CHAPTER 13
Simple Harmonic Motion

13 Oscillations

Oscillations arise in many physical systems, and can be observed at both the microscopic and macroscopic level.

The study of oscillations is confined to simple harmonic motion. Equations that describe simple harmonic oscillations are developed in this topic.

Damping and resonance are introduced, and consideration given to situations where this can be either an advantage or a disadvantage.

	Learning outcomes Candidates should be able to:
13.1 Simple harmonic oscillations	**a) describe simple examples of free oscillations** **b) investigate the motion of an oscillator using experimental and graphical methods** **c) understand and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency** **d) recognise and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion**

	e)	recall and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$
	f)	recognise and use the equations $v = v_0 \cos \omega t$ and $v = \pm\, \omega \sqrt{(x_0^2 - x^2)}$
	g)	describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion
13.2 Energy in simple harmonic motion	a)	describe the interchange between kinetic and potential energy during simple harmonic motion
13.3 Damped and forced oscillations, resonance	a)	describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and the importance of critical damping
	b)	describe practical examples of forced oscillations and resonance
	c)	describe graphically how the amplitude of a forced oscillation changes with frequency near to the natural frequency of the system, and understand qualitatively the factors that determine the frequency response and sharpness of the resonance
	d)	appreciate that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided

Oscillations

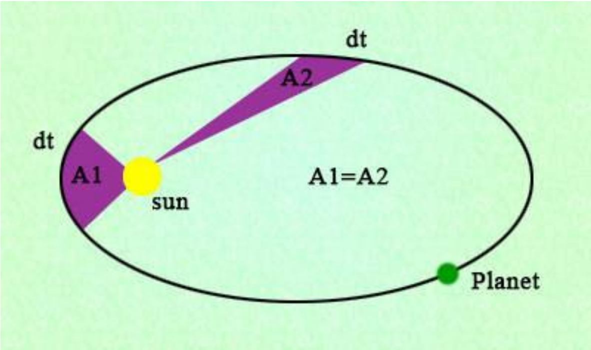
An object oscillates when it moves back and forth repeatedly, on either side of some equilibrium position. If the object is stopped from oscillating, it returns to the equilibrium position.

Oscillations can be observed in different ways:

1) Time keeping

Source: Pixabay

2) Planetary Motion



Source: flickr

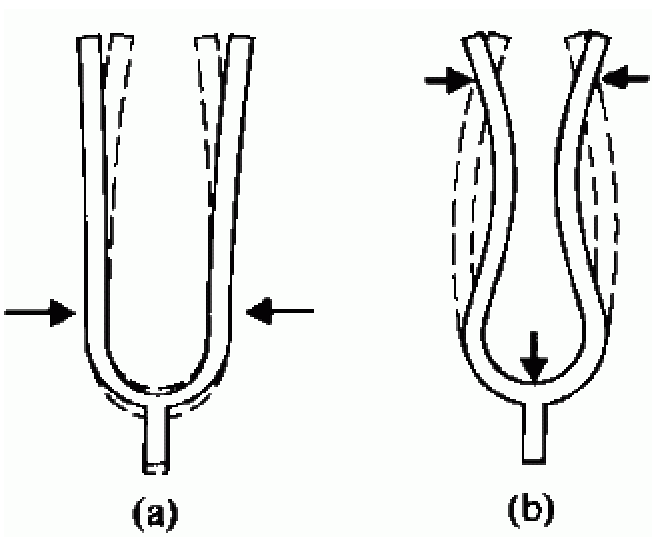
3) Musical Instruments

Source: Pixabay

Natural (free) Oscillations

The easiest oscillations to understand are **natural oscillations**, which every oscillator possesses. Every oscillator has a **natural frequency f_0** of vibration, the frequency with which it vibrates freely after an initial disturbance.

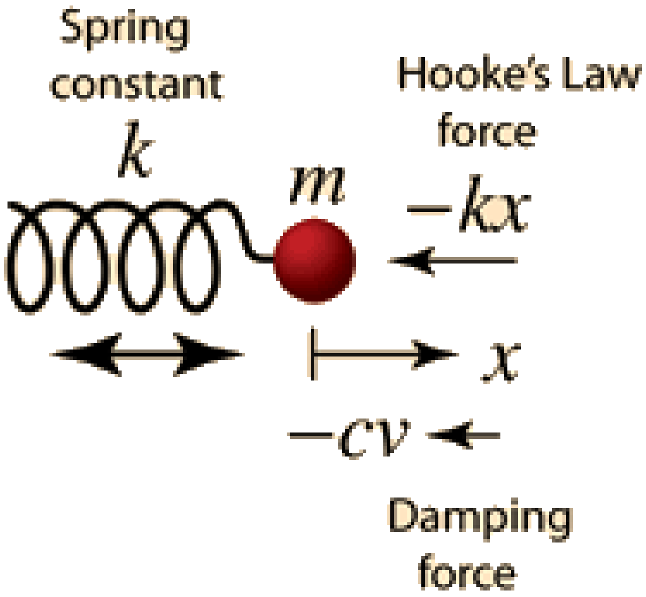
Simplest examples of natural oscillations: tuning fork and pendulum.

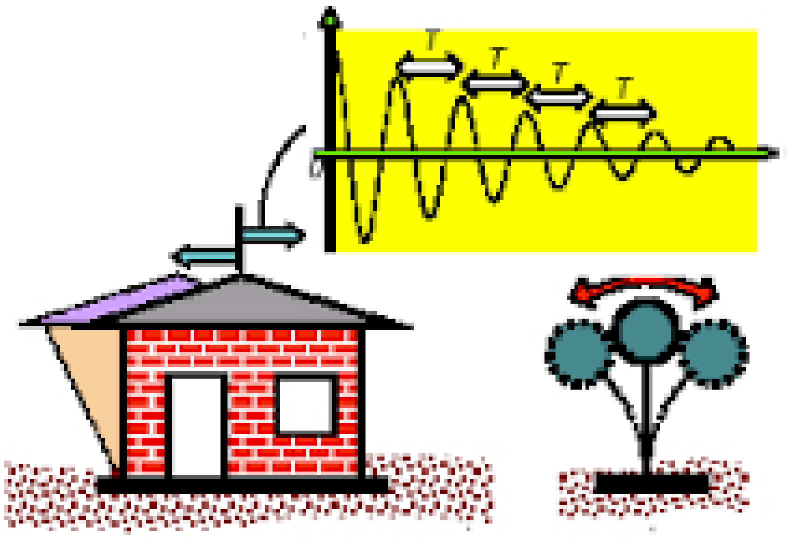


https://commons.wikimedia.org/wiki/File:Newtons_cradle.gif

Forced Oscillations

If a force is applied continuously to an oscillator, this will cause the oscillator to vibrate in a **forced oscillation**. The frequency of a forced oscillation is not the natural frequency of the oscillator but rather the forcing frequency of the external force.





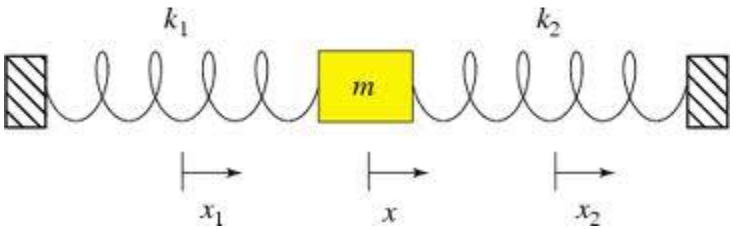
Simple Harmonic Motion

A simple harmonic motion is a special kind of oscillations.

Definition:
A simple harmonic motion is defined as the motion of a particle about a fixed point such that its acceleration a is proportional to its displacement x from the fixed point, and is directed towards the point.

Simple Harmonic Oscillators

1) Mass – Spring System



2) A Pendulum

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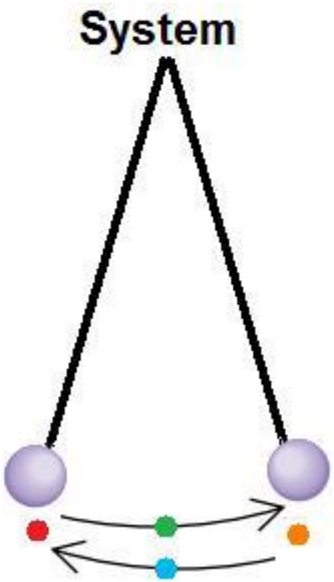

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Amplitude, Period and Frequency

The displacement changes between positive
and negative values.

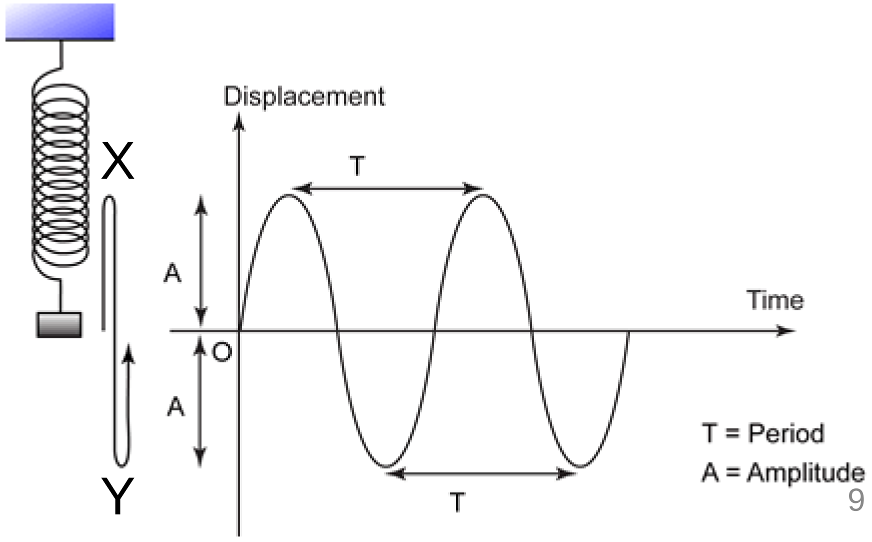
The maximum displacement is called **Amplitude,** $\boldsymbol{x_0}$ or ***A***.

The **period, *T*** is the time for one complete oscillation.

The **frequency,*f*** is the number of oscillations per unit time, a reciprocal of T.

O: Equilibrium Position

X, Y:Extreme Points



Amplitude, Period and Frequency

Recall:

Frequency, $f = \frac{1}{T}$

Angular frequency,

$$\omega = 2\pi f$$

Or

$$\omega = \frac{2\pi}{T}$$

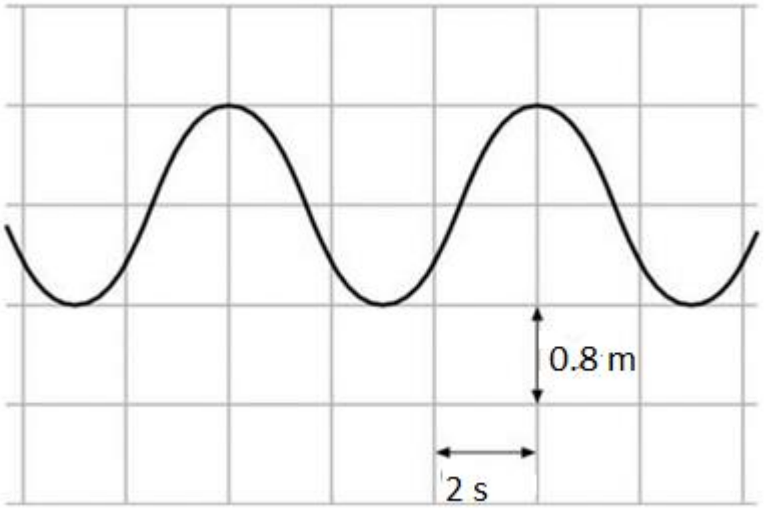
Example 1

For the graph below, determine the amplitude, frequency and time period of the oscillation.

Answer:
Amplitude, $A = 0.8$ m

Time period, $T = 3 \times 2$
$= 6$ s

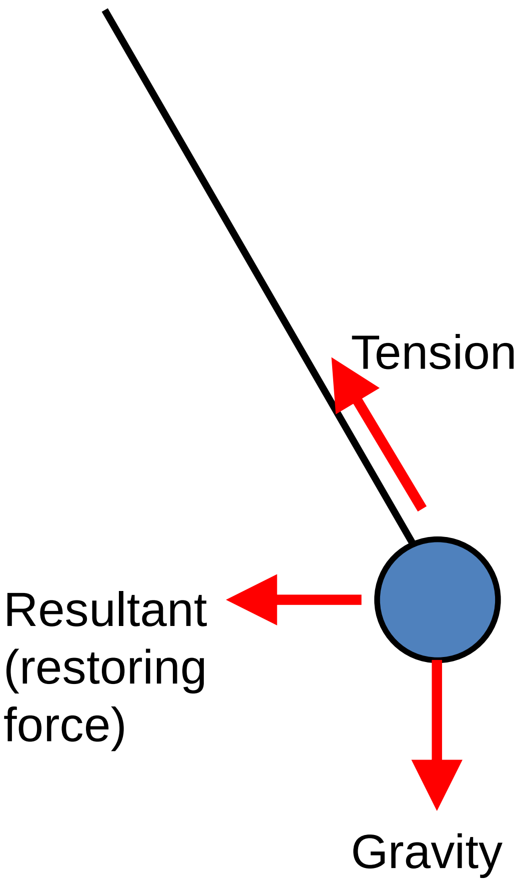
Frequency, $f = \frac{1}{6}$
$= 1.67$ Hz



Simple Harmonic Motion

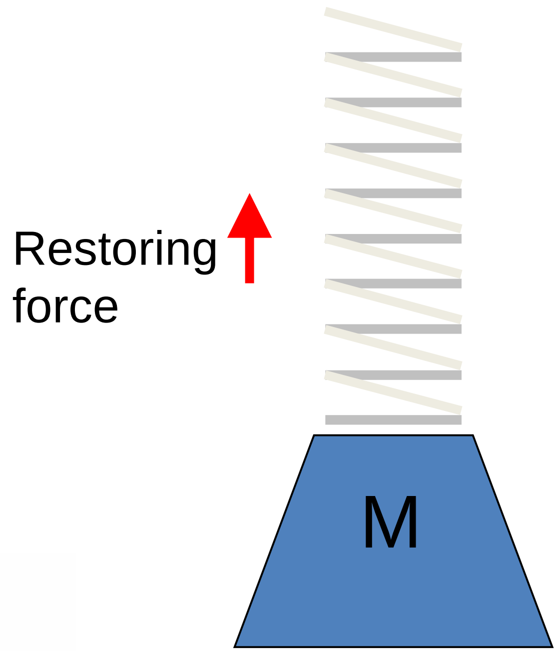
A system will oscillate if there is a force acting on it that tends to pull it back to its equilibrium position – a restoring force.

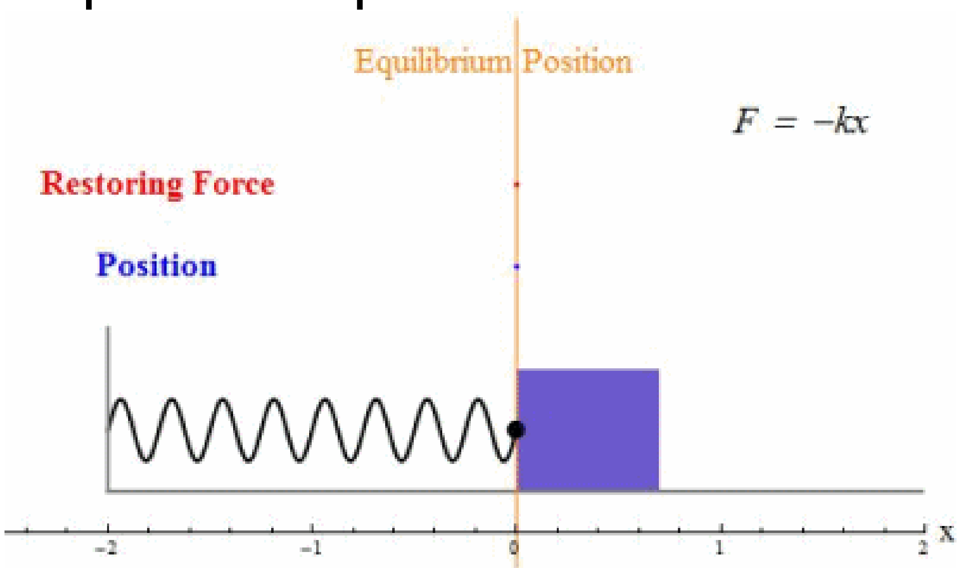
In a swinging pendulum the combination of gravity and the tension in the string that always act to bring the pendulum back to the centre of its swing.



Simple Harmonic Motion

For a spring and mass the combination of the gravity acting on the mass and the tension in the spring means that the system will always try to return to its equilibrium position.





Simple Harmonic Motion

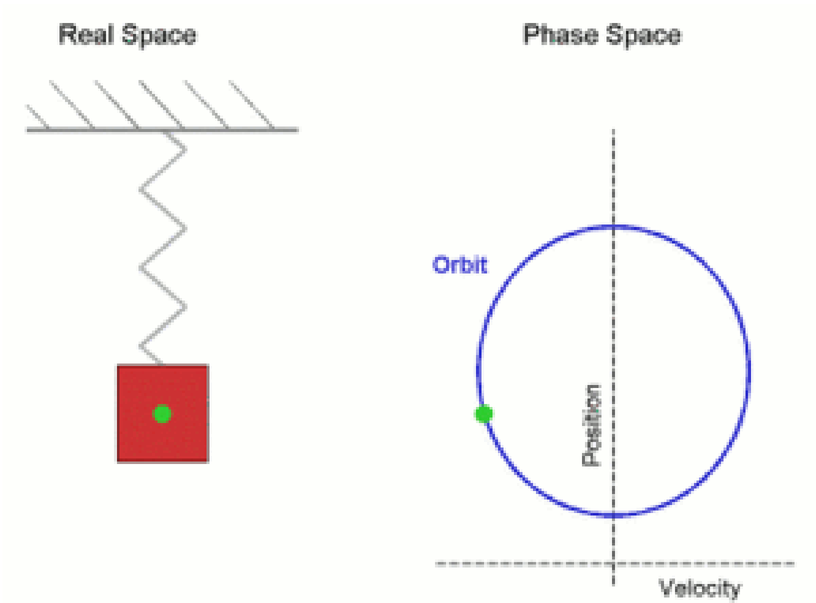
When the force acting on the particle is **directly proportional** to the displacement and opposite in direction, the motion is said to be simple harmonic motion.

$$\boldsymbol{F \alpha - x}$$

$$F = -kx$$

Relation of SHM with Circular Motion

The pattern of SHM is the same as the side view of an object moving at a constant speed around a circular path



Source: Wikimedia Commons

The amplitude of the oscillation is equal to the radius of the circle.

The object moves quickest as it passes through the central equilibrium position.

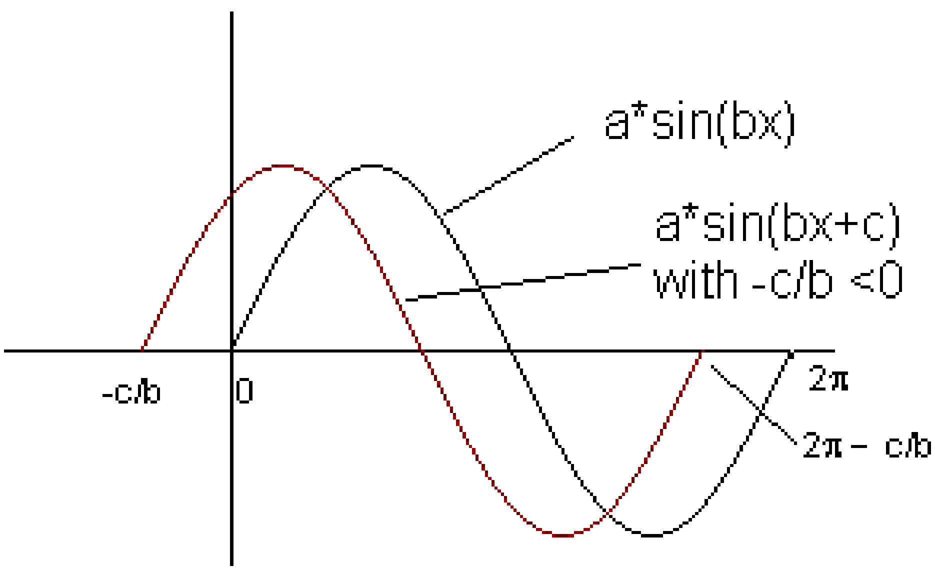
The time period of the oscillation is equal to the time taken for the object to complete the circular path.

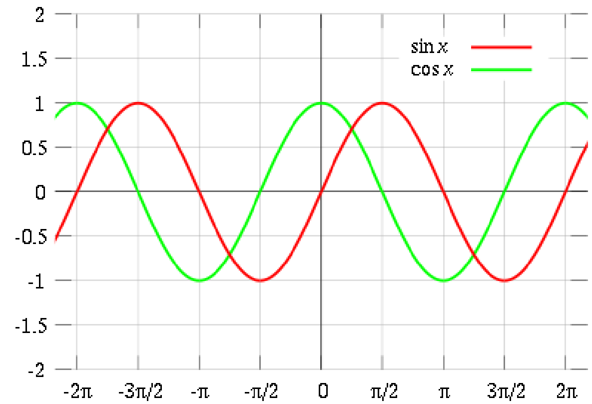
General SHM Equations

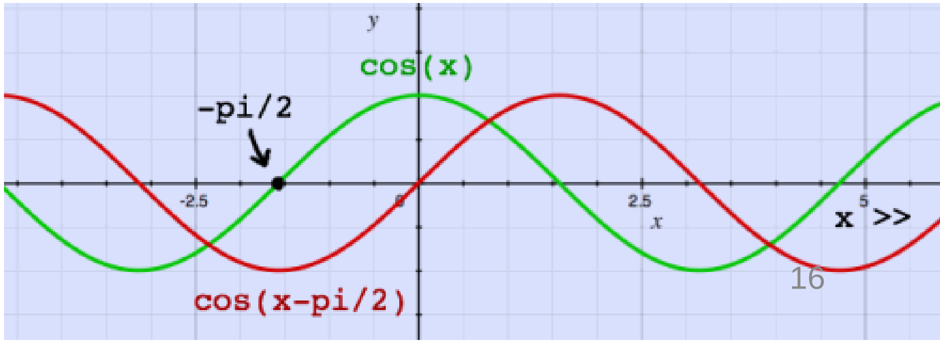
Most of the time, simple harmonic motion can be described by these two basic trigonometric functions:

$$x = x_0 \sin(\omega t + \phi) \text{ or } x = x_0 \cos(\omega t + \phi)$$

Where $(\omega t + \phi)$ denotes the **phase** of SHM. ω denotes the **angular frequency** and ϕ is the **phase constant**.



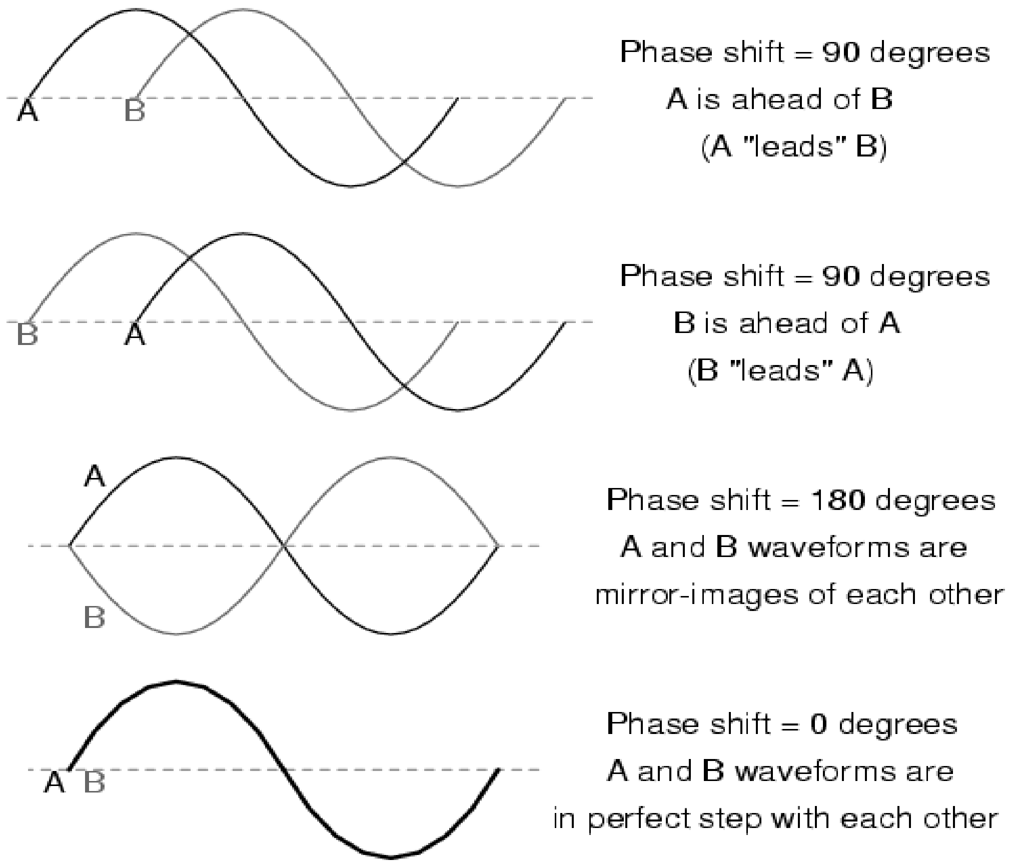




Phase Difference

The term phase describes the point that an oscillating mass has reached within the complete cycle of an oscillation.

It is often important to describe the **phase difference** between two oscillations.



Calculating Phase Difference

To calculate the phase difference, follow these steps:

1) Measure the time interval t between two corresponding points on the graphs.

2) Determine the period T for one complete oscillation.

3) Calculate the phase difference as a fraction of an oscillation.

4) Convert to degrees or radians.

Question 2

A body oscillating with SHM has a period of 1.5s and amplitude of 5cm. Calculate its frequency and maximum acceleration.

(a) $f = 1 / T \;\; = 1 / 1.5s$

frequency = 0.67 Hz

(b) $a = -(2\pi f)^2 x$

maximum acceleration is when x = A (the amplitude)

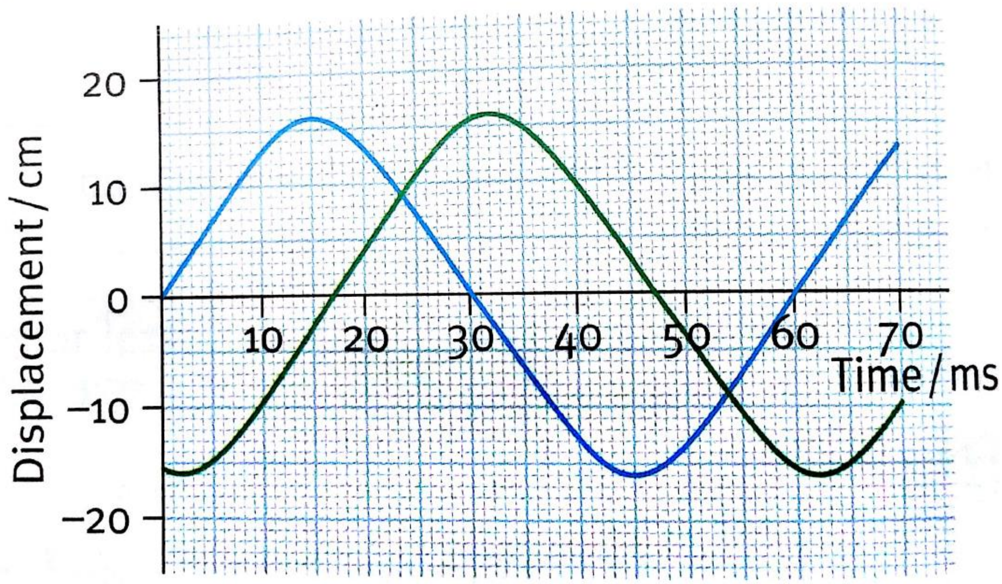
$a = -(2\pi f)^2 A$

$= -(2\pi \times 0.6667Hz)^2 \times 0.015m$

maximum acceleration = 0.88 ms^{-2}

Question 3:

Calculate the phase difference between the two oscillations in degrees and in radians.



Answer:

The time interval, t, is 17 ms.

Period, $T = 60$ ms.

Phase difference,

$\phi = \frac{t}{T} = \frac{17}{60} = 0.283$ oscillation

Phase difference in degree:

$\phi = 0.283 \times 360° = 102°$

Phase difference in radian:

$\phi = 0.283 \times 2\pi = 1.78$ rad

Question 4

The motion of a body in simple harmonic motion is described by the equation:

$$x = 4.0\cos(2\pi t + \frac{\pi}{3})$$

Where x is in metres, t is in seconds and $(2\pi t + \frac{\pi}{3})$ is in radians.

(a) What is (i) the displacement, (ii) the velocity, (iii) the acceleration and (iv) the phase when $t = 2.0$ s?

(b) Find (i) the frequency and (ii) the period of the motion.

Solution:

(a) (i) Displacement, $x = 4.0\cos(2\pi t + \frac{\pi}{3})$

when, $t = 2.0$ s,

$x = 4.0\cos(4\pi + \frac{\pi}{3})$

=2.0 m.

(ii) Velocity, $v = \frac{dx}{dt} = -2\pi(4.0)\sin(2\pi t + \pi/3)$

When $t = 2.0$s,

$v = -8.0\pi\sin(4\pi + \frac{\pi}{3})$

$= -21.8$ ms^{-1}

Solution (continued):

(iii)Acceleration, $a = \frac{dv}{dt} = -(4\pi^2)4.0\cos(2\pi t + \frac{\pi}{3})$

When $t = 2.0$ s,

$a = -(4\pi^2)4.0\cos\left(4\pi + \frac{\pi}{3}\right) = -79.0$ ms^{-2}.

(iv) Phase $= (2\pi t + \frac{\pi}{3})$

When t=2.0s, phase $= (4\pi + \frac{\pi}{3})$ rad $= 13.6$ rad.

(b) (i) $\omega = 2\pi$, $f = \frac{\omega}{2\pi} = \frac{2\pi}{2\pi} = 1$ Hz

(ii) $T = \frac{1}{f} = 1$s

Simple Harmonic Motion

Consider a simple harmonic oscillation, described by the formula
$x = x_0 \cos(\omega t)$

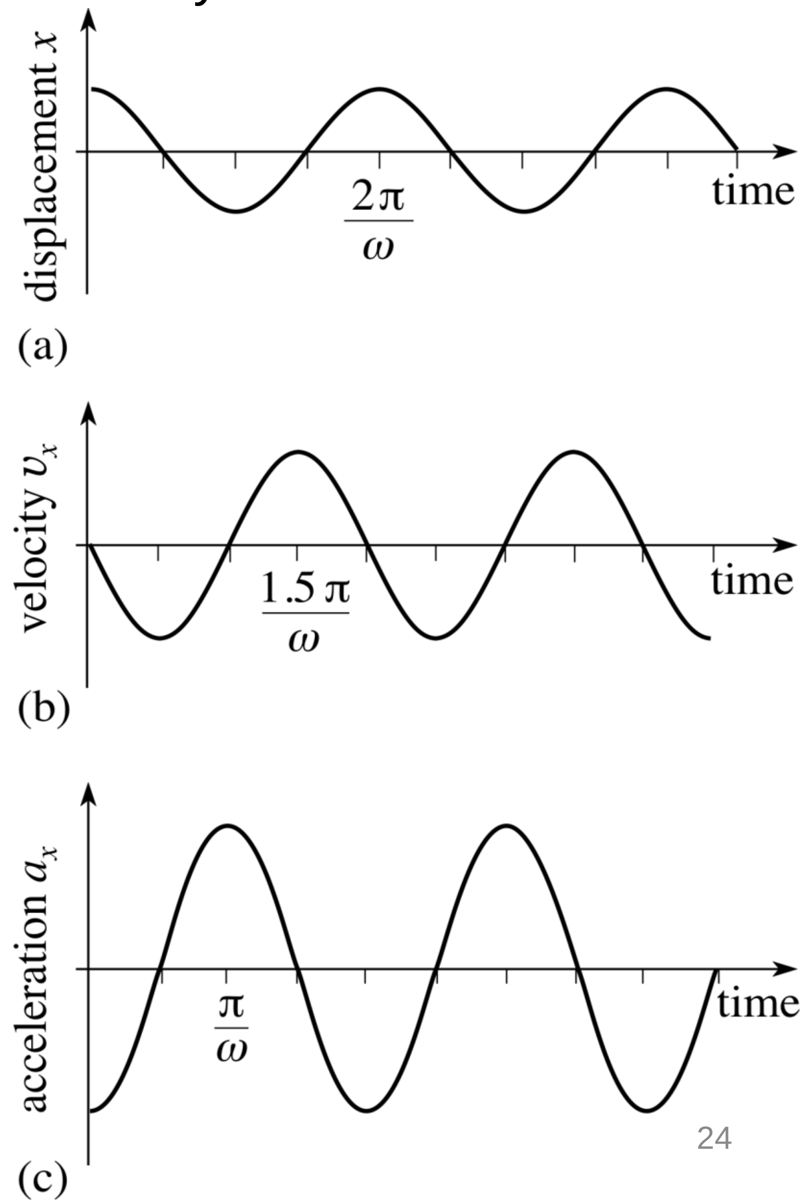
The velocity of its vibration is described as
$v = \frac{dx}{dt} = -\omega x_0 \sin \omega t$

The acceleration
$a = \frac{dv}{dt} = -\omega^2 x_0 \cos \omega t$

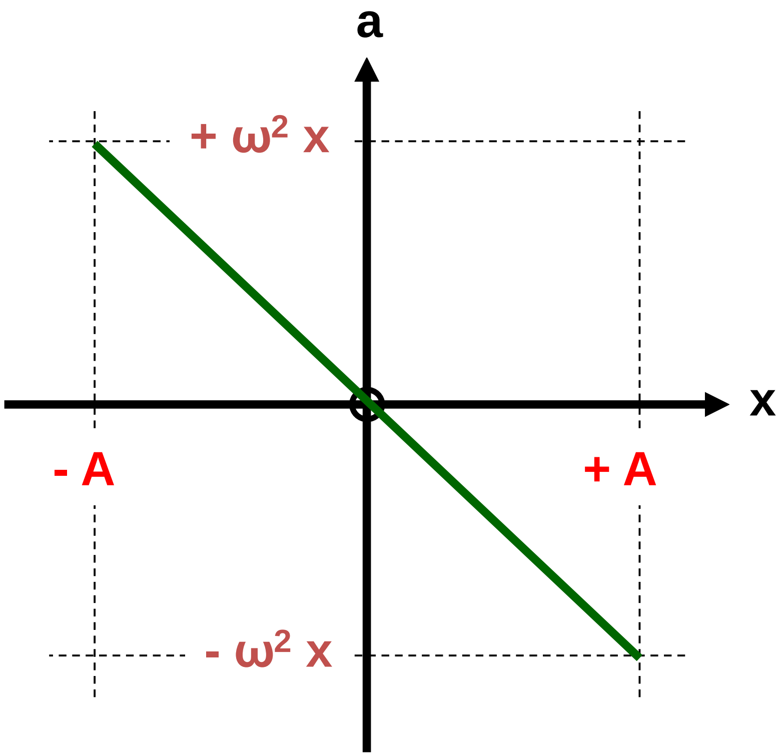
But $x_0 \cos \omega t = x$.

Hence,

$\boxed{a = -\omega^2 x.}$



Acceleration in SHM



gradient = $-\ \omega^2$ or $-\ (2\pi f)^2$

Simple Harmonic Motion

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$v\frac{dv}{dx} = -\omega^2 x$$

By separation of variables,

$$vdv = -\omega^2 x dx$$

Integrating them,

$$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c$$

When $x = \pm x_0$, $v = 0$.

$$c = \frac{\omega^2 x_0^2}{2}$$

Hence, $v^2 = \omega^2(x_0^2 - x^2)$

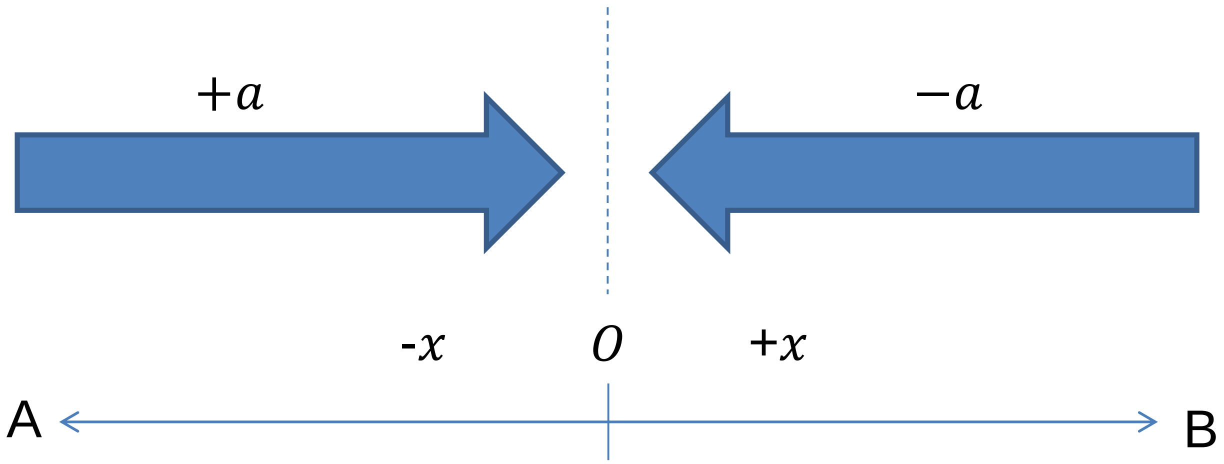
Or

$$\boxed{v = \pm\omega\sqrt{x_0^2 - x^2}}$$

When *v*= max,

$$\boxed{v_0 = \pm\omega x_0}$$

Simple Harmonic Motion



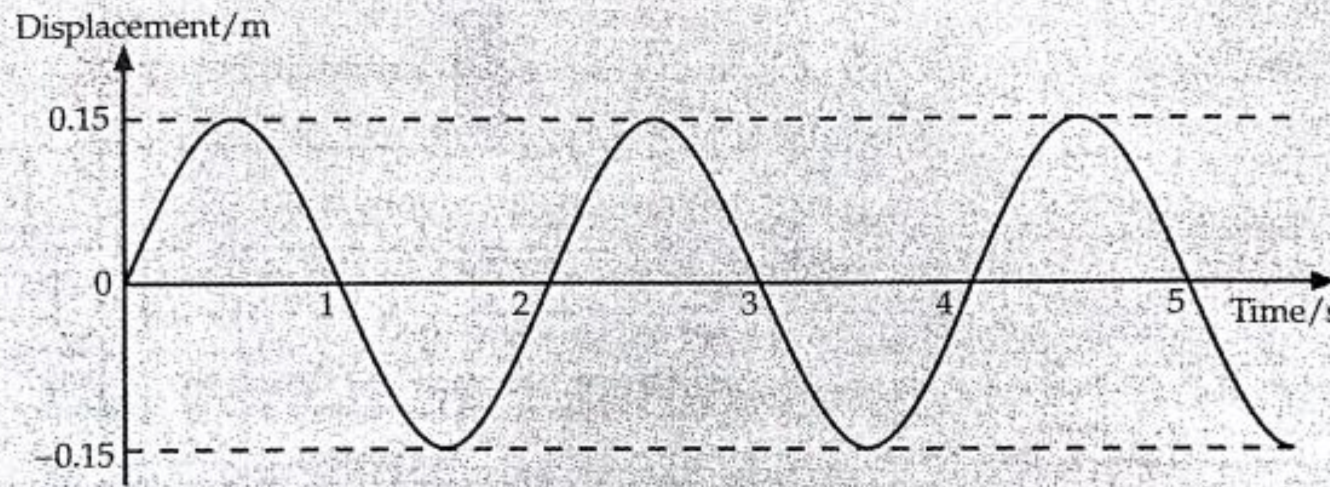
$x = -x_0$
$v = 0$
$$\frac{d^2x}{dt^2} = \omega^2|x|$$
Max. PE
Zero KE

$x = 0$
$v = \pm\omega a$
$$\frac{d^2x}{dt^2} = 0$$
Zero PE
Max KE

$x = +x_0$
$v = 0$
$$\frac{d^2x}{dt^2} = -\omega^2|x|$$
Max. PE
Zero KE

Example:

In a laboratory experiment, a pendulum bob oscillates with negligible damping. Its displacement from its equilibrium position with respect to time is shown in Fig 8.6.



▶ Fig 8.6

Determine

(a) the amplitude;

(b) the period;

(c) the frequency;

(d) the angular frequency;

(e) the acceleration

 (i) when the displacement is zero,

 (ii) when the displacement is at its maximum;

(f) the maximum velocity of the pendulum bob. $\left[\text{Hint: } v = \pm\omega\sqrt{x_0^2 - x^2}\right]$

Answer:

(a) Amplitude =0.15 m

(b) Period $T = 2.0$s

(c) Frequency $f = \frac{1}{T} = \frac{1}{2.0} = 0.5$ Hz

(d) Angular velocity $\omega = 2\pi f = 3.14 \text{ rad s}^{-1}$

(e) (i) When $x = 0$, $\left|\frac{d^2x}{dt^2}\right| = \omega^2(0) = 0$

(ii) When $x = 0.15$ m, $\left|\frac{d^2x}{dt^2}\right| = \omega^2 x = (3.14)^2(0.15)$

(f) Maximum velocity occurs when $x = 0$, i.e. $v_{max} = \omega x_0 = (3.14)(0.15) = 0.47$ms^{-1}

Note: always use radian in calculations involving SHM unless otherwise required by the questions.

The Period of a Simple Spring-Mass System

Consider a spring suspended with a mass m. If the elastic limit is not exceeded,

$$F = ke = mg$$

where k is the spring constant.

If the mass is pulled down and released, it oscillates with simple harmonic motion. The tension F' is equal to $k(e+x)$.
The restoring force on the mass is given by

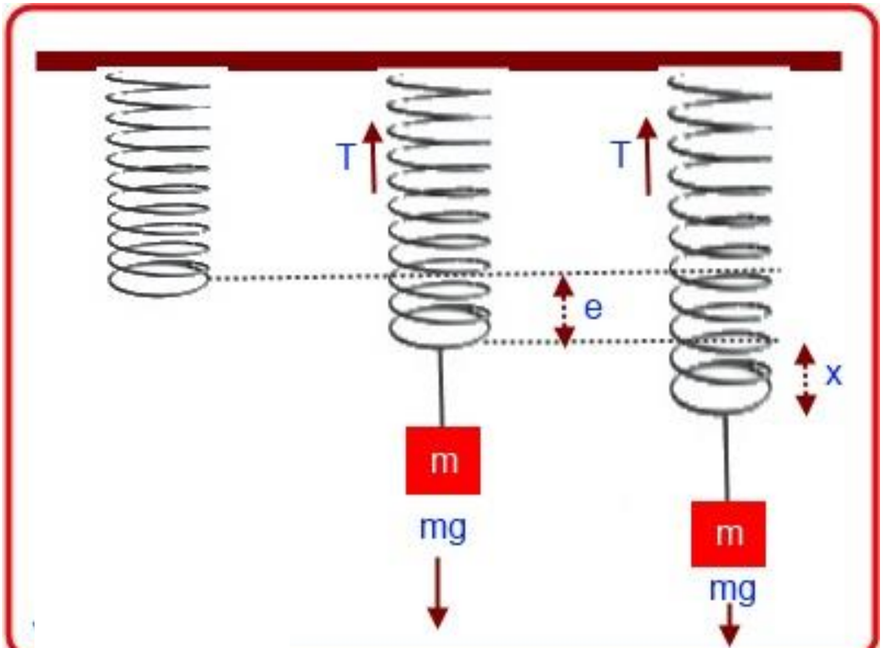
$$m\frac{d^2x}{dt^2} = mg - F' = mg - k(e+x)$$

But $ke = mg$, hence

$$m\frac{d^2x}{dt^2} = -kx$$

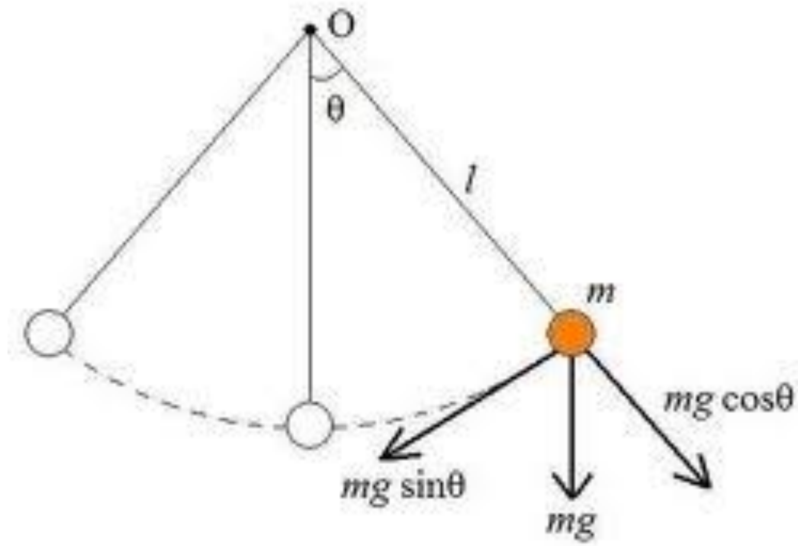
$$\boxed{\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x}$$

$$\boxed{\omega = \sqrt{\frac{k}{m}} \Rightarrow \mathrm{T} = 2\pi\sqrt{\frac{m}{k}}}$$



The Period of a Simple Pendulum

Consider a simple pendulum:



$$F_{res} = ma_{tan} mg \sin\theta = -ks$$
$$= -k(l\theta)$$

$$k = \frac{mg \sin\theta}{l\theta}$$

Small angle approximation:

$$\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1\,;$$

Equation becomes: $k = \frac{mg}{l}$; Substitute into $\mathrm{T} = 2\pi\sqrt{\frac{m}{k}}$:

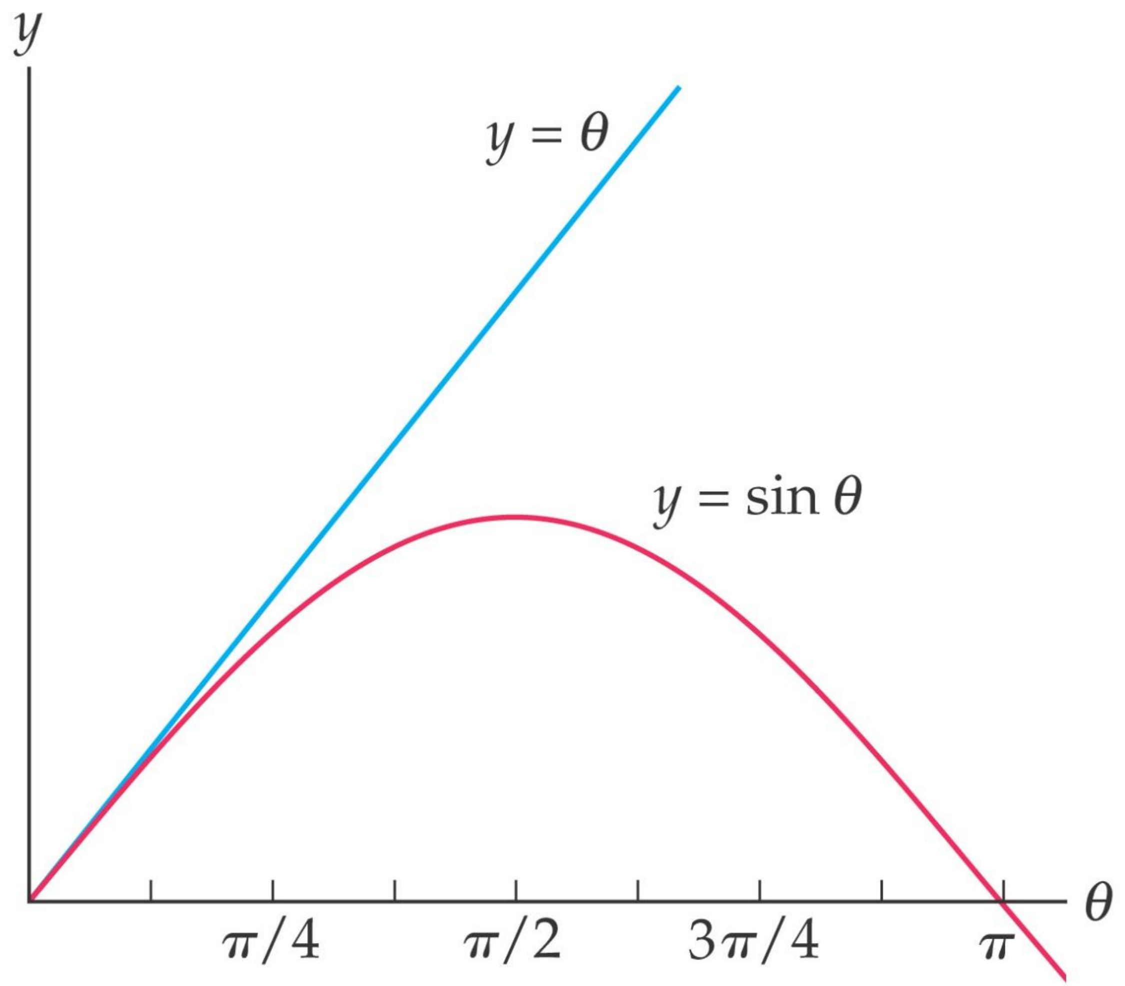
$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Christiaan Huygen’s law

$$\boxed{\omega = \sqrt{\frac{g}{l}}}$$

Small Angle Approximation

However, for small angles, sin *θ* and *θ* are approximately equal.



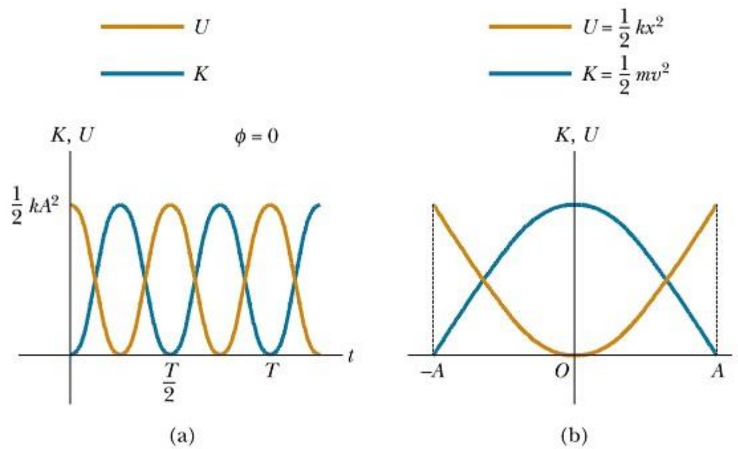
Energy in Simple Harmonic Motion

Kinetic Energy:

$$E_k = \frac{1}{2}mv^2$$

But $v = \pm\omega\sqrt{x_0^2 - x^2}$

$$\Rightarrow \boxed{E_k = \frac{1}{2}m\omega^2(x_0^2 - x^2)}$$



Potential Energy:

$F_{res} = -m\omega^2 x$

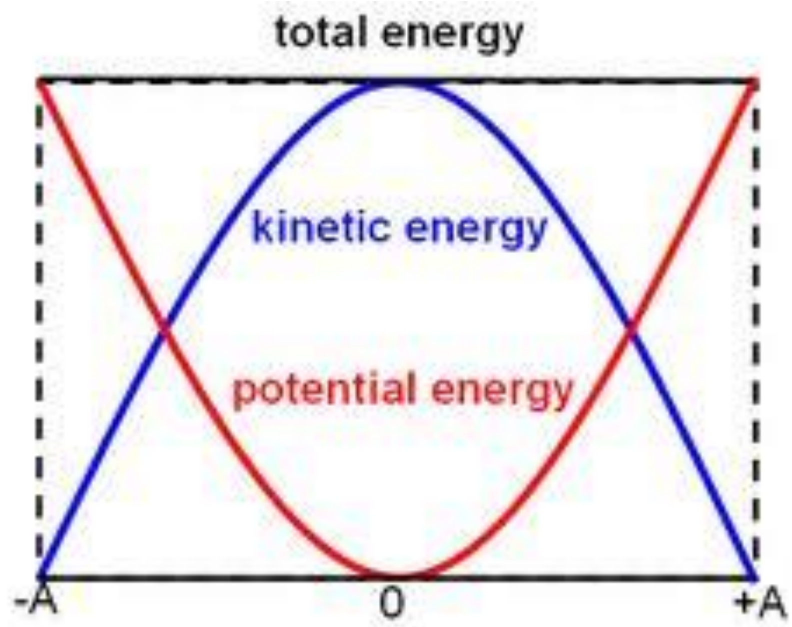
Potential Energy = average restoring force × displacement

$$\boxed{E_P = \frac{1}{2}m\omega^2 x^2}$$

Total Energy in Simple Harmonic Motion

Consider the oscillation of a spring,
The potential energy of a spring is given by:

$$U = \frac{1}{2}kx^2$$
$$= \frac{1}{2}kx_0^2 \sin^2(\omega t + \phi)$$



The kinetic energy of the mass is given by:

$$K = \frac{1}{2}mv^2$$
$$= \frac{1}{2}mx_0^2\omega^2 \cos^2(\omega t + \phi)$$
$$= \frac{1}{2}kx_0^2 \cos^2(\omega t + \phi)$$

The total mechanical energy,

$$\boxed{E = U + K = \frac{1}{2}kx_0^2}$$

OR

$$E = \frac{1}{2}\left(m\omega^2(x_0^2 - x^2)\right) + \frac{1}{2}m\omega^2x^2$$
$$\boxed{= \frac{1}{2}m\omega^2x_0^2}$$

The **total** mechanical energy **is constant** and is independent of the time.

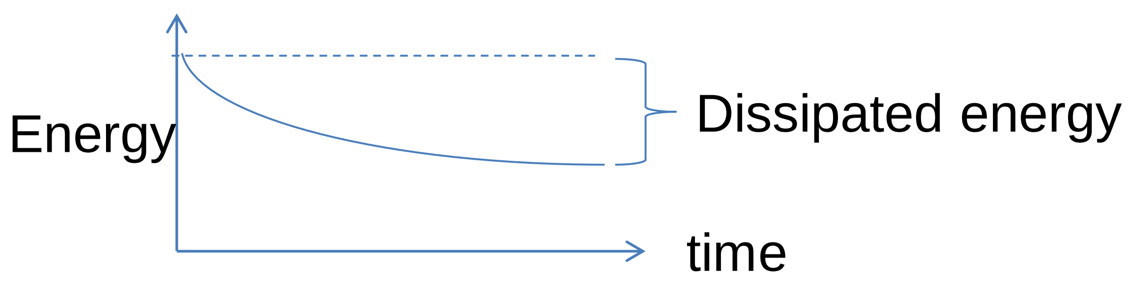
Free and Damped Oscillations

Recall:

A particle is said to be undergoing **free** oscillations when the only **external** force acting on it is the restoring force.

In real situations, however, frictional and other resistive forces cause the oscillator's energy to be dissipated, and this energy is converted eventually into thermal energy. The oscillations are said to be damped.

A displacement-time graph of the system shows an exponential decay of the amplitude with time.



Example:

A particle of mass 0.40 kg oscillates in simple harmonic motion with frequency 5.0 Hz and amplitude 12 cm. Calculate, for the particle at displacement 10 cm:

(a) The kinetic energy

$$E_k = \frac{1}{2}m\omega^2(x_0^2 - x^2) = \frac{1}{2}(0.40)\big(2\pi(5.0)\big)^2(0.12^2 - 0.10^2)$$

$$= 0.87\ \mathrm{J}$$

(b) The potential energy

$$E_p = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}(0.40)\big(2\pi(5.0)\big)^2(0.10^2)$$

$$= 1.97\mathrm{J}$$

(c) Total energy

$$E_{tot} = \frac{1}{2}m\omega^2 x_0^2$$

$$= \frac{1}{2}(0.40)\big(2\pi(5.0)\big)^2(0.12^2)$$

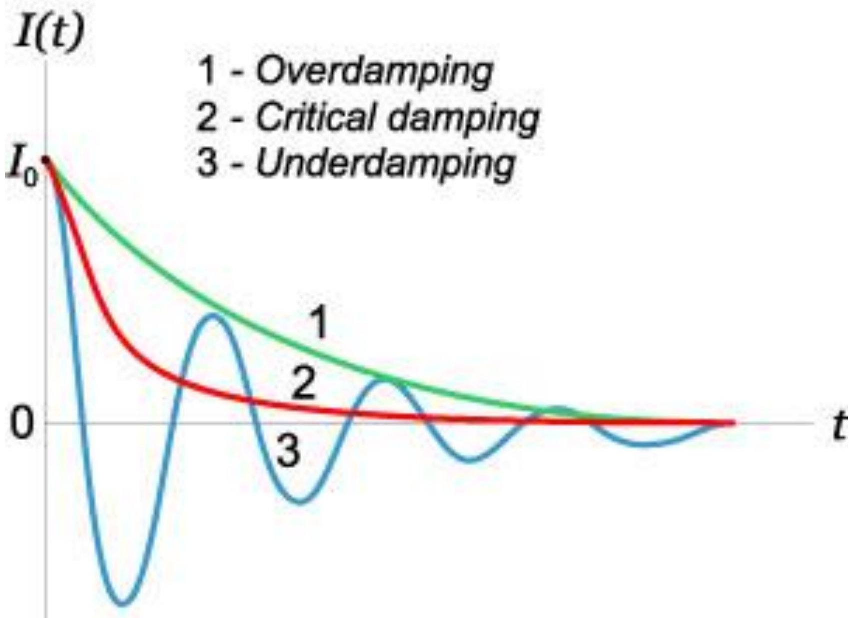
$$= 2.84\mathrm{J}$$

Damped Oscillations

(a) Light damping – The system oscillates about the equilibrium position with decreasing amplitude over a period of time. (ex: a mass and spring in air)

(b) Critical damping – The system does not oscillate but returns to the equilibrium position in the shortest time. (Ex: the suspension system of a car)

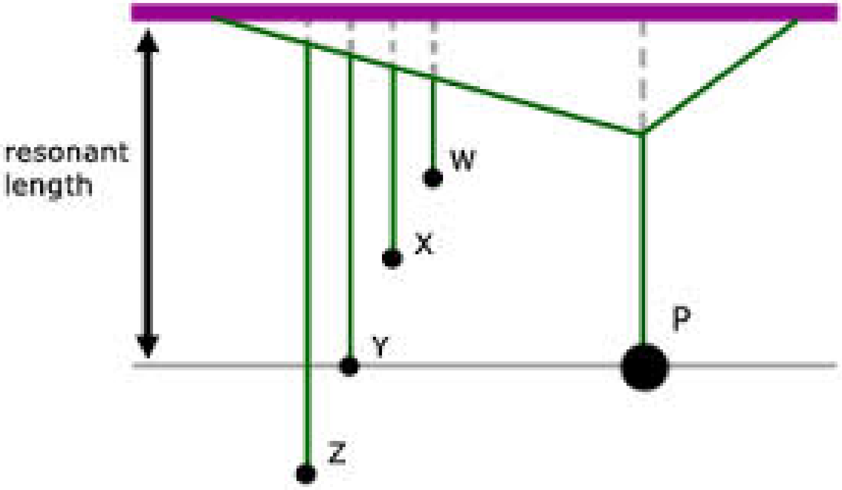
(c) Heavy damping – the system does not oscillate and damping is so large that it will take a long time to reach the equilibrium position. (example: coiled spring mattress)



Forced Oscillations and Resonance

Resonance is a **special** case of the **forced** oscillation. If the forcing frequency of an external force happens to **match** the **natural frequency** of oscillation of the system, the amplitude of the resulting oscillations can build up to become very large.

Barton's pendulum is a demonstration of the resonance phenomenon. Several pendulums of different lengths hang from a horizontal string. The 'driver' pendulum P at the right has a large mass. If it is set swinging, the pendulum Y will oscillate with the largest amplitude due to the fact that its natural frequency matches the natural frequency of the driver pendulum (their lengths are the same).

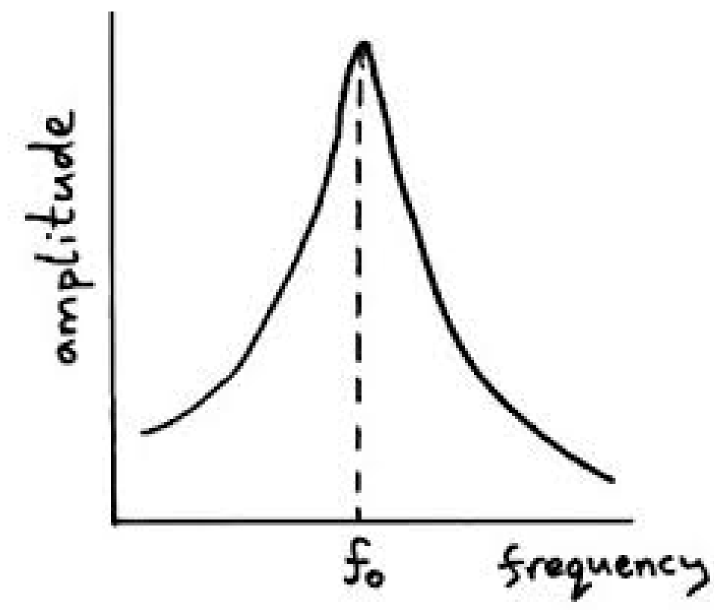


Resonance

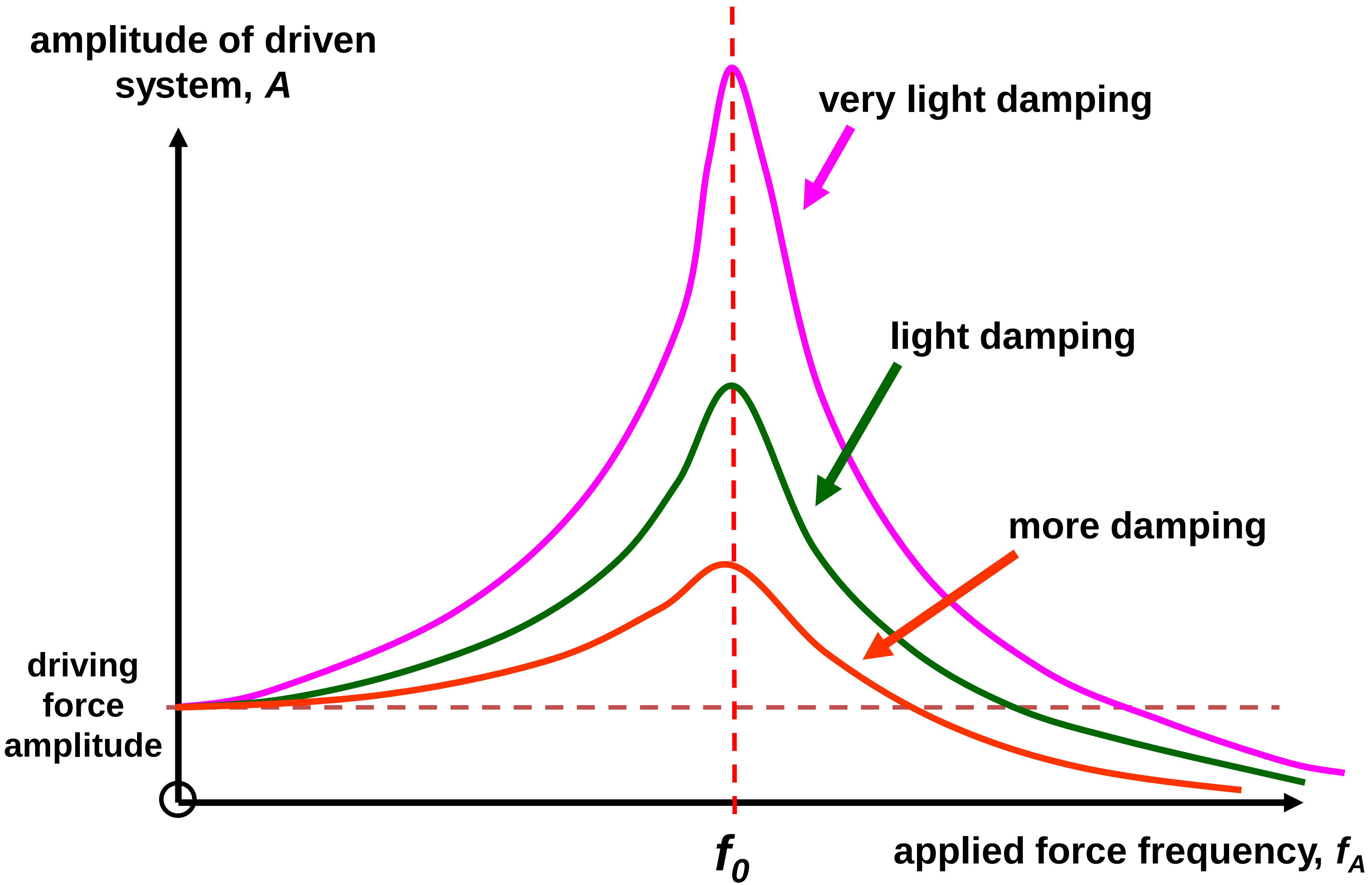
In resonance, energy is transferred from the driver to the resonating system much more efficiently.

For any system in resonance:

a) Its natural frequency is equal to the frequency of the driver.
b) Its amplitude is maximum
c) It absorbs the greatest possible energy from the driver.



Resonance Curves



Notes on the resonance curves

If damping is increased then the amplitude of the driven system is decreased at all driving frequencies.

If damping is decreased then the sharpness of the peak amplitude part of the curve increases.

The amplitude of the driven system tends to be:

- Equal to the driving system at very low frequencies.
- Zero at very high frequencies.
- Infinity (or the maximum possible) when f_A is equal to f_0 as damping is reduced to zero.

The Tacoma Narrows Bridge Collapse

Source: Picryl

An example of resonance caused by wind flow.

Washington State USA, November 7th 1940.

Review Questions

1) The pendulum of a clock is displaced by a distance of 4.0 cm and it oscillates in s.h.m. with a frequency of 1.0 s.

a) Write down an equation to describe the displacement x of the pendulum bob with time t.

b) Calculate:

i) The maximum velocity of the pendulum bob.

ii) Its velocity when its displacement is 2.0 cm.

Review Questions

2) (a) Describe the energy changes during one complete oscillation of an undamped simple pendulum.

(b) Explain why a lightly damped simple pendulum experiences a slow decrease in its amplitude.

Review Questions

3) The simple harmonic motion of a particle is given by

$$a = -bx$$

Where a is the acceleration,
x is the displacement of the particle and
b is a constant.

Write an expression of the time period of oscillation in term of b.

Review Questions

4) A 50 g mass is attached to a securely clamped spring. The mass is pulled downwards by 16 mm and released which causes it to oscillate with s.h.m. of time period of 0.84 s.

a) Calculate the frequency of the oscillation.
b) Calculate the maximum velocity of the mass.
c) Calculate the maximum kinetic energy of the mass and state at which point in the oscillation it will have this velocity.
d) Write down the maximum gravitational potential energy of the mass (relative to its equilibrium position). You may assume the damping is negligible.

Review Question

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5) A ball is held between two fixed points A and B by means of two stretched springs, as shown in Fig. 3.1.

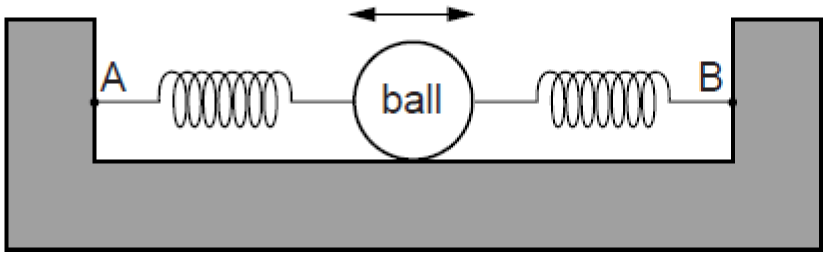


Fig. 3.1

The ball is free to oscillate along the straight line AB. The springs remain stretched and the motion of the ball is simple harmonic.

The variation with time t of the displacement x of the ball from its equilibrium position is shown in Fig. 3.2.

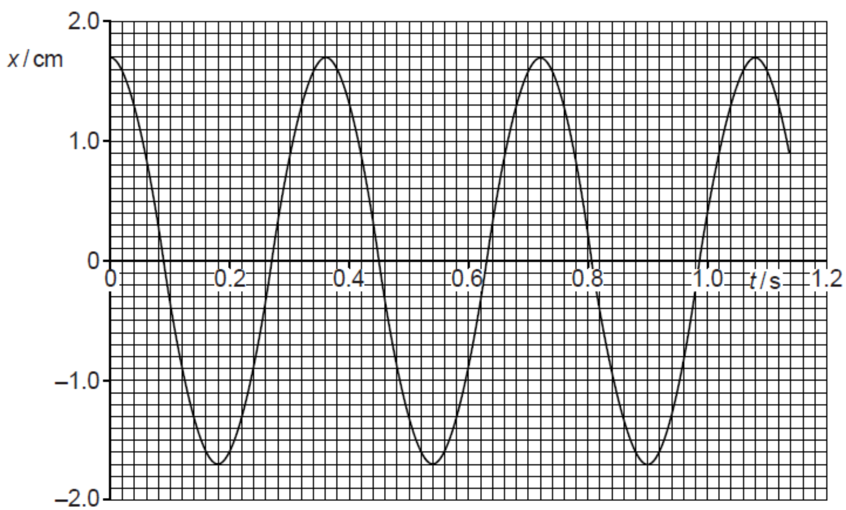


Fig. 3.2

Review Question 5(continued)

(a) **(i)** Use Fig. 3.2 to determine, for the oscillations of the ball,

1. the amplitude,

amplitude = ... cm [1]

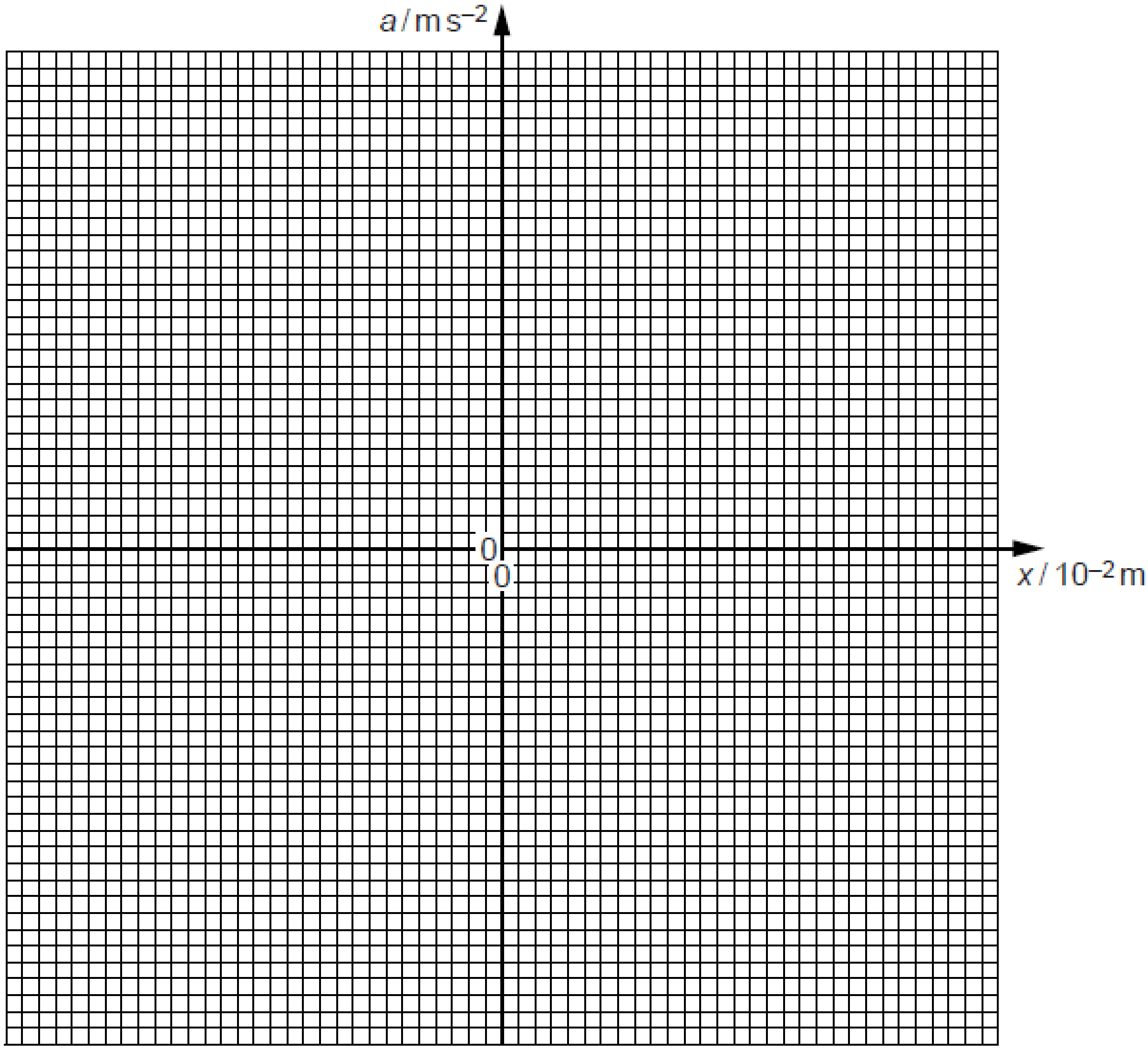
2. the frequency.

frequency = ... Hz [2]

(ii) Show that the maximum acceleration of the ball is $5.2\,m\,s^{-2}$.

Review Question 5) (continued)

(b) Use your answers in **(a)** to plot, on the axes of Fig. 3.3, the variation with displacement x of the acceleration a of the ball.



(c) Calculate the displacement of the ball at which its kinetic energy is equal to one half of the maximum kinetic energy.

displacement = ... cm [3]

Extra References:

http://www.acoustics.salford.ac.uk/feschools/waves/shm3.php

http://www.physbot.co.uk/further-mechanics.html

http://hyperphysics.phy-astr.gsu.edu/hbase/shm2.html#c3