AESoP

- **A**dvanced **E**quation **So**lver for **P**arallel Systems
  - Solver for a least-squares system of equations
  - Brian Gunter, Delft University of Technology
- Written in C
- Uses PLAPACK for parallel processing
  - Robert van de Geijn, UT Austin
  - High-level language for parallel linear algebra
  - www.cs.utexas.edu/~plapack
  - PLAPACK uses BLAS for the computations on each core

AESoP

- Solution Method
- Parameter "Leveling"
- Efficiency and Speed
- Optimal Weighting
- Future Gravity Fields

AESoP Least Squares Solution Method

- Block Householder Orthogonal Transformation
- Numerically superior to Normal Equations method

Weighted Least Squares

$$y = Hx + \varepsilon$$

m observations

n estimated parameters

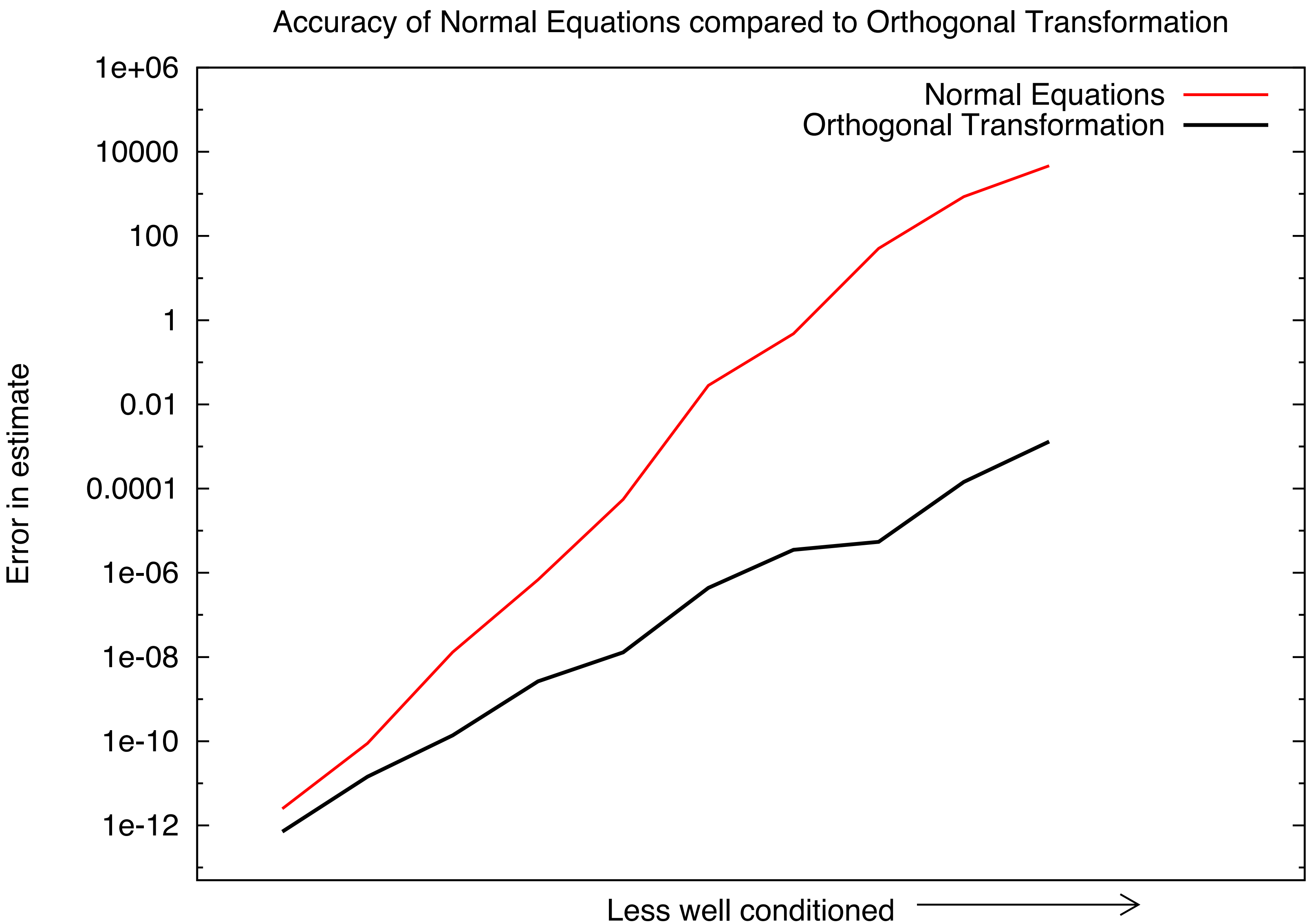
$m \times n$ $\quad$ $m > n$

$$W = m \times m \text{ diagonal weight matrix}$$

- Normal Equation method "squares" H. That is, you have to form $(H^T W H)x = H^T W y$
  – Forming $H^T W H$ doubles the range in the order of magnitude of the numbers, compared to H
  – Increases computer roundoff error.
- Orthogonal Transformation uses H:

$$QW^{1/2}H = \begin{bmatrix} R \\ 0 \end{bmatrix} \qquad QW^{1/2}y = \begin{bmatrix} b \\ e \end{bmatrix} \qquad Rx = b$$

  – Mathematically equivalent to Normal Equations
  – Better numerically because it uses H instead of $H^T W H$


Accuracy of Normal Equations compared to Orthogonal Transformation
Normal Equations
Orthogonal Transformation
Error in estimate
1e+06
10000
100
1
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
Less well conditioned

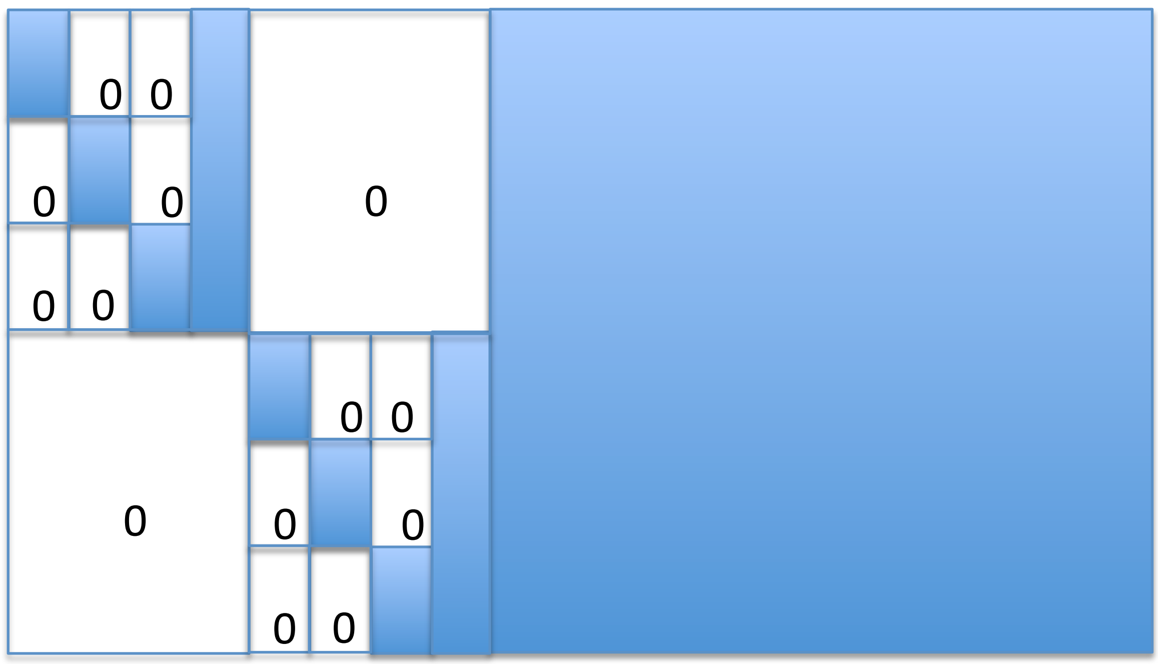
Parameter “Leveling”

- Local Parameters
  - For one type of data (we have 3 types) for one day
  - Example: data biases
- Common Parameters
  - For all 3 types of data for one day
  - Example: Satellite position and velocity
- Global Parameters
  - For all data for all days
  - Example: Gravity model

Parameter Leveling

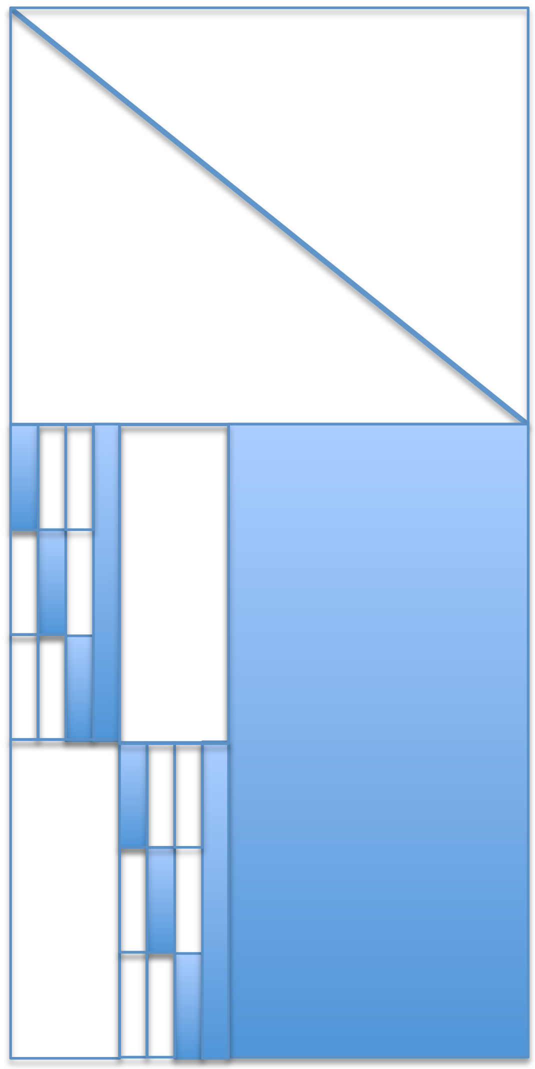
- Results in an H matrix with blocks of zeroes.

  For example, for 2 days of data:



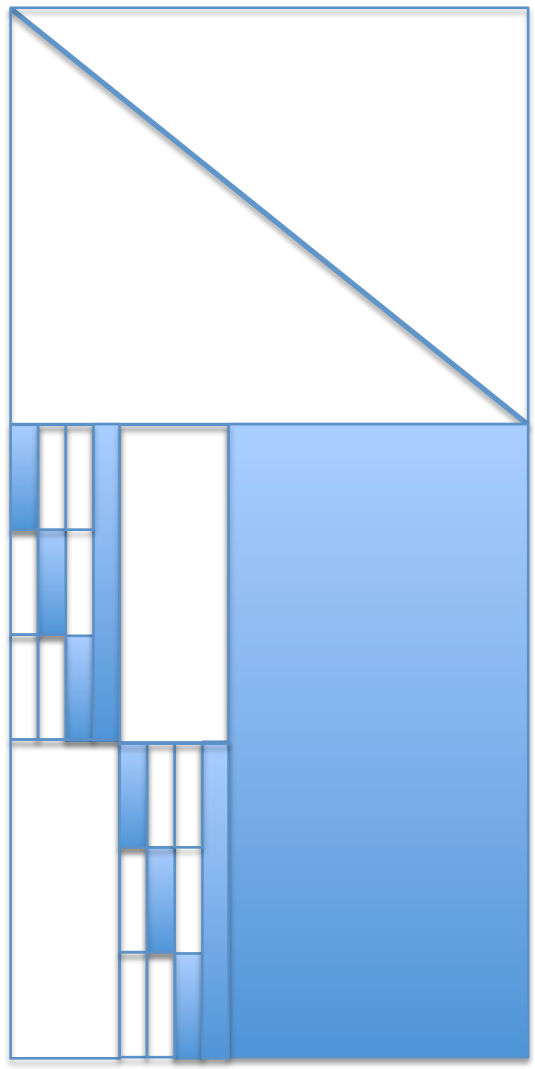
- AESoP builds this H matrix based on a user-supplied INPUT file.

Initial diagonal covariance is added on top



AESoP performs Orthogonal Transformation on this matrix

Q



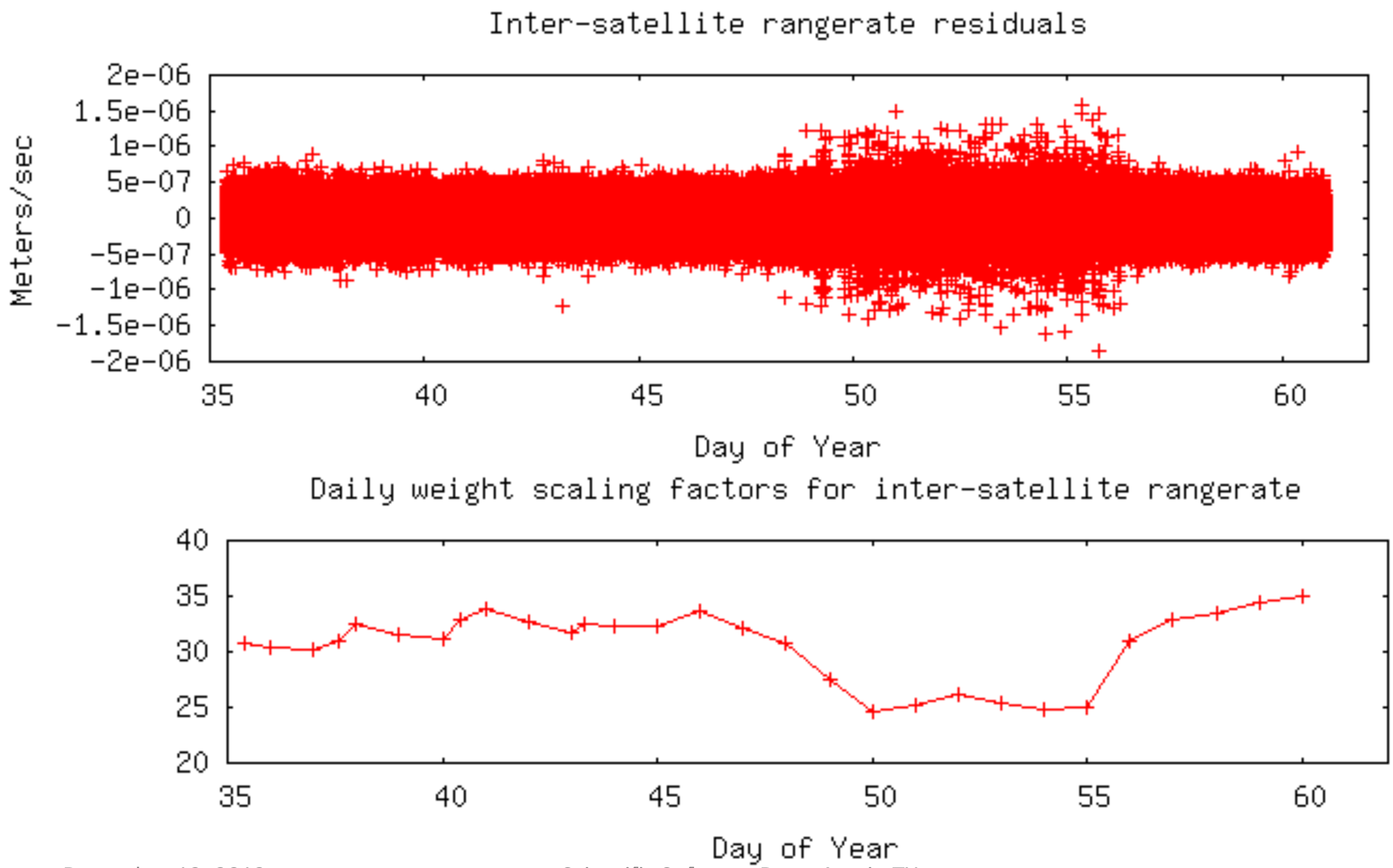
$$= \begin{bmatrix} R \\ 0 \end{bmatrix}$$

Efficiency and Speed

- Compute Efficient
  - Avoids multiplying zeros as much as possible
- Memory Efficient
  - Has only parts of matrix in memory at any one time.
- Fast Compute Speed
  - Uses a block Householder algorithm to take advantage of BLAS 3 speed
  - Up to 8 Gflops per core on lonestar using 4 nodes.

mal Weighting

Optimal Weighting



Future Gravity Models Will Be Larger

- Future missions (GRAIL, for example) will have higher precision data
- Solve for larger gravity field models
  - Degree 1000. Current monthly GRACE solution is degree 180, so a factor of ~5 increase in degree.
- Date file size would increase by a factor of ~25
  - Largest GRACE data file is currently 5Gb. So this could increase to > 100 Gb.
- Flops would increase by a factor of ~600

Weighted Least Squares

$$y = Hx + \varepsilon$$

m observations $\quad$ n estimated parameters $\quad$ $m > n$

J = weighted least squares performance index

$$= \sum_{i=1}^{m} \mathrm{w}_i \varepsilon_i^2$$

$$= (y - Hx)^T W (y - Hx)$$

Normal Equations

Obtain Normal Equations by minimizing J with respect to x

$$(H^T WH)x = (H^T Wy)$$

The solution to the Normal Equations gives the estimate for x, which we call $\hat{x}$:

$$\hat{x} = (H^T WH)^{-1}(H^T Wy)$$

Orthogonal Transformation

Insert an orthogonal matrix Q into J, where

$$Q^T Q = I$$

$$
\begin{aligned}
J &= (y - Hx)^T W (y - Hx) \\
&= (y - Hx)^T W^{1/2} W^{1/2} (y - Hx) \\
&= (y - Hx)^T W^{1/2} Q^T Q W^{1/2} (y - Hx) \\
&= \left\| Q W^{1/2} (y - Hx) \right\|^2
\end{aligned}
$$

Orthogonal Transformation

$$J = \left\| QW^{1/2}(y - Hx) \right\|^2 = \left\| QW^{1/2}(Hx - y) \right\|^2$$

Choose Q such that

$$QW^{1/2}H = \begin{bmatrix} R \\ 0 \end{bmatrix} \text{ and define } QW^{1/2}y = \begin{bmatrix} b \\ e \end{bmatrix}$$

where

R is an $n \times n$ upper - triangular matrix

Orthogonal Transformation

$$J = \left\| QW^{1/2}(Hx - y) \right\|^2 = \left\| \begin{bmatrix} R \\ 0 \end{bmatrix} x - \begin{bmatrix} b \\ e \end{bmatrix} \right\|^2$$

$$= \left\| Rx - b \right\|^2 + \left\| e \right\|^2$$

Minimize J by choosing x such that $Rx = b$. This value of x is the weighted least squares estimate $\hat{x}$.