Algebraic and Complex Number Problem-Solving Theorems

Inter IIT 14.0 Maths Bootcamp – L3

IIT Kanpur SciMath Society

Basic Bivariate Identities

I have compiled a standard list of identities that come in handy in solving 90% of the problems without proof

- $(a+b)^2 = a^2 + b^2 + 2ab$
- $(a-b)^2 = a^2 + b^2 - 2ab$
- $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
- $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
- $a^n - b^n = (a-b)(\sum_{i=0}^{n-1} a^i b^{n-i}) \forall$ natural n
- $a^n + b^n = (a+b)(\sum_{i=0}^{n-1} a^i (-b^{n-i})) \forall$ odd n

Useful trivariate polynomial identities

For real a, b, c:

- $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
- $(a + b + c)^3 = a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a)$
- If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$
- $a^2b + b^2c + c^2a - (ab^2 + bc^2 + ca^2) = (a - b)(b - c)(c - a)$
- $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
- $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\left((a - b)^2 + (b - c)^2 + (c - a)^2\right)$

Useful fourth degree polynomial identities

- $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$
- $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2$
- $x^4 + 4y^4 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$ (Sophie Germain)
- $a + b + c = 0 \iff a^4 + b^4 + c^4 = 2(a^2b^2 + b^2c^2 + c^2a^2)$

Application: Common in factorization problems and Diophantine equations.

Inside Bracket Trigonometric Identities

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
- $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
- $\sin 3x = 3 \sin x - 4 \sin^3 x$
- $\cos 3x = 4 \cos^3 x - 3 \cos x$

Outside Bracket Trigonometric Identities

- $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
- $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$
- $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$
- $\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$

Auxiliary and Strategy

$$a \sin x + b \cos x = R \sin(x + \alpha), \quad R = \sqrt{a^2 + b^2},\ \tan \alpha = \frac{b}{a}$$

- Reduce products or powers to linear forms using identities.
- Convert mixed terms via $t = \tan(x/2)$ when rationalization helps.
- Exploit symmetry: $\sin(\pi - x) = \sin x,\ \cos(\pi - x) = -\cos x$.

Factor and Remainder Theorems

- **Factor Theorem:** $f(a) = 0 \iff (x - a)$ divides $f(x)$.
- **Remainder Theorem:** Remainder when $f(x)$ divided by $(x - a)$ equals $f(a)$.

Use: Quickly extract factors or check divisibility among polynomials.

$$f(x) = (x - a)q(x) + f(a)$$

Vieta's Relations

For $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = a_n(x - r_1)\cdots(x - r_n)$:

$$\sum r_i = -\frac{a_{n-1}}{a_n}, \quad \sum_{i<j} r_i r_j = \frac{a_{n-2}}{a_n}, \quad r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}.$$

Application: Simplify expressions involving symmetric sums or powers of roots.

Descartes' Rule of Signs

For polynomial $f(x)$ with real coefficients:

- Number of positive roots $\leq$ number of sign changes in coefficients.
- Number of negative roots $\leq$ number of sign changes in $f(-x)$.

Idea: Each sign change corresponds to a possible crossing of the x-axis.

Fundamental Theorem of Algebra

Every nonconstant polynomial with complex coefficients has at least one complex root.

$$f(x) \in \mathbb{C}[x],\ \deg(f) \geq 1 \ \Rightarrow\ \exists z \in \mathbb{C} :\ f(z) = 0.$$

Consequence: Every real polynomial factors into linear and quadratic terms over $\mathbb{R}$.

Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_0$ with integer coefficients and $\frac{p}{q}$ (in lowest terms) is a root:

$$p \mid a_0, \quad q \mid a_n.$$

Usage: Narrow down rational roots before applying synthetic division.

Conjugate Root Theorem

If $f(x)$ has real coefficients and a complex root $z = a + bi$, then $\bar{z} = a - bi$ is also a root.
Extension: For rational coefficients, if $a + b\sqrt{d}$ is a root, then $a - b\sqrt{d}$ is too.
Use: Ensures real-coefficient polynomials have complex roots in pairs.

Newton's Sums

Let $S_k = r_1^k + r_2^k + \cdots + r_n^k$ for roots r_i of $f(x)$. Then

$$a_n S_k + a_{n-1} S_{k-1} + \cdots + a_{n-k+1} S_1 + k a_{n-k} = 0.$$

Use: Find power sums of roots recursively from coefficients.

Lagrange Interpolation

Given n points (x_i, y_i) with distinct x_i, there is a unique polynomial P of degree $< n$ such that $P(x_i) = y_i$:

$$P(x) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

Use: Construct explicit polynomials or prove identities by interpolation.

Eisenstein's Criterion

If a prime p satisfies

$$p \mid a_0, a_1, \ldots, a_{n-1}, \quad p \nmid a_n, \quad p^2 \nmid a_0,$$

then $f(x)$ is **irreducible** over $\mathbb{Q}$. **Example:** $x^p - p$ is irreducible for prime p.

Important Property about Argument :

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Applications: Simplify powers of complex numbers and derive trigonometric identities.
Solutions of $x^n = 1$ are:

$$\omega_k = e^{2\pi ik/n}, \quad k = 0, 1, \ldots, n-1.$$

Properties:

- $\sum_{k=0}^{n-1} \omega_k = 0$
- $\omega_k^n = 1$, $\omega_k \neq \omega_j$ for $k \neq j$

Use: Common in cyclic symmetry and polynomial factorization.

Geometric Interpretation in $\mathbb{C}$

- Multiplying by $e^{i\theta}$ rotates by θ.
- $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality).
- $|z_1 - z_2|$ gives Euclidean distance in the complex plane.

Use: Geometry of transformations and inequalities in problems involving modulus and argument.

Rouche's Theorem

If f and g are analytic on and inside a contour C and satisfy

$$|f(z) - g(z)| < |f(z)| \text{ on } C,$$

then f and g have the same number of zeros inside C. **Use:** Counting zeros of polynomials and bounding their locations.

Argument Principle

For meromorphic f,

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz$$

equals number of zeros minus poles inside C. **Application:** Root-counting and complex analysis in competition problems.

Cauchy's Root Bound and Gauss–Lucas and Rolle's Theorems

For $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, all real or complex roots r satisfy:

$$|r| \leq 1 + \max_{i<n} \left| \frac{a_i}{a_n} \right|.$$

Application: Restricts possible intervals for root search.

Some Important Theorems :

- **Rolle's Theorem:** Between any two distinct real roots of $f(x)$ lies a root of $f'(x)$.
- **Gauss–Lucas:** The roots of $f'(z)$ lie in the convex hull of the roots of $f(z)$ in $\mathbb{C}$.

Use: Geometric reasoning for locating or bounding derivative roots.

Telescopic Series Problem-Solving Strategies

- Express each term as a **difference of successive terms**, e.g. $T_k = f(k) - f(k+1)$.
- Identify the **cancelling pattern** — most middle terms vanish, leaving only boundary terms.
- Simplify $S_n = T_1 + T_2 + \cdots + T_n$ to its **first and last surviving terms**.
- When differences are not obvious, use **partial fractions** or **recursive subtraction**.
- Check convergence for infinite telescoping sums by examining the limit of $f(n)$ as $n \to \infty$.
- For alternating or product forms, seek a transformation that creates **pairwise cancellation**.

Polynomial Factorization Strategies

- Look for **common factors** or **grouping patterns** to split terms naturally.
- Check for **substitution or symmetry**, e.g. $x + \frac{1}{x} = t$ to reduce degree.
- Use **Rational Root Theorem** or simple integer trials to find linear factors.
- Apply **special identities**: $(a^3 + b^3)$, $(a^4 + 4b^4)$, $(x^n - y^n)$ for quick factorization.
- For symmetric polynomials, express in **elementary symmetric functions** (s_1, s_2, s_3).
- If coefficients follow a pattern, test **roots of unity**, **geometric progressions**, or interpolation.
- Verify by expansion or substitution to ensure completeness of factors.

Inequality Problem-Solving Strategies

- **Normalize** variables (e.g. fix sum or product) to reduce parameters and simplify symmetry.
- Check for **equality cases** early — they often reveal the correct approach or substitution.
- Use **homogenization** to balance degrees of terms before applying known inequalities.
- Apply classical tools appropriately: AM–GM, Cauchy–Schwarz, Jensen, or Rearrangement inequalities.
- For symmetric expressions, assume $a \geq b \geq c$ to simplify comparisons.
- Test with **extreme or boundary values** to verify monotonicity or sharpness.
- When algebraic manipulation is complex, square both sides or set variables as ratios.

Substitution Trick in Problem Solving

- Use substitution to **simplify structure** — replace complex expressions with a single variable.
- Common examples:
  - $x + \frac{1}{x} = t$ to reduce reciprocal symmetry.
  - $\sin^2\theta$ or $\cos^2\theta$ expressed using $\cos 2\theta$.
  - $a = b + k$ to exploit differences in symmetric inequalities.
- Preserve **domain constraints** — ensure the new variable's range is valid.
- Aim to make the equation **polynomial, quadratic, or linear** in the new variable.
- Useful in:
  - Solving higher-degree or symmetric equations.
  - Simplifying rational or trigonometric expressions.
  - Reducing telescoping or recurrence relations.
- After solving, **back-substitute carefully** to recover all possible original solutions.

Finding a Hidden Polynomial Constraint

- Express the given condition as a **relation among roots** — often through Vieta's formulas or symmetric sums.
- Eliminate parameters by **substituting root relationships** (e.g. $r_1 + r_2$, $r_1 r_2$) into the constraint.
- If an equation holds for all roots, reconstruct the **minimal polynomial** consistent with that condition.
- Use **resultants** or **elimination** to remove variables and reveal implicit polynomial links.
- Look for **invariance under shifts or scaling** — this often exposes hidden algebraic structure.
- Test simple numeric cases to guess the general pattern before formal proof.

Other Real Polynomial Problem-Solving Strategies

- Simplify the expression to its **lowest possible degree** or a symmetric equivalent in roots.
- Use **Vieta's relations** to replace powers of roots with elementary symmetric functions.
- Test **low-degree polynomials** (quadratic, cubic) to identify consistent algebraic patterns.
- Examine **sign changes and parity** in coefficients to infer the number and nature of real roots.
- Apply **Descartes' Rule of Signs** and **Intermediate Value Theorem** to bound root locations.
- For repeated structures, explore **recurrence relations**, **factorable patterns**, or interpolation.
- When possible, reduce to **geometric or inequality interpretations** for clearer insight.

Other Complex Polynomial Problem-Solving Strategies

- Represent complex numbers in **exponential form** $z = re^{i\theta}$ to simplify multiplication and rotation.
- Interpret equal moduli or arguments **geometrically** — they correspond to circles and rays in the complex plane.
- Visualize loci of points satisfying conditions like $|z - a| = r$ (circle) or $\arg(z) = \theta$ (ray).
- For equations involving sums or roots of unity, use **regular polygon** or **rotational symmetry** arguments.
- Exploit conjugate or reciprocal symmetry: if z is a root, often $\bar{z}$ or $1/z$ is related by structure.
- Use **argument addition** and **modulus product** properties to convert geometric constraints into algebraic ones.