

1. A machine starts operating with an empty set. It operates for 2025 minutes, where after each i^{th} minute ($1 \leq i \leq 2025$), it either adds the number i to the set or does not, each event with equal probability. Let p be the probability that the sum of the elements of the final set is divisible by 5 and the number of elements in the set is divisible by 3. Let q be the probability that the sum of the elements of the final set is divisible by 3 and the number of elements in the set is divisible by 5. Arrange p , q and $\frac{1}{15}$ in ascending order, and give a clear reasoning for your answer.
2. Denote by \mathbb{Q}^+ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f(f(x)^2y) = x^3 f(xy).$$

3. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1$$

4. Let k be a **positive integer**. Find the **smallest positive integer** n for which there exist k nonzero vectors v_1, \dots, v_k in \mathbb{R}^n such that for every pair i, j of indices with $|i - j| > 1$ the **vectors v_i and v_j are orthogonal**.
5. Let a and b be **real numbers** with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that

$$\int_a^b f(x) dx = \int_a^b g(x) dx \quad \text{but } f \not\equiv g.$$

For **every positive integer** n , define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n} dx.$$

Show that I_1, I_2, I_3, \dots is an **increasing sequence** with $\lim_{n \rightarrow \infty} I_n = \infty$.

6. Let s be a positive integer and let the non-negative integers $s \geq t_1 \geq t_2 \cdots \geq t_s \geq d_1 \geq d_2 \cdots \geq d_n$ denote the degrees of the vertices of a graph. Prove that the integers $t_1 - 1, t_2 - 1, \dots, t_s - 1, d_1, d_2, \dots, d_n$ are also the degrees of the vertices of a graph.
7. For any non-negative integer k , let $f(k)$ denote the number of 1s in the base-3 representation of k . Find all complex numbers z such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)} (z+k)^{2023} = 0$$