

## Important Instructions :

1. There are a total of **7 questions**:
  - All questions are worth 20 marks
  - Only 5 out of 7 questions have to be done.
2. The paper is of **3 hours long**.
3. Before submission, you must clearly indicate the 5 questions that you want to have checked. We will consider only those five questions mentioned.
4. **NOTE:** Solution of each problem must be in a separate page. Don't start the solution from the mid of the page.
5. **Multiple solutions** for the same problem are **allowed and encouraged**. Note that for multiple submissions we will consider the solution with highest mark.
6. Write neat proofs for each question attempted, even if you're not entirely sure. Partial marks will be given generously to some logical conclusions.

Important : Leave this table blank

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total
Marks								

1. A machine starts operating with an empty set. It operates for 2025 minutes, where after each  $i^{\text{th}}$  minute ( $1 \leq i \leq 2025$ ), it either adds the number  $i$  to the set or does not, each event with equal probability. Let  $p$  be the probability that the sum of the elements of the final set is divisible by 5 and the number of elements in the set is divisible by 3. Let  $q$  be the probability that the sum of the elements of the final set is divisible by 3 and the number of elements in the set is divisible by 5. Arrange  $p$ ,  $q$  and  $\frac{1}{15}$  in ascending order, and give a clear reasoning for your answer.
2. Denote by  $\mathbb{Q}^+$  the set of all positive rational numbers. Determine all functions  $f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$  which satisfy the following equation for all  $x, y \in \mathbb{Q}^+$ :

$$f(f(x)^2y) = x^3f(xy).$$

3. Determine all pairs  $(x, y)$  of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1$$

4. Let  $k$  be a **positive integer**. Find the **smallest positive integer**  $n$  for which there exist  $k$  nonzero vectors  $v_1, \dots, v_k$  in  $\mathbb{R}^n$  such that for every pair  $i, j$  of indices with  $|i - j| > 1$  the **vectors  $v_i$  and  $v_j$  are orthogonal**.
5. Let  $a$  and  $b$  be **real numbers** with  $a < b$ , and let  $f$  and  $g$  be continuous functions from  $[a, b]$  to  $(0, \infty)$  such that

$$\int_a^b f(x) dx = \int_a^b g(x) dx \quad \text{but } f \not\equiv g.$$

For **every positive integer**  $n$ , define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n} dx.$$

Show that  $I_1, I_2, I_3, \dots$  is an **increasing sequence** with  $\lim_{n \rightarrow \infty} I_n = \infty$ .

6. Let  $s$  be a positive integer and let the non-negative integers  $s \geq t_1 \geq t_2 \cdots \geq t_s \geq d_1 \geq d_2 \cdots \geq d_n$  denote the degrees of the vertices of a graph. Prove that the integers  $t_1 - 1, t_2 - 1, \dots, t_s - 1, d_1, d_2, \dots, d_n$  are also the degrees of the vertices of a graph.
7. For any non-negative integer  $k$ , let  $f(k)$  denote the number of 1s in the base-3 representation of  $k$ . Find all complex numbers  $z$  such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)} (z + k)^{2023} = 0$$