Inter IIT 25' Practice Test-3 (200 points)
SciMathSoc
November 11, 2025 Roll Number(s): ______________________________

Important Instructions :

1. There are a total of **8 questions**:
   - All questions are worth 25 marks
2. The paper is of **3 hours long**.
3. **There are no choices in this paper. You are advised to attempt the questions in an order that maximizes payoff**
4. **NOTE:** Solution of each problem must be in a separate page. Don't start the solution from the mid of the page.
5. **We will be tracking teamwork** in this paper.
6. Write neat proofs for each question attempted, even if you're not entirely sure. Partial marks will be given generously to some logical conclusions.

Important : Leave this table blank

Question	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Total**
Marks									

1. Find all positive integers $n > 2$ such that

$$n! \mid \prod_{p<q<n} (p+q)$$

Where p, q are primes

Answer. We claim that the only such integer is $n = 7$.

Assume that n satisfies

$$n! \Bigg| \prod_{\substack{p<q\le n \\ p,q \text{ primes}}} (p+q),$$

and let the primes in $\{1, 2, \ldots, n\}$ be

$$2 = p_1 < p_2 < \cdots < p_m \le n.$$

Each p_i divides $n!$, so in particular p_m must divide one of the factors $(p_i + p_j)$ in the product, for some $p_i < p_j \le n$. Thus

$$p_m \mid p_i + p_j \quad \text{for some } i < j.$$

But then

$$0 < \frac{p_i + p_j}{p_m} < \frac{p_m + p_m}{p_m} = 2,$$

so $\dfrac{p_i + p_j}{p_m}$ must be the integer 1, i.e.

$$p_m = p_i + p_j.$$

Because p_m is the largest prime $\le n$, this forces $p_i = 2$ and $p_j = p_m - 2$, so

$$m \ge 3, \quad p_1 = 2, \quad p_m = 2 + p_{m-1}.$$

Now consider the prime p_{m-1}. As above, p_{m-1} divides $n!$, so p_{m-1} must also divide some sum $p_k + p_\ell$ with $p_k < p_\ell \le n$. Then

$$0 < \frac{p_k + p_\ell}{p_{m-1}} \le \frac{p_m + p_{m-1}}{p_{m-1}} = \frac{(p_{m-1} + 2) + p_{m-1}}{p_{m-1}} = \frac{2p_{m-1} + 2}{p_{m-1}} < 3,$$

so the integer $\dfrac{p_k + p_\ell}{p_{m-1}}$ is either 1 or 2.

Case 1: $\dfrac{p_k + p_\ell}{p_{m-1}} = 1.$

Then $p_{m-1} = p_k + p_\ell$. As before, this forces $p_k = 2$ and $p_\ell = p_{m-2}$, so

$$p_{m-1} = 2 + p_{m-2}.$$

Case 2: $\dfrac{p_k + p_\ell}{p_{m-1}} = 2.$

Then

$$2p_{m-1} = p_k + p_\ell.$$

Since $p_k < p_\ell \le p_m$ and $p_m = p_{m-1} + 2$, we have

$$2p_{m-1} = p_k + p_\ell \le p_m + p_{m-1} = 2p_{m-1} + 2,$$

so $p_k > p_{m-1}$, hence $p_k = p_m$. Thus

$$2p_{m-1} = p_m + p_\ell = (p_{m-1} + 2) + p_\ell,$$

which yields

$$p_{m-1} = p_\ell + 2.$$

Again, p_ℓ must be the previous prime, so $p_\ell = p_{m-2}$ and

$$p_{m-1} = p_{m-2} + 2.$$

In both cases we have

$$p_{m-2} > 2, \qquad p_{m-1} = p_{m-2} + 2, \qquad p_m = p_{m-2} + 4.$$

Thus the three consecutive odd integers

$$p_{m-2},\ p_{m-2} + 2,\ p_{m-2} + 4$$

are all prime. Among any three consecutive integers exactly one is divisible by 3, so 3 divides at least one of these three primes. Hence $p_{m-2} = 3$, which gives

$$p_{m-1} = 5, \quad p_m = 7.$$

Therefore $p_m = 7$, and since p_m is the largest prime $\le n$, we have

$$7 \le n < 11.$$

It remains to check $n = 7, 8, 9, 10$:

- For $n = 7$, the primes are $2, 3, 5, 7$, and one computes directly that

$$\prod_{\substack{p<q\le 7 \\ p,q \text{ primes}}} (p+q) = 5 \cdot 7 \cdot 9 \cdot 8 \cdot 10 \cdot 12 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7,$$

  while

$$7! = 2^4 \cdot 3^2 \cdot 5 \cdot 7.$$

  Thus $7! \mid \prod_{p<q\le 7}(p+q)$.

- For $n \in \{8, 9, 10\}$, the set of primes $\le n$ is still $\{2, 3, 5, 7\}$, so the product

$$\prod_{\substack{p<q\le n \\ p,q \text{ primes}}} (p+q)$$

  is the same as for $n = 7$. However,

$$8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7, \qquad 9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7, \qquad 10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7,$$

  and in each case the exponent of 2 or 3 in $n!$ is larger than in the product, so $n!$ does not divide the product.

Hence the only possible value is $n = 7$.

2. Let $A, B \in M_{1012\times 1012}(\mathbb{Z})$ be such that $A, A+B, A+2B....., A+2023B, A+2024B$ are all invertible and each has an inverse with integer entries. Show that $A+2025B$ is invertible and that its inverse has integer entries.

Answer. Suppose that the $n \times n$ matrix M has integer entries and that its inverse M^{-1} also has integer entries. Then

$$M \cdot M^{-1} = I$$

implies

$$\det(M)\ \det(M^{-1}) = \det(I) = 1.$$

Thus $\det(M) = 1$ or $\det(M) = -1$.

Now set

$$M(t) = A + tB.$$

The determinant of $M(t)$,

$$\det M(t) = \det(A + tB),$$

is a polynomial in t of degree at most n:

$$\det(A + tB) = \det(A) + \cdots + t^n \det(B).$$

The polynomial $\det M(t)$ takes the values 1 or -1 at the integer points

$$t = 0, 1, 2, \ldots, 2n.$$

Hence $\det M(t)$ takes the value 1 or the value -1 at least $n+1$ times. This implies that $\det M(t)$ is a constant polynomial; that is,

$$\det M(t) = 1 \quad \text{or} \quad \det M(t) = -1$$

for all t.

Consequently,

$$\det M(2n+1) = \det\bigl(A + (2n+1)B\bigr) = \pm 1.$$

Hence the matrix $A + (2n+1)B$ is invertible. By Cramer's rule, the inverse matrix has integer entries, since its denominator is $\det\bigl(A + (2n+1)B\bigr) = \pm 1$.

Now in this question, $n = 1012$ and we're done!

3. Find all pairs of non-negative integers (x, y) that satisfy the equation

$$p^x - y^p = 1,$$

where p is an odd prime.

Answer. If $x = 0$, then $1 - y^p = 1$, giving $y = 0$. Hence $(x, y) = (0, 0)$ is a solution. Assume $x \geq 1$ and $y \geq 0$.

If $y = 0$, the equation becomes $p^x = 1$, impossible for $x \geq 1$. If $y = 1$, then $p^x - 1 = 1$, giving $p^x = 2$, which is impossible for odd p. Thus, we take $y \geq 2$.

From $p^x = y^p + 1$, we have $y^p \equiv -1 \pmod{p}$, hence $y \equiv -1 \pmod{p}$, so $p \mid (y + 1)$. By the Lifting The Exponent (LTE) lemma, since p is odd and $p \mid (y + 1)$,

$$v_p(y^p + 1) = v_p(y + 1) + v_p(p) = v_p(y + 1) + 1.$$

But $v_p(y^p + 1) = v_p(p^x) = x$, so

$$x = v_p(y + 1) + 1 \quad \Longrightarrow \quad v_p(y + 1) = x - 1.$$

Let $y + 1 = p^{x-1}m$ with $\gcd(m, p) = 1$. Then

$$y^p + 1 = (y + 1)\left(y^{p-1} - y^{p-2} + \cdots + 1\right) = p^{x-1}m\, S,$$

where $S = \dfrac{y^p + 1}{y + 1} = y^{p-1} - y^{p-2} + \cdots + 1$. Since $v_p(y^p + 1) = x$, we must have $v_p(S) = 1$, so $S = pk$ with $p \nmid k$. Hence

$$p^x = p^{x-1}m \cdot pk \quad \Longrightarrow \quad mk = 1.$$

Thus $m = k = 1$, giving

$$y + 1 = p^{x-1}, \qquad S = p.$$

Now $y = p^{x-1} - 1$, and substituting in $S = p$ gives

$$\frac{(p^{x-1} - 1)^p + 1}{p^{x-1}} = p.$$

If $x = 1$, then $y = 0$, already found. If $x = 2$, then $y = p - 1$, and

$$\frac{(p - 1)^p + 1}{p} = S.$$

For $p = 3$, $S = (2^3 + 1)/3 = 9/3 = 3 = p$, giving the solution $(p, x, y) = (3, 2, 2)$. For any $p \geq 5$,

$$S = \frac{(p - 1)^p + 1}{p} > p,$$

so there are no solutions. If $x \geq 3$, then $y = p^{x-1} - 1 \geq p^2 - 1 \geq 8$, and

$$S = y^{p-1} - y^{p-2} + \cdots + 1 > y^{p-2}(y - 1) \geq (p^2 - 1)^{p-2}(p^2 - 2) > p,$$

hence no solutions occur.

Therefore, the only solutions in non-negative integers are

$$(x, y) = (0, 0) \quad \text{for all odd primes } p,$$

and

$$(p, x, y) = (3, 2, 2),$$

since $3^2 - 2^3 = 1$.

4. Let k and n be positive integers. A sequence $(A_1, A_2, \ldots, A_k)$ of $n \times n$ real matrices is said to be good if
$$A_i^3 \neq 0 \quad \text{for all } 1 \leq i \leq k,$$
but
$$A_i A_j = 0 \quad \text{for all } 1 \leq i, j \leq k \text{ with } i \neq j.$$
Show that $k \leq n$ in all good sequences, and give an example of a good sequence with $k = n$ for each n.

Answer. Note that $M^3 \neq 0 \implies M^2 \neq 0$ We will provide 2 approaches

Approach 1

Let $(A_1, \ldots, A_k)$ be a nice sequence of $n \times n$ real matrices, i.e. $A_i^2 \neq 0$ for all i and $A_i A_j = 0$ for $i \neq j$. For each i, since $A_i^2 \neq 0$, there exists a column vector $v_i \in \mathbb{R}^n$ such that $A_i v_i \neq 0$. We claim that $v_1, \ldots, v_k$ are linearly independent. Suppose $c_1 v_1 + \cdots + c_k v_k = 0$. Fix i. Using $A_i A_j = 0$ for $j \neq i$, we get

$$0 = A_i\big(c_1 v_1 + \cdots + c_k v_k\big) = \sum_{j=1}^{k} c_j (A_i v_j) = c_i (A_i v_i).$$

Since $A_i v_i \neq 0$, it follows that $c_i = 0$. Hence all $c_i = 0$ and the vectors are independent, so $k \leq n$.

For $k = n$, an example is given by diagonal idempotents: let A_i be the diagonal matrix with a single 1 at the (i,i) entry and zeros elsewhere. Then $A_i^2 = A_i \neq 0$ and $A_i A_j = 0$ for $i \neq j$, so $(A_1, \ldots, A_n)$ is a nice sequence.

Approach 2

Let $U_i = \operatorname{Im}(A_i)$ and $K_i = \ker(A_i)$. For $i \neq j$, the relation $A_i A_j = 0$ implies $U_j \subseteq K_i$. Define $X_0 = \mathbb{R}^n$ and $X_i = K_1 \cap \cdots \cap K_i$ for $i = 1, \ldots, k$. Then $U_i \subseteq X_{i-1}$ (since $U_i \subseteq K_j$ for all $j < i$), but $U_i \nsubseteq X_i$ (because $U_i \nsubseteq K_i$ as $A_i^2 \neq 0$). Hence X_i is a proper subspace of X_{i-1} for each i, giving a strictly descending chain

$$\mathbb{R}^n = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_k.$$

Therefore $k \leq n$.

5. For every positive integer n, let $\sigma(n)$ denote the sum of all its positive divisors. A number n is called *weird* if

$$\sigma(n) \geq 2n$$

and there exists no representation

$$n = d_1 + d_2 + \cdots + d_r,$$

where $r > 1$ and $d_1, d_2, \ldots, d_r$ are pairwise distinct positive divisors of n.
Prove that there are infinitely many weird numbers.

Answer. Recall that for every positive integer n, $\sigma(n)$ denotes the sum of all positive divisors of n. A number n is called *weird* if

$$\sigma(n) \geq 2n$$

and there is no representation

$$n = d_1 + d_2 + \cdots + d_r,$$

where $r > 1$ and $d_1, \ldots, d_r$ are pairwise distinct positive divisors of n.

The idea is to show that, given one weird number, we can construct an infinite sequence of weird numbers tending to infinity.

Claim. Let n be a weird number and let p be a prime such that $p > \sigma(n)$ and $\gcd(p, n) = 1$. Then pn is also weird.

Proof of the claim. Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all positive divisors of n. Since $\gcd(p, n) = 1$, the positive divisors of pn are exactly

$$d_1, d_2, \ldots, d_k, \; pd_1, pd_2, \ldots, pd_k,$$

and they are pairwise distinct.

Suppose, for a contradiction, that pn is not weird. Then there exists a representation

$$pn = d_{i_1} + \cdots + d_{i_r} + p\,(d_{j_1} + \cdots + d_{j_s}),$$

where $r + s > 1$ and all d_{i_ℓ}, d_{j_m} are positive divisors of n (not necessarily distinct among themselves, but chosen from $\{d_1, \ldots, d_k\}$).

Rearranging, we obtain

$$d_{i_1} + \cdots + d_{i_r} = p\big(n - d_{j_1} - \cdots - d_{j_s}\big).$$

Note that $n \notin \{d_{j_1}, \ldots, d_{j_s}\}$, since otherwise we would have $pn = n$ plus other nonnegative terms, which is impossible. Thus

$$n - d_{j_1} - \cdots - d_{j_s} \neq 0,$$

because n itself is weird and therefore cannot be expressed as a sum of pairwise distinct proper divisors.

Hence the right-hand side is a nonzero multiple of p, so p divides $d_{i_1} + \cdots + d_{i_r}$. But every d_{i_ℓ} divides n and $\gcd(p, n) = 1$, so no d_{i_ℓ} is divisible by p. Therefore their sum cannot be divisible by p, a contradiction.

Thus pn admits no such representation, and one checks easily that $\sigma(pn) \geq 2pn$ as well. Therefore pn is weird, proving the claim.

It remains to find at least one weird number n.

One possible strategy is to search for n with

$$\sigma(n) = 2n + 4,$$

not divisible by 3 or 4. Then the smallest possible positive divisors are $1, 2, 5$, so it becomes impossible to represent 4 (and hence n) as a sum of pairwise distinct divisors of n.

Consider numbers with three distinct prime factors $2, p, q$, i.e. $n = 2pq$. Then

$$\sigma(2pq) = (1 + 2)(1 + p)(1 + q) = 3(p + 1)(q + 1).$$

We require

$$\sigma(2pq) = 2 \cdot 2pq + 4 = 4pq + 4,$$

so

$$3(p + 1)(q + 1) = 4pq + 4.$$

Expanding and rearranging gives

$$3pq + 3p + 3q + 3 = 4pq + 4 \quad \iff \quad (p - 3)(q - 3) = 8.$$

The integer solutions are $(p, q) = (5, 7)$ (up to symmetry), which yields $n = 2 \cdot 5 \cdot 7 = 70$. One checks that 70 is indeed weird.

By the claim, starting from $n = 70$ and choosing infinitely many primes p with $p > \sigma(70)$ and $\gcd(p, 70) = 1$, we obtain infinitely many weird numbers pn tending to infinity.

Hence there are infinitely many weird numbers.

6. Let $A \in M_{n\times n}(\mathbb{R})$ be a matrix. Denote A^R its counter-clockwise 90 rotation. For example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^R = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}$$

Prove that if $A = A_R$, then for any eigenvalue λ of A, we have $Re(\lambda) = 0$ or $Im(\lambda) = 0$.

Answer. If $\lambda = 0$, the claim is clear. Assume $\lambda \neq 0$ is an eigenvalue of A with eigenvector $x \in \mathbb{C}^n \setminus \{0\}$.

We first express the rotation algebraically. The element at position (i, j) in A moves to position $(n+1-j,\, i)$ in A^R, so $(A^R)_{i,j} = A_{j,n+1-i}$.

Let J be the matrix with entries $J_{i,j} = \delta_{i,n+1-j}$. Then transposing A and reversing rows gives $JA^\top$. Indeed,

$$(JA^\top)_{i,j} = \sum_{k=1}^{n} J_{i,k}(A^\top)_{k,j} = (A^\top)_{n+1-i,j} = A_{j,n+1-i},$$

which matches the definition of A^R. Hence,

$$A^R = JA^\top.$$

The matrix J is symmetric ($J = J^\top$) and involutory ($J^2 = I$). The condition $A = A^R$ therefore becomes $A = JA^\top$. Multiplying both sides by J on the left gives

$$JA = A^\top. \tag{$*$}$$

Now consider the standard Hermitian inner product $(u, v) = v^* u$ on $\mathbb{C}^n$. We compute (Ax, Ax) in two ways.

Since $Ax = \lambda x$,

$$(Ax, Ax) = (\lambda x, \lambda x) = |\lambda|^2 \|x\|^2.$$

Using the relation $(*)$,

$$(Ax, Ax) = (A^* Ax, x) = (A^\top Ax, x) = (JA(\lambda x), x) = \lambda(JAx, x) = \lambda^2 (Jx, x).$$

Thus,

$$|\lambda|^2 \|x\|^2 = \lambda^2 (Jx, x).$$

Since J is real symmetric, (Jx, x) is real. The left-hand side is a positive real number, so $\lambda^2 (Jx, x)$ must also be real and positive. Hence $\lambda^2 \in \mathbb{R}$.

Therefore, either λ is real (if $\lambda^2 > 0$) or purely imaginary (if $\lambda^2 < 0$). That is, $\mathrm{Re}\,\lambda = 0$ or $\mathrm{Im}\,\lambda = 0$.

7. Compute the 1000^{th} smallest positive multiple of 6 whose digits in base 10 are all strictly less than 4.

Answer. The answer is 1131300
We construct an order-preserving bijection between positive multiples of 6 in base 10 whose digits are all less than 4 and positive multiples of 6. For any multiple of 6 in base 10 with digits all less than 4, interpret it as a base 4 number and convert it to a base 10 decimal, which will be a multiple of 6. For example, 300 would map to $300_4 = 48_{10}$.

We first show that this is a valid mapping. Let $a_n \dots a_1 a_0$ be an arbitrary multiple of 6 (in base 10) whose digits are all less than 4. This has a natural interpretation in base 4, and, converting this interpretation to base 10 gives

$$(a_n \dots a_1 a_0)_4 = a_n \cdot 4^n + \cdots + a_1 \cdot 4^1 + a_0 \cdot 4^0 \equiv a_n \cdot 10^n + \cdots + a_1 \cdot 10^1 + a_0 \cdot 10^0 = (a_n \dots a_1 a_0)_{10} \pmod 6,$$

which is a multiple of 6.

This mapping is an injection because base conversion is an injection, and this mapping is also a surjection because for any multiple of 6 in base 10, we can convert it to base 4 and interpret this number as a base 10 decimal. (Reversing the above steps, we see the resulting decimal number is a multiple of 6.) Furthermore, for two positive multiples of 6 whose digits are less than 4, the larger one will have a larger base 4 representation, so this mapping is order-preserving.

Thus, this mapping is an order-preserving bijection between multiples of 6 in base 10 with digits less than 4 and multiples of 6 in base 4. Thus the answer is $(6000)_{10} = \mathbf{1131300}_4$

8. Denote by S the set of all real symmetric 2025×2025 matrices of rank 1 whose entries take values -1 or $+1$. Let $A, B \in S$ be matrices chosen independently and uniformly at random. Find the probability that A and B commute, i.e. that $AB = BA$.

Answer. Let $n = 2025$. We first characterize matrices in S.

Suppose $A = (a_{ij})_{i,j=1}^n \in S$. Since $\operatorname{rank}(A) = 1$, for all $1 < i, j \leq n$,

$$\det \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix} = a_{11}a_{ij} - a_{i1}a_{1j} = 0.$$

Hence, if $a_{11} = 1$, we have $a_{ij} = a_{i1}a_{1j}$. Let

$$u = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix},$$

then $A = uu^\top$.

If $a_{11} = -1$, we get $a_{ij} = -a_{i1}a_{1j}$. In this case, setting

$$u = \begin{pmatrix} -a_{11} \\ -a_{21} \\ \vdots \\ -a_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ -a_{21} \\ \vdots \\ -a_{n1} \end{pmatrix}$$

gives $A = -uu^\top$.

Hence every matrix in S can be uniquely written as $\pm uu^\top$, where $u = (1, u_2, \ldots, u_n)^\top$ with $u_i \in \{\pm 1\}$. Thus $|S| = 2^n$, since the sign and the entries $u_2, \ldots, u_n$ can be chosen independently.

Now take $A = \pm uu^\top$ and $B = \pm vv^\top$. Then

$$AB = \pm(uu^\top)(vv^\top) = \pm u(u^\top v)v^\top = \pm\langle u, v\rangle\, uv^\top,$$

and similarly,

$$BA = \pm\langle u, v\rangle\, vu^\top.$$

Since $n = 2025$ is odd, we have $\langle u, v\rangle \neq 0$. The first columns of $uv^\top$ and $vu^\top$ are u and v, respectively. Hence $AB = BA$ if and only if $u = v$, that is, $A = \pm B$.

For each $A \in S$, there are exactly two matrices $B \in S$ (namely A and $-A$) such that $AB = BA$. Therefore, the required probability is

$$\frac{2}{|S|} = \frac{1}{2^{n-1}} = \frac{1}{2^{2024}}.$$