Count Efficiently!

Effective Strategies for Solving Permutations & Combinations

Inter IIT 14.0 Maths Bootcamp – L2

SciMathSoc
Indian Institute of Technology Kanpur

Common Known Results

1. **Arrangement of k objects chosen from n:**

$$P(n,k) = \frac{n!}{(n-k)!}$$

2. **Choosing k elements from n:**

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

3. **Non–negative integer solutions of:**

$$x_1 + x_2 + \cdots + x_n = m, \quad x_i \geq 0 \Rightarrow \binom{m+n-1}{n-1}$$

These form the algebraic foundation for combinatorial enumeration.

Cyclic Permutations in $\{1, 2, \ldots, n\}$

A **cycle** $(a_1\, a_2\, \ldots\, a_k)$ denotes a permutation sending

$$a_1 \to a_2,\; a_2 \to a_3,\; \ldots,\; a_k \to a_1.$$

Every permutation can be uniquely expressed as a product of disjoint cycles.
Number of n-cycles:

$$(n-1)! \quad \text{(since fixing 1, the rest can be arranged circularly).}$$

Total permutations: sum over all possible cycle decompositions,

$$n! = \sum_{k=1}^{n} S(n,k)\,(k!),$$

where $S(n,k)$ are Stirling numbers of the second kind.

Odd and Even Permutations

Every permutation can be expressed as a product of transpositions (2-cycles).
Sign of a permutation:

$$\mathrm{sgn}(\pi) = \begin{cases} +1, & \text{if } \pi \text{ is even (even number of transpositions)} \\ -1, & \text{if } \pi \text{ is odd} \end{cases}$$

Properties:

- Exactly half of all $n!$ permutations are even, half are odd.
- Even permutations form the **alternating group** A_n, of size $\frac{n!}{2}$.
- Parity is invariant under composition: $\mathrm{sgn}(\pi\sigma) = \mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma)$.

Understanding parity aids in combinatorial proofs involving determinants and orientation.

Permutations of Objects Around a Circle

Circular arrangement (rotations identical):

$$\text{Distinct circular permutations of } n \text{ distinct objects} = (n-1)!$$

If reflection symmetry also considered (bracelet arrangement):

$$\text{Distinct arrangements} = \begin{cases} \dfrac{(n-1)!}{2}, & n \text{ odd} \\ \dfrac{n!}{2n}, & n \text{ even} \end{cases}$$

Example. Seating n people around a round table where rotations are identical but reflections are distinct uses $(n-1)!$.

Counting under Symmetry — Burnside's Lemma

When symmetry operations (rotations/reflections) make configurations equivalent, use **Burnside's Lemma**:

$$N_{\text{distinct}} = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|$$

Example. Coloring vertices of a square with 3 colors (rotational symmetry group of size 4):

$$N = \frac{1}{4}\left(3^4 + 2 \cdot 3^3 + 3^2\right) = 24$$

Counting symmetrically eliminates overcounting caused by equivalent rotations or reflections.

Derangements

Definition. A derangement is a permutation in which no element appears in its original position.

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

Approximation: $D_n \approx \frac{n!}{e}$.

Example. Four letters in wrong envelopes: $D_4 = 9$.

Pascal's Triangle and its Implications

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

- Arises naturally from choosing whether the n^{th} element is included.
- Generates many classical identities and recurrences.
- The sum of row n equals 2^n, corresponding to all subsets of an n-element set.

Understanding Pascal's rule gives immediate insight into recursive combinatorial structures.

Representative Identities

$$\text{Hockey-stick: } \sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

$$\text{Vandermonde: } \sum_{k} \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n}$$

$$\text{Binomial Theorem: } (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

These relationships reappear in most enumeration proofs and generating function derivations.

Ordinary Generating Functions (OGFs)

Definition. For sequence $(a_n)_{n\geq 0}$,

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

Examples.

- Constant sequence: $\dfrac{1}{1-x}$
- Arithmetic sequence $a_n = n$: $\dfrac{x}{(1-x)^2}$
- Fibonacci sequence: $\dfrac{x}{1-x-x^2}$

Key operations:

- Product $\to$ convolution of coefficients.
- Differentiation/integration $\to$ shifts or weightings by n.
- Useful for recurrence solving and coefficient extraction.

Exponential Generating Functions (EGFs)

Definition. For sequence (a_n),

$$E(x) = \sum_{n\geq 0} \frac{a_n x^n}{n!}.$$

Purpose. Encodes labelled combinatorial structures where order matters.
Examples.

- $a_n = n! \Rightarrow E(x) = \sum x^n = \frac{1}{1-x}$
- $a_n = 1 \Rightarrow E(x) = e^x$
- $a_n = 2^n \Rightarrow E(x) = e^{2x}$

Applications.

- Counting set partitions, labelled graphs, and permutations with restrictions.
- The product of EGFs corresponds to labelled structure composition.

Fundamental Binomial Identities

1. Symmetry:

$$\binom{n}{r} = \binom{n}{n-r}$$

Choosing which r to include is equivalent to choosing which $n-r$ to exclude.

2. Pascal's Rule:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Proof (Combinatorial): Decide whether the n^{th} element is included; in one case it contributes to $\binom{n-1}{r-1}$, in the other to $\binom{n-1}{r}$.

3. Boundary Values:

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = n$$

Summation Identities

1. Binomial Sum:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

Each element of a set of size n can be either included or excluded.

2. Alternating Sum:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0 \quad (n \geq 1)$$

Equal number of subsets with even and odd cardinalities.

3. Weighted Sum:

$$\sum_{r=0}^{n} r \binom{n}{r} = n2^{n-1}$$

Idea: Differentiate $(1+x)^n$ and set $x = 1$.

Convolution and Product Forms

1. Vandermonde's Convolution:

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

Interpretation: Choose r elements from two disjoint sets of sizes m and n.

2. Hockey-Stick Identity:

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

Proof: Summing along a diagonal of Pascal's Triangle; corresponds to all subsets whose largest element is fixed.

3. Binomial Inversion:

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \iff a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k$$

Transforms cumulative-type relationships between sequences.

Extended Identities and Applications

1. Negative Index:

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

Useful for series like $(1-x)^{-n} = \sum_{r \geq 0} \binom{n+r-1}{r} x^r$.

2. Central Binomial Coefficient:

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}, \quad \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$$

(by Stirling's approximation)

3. Catalan Connection:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Counts Dyck paths, balanced parentheses, and noncrossing matchings.

Strategy 1: Symmetry and Complementarity

When two configurations are equivalent under symmetry or complement, count only one representative.

- Divide by the size of the symmetry group when appropriate.
- Count the complement when direct counting is cumbersome.

Example. Distinct bracelets of 6 beads in two colors:

$$N = \frac{1}{12}(2^6 + 3 \cdot 2^3 + 2^4 + 6 \cdot 2^2) = 13.$$

Strategy 2: Constructive Counting

Enumerate possibilities step by step; multiply independent decisions, add disjoint cases.

Example. Five-digit numbers with exactly two 7s:

$$\binom{5}{2} \cdot 9^3 - \text{(leading zero cases)}.$$

This process-based approach generalizes easily to conditional constraints.

Strategy 3: Inclusion–Exclusion

Correct for overlaps by alternately adding and subtracting intersections:

$$|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \ldots$$

Example. Four-digit numbers with at least one repeated digit:

$$9000 - (9 \cdot 9 \cdot 8 \cdot 7) = 4464.$$

Strategy 4: Recurrence and Generating Functions

Derive relations among smaller instances by conditioning on an initial decision.
Example. Binary strings of length n with no consecutive 1s:

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 2, a_2 = 3 \Rightarrow a_n = F_{n+2}.$$

Generating Function:

$$A(x) = \frac{1+x}{1-x-x^2}.$$

Strategy 5: Bijections and Invariants

- **Bijection:** Map the problem to a known combinatorial structure.
- **Invariant:** Identify a property that remains unchanged during transformations.

Example. Dyck paths $\leftrightarrow$ balanced parentheses:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Key Takeaways

- Counting efficiency arises from structural understanding, not memorization.
- Recognize symmetry, recurrence, bijection, or algebraic representation.
- Generating functions serve as algebraic blueprints for combinatorial logic.

Essential Principle

Do not memorize — generalize.
Every combinatorial identity corresponds to a deeper structure.

Count efficiently by reasoning systematically.