

Winglets: Visualizing Association with Uncertainty in Multi-class Scatterplots

Min Lu, Shuaiqi Wang, Joel Lanir, Noa Fish, Yang Yue, Daniel Cohen-Or, and Hui Huang

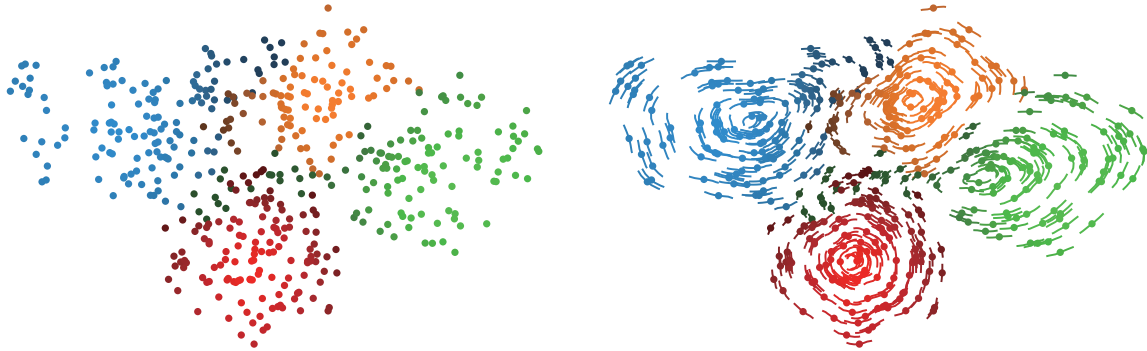


Fig. 1. (Left) A common visualization of a multi-class scatterplot uses colors to associate the points to their clusters. The uncertainty is visually encoded by darkening the colors. (Right) Little wings attached to the points have a stronger descriptive power of association based on the Gestalt principles. The wings length expresses the associating uncertainty of a point.

Abstract—This work proposes *Winglets*, an enhancement to the classic scatterplot to better perceptually pronounce multiple classes by improving the perception of association and uncertainty of points to their related cluster. Designed as a pair of dual-sided strokes belonging to a data point, *Winglets* leverage the Gestalt principle of *Closure* to shape the perception of the form of the clusters, rather than use an explicit divisive encoding. Through a subtle design of two dominant attributes, length and orientation, *Winglets* enable viewers to perform a mental completion of the clusters. A controlled user study was conducted to examine the efficiency of *Winglets* in perceiving the cluster association and the uncertainty of certain points. The results show *Winglets* form a more prominent association of points into clusters and improve the perception of associating uncertainty.

Index Terms—Scatterplot, Gestalt laws, Association, Uncertainty

1 INTRODUCTION

Scatterplots are widely used to represent objects in a dataset on two orthogonal dimensions. When the data carries a division to multiple classes (i.e., clustered), the challenge of how to express such a grouping arises [33]. Normally, standard visual cues such as different colors and shapes are utilized, creating a distinct separation between classes, and associating data points with their cluster.

Common multi-class scatterplot visualization techniques often present the discrete result of a particular clustering algorithm, and do not incorporate a visual encoding of the confidence of the underlying method in its chosen association of the data points to their clusters. Specifically, the visualization of high dimensional data is often realized with a scatterplot after performing dimension reduction, conveying the population of data points within an embedding space. It is often heuristics and parametric tuning which govern the decision made by the employed algorithm, resulting in a sensitivity to different settings, the most notable of which is the declared number of clusters. Operating within these confines, a corresponding scatterplot chart reflects the discrete, hard decision made by the algorithm.

In this work, we present an enhancement to the classic scatterplot chart to better highlight the association of points to their clusters and convey their confidence according to the clustering decision. We introduce a new means for visual encoding, which augments the geometric appearance of a data point with a dual-sided trail akin to a pair of wings (see Figure 1). *Winglets* are short strokes designed to enrich the visual information associated with a data point. The geometry of the wings (length and orientation) expresses the affiliation of a point to the other members in the cluster, and the uncertainty involved in this clustering decision.

A key attribute of *Winglets*, making them an attractive enhancement to a standard scatterplot, is that they constitute a visualization means, that suggest or hint at visual cues, rather than enforcing them. *Winglets* utilize the Gestalt principles of grouping, and specifically the Gestalt principle of *Closure* to shape the perception of the form of the clusters, avoiding an explicit divisive encoding. The strength and competence of the grouping principles of Gestalt is illustrated in Figure 3, when a colorless scatterplot is still able to deliver the underlying clustering structure with the use of winged data points. The *Closure* principle, together with other Gestalt principles of perceptual grouping such as *Good Continuation* that *Winglets* form in close proximity, aid the viewer to perform a mental completion of the clusters, assisted by the depicted level of confidence in the affiliation of the points. Specifically, the length of *Winglets* corresponds to the level of associating confidence, such that points with a high confidence are adorned with longer wings. As can be seen in Figure 1, points that reside within ambiguous, in-between regions and are, accordingly, of low association confidence to their cluster, are enhanced with shorter wings. However, the subtle hints given by these short wings, together with the Gestalt *Closure*, form a more prominent association of points into clusters.

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Winglets can be enhanced with various visual attributes, such as color, shape, length, thickness, etc., spanning a large design space. In this work, we explore two dominant attributes - length and orientation. The *Winglets* orientation of a point visually encodes the local trend of the cluster it is associated with, and the length reflects the uncertainty of its association to its cluster.

We explore and analyze the perceptual power of *Winglets* as a visual cue and evaluate their added value in a controlled user study where they are compared to a standard scatterplot that only uses color. Results of the study validate the advantage of *Winglets* in better perceiving the cluster association, and suggest *Winglet* as a useful tool for better distinguishing between clusters in multi-class scatterplots.

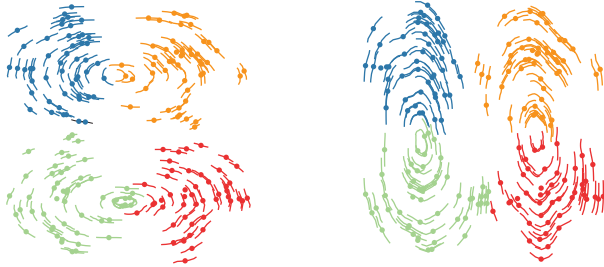


Fig. 2. The left and right scatterplots use the exact same configuration of points and coloring. The figure emphasizes the strength of *Winglets* in shaping the perception of clusters, by forming horizontal clusters (left) and vertical clusters (right) on the exact same point configuration. The two examples also show the interplay between coloring and *Winglets*.

2 WINGED DATA POINTS AS VISUAL CUES

Perceptual grouping. The perception of association of data points in multi-class scatterplots is based on the Gestalt principles of perceptual grouping [43]. The Gestalt principle of *Proximity* is of very high priority in grouping perception - points that are close to each other tend to be grouped together. In multi-class scatterplots, the position of points within the chart is highly indicative of their association, and when using dimension reduction techniques, it is usually determined by embedding algorithms such as t-SNE, MDS, and PCA.

Common multi-class scatterplots feature color coding to indicate cluster association. This approach is in line with the Gestalt principle of *Similarity* in which items that share the same visual attributes (such as color) are grouped together. Conversely, the visual cuing introduced in this paper mainly utilizes the Gestalt principle of *Closure* to create a sense of spatial association.

To demonstrate the power of *Winglets* for grouping perception, Figure 2 presents an example containing two clusters for which *Winglets* shape the perception of group association. The left and right scatterplots in Figure 2 use the exact same configuration of points and coloring, the only difference is the grouping association created by *Winglets*. As can be seen in this example, the mere use of *Winglets* entirely changes our perception of the grouping. Figure 3 shows the effect of *Winglets* in enhancing both colorless and color scatterplots. In a colorless scatterplot (top), *Winglets* can be used as an alternative visual tool to highlight clustering without the need to use colors. In a colored scatterplot (bottom), *Winglets* are able to enhance the current structure and give a clearer sense of the grouping.

The closure principle, realized here by the added *Winglets*, shares ideas with the enclosure technique [5] (see Figure 4) that makes use of contours to indicate grouping by locating points in common regions [29]. The key difference lies in the nature of these contours, which are global and indicate a fixed, discrete division of clusters that does not express individual association. Our added wings, on the other hand, are local additions that convey the information of which cluster the point is associated with. For example, the association of points in the joint region of contours can be ambiguous, while with the addition of wings, albeit very short ones, their association could still be clearly perceived.

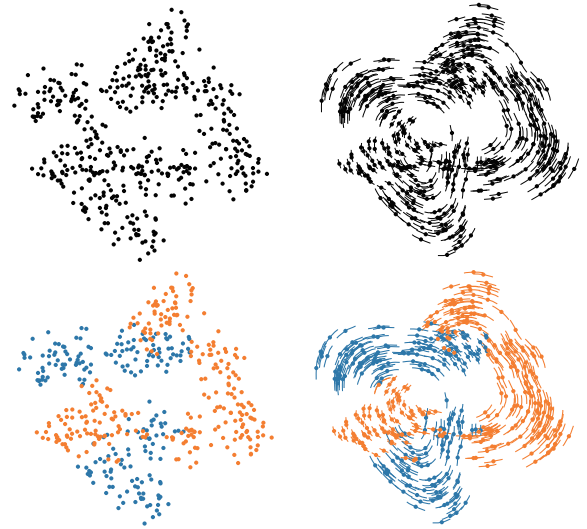


Fig. 3. An example showing how *Winglets* can enhance the grouping perception in a scatterplot, both with and without colors

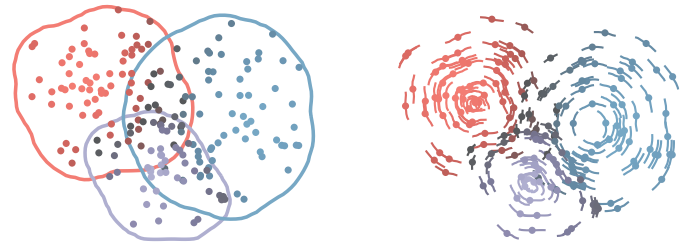


Fig. 4. The enclosure technique draws contours to visually encode hard association's boundaries. *Winglets* suggest the association in a local soft manner, with fine-grain depiction on association, especially removing the associating ambiguity of points in the joint region.

Uncertainty. Typical clustering algorithms associate each point with one cluster. However, although the algorithm makes a discrete final choice, the uncertainty of the association is not uniform throughout. There are various measures to assess the confidence of the association of a point to its cluster, independently of the particular clustering algorithm that was used. In our work, we express the association uncertainty using the *Silhouette Index* [32], which measures how similar a point is to points in its own cluster vs. points in the other clusters.

A common means to visually encode the silhouette value of a point is to modulate its lightness value, making uncertain points darker (lower lightness of their color). Typically, points residing within the overlap between two clusters have a lower certainty. However, not every overlap in the embedding space is necessarily a true overlap in the original high-dimensional space. This is demonstrated in Figure 5, where the green cluster overlaps with two other clusters in the embedding space, but not in the original primal space. The blue and orange clusters, on the other hand, do overlap, and the points in the overlap have a low silhouette indexing.

To visually encode the uncertainty using winged data points, we modulate stroke length, where shorter strokes express a higher uncertainty. However, even significantly uncertain points are decorated with a minimal set of wings which hints at their distinct association. This soft association of points with uncertainty is a unique feature of this visual encoding. *Winglets* help to perceive the global structure of the clusters, despite the uncertainty and overlap. In Figure 5, note the manner in which without the *Winglets*, the dark points in the overlap seem cluttered, while with the addition of *Winglets*, albeit very short ones, they could be perceived as an integral part of a corresponding association.

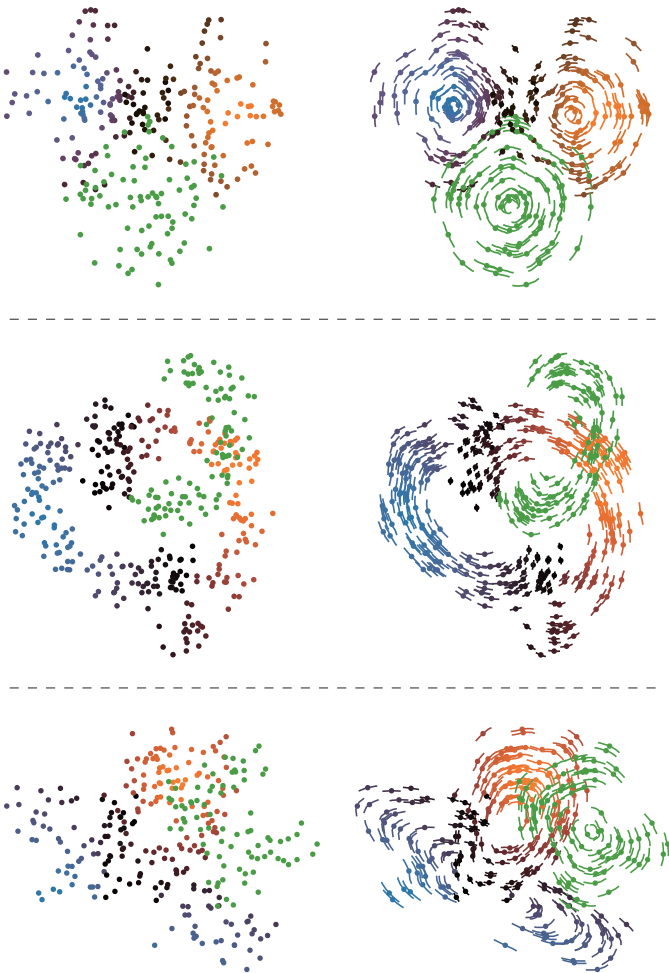


Fig. 5. In these examples, the Green cluster does not overlap with the two clusters in the original high dimensional space, while the Orange and Blue ones do. The wings help to perceive the global structure of the clusters, despite the uncertainty and overlap.

3 RELATED WORK

In Section 2 we discuss multi-class scatterplot visualization in relation to the perception of groupings of items. In this section, we follow this approach and review scatterplot-like related work in the perspective of the Gestalt principles of grouping.

Similarity principle. Many scatterplot techniques are based on the *Gestalt Similarity Principle*, where points with similar visual encoding are perceived as a group. Color and shape are the most widely used visual encoding [42]. Color was shown to have better performance in some visual aggregation tasks, such as perceiving the average value in multiclass scatterplots [16]. As the data scales and overlap increases, research works optimize the coloring strategies to maintain the separability among classes. For example, to improve the visibility of different overlapping graphical objects, Luboschik et al. [25] present a color weaving technique to define the color interlacing pattern in the overlap region. Wang et al. [41] optimize the color assignment strategy for better class separation. Stone et al. [37] study how color discriminability changes as a function of the symbol size. Some other studies focus on specific aspects related to color. Li et al. [23] study the lightness in the discrimination tasks and model the discriminability scales of lightness. Matejka et al. [26] identify the mean opacity of utilized pixels to discern the underlying structure of the scatterplot. Still, the number of colors or shapes that can be used effectively to distinguish symbols is quite limited. Colors are reported to have a fidelity of around 12 distinct hues only [1]. In our work,

we present *Winglets* as a geometric visual technique, which enhances multi-class perception as an orthogonal design to color and shape. In some cases, like the colorless example in Figure 3, *Winglets* could even alleviate the demand on color or shape discriminability.

Proximity principle. Another branch of research focuses on optimizing the point layout in a scatterplot, aiming to bring points of a group closer to each other according to the *Gestalt Principle of Proximity*. In a typical scatterplot where data are displayed in a Cartesian coordinates of two variables, layout is less flexible because positions are intrinsically determined. Some work focus on proposing proximity-based metrics to evaluate the quality of class separation in scatterplot. For example, Sips et al. [36] propose distance-based Class Consistency to evaluate the difference between scatterplot in subspace to the distribution in the origin high dimensional dataset, and then select the good subspace views. Tatu et al. [38] propose perceptual quality metrics for cluster separation as means for selecting the best scatterplot visualization. To alleviate the visual complexity and increase the structure awareness, some works perform coordinate distortion and down-sampling. Keim et al. [20] present an overlap-free distortion technique to generate the best-possible view of a scatterplot. Chen et al. [9] propose a hierarchical multi-class down sampling method that maintains the features, such as clusters, outliers, etc.

When the dataset to be displayed is in high dimensions, researchers utilize the *Proximity Principle* more by using dimension reduction techniques. For example, Principal Component Analysis (PCA) embeds the points into a linear subspace of lower dimensionality [19]. Multidimensional Scaling (MDS) [12] adapts scaling cost function from the original space to output coordinate matrix and compute the matrix which minimizes the cost function. t-SNE [13] minimizes the divergence between the distributions measuring pairwise similarities of input and the low-dimensional points in the embedding, which is usually used to recover well-separated clusters. For different dimension reduction techniques, Sedlmair et al. [35] presents an empirical user experiment to study their combination with scatterplot visualization techniques in terms of the class separability. *Winglets* are proposed as a design tool orthogonal to various layout techniques.

Continuity principle. A class of scatterplot techniques follow the *Gestalt Principle of Continuity*. The idea is to show trends in scatterplots, based on the human visual tendency to organize symbols following an established direction. One application of continuity is to show trends in scatterplots, by arranging symbols in the same way along lines or curves. Chan et al. [6] draw small lines to points which depict the local partial derivatives (i.e., sensitivity) from one variable to the other. In [7], they extend the sensitivity computation to a higher dimension space, exposing a relationship in a third dimension which would otherwise be hidden. In [8], they further present a 3D interactive scatterplot cube. The above tangent lines tracing how one variable changed to another demonstrate the potential of indicating the global trend by using local strokes. In our work, we further explore the potential benefit of the Gestalt *Closure* principle, exploring and leveraging its ability in the visual representation of association of the data points.

Common region principle. Some other scatterplot techniques locate points within a common region [29], such as by boundary with which groups can be perceived together. Collins et al. [11] designed Bubble Sets which enclose the points of a set. Splatterplot [27] fence the main distribution of a class with contours while retaining the visibility of outliers. Jo et al. [18] summarize the design space of aggregated multiclass maps and present a declarative grammar to construct the aggregation. Our method shares the idea of enclosure but uses small local strokes rather than a global common region to suggest the enclosure.

Common fate principle. Besides the static scatterplot improvements for multiple class interpretation, some works propose dynamic solutions based on the *Gestalt Principle of Common Fate*. The idea is to animate points of a group with a coordinated motion so that they are perceived with a strong association. Wang et al. [40] introduce non-linear transitions for clustered data over time to improve class tracking

accuracy. Flicker synchrony can be viewed as a special case of common fate, when points stay stationary but with on-and-off dynamic patterns. Chen et al. [10] verify that per-class flickering can improve the interpretation of multiple classes in Scatterplot Matrices with overlapping points.

Combining multiple gestalt principles. When multiple Gestalt principles are applied at the same time in one visualization, there is a joint and sometimes competing perceptual effect [43]. Kubovy et al. [21] investigate the effects of similarity and proximity principles and find that when the two are working together, they have the same effect as when each one is used separately. Feldman et al. [15] propose a grouping approach based on a Gestalt Principles named Minimal Model theory, aimed at grouping objects in a logic way. Desolneux et al. [14] quantify distinctive Gestalt rules and utilize them for detecting collinearity, regularity and proximity in images. This work focuses more on studying how *Winglets* strengthen the perception of association and uncertainty with color [4].

4 DESIGN CHOICES AND *Winglets* CONSTRUCTION

Winglets are defined as the stroke/curve attached to both sides of points in scatterplots. *Winglets* can have various design variables, such as length, color, or thickness, however, in this work, we focus only on the core expressiveness of *Winglets*, i.e., orientation and length (see Figure 6). The orientation of the wings conforms to a global form of the respective cluster, depicting the associating relationship among the points. The length of the wings, i.e., the extent of wings along their orientation, is driven by the associating uncertainty of the data point. The shorter the wings, the less certain the point is associated with the other points in its cluster.

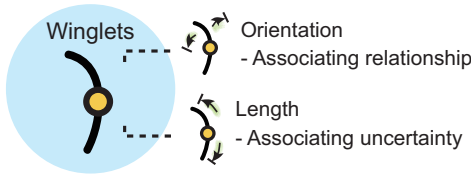


Fig. 6. *Winglets* encode the association and its uncertainty via its orientation and length.

4.1 Wing Orientation

We considered several design choices for the orientation of the wings, where they conform to some sort of global form in order to indicate their associations (see Figure 8). In the figure, we show three examples of multi-class scatterplots (each one presented in a separate row). For each example, we show how it is represented according to four different possible design choices (each design choice is shown in one column).

The design choices can be categorized into two general types: *Opened* and *Closed*, according to the conformed global form.

Open. In open form, *Winglets* conform to an acyclic global form. The two leftmost columns in Figure 8 depict two typical examples of open orientation: (i) *Centroid*, where the line segments are pointing towards a common point (the cluster's centroid) and (ii) *Directional*, where the line segments share a common direction which follows a representative direction of the cluster.

Note that the *Centroid* representation does not seem to represent the clusters well in example (a) (the one highlighted with dashed line box), in which the clusters are crescent shaped. Likewise, it would be difficult to distinguish between the clusters using the *Centroid* representation when the clusters have centroids that are near each other as in example (c) (notice that the two clusters in this example have nearby centroids because of the outliers). In the *Directional* representation, notice that in (b), the represented directed lines fail to represent the tendencies of the cluster, and the line segment grouping is ambiguous.

Closed. In closed form, the wings hint at their constituent point association with a global cyclic structure. The two rightmost columns

in Figure 8 depict two possible closed orientation choices. *Boundary Circle* is the circumscribed circle that fences the points of a class. *Contour* is the outline enclosing shape of points. Compared with *Contour*, *Boundary Circle* is more simplified and regulated, increasing the possibility for different classes to have similar global forms.

Looking at example (c) in Figure 8, we can see that in the *Boundary Circle* representation, there seems no distinction between the two classes, since, due to outliers, their boundary circles are almost the same. This is corrected in the *Contour* case, where the two classes could be easily distinguishable. Taking this into account, we favor the usage of *Contour* in *Winglets*.

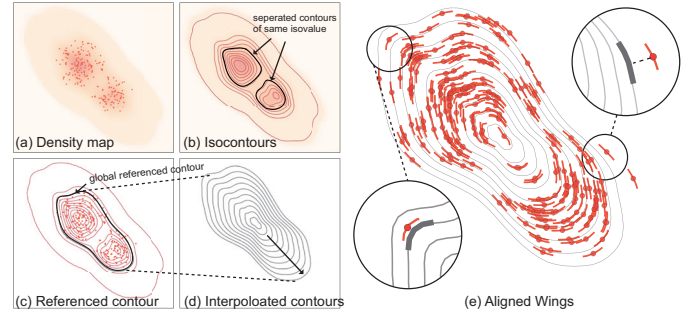


Fig. 7. Orienting Procedure: (a) Gaussian kernel density map is calculated for the plot. (b) isocontours of sampled densities are extracted by Marching Squares algorithm. (c) a global reference contour is picked for the coherent perception of grouping, before splitting into multiple contour siblings (annotated in (b)). (d) contours are interpolated from outside to inside. (e) points grow their wings along the orientation as their nearest points on the contours.

Figure 7 illustrates our *Winglets* orienting method. First, for each class, its density field is computed by kernel density estimation (KDE) [31] on a structured grid of 100x100 size. Gaussian kernel is used, with an automatic bandwidth determination based on Scott's Rule [34]. All of the points have equal weights in the density estimation. Next, isocontours are generated in the 2D grid according to a list of density values. Those density values are equidistantly sampled from low to high. For each density value (i.e., isovalue), Marching Squares algorithm [24] is applied to interpolate contour lines through the grid corresponding to this value. It is not uncommon for points distributed among several dense clouds but sharing the common isovalues, as a result, there can be separated multiple contour siblings (e.g., the two main clouds in Figure 7). To ensure a coherent perception of closure, a common global referenced contour needs to be determined. Therefore the third step is to determine the reference contour. The global contour should enclose as many points of the class as possible, but should also be free from extreme outliers that significantly influence and alter the contour. Our strategy is to trace isocontours from low density to high (i.e., outside to inside), and halt at a significant drop in the magnitude of contained points, picking the contour right before it. The drop is heuristically set to 5%. With the reference contour, contours are smoothly interpolated towards its centroid center. Lastly, each point is aligned to its nearest point on the contour and *Winglets* grow in the same orientation as the nearest point on the contour.

4.2 Wing Length

The uncertainty attributed to the association of a point to a cluster is mapped to the length of *Winglets*: $\tau : S \mapsto L$, inducing the consideration of two main design variables: range and mapping function.

Range. The choice of *Winglets* length range scales *Winglets* from conventional scatterplot dots to full contour enclosures (Figure 9(top)). On one extreme end, winged points are degraded to dots when wing length is minimal. On the other end, as wing length increases, strokes connect to one another and form full contours. Range selection must consider the trade-off between association indication and visual clutter minimization. Longer wings provide stronger cues to the global form

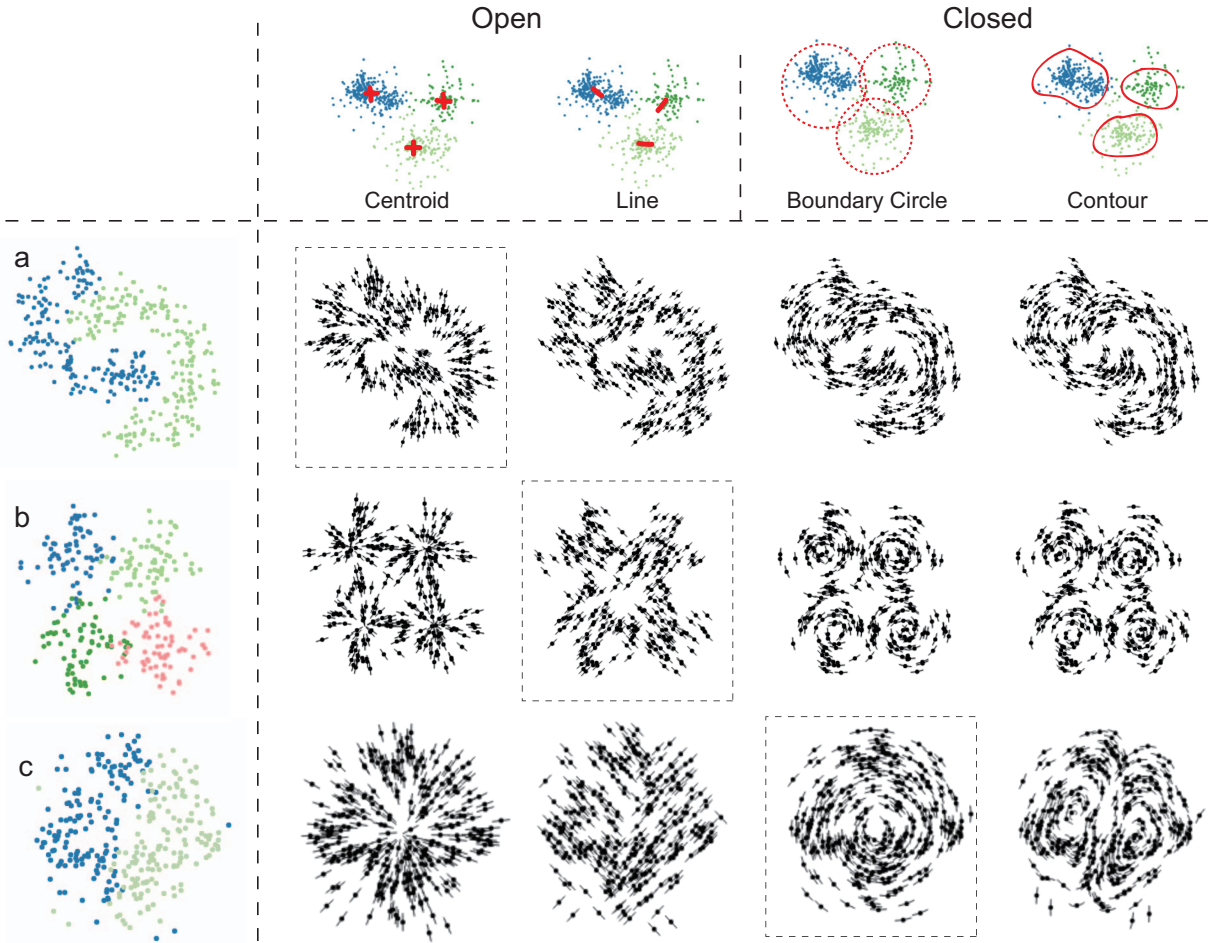


Fig. 8. Orientation Choice of *Winglets*: there are two main types of wings' orientation according to whether the global form is open or closed. Compared with Contour, orienting towards Centroid, Line or Boundary Circle may fail in some cases (marked by dashed line boxes).

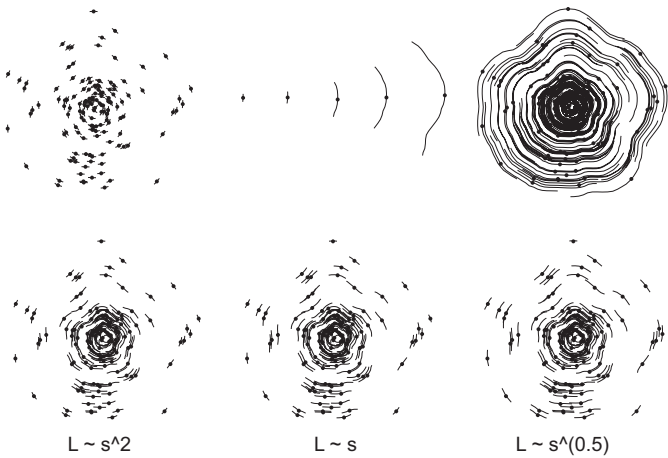


Fig. 9. Length of *Winglets* is encoded to the confidence a point is associated to its cluster. *Winglets* can vary with different choices of length (top) and mapping function (bottom).

but at the cost of increased visual clutter. Conversely, when wings are made too short, association perception is compromised. In this work, we heuristically set the minimum bound for *Winglets* range to be two pixels wider than the diameter of the point, to ensure wing visibility.

Mapping Function. This function maps the degree of uncertainty

to the length of the *Winglets*. There exist various choices for this function serving different visual expressive goals. Generally, it can be defined as $l(i) = a * s_i^n + b$, where s_i is the Silhouette Index of point i , $s_i \in [0, 1]$. When $n > 1$ (Figure 9 left), wing length is shortened rapidly as uncertainty increases. When $n < 1$, the length difference between high and low uncertain points becomes smaller, which might therefore be used when we seek to emphasize association relationship. Whichever mapping function is chosen, it ought to be uniform among clusters within a scatterplot. In this work, we choose the modest $n = 1$.

5 APPLICATIONS

This section presents two applications of *Winglets*. In the first, we apply *Winglets* on a real dataset, and in the second, we demonstrate the design choices of *Winglets* with other scatterplot visualization means. All examples are implemented in JavaScript for rendering, and Python for computation. KDE is accomplished with the third-party python library SciPy and isocontour extraction with scikit-image.

MNIST Dataset. We make use of a subset of the MNIST (Modified National Institute of Standards and Technology) dataset to demonstrate the application of *Winglets* to real data for the purpose of cluster indication. In MNIST, each instance is an image of a handwritten digit, thus it contains 10 classes, one per 0-9 digit. Each image instance is of resolution 28x28, amounting to 784 high-level dimensions. We randomly select 1200 instances to form the experiment dataset.

Figure 10 (top) plots the data using t-SNE embedding, where the colors are picked from ColorBrewer [3]. In the projection, some classes are embedded separately from others and can be easily perceived as distinct clusters, e.g., the light blue cluster at the bottom

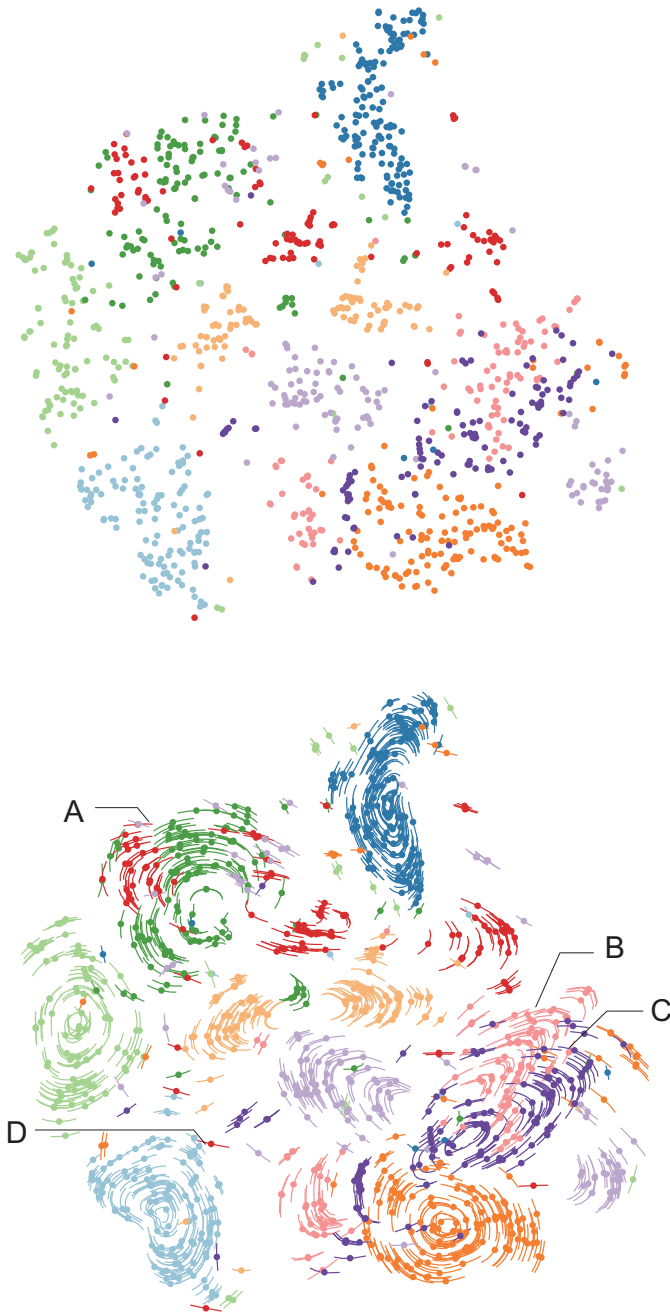


Fig. 10. *Winglets* in MNIST: 1200 samples are embedded in a 2D plot using t-SNE projection. Each class is assigned a unique color; with *Winglets*, associations are perceptually more pronounced when clusters are broken into parts (A), or are overlapping with one another (B, C). *Winglets* convey the association uncertainty, such as high association certainty points situated far away from the majority of their cluster (D).

left, or the blue class at the top. However, most classes are less distinguishable, either broken into several parts (e.g., the red class), or overlapping with others (e.g., the pink and purple classes).

In Figure 10 (bottom), the plot is enhanced with *Winglets*. We note that the association is perceptually clearer, and discerning full clusters is made easier. Points within a cluster are better integrated visually, even when they are situated far away from one another (e.g., the red class (A)). Overlapping clusters can also be easily perceived, e.g., the pink (B) and purple (C) classes.

Another important visual cue given by *Winglets* is the degree of association uncertainty. Dimensionality reduction of inherently high

dimensional data results in an oftentimes false impression of stronger association between close-by, rather than far away, points, in the projected space. By encoding the association uncertainty (e.g., Silhouette Index computed based on Euclidean distance in the original space) to wing length, *Winglets* indicate the true degree of association of points to their clusters. For example, point (D) is located far away from the majority of the points associated with its cluster, but its long wings convey the high certainty with which it actually associates to the cluster.

Winglets Joint Design. In this example, we combine *Winglets* with other design tools in a scatterplot. In Figure 11 (a), color is added as an enhancement to *Winglets*. Without color or with weak color separation (b), *Winglets* indicate the association to some extent. In scatterplots with high visual clutter, *Winglets* can be added after data down-sampling (e.g., reducing density of green dots) with an added density map to compensate for the loss of information (c). In another case, aggregation is performed to simplify the scatterplot (d). Following Splatplots [28], outliers outside the aggregation region are enhanced with *Winglets*, while within the region, *Winglets* are maintained to show the point distribution. Note that in this work we do not expand on the design possibility of *Winglets*, but focus more on the idea of local strokes to indicate association according to the Gestalt *Closure* principle.

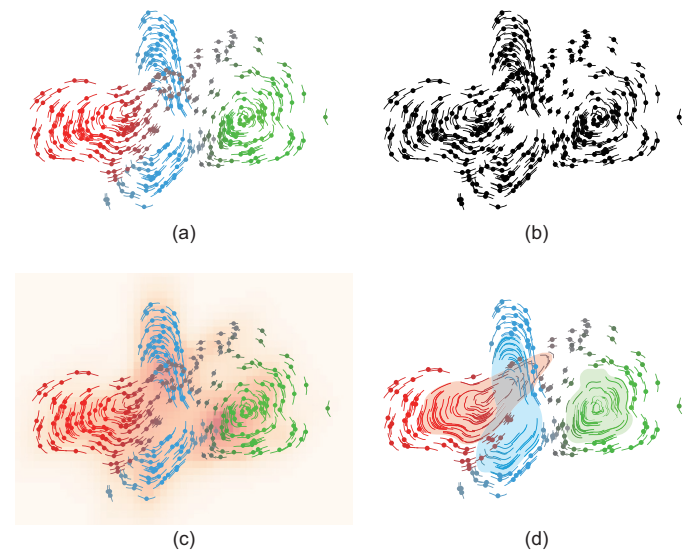


Fig. 11. *Winglet* joint use with other design tools in a scatterplot: (a) *Winglets* with color; (b) colorless *Winglets*; (c) *Winglets* with a density map to compensate for down-sampling; (d) *Winglets* with aggregation.

6 EVALUATION

We evaluate the performance of *Winglets* and their contribution to the perception of association and uncertainty in a formal user study. We conducted a controlled user study in which participants performed 4 tasks with both regular scatterplots and ones enhanced with *Winglets* (See Figure 1). The goal of the study was to examine whether *Winglets* enhance the perception of association and uncertainty beyond that of a regular scatterplot. To better understand the effect of *Winglets* given different scatterplot complexity levels, we examined two independent variables that might affect the way users perceive *Winglets*: the number of clusters in a given scatterplot, and the amount of cluster overlap.

6.1 Methodology

The main independent variable explored in the study was with/without *Winglets*. The experimental design was a 2x3x3 within-subject design with three main independent variables: *Winglets* (with, without), amount of *Clusters* (3, 5 and 8), and the amount of *Overlap* between clusters (low, medium, high). Figure 12 by columns shows examples of scatterplots with different cluster levels and Figure 12 by rows

shows examples of different overlap levels as examined in the study. Note that in the figures the scatterplots are aligned. In the experiment, a random rotation was performed to each of the scatterplots.

In addition, we examined four different user tasks. Because of the complexity of the experimental design, we analyze and report on the result of each task separately, and thus, do not include the tasks as an independent variable in our analysis.

Participants. We recruited 44 participants from a local university (16 Females) with an average age of 23.6 ($SD = 2.4$). All were science engineering students mostly at the post-graduate level. 17 participants reported no prior visualization knowledge at all, 22 participants reported minimal knowledge, and only 5 participants reported having a medium to high level of knowledge in visualization. Participants were rewarded with a monetary sum for their participation. All participants gave informed consent and the study conformed to the ethics procedure of our university.

Materials. We prepared two similar, but not identical, scatterplots for each level of Cluster X Overlap (9 levels). Half of these were left as regular scatterplots, while the other half were enhanced with *Winglets*. The use of colors in both conditions was the same - we chose a qualitative color scheme from Colorbrewer [3] to distinguish between the classes. Within a class, we used lightness to encode the uncertainty. The shortest wing length was adjusted to the minimal noticeable length, as explained above.

Tasks. We used four different tasks to evaluate the performance of *Winglets*:

T1. How many clusters are there in the graph?

T2. To which cluster does a given point belong to?

T3. Given two clusters, which cluster has a larger overall uncertainty?

T4. Given two points [of the same cluster], which point belongs to the cluster with a higher certainty?

Tasks are divided according to two dimensions, as listed in Figure 13. The first dimension is the *Content*, examining either *Association* perception (tasks 1 and 2), i.e., whether a point can be easily associated with its group, or *Uncertainty* perception (tasks 3 and 4), i.e., how participants perceive the association uncertainty of a given point. The second dimension examines the *Scope* of the task. That is, the level of perception of the scatterplot - either *Global* (Tasks 1 and 3), i.e., the entire scatterplot, or *Local* (tasks 2 and 4), i.e., a single point.

Measurements. For each test, we measured performance in terms of *Completion Time* and *Accuracy* (error rate).

Procedure. The experiment was performed one participant at a time. Each participant was seated in a quiet room in front of a 24-inch display screen. The experiment was divided into three parts: the introductory part, the main experiment and the followup. In the introductory part, the administrator first briefly informed the participants about the experiment structure and collected their demographic information. Next, the administrator introduced the experiment, showed some scatterplot examples, and explained the important concepts such as point association uncertainty, the average and overall certainty of a group (the sum of uncertainties of a group's points), etc. Participants were encouraged to ask any questions they may have, and were then asked to complete a series of 8 practice trials, in order to let them familiarize themselves with the tasks, and to minimize possible learning impediments. Example and practice trials used different scatterplots than were later used in the experimental trials.

The main part of the experiment included a series of trials, each containing the task question followed by the corresponding scatterplot. For each trial, the question was first shown on a front-up page and was not time restricted, such that the participant was free to press "next" at their leisure, in order to reveal the scatterplot at the center of the screen, along with multiple choice answers to the task question. Participants were instructed to select the best answers to the given questions, and to perform their tasks as quickly and accurately as possible. Completion time, from the moment the scatterplot was presented until

a selection was made, as well as the selected answer, were automatically recorded. At the end of the trials, the experiment administrator interviewed participants as per their personal preference among the two types of graphs.

We assigned each participant to perform only two tasks. We randomly divided participants into two groups, 22 in each. The first group completed tasks 1 and 3, while the second group completed tasks 2 and 4. Overall, each participant completed 36 trials: 3 (number of clusters) \times 3 (overlap level) \times 2 (with/without wings) \times 2 (tasks). The order of the trials was randomized for each participant.

Analysis. We conducted a 3-way analysis of variance (ANOVA) for each task on *completion time*, with *Winglets* (with/without), cluster quantity (3, 5, 8) and overlap (low, medium, high) being within-subject variables. In addition to reporting statistical significance, we report effect size, partial eta-squared η^2 , which is a measure of the magnitude of the effect of a difference that is independent of sample size. Landauer [22] notes that effect size is often more appropriate than statistical significance in applied research. The metric for interpreting eta-squared is: 0.01 is a small effect, 0.06 is medium, and 0.14 is large.

6.2 Results

Figure 14 summarizes the results comparing the average completion time with and without *Winglets* for the different cluster quantities and overlap levels over the four tasks.

As expected, for all tasks, a main effect was found for both cluster quantity and overlap in which an increase in quantity and/or overlap is directly proportional to the amount of time participants spent on completing the trial. Next, we report the statistical results per task separately.

Task 1. Task 1 asked to detect how many clusters there were in each presented scatterplot. Results indicate a strong main effect for *Winglets*, in which performance with *Winglets* ($M=6.36$, $SD=3.28$) were faster than performance without ($M=8.81$, $SD=5.33$), $F(1,21) = 44.90$; $p < 0.001$; $\eta^2 = .681$. In addition, a main effect was found for both Cluster and Overlap, in which the more clusters there were, the more time it took participants $F(2,42)=81.6$; $p < 0.001$; $\eta^2 = .795$, and the more overlap there was, the higher it took participants, $F(2,42)=33.16$; $p < 0.001$, $\eta^2 = .612$;

Interestingly, an interaction effect was found between *Winglets* and Cluster, $F(2,42)=4.307$; $p=0.020$; $\eta^2 = .17$. An examination of differences between the completion times with and without *Winglets* indicates that the difference gets larger as there are more clusters. That is, the interaction effect suggests that the more clusters there are, the stronger the effect that *Winglets* have (in improving detection time over non-*Winglets*). An interaction effect was also found between *Winglets* and Overlap level, $F(2,42)=3.88$; $p=0.028$; $\eta^2=.156$, suggesting that the more overlap there is, the stronger the effect that *Winglets* have. Finally, the 3-way interaction was not significant, $F(4,84)=2.13$, $p=.08$.

Looking at error rate, task 1 was relatively simple and there were almost no cases of error in this task (only three instances, two with and one without *Winglets*).

Task 2. Task 2 asked to which cluster does a given point belong to. Looking at the main effect, the with-*Winglets* condition ($M=7.73$, $SD=3.89$) was found to be significantly faster than without-*Winglets* ($M=11.01$, $SD=6.58$), $F(1,21) = 27.8$, $p < .001$; $\eta^2 = .57$. In addition, a main effect was found for Cluster number, $F(2,42)=21.4$; $p < 0.001$; $\eta^2 = .501$, and for Overlap $F(2,42)=9.58$; $p < 0.001$; $\eta^2 = .313$

Looking at the interaction effects, an interaction effect for *Winglets* X Cluster was found, $F(2,42) = 3.25$, $p=.049$; $\eta^2=.134$. No interaction effect was found for *Winglets* X Overlap ($p > .6$) or for the three-way interaction ($p > .1$).

Looking at accuracy, there was a large difference in error rate between the two conditions. While the without-*Winglets* condition had a total of 42 errors (out of 198 trials adding up to a 21.2% error rate) the with-*Winglets* condition only had 7 errors (3.5%). Results of a paired t-test on the total correct values for each participant showed that these differences were significant, $t(21) = -5.20$, $p < .001$.

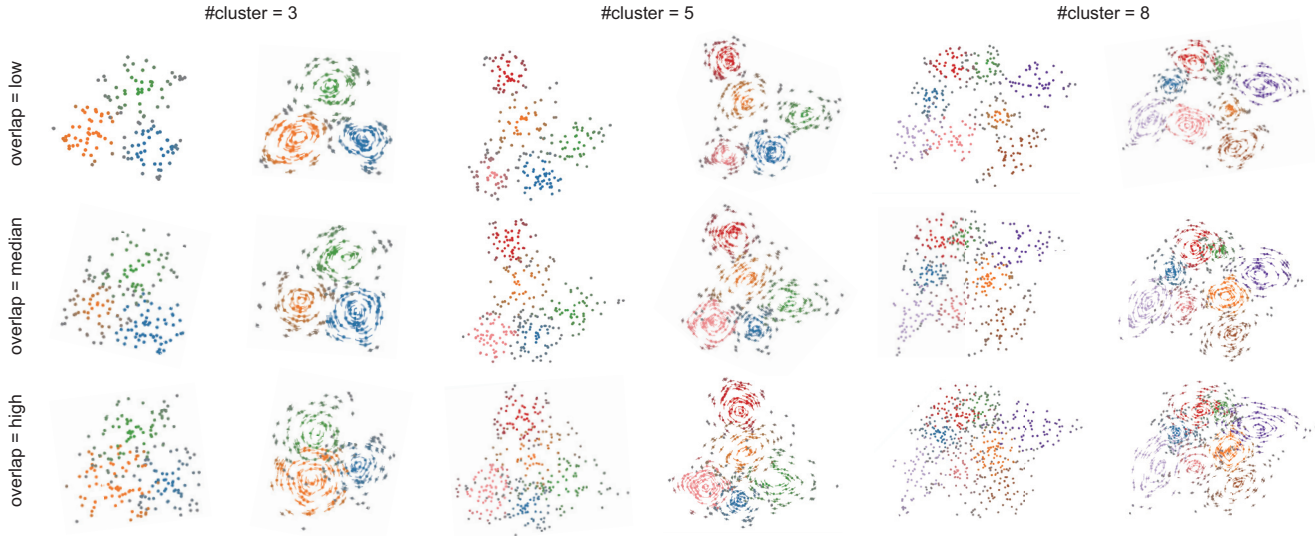


Fig. 12. Cluster and Overlap conditions: three options for the amount of clusters with and without *Winglets*; three levels of overlap with and without *Winglets*. Note that in the figures the scatterplots are aligned. In the experiment, a random rotation was performed to vary each of the scatterplots.

Content	Scope	Tasks
Association	Global	T1. How many clusters are there in the graph?
	Local	T2. To which cluster does a given point belong to?
Uncertainty	Global	T3. Given two clusters, which cluster has a larger average overall uncertainty?
	Local	T4. Given two points, which point belongs to the cluster with a higher certainty?

Fig. 13. Tasks used in the study. Tasks are divided according to two dimensions: *Content* and *Scope*

Task 3. Task 3 asked to determine which cluster had an overall larger uncertainty. While with-*Winglets* ($M=8.94$, $SD=6.00$) was overall faster than without-*Winglets* ($M=9.75$, $SD=5.08$), the differences for the main effect of *Winglets* were not significant in Task 3, $F(1,21) = 2.82$, $p>.1$. As expected, a main effect was found for Cluster number, $F(2,42)=31.2$, $p<0.001$; $\eta^2 = .598$, and for Overlap level, $F(2,42)=14.9$, $p<0.001$; $\eta^2 = .416$.

An interaction effect was found between *Winglets* and Cluster, $F(2,42) = 6.95$; $p=.002$; $\eta^2 = .249$. To examine this interaction more carefully, we compared the with and without conditions for each Cluster number separately. Results indicate that while for 3 clusters, without-*Winglets* was faster than with-*Winglets* ($F(1,21)=9.58$; $p=.005$; $\eta^2=.313$), for 5 ($F(1,21)=19.2$; $p<.001$; $\eta^2=.47$) and 8 clusters ($F(1,21)=11.6$; $p=.003$; $\eta^2=.35$), this was the other way around, with a stronger increase in difference for the 8 clusters, suggesting that the more clusters there are, the stronger the impact of *Winglets*. No interaction was found between *Winglets* and overlap ($p>.17$) and no three-way interaction was found ($p>.1$).

Looking at error rate, there were overall 34 errors in the no-*Winglet* condition compared to 24 errors in the with-*Winglets* condition. Results of a paired t-test showed that these differences were not statistically significant, $t(21) = -1.60$, $p=.13$.

Task 4. Finally, task 4 asked given two points, which belongs to its cluster with a higher uncertainty. No significant main effect for *Winglets* was found for task 4 although with *Winglets* ($M=7.51$, $SD=3.94$) were slightly faster than without-*Winglets* ($M=8.04$, $SD=3.41$). As before, a main effect was found for Clus-

ter number, $F(2,42)=8.35$, $p=.001$; $\eta^2 = .285$ and for Overlap level, $F(2,42)=2.08$, $p=0.01$; $\eta^2 = .813$.

Looking at the interactions, no interaction effects were found ($p>.17$).

Looking at accuracy, there were overall 6 errors in the with-*Winglets* condition (3.0%) vs. 12 overall errors in the without-*Winglets* condition (6.1%). These differences were not significant, $t(21) = -1.37$, $p=.19$.

Preference. At the end of each experimental session, we performed open-ended interviews to gather individual opinions and preferences with respect to the two conditions. Overall, 40 out of the 44 participants preferred *Winglets* over the alternative. Three participants favored the scatterplot without *Winglets*, and one participant remained neutral. Generally, participants liked *Winglets*, and thought it enhances and organizes the scatterplot better.

When asked to elaborate, most participants praised *Winglets* for its clear association representation. Many commented that *Winglets* help to bind the classes, alleviating cluster identification, especially when clusters are highly overlapping. As one participant commented: ‘I can better see the outline of a cluster’. Some participants pointed out that with color alone, it is difficult to distinguish between classes in overlap regions. Others commented that the lightness of the color interferes with the judgement of color hue, for example, dark orange and brown. For the uncertainty analysis tasks (3 and 4), several participants commented that with *Winglets*, it is significantly easier to perceive the association uncertainty according to wing length, than only by color. As one participant said: ‘I felt more confident with my answer with *Winglets*’. Conversely, a couple of participants felt that *Winglets* may induce visual clutter, especially when the scatterplot is dense. Another participant commented that *Winglets* may make it more difficult to distinguish a single point.

6.3 Summary

The results of the user study clearly show the perceptual benefits of *Winglets*. The addition of *Winglets* shortened the overall task completion time and reduced the overall error count. As can be seen in Figure 14, the perceptual benefits of *Winglets* over the standard scatterplot were identified across tasks, cluster quantity and overlap level. Specifically, the association tasks (tasks 1 and 2) showed a statistically significant improvement with *Winglets*. In addition, there were less errors using *Winglets* and overall, participants preferred *Winglets* over the non-*Winglets* alternative.

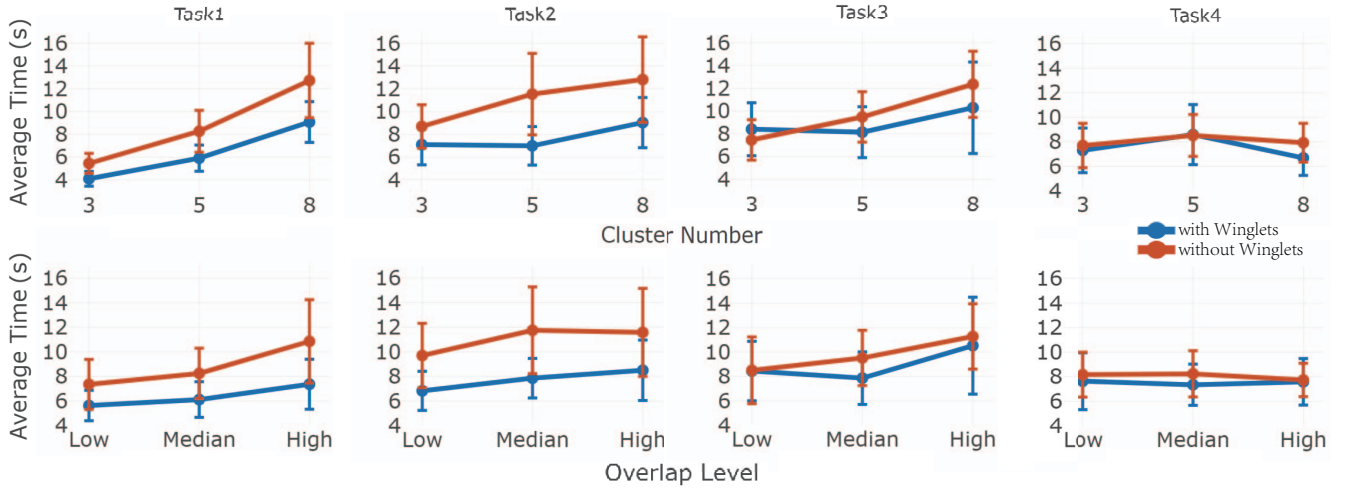


Fig. 14. Time cost with and without *Winglets* averaged within the three options of cluster quantity (top) and the three levels of overlap (bottom) for the four tasks, with the error bars of standard deviation.



Fig. 15. Error percentage with and without *Winglets* by task.

Our results show a significant advantage for *Winglets* over the regular scatterplot alternative in the association tasks (tasks 1 and 2), indicating that they promote faster and clearer association of clusters. The results also indicate that for the global tasks (1 and 3), the cluster quantity and overlap levels are directly proportional to the prominence of the advantage of *Winglets*. This can be attributed to the global nature of the task which calls for scrutinization of the entire graph, thus leaving more room for *Winglets* to shape and guide viewer perception in visually complex scatterplots.

A limitation of the study is using a single scatterplot for each of the 3x3 (cluster x overlap) conditions. This may cause conditional differences to be related to idiosyncrasies that occur between the single plots in each condition. While this has a lesser effect on the main effect of *Winglets* which are examined across the other independent variables, it may limit our results pertaining to the interaction effects. Future studies should conduct a more thorough evaluation, having multiple different scatterplots in each condition.

7 CONCLUSION

In this paper we proposed *Winglets* as a means to guide and direct the perception of clusters in a multi-class scatterplot. *Winglets* utilize the power of the Gestalt grouping principles, and specifically, the Gestalt principle of *Closure*. Designed with two dominant variables, orientation and length, *Winglets* provide a visual indication of the association of points to clusters, as well as the uncertainty involved in this association. Our evaluation shows that *Winglets* enhance viewer perception of multi-class scatterplots in terms of association, and suggests that this effect might be enhanced under increased graph complexity or clutter.

The addition of *Winglets* necessarily increases “ink” usage which induces more visual clutter than is normally observed within a plain scatterplot. To alleviate the clutter when scaling up, *Winglets* can be combined with scatterplot simplification methods, such as aggregation [17, 28], multi-class down-sampling [7], etc. These methods can be applied to the content for simplification, prior to proceeding with *Winglets* for enhancement. In Section 5, we demonstrate the joint

usage of *Winglets* with other scatterplot means. One interesting derivative is drawing *Winglets* alone within the aggregated shapes (Figure 11 (d)), where the thin wings complement the aggregated shapes with finer information of point distribution, incurring considerably less visual clutter than the original dots. Other means of joint operators, such as balancing between the visual inks induced by *Winglets* and information loss by other techniques, are interesting topics for future research.

The idea of *Winglets* is framed in perceptual grouping and is closely related to the Gestalt principle of *Closure*. However, there could be other emergent features [39] that are involved when *Winglets* are combined into a whole group, such as collinearity or parallelism formed by wings in close proximity that facilitate or hinder the grouping perception of *Winglets*. One future research direction is to more closely examine the perception mechanism behind *Winglets*, to gain a better understanding of how people perceive them and what perceptual mechanisms are involved in the grouping of its items.

Winglets are related to the research topic of glyph-based visualization [2], where small symbols of multiple visual encodings are composed to depict attributes of data items. Unlike glyphs which are mostly designed independently from each other, *Winglets* orient collectively to form a global shape. Other glyph designs share a similar idea, where the glyph is utilized to indicate flow in vector field visualization [30], or forms small tangent lines for points in a scatterplot encoding sensitivity [6]. Unlike these designs that utilize the Gestalt Principle of *Continuity* to suggest a continuous trend, *Winglets* are designed to convey association using the *Closure* Principle. It may therefore be interesting to extend glyph designs to encode global trends in the spirit of *Winglets*.

In this work, we explore two design variables of *Winglets* employed to promote the notions of association and uncertainty. Other design variables of *Winglets* can be further explored, such as color, thickness or shape (not necessarily a curve), and may possibly be designed per specific usage scenarios. Specifically for color, Figure 3 suggests potential benefit for colorless *Winglets*. However, the general advantages and use of colorless *Winglets* are left to be examined in future works. In general, we believe that *Winglets* may inspire further development of advanced means for scatterplot enhancement, in particular that of cluster perception for data analysis facilitation.

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