Inter IIT 25' Practice Test-2 (100 points)
SciMathSoc
November 1, 2025 Roll Number: ______________________________

Important Instructions :

1. There are a total of **7 questions**:
   - All questions are worth 20 marks
   - Only 5 out of 7 questions have to be done.
2. The paper is of **3 hours long**.
3. Before submission, you must clearly indicate the 5 questions that you want to have checked. We will consider only those five questions mentioned.
4. **NOTE:** Solution of each problem must be in a separate page. Don't start the solution from the mid of the page.
5. **Multiple solutions** for the same problem are **allowed and encouraged**. Note that for multiple submissions we will consider the solution with highest mark.
6. Write neat proofs for each question attempted, even if you're not entirely sure. Partial marks will be given generously to some logical conclusions.

Important : Leave this table blank

Question	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Total**
Marks											

Functional Equations(2 out of 4)

1. Consider a function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n+1) > f(f(n))$ for every positive integer n Find the function $f(n)$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x^2 + f(y)) = y + f(x)^2, \quad \text{for all } x, y \in \mathbb{R}.$$

3. Find all functions $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ that satisfy:

   (a) $f(xf(y))f(y) = f(x+y), \quad \text{for all } x, y \geq 0.$

   (b) $f(2) = 0.$

   (c) $f(x) = 0, \quad \text{for all } x \text{ such that } 0 \leq x < 2.$

4. Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function such that for all $x, y \in \mathbb{R}^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

   Prove that $f(f(x)) = x$.

Algebra(3 out of 6)

1. Let F be the set of all polynomials $f(x)$ with integer coefficients such that $f(x) = 1$ has at least one integer root.
   For each integer $k > 1$, find m_k, the least integer greater than 1 for which there exists an $f \in F$ such that $f(x) = m_k$ has exactly k distinct integer roots.

2. Let $\{a_n\}$ be the sequence of real numbers defined by

$$a_1 = t, \quad \text{and} \quad a_{n+1} = 4a_n(1 - a_n) \text{ for } n \geq 1.$$

   For how many distinct values of t do we have $a_{1998} = 0$?

3. (a) Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^3, \quad \forall x \in \mathbb{R}?$$

   (b) Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^4, \quad \forall x \in \mathbb{R}?$$

4. Mr. Fat and Mr. Taf play a game with a polynomial of degree at least 4:

$$x^{2n} + \_x^{2n-1} + \_x^{2n-2} + \cdots + \_x + 1.$$

   They fill in real numbers in the empty spaces in turn. If the resulting polynomial has no real root, Mr. Fat wins; otherwise, Mr. Taf wins. If Mr. Fat goes first, who has a winning strategy?

5. The Fibonacci sequence F_n is given by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}).$$

Prove that

$$F_{2n} = \frac{F_{2n+2}^3 + F_{2n-2}^3}{9} - 2F_{2n}^3$$

for all $n \geq 2$.

6. A polynomial $P(x)$ of degree $n > 5$ with integer coefficients and n distinct integer roots is given. Find all integer roots of $P(P(x))$ given that 0 is a root of $P(x)$.