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On the Reeb Spaces of Definable Maps

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Abstract
We prove that the Reeb space of a proper definable map $f\colon X \to Y$ in an arbitrary o-minimal expansion of a real closed field is realizable as a proper definable quotient. This result can be seen as an o-minimal analog of Stein factorization of proper morphisms in algebraic geometry. We also show that the Betti numbers of the Reeb space of f can be arbitrarily large compared to those of X, unlike in the special case of Reeb graphs of manifolds. Nevertheless, in the special case when $f\colon X \to Y$ is a semi-algebraic map and X is closed and bounded, we prove a singly exponential upper bound on the Betti numbers of the Reeb space of f in terms of the number and degrees of the polynomials defining X, Y, and f.

Keywords Reeb spaces · O-minimal structures · Betti numbers · Semi-algebraic maps

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1 Introduction

Given a topological space X and a continuous function $f\colon X \to \mathbb{R}$, define an equivalence relation $\sim$ on X by setting $x \sim x'$ if and only if $f(x) = f(x')$ and x and x' are in the same connected component of $f^{-1}(f(x)) = f^{-1}(f(x'))$. The space $X/{\sim}$ is called the *Reeb graph* of f, denoted $\mathrm{Reeb}(f)$. The concept of the Reeb graph was introduced by Georges Reeb in [31] as a tool in Morse theory. The notion of the Reeb graph can be generalized to the notion of *Reeb space* by letting $f\colon X \to Y$, where Y is any topological space. Burlet and de Rham first introduced the Reeb space in [9] as the *Stein factorization* of a map f, but their work was limited to bivariate, generic, smooth mappings. Existence of Stein factorization for more general morphisms in algebraic geometry is proved in [23, III, Cor. 11.5], and is closely related to the well-known Zariski's Main Theorem [23, III, Cor. 11.4] (see Remarks 3.5 and 5.12 for the connection between Stein factorization in algebraic geometry and the results of the current paper).

From the point of view of applied topology, Reeb spaces have been investigated from both a theoretical and practical perspective. Edelsbrunner et al. defined the Reeb space of a multivariate piecewise linear mapping on a combinatorial manifold in [18], and they proved results regarding the local and global structure of such spaces. Expanding on this work, Patel [27] produced an algorithm to construct the Reeb space of a mapping f. Mapper, introduced in [34], gives a discrete approximation of the Reeb space of a multivariate mapping; this allows for more efficient computation of the underlying data structure. Munch et al. [26] define the *interleaving distance* for Reeb spaces to show the convergence between the Reeb space and Mapper.

In this paper, we investigate Reeb spaces from the point of view of topological complexity. Our motivation is to understand how topologically complicated the Reeb space of a map can become in terms of the complexity of the map itself. In order to obtain meaningful results we restrict ourselves to the category of maps *definable in an o-minimal expansion of a real closed* field R (for example, one can take $\mathrm{R} = \mathbb{R}$) and in particular, to *semi-algebraic* maps (see Sect. 2 for a quick overview of o-minimality).

Remark 1.1 We remark here that in [33] Reeb graphs are considered for so-called constructible $\mathbb{R}$-spaces that the authors introduce. Compact definable sets in any o-minimal expansion of $\mathbb{R}$ are examples of constructible $\mathbb{R}$-spaces. Our choice of studying Reeb spaces within the framework of o-minimal geometry stems from the fact that o-minimality is now a widely accepted framework for studying tame geometry and also allows us to prove effective upper bounds on the topology of Reeb spaces in special situations of interest—for instance, for Reeb spaces of semi-algebraic maps.

The notion of o-minimal structures has its origins in model theory but has since become a widely accepted framework for studying "tame geometry". The definable sets and maps of an o-minimal structure satisfy many uniform finiteness properties (similar to those of semi-algebraic sets) while allowing much richer families of sets and maps. We refer the reader to the survey by Wilkie [38] for the origin and motivation of this notion of tameness. The reader will also find many applications of interest.

Our first result is that the Reeb spaces of "tame" maps are themselves tame. More precisely, we prove that the quotient map corresponding to the Reeb space of a proper definable map can be realized as a proper definable map (Theorem 3.3 below). This implies as a special case that the Reeb spaces of proper semi-algebraic maps can be realized as semi-algebraic quotients. Theorem 3.3 can be viewed as the definable analog of the theorem [23, III, Cor. 11.5] on the existence of Stein factorization for proper morphisms in algebraic geometry (see Remark 3.5 below). Another significance of this result is that it makes it possible to ask for an algorithm to semi-algebraically describe this semi-algebraic quotient using results from the well-developed area of algorithmic semi-algebraic geometry [6]. A naive approach would be to make the proof of Proposition 2.4 in [13] (in the semi-algebraic case) algorithmic. However, the proof of Proposition 2.4 makes heavy use of definable (in this case semi-algebraic) triangulations. Algorithms with the best known complexity for computing semi-algebraic triangulations (see for instance [6, Chap. 5]) use the technique of *cylindrical algebraic decomposition* which has intrinsically a doubly exponential complexity. So this approach is unlikely to yield anything better than an algorithm with a doubly exponential complexity. However, it must be noted that the problem of computing a description of the Reeb space is much more special than the problem of computing the description of quotients by an arbitrary semi-algebraic equivalence relations, since the equivalence relation in the case of the Reeb space is very geometric. Many problems in algorithmic semi-algebraic geometry with a geometric or topological flavor—such as the problem of computing semi-algebraic descriptions of the semi-algebraically connected components of a given semi-algebraic set, computing certain topological invariants like the Euler–Poincaré characteristics etc. admit algorithms with singly exponential complexity (see for examples [6, Chaps. 13–16]). They rely on a common technique called the *critical point method*.

In this paper we prove a singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map. Driven by the analogy with the problems mentioned above, this singly exponential upper bound suggests that an algorithm having a singly exponential complexity should exist for computing a description of the Reeb space using the critical point method. However, the technique introduced in this paper is not sufficient for this purpose, since the upper bound that we prove comes from approximating the cohomology of the Reeb space by the E_1-term of a spectral sequence converging to it. But this is not sufficient to describe the Reeb space itself. Doing so would require additional techniques and we do not pursue this question further in this paper, leaving it for future work.

It is known [17, p. 141] that the sum of the Betti numbers of the *Reeb graph* of a map $f\colon X \to \mathrm{R}$ is bounded from above by the sum of the Betti numbers of X. We show that this is false for more general maps by exhibiting a couple of natural examples of sequences of maps $(f_n\colon X_n \to Y_n)_{n>0}$, such that the sum of the Betti numbers of the Reeb space of f_n is arbitrarily large compared to that of X_n. In view of these examples, it makes sense to ask whether it is still possible to bound the Betti numbers of the Reeb space of a map f in terms of some measure of the "complexity" of the map f. In particular, if the map is semi-algebraic, then one can measure the complexity of the map by the number and degrees of the polynomials defining the map. We are then led to the problem of studying the topological complexity of Reeb spaces of semi-algebraic maps.

While studying the topological complexity of Reeb spaces of semi-algebraic maps is a natural mathematical question on its own, another motivation is related to the algorithmic question mentioned earlier concerning the design of efficient algorithms for computing a semi-algebraic description of the Reeb space of a semi-algebraic map. It is a meta theorem in algorithmic semi-algebraic geometry that upper bounds on topological complexity of objects are closely related to the worst-case complexity of algorithms computing the topological invariants of such objects. Thus, a singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map opens up the possibility of being able to compute the Betti numbers of the Reeb space. The singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map may also hint that one could compute a semi-algebraic description of the Reeb space with an algorithm having a singly exponential complexity bound.

The problem of bounding the topological complexity (for example measured in terms of Betti numbers or the number of homotopy types of fibers) of semi-algebraic sets or maps in terms of the parameters of the formula defining them has a long history (see [4] for a survey). Bounds on these quantities which are doubly exponential in the dimension or the number of variables usually follow from the fact that semi-algebraic sets admit semi-algebraic triangulations of at most doubly exponential size. Singly exponential upper bounds are more difficult and usually involve more careful arguments involving Morse inequalities and other inequalities coming from certain spectral sequences [5, 6, 19, 24, 28, 35]. To the best of our knowledge, the problem of bounding the Betti numbers of the Reeb space of a semi-algebraic map has not been considered before. In this paper we prove a singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map $f\colon X \to Y$, where X is a closed and bounded semi-algebraic set, in terms of the number and the degrees of the polynomials defining X, Y, and f (cf. Theorem 5.2 below).

The rest of the paper is organized as follows: In Sect. 2, we recall the basic definitions related to o-minimality. In Sect. 3, we prove the definability of Reeb spaces of proper definable maps. In Sect. 4, we describe examples showing that the Betti numbers of the Reeb space of a definable map $f\colon X \to Y$ can be arbitrarily large compared to those of X. We also give a proof of the inequality $b_1(\mathrm{Reeb}(f)) \leq b_1(X)$ for definable proper maps $f\colon X \to Y$ with X connected, using a spectral sequence that plays an important role in this paper (this inequality was proved previously using alternative techniques by Dey et al. [11]). Finally, in Sect. 5, we prove a singly exponential upper bound on the sum of the Betti numbers of the Reeb space of a proper semi-algebraic map in terms of the number and degrees of the polynomials defining the map. We defer the proof of a key intermediate result (Proposition 5.6) to the appendix since it is long and technical. We conclude in Sect. 6 with some open problems.

2 Basic Definitions

We first recall the important model theoretic notion of o-minimality which plays an important role in what follows.

2.1 O-Minimal Structures

O-minimal structures were invented and first studied by Pillay and Steinhorn in the pioneering papers [29, 30], motivated by the prior work of van den Dries [12]. Later, the theory was further developed through contributions of other researchers, most notably van den Dries, Wilkie, Rolin, and Speissegger, amongst others [14–16, 32, 36, 37]. We particularly recommend the book by van den Dries [13] and the notes by Coste [10] for an easy introduction to the topic as well as for the proofs of the basic results that we use in this paper.

Definition 2.1 (*o-minimal structure*) An o-minimal structure over a real closed field R (or equivalently an o-minimal expansion of R) is a sequence $\mathcal{S}(\mathrm{R}) = (\mathcal{S}_n)_{n\in\mathbb{N}}$ where each $\mathcal{S}_n$ is a collection of subsets of R^n (called the *definable sets* in the structure) satisfying the following axioms (following the exposition in [10]):

(A) All algebraic subsets of R^n are in $\mathcal{S}_n$.
(B) The class $\mathcal{S}_n$ is closed under complementation and finite unions and intersections.
(C) If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$ then $A \times B \in \mathcal{S}_{m+n}$.
(D) If $\pi : \mathrm{R}^{n+1} \to \mathrm{R}^n$ is the projection map on the first n coordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.
(E) The elements of $\mathcal{S}_1$ are finite unions of points and intervals. (Note that these are precisely the subsets of R which are definable by a first-order formula in the language of the reals with one free variable.)

A map $f : X \to Y$ between two definable sets X and Y is *definable* if its graph is a definable set. Note that for any definable map $f : X \to Y$, there exists a finite partition $(X_i)_{i\in I}$ of X into definable subsets such that f restricted to each X_i is continuous. In light of this, for rest of this paper we use the term "definable map" to mean a map that is definable and continuous.

The class of semi-algebraic subsets of R^n, $n > 0$, where R is a real closed field, is one obvious example of an o-minimal structure, but in fact there are much richer classes of sets which have been proven to be o-minimal. The class of *sub-analytic sets* is one such example [37].

We now consider quotients by definable equivalence relations.

Definition 2.2 Let $E \subset X \times X$ be a definable equivalence relation on a definable set X. A *definable quotient* of X by E is a pair (p, Y) consisting of a definable set Y and a definable surjective map $p: X \to Y$ such that

(i) $(x_1, x_2) \in E \;\Leftrightarrow\; p(x_1) = p(x_2)$ for all $x_1, x_2 \in X$;
(ii) p is definably identifying; that is, for all definable $K \subset Y$, if $p^{-1}(K)$ is closed in X, then K is closed in Y.

We say that the definable quotient (p, Y) is *definably proper* if p is a definably proper map. This means that if $X \subset \mathrm{R}^m$, and $Y \subset \mathrm{R}^n$, then for every definable subset $K \subset Y$ with K closed and bounded in R^n, $p^{-1}(K) \subset X$ is closed and bounded in R^m.

Definition 2.3 A definable equivalence relation $E \subset X \times X$ is said to be *definably proper* if the two maps $pr_1, pr_2 : E \to X$ are definably proper.

We will use the following proposition which appears in [13]:

Proposition 2.4 [13, p. 166] *Let X be a definable set and $E \subset X \times X$ a definably proper equivalence relation on X. Then X/E exists as a definably proper quotient of X.*

3 The Reeb Space of a Definable Map $f : X \to Y$

We now fix an o-minimal expansion of a real closed field R. Let $X \subset \mathrm{R}^n$ be a closed and bounded definable set, and $f : X \to Y$ be a definable map.

Definition 3.1 The Reeb space of the map f, henceforth denoted $\mathrm{Reeb}(f)$, is the topological space $X/{\sim}$, equipped with the quotient topology, where $x \sim x'$ if and only if $f(x) = f(x')$, and x, x' belong to the same connected component of $f^{-1}(f(x))$. (Here the topology on X is the definable topology whose open sets are the definable open subsets of X).

Remark 3.2 Note that a definable (resp. semi-algebraic) set $S \subset \mathrm{R}^k$ is connected if and only if S is definably (resp. semi-algebraically) path-connected, i.e., for all $x, y \in S$, there exists a definable (resp. semi-algebraic) path $\gamma : [0, 1] \to S$ with $\gamma(0) = x$, $\gamma(1) = y$.

Theorem 3.3 *Let $X \subset \mathrm{R}^n$ be a closed and bounded definable set, and $f : X \to Y$ be a definable map with Y a definable set. Then, the space* $\mathrm{Reeb}(f) \triangleq X/{\sim}$ *is a definably proper quotient. In other words, let $X \subset \mathrm{R}^n$ be a closed and bounded definable set, and $f : X \to Y$ be a definable map. Then there exists a definable set Z, and a proper definable map $\psi : X \to Z$ and a homeomorphism $\theta :$ $\mathrm{Reeb}(f) \to Z$ such that the following diagram commutes*

$$\begin{tikzcd}
 & X \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
\mathrm{Reeb}(f) = X/{\sim} \arrow[rr, "\theta"] & & Z
\end{tikzcd}$$

(*here ϕ is the quotient map*). *In particular,* $\mathrm{Reeb}(f)$ *is homeomorphic to a definable set.*

Remark 3.4 The assumption of that X is closed and bounded is needed. For example, suppose $X = \mathrm{R}^2 \setminus \mathbf{0}$ and $f : X \to \mathrm{R}$ is the projection map forgetting the second coordinate. Then, each fiber $f^{-1}(x)$ has one connected component if $x \neq 0$, and $f^{-1}(0)$ has two connected components. The Reeb space of f is homeomorphic to R with a doubled point, and is not definable.

Remark 3.5 Theorem 3.3 can also be seen as a definable analog of Stein factorization for projective morphisms [23, III, Cor. 11.5] which states that "every projective morphism $f : X \to Y$ of Noetherian schemes factors as $f = g \circ f'$, with $g : Y' \to Y$ a finite morphism, and $f' : X \to Y'$ a projective morphism with connected fibers." Note that this result is valid for Noetherian schemes over fields of any characteristic and the connectivity of the fibers is with respect to the the Zariski topology of the underlying topological space of the corresponding scheme. Notice also that the scheme Y' plays the role of Reeb space of f. More precisely, the underlying topological space of the scheme Y' is the Reeb space of the map induced by the morphism f between the underlying topological spaces of the schemes X and Y.

We discuss below a simple illustrative example assuming only a minimum familiarity with some concepts from algebraic geometry, such as affine and projective schemes over a field K, homogeneous coordinates etc. (the reader can consult [23] for definitions).

Example 3.6 Let K be any field. Recall that a morphism $f\colon X \to Y$ between K-schemes is projective if it factors as $f = g \circ f'$, where $f'\colon X \to \mathbb{P}^N_K \times Y$ is a closed immersion for some $N > 0$, and $g\colon \mathbb{P}^N_K \times Y \to Y$ is the projection [23, p. 103].

Consider in $\mathbb{A}^3_K$ the subscheme consisting of the union two lines defined by the equations parametrized by T: $X_3 = T$, $X_2 = 0$, and $X_3 = 0$, $X_1 = 0$. Then for $T \neq 0$, the two lines are disjoint, but at $T = 0$ the subscheme is connected. In order to produce an example of a projective morphism we need to projectivize the picture.

We will use $(X_0 : X_1 : X_2 : X_3)$ to denote the homogeneous coordinates of $\mathbb{P}^3_K$. Let X denote the subscheme of $\mathbb{P}^3_K \times \mathbb{A}^1_K$ defined by intersection of the ideals $(X_2, X_3 - X_0 T)$ and (X_1, X_3). The fibers of X with respect to the projection to $\mathbb{A}^1_K$ consist generically of two skew lines in $\mathbb{P}^3_K$ except over the point defined by $T = 0$. Over $T = 0$, the two lines meet, and so the fiber is connected.

Let $Y' \subset \mathbb{P}^1_K \times \mathbb{A}^1_K$ be the subscheme defined by the polynomial

$$W_1(W_1 - W_0 T)$$

and let $g\colon Y' \to \mathbb{P}^1_K = Y$ be the restriction to Y' of the projection morphism $\mathbb{P}^1_K \times \mathbb{A}^1_K \to \mathbb{A}^1_K$. Note that the fibers of g consist of two points except over $T = 0$ where it is a doubled point. Let

$$f'\colon X \to Y'$$

be the morphism defined by

$$((X_0 : X_1 : X_2 : X_3), T) \mapsto ((X_0 : X_3), T).$$

Note that X_0 and X_3 cannot simultaneously be 0 on X and so f' is well defined. Note also that f' factors through the closed embedding

$$f''\colon X \to \mathbb{P}^3_K \times Y'$$

defined by

$$((X_0 : X_1 : X_2 : X_3), T) \mapsto ((X_0 : X_1 : X_2 : X_3), (X_0 : X_3), T),$$

followed by the projection to Y'. Hence, f' is a projective morphism. It can now be verified that:

- $f = g \circ f'$,
- g is a finite morphism, and
- f' has connected fibers.

We will now prove Theorem 3.3.

Proof of Theorem 3.3 We first claim that the relation, "$x \sim x'$ if and only if $f(x) = f(x')$, and x, x' belong to the same connected component of $f^{-1}(f(x))$" is a definably proper equivalence relation. Using Hardt's triviality theorem for o-minimal structures [10, 13], we have that there exists a finite definable partition of Y into locally closed definable sets $(Y_\alpha)_{\alpha\in I}$, $y_\alpha \in Y_\alpha$, and definable homeomorphisms $\phi_\alpha : Y_\alpha \times f^{-1}(y_\alpha) \to f^{-1}(Y_\alpha)$ such that the following diagram commutes for each $\alpha \in I$:

$$\begin{tikzcd} Y_\alpha \times f^{-1}(y_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi_1"'] & & f^{-1}(Y_\alpha) \arrow[dl, "f|_{f^{-1}(Y_\alpha)}"] \\ & Y_\alpha & \end{tikzcd}$$

(here π_1 is the projection to the first factor in the direct product). For each $\alpha \in I$, let $(C_{\alpha,\beta})_{\beta\in J_\alpha}$ be the connected components of $f^{-1}(y_\alpha)$, and for each $\alpha \in I$, $\beta \in J_\alpha$, let $D_{\alpha,\beta} = \phi_\alpha(Y_\alpha \times C_{\alpha,\beta})$. Let

$$E = \bigcup_{\alpha\in I,\beta\in J_\alpha} (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta})),$$

where $(Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta})$ is the definable subset of $(Y_\alpha \times f^{-1}(y_\alpha)) \times (Y_\alpha \times f^{-1}(y_\alpha))$ defined by

$$((y,x),(y',x')) \in (Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta}) \quad \Leftrightarrow \quad y = y',\ \ x, x' \in C_{\alpha,\beta}.$$

It is clear that E is a definable subset of $X \times X$, and that $x \sim x'$ if and only if $(x, x') \in E$.

Since X is assumed to be closed and bounded, if we can show that E is closed in $X \times X$, it would follow that E is a definably proper equivalence relation, and we can apply Proposition 2.4. The rest of the proof is devoted to showing that E is a closed definable subset of $X \times X$. For each $\alpha \in I$, $\beta \in J_\alpha$, let

$$E_{\alpha,\beta} = (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta})).$$

Since $E = \bigcup_{\alpha\in I,\beta\in J_\alpha} E_{\alpha,\beta}$, in order to prove that E is closed it suffices to prove that for each $\alpha \in I$, $\beta \in J_\alpha$,

$$\overline{E_{\alpha,\beta}} \subset E,$$

where $\overline{E_{\alpha,\beta}}$ is the closure of $E_{\alpha,\beta}$ in $X \times X$.

It follows from the curve selection lemma for o-minimal structures [10] that for every $z \in \overline{E_{\alpha,\beta}}$ there exists a definable curve $\gamma\colon [0,1] \to E_{\alpha,\beta}$ with $\gamma(0) = z$, $\gamma((0,1]) \subset E_{\alpha,\beta}$. Thus, in order to prove that $\overline{E_{\alpha,\beta}} \subset E$, it suffices to show that for each definable curve $\gamma\colon (0,1] \to E_{\alpha,\beta}$, $z_0 = \lim_{t\to 0}\gamma(t) \in E$.

Let $\gamma\colon (0,1] \to E_{\alpha,\beta}$ be a definable curve, and suppose that $\lim_{t\to 0}\gamma(0) \notin E_{\alpha,\beta}$. Otherwise, $\lim_{t\to 0}\gamma(0) \in E_{\alpha,\beta} \subset E$, and we are done. For $t \in (0,1]$, let $y_t = f(\gamma(t))$ and let $(x_t, x'_t) \in (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta}))$ be such that $\gamma(t) = (x_t, x'_t)$. Note that $f(x_t) = f(x'_t) = y_t$. Finally, let $z_0 = (x_0, x'_0) = \lim_{t\to 0}\gamma(t)$.

Since $z_0 \notin E_{\alpha,\beta}$ by assumption and $\gamma((0,1]) \subset E_{\alpha,\beta}$, there exists $t_0 > 0$ such that $\lambda = f \circ \gamma|_{(0,t_0]}\colon (0,t_0] \to Y_\alpha$ is an injective definable map and $\lim_{t\to 0}\lambda(t) = y_0 = f(x_0) = f(x'_0) \in Y_{\alpha'}$ for some $\alpha' \in I$. We need to show that x_0 and x'_0 belong to the same connected component of $f^{-1}(y_0)$, which would imply that $(x_0, x'_0) \in E$.

Let $D_{\alpha,\beta,\gamma} = f^{-1}(\lambda((0,t_0])) \cap D_{\alpha,\beta}$ and let $g\colon D_{\alpha,\beta,\gamma} \to (0,t_0]$ be defined by $g(x) = \lambda^{-1}(f(x))$ (which is well defined by the injectivity of λ). Note that for each $t \in (0,t_0]$, $g^{-1}(t)$ is definably homeomorphic to $C_{\alpha,\beta}$, and hence is connected. It also follows from Hardt's triviality theorem that there exists $t'_0 \in (0,t_0]$ and a definable homeomorphism $\theta\colon g^{-1}(t'_0) \times (0,t'_0] \to g^{-1}((0,t'_0])$ such that the following diagram commutes:

$$\begin{tikzcd}
g^{-1}(t'_0) \times (0,t'_0] \arrow[rr, "\theta"] \arrow[dr, "\pi_2"'] & & g^{-1}((0,t'_0]) \arrow[dl, "g"] \\
& (0,t'_0]. &
\end{tikzcd}$$

Extend θ continuously to a definable map $\overline{\theta}\colon g^{-1}(t'_0) \times [0,t_0] \to \overline{g^{-1}((0,t'_0])}$ by setting $\overline{\theta}(x,0) = \lim_{t\to 0}\theta(x,t)$. Finally, let $\theta'\colon g^{-1}(t'_0) \to f^{-1}(y_0)$ be the definable map obtained by setting $\theta'(x) = \overline{\theta}(x,0)$. Note that since $g^{-1}(t'_0)$ is connected, $\theta'(g^{-1}(t'_0))$ is connected as well, since it is the image of a connected set under a continuous map. Also note that for each $t \in (0,t'_0]$, we have that $x_t, x'_t \in D_{\alpha,\beta,\gamma}$ and $f(x,t) = f(x'_t) = \lambda(t)$, hence $x_t, x'_t \in g^{-1}(t)$, and thus $x_0, x'_0 \in \theta'(g^{-1}(t'_0))$. Moreover, $f(x_0) = f(x'_0) = y_0$. Therefore, since $\theta'(g^{-1}(t'_0))$ is connected, x_0 and x'_0 belong to the same connected component of $f^{-1}(y_0)$. This shows that $(x_0, x'_0) \in E$, which in turn implies that E is closed in $X \times X$. The fact that $\mathrm{Reeb}(f)$ exists as a definably proper quotient now follows from Proposition 2.4. $\square$

Remark 3.7 Theorem 3.3 opens up an algorithmic problem of actually realizing the Reeb space as a definable quotient in the special case where the o-minimal structure is that of semi-algebraic sets and maps. More precisely, the problem is to design an algorithm that, given a proper semi-algebraic map $f\colon X \to Y$, will compute a description of a semi-algebraic map $g\colon X \to Z$ such that following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "g"] & \\ \mathrm{Reeb}(f) = X/{\sim} \arrow[rr, "\theta"] & & Z \end{tikzcd}$$

(here ϕ is the quotient map and θ is a homeomorphism). The complexity of the algorithm will then depend on the number and degrees of the polynomials defining X, Y, and the graph of the semi-algebraic map f. In this paper, we do not pursue this algorithmic problem any further leaving it for future work.

4 The Betti Numbers of the Reeb Space of $f\colon X \to Y$ Can Exceed That of X

Definition 4.1 Any closed and bounded definable set is finitely triangulable [10, Thm. 4.4]. For any closed and bounded definable set X and $i \geq 0$, let $b_i(X)$ denote the ith Betti number (that is, the dimension of the ith simplicial homology group of a definable triangulation of X with coefficients in $\mathbb{Q}$), and we denote $b(X) = \sum_{i\geq 0} b_i(X)$. It is well known that the Betti numbers so defined do not depend on the chosen triangulation of X.

In [17, p. 141] it is noted that the inequality $b(\mathrm{Reeb}(f)) \leq b(X)$ holds for arbitrary maps $f\colon X \to \mathrm{R}$. We first show that the same is not true for Reeb spaces of more general maps by giving several examples.

Example 4.2 Consider the closed n-dimensional disk $\mathbf{D}^n$ with $n \geq 1$, and let $\sim$ be the equivalence relation identifying all points on the boundary of $\mathbf{D}^n$. Then $\mathbf{D}^n/{\sim} \cong \mathbf{S}^n$, where $\mathbf{S}^n$ is the n-dimensional sphere. Let f_n denote the quotient map $f_n\colon \mathbf{D}^n \to \mathbf{S}^n$. The fibers of f_n consist of either one point or the boundary $\mathbf{S}^{n-1}$ of $\mathbf{D}^n$, and hence $\mathrm{Reeb}(f_n) \cong \mathbf{S}^n$ for all $n > 1$. Note that $b_0(\mathbf{D}^n) = 1$ and $b_i(\mathbf{D}^n) = 0$ for all $i > 0$. Moreover, $b_0(\mathbf{S}^n) = 1$, $b_n(\mathbf{S}^n) = 1$, and $b_i(\mathbf{S}^n) = 0$, $i \neq 0, n$. Thus, we have for $n > 1$,

$$b(\mathbf{D}^n) = 1, \qquad b(\mathrm{Reeb}(f_n)) = 2.$$

More generally, for $k \geq 0$, let

$$f_{n,k} = \underbrace{f \times \cdots \times f}_{k \text{ times}}\colon \underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}} \longrightarrow \underbrace{\mathbf{S}^n \times \cdots \times \mathbf{S}^n}_{k \text{ times}}.$$

Using the same argument as before, for $n > 1$ and $k > 0$,

$$\mathrm{Reeb}(f_{n,k}) \cong \underbrace{\mathbf{S}^n \times \cdots \times \mathbf{S}^n}_{k \text{ times}}.$$

Thus,

$$b_0(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) = 1, \qquad b_i(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) = 0, \quad i > 0,$$

and hence

$$b(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) = 1.$$

Moreover, for $n > 1$,

$$b_i(\mathrm{Reeb}(f_{n,k})) = \begin{cases} 0 & \text{if } n \not| i \text{ or if } i > nk, \\ \binom{k}{i/n} & \text{otherwise,} \end{cases}$$

and hence for $n > 1$,

$$b\,(\mathrm{Reeb}(f_{n,k})) = 2^k.$$

This example shows that even for definably proper maps $f : X \to Y$, the individual as well as the total Betti numbers of $\mathrm{Reeb}(f)$ can be arbitrarily large compared to those of X.

Our second example comes from the topology of compact Lie groups, in particular the complex unitary group:

Example 4.3 For $n > 0$, let $U(n)$ denote the group of $n \times n$ complex unitary matrices, and let $T^n \subset U(n)$ denote the maximal torus. (Note that T^n is the group of $n \times n$ unitary diagonal matrices $\mathrm{diag}\,(z_1, \ldots, z_n)$ with $|z_i| = 1$, $1 \le i \le n$, and is thus homeomorphic to the product of n circles.) Denote the quotient map by $\pi_n : U(n) \to U(n)/T^n$. We have that

$$b(U(n)/T^n) = n! \qquad \text{(see [25, Thm. 4.6])}, \tag{4.1}$$
$$b(U(n)) = 2^n \qquad \text{(see [25, Cor. 3.11])}. \tag{4.2}$$

Observing that the fibers of π_n are all connected, one has that $\mathrm{Reeb}(\pi_n) \cong U(n)/T^n$, and it follows from (4.1) and (4.2) that for all $n \ge 4$,

$$b(\mathrm{Reeb}(\pi_n)) = n! \ge 2^n = b(U(n)).$$

We note that recently Dey et al. [11] have shown that

$$b_1(\mathrm{Reeb}(f)) \le b_1(X) \tag{4.3}$$

if $f: X \to Y$ is a proper map and X is connected. Notice that the examples given above do not violate this bound since the stated inequalities involve only the sum rather than the individual Betti numbers.

We sketch below an alternative proof of the inequality (4.3) for a proper definable map $f: X \to Y$, with X connected, using an inequality coming from a spectral sequence associated to the quotient map $\phi: X \to \mathrm{Reeb}(f)$. This spectral sequence also plays a key role in the proof of the main result in this paper.

More precisely, for a proper definable surjective map $g: A \to B$, Gabrielov et al. [21] proved that there exists a spectral sequence (which we write as a cohomological spectral sequence for convenience) which converges to $\mathrm{H}^*(B)$. This spectral sequence is referred to as the *descent spectral sequence* of g below and its E_1-term is given by

$$E_1^{p,q} = \mathrm{H}^q(\underbrace{A \times_g \cdots \times_g A}_{p+1}),$$

where

$$\underbrace{A \times_g \cdots \times_g A}_{p+1} = \{(a_0, \dots, a_p) \in A^{p+1} \mid g(a_0) = \cdots = g(a_p)\}.$$

Theorem 4.4 [11] *Let X be a connected definable set, and $f: X \to Y$ a proper definable map. Then*

$$b_1(\mathrm{Reeb}(f)) \leq b_1(X).$$

Alternative proof of Theorem 4.4 using a spectral sequence argument First note that if X is connected, then so is

$$\underbrace{X \times_\phi \cdots \times_\phi X}_{p+1},$$

and $\dim(E_1^{p,0}) = 1$ for all $p \geq 0$. Moreover, the differential $d_1^{p,0}: E_1^{p,0} \to E_1^{p+1,0}$ has rank 0 or 1 depending on whether p is even or odd, respectively. This implies that $E_2^{p,0} = 0$ for all $p > 0$ in the descent spectral sequence of the quotient map $\phi: X \to \mathrm{Reeb}(f)$. Moreover, notice that $E_1^{0,1} \cong \mathrm{H}^1(X)$, and hence

$$\dim(E_1^{0,1}) = b_1(X).$$

Since the spectral sequence converges to $\mathrm{H}^{p+q}(\mathrm{Reeb}(f))$, the following inequality holds for each $n \geq 0$ and $r \geq 1$:

$$\mathrm{H}^n(\mathrm{Reeb}(f)) \leq \sum_{p+q=n} \dim(E_r^{p,q}). \tag{4.4}$$

Moreover, for $r \geq r'$ and for any p, q,

$$\dim(E_r^{p,q}) \leq \dim(E_{r'}^{p,q}), \tag{4.5}$$

since $E_r^{p,q}$ is a sub-quotient of $E_{r'}^{p,q}$. It follows from the inequalities (4.4) and (4.5) with $n = 1$, $r' = 1$, and $r = 2$, that

$$\begin{aligned} b_1(\mathrm{Reeb}(f)) &\leq \dim(E_2^{0,1}) + \dim(E_2^{1,0}) \\ &\leq \dim(E_1^{0,1}) + \dim(E_2^{1,0}) = b_1(X) + 0 = b_1(X). \end{aligned}$$ □

Remark 4.5 We note here that an inequality (cf. inequality (5.4)) coming from the consideration of the E_1-term of the spectral sequence of the map ϕ plays a key role in the proof of Theorem 5.2, which is the main result of this paper.

5 Quantitative Bounds

We now consider the problem of bounding effectively from above the Betti numbers of the Reeb space of a definable continuous map. We have seen from Example 4.2 that, given a continuous semi-algebraic map $f: X \to Y$, $b(\mathrm{Reeb}(f))$ can be arbitrarily large compared to $b(X)$, unlike in the case of Reeb graphs (i.e., when $\dim(Y) \leq 1$). In this section, we prove an upper bound on $b(\mathrm{Reeb}(f))$ in terms of the "semi-algebraic" complexity of the map f. We first introduce some more notation.

Notation 5.1 Let R be a real closed field. For any finite family of polynomials $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, we call an element $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$ a *sign condition* on $\mathcal{P}$. For any semi-algebraic set $Z \subset \mathrm{R}^k$ and sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, we denote by $\mathcal{R}(\sigma, Z)$ the semi-algebraic set defined by

$$\{\mathbf{x} \in Z \mid \mathbf{sign}(P(\mathbf{x})) = \sigma(P),\ P \in \mathcal{P}\},$$

and call it the *realization* of σ on Z. More generally, we call any Boolean formula Φ with atoms $P\,\{=, >, <\}\,0$, $P \in \mathcal{P}$, a $\mathcal{P}$-*formula*. We call the realization of Φ, namely the semi-algebraic set

$$\mathcal{R}(\Phi, \mathrm{R}^k) = \{\mathbf{x} \in \mathrm{R}^k \mid \Phi(\mathbf{x})\},$$

a $\mathcal{P}$-*semi-algebraic set*. Finally, we call a Boolean formula without negations and with atoms $P\,\{\geq, \leq\}\,0$ (resp. $P\,\{>, <\}\,0$), $P \in \mathcal{P}$, a $\mathcal{P}$-*closed* (resp. $\mathcal{P}$-*open*) formula, and we call the realization, $\mathcal{R}(\Phi, \mathrm{R}^k)$, a $\mathcal{P}$-*closed* (resp. a $\mathcal{P}$-*open*) semi-algebraic set.

We will denote by $\mathrm{SIGN}(\mathcal{P})$ the set of *realizable sign conditions of* $\mathcal{P}$, i.e.,

$$\mathrm{SIGN}(\mathcal{P}) = \{\sigma \in \{0, 1, -1\}^{\mathcal{P}} \mid \mathcal{R}(\sigma, \mathrm{R}^k) \neq \emptyset\}.$$

Finally, for any semi-algebraic set S, we will denote the set of its semi-algebraically connected components by $\mathrm{Cc}(S)$.

We prove the following theorem.

Theorem 5.2 *Let $S \subset \mathrm{R}^n$ be a bounded $\mathcal{P}$-closed semi-algebraic set, and $f = (f_1, \ldots, f_m)\colon S \to \mathrm{R}^m$ be a polynomial map. Suppose that $s = \mathrm{card}(\mathcal{P})$ and the maximum of the degrees of the polynomials in $\mathcal{P}$ and $f_1, \ldots, f_m$ is bounded by d. Then*

$$b(\mathrm{Reeb}(f)) \leq (sd)^{(n+m)^{O(1)}}.$$

The rest of the paper is devoted to the proof of Theorem 5.2. We first outline the main idea behind the proof.

5.1 Outline of the Proof of Theorem 5.2

We first replace the map $f\colon S \to \mathrm{R}^m$, by a new map $\tilde{f}\colon \tilde{S} \to \mathrm{R}^m$, where $\tilde{S} \subset \mathrm{R}^n \times \mathrm{R}^m$ and $\tilde{f}$ is the restriction to $\tilde{S}$ of the projection map to R^m, such that the following diagram commutes:

$$\begin{tikzcd}
\tilde{S} \arrow[rr, hook] \arrow[dr, "\tilde{f}"'] & & \mathrm{R}^n \times \mathrm{R}^m \arrow[dl, "\pi_m"] \\
& \mathrm{R}^m &
\end{tikzcd}$$

From the definitions it is evident that $\mathrm{Reeb}(f)$ and $\mathrm{Reeb}(\tilde{f})$ are homeomorphic. We next prove that there exists a semi-algebraic partition of R^m of controlled complexity (more precisely given by the connected components of the realizable sign conditions of a family of polynomials of singly exponentially bounded degrees and cardinality) into connected semi-algebraic sets C, such that the connected components of the fibers $\tilde{f}^{-1}(z)$ are in $1-1$ correspondence with each other as z varies over C. Moreover each of these connected components C is described by a quantifier-free first order formula and the complexity of these formulas (i.e., the number of polynomials appearing in the formula and their respective degrees) is bounded singly exponentially (see Proposition 5.4 below for the precise formulation of this statement).

The proof of this result (Proposition 5.4) uses an intermediate result—namely Proposition 5.6—whose long and somewhat technical proof is deferred to the appendix. The fact that the semi-algebraically connected components of a semi-algebraic set can be described efficiently (with singly exponential complexity) is a consequence of a result in [6] (Proposition 5.5 below).

Next, we use the fact that the canonical surjection $\phi\colon \tilde{S} \to \mathrm{Reeb}(\tilde{f})$ is a proper semi-algebraic map. We then use an inequality proved in [21] (see Proposition 5.8 below) to obtain an upper bound on the Betti numbers of the image of a proper semi-algebraic map $F\colon X \to Y$ in terms of the sum of the Betti numbers of various fiber products $X \times_F \cdots \times_F X$ of the same map. Recall that for $p \geq 0$, the $(p+1)$-fold fiber product is given by

$$\underbrace{X \times_F \ldots \times_F X}_{p+1 \text{ times}} \triangleq \{(x^{(0)}, \ldots, x^{(p)}) \in X^{p+1} | F(x^{(0)}) = \ldots = F(x^{(p)})\}.$$

Proposition 5.4 provides us with a well-controlled description (i.e., by quantifier-free first order formulas involving singly exponentially any polynomials of singly exponentially bounded degrees) of the fibered products $\tilde{S} \times_{\tilde{f}} \cdots \times_{\tilde{f}} \tilde{S}$. Finally, using these descriptions and results on bounding the Betti numbers of general semi-algebraic sets in terms of the number and degrees of polynomials defining them (cf. Proposition 5.10 below) we obtain the claimed bound on $\mathrm{Reeb}(f)$. In order to make the above summary precise we first need to state some preliminary results.

5.2 Parametrized Description of Connected Components

The following proposition, which states that given any finite family of polynomials

$$\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell],$$

where R is a real closed field, there exists a semi-algebraic partition of R^ℓ of controlled complexity which has good properties with respect to $\mathcal{P}$, will play a crucial role in the proof of Theorem 5.2. We will use the following notation in the rest of the paper.

Notation 5.3 We will denote by $\pi_Y : \mathrm{R}^{k+\ell} \to \mathrm{R}^\ell$ the projection to the last ℓ (denoted by $Y = (Y_1, \ldots, Y_\ell)$) coordinates. For any semi-algebraic subset $S \subset \mathrm{R}^{k+\ell}$ and $T \subset \mathrm{R}^\ell$, we will denote by $S_T = S \cap \pi_Y^{-1}(T)$. If $T = \{\mathbf{y}\}$, we will write $S_\mathbf{y}$ instead of $S_{\{\mathbf{y}\}}$.

Proposition 5.4 *Let* R *be a real closed field, and let* $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$ *be a finite set of polynomials of degrees bounded by* d, *with* $\mathrm{card}(\mathcal{P}) = s$. *Let* $S \subset \mathrm{R}^k \times \mathrm{R}^\ell$ *be a* $\mathcal{P}$*-semi-algebraic set. Then there exists a finite set of polynomials* $\mathcal{Q} \subset \mathrm{R}[Y_1, \ldots, Y_\ell]$ *such that* $\mathrm{card}(\mathcal{Q})$ *and the degrees of polynomials in* $\mathcal{Q}$ *are bounded by* $(sd)^{(k+\ell)^{O(1)}}$, *and* $\mathcal{Q}$ *has the following additional property: For each* $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$ *and* $C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathrm{R}^\ell))$, *there exist*

(i) *an index set* $I_{\sigma,C}$,
(ii) *a finite family of polynomials* $\mathcal{P}_{\sigma,C} \subset \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$, *and*
(iii) $\mathcal{P}_{\sigma,C}$*-formulas,* $(\Theta_\alpha(\overline{X}, \overline{Y}))_{\alpha \in I_{\sigma,C}}$,

such that

(A) $\Theta_\alpha(\mathbf{x}, \mathbf{y}) \Rightarrow \mathbf{y} \in C$;
(B) *for each* $\mathbf{y} \in C$ *and each* $D \in \mathrm{Cc}(S_C)$, *there exists a unique* $\alpha \in I_{\sigma,C}$ *such that* $\mathcal{R}(\Theta_\alpha(\,\cdot\,, \mathbf{y})) = D_\mathbf{y}$ *and* $D_\mathbf{y} \in \mathrm{Cc}(S_\mathbf{y})$;
(C) *the cardinalities of* $I_{\sigma,C}$ *and* $\mathcal{P}_{\sigma,C}$ *and the degrees of the polynomials in* $\mathcal{P}_{\sigma,C}$ *are all bounded by* $(sd)^{(k+\ell)^{O(1)}}$.

The proof of Proposition 5.4 will use the following result on efficient descriptions of the connected components of semi-algebraic sets which can easily be deduced from [6, Thm. 16.3] and which we state without proof.

Proposition 5.5 *Let* R *be a real closed field and let* $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k]$ *with* $\deg(P_i) \le d$ *for* $1 \le i \le s$ *and let a semi-algebraic set* S *be defined by a* $\mathcal{P}$*-formula. Then there exists an algorithm that outputs quantifier-free semi-algebraic descriptions of all the connected components of* S*. The number of polynomials that appear in the output is bounded by* $s^{k+1}d^{O(k^4)}$*, while the degrees of the polynomials are bounded by* $d^{O(k^3)}$.

In order to prove Proposition 5.4 we will also need the following proposition.

Proposition 5.6 *Let* R *be a real closed field. Let* $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$ *be a finite set of polynomials with degrees bounded by* d *and with* $\mathrm{card}(\mathcal{P}) = s$*, and let* $S \subset \mathrm{R}^k \times \mathrm{R}^\ell$ *be a bounded* $\mathcal{P}$*-closed semi-algebraic set. Then there exists a finite set of polynomials,* $\mathcal{Q} \subset \mathrm{R}[Y_1, \ldots, Y_\ell]$*, with*

$$\mathrm{card}(\mathcal{Q}) \le (sd)^{(k+\ell)^{O(1)}}, \qquad \max_{Q \in \mathcal{Q}} \deg(Q) \le d^{(k+\ell)^{O(1)}},$$

such that for each $\sigma \in \mathrm{SIGN}(\mathcal{Q})$*,* $C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathrm{R}^\ell))$*,* $\mathbf{y} \in C$*, and* $D \in \mathrm{Cc}(S_C)$*,* $D_{\mathbf{y}} \in \mathrm{Cc}(S_{\mathbf{y}})$.

Proof See the appendix. □

Remark 5.7 In the case $\mathrm{R} = \mathbb{R}$, Proposition 5.6 can be deduced from a somewhat more general theorem [3, Thm. 4.21] about constructible sheaves. However, to explain this properly one would need to recall basic definitions and functorial properties of constructible sheaves which would put an unreasonable burden on the reader. So we prefer to give a more self-contained proof avoiding the sheaf-theoretic formalism which can be found in the appendix.

The connection of Proposition 5.6 with the theory of constructible sheaves is the following. We assume that the reader is familiar with familiar with basic definitions sheaf theory (which can be found for example in [22]). Let $\mathcal{F}$ be the sheaf on R^ℓ defined as the sheafification of the presheaf that associates to each open subset $U \subset \mathrm{R}^\ell$, the finite-dimensional $\mathbb{Q}$-vector space $\mathrm{H}^0(\pi_Y^{-1}(U \cap S, \mathbb{Q}))$. For each $\mathbf{y} \in \mathrm{R}^\ell$, the stalk $\mathcal{F}_{\mathbf{y}}$ of the sheaf $\mathcal{F}$ at $\mathbf{y}$ is then isomorphic to $\mathrm{H}^0(S_{\mathbf{y}}, \mathbb{Q})$. The property of the finite set of polynomials $\mathcal{Q}$ guaranteed by Proposition 5.6—namely, that for each $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$, $C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathrm{R}^\ell))$, $\mathbf{y} \in C$, and for $D \in \mathrm{Cc}(C_S)$, $D_{\mathbf{y}} \in \mathrm{Cc}(S_{\mathbf{y}})$—can be expressed in the language of sheaves as follows.

The set $\mathcal{Q}$ induces a semi-algebraic partition of R^ℓ into a finite number of locally closed semi-algebraic sets, $\mathcal{R}(\sigma, \mathrm{R}^\ell), \sigma \in \mathrm{SIGN}(\mathcal{Q})$, such that for each $\sigma \in \mathrm{SIGN}(\mathcal{Q})$, the stalks $\mathcal{F}_{\mathbf{y}}$ are locally constant. This implies in particular that the sheaf $\mathcal{F}$ is a constructible sheaf.

Let j denote the inclusion $j \colon S \hookrightarrow \mathrm{R}^{k+\ell}$, and $\mathbb{Q}_S = j_* j^* \mathbb{Q}_{\mathrm{R}^{k+\ell}}$, where $\mathbb{Q}_{\mathrm{R}^{k+\ell}}$ denotes the constant sheaf with stalks isomorphic to $\mathbb{Q}$ on $\mathrm{R}^{k+1\ell}$. Then

$$\mathcal{F} \cong R^0 \pi_{Y,*} \mathbb{Q}_S$$

(i.e., $\mathcal{F}$ is isomorphic to the zeroth higher direct image of the constant sheaf supported on S under the map π_Y). The general theorem [3, Thm. 4.21] alluded to in the first paragraph of this remark gives a singly exponential bound on the "complexity" of the higher direct images of a constructible sheaf in terms of the complexity of the sheaf itself. The complexity of a constructible sheaf is defined in terms of the complexity of the smallest semi-algebraic partition on which the stalks are locally constant. Proposition 5.6 could in principle be deduced from [3, Thm. 4.21], since the partition $\mathrm{R}^{k+\ell}$ into the semi-algebraic set S and its complement has the property that the stalks of $\mathbb{Q}_S$ are locally constant on the sets of this partition, and applying the theorem only for the 0th higher direct image $\mathbb{Q}_S$ under the (proper) map π_Y.

We are now in a position to prove Proposition 5.4.

Proof of Proposition 5.4 Let $\Phi(\overline{X}, \overline{Y})$ be the $\mathcal{P}$-closed formula describing S. First apply Proposition 5.6 to obtain a set of polynomials $\mathcal{Q} \subset \mathrm{R}[Y_1, \ldots, Y_\ell]$ with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$, and for each connected component C of each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$, each $y \in C$, and for each connected component of D of $\pi_Y^{-1}(C) \cap S$, $D_y = \pi_Y^{-1}(y) \cap D$ is a connected component of $S_y = \pi_Y^{-1}(y) \cap S$.

Next using Proposition 5.5 obtain for each realizable sign condition σ of $\mathcal{Q}$, and for each connected component of C of $\mathcal{R}(\sigma, \mathrm{R}^\ell)$, a quantifier-free formula $\Phi_{\sigma,C}(\overline{Y})$ describing C.

Now using Proposition 5.5 one more time, obtain for each σ, C, and each connected component D_α of the semi-algebraic set defined by $\Phi_{\sigma,C}(\overline{Y}) \wedge \Phi(\overline{X}, \overline{Y})$, a quantifier-free formula $\Theta_\alpha(\overline{X}, \overline{Y})$ describing D_α. □

5.3 Bounding the Topology of the Image of a Polynomial Map

The following proposition proved in [21] allows one to bound the Betti numbers of the image of a closed and bounded definable set X under a definable map F in terms of the Betti numbers of the iterated fibered product of X over F. More precisely:

Proposition 5.8 [21] *Let $F\colon X \to Y$ be a definable continuous map, and X a closed and bounded definable set. Then, for for all $p \geq 0$,*

$$b_p(F(X)) \leq \sum_{\substack{i,j\geq 0\\ i+j=p}} b_i(\underbrace{X \times_F \cdots \times_F X}_{j+1}).$$

5.4 Bounds on the Betti Numbers of Semi-Algebraic Sets

Finally, in order to prove Theorem 5.2, we will need singly exponential upper bounds on the Betti numbers of semi-algebraic sets in terms of the number and degrees of the polynomials appearing in any quantifier-free formula defining the set. We give a brief overview of these results. The key result that we will need in the proof of Theorem 5.2 is Proposition 5.10. (We refer the reader to [6, Chap. 6] for the definition of homology groups of semi-algebraic sets which are only locally closed and not necessarily closed and bounded.)

5.4.1 General Bounds

The first results on bounding the Betti numbers of real varieties were proved by Oleĭnik and Petrovskiĭ [28], Thom [35], and Milnor [24]. Using a Morse-theoretic argument and Bézout's theorem they proved the following proposition which appears in [5] and makes more precise an earlier result which appeared in [2]:

Proposition 5.9 [5] *If $S \subset \mathrm{R}^k$ is a $\mathcal{P}$-closed semi-algebraic set, then*

$$b(S) \le \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{s+1}{j} 6^j d(2d-1)^{k-1}, \tag{5.1}$$

where $s = \mathrm{card}(\mathcal{P}) > 0$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Using an additional ingredient (namely, a technique to replace an arbitrary semi-algebraic set by a locally closed one with a very controlled increase in the number of polynomials used to describe the given set), Gabrielov and Vorobjov [19] extended Proposition 5.9 to arbitrary $\mathcal{P}$-semi-algebraic sets with only a small increase in the bound. Their result in conjunction with Proposition 5.9 gives the following proposition.

Proposition 5.10 [6, 20] *If $S \subset \mathrm{R}^k$ is a $\mathcal{P}$-semi-algebraic set, then*

$$b(S) \le \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{2ks+1}{j} 6^j d(2d-1)^{k-1}, \tag{5.2}$$

where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

We will also use the following bound on the number of connected components of the realizations of all realizable sign conditions of a family of polynomials proved in [5].

Proposition 5.11 *Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]_{\le d}$ and let $s = \mathrm{card}(\mathcal{P})$. Then*

$$\mathrm{card}\left(\bigcup_{\sigma \in \mathrm{SIGN}(\mathcal{P})} \mathrm{Cc}(\mathcal{R}(\sigma, \mathrm{R}^k))\right) \le \sum_{1 \le j \le k} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

We now have all the ingredients needed to prove Theorem 5.2.

5.5 Proof of Theorem 5.2

Proof of Theorem 5.2 Let Φ be the $\mathcal{P}$-closed formula defining S. Introducing new variables $Z_1, \ldots, Z_m$, let $\tilde{S} \subset \mathrm{R}^n \times \mathrm{R}^m$ be the $\tilde{\mathcal{P}}$-formula

$$\Phi \wedge \bigwedge_{1 \le i \le m} (Z_i - f_i = 0).$$

Let $\tilde{f} : \tilde{S} \to \mathrm{R}^m$ be the restriction to $\tilde{S}$ of the projection map $\pi_Z : \mathrm{R}^m \times \mathrm{R}^n \to \mathrm{R}^m$ to the Z-coordinates. Then clearly S is semi-algebraically homeomorphic to $\tilde{S}$, $f(S) = \tilde{f}(\tilde{S})$, and $\mathrm{Reeb}(f)$ is semi-algebraically homeomorphic to $\mathrm{Reeb}(\tilde{f})$. We have the following commutative square where the horizontal arrows are homeomorphisms and the vertical arrows are the quotient maps.

$$\begin{tikzcd} S \arrow[r, "\cong"] \arrow[d, "\phi"] & \tilde{S} \arrow[d, "\tilde{\phi}"] \\ \mathrm{Reeb}(f) \arrow[r, "\cong"] & \mathrm{Reeb}(\tilde{f}). \end{tikzcd}$$

Now it follows from Proposition 5.4 that there exists a finite set of polynomials $\mathcal{Q} \subset \mathrm{R}[Z_1, \ldots, Z_m]$, with

$$\mathrm{card}(\mathcal{Q}),\ \max_{Q \in \mathcal{Q}} \deg(Q) \le (sd)^{(n+m)^{O(1)}} \tag{5.3}$$

having the following property: for each $\sigma \in \mathrm{SIGN}(\mathcal{Q})$ and each $C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathrm{R}^m))$, there exists an index set $I_{\sigma,C}$, a finite family of polynomials

$$\mathcal{P}_{\sigma,C} \subset \mathrm{R}[X_1, \ldots, X_n, Z_1, \ldots, Z_m],$$

and $\mathcal{P}_{\sigma,C}$ formulas $(\Theta_\alpha(\overline{X}, \overline{Z}))_{\alpha \in I_{\sigma,C}}$ such that $\Theta_\alpha(x, z) \Rightarrow z \in C$, and for each $z \in C$, and each connected component D of $\pi_Z^{-1}(C) \cap \tilde{S}$, there exists a unique $\alpha \in I_{\sigma,C}$ (which does not depend on z) with $\mathcal{R}(\Theta_\alpha(\cdot, z)) = \pi_Z^{-1}(z) \cap D$. Moreover, the cardinalities of $I_{\sigma,C}$ and $\mathcal{P}_{\sigma,C}$ and the degrees of the polynomials in $\mathcal{P}_{\sigma,C}$ are all bounded by $(sd)^{(n+m)^{O(1)}}$.

Let ϕ (resp. $\tilde{\phi}$) be the canonical surjection $\phi : S \to \mathrm{Reeb}(f) \cong S/{\sim}$ (resp. $\tilde{\phi} : \tilde{S} \to \mathrm{Reeb}(\tilde{f}) \cong \tilde{S}/{\sim}$). From Theorem 3.3 it follows that we can assume that ϕ is a proper semi-algebraic map. For each $i \ge 0$, we have the inequality (cf. Proposition 5.8)

$$b_i(\mathrm{Reeb}(f)) \le \sum_{p+q=i} b_q(\underbrace{S \times_\phi \cdots \times_\phi S}_{p+1 \text{ times}}). \tag{5.4}$$

Now observe that

$$\underbrace{\tilde{S} \times_{\tilde{\phi}} \cdots \times_{\tilde{\phi}} \tilde{S}}_{p+1 \text{ times}}, \quad \text{and hence} \quad \underbrace{S \times_{\phi} \cdots \times_{\phi} S}_{p+1 \text{ times}},$$

is semi-algebraically homeomorphic to the semi-algebraic set defined by the formula

$$\Theta\big(\overline{X}^{(0)}, \ldots, \overline{X}^{(p)}, \overline{Z}\big) = \bigvee_{\substack{\sigma \in \mathrm{SIGN}(\mathcal{Q}) \\ C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathrm{R}^m)) \\ \alpha \in I_{\sigma,C}}} \bigwedge_{0 \le j \le p} \Theta_\alpha(\overline{X}^{(j)}, \overline{Z}). \tag{5.5}$$

To see this observe that

$$\big((x^{(0)}, z^{(0)}), \ldots, (x^{(p)}, z^{(p)})\big) \in \underbrace{\tilde{S} \times_{\tilde{\phi}} \cdots \times_{\tilde{\phi}} \tilde{S}}_{p+1 \text{ times}}$$

if and only if

$$z^{(0)} = \ldots = z^{(p)} = z,$$

for some z, and $x^{(0)}, \ldots, x^{(p)}$ belong to the same connected component of $\tilde{f}^{-1}(z)$. It is easy to verify this last equivalence using the properties of the decomposition given by Proposition 5.4.

We now claim that each of the formulas

$$\Theta\big(\overline{X}^{0}, \ldots, \overline{X}^{(p)}, \overline{Z}\big), \qquad 0 \le p \le m,$$

is a $\tilde{\mathcal{P}}_p$-formula for some finite set $\tilde{\mathcal{P}}_p \subset \mathrm{R}[\overline{X}^{0}, \ldots, \overline{X}^{(p)}, \overline{Z}]$ with $\mathrm{card}(\tilde{\mathcal{P}}_p)$ and the degrees of the polynomials in $\tilde{\mathcal{P}}_p$ being bounded singly exponentially. In order to prove the claim first observe that the cardinality of the set

$$\bigcup_{\sigma \in \mathrm{SIGN}(\mathcal{Q})} \mathrm{Cc}(\mathcal{R}(\sigma, \mathrm{R}^m))$$

is bounded singly exponentially, once the number of polynomials in $\mathcal{Q}$, and their degrees are bounded singly exponentially (using Proposition 5.11). The fact that the number of polynomials in $\mathcal{Q}$ and their degrees are bounded singly exponentially follows from (5.3). Moreover, for similar reasons the cardinalities of the index sets $I_{\sigma,C}$ are also bounded singly exponentially. The claim now follows from (5.5). Finally, to prove the theorem we first apply inequality (5.4) and then apply Proposition 5.10 to bound the right hand side of the inequality (5.4). □

Remark 5.12 Given the analogy between Reeb spaces and Stein factorization (cf. Remark 3.5) it could be interesting to investigate (in the context of algebraic geometry) Stein factorization for projective morphisms from the point of view of complexity in analogy with Theorem 5.2. To the best of our knowledge this has not yet been investigated.

6 Conclusion

In this paper we have proved the realizability of the Reeb space of proper definable maps in an o-minimal structure as a proper definable quotient. We have exhibited examples where the Reeb spaces of maps can have arbitrarily complicated topology compared to that of the domains of the maps, a sharp contrast with the behavior of Reeb graphs. Nevertheless, we have proved singly exponential upper bounds on the Betti numbers of the Reeb spaces of proper semi-algebraic maps.

Declarations

Conflict of interest and data sharing On behalf of all authors, the corresponding author states that there is no conflict of interest. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix

This section is devoted to the proof of Proposition 5.6. For the convenience of the reader we first recall the proposition.

Proposition *Let* R *be a real closed field. Let* $\mathcal{P} \subset \mathrm{R}[X_1, \dots, X_k, Y_1, \dots, Y_\ell]$ *be a finite set of polynomials with degrees bounded by* d *and with* $\mathrm{card}(\mathcal{P}) = s$*, and let* $S \subset \mathrm{R}^k \times \mathrm{R}^\ell$ *be a bounded* $\mathcal{P}$*-closed semi-algebraic set. Then there exists a finite set of polynomials,* $\mathcal{Q} \subset \mathrm{R}[Y_1, \dots, Y_\ell]$*, with*

$$\mathrm{card}(\mathcal{Q}) \le (sd)^{(k+\ell)^{O(1)}}, \qquad \max_{Q \in \mathcal{Q}} \deg(Q) \le d^{(k+\ell)^{O(1)}},$$

such that for each $\sigma \in \mathrm{SIGN}(\mathcal{Q})$*,* $C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathrm{R}^\ell))$*,* $\mathbf{y} \in C$*, and* $D \in \mathrm{Cc}(S_C)$*,* $D_{\mathbf{y}} \in \mathrm{Cc}(S_{\mathbf{y}})$*.*

The rest of this section is devoted to the proof of Proposition 5.6. Since the proof is technical and relies on several intermediate results from algorithmic semi-algebraic geometry we first sketch an outline of the proof for the benefit of the reader.

A.1 Outline of the Proof of Proposition 5.6

The core idea behind the proof comes from [4] where a singly exponential upper bound is proved on the number of homotopy types of the fibers of a semi-algebraic map. We explain this idea informally below.

Given $S \subset \mathrm{R}^{k+\ell}$ a $\mathcal{P}$-closed and bounded semi-algebraic set, $\pi_Y : \mathrm{R}^{k+\ell} \to \mathrm{R}^\ell$ the projection map onto the last ℓ-coordinates, we first introduce infinitesimal elements

$\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_s)$ (where $s = \mathrm{card}(\mathcal{P})$), and define a semi-algebraic subset, $S^\star(\bar{\varepsilon}) \subset \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell}$ which is infinitesimally larger than S. We prove (see Lemma A.26) that for each $\mathbf{y} \in \mathrm{R}^\ell$, the extension of the fiber, $S_\mathbf{y} = S \cap \pi_Y^{-1}(\mathbf{y}) \cap \mathrm{R}^\ell$, to $\mathrm{R}\langle\bar{\varepsilon}\rangle$ (see Definition A.3 below for the definition of extension) is a semi-algebraic deformation retract of the fiber $S^\star(\bar{\varepsilon})_\mathbf{y} = S^\star(\bar{\varepsilon}) \cap \pi_Y^{-1}(\mathbf{y})$—and in particular, these two sets fibers are semi-algebraically homotopy equivalent. (The field $\mathrm{R}\langle\bar{\varepsilon}\rangle$ is a real closed extension of R consisting of algebraic Puiseux series with coefficients in R and is defined below in Sect. 1.) It is important to note that in order to obtain the semi-algebraic homotopy equivalence between the the extension of the fiber $S_\mathbf{y}$ to $\mathrm{R}\langle\bar{\varepsilon}\rangle$ and $S^\star(\bar{\varepsilon})_\mathbf{y}$, we need the fact that all the coordinates of $\mathbf{y}$ belong to R (rather than $\mathrm{R}\langle\bar{\varepsilon}\rangle \setminus \mathrm{R}$). Note that R is a subfield of $\mathrm{R}\langle\bar{\varepsilon}\rangle$, and there is thus a natural inclusion of R^ℓ into $\mathrm{R}\langle\bar{\varepsilon}\rangle^\ell$ as well.

Next, we identify a semi-algebraic subset $G(\bar{\varepsilon}) \subset \mathrm{R}\langle\bar{\varepsilon}\rangle^\ell$ (cf. Definition A.19) of codimension at least one in $\mathrm{R}\langle\bar{\varepsilon}\rangle^\ell$, consisting of the critical values of the projection map restricted to the various algebraic sets defined by the polynomials defining $S^\star(\bar{\varepsilon})$. We prove (Lemma A.20) that the map π_Y restricted to the set $\pi_Y^{-1}(\mathrm{R}\langle\bar{\varepsilon}\rangle^\ell \setminus G(\bar{\varepsilon})) \cap S^\star(\bar{\varepsilon})$ is a *locally trivial semi-algebraic fibration* (see Sect. 1 for the definition of semi-algebraic fibrations). We also observe that $G(\bar{\varepsilon}) \cap \mathrm{R}^\ell$ is empty and hence $\mathrm{R}^\ell \subset \mathrm{R}\langle\bar{\varepsilon}\rangle^\ell \setminus G(\bar{\varepsilon})$.

Moreover, for each $C \in \mathrm{Cc}(\mathrm{R}\langle\bar{\varepsilon}\rangle^\ell \setminus G(\bar{\varepsilon}))$, $C \cap \mathrm{R}^\ell$ is a semi-algebraic subset of R^ℓ. If the projection map π_Y restricted to $\pi_Y^{-1}(C) \cap S^\star(\bar{\varepsilon})$ is a *trivial* semi-algebraic fibration (not just a *locally trivial* one), then we would be done at this point, because we could take $\mathcal{Q} \subset \mathrm{R}[Y_1, \ldots, Y_\ell]$ to be a finite set of polynomials such that the semi-algebraic sets

$$C \cap \mathrm{R}^\ell, \quad C \in \mathrm{Cc}(\mathrm{R}\langle\bar{\varepsilon}\rangle^\ell \setminus G(\bar{\varepsilon}))$$

are all $\mathcal{Q}$-semi-algebraic sets. (It follows from certain preliminary results (Propositions 5.5 and A.8) that such a set $\mathcal{Q}$ exists satisfying the singly exponential estimates in Proposition 5.6.)

In order to arrive at the desired situation (i.e., to have a trivial semi-algebraic fibration instead of just a locally trivial one) we need an additional step of covering the complement of $G(\bar{\varepsilon})$ using semi-algebraically contractible sets noting that a locally trivial semi-algebraic fibration over a contractible semi-algebraic set is automatically trivial (see Proposition A.11). We use a result from [6] which gives an algorithm for constructing a cover of a given semi-algebraic set by contractible ones (see Proposition A.6) whose complexity is bounded singly exponentially. This introduces some additional technicalities since we need to consider an infinitesimal thickening of $G(\bar{\varepsilon})$ before taking a cover by contractible semi-algebraic sets of its complement and this is explained in the proof.

One important technical detail is that we need to keep track of the dependence of each polynomial appearing in the definition of the cover on the various infinitesimals introduced in the course of the construction. It is important that each such polynomial depends on at most polynomially many of these infinitesimals and the degrees of the polynomials in the infinitesimals are bounded singly exponentially (cf. parts (c) and (d) of Lemma A.23). This is crucial for ensuring the singly exponential bounds on the set of polynomials $\mathcal{Q}$ that is produced at the end.

A.2 Preliminaries

Before proving Proposition 5.6 we need a few preliminary results.

A.2.1 Real Closed Extensions and Puiseux Series

We will need some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [6] for further details.

Notation A.1 For R a real closed field we denote by $\mathrm{R}\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in ε with coefficients in R. We use the notation $\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$ to denote the real closed field $\mathrm{R}\langle\varepsilon_1\rangle\langle\varepsilon_2\rangle\ldots\langle\varepsilon_m\rangle$. Note that in the unique ordering of the field $\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$, $0<\varepsilon_m\ll\varepsilon_{m-1}\ll\ldots\ll\varepsilon_1\ll 1$.

Notation A.2 For elements $x\in\mathrm{R}\langle\varepsilon\rangle$ which are bounded over R we denote by $\lim_\varepsilon x$ to be the image in R under the usual map that sets ε to 0 in the Puiseux series x. If $\bar{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_m)$, and $x\in\mathrm{R}\langle\bar{\varepsilon}\rangle$ bounded over R, then we will denote

$$\lim_{\bar{\varepsilon}} x=\lim_{\varepsilon_1}\left(\ldots\left(\lim_{\varepsilon_m} x\right)\right).$$

Definition A.3 If R' is a real closed extension of a real closed field R, and $S\subset\mathrm{R}^k$ is a semi-algebraic set defined by a first-order formula with coefficients in R, then we will denote by $\mathrm{Ext}(S,\mathrm{R}')\subset\mathrm{R}'^k$ the semi-algebraic subset of R'^k defined by the same formula. It is well known that $\mathrm{Ext}(S,\mathrm{R}')$ does not depend on the choice of the formula defining S [6].

For ϕ a $\mathcal{P}$-closed formula, with $\mathcal{P}=\{P_1,\ldots,P_s\}$, we will denote by $\phi^\star(\bar{\delta})$ the formula obtained from ϕ by replacing any occurrence of $P_i\geq 0$ with $P_i\geq-\delta_i$, and any occurrence of $P_i\leq 0$ with $P_i\leq\delta_i$, for each i, $1\leq i\leq s$.

Lemma A.4 *Let* R *be a real closed field and* $R\in\mathrm{R}$ *with* $R>0$. *The semi-algebraic set* $\mathrm{Ext}(\mathcal{R}(\phi,\mathrm{R}^k)\cap[-R,R]^k,\mathrm{R}\langle\bar{\delta}\rangle)$ *is semi-algebraically homotopy equivalent to* $\mathcal{R}(\phi^\star(\bar{\delta}),\mathrm{R}\langle\bar{\delta}\rangle^k)\cap[-R,R]^k$.

A.2.2 Covering by Semi-Algebraic Contractible Sets

As explained in the outline we will need an auxiliary result on covering of semi-algebraic sets by contractible ones with singly exponential complexity. In order to state this result we first need the notion of being in *strong general position*.

Definition A.5 Let R be a real closed field and $\mathcal{P}\subset\mathrm{R}[X_1,\ldots,X_k]$ be a finite set. We say that the family $\mathcal{P}$ is in *ℓ-general position*, if no more than ℓ polynomials belonging to $\mathcal{P}$ have a zero in R^k. The family $\mathcal{P}$ is in *strong ℓ-general position* if moreover any ℓ polynomials belonging to $\mathcal{P}$ have at most a finite number of zeros in R^k.

We will use the following proposition which follows from the correctness and complexity analysis of Algorithm 16.14 (Covering by Contractible Sets) in [6].

Proposition A.6 *Let* R *be a real closed field and* D $\subset$ R *and ordered domain. There exists an algorithm that takes as input*

(i) *a finite set of s polynomials* $\mathcal{G} = \{G_1, \ldots, G_t\} \subset \mathrm{D}[\bar{\varepsilon}, \bar{\delta}][Y_1, \ldots, Y_\ell]$*, where* $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_s)$*,* $\bar{\delta} = (\delta_1, \ldots, \delta_t)$*, in strong* ℓ*-general position on* $\mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}\rangle^\ell$*, and such that each polynomial in* $\mathcal{G}$ *depends on at most* $k + \ell$ *of the* ε_i*'s and at most one of the* δ_i*'s;*
(ii) *a* $\mathcal{G}$*-closed formula* ψ*;*
(iii) $R > 0$*,* $R \in \mathrm{D}$*;*

and outputs

(a) *a finite set of polynomials* $\mathcal{H} \subset \mathrm{D}[\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}][Y_1, \ldots, Y_\ell]$*,* $\bar{\zeta} = (\zeta_1, \ldots, \zeta_{2\mathrm{card}(\mathcal{H})})$*;*
(b) *a tuple of* $\mathcal{H}$*-formulas* $(\theta_\alpha)_{\alpha \in I}$ *such that each* $\mathcal{R}(\theta_\alpha, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}\rangle,^\ell)$*,* $\alpha \in I$*, is a closed semi-algebraically contractible set; and*
(c) $\displaystyle\bigcup_{\alpha \in I} \mathcal{R}(\theta_\alpha, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}\rangle^\ell) = \mathcal{R}(\psi, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}\rangle^\ell) \cap [-R, R]^\ell$.

Moreover,

$$\mathrm{card}(I), \mathrm{card}(\mathcal{H}) \le (tD)^{\ell^{O(1)}},$$
$$\deg_{\bar{Y}}(H), \deg_{\bar{\varepsilon}}(H), \deg_{\bar{\delta}}(H), \deg_{\bar{\zeta}}(H) \le D^{\ell^{O(1)}},$$

where $D = \max_{1 \le i \le t} \deg(G_i)$*. Moreover, each polynomial appearing in* $\mathcal{H}$ *depends on at most* $(k + \ell)(\ell + 1)^2$ *of* ε_i*'s, at most* $(\ell + 1)^2$ *of the* δ_i*'s, and on at most one of the* ζ_i*'s.*

Remark A.7 Note that the last claim in Proposition A.6, namely that each polynomial appearing in any of the formulas θ_α depends on at most $(k+\ell)(\ell+1)^2$ of ε_i's, at most $(\ell+1)^2$ of the δ_i's and on at most one of the ζ_i's, does not appear explicitly in [6], but is evident on a close examination of the algorithm. It is also reflected in the fact that the combinatorial part (i.e., the part depending on $\mathrm{card}(\mathcal{G})$) of the complexity of [6, Algorithm 16.14] is bounded by $\mathrm{card}(\mathcal{G})^{(\ell+1)^2}$. This is because [6, Algorithm 16.14] has a "local property", namely that all computations involve at most a small number (in this case $(\ell + 1)^2$) polynomials in the input at a time.

A.2.3 Effective Quantifier Elimination

We will also need the following effective bound on the complexity of eliminating one block of existential quantifiers in the theory of real closed fields. It is a direct consequence of [6, Thm. 14.16].

Proposition A.8 *Let* R *be a real closed field,* D $\subset$ R *an ordered domain,* $\mathcal{Q} \subset \mathrm{D}[\varepsilon_1, \ldots, \varepsilon_m][X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$ *a finite set of polynomials, and* ψ *a* $\mathcal{Q}$*-formula. Suppose that the degrees of the polynomials in* $\mathcal{Q}$ *in* ε_h, X_i, Y_j *for* $1 \le h \le m$*,* $1 \le i \le k$*,* $1 \le j \le \ell$*, are all bounded by* D*. Then, there exists a finite set of polynomials* $\mathcal{G} \subset \mathrm{D}[\varepsilon_1, \ldots, \varepsilon_m][Y_1, \ldots, Y_\ell]$*, and a* $\mathcal{G}$*-formula* θ *satisfying the following conditions:*

(a) $\mathrm{card}(\mathcal{G}) = (\mathrm{card}(\mathcal{Q})D)^{O((k+1)(\ell+1))}$*;*

(b) $\max_{G\in\mathcal{G},1\le i\le m,1\le j\le \ell}\big(\deg_{\varepsilon_i}(G),\deg_{Y_j}(G)\big)=D^{O(k)}$;
(c) $\mathcal{R}(\theta,\mathrm{R}\langle\bar\varepsilon\rangle^{\ell})=\pi_Y(\mathcal{R}(\psi,\mathrm{R}\langle\bar\varepsilon\rangle^{k+\ell}))$.

A.2.4 Trivial and Locally Trivial Semi-Algebraic Fibrations

As explained in the outline we will also need the notion of trivial and locally trivial semi-algebraic fibrations.

Definition A.9 Let R be a real closed field. Let $p\colon E\to B$ be a semi-algebraic map. We say that p is a *trivial semi-algebraic fibration* if there exists $b\in B$ and a semi-algebraic homeomorphism $h\colon E\to p^{-1}(b)\times B$, such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & p^{-1}(b)\times B \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

(here π is the projection to the second factor). We say that the map p is a *locally trivial semi-algebraic fibration* if there exists a finite covering $(U_i)_{i\in I}$ of B by semi-algebraic subsets which are open in B, such that for each $i\in I$, $p|_{p^{-1}(B_i)}$ is a semi-algebraic trivial fibration.

Definition A.10 (*pull-back of a semi-algebraic locally trivial fibration under a semi-algebraic map*) Let $p\colon E\to B$ be a locally trivial semi-algebraic fibration, and $f\colon B'\to B$ be a continuous semi-algebraic map. Then the map $f^*p\colon E\times_B B'\to B'$, $(e,b')\mapsto b'$, is again a semi-algebraic locally trivial fibration (called the *pull-back of p under f*).

The main property of semi-algebraic fibrations that we will need is the following.

Proposition A.11 *Let $p\colon E\to B$ be a locally trivial semi-algebraic fibration. Suppose B' is a semi-algebraically contractible subset of B and B'' any semi-algebraic subset of B'. Then $p|_{p^{-1}(B'')}$ is a trivial semi-algebraic fibration.*

Proposition A.11 is a corollary of the following more general proposition.

Proposition A.12 *Let $p\colon E\to B$ be a locally trivial semi-algebraic fibration, and let $f,g\colon B'\to B$ be two continuous semi-algebraic maps which are semi-algebraically homotopic. The, f^*p and g^*p are isomorphic as locally trivial semi-algebraic fibrations.*

Proof The corresponding statement in the category of topological spaces and maps appear in [1, Thm. 4.6.4] with the extra assumption that B' is paracompact. We do not give a complete proof of Proposition A.12 here but just indicate the modifications needed in the proof of [1, Thm. 4.6.4] to carry it over to the semi-algebraic setting over an arbitrary real closed field R.

The key result used in the proof of Theorem 4.6.4 in [1] is Proposition 4.6.3, whose statement is reproduced below using the notation of the current paper.

> Let $p\colon E \to B \times [0,1]$ be a locally trivial fibration, where B is a paracompact space. Let $r\colon B \times [0,1] \to B \times [0,1]$ be the retraction $r(b,t) = (b,1)$. Then, p is isomorphic to $r^* p$.

We will need the following semi-algebraic version of the above statement, namely:

Claim A.13 *Let $p\colon E \to B \times [0,1]$ be a locally trivial semi-algebraic fibration. Let $r\colon B' \times [0,1] \to B \times [0,1]$ be the retraction defined by $r(b,t) = (b,1)$. Then, p is isomorphic to $r^* p$.*

All the modifications needed are in the proof of the above claim. In the proof of [1, Prop. 4.6.3] replace the open cover $(U_\alpha)_{\alpha\in\Lambda}$ of B (which is assumed to be only locally finite using the paracompactness of B) by a finite open cover of $(U_i)_{i\in I}$ of the semi-algebraic set B such that p restricted to each $U_\alpha \times [0,1]$ is a trivial semi-algebraic fibration. Such a finite cover exists if p is a locally trivial semi-algebraic fibration. Next, replace the partition of unity $\{\eta_\alpha\}_{\alpha\in\Lambda}$ subordinate to the partition $(U_\alpha)_{\alpha\in\Lambda}$ by a a semi-algebraic partition of unity $\{\eta_i\}_{i\in I}$ subordinate to the (finite) cover $(U_i)_{i\in I}$ which is known to exist (see [8, Lem. 12.7.3]). These two modifications to the proof of [1, Prop. 4.6.3] produce a proof of Claim A.13. Proposition A.12 now follows from Claim A.13 in exactly the same way as Theorem 4.6.4 is deduced from Proposition 4.6.3 in [1]. □

Proof of Proposition A.11 First observe that it is obvious that the restriction of a trivial semi-algebraic fibration $p'\colon E' \to B'$ to to a semi-algebraic subset $B'' \subset B'$ is again a trivial semi-algebraic fibration. Let $p'\colon E' \to B'$ be the restriction of the locally trivial semi-algebraic fibration p to B'. Now since B' is assumed to semi-algebraically contractible, there exists a semi-algebraic homotopy between $f = \mathrm{id}_{B'}$ and a constant map $g\colon B' \to B'$. We obtain that $p' \cong f^* p' \cong g^* p'$, where the first isomorphism is due to the fact that f is the identity map, and the second isomorphism is a consequence of Proposition A.12 and the fact that f and g are homotopic. Now since $g^* p'$ is the pull-back of p' under a constant map, it is clearly a trivial semi-algebraic fibration. Finally, the observation at the beginning of the proof implies that p' restricted to $B'' \subset B'$ is also trivial. □

A.3 Proof of Proposition 5.6

For the rest of this section we fix a real closed field R, a finite set of polynomials

$$\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell],$$

as well as a $\mathcal{P}$-closed formula ϕ. We denote $S = \mathcal{R}(\phi, \mathrm{R}^{k+\ell})$.

Notation A.14 For $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_s)$, we denote by $\phi^{\star,c}(\bar{\varepsilon})$, the $\mathcal{P}^\star(\bar{\varepsilon})$-closed formula obtained by replacing each occurrence of $P_i \geq 0$ in ϕ by $P_i + \varepsilon_i \geq 0$ (resp. $P_i \leq 0$ in ϕ by $P_i - \varepsilon_i \leq 0$) for $1 \leq i \leq s$, where

$$\mathcal{P}^\star(\bar{\varepsilon}) = \bigcup_{1\leq i\leq s} \{P_i + \varepsilon_i, P_i - \varepsilon_i\}.$$

Observe that

$$S^{\star,c}(\bar{\varepsilon}) := \mathcal{R}(\phi^{\star,c}(\bar{\varepsilon}), \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell}) \tag{A.1}$$

is a $\mathcal{P}^\star(\bar{\varepsilon})$-closed semi-algebraic set.

Remark A.15 We will use a few results whose proofs can be found in [4]. These results are stated in [4] not using the language of Puiseux extensions but rather with the ε_i's occurring in the statement as small enough positive elements of the ground field R. However, the statements in [4] imply the corresponding statements of the current paper in terms of Puiseux series by an application of a standard argument using the Tarski–Seidenberg transfer principle.

More precisely, the following fact which is a consequence of the Tarski–Seidenberg transfer principle will be used repeatedly. If $\phi(T)$ is a first order formula with constants in a real closed field R, then the first order sentence

$$(\exists T_0)(T_0 > 0) \wedge (\forall T)((0 < T) \wedge (T < T_0)) \implies \phi(T)$$

is true over R if and only if the sentence $\phi(\varepsilon)$ is true over $\mathrm{R}\langle\varepsilon\rangle$.

In order to see this observe that $\mathcal{R}(\phi(T), \mathrm{R})$ is a semi-algebraic subset of R which contains an interval $(0, t_0)$ for some $t_0 > 0$. Then by the Tarski–Seidenberg transfer principle, $(0, t_0) \subset \mathcal{R}(\phi(T), \mathrm{R}\langle\varepsilon\rangle)$ as well. Now, since ε is positive and smaller than all positive elements of R in the unique ordering of the real closed field $\mathrm{R}\langle\varepsilon\rangle$, $0 < \varepsilon < t_0$, and so $\varepsilon \in \mathcal{R}(\phi(T), \mathrm{R}\langle\varepsilon\rangle)$. Hence, $\phi(\varepsilon)$ is true over $\mathrm{R}\langle\varepsilon\rangle$.

Lemma A.16 *For each $\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon})$, $\mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell})$ is either empty or is a non-singular $(k + \ell - \mathrm{card}(\mathcal{Q}))$-dimensional real variety such that at every point*

$$(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_k, y_1, \ldots, y_\ell) \in \mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell}),$$

the $(\mathrm{card}(\mathcal{Q}) \times (k + \ell))$-Jacobian matrix,

$$\left(\frac{\partial P}{\partial X_i}, \frac{\partial P}{\partial Y_j}\right)_{\substack{P \in \mathcal{Q} \\ 1 \le i \le k, 1 \le j \le \ell}}$$

has the maximal possible rank.

Proof See the proof of [4, Lem. 3.3] and use Remark A.15. □

Lemma A.17 *For each $\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon})$, and $\mathbf{y} \in \mathrm{R}^\ell$, $\mathrm{Z}(\mathcal{Q}(\cdot, \mathbf{y}), \mathrm{R}\langle\bar{\varepsilon}\rangle^k)$ is either empty or is a non-singular $(k - \mathrm{card}(\mathcal{Q}))$-dimensional real variety such that at every point*

$$(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_k, y_1, \ldots, y_\ell) \in \mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell}),$$

the $(\mathrm{card}(\mathcal{Q}) \times k)$-Jacobian matrix,

$$\left(\frac{\partial P}{\partial X_i}\right)_{\substack{P \in \mathcal{Q} \\ 1 \le i \le k}}$$

has the maximal possible rank.

Proof Similar to the proof of Lemma A.16. □

Notation A.18 *[critical points and critical values]* For $\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon})$, we denote by $\mathrm{Crit}(\mathcal{Q})$ the subset of $\mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell})$ at which the Jacobian matrix,

$$\left(\frac{\partial P}{\partial X_i}\right)_{\substack{P \in \mathcal{Q} \\ 1 \le i \le k}}$$

is not of the maximal possible rank. We denote $\mathrm{crit}(\mathcal{Q}) = \pi_Y(\mathrm{Crit}(\mathcal{Q}))$.

Definition A.19 Let

$$G_1(\bar{\varepsilon}) = \bigcup_{\substack{\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon}) \\ 0 < \mathrm{card}(\mathcal{Q}) \le k}} \mathrm{crit}(\mathcal{Q}), \qquad G_2(\bar{\varepsilon}) = \bigcup_{\substack{\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon}) \\ k < \mathrm{card}(\mathcal{Q}) \le k+\ell}} \pi_Y(\mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell})),$$
$$\text{and} \quad G(\bar{\varepsilon}) = G_1(\bar{\varepsilon}) \cup G_2(\bar{\varepsilon}).$$

Lemma A.20 *Let*

$$\widehat{S} := \pi_Y^{-1}(\mathrm{R}\langle\bar{\varepsilon}\rangle^{\ell} \setminus G(\bar{\varepsilon})) \cap S^{\star,c}(\bar{\varepsilon}).$$

Then, the map $\pi_Y|_{\widehat{S}}$ is a locally trivial semi-algebraic fibration.

Proof See the proof of [4, Lem. 3.8] and use Remark A.15. □

Lemma A.21 *There is a finite set of polynomials $\mathcal{G}\{G_1, \ldots, G_t\} \subset \mathrm{R}[\bar{\varepsilon}][Y_1, \ldots, Y_\ell]$, and a $\mathcal{G}$-formula ψ satisfying the following.*

(a) $G(\bar{\varepsilon}) = \mathcal{R}(\psi, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\ell})$,
(b) $\mathrm{card}(\mathcal{G}) \le (ksd)^{O(k\ell)}$,
(c) $\max_{1 \le i \le t} \left(\deg_{\bar{Y}}(G_i), \deg_{\bar{\varepsilon}} \deg(G_i)\right) \le d^{O(k\ell)}$.
(d) *Moreover, each polynomial in $\mathcal{G}$ depends on at most $k + \ell$ of the ε_i's.*

Proof For each subset $\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon})$ with $0 < \mathrm{card}(\mathcal{Q}) \le k$, use Proposition A.8 to obtain a set of polynomials $\mathcal{G}_{1,\mathcal{Q}} \subset \mathrm{R}[\bar{\varepsilon}][Y_1, \ldots, Y_\ell]$ such that $\mathrm{crit}(\mathcal{Q})$ is a $\mathcal{G}_{1,\mathcal{Q}}$-semi-algebraic set defined by a $\mathcal{G}_{1,\mathcal{Q}}$-formula $\psi_{1,\mathcal{Q}}$. For each subset $\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon})$ with $k < \mathrm{card}(\mathcal{Q}) \le k + \ell$, use Proposition A.8 to obtain a set of polynomials $\mathcal{G}_{2,\mathcal{Q}} \subset \mathrm{R}[\bar{\varepsilon}][Y_1, \ldots, Y_\ell]$ such that $\pi_Y(\mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\ell}))$ is a $\mathcal{G}_{2,\mathcal{Q}}$-semi-algebraic set defined by a $\mathcal{G}_{2,\mathcal{Q}}$-formula $\psi_{2,\mathcal{Q}}$. Let

$$\mathcal{G} = \{G_1, \ldots, G_t\} = \bigcup_{\substack{\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon}) \\ 0 < \mathrm{card}(\mathcal{Q}) \le k}} \mathcal{G}_{1,\mathcal{Q}} \cup \bigcup_{\substack{\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon}) \\ k < \mathrm{card}(\mathcal{Q}) \le k+\ell}} \mathcal{G}_{2,\mathcal{Q}},$$

and

$$\psi = \bigvee_{\substack{\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon}) \\ 0 < \mathrm{card}(\mathcal{Q}) \le k}} \psi_{1,\mathcal{Q}} \vee \bigvee_{\substack{\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon}) \\ k < \mathrm{card}(\mathcal{Q}) \le k+\ell}} \psi_{2,\mathcal{Q}}.$$

Then

$$\begin{aligned}\mathcal{R}(\psi, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\ell}) &= \bigcup_{\substack{\mathcal{Q}\subset\mathcal{P}^{\star}(\bar{\varepsilon})\\ 0<\mathrm{card}(\mathcal{Q})\le k}} \mathcal{R}(\psi_{1,\mathcal{Q}}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\ell}) \cup \bigcup_{\substack{\mathcal{Q}\subset\mathcal{P}^{\star}(\bar{\varepsilon})\\ k<\mathrm{card}(\mathcal{Q})\le k+\ell}} \mathcal{R}(\psi_{2,\mathcal{Q}}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\ell})\\ &= \bigcup_{\substack{\mathcal{Q}\subset\mathcal{P}^{\star}(\bar{\varepsilon})\\ 0<\mathrm{card}(\mathcal{Q})\le k}} \pi_Y(\mathrm{Crit}(\mathcal{Q})) \cup \bigcup_{\substack{\mathcal{Q}\subset\mathcal{P}^{\star}(\bar{\varepsilon})\\ k<\mathrm{card}(\mathcal{Q})\le k+\ell}} \pi_Y(\mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{\ell}))\\ &= G_1(\bar{\varepsilon}) \cup G_2(\bar{\varepsilon}) = G(\bar{\varepsilon}).\end{aligned}$$

Observe that each of the sets $\mathcal{G}_{i,\mathcal{Q}}$, where $i = 1, 2$ and $\mathcal{Q} \subset \mathcal{P}^{*}(\bar{\varepsilon})$, depends on at most $k + \ell$ of the ε_i's. Moreover, using (b) and (c) of Proposition A.8 we can assume that

$$\mathrm{card}(\mathcal{G}) \le (ksd)^{O(k\ell)}, \quad \max_{G\in\mathcal{G}} \bigl(\deg_Y(G), \deg_{\bar{\varepsilon}} \deg(G)\bigr) \le d^{O(k\ell)}$$

(where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P\in\mathcal{P}} \deg(P)$). □

Without loss of generality we can assume that

$$\psi = \bigvee_{\sigma\in\Sigma} \psi_\sigma,$$

for some subset $\Sigma \subset \mathrm{SIGN}(\mathcal{G})$, where for $\sigma \in \Sigma$, ψ_σ is the formula defined by

$$\psi_\sigma = \bigwedge_{1\le i\le t} (G_i \sigma_i 0),$$

where σ_i equals $>, <, 0$ according to $\sigma(G_i) = 1, -1, 0$, respectively.

Denote by $\psi^{\star,o}(\bar{\varepsilon}, \bar{\delta})$, the $\mathcal{G}^{\star}(\bar{\varepsilon}, \bar{\delta})$-open formula obtained by replacing each occurrence of $G_i = 0$ in ψ by $(G_i - \delta_i < 0) \wedge (G_i + \delta_i > 0)$, $G_i > 0$ in ψ by $G_i + \delta_i > 0$, and each occurrence of $G_i < 0$ by $G_i - \delta_i < 0$, for $1 \le i \le t$, where

$$\mathcal{G}^{\star}(\bar{\varepsilon}, \bar{\delta}) = \bigcup_{1\le i\le t} \{G_i + \delta_i, G_i - \delta_i\}.$$

Let

$$G^{+,o}(\bar{\varepsilon}, \bar{\delta}) := \mathcal{R}(\psi^{\star,o}(\bar{\varepsilon}, \bar{\delta}), \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}\rangle^{\ell}).$$

Then

$$G^{+,o}(\bar{\varepsilon}, \bar{\delta})^{c} := \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}\rangle^{\ell} \setminus G^{+,o}(\bar{\varepsilon}, \bar{\delta})$$

is a $\mathcal{G}^{\star}(\bar{\varepsilon}, \bar{\delta})$-closed semi-algebraic set.

Lemma A.22 *The set $\mathcal{G}^\star(\bar{\varepsilon}, \bar{\delta})$ is in strong ℓ-general position over $\mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}\rangle^\ell$.*

Proof Similar to the proof of Lemma A.16. □

Lemma A.23 *For each $R > 0$, $R \in \mathrm{R}$, there exists a finite set of polynomials $\mathcal{H} \subset \mathrm{R}[\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}][Y_1, \ldots, Y_\ell]$, where $\bar{\zeta} = (\zeta_1, \ldots, \zeta_{2\mathrm{card}(\mathcal{H})})$, and a tuple of $(C_\alpha)_{I\in\alpha}$ of $\mathcal{H}$-closed semi-algebraic sets satisfying the following.*

(a) *Each C_α is an $\mathcal{H}$-semi-algebraic set which is moreover semi-algebraically contractible.*
(b) $\displaystyle\bigcup_{\alpha\in I} C_\alpha = G^{+,o}(\bar{\varepsilon}, \bar{\delta})^c \cap [-R, R]^\ell$.
(c) $\mathrm{card}(I), \mathrm{card}(\mathcal{H}) \le (sd)^{k^{O(\ell)}}$; $\deg_{\bar{Y}}(H), \deg_{\bar{\varepsilon}}(H), \deg_{\bar{\delta}}(H), \deg_{\bar{\zeta}}(H) \le d^{k^{O(\ell)}}$.
(d) *Moreover, each polynomial in $\mathcal{H}$ depends on at most $(k+\ell)(\ell+1)^2$ of ε_i's, at most $(\ell+1)^2$ of the δ_i's and on at most one of the ζ_i's.*

Proof Use Lemma A.21 and Proposition A.6. □

Since the semi-algebraic set $S = \mathcal{R}(\phi, \mathrm{R}^{k+\ell})$ is assumed to be bounded over R, there exists $R \in \mathrm{R}$, $R > 0$, such that $S \subset [-R, R]^{k+\ell}$. We fix $R > 0$, such that $S \subset [-R, R]^{k+\ell}$ in what follows. We also fix the cover $(C_\alpha)_{\alpha\in I}$ and the finite family of polynomials $\mathcal{H} \subset \mathrm{R}[\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}][Y_1, \ldots, Y_\ell]$ given by Lemma A.23.

Lemma A.24 *For each $\sigma \in \mathrm{SIGN}(\mathcal{H})$, such that $\mathcal{R}(\sigma, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}\rangle^\ell) \subset C_\alpha$ for some $\alpha \in I$, let*

$$S_\sigma := \pi_Y^{-1}(\mathcal{R}(\sigma, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}\rangle^\ell)) \cap \mathrm{Ext}(S^\star(\bar{\varepsilon}), \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}\rangle).$$

Then the map $\pi_Y|_{S_\sigma}$ is a trivial semi-algebraic fibration.

Proof Follows from Lemma A.20 and Proposition A.11 noting that the semi-algebraic set C_α is semi-algebraically contractible using part (a) of Lemma A.23. □

Using the same notation as above we have:

Lemma A.25 (a) $\displaystyle \mathrm{R}^\ell \cap [-R, R]^\ell \subset \bigcup_{\substack{\sigma\in\mathrm{SIGN}(\mathcal{H})\\ \mathcal{R}(\sigma)\subset\bigcup_{\alpha\in I} C_\alpha}} \mathcal{R}(\sigma, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}\rangle^\ell).$

(b) *There exists a finite set of polynomials, $\mathcal{Q} \subset \mathrm{R}[Y_1, \ldots, Y_\ell]$, with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$, such that for each $\sigma \in \mathrm{SIGN}(\mathcal{Q})$, there exists $\alpha \in I$ such that $\mathcal{R}(\sigma, \mathrm{R}^\ell) \subset C_\alpha$.*

Proof First we prove (a). It follows from the property guaranteed by (b) of Lemma A.23 that for each $\alpha \in I$,

$$C_\alpha \subset G^{+,o}(\bar{\varepsilon}, \bar{\delta})^c \cap [-R, R]^\ell,$$
$$\bigcup_{\alpha\in I} C_\alpha = G^{+,o}(\bar{\varepsilon}, \bar{\delta})^c \cap [-R, R]^\ell = \bigcup_{\substack{\sigma\in\mathrm{SIGN}(\mathcal{H})\\ \mathcal{R}(\sigma)\subset\bigcup_{\alpha\in I} C_\alpha}} \mathcal{R}(\sigma, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}, \bar{\zeta}\rangle^\ell).$$

So it suffices to prove that $G^{+,o}(\bar{\varepsilon},\bar{\delta})\cap[-R,R]^\ell\cap \mathrm{R}^\ell=\emptyset$. Since, $\lim_{\bar{\delta}} G^{+,o}(\bar{\varepsilon},\bar{\delta})\cap[-R,R]^\ell = G(\bar{\varepsilon})\cap[-R,R]^\ell$, in order to prove that $G^{+,o}(\bar{\varepsilon},\bar{\delta})\cap[-R,R]^\ell\cap\mathrm{R}^\ell=\emptyset$, it is enough to show that $G(\bar{\varepsilon})\cap[-R,R]^\ell\cap\mathrm{R}^\ell=\emptyset$. Suppose there exists $\mathbf{y}\in G(\bar{\varepsilon})\cap[-R,R]^\ell\cap\mathrm{R}^\ell$. Then there are two cases:

Case 1 There is $\mathcal{Q}\subset\mathcal{P}^\star(\bar{\varepsilon})$, $0<\mathrm{card}(\mathcal{Q})\le k$, such that $\mathbf{y}\in\mathrm{crit}(\mathcal{Q})$. But this would imply that there exists $(\mathbf{x},\mathbf{y})\in\mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell}$ such that the rank of the corresponding Jacobian matrix

$$\left(\frac{\partial P}{\partial X_i}\right)_{\substack{P\in\mathcal{Q}\\ 1\le i\le k}}$$

at the point $(\mathbf{x},\mathbf{y})$ is not full, which contradicts Lemma A.17.

Case 2 There is $\mathcal{Q}\subset\mathcal{P}^\star(\bar{\varepsilon})$, $k<\mathrm{card}(\mathcal{Q})\le k+\ell$, such that $\mathbf{y}\in\pi_Y(\mathrm{Z}(\mathcal{Q},\mathrm{R}\langle\bar{\varepsilon}\rangle^{k+\ell}))$. But since $\mathrm{card}(\mathcal{Q})>k$, and $\mathbf{y}\in\mathrm{R}^\ell$, by Lemma A.17, $\mathrm{Z}(\mathcal{Q}(\,\cdot\,,\mathbf{y}),\mathrm{R}\langle\bar{\varepsilon}\rangle^k)=\emptyset$, which is a contradiction.

We turn to the proof of (b). Let $\eta=(\bar{\varepsilon},\bar{\delta},\bar{\zeta})$. For each $H\in\mathcal{H}$, write

$$H=\sum_\alpha H_\alpha\eta^\alpha,$$

where $\alpha\in\mathbb{N}^{|\bar{\varepsilon}|+|\bar{\delta}|+|\bar{\zeta}|}$ is a multi-index exponent, and each $H_\alpha\in\mathrm{R}[Y_1,\dots,Y_\ell]$. Denote by $\mathrm{Supp}(H)=\{\alpha\,|\,H_\alpha\ne0\}$. It follows from the bounds on the set $\mathcal{H}$ stated in parts (c) and (d) of Lemma A.23 that $\mathrm{card}(\mathrm{Supp}(H))\le(sd)^{(k+\ell)^{O(1)}}$. Let

$$\mathcal{Q}=\bigcup_{H\in\mathcal{H}}\{H_\alpha\mid H\in\mathrm{Supp}(H)\}.$$

It follows from the definition of the ordering of the real closed field $\mathrm{R}\langle\eta\rangle$ (cf. Notation A.1), that for any $\mathbf{y}\in\mathrm{R}^\ell$, and $H\in\mathcal{H}$, $\mathbf{sign}(H(\mathbf{y}))$ is determined by the tuple of signs $(\mathbf{sign}(H_\alpha(\mathbf{y})))_{\alpha\in\mathrm{Supp}(H)}$. It is now easy to see that $\mathcal{Q}$ satisfies the claimed properties. □

Lemma A.26 *Let* $\mathbf{y}\in\mathrm{R}^\ell$. *Then, there exists a semi-algebraic deformation retraction* $S^\star(\bar{\varepsilon})_{\mathbf{y}}\to\mathrm{Ext}(S_{\mathbf{y}},\mathrm{R}\langle\bar{\varepsilon}\rangle)$. *In particular,* $\mathrm{Ext}(S_{\mathbf{y}},\mathrm{R}\langle\bar{\varepsilon}\rangle)$ *is semi-algebraically homotopy equivalent to* $S^\star(\bar{\varepsilon})_{\mathbf{y}}$.

Proof This is a consequence of [6, Lem. 16.17] noting that the set $S^\star(\bar{\varepsilon})_{\mathbf{y}}$ is bounded over R. □

Proof of Proposition 5.6 Let $\mathcal{Q}$ be as in Lemma A.25, and let $\sigma\in\mathrm{SIGN}(\mathcal{Q})$, such that $\mathcal{R}(\sigma,\mathrm{R}^\ell)\subset C_\alpha$ for some $\alpha\in I$ (following the same notation as in Lemma A.25). Let $C\in\mathrm{Cc}(\mathcal{R}(\sigma,\mathrm{R}^\ell))$. Now, let $D_1^\alpha,\dots,D_N^\alpha\subset\mathrm{R}\langle\bar{\varepsilon},\bar{\delta}\rangle^\ell$ be the elements of $\mathrm{Cc}(D_\alpha)$ where $D_\alpha=\pi_Y^{-1}(C_\alpha)\cap\mathrm{Ext}(S^\star(\bar{\varepsilon})\cap[-R,R]^{k+\ell},\mathrm{R}\langle\bar{\varepsilon},\bar{\delta}\rangle)$. The proposition will follow from the following two claims. □

Claim A.27 $\left(\lim_{\bar{\varepsilon},\bar{\delta}} D_\alpha\right)_C=S_C$.

Claim A.28 $\left(\lim_{\bar\varepsilon,\bar\delta} D^i_\alpha\right)_C$, $i = 1, \ldots, N$, *are the semi-algebraically connected components of* S_C.

Proof of Claim A.27 First observe that since $S^{\star,c}(\bar\varepsilon) \cap [-R, R]^{k+\ell}$ is bounded over R, so is D_α. Also, it is clear that $S_C \subset \left(\lim_{\bar\varepsilon,\bar\delta} D_\alpha\right)_C$. To prove the reverse inclusion, let $(\mathbf{x}, \mathbf{y}) \in \left(\lim_{\bar\varepsilon,\bar\delta} D_\alpha\right)_C$. Then, there exists $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in D_\alpha$, such that

$$\lim_{\bar\varepsilon,\bar\delta} (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = (\mathbf{x}, \mathbf{y}).$$

Since $S^{\star,c}(\bar\varepsilon) \cap [-R, R]^{k+\ell}$ (cf. (A.1)) is a closed semi-algebraic subset of $\mathrm{R}\langle\bar\varepsilon\rangle^{k+\ell}$ bounded over R, it follows that

$$\lim_{\bar\delta} \mathrm{Ext}\left(S^{\star,c}(\bar\varepsilon) \cap [-R, R]^{k+\ell}, \mathrm{R}\langle\bar\varepsilon, \bar\delta\rangle\right) = S^{\star,c}(\bar\varepsilon) \cap [-R, R]^{k+\ell}.$$

This implies

$$\lim_{\bar\delta} (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in S^{\star,c}(\bar\varepsilon)_{\lim_{\bar\delta} \tilde{\mathbf{y}}}.$$

Finally, it follows from

$$\lim_{\bar\varepsilon} (S^{\star,c} \cap [-R, R]^{k+\ell}) = S$$

that

$$\lim_{\bar\varepsilon,\bar\delta} (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in S_{\mathbf{y}} \subset S_C. \qquad \square$$

Proof of Claim A.28 Using Claim A.27 it suffices to prove that for each i, $1 \le i \le N$, $\left(\lim_{\bar\varepsilon,\bar\delta} D^i_\alpha\right)_C$ is semi-algebraically connected, and for $1 \le i < j \le N$,

$$\left(\lim_{\bar\varepsilon,\bar\delta} D^i_\alpha\right)_C \cap \left(\lim_{\bar\varepsilon,\bar\delta} D^j_\alpha\right)_C = \emptyset. \tag{A.2}$$

In order to prove that $\left(\lim_{\bar\varepsilon,\bar\delta} D^i_\alpha\right)_C$ is semi-algebraically connected, let

$$(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \left(\lim_{\bar\varepsilon,\bar\delta} D^i_\alpha\right)_C.$$

Let $\gamma : [0, 1] \to C$ be a semi-algebraic path, with $\gamma(0) = \mathbf{y}$, $\gamma(1) = \mathbf{y}'$ (which exists since C is semi-algebraically connected). First observe, that since $\mathbf{y}, \mathbf{y}' \in C \subset \mathrm{R}^\ell$, it follows from Lemma A.26, that $\mathbf{x}, \mathbf{x}' \in D^i_\alpha$. Then, using Lemma A.24, there exists a lift of $\mathrm{Ext}(\gamma, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}\rangle)$ to a semi-algebraic path,

$$\widehat{\gamma}\colon [0,1] \to D^i_\alpha \cap \pi_Y^{-1}(\mathrm{Ext}(C, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}\rangle)),$$
$$\widehat{\gamma}(0) = (\mathbf{x}, \mathbf{y}), \quad \widehat{\gamma}(1) = (\mathbf{x}', \mathbf{y}'), \quad \pi_Y(\widehat{\gamma}(t)) = \mathrm{Ext}(\gamma, \mathrm{R}\langle\bar{\varepsilon}, \bar{\delta}\rangle)(t), \quad 0 \le t \le 1.$$

Then

$$\lim_{\bar{\varepsilon},\bar{\delta}} \circ \widehat{\gamma}\colon [0,1] \to \left(\lim_{\bar{\varepsilon},\bar{\delta}} D^i_\alpha\right)_C$$

is a semi-algebraic path connecting $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{x}', \mathbf{y}')$ with image in $\left(\lim_{\bar{\varepsilon},\bar{\delta}} D^i_\alpha\right)_C$, proving that $\left(\lim_{\bar{\varepsilon},\bar{\delta}} D^i_\alpha\right)_C$ is semi-algebraically connected.

We now prove that for $1 \le i < j \le N$, (A.2) holds. Suppose that

$$(\mathbf{x}, \mathbf{y}) \in \left(\lim_{\bar{\varepsilon},\bar{\delta}} D^i_\alpha\right)_C \cap \left(\lim_{\bar{\varepsilon},\bar{\delta}} D^j_\alpha\right)_C.$$

Using Lemma A.26 and the fact that $\mathbf{y} \in C \subset \mathrm{R}^\ell$, this would imply that $(\mathbf{x}, \mathbf{y}) \in D^i_\alpha \cap D^j_\alpha$. But this is impossible, since D^i_α and D^j_α are distinct semi-algebraically connected components of D_α. The proposition now follows from Claim A.28. □

References

1. Aguilar, M., Gitler, S., Prieto, C.: Algebraic Topology from a Homotopical Viewpoint. Universitext. Springer, New York (2002)
2. Basu, S.: On bounding the Betti numbers and computing the Euler characteristic of semi-algebraic sets. Discrete Comput. Geom. **22**(1), 1–18 (1999)
3. Basu, S.: A complexity theory of constructible functions and sheaves. Found. Comput. Math. **15**(1), 199–279 (2015)
4. Basu, S., Pollack, R., Roy, M.-F.: Betti number bounds, applications and algorithms. In: Combinatorial and Computational Geometry. Math. Sci. Res. Inst. Publ., vol. 52, pp. 87–96. Cambridge University Press, Cambridge (2005)
5. Basu, S., Pollack, R., Roy, M.-F.: On the Betti numbers of sign conditions. Proc. Am. Math. Soc. **133**(4), 965–974 (2005)
6. Basu, S., Pollack, R., Roy, M.-F.: Algorithms in Real Algebraic Geometry. Algorithms and Computation in Mathematics, vol. 10. Springer, Berlin (2006)
7. Basu, S., Vorobjov, N.: On the number of homotopy types of fibres of a definable map. J. Lond. Math. Soc. **76**(3), 757–776 (2007)
8. Bochnak, J., Coste, M., Roy, M.-F.: Géométrie Algébrique Réelle. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 12. Springer, Berlin (1987)
9. Burlet, O., de Rham, G.: Sur certaines applications génériques d'une variété close à 3 dimensions dans le plan. Enseign. Math. **20**, 275–292 (1974)
10. Coste, M.: An Introduction to o-Minimal Geometry. Lecture notes (1999). https://perso.univ-rennes1.fr/michel.coste/polyens/OMIN.pdf
11. Dey, T.K., Mémoli, F., Wang, Y.: Topological analysis of nerves, Reeb spaces, mappers, and multiscale mappers. In: 33rd International Symposium on Computational Geometry (Brisbane 2017). Leibniz Int. Proc. Inform., vol. 77, # 36. Leibniz-Zent. Inform., Wadern (2017)
12. van den Dries, L.: Remarks on Tarski's problem concerning ($\mathbf{R}, +, \cdot, \exp$). In: Logic Colloquium '82 (Florence 1982). Stud. Logic Found. Math., vol. 112, pp. 97–121. North-Holland, Amsterdam (1984)

13. van den Dries, L.: Tame Topology and o-Minimal Structures. London Mathematical Society Lecture Note Series, vol. 248. Cambridge University Press, Cambridge (1998)
14. van den Dries, L., Miller, Ch.: Geometric categories and o-minimal structures. Duke Math. J. **84**(2), 497–540 (1996)
15. van den Dries, L., Speissegger, P.: The real field with convergent generalized power series. Trans. Am. Math. Soc. **350**(11), 4377–4421 (1998)
16. van den Dries, L., Speissegger, P.: The field of reals with multisummable series and the exponential function. Proc. Lond. Math. Soc. **81**(3), 513–565 (2000)
17. Edelsbrunner, H., Harer, J.L.: Computational Topology. American Mathematical Society, Providence (2010)
18. Edelsbrunner, H., Harer, J., Patel, A.K.: Reeb spaces of piecewise linear mappings. In: 24th Annual Symposium on Computational Geometry (College Park 2008), pp. 242–250. ACM, New York (2008)
19. Gabrielov, A., Vorobjov, N.: Betti numbers of semialgebraic sets defined by quantifier-free formulae. Discrete Comput. Geom. **33**(3), 395–401 (2005)
20. Gabrielov, A., Vorobjov, N.: Approximation of definable sets by compact families, and upper bounds on homotopy and homology. J. Lond. Math. Soc. **80**(1), 35–54 (2009)
21. Gabrielov, A., Vorobjov, N., Zell, T.: Betti numbers of semialgebraic and sub-Pfaffian sets. J. Lond. Math. Soc. **69**(1), 27–43 (2004)
22. Godement, R.: Topologie Algébrique et Théorie des Faisceaux. Publ. Math. Univ. Strasbourg, vol. 13. Hermann, Paris (1958)
23. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics, vol. 52. Springer, New York (1977)
24. Milnor, J.: On the Betti numbers of real varieties. Proc. Am. Math. Soc. **15**, 275–280 (1964)
25. Mimura, M., Toda, H.: Topology of Lie Groups. I, II. Translations of Mathematical Monographs, vol. 91. American Mathematical Society, Providence (1991)
26. Munch, E., Wang, B.: Convergence between categorical representations of Reeb space and mapper (2015). arXiv:1512.04108
27. Patel, A.: Reeb Spaces and the Robustness of Preimages. PhD thesis, Duke University (2010). https://dukespace.lib.duke.edu/dspace/handle/10161/2447
28. Petrovskiĭ, I.G., Oleĭnik, O.A.: On the topology of real algebraic surfaces. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**(5), 389–402 (1949). (in Russian)
29. Pillay, A., Steinhorn, C.: Definable sets in ordered structures. I. Trans. Am. Math. Soc. **295**(2), 565–592 (1986)
30. Pillay, A., Steinhorn, C.: Definable sets in ordered structures. III. Trans. Am. Math. Soc. **309**(2), 469–476 (1988)
31. Reeb, G.: Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique. C. R. Acad. Sci. Paris **222**, 847–849 (1946)
32. Rolin, J.-P., Speissegger, P., Wilkie, A.J.: Quasianalytic Denjoy–Carleman classes and o-minimality. J. Am. Math. Soc. **16**(4), 751–777 (2003)
33. de Silva, V., Munch, E., Patel, A.: Categorified Reeb graphs. Discrete Comput. Geom. **55**(4), 854–906 (2016)
34. Singh, G., Mémoli, F., Carlsson, G.: Topological methods for the analysis of high dimensional data sets and 3D object recognition. In: Eurographics Symposium on Point-Based Graphics (Prague 2007), pp. 99–100. The Eurographics Association, Aire-la-Ville (2007)
35. Thom, R.: Sur l'homologie des variétés algébriques réelles. In: Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), pp. 255–265. Princeton University Press, Princeton (1965)
36. Wilkie, A.J.: Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. J. Am. Math. Soc. **9**(4), 1051–1094 (1996)
37. Wilkie, A.J.: A theorem of the complement and some new o-minimal structures. Sel. Math. **5**(4), 397–421 (1999)
38. Wilkie, A.J.: o-minimal structures. Astérisque **326**, 131–142 (2009)

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