# Linear Algebra Problem-Solving Theorems and Strategies

Inter IIT 14.0 Maths Bootcamp – L5

IIT Kanpur SciMath Society

# Vector Space Axioms

A **vector space** $V$ over a field $\mathbb{F}$ is a set with addition and scalar multiplication satisfying:

- $(V, +)$ is an abelian group.
- Scalar multiplication distributes:

$$a(u+v) = au + av, \quad (a+b)u = au + bu$$

- $(ab)u = a(bu), \quad 1u = u$

**Examples:** $\mathbb{R}^n$, space of polynomials $\mathbb{R}[x]$, matrices $M_{m\times n}(\mathbb{R})$.

# Subspaces and Linear Independence

- $W \subseteq V$ is a **subspace** if $u, v \in W$ and $\alpha, \beta \in \mathbb{F} \Rightarrow \alpha u + \beta v \in W$.
- **Span:** span$(S)$ is the smallest subspace containing $S$.
- **Linear Independence:** $\sum c_i v_i = 0 \Rightarrow c_i = 0$ for all $i$.
- **Basis:** A minimal generating independent set. **Dimension:** Number of basis elements.

**Important:** Any two bases of a finite-dimensional space have the same number of vectors.

# Direct Sums and Coordinates

- $V = U \oplus W$ if every $v \in V$ can be written uniquely as $v = u + w$.
- **Coordinate Representation:** Once basis $\mathcal{B}$ fixed, $v \leftrightarrow [v]_{\mathcal{B}}$.
- **Change of Basis:** If $P$ is transition matrix from $\mathcal{B}$ to $\mathcal{B}'$, then $[v]_{\mathcal{B}'} = P^{-1}[v]_{\mathcal{B}}$.

**Use:** Converts geometric interpretation to matrix form for computation.

# Rank and Nullity

For $A \in M_{m \times n}(\mathbb{F})$:

- **Rank:** $\dim(\text{Col}(A)) = \dim(\text{Row}(A))$.
- **Nullity:** $\dim(\ker(A))$.

$$\textbf{Rank–Nullity Theorem:} \quad \text{rank}(A) + \text{nullity}(A) = n.$$

**Use:** Relates number of independent equations to degrees of freedom in solution.

# Determinant Properties

- $\det(AB) = \det A \cdot \det B, \quad \det A^T = \det A.$
- Swapping two rows $\Rightarrow$ sign change.
- $\det(A) = 0 \iff$ rows/columns are linearly dependent.
- $\det(A^{-1}) = (\det A)^{-1}$ when $\det A \neq 0$.
- **Cofactor Expansion:** $\det A = \sum_j (-1)^{i+j} a_{ij} M_{ij}$.

**Geometric meaning:** Volume scaling factor of linear transformation.

For $A\mathbf{x} = \mathbf{b}$:

- Solution exists iff $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$.
- Unique solution iff rank $= n$.
- Use **Gaussian Elimination** or **LU Decomposition**.

**Cramer's Rule:**

$$x_i = \frac{\det A_i}{\det A}, \quad A_i = \text{matrix obtained by replacing column } i \text{ with } \mathbf{b}.$$

**Use:** Symbolic computation for small systems.

# Definition and Kernel/Image

A linear transformation $T : V \to W$ satisfies:

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

- $\ker(T) = \{v : T(v) = 0\}, \quad \mathrm{im}(T) = \{T(v) : v \in V\}.$
- $\dim(\ker T) + \dim(\mathrm{im}\, T) = \dim V$ (Rank–Nullity again).

**Use:** Reduces analysis of abstract maps to matrix problems under chosen bases.

# Matrix Representation and Composition

If $\mathcal{B}_V, \mathcal{B}_W$ are bases:

$$[T]_{\mathcal{B}_V,\mathcal{B}_W} = [T(v_1) \ \cdots \ T(v_n)]$$

$$[T_2 \circ T_1] = [T_2][T_1], \quad [I_V] = I.$$

**Idea:** Composition of transformations corresponds to matrix multiplication.

# Eigenvalues and Eigenvectors

- $A\mathbf{v} = \lambda\mathbf{v}$ with $\mathbf{v} \neq 0$.
- $\lambda$ satisfies $\det(A - \lambda I) = 0$.
- **Eigenspace:** $\ker(A - \lambda I)$.
- **Diagonalizable:** If basis of eigenvectors exists.

**Trace–Determinant Relation:**

$$\mathrm{tr}(A) = \sum \lambda_i, \quad \det(A) = \prod \lambda_i$$

**Use:** Simplifies power computations and dynamic systems.

# Similarity and Canonical Forms

- $A, B$ are **similar** if $\exists P$ invertible with $B = P^{-1}AP$.
- Similar matrices represent the same linear map under different bases.
- **Diagonal Form:** For distinct eigenvalues or complete set of eigenvectors.
- **Jordan Form:** Block-diagonal form with Jordan blocks for each eigenvalue.
- **Rational Canonical Form:** Based on companion matrices of invariant factors.

**Use:** Structure theorem for matrices — classifies transformations up to similarity.

# Cayley–Hamilton Theorem

**Statement:** Every square matrix satisfies its own characteristic equation. If $A$ is an $n \times n$ matrix and

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$

then

$$p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0.$$

# Cayley–Hamilton Theorem (Contd.)

**Consequences:**

- Provides a polynomial relation that every matrix satisfies.
- Allows computation of $A^{-1}$, powers, and matrix functions without diagonalization:

$$A^{-1} = -\frac{1}{c_0}(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I), \text{ if } c_0 \neq 0.$$

- Key step in deriving **Minimal Polynomial** and **Rational Canonical Form**.
- Basis for efficient computation of powers and recurrence relations involving matrices.

**Use:** Reduces higher powers of $A$ to linear combinations of lower powers — essential in theoretical proofs and matrix exponentiation problems.

# Bilinear Forms

A map $B : V \times V \to \mathbb{F}$ is **bilinear** if linear in each argument:

$$B(au_1+bu_2, v) = aB(u_1, v)+bB(u_2, v), \quad B(u, av_1+bv_2) = aB(u, v_1)+bB(u,$$

- **Matrix form:** $B(u, v) = u^T Av$.
- **Symmetric:** $A = A^T$; **Skew-symmetric:** $A = -A^T$.
- **Rank and signature** characterize equivalence up to congruence.

# Quadratic Forms and Diagonalization

Given $Q(x) = x^T Ax$ with symmetric $A$:

- **Canonical (Diagonal) Form:** via orthogonal (real) or congruence transformations.
- **Sylvester's Law of Inertia:** Number of $+$, $-$, 0 entries invariant under congruence.
- **Definiteness:**

$$Q(x) > 0, \forall x \neq 0 \iff A \text{ positive definite.}$$

- Check via eigenvalues or leading principal minors.

**Use:** Optimization, conic classification, stability analysis.

# Inner Product Spaces

- $\langle u, v \rangle$ satisfies linearity, symmetry, and positivity.
- $\|v\| = \sqrt{\langle v, v \rangle}$.
- **Cauchy–Schwarz:** $|\langle u, v \rangle| \leq \|u\| \|v\|$.
- **Orthogonality:** $\langle u, v \rangle = 0$.
- **Gram–Schmidt:** Construct orthonormal basis from any independent set.
- **Projection:** $\operatorname{proj}_u v = \frac{\langle v, u \rangle}{\langle u, u \rangle} u$.

**Use:** Least squares, orthogonal decompositions, and PCA.

# Matrix Decompositions

- **LU Decomposition:** $A = LU$ (Lower–Upper triangular).
- **QR Decomposition:** $A = QR$ with $Q$ orthogonal.
- **Spectral Decomposition:** $A = Q\Lambda Q^{-1}$ for symmetric $A$.
- **SVD:** $A = U\Sigma V^T$ with orthogonal $U, V$ and $\Sigma$ diagonal with singular values.

**Use:** Numerical stability, dimension reduction, and rank estimation.

# Trace, Adjoint, and Inverse

- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- **Adjugate:** $A\,\mathrm{adj}(A) = \det(A)I$.
- $A^{-1} = \dfrac{\mathrm{adj}(A)}{\det(A)}$ when invertible.
- For Hermitian $A$, eigenvalues are real and eigenvectors orthogonal.

**Use:** Matrix identities and simplifications in algebraic manipulations.

# Vector Space Problem-Solving Strategies

- Identify the **ambient field and dimension**.
- Convert set definitions to **linear equations** to test subspace criteria.
- For independence, write $\sum c_i v_i = 0$ and solve for $c_i$.
- Use basis vectors to express general elements and dimension counts.
- Look for symmetry or constraint patterns — e.g. $x + y + z = 0$ subspaces.

# Matrix and System Solving Strategies

- Reduce to **row echelon** or **reduced echelon form**.
- Track pivot positions for rank, free variables for nullity.
- For parameterized systems, represent the solution set as affine subspace.
- Test determinant and minors for dependence.
- In optimization problems, link rank conditions to constraint consistency.

# Eigenvalue and Transformation Strategies

- For small matrices, compute $\det(A - \lambda I)$ symbolically.
- If matrix is triangular, eigenvalues are diagonal entries.
- For repeated eigenvalues, test dimension of eigenspace for diagonalizability.
- For transformations, check invariance of subspaces to infer matrix structure.
- Use spectral decomposition for powers and exponentials of matrices.

# Quadratic and Inner Product Strategies

- Translate geometric or optimization constraints into quadratic form.
- Diagonalize using orthogonal change of variables to identify nature (ellipse, hyperbola, etc.).
- For inequalities, apply Cauchy–Schwarz or Bessel's inequality.
- When vectors form non-orthogonal bases, compute projection coefficients via Gram matrices.
- Normalize whenever possible to reduce dependence on scaling.

# General Problem-Solving Insights

- Convert algebraic problems to matrix language — linear algebra unifies structure.
- Focus on invariants: rank, determinant, trace, eigenvalues remain stable under transformation.
- When symbolic methods fail, move to geometric or orthogonal interpretation.
- Check degenerate cases (zero vectors, repeated eigenvalues) early.
- Exploit symmetry and orthogonality — they often hide the simplest route.