Functions and Functional Equations

Inter IIT 14.0 Maths Bootcamp – L4

IIT Kanpur SciMath Society

What are Functional Equations?

A **functional equation** is an equation where the unknown is a function and the relation involves its values at different points:

$$\text{Find } f : \mathbb{R} \to \mathbb{R} \text{ satisfying } \Phi(x, y, f(x), f(y), f(x+y), \dots) = 0.$$

Goal: Determine all possible functions that satisfy the relation.
Common types:

- Additive, multiplicative, or composite relations.
- Equations involving symmetry, scaling, or translation.
- Iterative and recurrence-based definitions.

Additive and Cauchy-Type Equations

$$f(x+y) = f(x) + f(y)$$

Standard solutions: $f(x) = kx$ (for continuous or bounded f).
Variants:

- $f(x+y) = f(x) + f(y) + c \Rightarrow f(x) = kx + c/2$
- $f(x+y) = f(x) + f(y) + xy \Rightarrow f(x) = kx + \frac{1}{2}x^2$
- $f(x-y) = f(x) - f(y)$ — implies oddness: $f(-x) = -f(x)$.

Key idea: Plug $x = 0$, $y = 0$, or $x = y$ to reveal structure and constants.

Multiplicative and Homogeneous Equations

$$f(xy) = f(x)f(y)$$

Typical solutions:

- $f(x) = x^c$, $f(x) = |x|^c$, or $f(x) = 0, 1$ (depending on domain).
- If also $f(x+y) = f(x) + f(y)$, only possible $f(x) = x$ or 0.

Homogeneity:

$$f(kx) = k^n f(x)$$

$\Rightarrow$ function of degree n; often implies polynomial or power-law behavior.

Periodic and Symmetric Equations

- $f(x+T) = f(x)$ defines periodicity, smallest $T > 0$ is period.
- $f(x) = f(-x)$ (even), $f(-x) = -f(x)$ (odd) — crucial in symmetry reasoning.
- If $f(x+y) = f(x-y)$ for all x, y, then f is even.

Useful trick: Plug $x \leftrightarrow -x$ or swap x, y to expose even/odd structure or constancy.

Composite and Iterative Equations

$$f(f(x)) = g(x), \quad \text{or} \quad f(x+1) = af(x)$$

Common solutions:

- $f(f(x)) = x \Rightarrow$ involutions like $f(x) = \pm x + c$.
- $f(x+1) = af(x) \Rightarrow f(x) = ka^x$ if f is exponential.
- $f(x+y) = f(x)f(y) \Rightarrow f(x) = a^x$.

Idea: Check for fixed points ($f(x_0) = x_0$), or use iteration $f^n(x)$ to find recurrence structure.

Functional Equations with Absolute Value or Modulus

- $f(|x|) = |f(x)|$ often implies evenness or linearity.
- $f(xy) = f(x)f(y)$ and $f(-1) = -1 \Rightarrow f(x) = x$ for $x > 0$, $-x$ for $x < 0$.
- Problems involving $|f(x)| \leq M$ often force bounded additive functions to linear forms.

Technique: Separate into $x \geq 0$ and $x \leq 0$ regions, then check continuity or parity.

Strategic Substitutions

- **Set** $x = 0$ or $y = 0$ to determine constants like $f(0), f(1)$.
- **Set** $x = y$ or $x = -y$ to reveal even/odd or quadratic patterns.
- Replace x, y by expressions like $(x + y)/2$, $(x - y)$ to symmetrize.
- Introduce new function $g(x) = f(x) - f(0)$ to simplify constant terms.
- For scaling patterns: $f(kx)$ vs $kf(x)$ — check multiplicative consistency.

Tip: Always isolate $f(0)$ early — it often linearizes the equation.

Parameter Elimination and Functional Invariance

- Identify **invariant transformations**: $f(x+c) - f(x)$ constant implies linearity.
- Differentiate (if allowed) or take finite differences to extract recurrence form.
- If functional form is polynomial-like, assume $f(x) = ax^2 + bx + c$ and compare coefficients.
- For real-analytic f, expand both sides as series and match terms.

Example: $f(x+y) - f(x) - f(y) = xy \Rightarrow f(x) = \frac{x^2}{2} + kx$.

Cauchy, Jensen, and D'Alembert

- **Cauchy:** $f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx$ (if continuous).
- **Jensen:** $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \Rightarrow$ affine $f(x) = kx + c$.
- **D'Alembert:** $f(x+y) + f(x-y) = 2f(x)f(y) \Rightarrow f(x) = \cos(ax)$ or $\cosh(ax)$.

Trick: Set $x = y$ or $y = 0$ to derive recurrence; often leads to trigonometric/exponential forms.

Cauchy Exponential and Logarithmic Forms

- $f(x+y) = f(x)f(y) \Rightarrow f(x) = a^x$.
- $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \log_a x$.
- Combining: $f(a^x) = kf(x)$ or $f(\log x) = a^{f(x)}$ links exponential and logarithmic structures.

Hint: Check monotonicity or continuity to discard pathological (non-measurable) solutions.

Systematic Problem-Solving Pattern

1. Substitute special values to simplify constants.
2. Look for symmetry (even, odd, or homogeneous).
3. Find $f(0)$, $f(1)$, $f(-1)$ — anchor points.
4. Try $x = y$, $x = -y$, $y = 0$, $x = 0$ sequentially.
5. Define $g(x) = f(x) - f(0)$ to remove constants.
6. Test linear/quadratic guesses when patterns suggest polynomial.
7. Validate all solutions by re-substitution.

- Linear f: straight lines passing through $(0, f(0))$.
- Periodic f: repeating pattern – often bounded sine/cosine-type.
- Exponential f: constant ratio of increments.
- Polynomial-type f: fixed curvature across domain.
- Use symmetry or reflection about axes to test even/odd nature visually.

Insight: Graph intuition often hints whether f must be linear, exponential, or bounded.

Common Transformational Tricks

- Define auxiliary functions:

$$g(x) = f(x) - kx, \quad h(x) = f(x) - f(-x)$$

  to separate linear and parity components.

- Apply iterative substitution:

$$f(f(x)) = x \Rightarrow f(x) = x \text{ or } f(x) = -x.$$

- Replace $x \to x + y$ and compare to original to create a recurrence.
- For rational or periodic f, check behavior on integers then extend.

- Combine algebraic and functional reasoning (e.g. define $f(x) + f(y)$ as a symmetric polynomial).
- Introduce new functions ($P(x, y)$ notation) to organize substitutions systematically.
- Guess structure ($f(x) = kx^2 + c$ or a^x) and test using given relations.
- Use injectivity/surjectivity to rule out spurious constants.
- Exploit boundedness/monotonicity to force linear forms.

Key Skill: Recognize when the equation implies additivity, multiplicativity, or periodicity.

Problem-Solving Strategies Summary

- Start from trivial substitutions: $x = 0$, $y = 0$.
- Detect invariances under shift, reflection, or scaling.
- Normalize $f(0) = 0$ by redefining $g(x) = f(x) - f(0)$.
- Use difference method: subtract equations at (x, y) and (y, x).
- Apply functional iteration or inverse mapping if f is bijective.
- Test linearity or parity early; these simplify 70% of problems.
- If polynomial degree suspected, match coefficients explicitly.

Typical Problem Archetypes

- $f(x+y)+f(x-y)=2f(x)f(y) \Rightarrow$ cos or cosh structure.
- $f(x+y)-f(x)-f(y)=xy \Rightarrow$ quadratic.
- $f(xy)=f(x)+f(y) \Rightarrow$ logarithmic.
- $f(x+1)=f(x)+c \Rightarrow$ linear with slope c.
- $f(f(x))=x \Rightarrow$ involution.

Pattern Recognition: classify the given equation by similarity to these archetypes before starting.

Common Traps and Checks

- Don't assume continuity unless stated.
- Always verify the domain (reals, integers, positives, etc.).
- Check constant solutions separately ($f(x) = c$).
- If multiple forms possible, test edge values ($x = 1$, $x = -1$).
- Always back-substitute to confirm validity for all inputs.

Golden Rule: "Every substitution must simplify something."

Summary

- Identify type: additive, multiplicative, or symmetric.
- Simplify constants and parity early.
- Try polynomial or exponential ansatz if structure fits.
- Check edge cases and special inputs methodically.
- Validate all derived forms rigorously.

Functional equations reward pattern-recognition, persistence, and algebraic creativity.