Mathematical Proof Techniques and Problem Solving Heuristics

Inter IIT 14.0 Maths Bootcamp – L1

IIT Kanpur SciMath Society

Weak Mathematical Induction

The **Principle of Mathematical Induction** states that to prove a statement $P(n)$ true for all $n \geq n_0$, it suffices to show:

- **Base Case:** $P(n_0)$ is true.
- **Inductive Step:** Assuming $P(k)$ is true, prove $P(k+1)$.

Intuitively, once the domino at n_0 falls, the chain reaction ensures all others fall.
Example: Show that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Problem-Solving with Induction

- Identify a clear pattern or recursive relationship.
- Make the inductive hypothesis strong enough to reach the next step.
- Check small cases carefully — often, they reveal missing base conditions.
- Break complex identities into smaller ones; induction works well with algebraic summations, divisibility, and inequalities.
- Avoid circular reasoning — never assume what you need to prove directly.

Strong Induction

In **Strong Induction**, we assume $P(n_0), P(n_0+1), \ldots, P(k)$ are true to prove $P(k+1)$.
Useful when $P(k+1)$ depends on several previous cases.
Example: Every integer $n > 1$ can be written as a product of primes.
The base case $n = 2$ is prime. Assume all integers up to k can be factored into primes. If $k+1$ is composite, write $k+1 = ab$ with $a, b < k+1$, so both have prime factorizations.

Problem-Solving with Strong Induction

- Apply when the next term depends on multiple earlier terms (like in recurrence relations).
- Define base cases for the smallest few integers clearly.
- Typical in number theory proofs (divisibility, factorization).
- Often works naturally with combinatorial structures or recursively defined sequences.

Simple Pigeonhole Principle

If $n+1$ or more objects are placed into n boxes, then at least one box contains at least two objects.
Example: Among 13 people, at least two have birthdays in the same month.

Objects: 13 people, Boxes: 12 months.

Generic Pigeonhole and Techniques

- **General Form:** If N objects are distributed among k boxes, some box contains at least $\lceil N/k \rceil$ objects.
- Works well for problems involving remainders, parity, and modular arithmetic.
- Try to model the problem by identifying "objects" and "boxes."
- Combine with the extremal or invariant principles for stronger results.
- Useful heuristic: Bound differences using averages or floors/ceilings.

Principle of Invariance

An **invariant** is a quantity that remains unchanged during a sequence of operations.
To prove impossibility, show that every allowed move preserves an invariant which contradicts the desired end state.
Example: A checkerboard has alternate black and white squares. Removing two opposite corners (both white) makes it impossible to cover with dominoes — because each domino covers one white and one black square, the color count difference is invariant.

Problem-Solving via Invariance

- Identify a quantity (sum, parity, color count, etc.) that doesn't change.
- Invariant proofs are common in combinatorial games and parity arguments.
- If exact invariants are hard, look for **monovariants** — quantities that move monotonically toward a limit.
- Test small configurations to guess the invariant before proving it.

Proof by Contradiction

To prove a statement P:

1. Assume $\neg P$ (its negation).
2. Derive a contradiction — logical, numerical, or structural.

Example: Suppose $\sqrt{2}$ is rational, i.e., $\sqrt{2} = p/q$ with $\gcd(p, q) = 1$. Then $2q^2 = p^2$, implying p is even, so $p = 2r$. Then $2q^2 = 4r^2 \Rightarrow q$ is even. Contradiction!

Common Patterns in Contradiction Proofs

- Parity or divisibility conflicts (as in irrational proofs).
- Minimal counterexample leads to an even smaller one.
- Infinite descent arguments.
- Contradictions via inequality bounds (e.g., smallest element assumptions).

Proof by Construction

A statement is proven by explicitly constructing an example satisfying the conditions.
Example: There exist infinitely many primes. Construct $N = p_1 p_2 \cdots p_n + 1$ and show it introduces a new prime factor not in the list.

Problem-Solving with Construction

- When an existence statement is asked, think “Can I build it?”
- Begin with simpler cases; then generalize.
- Construction often involves sequences, sets, or graphs with specific properties.
- Greedy or recursive construction can be effective.

Extremal Principle

The **Extremal Principle** involves choosing an object with an extreme property (smallest, largest, shortest, etc.) to simplify reasoning.
Example: Among finitely many positive integers, choose the smallest counterexample. This often helps derive a contradiction or pattern.

Problem-Solving via Extremal Principle

- Identify an extremal element — e.g., minimal counterexample.
- Prove something holds for it and generalize.
- Combine with induction or contradiction for strong proofs.
- Common in geometry, combinatorics, and graph theory.