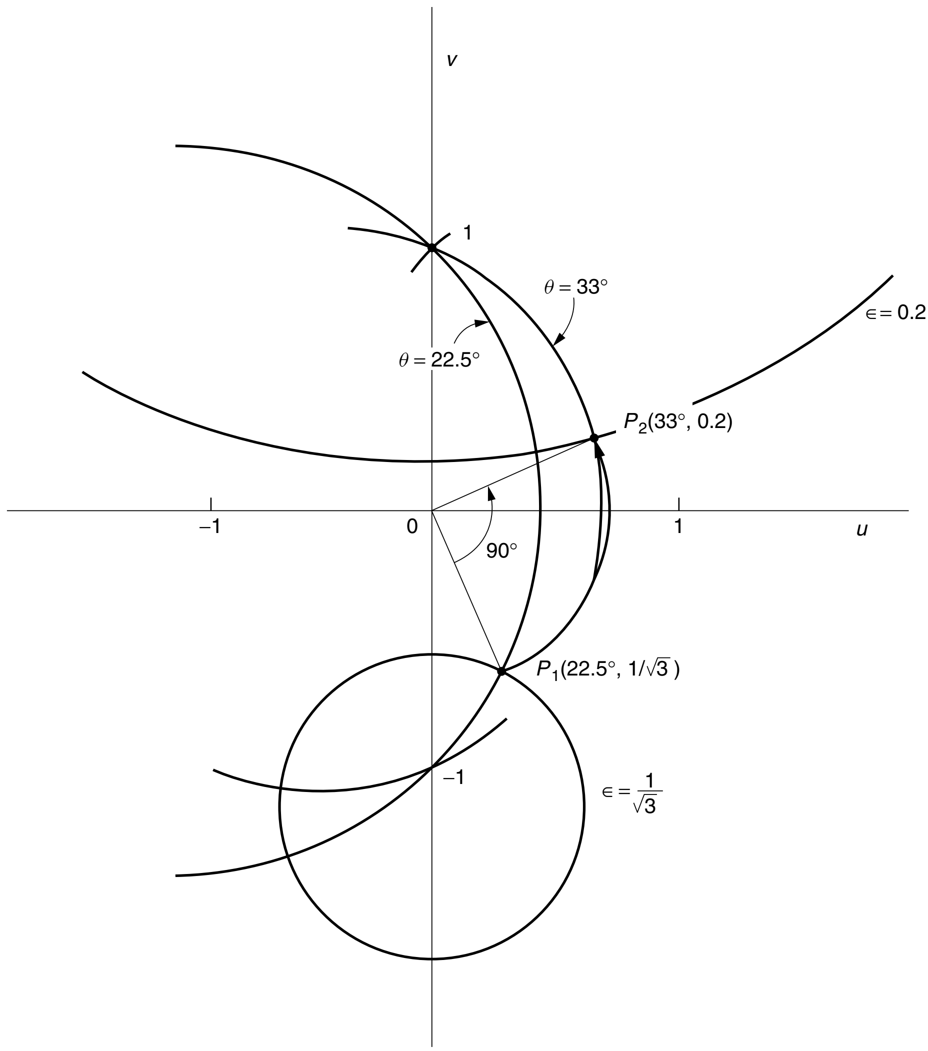


Figure 7.9 Solution of Example 7.4 using the Argand diagram.

Example 7.4 A right-handed elliptically polarized wave with $\theta = 22.5^\circ$ and $\epsilon = 1/\sqrt{3}$ is incident on a $\lambda/4$ plate whose fast axis is along the x axis. Find the state of polarization of the emergent light using the Argand diagram.

Solution Point P_1, which represents the state of polarization of the incident wave, is plotted at $(\theta, \epsilon) = (22.5^\circ, 1/\sqrt{3})$ in the lower half-plane as in Fig. 7.9. The $\lambda/4$ plate with its fast axis along the x axis moves P_1 by 90° counterclockwise along the circumference with radius $0P_1$ to point P_2. The coordinates of P_2 represent the state of polarization of the emergent wave. Point P_2 is read from the Argand diagram at approximately $(\theta, \epsilon) = (33^\circ, 0.2)$ in the upper half-plane. The emergent wave therefore has left-handed elliptical polarization with $\theta = 33^\circ$ and $\epsilon = 0.2$. □

Example 7.5 The same elliptically polarized wave is incident onto a $\lambda/4$ plate but this time its fast axis is not along the x axis but is tilted from the x axis by $\Theta = 45^\circ$. Find the state of polarization of the emergent wave.

Solution Now that the fast axis is not along the x axis, the coordinates have to be rotated by Θ; then the retardance is accounted for, followed by a rotation back by $-\Theta$. The state of polarization of the incident wave is represented by P_1 at $(\theta, \epsilon) = (22.5^\circ, 1/\sqrt{3})$ in the lower half-plane in Fig. 7.10. Due to the rotation of the coordinates, point P_1 moves along the $\epsilon = 1/\sqrt{3}$ line to point P_2 at $(\theta, \epsilon) = (-22.5^\circ, 1/\sqrt{3}) = (157.5^\circ, 1/\sqrt{3})$. The $\lambda/4$ plate rotates the point by $+90^\circ$ to point P_3 at $(\theta, \epsilon) = (34.2^\circ, 0.2)$. Finally, the coordinates are rotated back by $-\Theta = 45^\circ$. The azimuth of the emergent light in the rotated back coordinates increases the same amount. P_3 is brought to the final point P_4 at $(\theta, \epsilon) = (79.2^\circ, 0.2)$ in the lower half-plane. The emergent wave has right-handed elliptical polarization with $\theta = 79.2^\circ$ with $\epsilon = 0.2$. □

7.3.2 Orthogonality Between Constant θ and ϵ Lines

It will be shown that the constant θ lines and the constant ϵ lines are orthogonal to each other. Using this fact, a constant ϵ line can be drawn from a constant θ line or vice versa. In the next section, the orthogonality relationship is used to construct custom-made Argand diagrams for specific problems.

The slope of the constant θ line is the derivative of Eq. (7.13) with respect to u,

$$2u + 2vv' + 2\cot 2\theta = 0 \tag{7.27}$$

where $v' = dv/du$. Multiplying Eq. (7.27) by u and subtracting Eq. (7.13) gives

$$v' = -\frac{u^2 - v^2 + 1}{2uv} \tag{7.28}$$

Similarly, the slope of the line of constant ϵ is, from Eq. (7.20),

$$2u + 2vv' - 2v' \operatorname{cosec} 2\beta = 0 \tag{7.29}$$

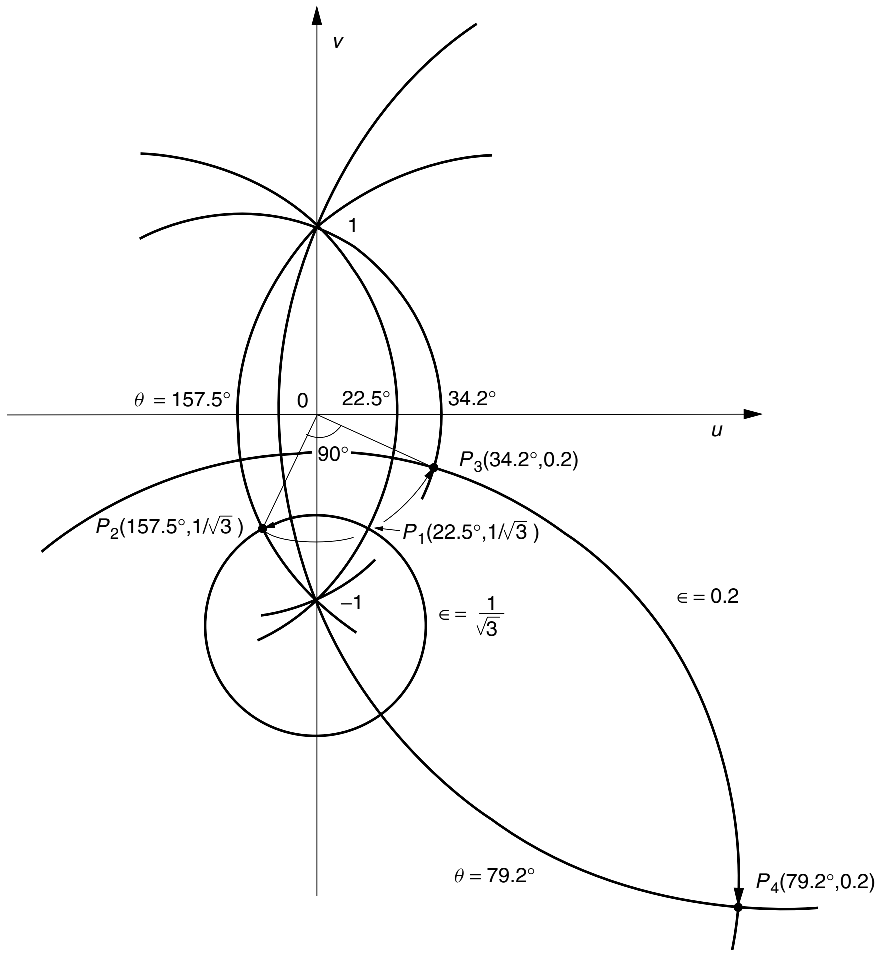


Figure 7.10 Solution of Example 7.5 using the Argand diagram.

Multiplying Eq. (7.29) by v and multiplying Eq. (7.20) by v', and then subtracting gives

$$v' = \frac{2uv}{u^2 - v^2 + 1} \tag{7.30}$$

From Eqs. (7.28) and (7.30), the slopes of the two curves are negative reciprocals and therefore the two curves are orthogonal to each other.

7.3.3 Solution Using a Custom-Made Argand Diagram

In using an already drawn Argand diagram, one frequently has to rely on interpolation between the lines, unless the point falls exactly on the line. What will be described here is a method of drawing specific constant θ and ϵ lines to solve a particular problem. The accuracy of a custom-made Argand diagram is higher than the accuracy of interpolating a ready-made diagram.

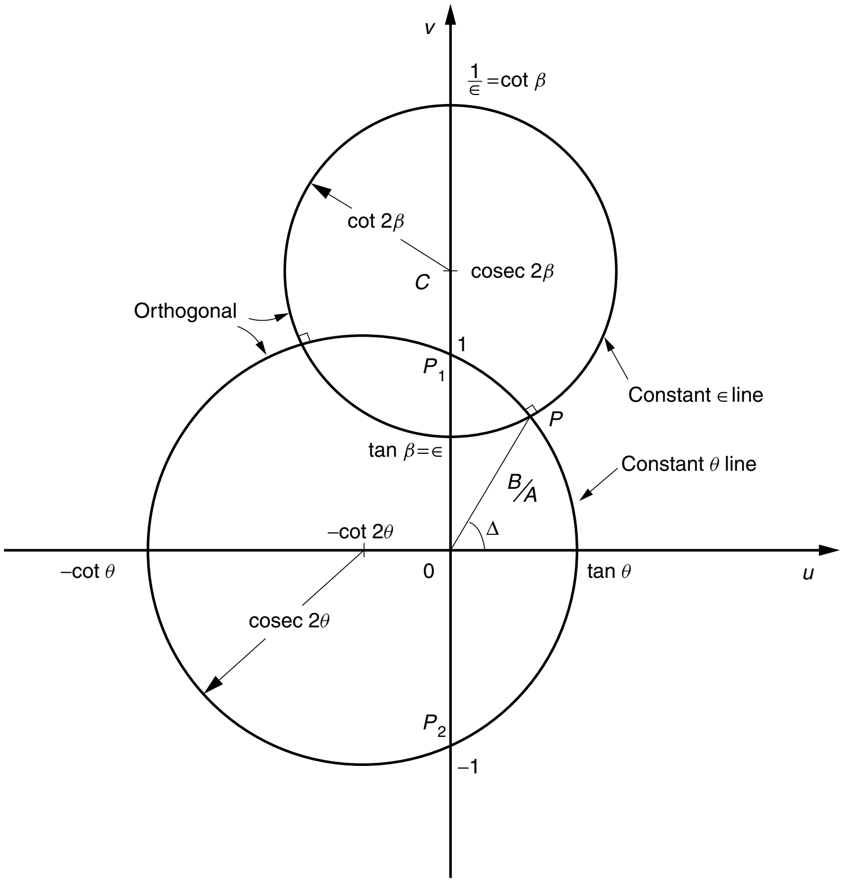
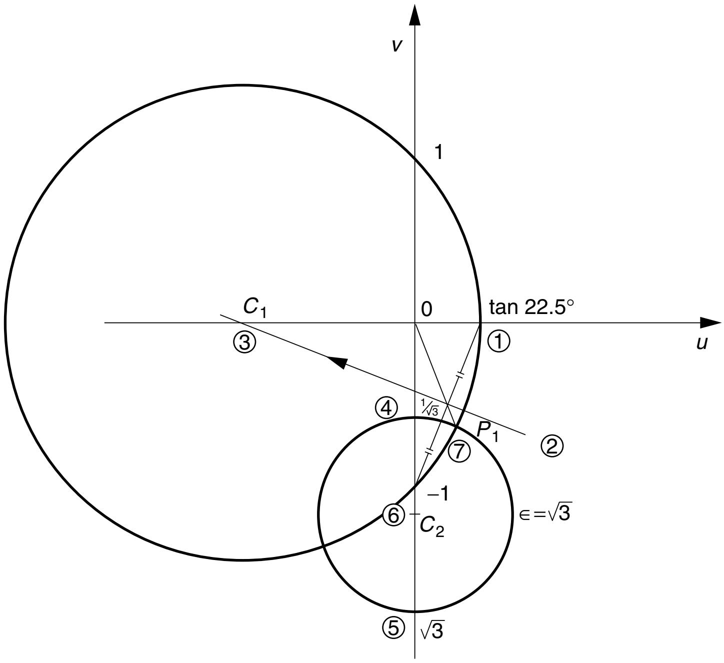


Figure 7.11 Curves for constant θ and ϵ.

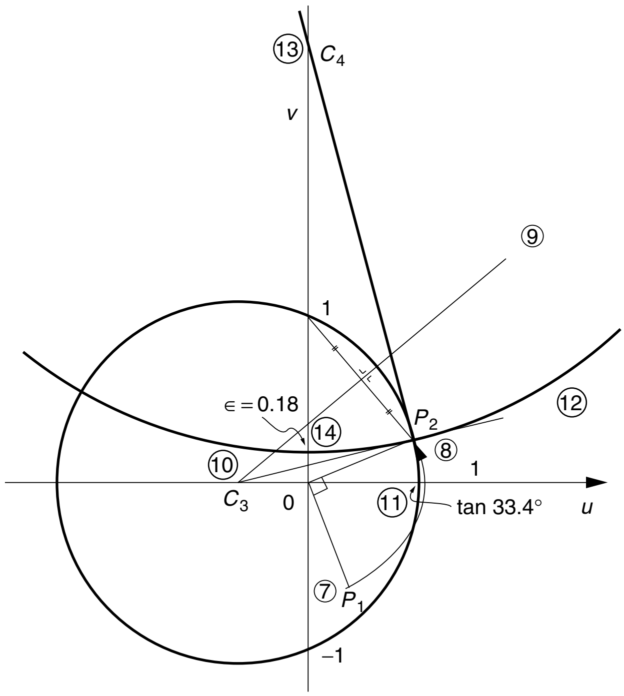
First of all, information about the intersections with the axes is useful in drawing lines. The information so far obtained is summarized in Fig. 7.11.

Figure 7.12 illustrates how to draw the constant θ and ϵ lines that were used to solve Example 7.4 in Fig. 7.9. The circled numbers in the figure correspond to the step numbers below. First, the circles associated with the state of polarization of the incident light are drawn. The effect of the $\lambda/4$ plate is accounted for to obtain the final results.

1. Find the point $(u, v) = (\tan 22.5^\circ, 0)$ on the u axes. If 5 cm is taken as the unit length in the drawing, (u, v) is located at 2.1 cm horizontally from the origin.
2. Draw the bisect of a line connecting $(\tan 22.5^\circ, 0)$ and $(0, -1)$. Extend the bisect to find intersection C_1 with the u axis.
3. Centered at C_1, draw the constant θ circle passing through points $(\tan 22.5^\circ, 0)$ and $(0, -1)$. This is the circle of $\theta = 22.5^\circ$.
4. Next, the line of $\epsilon = 1/\sqrt{3}$ will be found. Find the point $(u, v) = (0, -1/\sqrt{3})$, which is -2.9 cm vertically from the origin from Eq. (7.25).



(a)



(b)

Figure 7.12 How to draw the constant θ and ϵ lines. (a) The first seven steps for drawing your own Argand diagram. (b) Steps from 7 to 13 for drawing your own Argand diagram.

⑤ Find the point $(u, v) = (0, -\sqrt{3})$, which is -8.7 cm vertically from the origin from Eq. (7.24).

⑥ Find the center $C_2(0, -\frac{1}{2}(1/\sqrt{3} + \sqrt{3}))$. C_2 is the circle of $\epsilon = 1/\sqrt{3}$.

⑦ The intersection P_1 between the circles centered at C_1 and C_2 is found. The phasor $\overline{0P_1}$ represents the component field ratio $0.69e^{-j67.5^\circ}$ of the incident light.

⑧ Rotate point P_1 counterclockwise by 90° to point P_2 to account for the retardance $\Delta = 90^\circ$ as shown in Fig. 7.12b.

⑨ Next, the value of θ at P_2 will be found. Find the bisect of the line connecting P_2 and the point (0,1), and extend it to find the intersection C_3 with the u axis.

⑩ Centered at C_3 and passing through points P_2 and (0,1), draw the constant θ circle.

⑪ The intersection of the circle centered at C_3 with the u axis is measured as 3.3 cm horizontally from the origin. The value of u at this point is 0.66; $\tan\theta = 0.66$ or $\theta = 33.4^\circ$.

⑫ Finally, the circle of constant ϵ will be found. Draw the straight line $\overline{C_3P_2}$.

⑬ Draw the normal to the straight line $\overline{C_3P_2}$ from the point P_2, and this will locate the intersection C_4 with the v axis. Orthogonality between constant θ and ϵ lines is being invoked to construct the desired ϵ line from the $\theta = 33.4^\circ$ line.

⑭ Centered at C_4, draw the constant ϵ circle passing through P_2. The intersection of the circle with the v axis is 0.9 cm from the origin, which is $v = 0.18$. Thus, $\epsilon = 0.18$.

The state of polarization of the emergent light is left-handed circularly polarized with $(\theta, \epsilon) = (33.4^\circ, 0.18)$. The custom-made Argand diagram provides better accuracy than the results obtained in Example 7.4.

7.4 FROM ARGAND DIAGRAM TO POINCARÉ SPHERE

The Poincaré sphere is generated by back projecting the Argand diagram onto a unit diameter sphere. Figure 7.13 illustrates the relative orientation between the Poincaré sphere and the Argand diagram. The surface of the Poincaré sphere touches at the origin of the Argand diagram. The real axis u is back-projected onto the equator of the sphere and the imaginary axis v is back-projected onto the great circle passing through the north and south poles of the sphere. The plane of the *great circle* contains the center of the sphere. The diameter of the sphere being unity, points (0, 1) and (0, -1) on the Argand diagram back-project onto the north and south poles, respectively. Back-projections of other general points will be calculated using elementary analytic geometry in the next section.

7.4.1 Analytic Geometry of Back-Projection

As shown in Fig. 7.13, all the points on the Argand diagram are back-projected onto the single point $0'$, which is diametrically opposite to the tangent point 0. The concept of the Poincaré sphere boils down to finding the intersection of a straight line with the surface of a sphere.

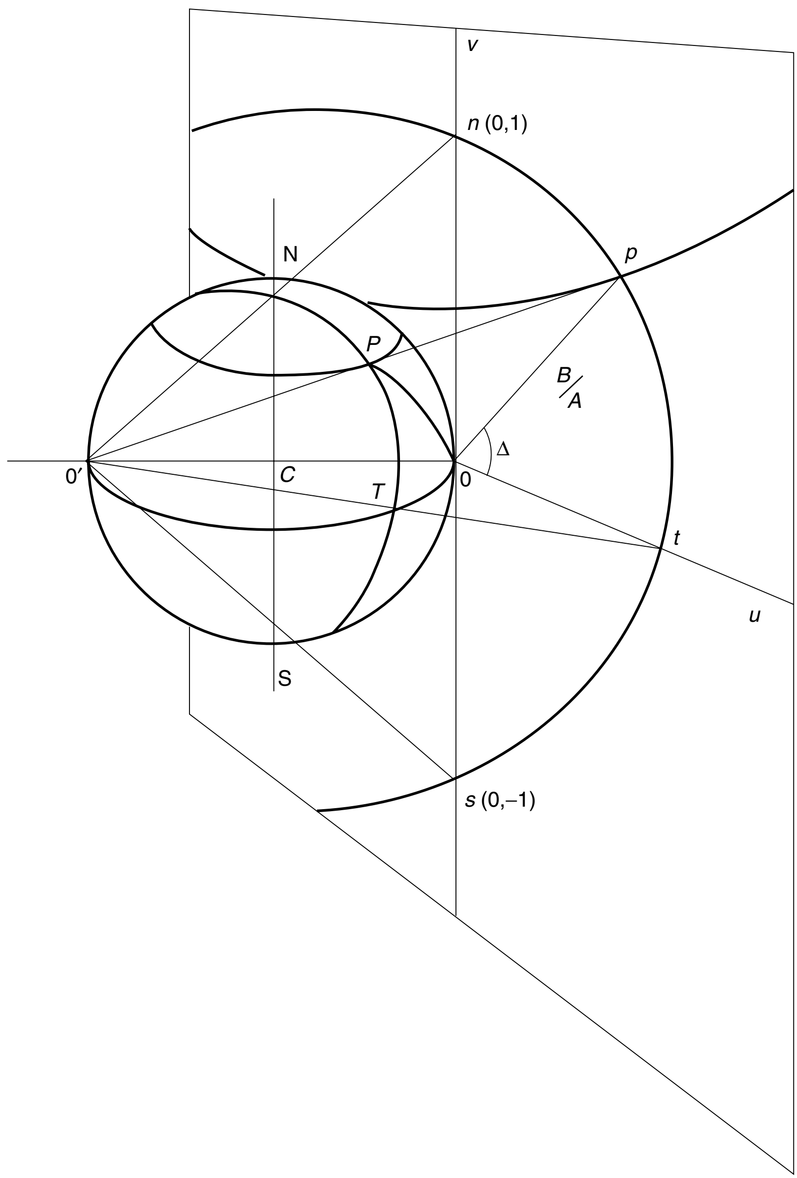


Figure 7.13 Poincaré sphere and Argand diagram.

Let us begin with the expression for a straight line passing through two specific points in space. Referring to Fig. 7.14, let the coordinates of these two points P_1 and P_2 be (a_1, b_1, c_1) and (a_2, b_2, c_2). If these two points are represented by position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, then the line segment $\overline{P_1P_2}$ is expressed by the vector $\mathbf{r}_2 - \mathbf{r}_1$. Let us pick another point P at (x, y, z). If all three points P_1, P_2, and P lie on the same straight line, then the line segments $\overline{P_1P}$ and $\overline{P_1P_2}$ must be parallel and share at least one point in common. The line segment $\overline{P_1P}$ is represented by the vector $\mathbf{r} - \mathbf{r}_1$. The line segments $\overline{P_1P}$ and $\overline{P_1P_2}$ share point P_1. Now, the condition that the two line segments

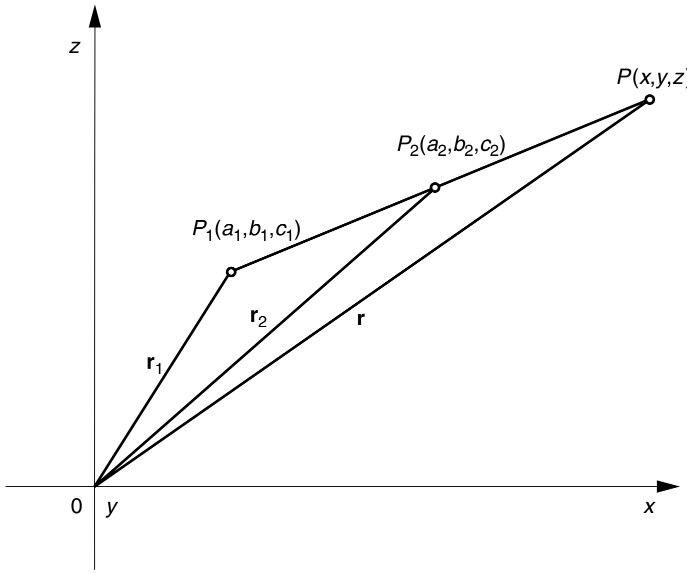


Figure 7.14 Geometry of a straight line in space.

are parallel is that the vector product is zero:

$$(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r} - \mathbf{r}_1) = 0 \tag{7.31}$$

Equation (7.31) leads to the following expression of a straight line passing through points P_1, P_2, and P:

$$\frac{x - a_1}{a_2 - a_1} = \frac{y - b_1}{b_2 - b_1} = \frac{z - c_1}{c_2 - c_1} \tag{7.32}$$

Both the sphere and the Argand diagram will be put into x, y, z coordinates, as shown in Fig. 7.15. Let the point where two surfaces touch be O. Let us pick the origin C of the x, y, z coordinates at the center of the Poincaré sphere, and let the radius of the Poincaré sphere be $\frac{1}{2}$. Let $P(x, y, z)$ be a point on the surface of the sphere. The choice of the senses of the coordinates should be noted. The positive x direction is from C to 0. The positive y direction is antiparallel to the u axis (indicated as $\overrightarrow{\mathrm{CW}}$ in the figure) so that the direction of the positive z axis is vertically upward in the right-hand rectangular coordinate system. In this arrangement the Argand diagram stands vertically in the plane of $x = \frac{1}{2}$ with its imaginary axis parallel to the z axis. The point $0'$ diametrically opposite to point 0 is at $(-\frac{1}{2}, 0, 0)$. All back-projections converge to point $0'$.

The coordinates of a point p on the Argand diagram are, from Eq. (7.6),

$$(a_2, b_2, c_2) = \left(\tfrac{1}{2}, -\tan\alpha\cos\Delta, \tan\alpha\sin\Delta\right)$$

and the coordinates of point $0'$ are

$$(a_1, b_1, c_1) = \left(-\tfrac{1}{2}, 0, 0\right)$$

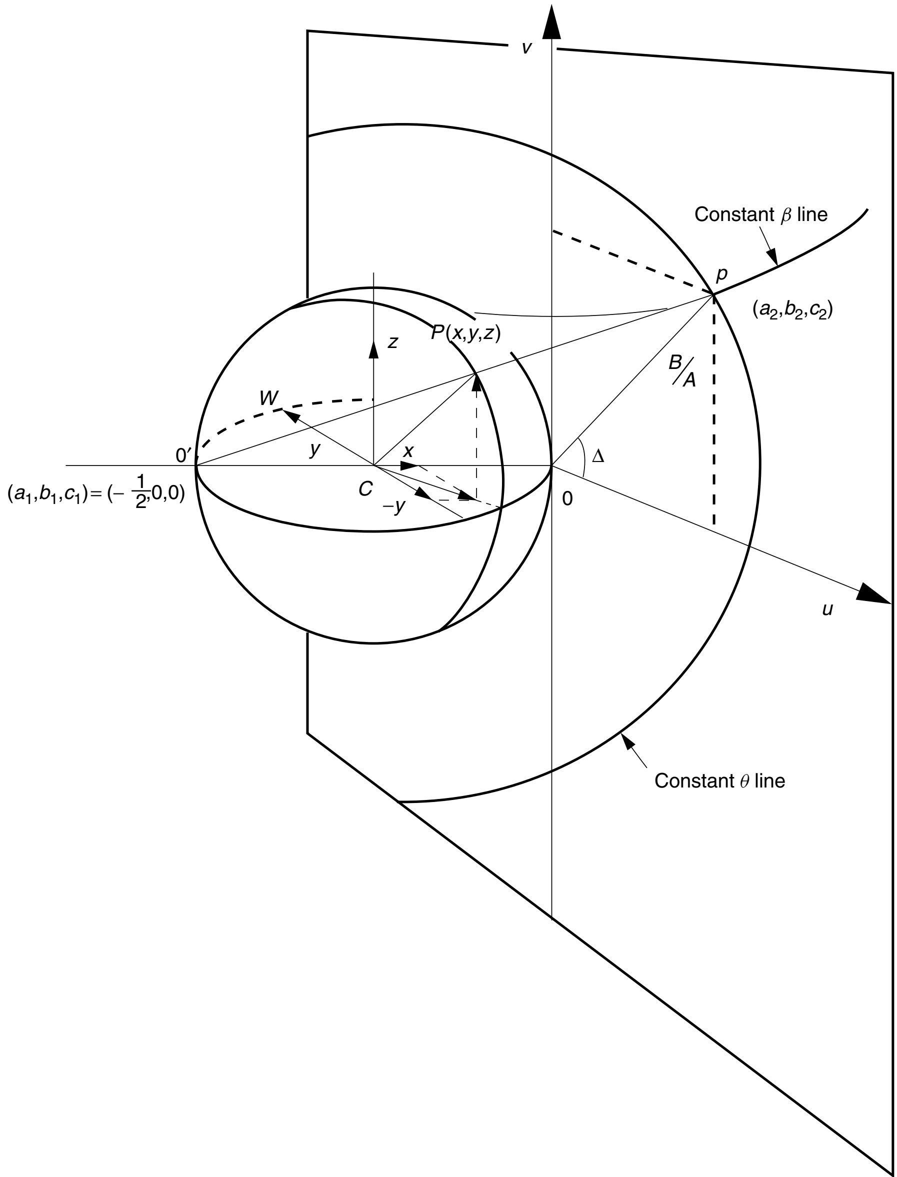


Figure 7.15 Projection of a point on the Argand diagram to the Poincaré sphere.

The expression for the straight line connecting $0'$ and p is, from Eq. (7.32),

$$\frac{x + \frac{1}{2}}{1} = \frac{y}{-\tan\alpha\cos\Delta} = \frac{z}{\tan\alpha\sin\Delta} \tag{7.33}$$

The expression for the unit diameter sphere is

$$x^2 + y^2 + z^2 = \tfrac{1}{4} \tag{7.34}$$

The solution of the simultaneous equations (7.33) and (7.34) gives the intersection of the line with the sphere. The calculation of the solution is simpler than it looks. First, the value of x will be found. Solving Eq. (7.33) for y and z in terms of x gives

$$y = -\left(x + \tfrac{1}{2}\right) \tan\alpha \cos\Delta \tag{7.35}$$

$$z = \left(x + \tfrac{1}{2}\right) \tan\alpha \sin\Delta \tag{7.36}$$

Insertion of Eqs. (7.35) and (7.36) into Eq. (7.34) gives

$$x^2 \cos^2\alpha + \left(x + \tfrac{1}{2}\right)^2 \sin^2\alpha = \tfrac{1}{4} \cos^2\alpha$$

which can be rewritten in terms of $\cos 2\alpha$ as

$$4x^2 + 2x(1 - \cos 2\alpha) - \cos 2\alpha = 0 \tag{7.37}$$

which can be factored as

$$(2x - \cos 2\alpha)(2x + 1) = 0 \tag{7.38}$$

The intersections are at

$$2x = \cos 2\alpha \tag{7.39}$$

and

$$2x = -1 \tag{7.40}$$

Equation (7.39) gives the intersection P, and Eq. (7.40) gives the point $0'$ on the sphere. Using Eq. (6.125), Eq. (7.39) becomes

$$2x = \cos 2\beta \cos 2\theta \tag{7.41}$$

Next, y is obtained by inserting Eq. (7.39) into (7.35) and expressing $\cos 2\alpha$ in terms of $\cos\alpha$ as

$$2y = -\sin 2\alpha \cos\Delta \tag{7.42}$$

Like Eq. (7.41), Eq. (7.42) will be expressed in terms of β and θ. In Eq. (6.99), $\tan 2\alpha$ is first expressed as $\sin 2\alpha / \cos 2\alpha$ and then Eq. (6.125) is used as a substitution for $\cos 2\alpha$. The resulting relationship is

$$\cos 2\beta \sin 2\theta = \sin 2\alpha \cos\Delta \tag{7.43}$$

Insertion of Eq. (7.43) into (7.42) gives

$$2y = -\cos 2\beta \sin 2\theta \tag{7.44}$$

Finally, the value of z will be obtained from Eqs. (7.36) and (7.39):

$$2z = \sin 2\alpha \sin\Delta \tag{7.45}$$

Using Eq. (6.112),

$$2z = \begin{cases} \sin 2\beta & \text{for } \sin\Delta > 0 \\ -\sin 2\beta & \text{for } \sin\Delta < 0 \end{cases} \tag{7.46}$$

In summary, the coordinates of the projected point P on the sphere are

$$x = \tfrac{1}{2}\cos 2\beta \cos 2\theta \tag{7.41}$$

$$y = -\tfrac{1}{2}\cos 2\beta \sin 2\theta \tag{7.44}$$

$$z = \pm\tfrac{1}{2}\sin 2\beta \tag{7.46}$$

where the $+$ sign of z is for left-handed rotation while the $-$ sign of z is for right-handed rotation. The Poincaré sphere can now be drawn from these results.

7.4.2 Poincaré Sphere

Let us first express point $P(x, y, z)$ in terms of the latitude l and longitude k on the sphere, rather than (x, y, z) coordinates. The latitude l (angle of elevation) of an arbitrary point (x, y, z) on the sphere of radius $\frac{1}{2}$ is, from Fig. 7.16,

$$l = \sin^{-1} 2z \tag{7.47}$$

The z coordinate of P, back-projected from point p, which is the intersection point of the constant θ and ϵ lines on the Argand diagram, is given by Eq. (7.46). Inserting Eq. (7.46) into (7.47) gives

$$l = 2\beta \tag{7.48}$$

Thus, the latitude l of point P is 2β and is linearly proportional to β. The constant β (or ϵ) lines on the Poincaré sphere are equally spaced in β, unlike the unequal spacing of the corresponding lines in the Argand diagram.

Moreover, as point p on the Argand diagram shown in Fig. 7.16 moves along the constant β line, its height in the v direction varies. The corresponding point P on the Poincaré sphere, however, stays at the same height or at the same latitude l, and the constant β lines are cylindrically symmetric.

Next, the longitude k of an arbitrary point (x, y, z) on the sphere is, from Fig. 7.17,

$$k = \tan^{-1}\left(\frac{-y}{x}\right) \tag{7.49}$$

The x and y coordinates of P back-projected from point p are given by Eqs. (7.41) and (7.44), and they are inserted into Eq. (7.49) to obtain

$$k = 2\theta \tag{7.50}$$

The longitude k is 2θ and is linearly proportional to θ. The constant θ lines are also equally spaced in θ on the Poincaré sphere. Thus, both constant β lines and constant θ lines are equally spaced on the Poincaré sphere. These equal spacings together with the above-mentioned cylindrical symmetry make the Poincaré sphere strikingly more versatile than the Argand diagram.

The back-projection of the other two quantities B/A and Δ will now be considered. An arc that connects two points on a spherical surface with the shortest distance along the surface is called a *geodesic*. In Fig. 7.18, the projection of $\overline{0p}$ in the Argand diagram is the geodesic $\overline{0P}$ on the Poincaré sphere. Since the Argand diagram is the tangent plane 0, the angle Δ on the Argand diagram is preserved when it is back-projected onto the sphere, and the angle of the geodesic $\overline{0P}$ with respect to the equator is also Δ.

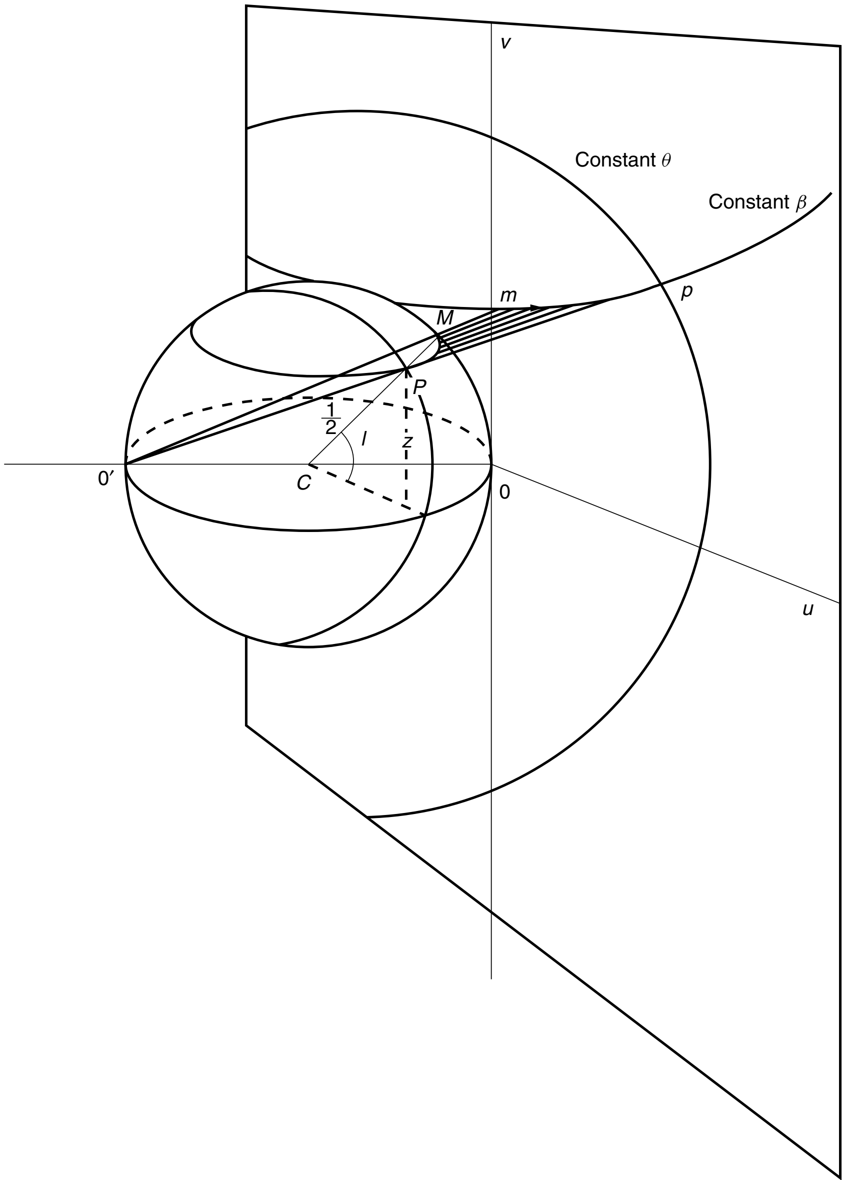


Figure 7.16 Projection of the constant ϵ curve onto the Poincaré sphere.

As shown in Fig. 7.18, $B/A(=\tan\alpha)$ is represented by $\overline{0p}$. Since $\overline{00}' = 1$,

$$\angle P0'0 = \alpha \tag{7.51}$$

$$\angle PC0 = 2\alpha \tag{7.52}$$

Thus, the Poincaré sphere provides a quick way of finding α from a given state of polarization. Referring to Fig. 7.19, it is interesting to note that, the radius of the sphere being $\frac{1}{2}$, geodesic $\overline{PT}$ is β, and geodesic $\overline{0P}$ is α.

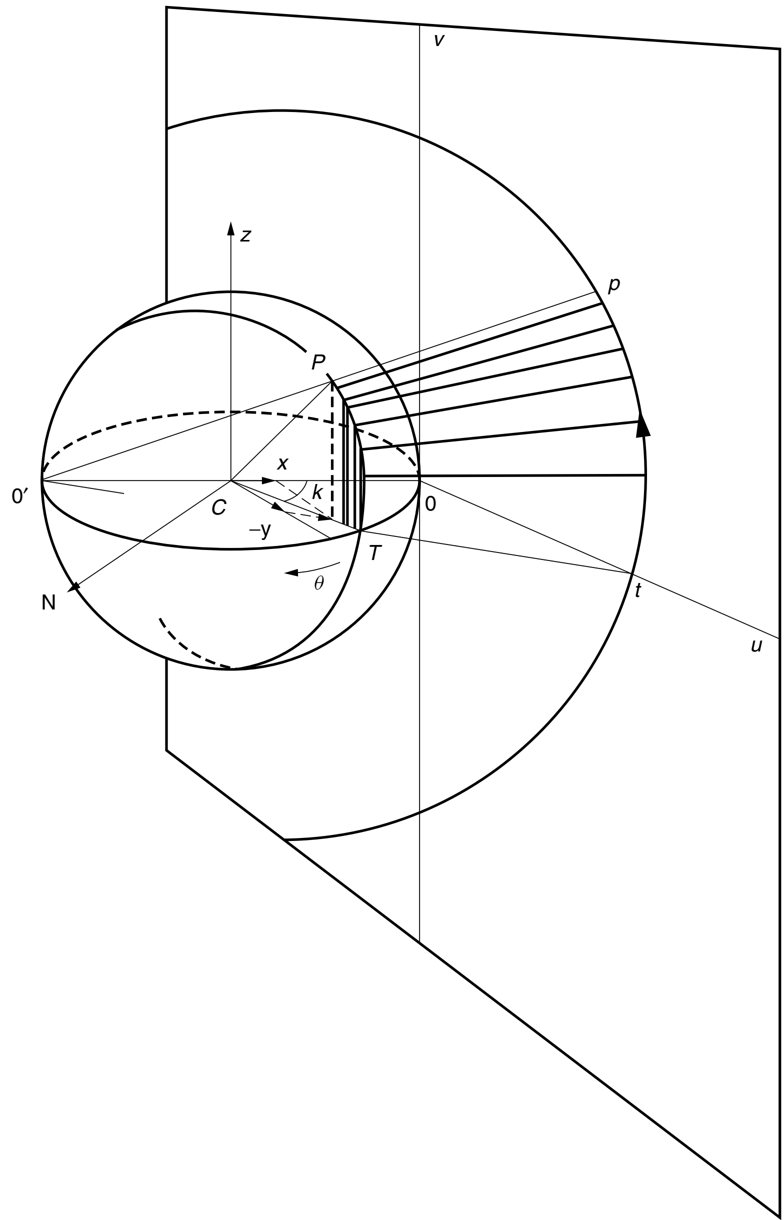


Figure 7.17 Projection of the constant θ line onto the Poincaré sphere.

Finally, the direction of the movement of the back-projected point on the sphere relative to that on the Argand diagram will be considered. Referring to Fig. 7.16, as the point moves from the left to the right, that is, from m to p on the Argand diagram, the projected point moves from the right to the left, that is, from M to P on the sphere, when observed from outside the sphere facing toward the center of the sphere. The points on the Argand diagram and on the sphere are more or less like mirror images of each other and are left and right reversed. This mirror image effect influences the

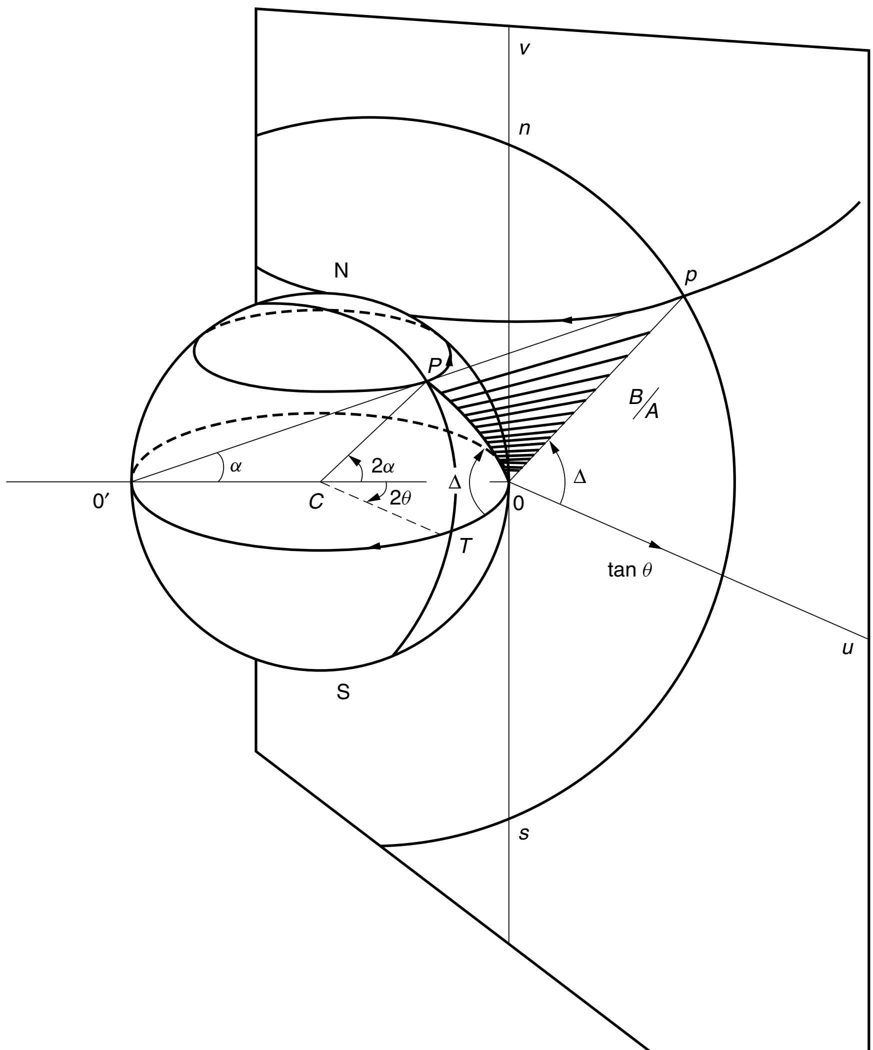


Figure 7.18 Directions of increasing Δ and θ.

direction of increasing Δ. The direction of increasing Δ is counterclockwise on the Argand diagram, whereas on the Poincaré sphere, the direction of increasing Δ is clockwise, as indicated when the sphere is viewed from the reader's vantage point in Fig. 7.18. Similarly, referring to Fig. 7.18, θ increases toward the right on the Argand diagram, and the direction of increasing θ is clockwise when the sphere is viewed from a point above the north pole. Whenever confusion about the direction arises, always go back to the Argand diagram, which is, after all, the basis for the Poincaré sphere.

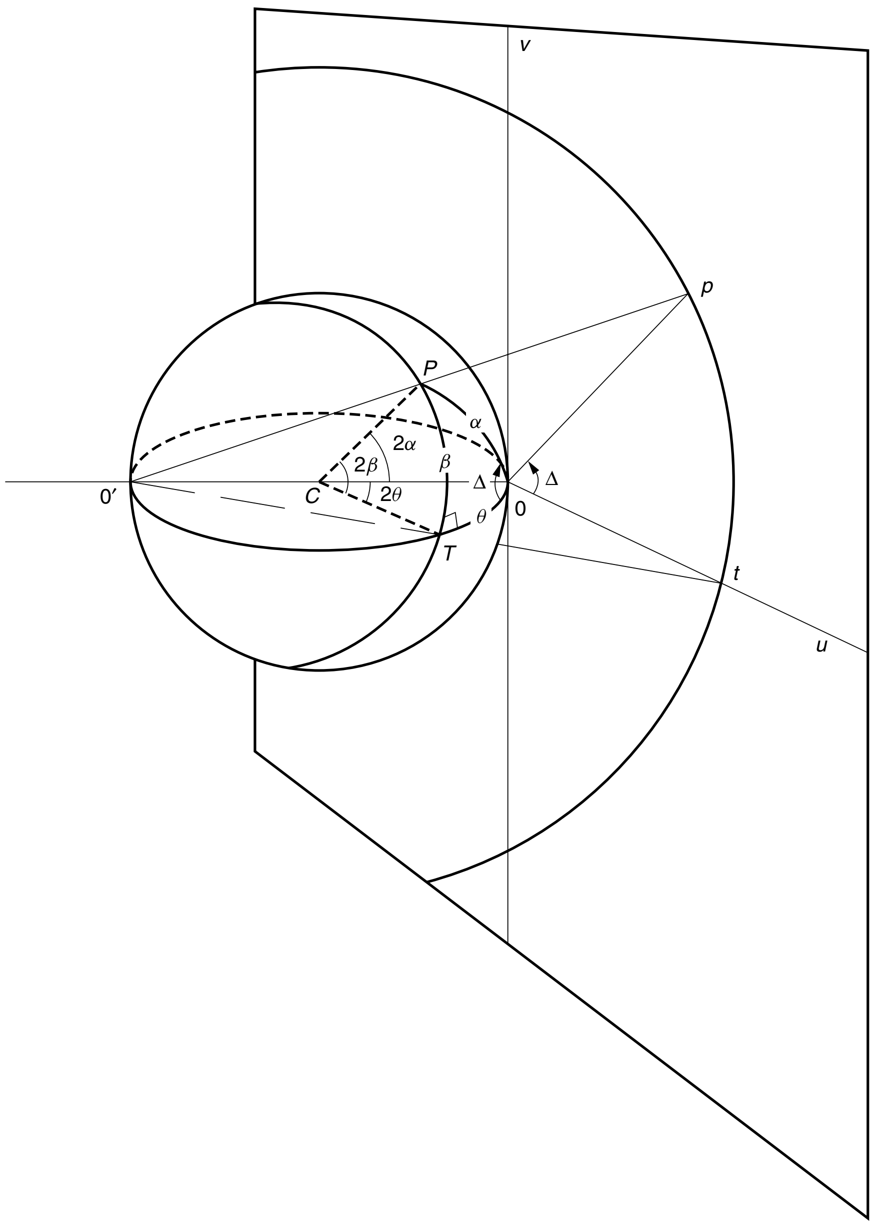


Figure 7.19 Geodesic triangle.

Figure 7.20 illustrates various states of polarization and their corresponding locations on the Poincaré sphere. The latitude lines are constant β (or ϵ) lines. Along the equator, $\epsilon = 0$ and the states of polarizations are linearly polarized waves with various azimuth angles. As higher latitudes are approached, the ellipticity increases and finally reaches unity at the poles. The states of polarization with β are represented by the latitude of 2β. The state of polarization $\beta = \pi/4$ (or $\epsilon = \tan \beta = 1$) is represented by the north pole of the sphere.

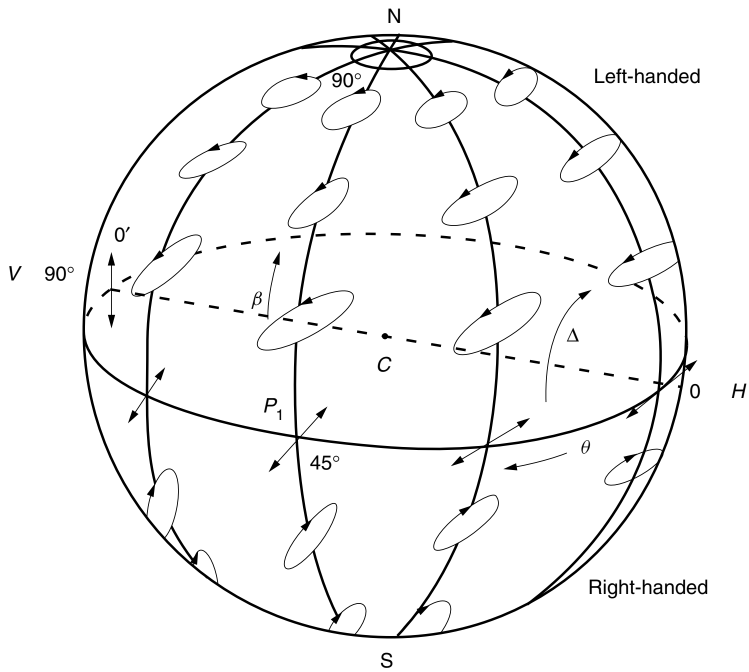


Figure 7.20 Poincaré sphere based on the convention of $e^{-j\omega t}$. The Poincaré sphere based on the $e^{j\omega t}$ convention is obtained by rotating in the plane of the page by 180°.

The longitudinal lines are constant θ (or azimuth) lines. The reference of the longitude is the longitudinal line passing through point 0. The state of polarization with θ is represented by the longitudinal line of 2θ. The state of polarization with $\theta = 90^\circ$ is represented by the longitudinal line that passes through point $0'$. The value of the azimuth increases in a clockwise sense when the sphere is viewed from a point above the north pole. The state of polarization with $\theta = 180^\circ$ is represented by the longitude line that passes through point 0 again. Point 0 represents horizontal linear polarization and point $0'$, vertical linear polarization. These points are often designated by points H and V. The azimuths of the ellipses along the same longitude stay the same. The states of polarization in the northern hemisphere are all left-handed rotation, while those in the southern hemisphere are right-handed rotation.

It is important to remember that the physical locations of the constant θ and β graduation lines are at 2θ and 2β, respectively, on the Poincaré sphere, while the retardance Δ° means a rotation of the point on the Poincaré sphere by Δ degrees, not 2Δ degrees.

Depending on which convention, $e^{-j\omega t}$ or $e^{j\omega t}$ is used, the same retardance can be expressed by either Eq. (6.8) or (6.10). The Poincaré sphere shown in Fig. 7.20 is based on the $e^{-j\omega t}$ convention of this book. The Poincaré sphere based on $e^{j\omega t}$, however, can be obtained by rotating the figure by 180° in the plane of the page.

7.5 POINCARÉ SPHERE SOLUTIONS FOR RETARDERS

Uses of the Poincaré sphere will be explained by a series of examples [4].

Example 7.6 Light linearly polarized at $\theta = 45^\circ$ is incident onto a $\lambda/4$ plate whose fast axis is along the x axis. Find the state of polarization of the emergent light using both the Argand diagram and the Poincaré sphere.

Solution The input light is characterized by $B/A = 1$ and $\epsilon = 0$ and is represented by point p_1 at $(u, v) = (1, 0)$ on the Argand diagram, as shown in Fig. 7.21. The $\lambda/4$ plate with its fast axis along the x axis rotates point p_1 by 90° to p_2 at $(u, v) = (0, 1)$, which represents left-handed circular polarization.

The same problem will be solved by using the Poincaré sphere. The back-projected point from point p_1 on the Argand diagram to the Poincaré sphere is P_1 at $(\theta, \epsilon) = (45^\circ, 0)$ in either Fig. 7.20 or 7.21.

Geodesic $\overline{0P_1}$ represents B/A. With point 0 as the center of rotation, point P_1 is rotated by 90° in a clockwise direction due to the $\lambda/4$ plate. The final point P_2 is on the north pole. The answer is left-handed circularly polarized light. The answers agree with the result in Fig. 6.4.

The next example deals with a retarder whose fast axis is at an arbitrary angle and is along neither the x nor y axis. A significant advantage of the Poincaré sphere over the Argand diagram is seen.

Example 7.7 A linearly polarized light with azimuth $\theta = 40^\circ$ is incident onto a $\lambda/4$ plate whose fast axis is oriented at $\Theta = 30^\circ$. Find the state of polarization of the emergent light.

Solution The orientation of the incident light and the $\lambda/4$ plate are represented by P_1 and R, respectively, in Fig. 7.22a. First, the coordinates have to be rotated by 30° so that the new x' axis lines up with the fast axis of the $\lambda/4$ plate. When the coordinates are rotated by $+30^\circ$, the values of θ and Θ are decreased by 30° as shown in Example 7.3. After the rotation of the coordinate system, point P_1 and R are transferred to P_1' and R' at $\theta = 10^\circ$ and 0°, respectively, as shown in Fig. 7.22b. The $\lambda/4$ plate rotates P_1' to P_2' around R'. The final result is obtained by rotating the coordinates back by -30° and the value of θ is increased by 30°. As shown in Fig. 7.22c, the final point is P_2''. Approximate values of $\theta = 30^\circ$, $\epsilon = 0.17$ can be read directly from the graduation on the Poincaré sphere.

An important feature of the Poincaré sphere is that rotation and rerotation of the coordinate system can be avoided because of the cylindrical symmetry of the Poincaré sphere with respect to the polar axis. Geodesic $\overline{P_1'P_2'}$, which was obtained following the rotation of the coordinates is the same as geodesic $\overline{P_1''P_2''}$. Rotation of the coordinates by 30° followed by rotation back by -30° is not necessary. The same result can be obtained by rotating P_1'' to P_2'' by 90° around point R'' from the very beginning owing to the symmetry that the Poincaré sphere has. This is one of the major advantages of the Poincaré sphere. Compare this with the steps needed when the Argand diagram was used in Example 7.3.

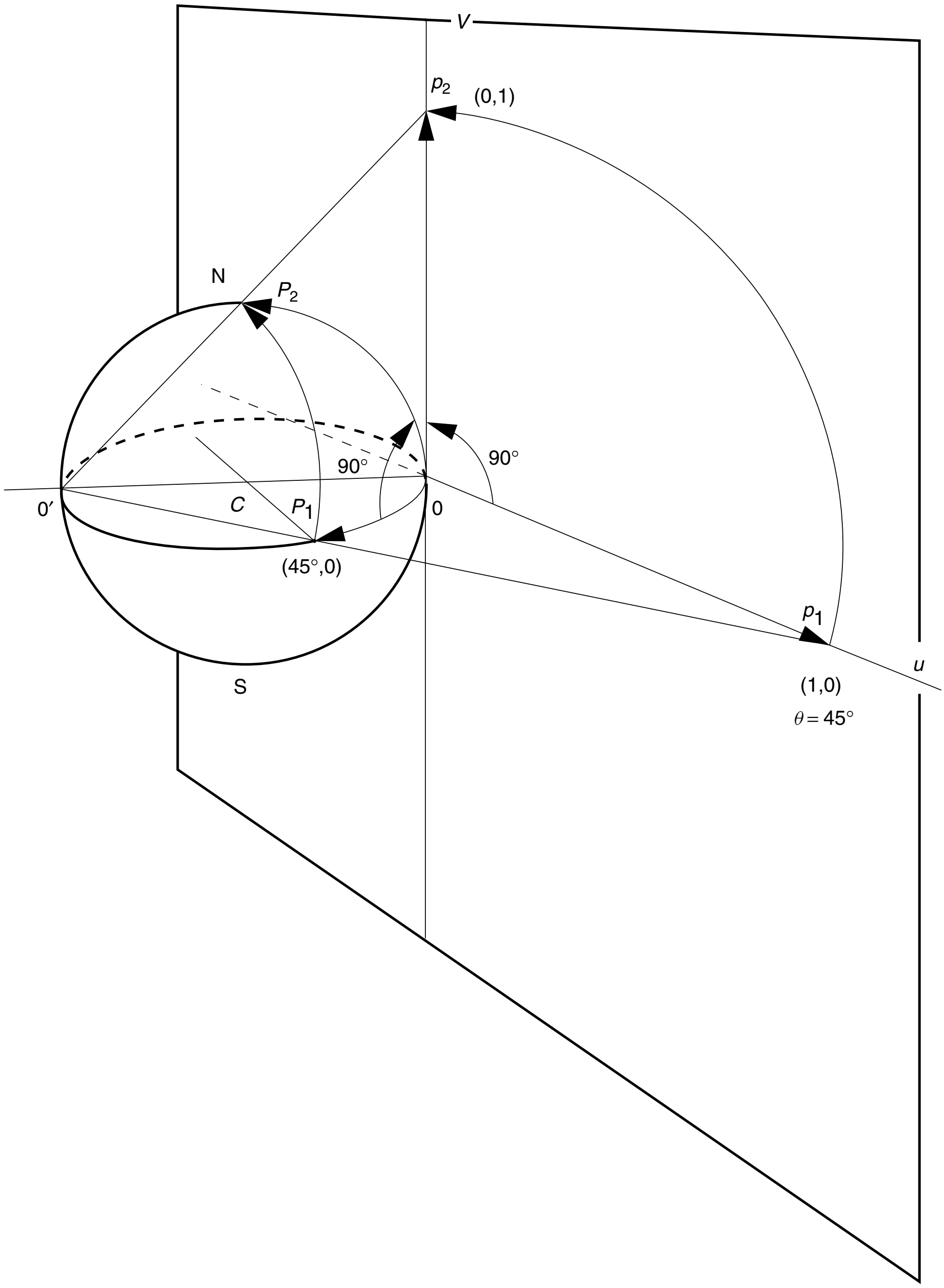


Figure 7.21 Linearly polarized light with $\theta = 45°$ is incident onto a $\lambda/4$ plate whose fast axis is along the x axis. The Poincaré sphere with Argand diagram is used to find the emergent wave.

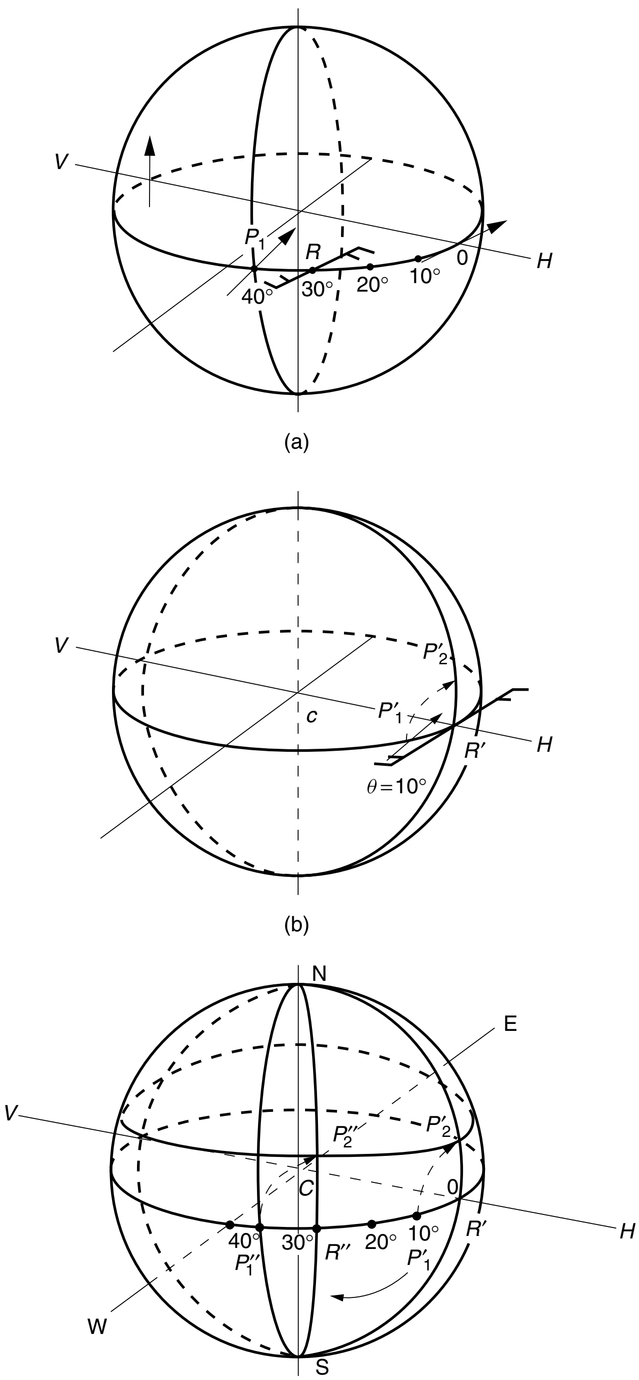


Figure 7.22 Linearly polarized light with azimuth $\theta = 40^\circ$ is incident onto a $\lambda/4$ plate whose fast axis is oriented at $\Theta = 30^\circ$. (a) State of polarization of the incident wave and the $\lambda/4$ plate are plotted on the Poincaré sphere. (b) Conversion of the state of polarization due to the $\lambda/4$ plate after rotation of the coordinates. (c) Similarity between the operations with and without the rotation of the coordinates.

A more accurate number can be obtained by noting the geometry. As already shown in Fig. 7.19, geodesic $\overline{P_2''R''}$ in Fig. 7.22c represents β. From the relationship $\overline{P_2''R''} = \overline{P_1''R''}$ combined with $\overline{P_1''R''} = (\pi/180^\circ) \times 10$, β is found to be $\beta = (\pi/180^\circ) \times 10$ rad and $\epsilon = \tan\beta = 0.176$. □

Example 7.8 Using the Poincaré sphere solve the following problems:

(a) Prove that a $\lambda/4$ plate can convert a left-handed or right-handed circularly polarized wave into a linearly polarized wave regardless of the orientation of the fast axis of the $\lambda/4$ plate.
(b) Identify the handedness of an incident circularly polarized wave by using a $\lambda/4$ plate of known fast axis orientation.
(c) Find the proportion of powers of each handedness when the incident light is a mixture of left-handed and right-handed circularly polarized waves.
(d) Consider the converse to part (a). Can a $\lambda/4$ plate convert a linearly polarized wave into a circularly polarized wave, regardless of the orientation of the fast axis of the $\lambda/4$ plate?

Solution

(a) As seen from Fig. 7.23, regardless of the orientation of the $\lambda/4$ plate, the 90° rotation from either pole brings the point onto the equator and the emergent light becomes linearly polarized. The direction of the emergent linear polarization, however, depends on the orientation of the fast axis of the $\lambda/4$ plate.

(b) As seen from Fig. 7.23, when the fast axis is oriented at Θ, the direction of the polarization of the emergent light is at $\theta = \Theta \pm 45^\circ$. If the direction of polarization of the emergent light is at $\theta = \Theta + 45^\circ$, the incident light has right-handed rotation, and if the direction is at $\theta = \Theta - 45^\circ$, the incident light has left-handed rotation. In this manner, the handedness of the incident wave is identified.

(c) As shown in Fig. 7.23, mixed left and right circularly polarized waves are incident on a $\lambda/4$ plate with fast axis at $\Theta = 45^\circ$. The north pole is brought to H and the south pole is brought to V. The emergent light is a combination of horizontal and vertical linear polarization, which can be separated using a polarization beamsplitter. Figure 7.24 illustrates the arrangement for measuring the ratio, between the two oppositely handed rotations.

(d) As illustrated in Fig. 7.25, if the orientation of the fast axis is other than $\pm 45^\circ$ with respect to the direction of the linear polarization of the incident light, represented by point P_1, then it is not possible for the emergent state, represented by P_1', to reach the pole. Therefore, the converse to part (a) is not true. A $\lambda/4$ plate cannot convert a linearly polarized wave into a circularly polarized wave when the fast axis is in an arbitrary orientation. A circularly polarized wave is obtainable only when the fast axis is at $\pm 45^\circ$ with respect to the direction of linear polarization of the incident light. □

Example 7.9 Fabricate your own Poincaré sphere by drawing the state of polarizations on flat one-eighth sectors that can later be taped together to form a sphere. Choose the length of $\sqrt{A^2 + B^2}$ to be $\pi/20$ of the radius of the Poincaré sphere. Draw the states of polarization at steps of $\Delta\theta = 22.5^\circ$ and $\Delta\beta = 11.25^\circ$.

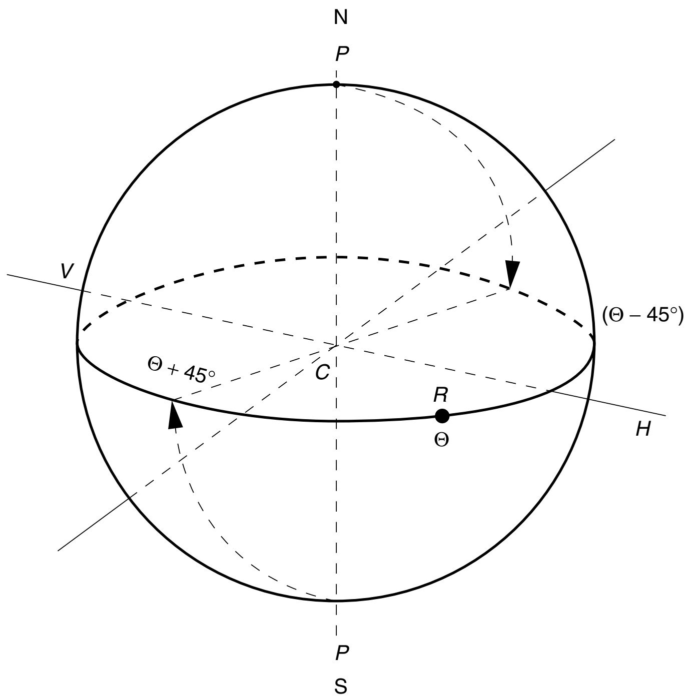


Figure 7.23 Regardless of the orientation of the fast axis, a $\lambda/4$ plate converts a circularly polarized wave into a linearly polarized wave.

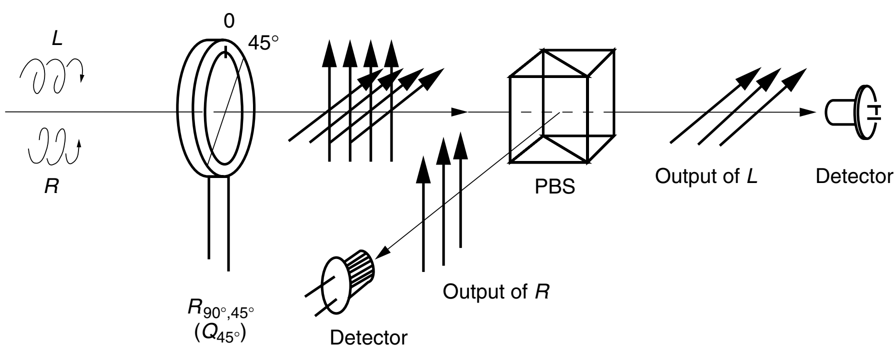


Figure 7.24 Determination of the ratio of powers of left- and right-handed circular polarizations.

Solution The expressions for the lengths of the major and minor axes of the ellipses for a given β are obtained from Eqs. (6.100), (6.107) and (6.113), as

$$a = \sqrt{\frac{A^2 + B^2}{1 + \tan^2 \beta}} \tag{7.53}$$

$$b = a \tan \beta \tag{7.54}$$

Figure 7.26 shows the finished pattern.

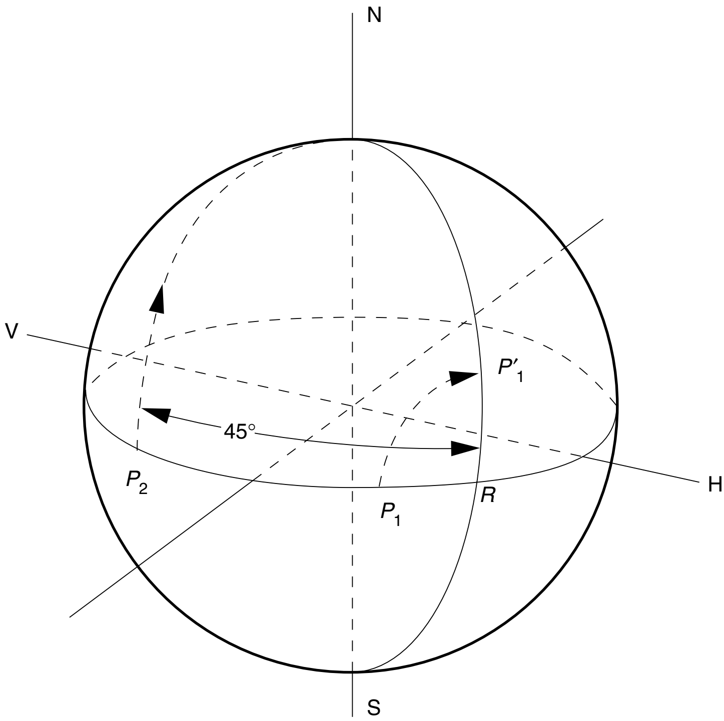


Figure 7.25 The difference in angle between the azimuth of the incident wave and the fast axis of the $\lambda/4$ plate has to be 45° to convert a linearly polarized wave into a circularly polarized wave.

Try to fabricate your own Poincaré sphere by photo copying the enlarged pattern. Firm materials such as acetate sheets for overhead projectors work best. Then cut and tape the sections to form a balloon. Such a balloon made out of thin colorful paper is called a Fusen and is a popular toy among Japanese children. □

7.6 POINCARÉ SPHERE SOLUTIONS FOR POLARIZERS

The Poincaré sphere will be used to find the power transmittance k of a polarizer [4], which is the ratio of the polarizer's transmitted to incident power.

Consider light of an arbitrary state of polarization incident onto a polarizer whose transmission axis is along the x axis as shown in Fig. 7.27a. Of the total power of the incident light $A^2 + B^2$, only the component in the x direction transmits through the polarizer. The power transmittance k of an ideal polarizer ($k_1 = 1, k_2 = 0$) is

$$k = \frac{A^2}{A^2 + B^2} \tag{7.55}$$

Equation (7.55) is true for any state of polarization as long as the direction of the transmission axis is along the x axis.

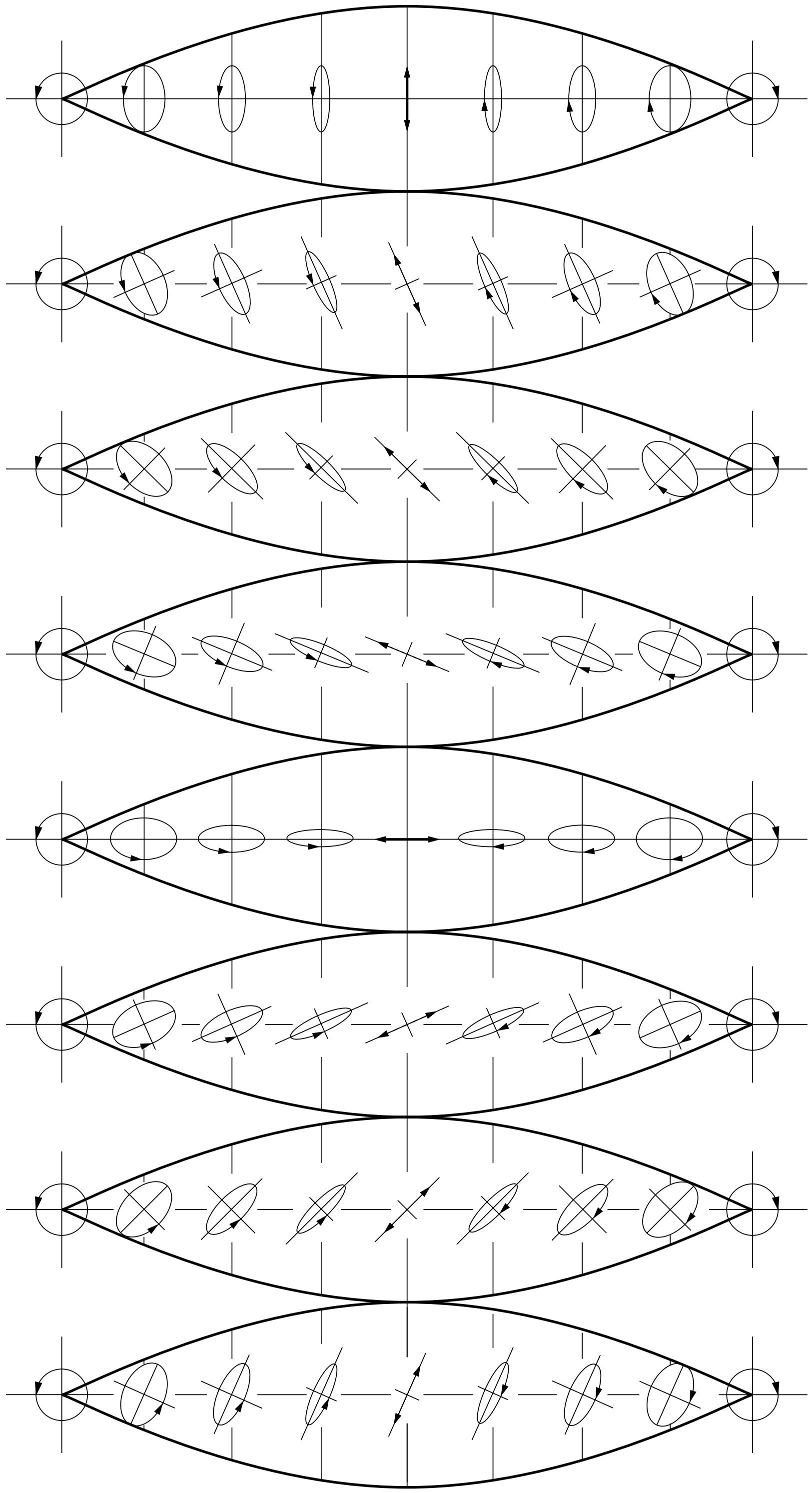
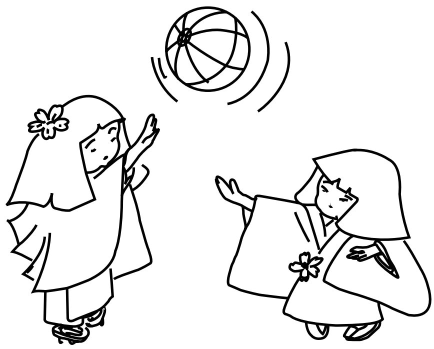


Figure 7.26 Pattern for making a Poincaré sphere.



Japanese children playing with a Fusen.

With Eq. (7.7), Eq. (7.55) becomes

$$k = \cos^2 \alpha \tag{7.56}$$

Thus, once α of the incident light is found by the Poincaré sphere, k can be calculated from Eq. (7.56).

A method of finding α using the Poincaré sphere will be explained. First, the transmission axis of the polarizer is assumed to be along the x axis. Let the state of polarization of the incident light be (θ, β), as represented by point P_1 on the Poincaré sphere shown in Fig. 7.27b.

In order to find 2α, $\angle P_0CP_1$ has to be found, where P_0 represents the azimuth of the polarizer transmission axis and is located at H for this case. The sphere is cut by a plane perpendicular to the $\overline{HV}$ axis and containing point $P_1(\theta, \beta)$. Point Q is any point on the circle made by the intersection of the perpendicular plane and the sphere, and $\angle P_0CQ$ is always equal to $\angle P_0CP_1$. Thus, when Q falls on the equator,

$$2\alpha = 2\theta' \quad \text{or} \quad \alpha = \theta'$$

where θ' is the value shown on the graduation line. Recall from Fig. 7.20 that the value shown on the graduation line, for instance, $\theta' = 45^\circ$, is at the longitudinal angle of 90° from point 0 or H.

In short, in order to find α:

1. Draw the cross-sectional circle that contains point $P_1(\epsilon, \theta)$ of the incident light and is perpendicular to $\overline{VH}$.
2. Find the intersection of the cross-sectional circle with the equator.
3. The value of the graduation line of the intersection is the desired α.

A few interesting observations are (1) $\theta \neq \theta'$, unless $\theta = 45^\circ$, and (2) $\angle CVP_1 = \angle CVQ = \alpha$, and this angle can also be used to find α.

Now consider the situation where the transmission axis of the polarizer is not along the x axis and the azimuth is Θ. This is represented by P_Θ in Fig. 7.27c. One way to solve this problem is to rotate the coordinates by Θ so that the x axis lines up with the transmission axis. The value α is found as in Fig. 7.27b. Once α is obtained, the coordinates are rotated back by $-\Theta$. A simpler approach is to use the symmetry of

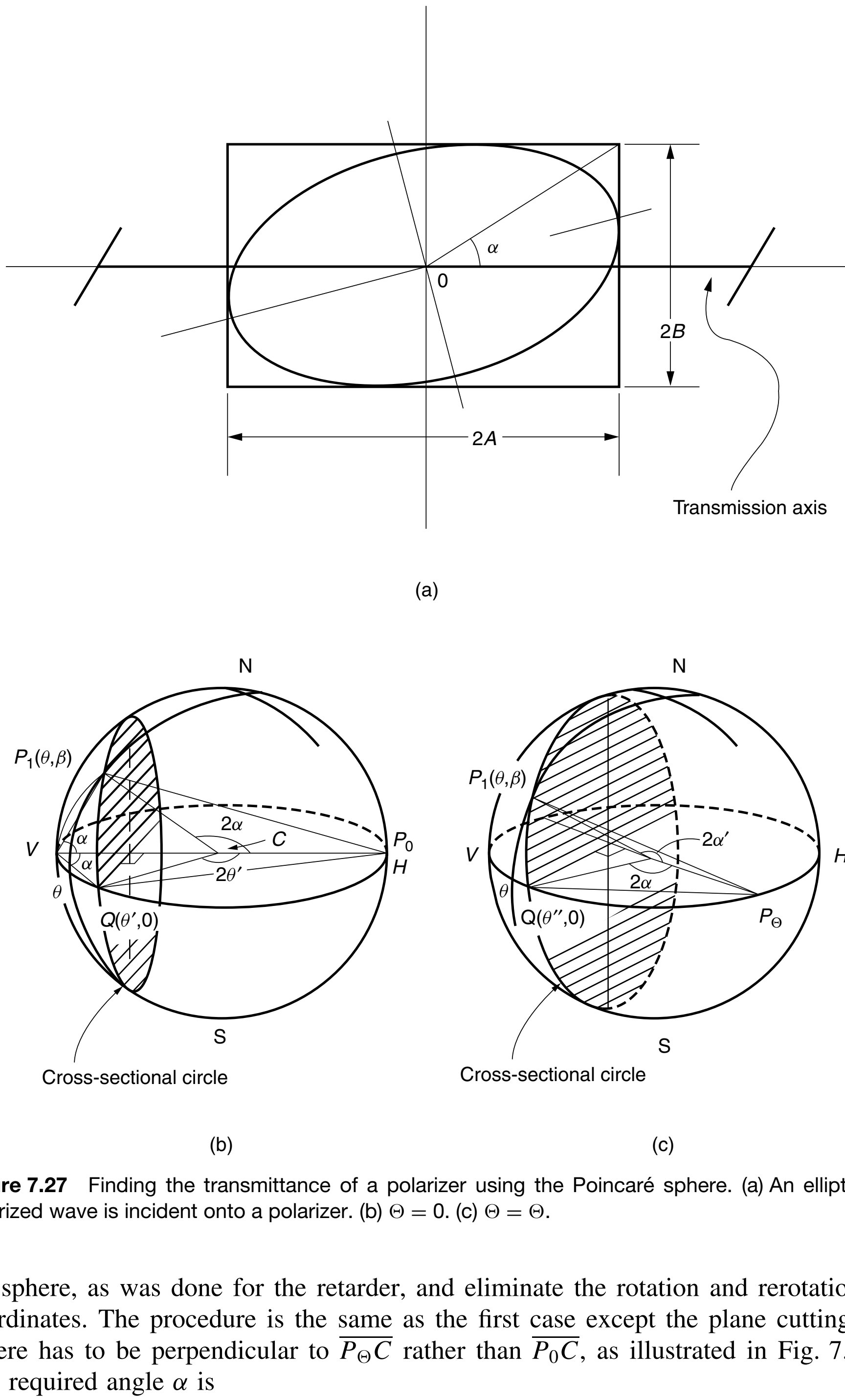


Figure 7.27 Finding the transmittance of a polarizer using the Poincaré sphere. (a) An elliptically polarized wave is incident onto a polarizer. (b) $\Theta = 0$. (c) $\Theta = \Theta$.

the sphere, as was done for the retarder, and eliminate the rotation and rerotation of coordinates. The procedure is the same as the first case except the plane cutting the sphere has to be perpendicular to $\overline{P_\Theta C}$ rather than $\overline{P_0 C}$, as illustrated in Fig. 7.27c. The required angle α is

$$\alpha = P_\Theta - \theta'' \tag{7.57}$$

where values are those shown on the graduation line.

Example 7.10 Using the Poincaré sphere, find the transmittance through an ideal polarizer with the following configurations:

(a) Vertically polarized light is incident onto a polarizer whose transmission axis is horizontal.
(b) Linearly polarized light with azimuth 45° is incident onto a polarizer whose transmission axis is along the x axis.
(c) A circularly polarized wave with left-hand rotation is incident onto a polarizer whose transmission axis is at Θ.
(d) Left-handed elliptically polarized light with $\epsilon = 0.414$ and $\theta = 60^\circ$ is incident onto a polarizer whose transmission axis is at $\Theta = 20^\circ$.

Solution

(a) With Fig. 7.27b, the cross-sectional circle is tangent to the sphere at V, and the intersection with the equator is at the graduation line of $\theta' = 90^\circ$, and thus $\alpha = 90^\circ$.

$$k = \cos^2 90^\circ = 0$$

(b) With Fig. 7.27b, the cross-sectional circle cuts the sphere into equal halves. The intersection with the equator is at the graduation line of $\theta = 45^\circ$ and $\alpha = 45^\circ$.

$$k = \cos^2 45^\circ = 0.5$$

(c) The cross-sectional circle is through the N–S axis and perpendicular to $\overline{CP_\Theta}$ and

$$\alpha = \theta' - P'_\Theta = 45^\circ$$
$$k = \cos^2 45^\circ = 0.5$$

(d) Referring to Fig. 7.28, the cross-sectional circle containing the point at ($\theta = 60^\circ$, $\epsilon = 0.414$) is drawn and is perpendicular to $\overline{CP_\Theta}$. The intersection with the equator is at the graduation line of

$$\theta' = 61^\circ,$$
$$\alpha = \theta' - P_\Theta = 61^\circ - 20^\circ = 41^\circ$$
$$k = 0.56$$

It should be noted that $\theta' \neq 60^\circ$. □

The analytical expression for k gives better accuracy. Rewriting Eq. (7.56) in terms of $\cos 2\alpha$, and using Eq. (6.125) gives

$$k = \tfrac{1}{2}(1 + \cos 2\theta \cos 2\beta) \tag{7.58}$$

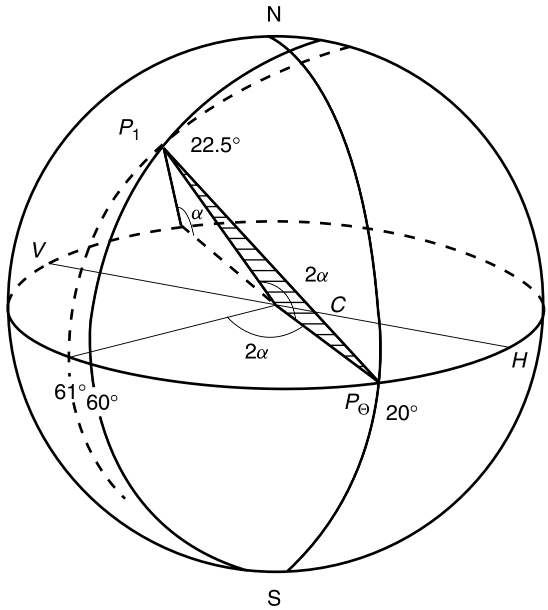


Figure 7.28 The Poincaré sphere is used to find the transmittance through a polarizer.

7.7 POINCARÉ SPHERE TRACES

For ease and accuracy, traces of the Poincaré sphere on a plane are often used [3]. Any convenient plane can be selected. The method will be explained using the example that a linearly polarized wave with $\theta = 50^\circ$ is transmitted through a compensator whose fast axis is oriented at $\Theta = 30^\circ$ with retardance $\Delta = 135^\circ$.

Figure 7.29 explains how the traces are drawn [8]. The top drawing is the projection onto the horizontal plane. In this plane, the azimuth angles are clearly seen. Point P_1 corresponds to the azimuth of the incident light, and point R corresponds to the azimuth of the compensator.

The left bottom trace is the projection onto plane 1_F. Plane 1_F is the frontal plane perpendicular to radius RC. (Rather than projecting perpendicular to $\overline{VH}$, it is projected off the orthogonal direction $\overline{CR}$ and denoted as 1_F.) In this plane, the true angle Δ of the retardance can be seen, and from Δ, point P_2 for the emergent light is drawn in. The corresponding point P_2 in the horizontal plane can be obtained.

The Poincaré sphere is also projected onto the profile plane 1_P, which is the side-view plane perpendicular to both H and 1_F planes. Point P_2 in this plane is obtained by extensions from the corresponding points in the H and 1_F planes.

The essentials for finding (θ, β) of the emergent light are now in place. Referring to the projection of P_2 onto the H plane, $2\theta = 30^\circ$ can be read directly from the diagram.

The angle P_2CP_1 in the 1_P plane is not the true 2β angle. In order to see the true angle, the sphere is rotated around the $\overline{\mathrm{NS}}$ axis until $\overline{P_2'C}$ falls in a plane parallel to the 1_P plane. The height from the equator does not change by this rotation because the axis of rotation is $\overline{\mathrm{NS}}$. The true angle is $2\beta = \angle P_2'CP_1 = 28^\circ$.

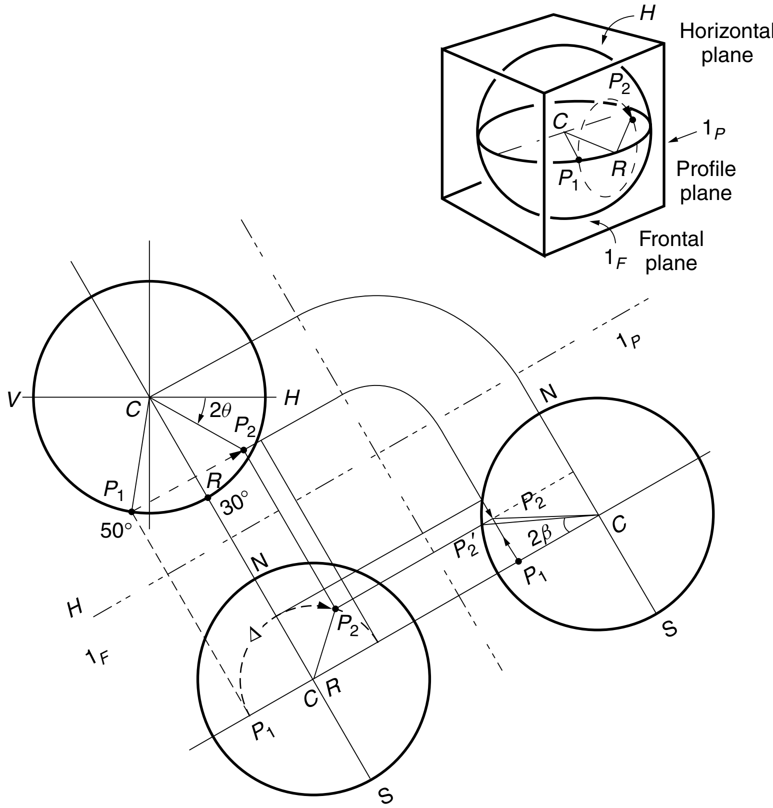


Figure 7.29 Poincaré sphere traces.

The emergent wave is therefore a left-handed elliptically polarized wave with $\theta = 15°$ and $\epsilon(= \tan\beta) = 0.25$.

Example 7.11 The retardance of a sample was measured by means of Senarmont's method in Section 6.4.3.3. The azimuth angle of the extinction axis (minor principal transmission axis) of the analyzer was at $\theta = 150°$. Find the retardance using the method of tracing onto the projected planes from the Poincaré sphere.

Solution As shown in Fig. 6.21 in the last chapter, Senarmont's method determines the retardance of a sample by measuring the ellipticity of the emergent light when linearly polarized light is incident with its direction of polarization at 45° with respect to the horizontally oriented fast axis of the sample. As already proved (Problem 6.11), 2β of the emergent light equals the retardance Δ. 2β is measured by means of a $\lambda/4$ plate with $\Theta = 45°$. The light from the $\lambda/4$ plate emerges as linearly polarized light with $\theta = 45° + \beta$. θ is measured by means of an analyzer, and β is determined.

The operation is shown on the Poincaré sphere in Fig. 7.30. P_1 represents linearly polarized incident light with $\theta = 45°$. The sample, whose fast axis is along the x axis, rotates P_1 centered around R_x by Δ to P_2. Point P_2 is further rotated by the $\lambda/4$ plate centered around $R(\theta = 135°)$ by 90° to P_3, which represents a linearly polarized wave

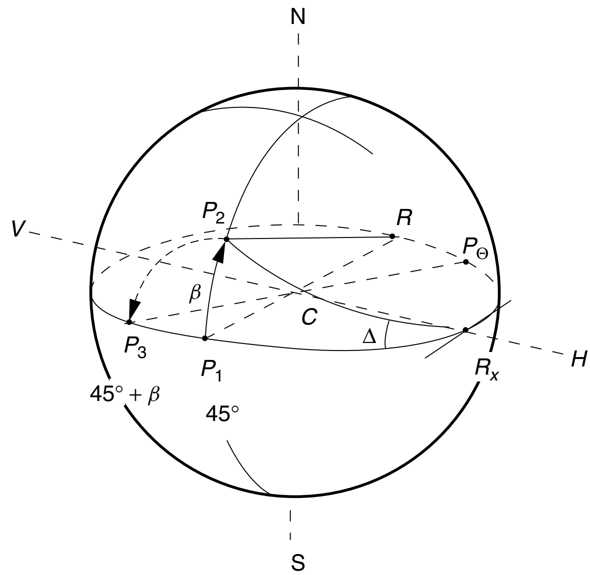


Figure 7.30 Senarmont's method represented on the Poincaré sphere.

with $\theta = 45° + \beta$. When the polarizer P_Θ is adjusted to $\theta = 135° + \beta$, null output is observed.

Now, the points on the Poincaré sphere in Fig. 7.30 will be transferred onto traces in Fig. 7.31. Projections are made in the horizontal, frontal, and profile planes. The trace in the profile plane shows the true retardance angle Δ. P_1 of the incident light is rotated around R_x by the retardance Δ of the sample. Point P_2 is further rotated by the $\lambda/4$ plate by 90° around R at $\theta = 135°$ to reach point P_3. The frontal projection shows the true angle of the rotation from P_2 to P_3.

Point P_Θ of the azimuth of the analyzer for extinction is diametrically opposite to point P_3. The true angle of P_Θ is seen in the horizontal plane.

Now in order to obtain the value of Δ from a given value of P_Θ, one has to follow the procedure backward. Keep in mind that the locations on the Poincaré sphere are graduated from $\theta = 0°$ to 180° and from $\beta = 0°$ to 45°. As was shown in Fig. 7.20, the relevant angles on the projected planes are 2θ, 2β, and Δ. The amount of Δ is not doubled. A retardance of 90° is represented by a rotation of 90°.

Now, with the given value of $P_\Theta = 150°$, Δ will be obtained. The point graduated as $\Theta = 150°$ is labeled as P_Θ in the horizontal plane in Fig. 7.31. The graduation of the diagonally opposite point P_3 is $150° - 90° = 60°$. If one goes from the horizontal plane to the profile plane, angle $\angle P_1CP_2$ indicates that the desired retardance is $\Delta = 30°$. □

The next example addresses (1) how to draw traces for a given state of polarization, (2) how to manipulate the operation of a $\lambda/4$ plate with an arbitrary orientation, and (3) how to read off the true value of the state of polarization from the trace.

Example 7.12 Light with $(\theta, \epsilon) = (77°, 0.34)$ is incident onto a $\lambda/4$ plate whose fast axis azimuth is $\Theta = 56°$.

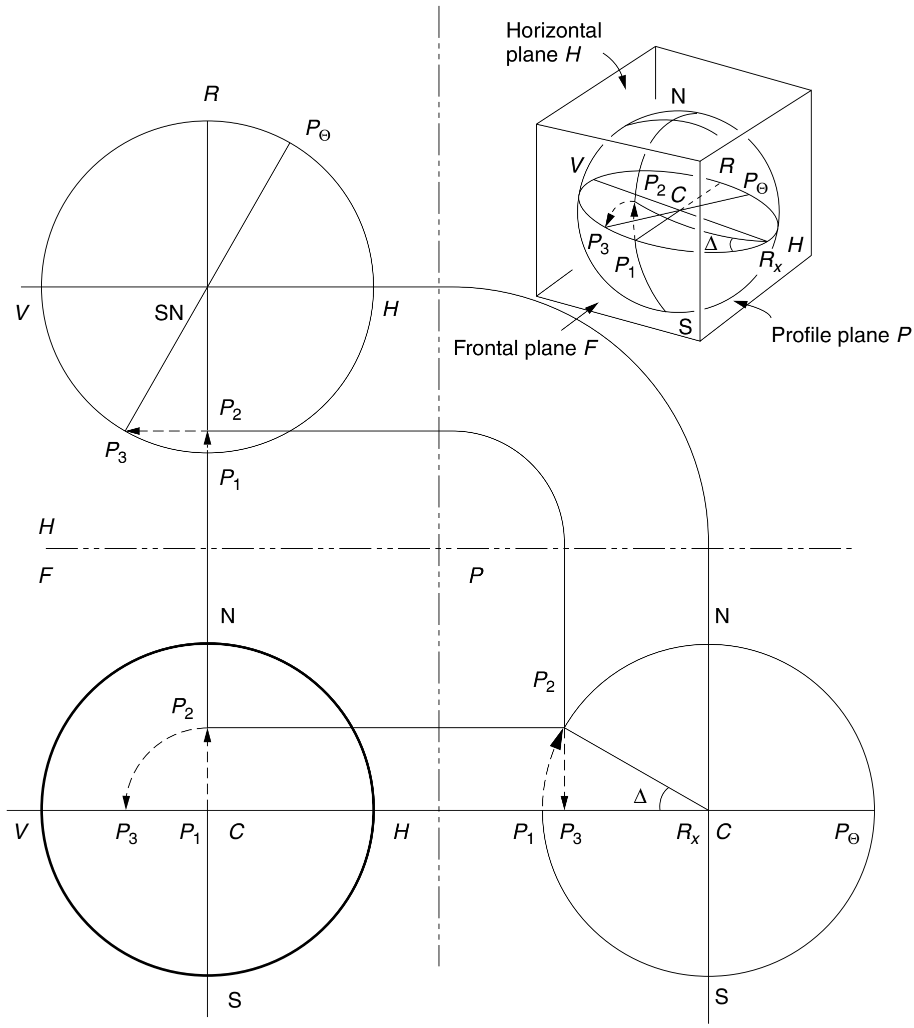


Figure 7.31 Traces of the Poincaré sphere to illustrate the Senarmont method.

(a) Find the state of polarization of the light emergent from the $\lambda/4$ plate.

(b) The above wave is further transmitted through an analyzer. The azimuth angle of the transmission axis of the analyzer is $\Theta = 15^\circ$. Find the value of α and k for the wave emergent from the analyzer.

Solution Dividing the procedure into separate steps makes it easier to follow.

Step 1. Represent the orientation of the fast axis of the $\lambda/4$ plate.

Step 2. Represent the state of polarization of the incident light.

Step 3. Draw the operation of the $\lambda/4$ plate.

Step 4. Read the value of (θ, ϵ) from the traces.

In this example, many subscripts are needed and new notations involving $A, B, C, \ldots$ are introduced to represent the states of polarization, in place of $P_1, P_2, P_3, \ldots$ used in previous examples. For instance, the projections of point A in the horizontal, frontal, and profile planes are designated as A_H, A_F, and A_P.

Step 1. The azimuth of the fast axis is $\Theta = 56°$ and is represented by the point R_H on the equator in the horizontal plane. The true azimuth angle is seen in the horizontal plane at $2\Theta = 112°$, as shown in Fig. 7.32a. The intersection between the projection from R_H and the equatorial line in the frontal plane determines point R_F in the frontal plane. Extensions from R_H and R_F determine R_P in the profile plane.

Step 2. Figure 7.32b illustrates point A of the incident light with $\beta = \tan^{-1}\epsilon = 19°$ and $\theta = 77°$. In the frontal plane, the true latitude angle is seen. A straight horizontal line with $2\beta = 38°$ is the line for the graduation line of $\beta = 19°$. In the horizontal plane, the latitude 2β sweeps out a circle of radius r. The longitude angle 2θ is seen in the horizontal plane, and the longitude of $2\theta = 154°$ represents the graduation line of $\theta = 77°$. The intersection of the circle with radius r and the longitude determines A_H in the horizontal plane. The projection of A_H onto the frontal plane intersects at A_F. Points A_H and A_F determine point A_P in the profile plane.

Step 3. Figure 7.32c outlines the operation of the $\lambda/4$ plate. An off-orthogonal frontal plane 1_F, which is perpendicular to CR_H, is drawn. The $\lambda/4$ plate rotates A_1 to B_1 by 90°, and A_H moves perpendicular to CR_H to B_H in the horizontal plane. B_F in the frontal plane is obtained by the projection from B_H and the height b in the 1_F plane, because the true height of B from the equator is seen both in the 1_F plane and the frontal plane. Point B_P in the profile plane is determined from the projections of B_H and B_F. Neither geodesic $\overline{A_F B_F}$ nor $\overline{A_P B_P}$ is circular.

Step 4. The state of polarization of the emergent wave (θ, ϵ) is read from point B. The three-dimensional representation of point B is shown in the inset to Fig. 7.32b. From the horizontal projection in Fig. 7.32c, the azimuth is $2\theta = 65°$ or $\theta = 32.5°$.

In order to find 2β of B in the frontal plane in Fig. 7.32c, the sphere is rotated around NS so that the true angle $\angle B'_F CH$ in the frontal plane is $2\beta = 32°$ or $\epsilon = 0.29$.

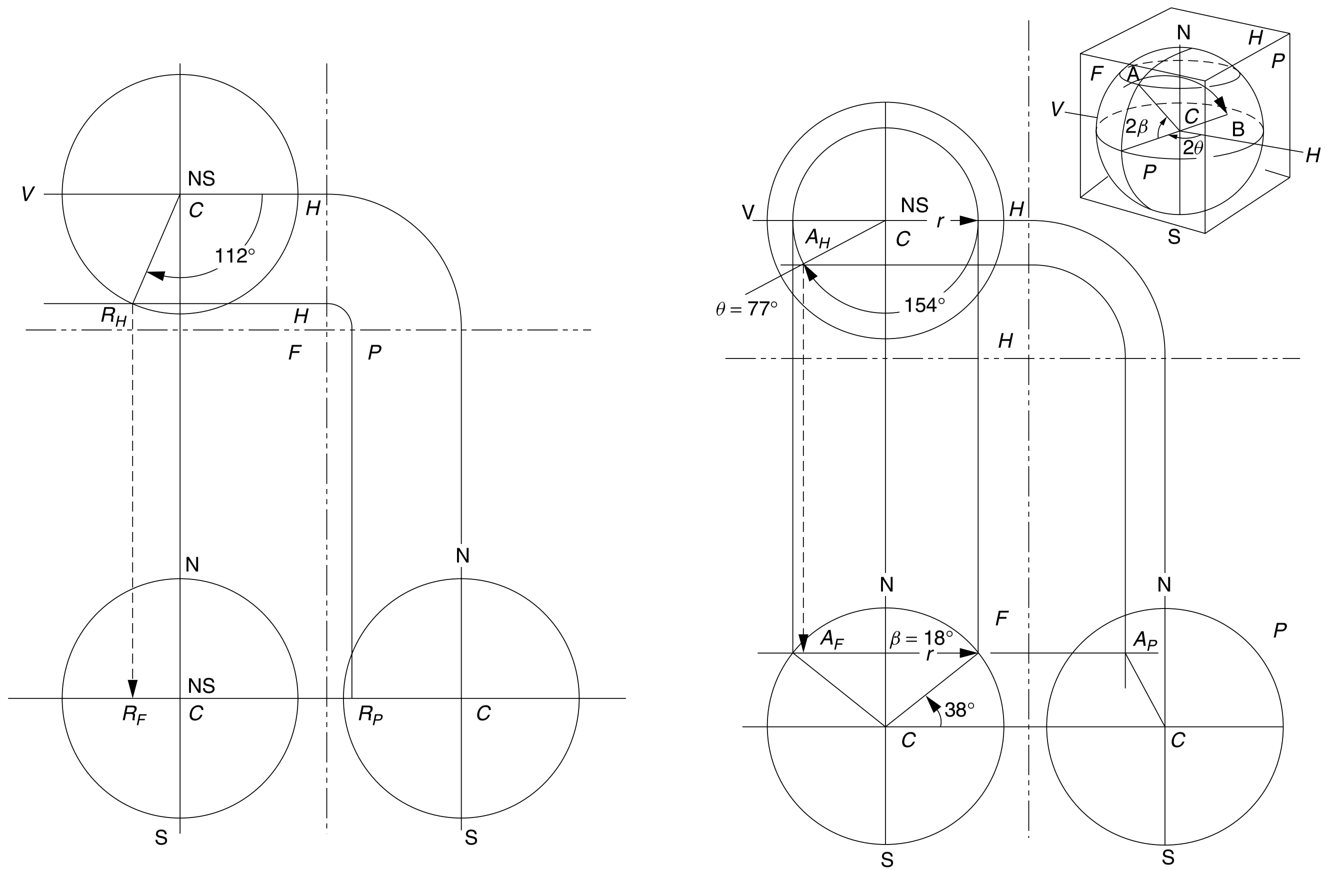
The output from the analyzer is found from the drawing in Fig. 7.32d. The value of α is 21.5°, and from Eq. (7.56), $k = 0.87$. □

7.8 MOVEMENT OF A POINT ON THE POINCARÉ SPHERE

In principle, any given state of polarization can be converted into any other state of polarization by moving along lines of constant longitude and latitude. Hence, any general movement along the Poincaré sphere [4] can be treated by decomposing the movement into these two directions. Although this way of decomposition has a simple conceptual appeal, it is not necessarily the simplest from the viewpoint of implementation.

7.8.1 Movement Along a Line of Constant Longitude (or Constant θ Line)

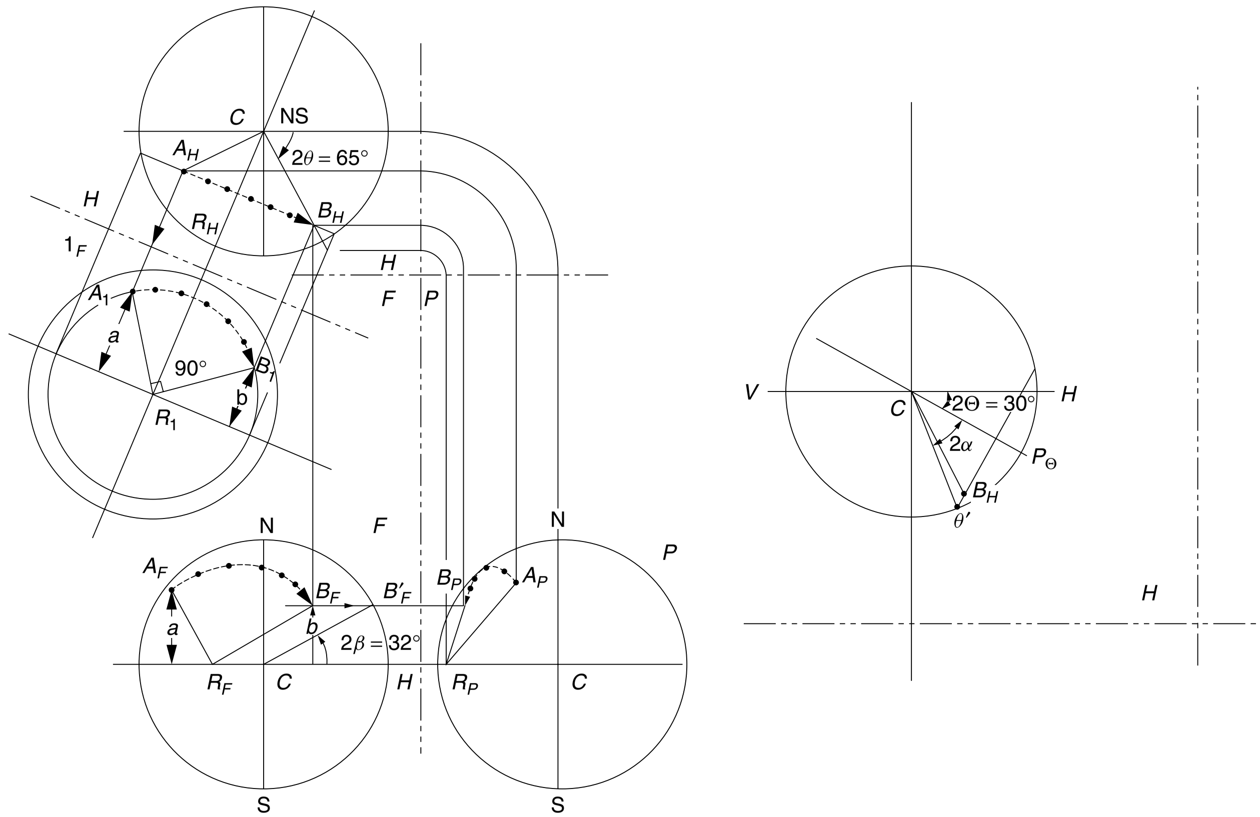
As shown in Fig. 7.33, the state of polarization of an incident wave is represented by P_1 on the Poincaré sphere. P_1 is to be moved along a line of constant longitude, which can be accomplished with a compensator. When the azimuth Θ of the fast axis is set



(a)

(b)

Figure 7.32 A step-by-step solution of Example 7.12. (a) Representation of the fast axis of the $\lambda/4$ plate with $\Theta = 56°$. (b) Representation of the incident light with $(\theta, \beta) = (77°, 19°)$. (c) Operation of the $\lambda/4$ plate. (d) Operation of a polarizer with $\Theta = 15°$ in the horizontal plane.



(c) (d)

Figure 7.32 (*Continued*)

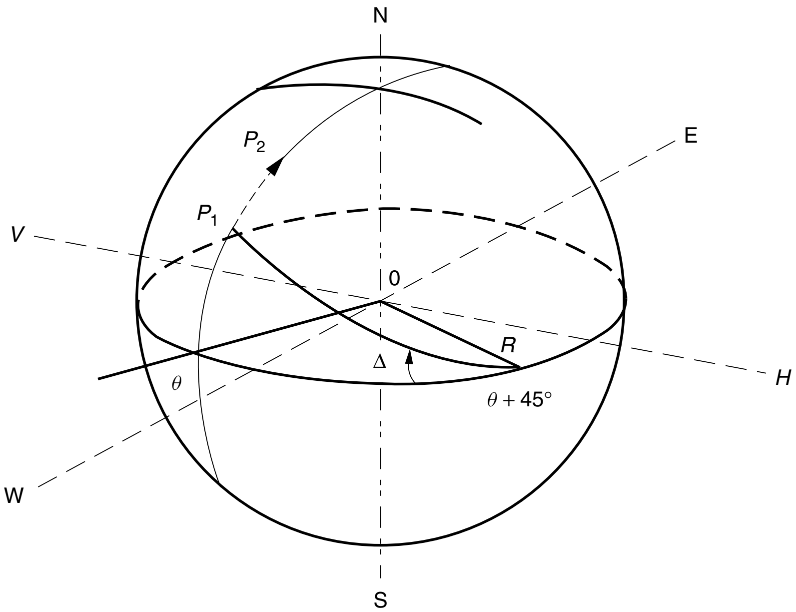


Figure 7.33 Moving the state of polarization along a line of constant longitude.

at 45° with respect to the azimuth θ of the incident elliptic light, point P_1 will move up or down the constant θ line as the retardance of the compensator is varied.

7.8.2 Movement Along a Line of Constant Latitude (or Constant β Line)

By using two half-waveplates, the state of polarization can be moved along a line of constant latitude. Let the initial state of polarization be represented by P_1 on the Poincaré sphere shown in Fig. 7.34. The first $\lambda/2$ plate rotates P_1 around R_1 by 180° to P_2. P_2 has the same ellipticity as P_1 but in the opposite direction of rotation with different azimuth. The second half-waveplate further rotates P_2 around R_2 to P_3. The ellipticity and the sense of rotation of P_3 are the same as that of the incident light P_1. Only the azimuth is changed from that of the incident light. If reversal of the handedness is acceptable, a solitary half-waveplate will suffice.

The last example of this chapter [9] is a comprehensive one and will serve as a good review of the material presented in Chapters 5, 6, and 7.

Example 7.13 Figure 7.35 shows the diagram of a TM−TE mode converter on a lithium niobate wafer. The direction of polarization of the TM mode is vertical inside a rectangular waveguide, as shown in Fig. 7.35, and that of the TE mode is horizontal inside the waveguide. In order to convert the direction of polarization inside the waveguide, a TM−TE mode converter is used. Chapters 9 and 10 are devoted to the topic of optical waveguides. In particular, TM and TE modes are discussed in Section 9.3 and mode converters are discussed in Section 10.9.

The converter in Fig. 7.35 consists of conversion retarder regions, where an external dc electric field is applied by the fingers of the interdigital electrodes, and modal retarder regions, which are located in between the conversion retarder regions and

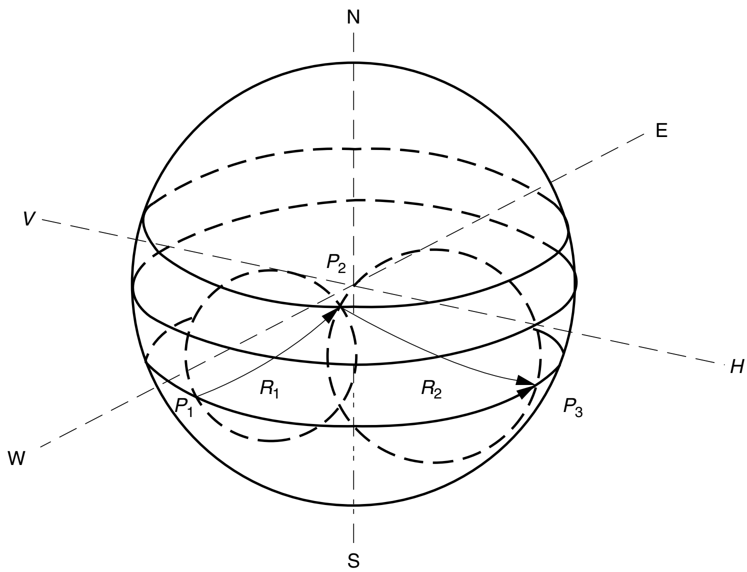


Figure 7.34 Moving the state of polarization along a line of constant latitude.

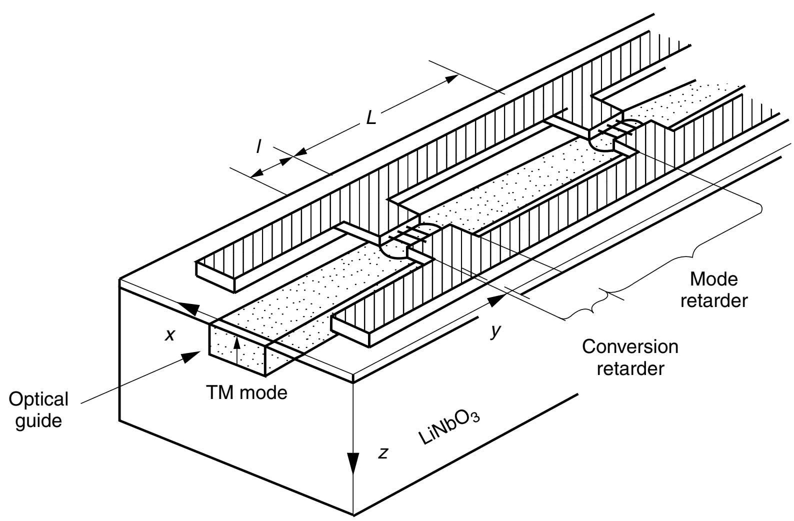


Figure 7.35 TM−TE mode converter.

whose function is based on the difference in the propagation constants of the TE and TM modes in the rectangular optical guide.

Use the Poincaré sphere to explain the principle of operation of the TM−TE mode converter. Assume the input light is in the TM mode.

Solution The function of the conversion retarder and modal retarder will be described separately, and then they will be combined to explain the principle of operation of the TM−TE converter.

(a) Let us first consider the conversion retarder. Interdigital electrodes are deposited on a $LiNbO_3$ wafer to create a periodic external dc field ε_x. The electric lines of flux of the external field are periodic and parallel to the x axis. The external E field ε_x rotates the indicatrix by θ degrees (in the case of lithium niobate, $A < C$ and θ in Eq. (5.32) becomes a negative quantity). The fast axis of the conversion retarder is, therefore, at $\Theta = \pi/2 + \theta$ as shown in Fig. 7.36a. The retardance Δ_i of each electrode with length l is

$$\Delta_i = 2(\Delta N)kl \tag{7.59}$$

where ΔN is the change in the index of refraction due to the application of the external E field and k is the free space propagation constant. Each interdigital electrode creates a small retardance, but as shown next, the retardance is accumulated constructively and the incident point P_1 moves from point V to H.

(b) The modal retarder refers to the region where no external electric field is present. The propagation constants β_{TE} and β_{TM} for the TE and TM modes are quite different due to both birefringence and geometry. Lithium niobate is birefringent, and $n_0 > n_e$. The geometry of the cross section of the optical guide seen by the TM and TE modes is different. These regions are considered as retarders with their fast axis along the z direction ($\Theta = \pi/2$). The length L of the modal retarder is chosen such that the converted component constructively accumulates, as given by the condition

$$(\beta_{\mathrm{TM}} - \beta_{\mathrm{TE}})L = 2\pi n \tag{7.60}$$

where n is an integer.

(c) Finally, the principle of operation of the TM–TE converter will be explained using the Poincaré sphere in Fig. 7.36b. The incident light in the TM mode is represented by P_1 located at point V. The fast axis of the finger retarder is at $\Theta = \pi/2 + \theta$, which is represented by point R on the Poincaré sphere. The retardance of the first finger will rotate point P_1 by Δ_1 around R to P_2. Then the modal retarder region whose length is designed to provide 360° retardance rotates point P_2 by 360° around point V to P_3. The second finger electrode provides a retardance of Δ_2 and moves point P_3 to point P_4. Point P_4 is further rotated by 360° by the modal retarder. The same process is repeated, and the state of polarization moves toward H, or the direction of polarization of the TE mode.

The retardance of the modal retarder must be close to 360°, otherwise the point does not proceed toward H effectively. For instance, if the retardance of the first modal retarder region were 180°, point P_4 would have been at point P_1 again after the second finger electrode. □

This example looked at the specific case of converting V to H polarization on the Poincaré sphere. Various other states of polarization are obtainable by adjusting the external electric field and the length of the modal converter.

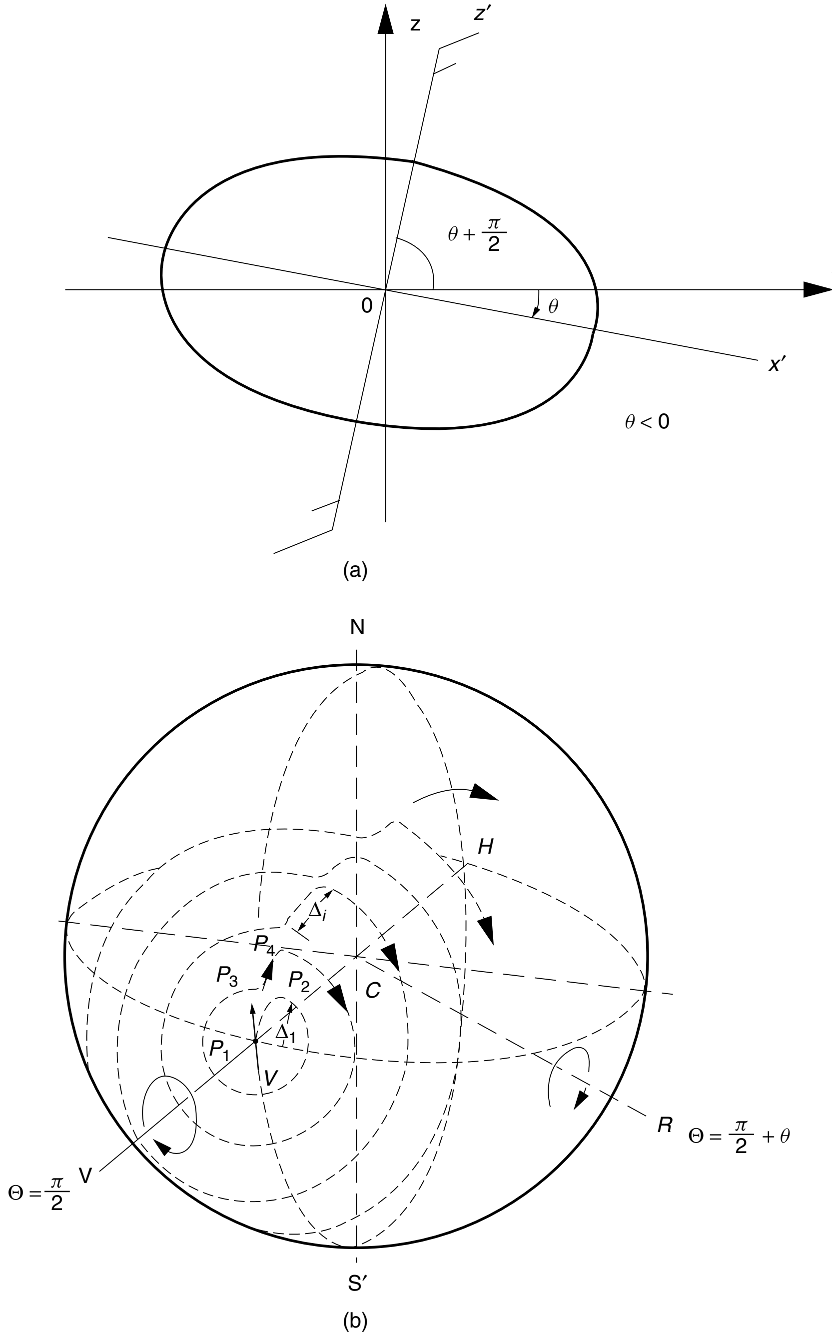


Figure 7.36 Solution of Example 7.13. (a) Indicatrix of the optical guide. (b) Movement of the state of polarization on the Poincaré sphere.

PROBLEMS

7.1 The Poincaré sphere is useful for quickly finding an approximate answer in the laboratory. Example 6.11 asked the calculation of the emergent wave from a $\lambda/4$ plate with $\Theta = 45^\circ$ when the incident wave is a right-handed elliptically polarized wave with $a = \sqrt{3}$ V/m and $b = 1$ V/m and $\theta = 22.5^\circ$. Draw a diagram of the operation on the Poincaré sphere.

7.2 **(a)** With the fast axis of a $\lambda/4$ plate fixed along the x axis, use the Poincaré sphere to obtain the emergent states of polarization for incident linearly polarized waves with azimuth angles of $\theta = 15^\circ$, 30°, and 45°.

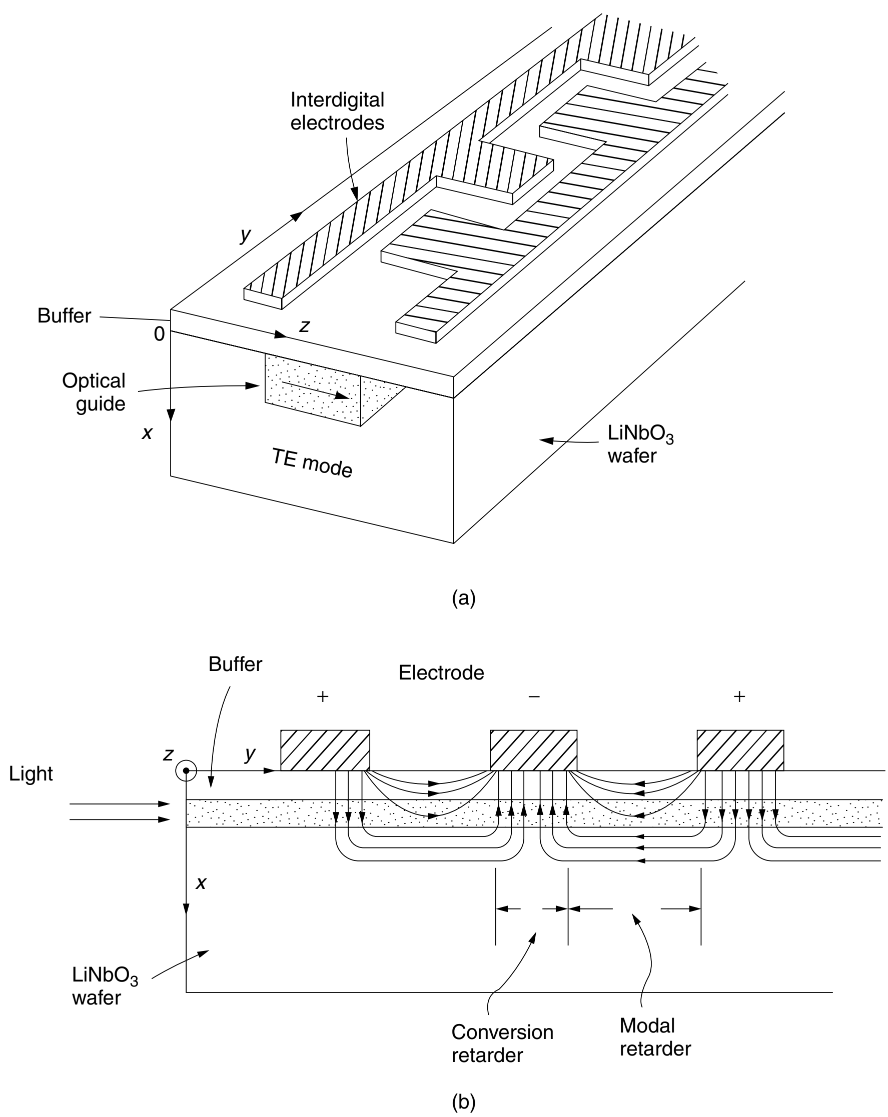


Figure P7.8 TE–TM converter on $LiNbO_3$ wafer. (a) Bird's eye view. (b) Profile view.

(b) With the azimuth of the incident linearly polarized wave fixed along the x axis, use the Poincaré sphere to obtain the states of polarization emergent from a $\lambda/4$ plate for three different azimuths of the fast axis at $\Theta = 15^\circ$, 30°, and 45°.

7.3 The diagram in Fig. 6.35 illustrates how the glare from a radar screen is suppressed by an antiglare circularly polarizing sheet. Indicate the state of polarization of the light at each step of the explanation on the Poincaré sphere.

7.4 Problem 6.11 asked one to prove that when linearly polarized light with $B/A = 1$ is incident onto a retarder whose fast axis is along the x axis, 2β of the emergent light is equal to Δ of the retarder. Solve this problem using the Poincaré sphere.

7.5 Problem 6.2 asked for the state of polarization emergent from a retarder whose fast axis is oriented along the x axis with $\Delta = 315^\circ$ when linearly polarized light is incident with $\theta = 63.4^\circ$. Answer the same problem using Poincaré sphere traces.

7.6 Use Poincaré sphere traces to find the state of polarization of the emergent wave from a $\lambda/4$ plate. The state of polarization of the incident light is $(\theta, \epsilon) = (100^\circ, 0.5)$ and the azimuth of the fast axis of the $\lambda/4$ plate is $\Theta = 35^\circ$.

7.7 **(a)** Example 6.9 asked one to obtain θ and ϵ of the emergent light from a retarder with $\Delta = 38^\circ$ and its fast axis along the x axis. The incident light was $E_x = 2.0$ V/m and $E_y = 3.1$ V/m. Verify the answer using the Poincaré sphere trace.

(b) The above system is followed by an analyzer with its transmission axis along the x axis. Find the emergent light power.

(c) Next, the transmission axis of the analyzer is rotated to $\Theta = 25^\circ$. Find the new transmittance from the analyzer.

7.8 A TE−TM converter is shown in Fig. P7.8. The converter uses Y propagating X-cut lithium niobate. The external field ε_x is vertical and bipolar. Explain the operation of the mode converter on the Poincaré sphere. Assume the incident light is in the TE mode.

7.9 One of the methods of laser cooling is polarization gradient cooling. This method needs to create a spatial change of the state of polarization in the cooling laser

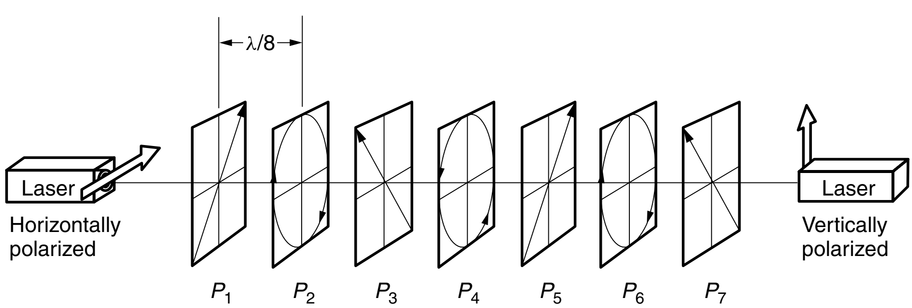


Figure P7.9 Polarization grating used for laser cooling.

beam. Such a laser beam is created by two oppositely propagating laser beams. Both laser beams are linearly polarized, but one is vertically polarized and the other is horizontally polarized. Such laser beams establish a spatial variation of the state of circular polarization, as shown in Fig. P7.9. Atoms moving along the laser beam are slowed down by the interaction with the magnetic sublevels of the ground state established by this circularly polarized laser beam [10]. Using the Poincaré sphere, verify the establishment of such a polarization gradient in the cooling laser beam.

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