Stable Fluids

by

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based on Stam, 1999

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We hereby declare that this article is based on our work, unless stated otherwise. No other person's work has been used without due acknowledgement in this thesis. All references and verbatim extracts have been quoted, and all sources of information have been specifically acknowledged.

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0 Abstract

1 Introduction

The concept of this solver is based on [2]. The method of solution has been implemented in this article. The program runs in the framework of JavaView [4].
We added
-¿Colors
-¿Fotoimport
-¿nice user interface
-¿Insertion in the framework of JavaView.

2 Mathematical Modelling

2.1 Basic Equations

Let $\Omega \subset \mathbb{R}^2$ be the domain of interest, e.g. $\Omega = (0,1)^2$. Now let $u \in C^2(\mathbb{R} \times \Omega, \mathbb{R}^2)$ denote the velocity vector field and $p \in C^2(\mathbb{R} \times \Omega, \mathbb{R})$ denote the pressure field. Both fields depend on the time $t \in \mathbb{R}$ and the position in space $x \in \Omega$. The evolution of these fields is given by the Navier-Stokes equations

$$\text{div } u = 0 \tag{2.1}$$

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u - \frac{1}{\rho}\nabla p + \nu \Delta u + f, \tag{2.2}$$

where ν, ρ are constants that determine the viscosity of the fluid and the density respectively. In $\mathbb{R}^2$, equation 2.2 can be written out as

$$\begin{pmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \end{pmatrix} = -\begin{pmatrix} u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} \\ u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} \end{pmatrix} - \frac{1}{\rho}\begin{pmatrix} \frac{\partial p}{\partial x_1} \\ \frac{\partial p}{\partial x_2} \end{pmatrix} + \nu \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} \\ \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

The pressure and velocity field that appears in the Navier-Stokes equations, are related. By combining equation 2.1 and 2.2 we obtain a singe equation as follows.

By the Helmholtz-Hodge decomposition theorem we have that any vector field $w = u + \nabla q + \text{res}$ uniquely decomposes into a divergence-free part u, a gradient field ∇q and a residual term depending on the genus of the surface. In our case, the residual term vanishes.

Let $P : C^2(\Omega, \mathbb{R}^2) \rightarrow \{f \in C^2(\Omega, \mathbb{R}^2), \text{div} f = 0\}$ denote the projection operator onto the divergence free part. Obviously, the operator P is implicitly defined by

$$\text{div } w = \Delta q. \tag{2.3}$$

With Neumann boundary condition ($\frac{\partial q}{\partial n} = 0$ on $\partial \Omega$, n is the outward normal), equation 2.3 is a Poisson equation. Let q denote the solution, then P is defined by $Pw = w - \nabla q$. If we now apply P to both sides of 2.2, the Navier-Stokes equation compress into our fundamental equation 2.4.

$$\frac{\partial u}{\partial t} = P\left(-(u \cdot \nabla)u - \frac{1}{\rho}\nabla p + \nu \Delta u + f\right) \tag{2.4}$$

2.2 Method of Solution and Discretization

Equation 2.4 consists of four parts, the **add force** term f, the **advection** term $(u \cdot \nabla)u$, the **diffusion** term $\nu \Delta u$ and the **projection** operator P. The equation is solved from an initial state $u^0 = u(0, x)$. Both time and space are discretized with time step ΔT and some equidistant grid points of distance $h = \frac{1}{n}$. Each of the four terms in equation 2.4 is applied

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successively to the initial state $u^0 \in C^2(\Omega, \mathbb{R})$. The general procedure is

$$u^0 \xrightarrow{\text{add force}} u^1 \xrightarrow{\text{advect}} u^2 \xrightarrow{\text{diffuse}} u^3 \xrightarrow{\text{project}} u^4$$

The solution at time $t + \Delta t$ is then given by $u(x, t + \Delta t) = u^4(x)$.

2.2.1 Add force

The add force step incorporates additional force by the user, or buoyancy force due to uplift of lighter gases and downlift of heavier gases.

$$u^1(x) = u^0(x) + \Delta t \ f(t, x)$$

The buoyancy force is computed using Archimedes' principle. In a simplified approach, heaviness is equal to the density of the smoke. After computing the average temperature of the fluid, the upward force is determined for each pixel separately depending on the difference with respect to the average temperature.

2.2.2 Advect

The advect step accounts for the advection or convection of the fluid itself, i.e. this step lets the fluid „flow" a little. The advection step is fundamental to this particular fluid solver. Thats why the solver is called „Stable"Fluids, as this solver will never blow up, independent of the size of the time step Δt.
The method can be understood intuitively: All particles in the fluid are moved by the velocity of the fluid itself. To obtain the velocity at the point x at time $t + \Delta t$ we backtrace the the point x through the velocity field at time t. This defines a path $p : (-\delta, \delta) \times \Omega \to \Omega$ corresponding to a streamline of the fluid. The velocity $u^2(x)$ is the set to be the velocity of $u^1(p(-\Delta t, x))$ at the previous time step:

$$u^2(x) = u^1(p(-\Delta t, x)).$$

[tba] include a figure that illustrates the approach. [tba] [tba] include a comparison to other solvers of the advect step. [tba]

2.2.3 Diffuse

This step solves the diffusion of the fluid itself, i.e. the „friction" between parts of the fluid with different velocity. This effect is equivalent to the diffusion equation

$$\frac{\partial u^2}{\partial t} = \nu \Delta u^2. \tag{2.5}$$

The most straightforward way would be to discretize the Laplacian and solve the resulting sparse linear system. However, this approach is unstable when the viscosity is large. For our implicit approach we proceed as follows, by approximating $\frac{\partial u}{\partial t}$ with the backward difference

quotient:

$$\frac{u(t + \Delta t, x) - u(x, t)}{\Delta t} = \nu \Delta u(t - \Delta t, x)$$

finally, this gives

$$(I - \nu \Delta t)u^3(x) = u^2(x). \tag{2.6}$$

We now discretize 2.6 using a finite difference method and obtain

$$\left(\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & & 1 & & \\ 1 & & & & & \\ & & & & & 1 \\ 1 & & & & & \\ & & & & & 1 \\ & & 1 & & 1 & -4 \end{bmatrix} \nu \frac{\Delta t}{h^2} \right) u^3 = u^2. \tag{2.7}$$

The resulting square matrix has $n * m$ rows and columns, where n, m denote the number of pixels in the x- and y-axis respectively. Solving such a system can be done efficiently by iterative schemes, e.g. Gauß-Seidel.

2.2.4 Project

The last step makes the vectorfield mass preserving, i.e. divergence free. We already discussed in the derivation of our fundamental equation that the projection is obtained by solving

$$\text{div } u = \Delta q.$$

When discretized, this equation becomes

$$\frac{1}{2h} \begin{bmatrix} 0 & 1 & & & \\ 1 & & & & \\ & \ddots & & & \\ & & & & 1 \\ & & & 1 & 0 \end{bmatrix} u_1^3 + \frac{1}{2h} \begin{bmatrix} 0 & & 1 & & \\ & & & & \\ & & & & 1 \\ 1 & & & & \\ & & 1 & & 0 \end{bmatrix} u_2^3 = \begin{bmatrix} -4 & 1 & & 1 & & \\ 1 & & & & & \\ & & & & & 1 \\ 1 & & & & & \\ & & & & & 1 \\ & & 1 & & 1 & -4 \end{bmatrix} q. \tag{2.8}$$

As in equation 2.7 we need to solve a sparse linear system.

2.3 Moving substances through the Fluid

Our solver enables us to compute the ambient fluid. However, we need to visualize the vector field. A substance, that is injected in the fluid and does not interact with it, will be advected by the vector field and diffuse at the same time. Let $d \in C^2(\mathbb{R}^+ \times \Omega, \mathbb{R})$ denote the density field

of such a substance. The evolution of this scalar field is given by

$$\frac{\partial d}{\partial t} = -u \cdot \nabla a + \kappa \Delta a - \alpha a + S \tag{2.9}$$

where κ is the diffusion constant, α the dissipation term and S is a source term. The dissipation term, which will be dropped in our modelling, describes the effect that kinetic energy is converted in thermal energy. The diffusion constant determines the effect of diffusion, i.e. the effect that density „interfuses" with nearby density and the overall picture becomes „softer".

The terms in equation 2.9 are quite similar to the terms in our fundamental equation 2.4. Thus the evolution of the density field can be computed analogously to the computation of the velocity field. In particular we need to perform the advect step and the diffuse step, which correspond to the movement of the density along the velocity field and the diffusion of the density itself.

2.4 Vorticity Confinement

For sufficiently number of grid points, the steps in the previous section 2 give the fluid a realistic behaviour, quite similar to the real life experience. In practice, computational power is limited and we have to use a relatively small number of grid points to provide real-time computation. It turns out that small scale details, in particular rotational turbulences are lost due to numerical dissipation.
The key idea of Vorticity Confinement is to add these details artificially in a way that the properties of a fluid are preserved and the effect of numerical dissipation is damped. In a follow-up paper [3], Stam proposes this idea based on previous work of Steinhoff [5].

The vorticity of a 3d vector field u is defined as

$$\omega = \nabla \times u,$$

i.e. in our two dimensional case, we have

$$\omega = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} u^1 \\ u^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial}{\partial x_1} u^2 - \frac{\partial}{\partial x_2} u^1 \end{pmatrix}.$$

We identify ω with a 1d scalar field on Ω. The vorticity measures the amount of small scale detail of the fluid. The normalized gradient of the absolute vorticity

$$N = \frac{\nabla |\omega|}{|\nabla |\omega||}$$

is a vector field on Ω that points from regions of low to high vorticity.

Finally the small scale detail is added as follows

$$f = \epsilon h (N \times \omega) = \begin{pmatrix} N^1 \\ N^2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \begin{pmatrix} N^2 \omega \\ -N^1 \omega \\ 0 \end{pmatrix} \qquad (2.10)$$

Again in equation 2.10 we identify 2d vector fields with 3d vector fields that are constant in the with respect to the direction of the x^3-axis. Here $\epsilon > 0$ is a factor that determines the effect of vorticity confinement. The dependence on h gives that small scale detail is added proportional to the grid size, i.e. to the numerical dissipation. Hence the convergence of the solver is preserved.

In fact the construction provides that small scale detail is added, exactly where it is needed.
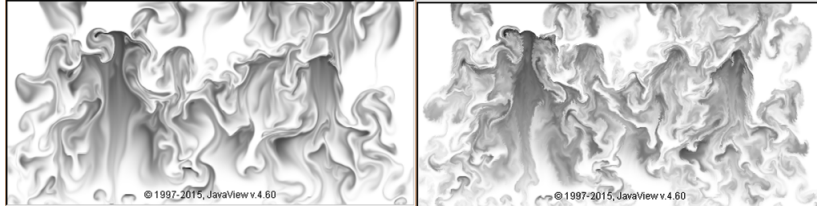

Figure 2.1: The left picture shows the fluid solver without vorticity confinement, whereas the right picture shows the same setting with medium vorticity confinement. Both pictures where computed with the same grid size and initial configuration. One can see clearly how the left picture misses small scale rotational turbulences. The right picture provides a more gaseous behaviour of the fluid.

2.5 Dependence on the Parameters

The solver is compatible with a wide range of parameters. Thus we can simulate different kinds of fluids, depending on viscosity, buoyancy force, diffusion, etc.

Figure 2.2: The left picture shows a fluid with high viscosity, e.g. honey or very thick water. The right picture shows a more liquid fluid like thin water or some gas. The left picture preserBoth pictures where created with identical initial condition and vary only in the viscosity constant. In both pictures there is no vorticity confinement to emphasize the effect of viscosity.

Figure 2.3: In the left picture there is no diffusion, whereas in the right picture we can clearly see how diffusion influences the nature of the fluid. There are less details and the picture looks overall smoother. Again both simulations vary only in the effect of diffusion and are computed with the same resolution.

2.6 Colors and Fotoimport

There are several ways to represent colors. One of the most common priciples are RGB or CMY color schemes. In the black and white solver there is only one density array which evolves according to the velocity field. To represent colors, we use three separate density fields, each represents cyan, magenta or yellow respectively and evolves according to the velocity field.

We are now able to import picture, which behave like liquid or smoke, i.e. the picture can be manipulated interactively and behaves like the ambient fluid.

One can experiment with different kinds of buoyancy forces, e.g. only specific colors generate an uplift. However

3 Implementation and Framework

4 Results

5 Extensions

6 Outlook

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Verlgleich der einzelnen Schritte vom Solver mit alternativen Lösungsansätzen -¿ Warum ist
der Solver STABLE
Darstellung der Farben mit Density-Arrays

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