Additional Files: Mathematical Proofs of Optimization and Related Codes

Appendix A

The Lagrangian function for optimization problem is:

$$\mathcal{L} = y - \lambda(w_1 S + w_2 I + w_3 P + w_4 N - m)$$
$$\mathcal{L} = (S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}} - \lambda(w_1 S + w_2 I + w_3 P + w_4 N - m)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial S} = (S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}-1} S^{\alpha-1} - \lambda w_1 = 0 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial I} = (S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}-1} I^{\alpha-1} - \lambda w_2 = 0 \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial N} = (S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}-1} N^{\alpha-1} - \lambda w_3 = 0 \tag{3}$$
$$\frac{\partial \mathcal{L}}{\partial I} = (S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}-1} P^{\alpha-1} - \lambda w_3 = 0 \tag{4}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(w_1 S + w_2 I + w_3 P + w_4 N - m) = 0 \tag{5}$$

Dividing (18),(19),(20) by (17)

$$\frac{w_2}{w_1} = (\frac{I}{S})^{\alpha-1}$$

Similar way

$$I = \sqrt[\alpha-1]{\frac{w_2}{w_1}} S \tag{6}$$
$$P = \sqrt[\alpha-1]{\frac{w_3}{w_1}} S \tag{7}$$
$$N = \sqrt[\alpha-1]{\frac{w_4}{w_1}} S \tag{8}$$

Substituting these values in equation (21)

$$w_1 S + W_2 \sqrt[\alpha-1]{\frac{w_2}{w_1} S} + W_3 \sqrt[\alpha-1]{\frac{w_3}{w_1} S} + w_4 \sqrt[\alpha-1]{\frac{w_4}{w_1}} S - m = 0$$

$$S = \frac{m w_1 \frac{1}{\alpha-1}}{w_1^{\frac{\alpha}{\alpha-1}} + w_2^{\frac{\alpha}{\alpha-1}} + w_3^{\frac{\alpha}{\alpha-1}} + w_4^{\frac{\alpha}{\alpha-1}}} \tag{9}$$

Similarly

$$I = \frac{m w_2 \frac{1}{\alpha-1}}{w_1^{\frac{\alpha}{\alpha-1}} + w_2^{\frac{\alpha}{\alpha-1}} + w_3^{\frac{\alpha}{\alpha-1}} + w_4^{\frac{\alpha}{\alpha-1}}} \tag{10}$$

$$N = \frac{m w_3 \frac{1}{\alpha-1}}{w_1^{\frac{\alpha}{\alpha-1}} + w_2^{\frac{\alpha}{\alpha-1}} + w_3^{\frac{\alpha}{\alpha-1}} + w_4^{\frac{\alpha}{\alpha-1}}} \tag{11}$$

$$P = \frac{m w_4 \frac{1}{\alpha-1}}{w_1^{\frac{\alpha}{\alpha-1}} + w_2^{\frac{\alpha}{\alpha-1}} + w_3^{\frac{\alpha}{\alpha-1}} + w_4^{\frac{\alpha}{\alpha-1}}} \tag{12}$$

Appendix B

$$\mathcal{L} = w_1 S + w_2 I + w_3 P + w_4 N - \lambda((S^\alpha + I^\alpha + P^\alpha + N^\alpha) - y_{tar})$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial S} = w_1 - \lambda S^{\alpha-1}(S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}-1} = 0 \tag{13}$$
$$\frac{\partial \mathcal{L}}{\partial I} = w_2 - \lambda I^{\alpha-1}(S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}-1} = 0 \tag{14}$$
$$\frac{\partial \mathcal{L}}{\partial P} = w_3 - \lambda P^{\alpha-1}(S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}-1} = 0 \tag{15}$$
$$\frac{\partial \mathcal{L}}{\partial N} = w_4 - \lambda N^{\alpha-1}(S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}-1} = 0 \tag{16}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = (S^\alpha + I^\alpha + P^\alpha + N^\alpha)^{\frac{1}{\alpha}} - y_t ar = 0 \tag{17}$$

Dividing (30),(31),(32) by (29)

$$I = \sqrt[\alpha-1]{\frac{w_2}{w_1}} S$$
$$P = \sqrt[\alpha-1]{\frac{w_3}{w_1}} S$$
$$N = \sqrt[\alpha-1]{\frac{w_4}{w_1}} S$$

Substituting the values in CES function

$$y_t ar = (S^\alpha + (\frac{w_2}{w_1}^{\frac{\alpha}{\alpha-1}} S^\alpha) + (\frac{w_3}{w_1}^{\frac{\alpha}{\alpha-1}} S^\alpha) + (\frac{w_4}{w_1}^{\frac{\alpha}{\alpha-1}} S^\alpha))$$

$$S = \frac{y_{tar} w_1^{\frac{1}{\alpha-1}}}{(w_1^{\frac{\alpha}{\alpha-1}} + w_2^{\frac{\alpha}{\alpha-1}} + w_3^{\frac{\alpha}{\alpha-1}} + w_4^{\frac{\alpha}{\alpha-1}})^{\frac{1}{\alpha}}} \tag{18}$$

$$I = \frac{y_{tar} w_2^{\frac{1}{\alpha-1}}}{(w_1^{\frac{\alpha}{\alpha-1}} + w_2^{\frac{\alpha}{\alpha-1}} + w_3^{\frac{\alpha}{\alpha-1}} + w_4^{\frac{\alpha}{\alpha-1}})^{\frac{1}{\alpha}}} \tag{19}$$

$$P = \frac{y_{tar} w_3^{\frac{1}{\alpha-1}}}{(w_1^{\frac{\alpha}{\alpha-1}} + w_2^{\frac{\alpha}{\alpha-1}} + w_3^{\frac{\alpha}{\alpha-1}} + w_4^{\frac{\alpha}{\alpha-1}})^{\frac{1}{\alpha}}} \tag{20}$$

The cost function can be rewritten as :

$$c = (\frac{y_{tar}}{w_1^{\frac{\alpha}{\alpha-1}} + w_2^{\frac{\alpha}{\alpha-1}} + w_3^{\frac{\alpha}{\alpha-1}} + w_4^{\frac{\alpha}{\alpha-1}}})^{\frac{1}{\alpha}-1}$$

APPENDIX C

The profit function is below
Profit=$(S^\rho + I^\rho + P^\rho + N^\rho)^{\frac{1}{\rho}} - (w_1 S + w_2 I + w_3 P + w_4 N)$
We want to maximize the profit subject to $w_1 S + w_2 I + w_3 P + w_4 N = c_{thresh}$
Using Lagrange multiplier

$$\mathcal{L} = (S^\rho + I^\rho + P^\rho + N^\rho)^{\frac{1}{\rho}} - (w_1 S + w_2 I + w_3 P + w_4 N) \\ +\lambda(w_1 S + w_2 I + w_3 P + w_4 N - c_{thresh}) \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial S} = (S^\rho + I^\rho + P^\rho + N^\rho)^{\frac{1}{\rho}-1} S^{\rho-1} - w_1 + \lambda w_1 = 0 \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial I} = (S^\rho + I^\rho + P^\rho + N^\rho)^{\frac{1}{\rho}-1} I^{\rho-1} - w_2 + \lambda w_2 = 0 \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial P} = (S^\rho + I^\rho + P^\rho + N^\rho)^{\frac{1}{\rho}-1} P^{\rho-1} - w_3 + \lambda w_3 = 0 \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial N} = (S^\rho + I^\rho + P^\rho + N^\rho)^{\frac{1}{\rho}-1} N^{\rho-1} - w_4 + \lambda w_4 = 0 \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w_1 S + w_2 I + w_3 P + w_4 N - c_{thresh} = 0 \tag{26}$$

Comparing the λ value from equation (60) an (61). We can get

$$I = \frac{w_2}{w_1}^{\frac{1}{\rho-1}} S$$

Similar way other values are

$$P = \frac{w_3}{w_1}^{\frac{1}{\rho-1}} S$$
$$N = \frac{w_4}{w_1}^{\frac{1}{\rho-1}} S$$

Putting the values of I, P, N in equation (64)

$$w_1 + w_2 \frac{w_2}{w_1}^{\frac{1}{\rho-1}} S + w_3 \frac{w_3}{w_1}^{\frac{1}{\rho-1}} S + w_4 \frac{w_4}{w_1}^{\frac{1}{\rho-1}} S - c_{thresh} = 0$$
$$S = \frac{c_{thresh} w_1^{\frac{1}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} + w_3^{\frac{\rho}{\rho-1}} + w_4^{\frac{\rho}{\rho-1}}}$$

Similar way other values are

$$I = \frac{c_{thresh} w_2^{\frac{1}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} + w_3^{\frac{\rho}{\rho-1}} + w_4^{\frac{\rho}{\rho-1}}}$$
$$P = \frac{c_{thresh} w_3^{\frac{1}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} + w_3^{\frac{\rho}{\rho-1}} + w_4^{\frac{\rho}{\rho-1}}}$$
$$N = \frac{c_{thresh} w_4^{\frac{1}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} + w_3^{\frac{\rho}{\rho-1}} + w_4^{\frac{\rho}{\rho-1}}}$$

APPENDIX D

CES function

$$\begin{aligned}
y &= (K^\rho + L^\rho)^{\frac{1}{\rho}} \\
\frac{\partial y}{\partial K} &= \frac{1}{\rho}(K^\rho + L^\rho)^{\frac{1}{\rho}-1} \rho K^{\rho-1} \\
\frac{\partial y}{\partial L} &= \frac{1}{\rho}(K^\rho + L^\rho)^{\frac{1}{\rho}-1} \rho L^{\rho-1} \\
\frac{\partial y}{\partial K \partial L} &= \rho K^{\rho-1} L^{\rho-1} (K^\rho + L^\rho)^{\frac{1}{\rho}-2} \\
\frac{\partial y}{\partial L \partial K} &= \rho K^{\rho-1} L^{\rho-1} (K^\rho + L^\rho)^{\frac{1}{\rho}-2} \\
\frac{\partial^2 y}{\partial^2 K} &= \rho(K^\rho - 1)^2 (K^\rho + L^\rho)^{\frac{1}{\rho}-2} + (\rho-1) K^{\rho-2} (K^\rho + L^\rho)^{\frac{1}{\rho}-1} \\
\frac{\partial^2 y}{\partial^2 L} &= \rho(L^{\rho-1})^2 (K^\rho + L^\rho)^{\frac{1}{\rho}-2} + (\rho-1) L^{\rho-2} (K^\rho + L^\rho)^{\frac{1}{\rho}-1}
\end{aligned}$$

Hessian Matrix

$$\begin{bmatrix} \begin{matrix} \rho(K^\rho - 1)^2 (K^\rho + L^\rho)^{\frac{1}{\rho}-2} \\ +(\rho-1) K^{\rho-2} (K^\rho + L^\rho)^{\frac{1}{\rho}-1} \end{matrix} & \rho K^{\rho-1} L^{\rho-1} (K^\rho + L^\rho)^{\frac{1}{\rho}-2} \\ \rho K^{\rho-1} L^{\rho-1} (K^\rho + L^\rho)^{\frac{1}{\rho}-2} & \begin{matrix} \rho(L^{\rho-1})^2 (K^\rho + L^\rho)^{\frac{1}{\rho}-2} \\ +(\rho-1) L^{\rho-2} (K^\rho + L^\rho)^{\frac{1}{\rho}-1} \end{matrix} \end{bmatrix}$$

$\Delta_1 = (K^\rho + L^\rho)^{\frac{1}{\rho}-1} K^{\rho-1} (\frac{\rho K^{\rho-1}}{K^\rho + L^\rho} + \frac{\rho-1}{K})$
As K,L, $\rho > 0$ $\Delta_1 > 0$

$\Delta_2 = \rho(\rho - 1)(K^{\rho-1})^2 (L^{\rho-2})(K^\rho + L^\rho)^{\frac{2}{\rho}-3} + \rho(\rho - 1)(L^{\rho-1})^2 (K^{\rho-2})(K^\rho + L^\rho)^{\frac{2}{\rho}-3} + (\rho - 1)^2 (K^{\rho-2})(L^{\rho-2})(K^\rho + L^\rho)^{\frac{2}{\rho}-2}$
$\Delta_2 \geq 0$ in case $\rho \geq 1$
As $\Delta_1 \geq 0$ and $\Delta_2 \geq 0$ in case $\rho \geq 1$. It will produce concave graph.
When $\rho < 1$, $\Delta_1 \geq 0$ and $\Delta_2 \leq 0$.
It is neither concave or convex.

APPENDIX E

In this appendix, we are going to explain the reason why Δ_1 from previous appendix is always positive.

Considering δ_1 value again:
$\Delta_1 = (K^\rho + L^\rho)^{\frac{1}{\rho}-1} K^{\rho-1} (\frac{\rho K^{\rho-1}}{K^\rho + L^\rho} + \frac{\rho-1}{K})$
Δ_1 will be negative if below two conditions are satisfied.
1)$\rho < 1$
2)$\frac{\rho-1}{K} \geq \frac{\rho K^{\rho-1}}{K^\rho + L^\rho}$

$$\frac{\rho-1}{K} \geq \frac{\rho K^{\rho-1}}{K^\rho + L^\rho}$$
$$=> (\rho-1)(K^\rho + L^\rho) \geq \rho K^\rho$$
$$=> \frac{\rho-1}{\rho} \geq \frac{K^\rho}{K^\rho + L^\rho}$$
$$=> 1 - \frac{1}{\rho} \geq \frac{K^\rho}{K^\rho + L^\rho}$$
$$=> 1 - \frac{K^\rho}{K^\rho + L^\rho} \geq \frac{1}{\rho}$$
$$=> \frac{L^\rho}{K^\rho + L^\rho} \geq \frac{1}{\rho}$$
$$=> \rho L^\rho \geq K^\rho + L^\rho$$
$$=> (\rho-1) \geq \frac{K^\rho}{L^\rho}$$
$$=> \rho - 1 \geq (\frac{K}{L})^\rho$$

As K,L¿0, hence $(\frac{K}{L})^\rho$ will be always positive.

$$(\rho-1) > 0$$
$$=> \rho > 1$$

Which is contradicting the first condition $\rho < 1$.
Hence Δ_1 will be always positive.

APPENDIX F

Below is the proposed model, which we want to fit using least square approach.

$$(k_1^r + k_2^r)^{\frac{1}{r}}$$

Matlab code when $r < 1$ The datsets k_1 ,k_2 and observed data ydata are given as

$$k_2 = [5, 5, 10, 10, 10, 15, 15, 15, 20, 20, 20, 30, 30, 30, 40, 40, 40]$$
$$k_1 = [62, 65, 62, 60, 65, 55, 45, 47, 50, 52, 55, 56, 57, 58, 59, 60, 60]$$
$$ydata = [19511.9, 20038.22, 26579.25, 26108.41, 27274.01, 30038.95, 27008.5, 27635.78, 32723.23, 33401.89, 34399.27, 42176.36, 42563.18, 42947.06, 49839.75, 50268.74, 50268.74]$$

The function which is needed to pass in the matlab lsqnonlin function is $fun = @(r)(k_1^r + k_2^r)^{\frac{1}{r}} - ydata$ In the case of without constraints, no upper and lower bound of r need to pass in the function. We will assume a initial value for r as 0.4. Next calling the lsqnonlin function
x = lsqnonlin(fun,x0)
x =0.1000; So we obtained the least square r value as 0.1000.
Say the upper bound for r is 0.9 and lower bound as 0.1. We will pass the same information in lsqnonlin function.

$$x = lsqnonlin(fun, x_0, 0, 0.9);$$
$$x = 0.1000;$$

We get the same output as we have obtained without constraints.
Matlab code when $r > 1$

$$k_2 = [5, 5, 10, 10, 10, 15, 15, 15, 20, 20, 20, 30, 30, 30, 40, 40, 40]$$
$$k_1 = [62, 65, 62, 60, 65, 55, 45, 47, 50, 52, 55, 56, 57, 58, 59, 60, 60]$$
$$ydata = [65.5239, 68.5078, 69.5306, 67.5538, 72.4972, 66.8546, 57.0674, 59.0214, 66.3455, 68.2907, 71.2122, 81.1220, 82.0859, 83.0503, 93.1262, 94.0819, 94.0819]$$
$$fun = @(r)(k_1^r + k_2^r)^{\frac{1}{r}} - ydata;$$
$$x_0 = 1.4;$$
$$x = lsqnonlin(fun, x_0)$$
$$x = 1.1000;$$

Say the upper bound for r is 1.9 and lower bound as 1.

$$x = lsqnonlin(fun, x_0, 1, 1.9);$$
$$x = 1.1000;$$

APPENDIX G

Code for Shapiro-Wilk goodness-of-fit test
Mupad command has been used to generate the code.

$$data := [62, 65, 62, 60, 65, 55, 45, 47, 50, 52, 55, 56, 57, 58, 59, 60, 60] :$$
$$stats :: swGOFT(data)$$
$$[PValue = 0.4011881607, StatValue = 0.9463348025]$$
$$data := [78, 60, 55, 44, 75, 62, 55, 49, 55, 57, 65, 46, 77, 68, 59, 48, 60] :$$
$$stats :: swGOFT(data)$$
$$[PValue = 0.3841895654, StatValue = 0.9451314104]$$
$$data := [5, 5, 10, 10, 10, 15, 15, 15, 20, 20, 20, 30, 30, 30, 40, 40, 40] :$$
$$stats :: swGOFT(data)$$

[

$$data := [15, 35, 18, 20, 30, 15, 25, 18, 25, 24, 35, 38, 27, 32, 16, 10, 30] :$$
$$stats :: swGOFT(data)$$
$$[PValue = 0.6499020509, StatValue = 0.9609763672]$$

0.05 is the threshold of pvalue. As all the observed Pvalue are greater than the threshold. The null hypothesis is accepted and datasets are belong to normal distribution.
Code for chi-square
data= [62,65,62,60,65,55,45,47,50,52,55,56,57,58,59,60,60];
h = chi2gof(data);
In case of normal distribution h=0 otherwise h=1.

Appendix H

The matlab fmincon code when $\rho < 1$:

$A = [1; -1];$
$b = [0.9; -0.1];$
$x0 = [0.4];$
$[x, fval] = fmincon(@myCES, x0, A, b)$
$functionf = myCES(x)$
$pow = 1/x(1);$
$f = -(62^x(1) + 5^x(1)).^pow;$
end

The matlab fmincon code when $\rho > 1$:

$A = [1; -1];$
$b = [1.9; -1.1];$
$x0 = [0.4];$
$[x, fval] = fmincon(@myCES, x0, A, b)$
$functionf = myCES(x)$
$pow = 1/x(1);$
$f = -(62^x(1) + 5^x(1)).^pow;$
end