Anadijiban Das · Andrew DeBenedictis

The General Theory of Relativity

A Mathematical Exposition

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Dedicated to the memory of
Professor J. L. Synge

Preface

General relativity is to date the most successful theory of gravity. In this theory, the gravitational field is not a conventional force but instead is due to the geometric properties of a manifold commonly known as space–time. These properties give rise to a rich physical theory incorporating many areas of mathematics. In this vein, this book is well suited for the advanced mathematics or physics student, as well as researchers, and it is hoped that the balance of rigorous mathematics and physical insights and applications will benefit the intended audience. The main text and exercises have been designed both to gently introduce topics and to develop the framework to the point necessary for the practitioner in the field. This text tries to cover all of the important subjects in the field of classical general relativity in a mathematically precise way.

This is a subject which is often counterintuitive when first encountered. We have therefore provided extensive discussions and proofs to many statements, which may seem surprising at first glance. There are also many elegant results from theorems which are applicable to relativity theory which, if someone is aware of them, can save the individual practitioner much calculation (and time). We have tried to include many of them. We have tried to steer the middle ground between brute force and mathematical elegance in this text, as both approaches have their merits in certain situations. In doing this, we hope that the final result is "reader friendly." There are some sections that are considered advanced and can safely be skipped by those who are learning the subject for the first time. This is indicated in the introduction of those sections.

The mathematics of the theory of general relativity is mostly derived from tensor algebra and tensor analysis, and some background in these subjects, along with special relativity (relativity in the absence of gravity), is required. Therefore, in Chapter 1, we briefly provide the tensor analysis in Riemannian and pseudo-Riemannian differentiable manifolds. These topics are discussed in an arbitrary dimension and have many possible applications.

In Chapter 2, we review the special theory of relativity in the arena of the four-dimensional flat space–time manifold. Then, we introduce curved space–time and Einstein's field equations which govern gravitational phenomena.

In Chapter 3, we explore spherically symmetric solutions of Einstein's equations, which are useful, for example, in the study of nonrotating stars. Foremost among these solutions is the Schwarzschild metric, which describes the gravitational field outside such stars. This solution is the general relativistic analog of Newton's inverse-square force law of universal gravitation. The Schwarzschild metric, and perturbations of this solution, has been utilized for many experimental verifications of general relativity within the solar system. General solutions to the field equations under spherical symmetry are also derived, which have application in the study of both static and nonstatic stellar structure.

In Chapter 4, we deal with static and stationary solutions of the field equations, both in general and under the assumption of certain important symmetries. An important case which is examined at great length is the Kerr metric, which may describe the gravitational field outside of certain rotating bodies.

In Chapter 5, the fascinating topic of black holes is investigated. The two most important solutions, the Schwarzschild black hole and the axially symmetric Kerr black hole, are explored in great detail. The formation of black holes from gravitational collapse is also discussed.

In Chapter 6, physically significant cosmological models are pursued. (In this arena of the physical sciences, the impact of Einstein's theory is very deep and revolutionary indeed!) An introduction to higher dimensional gravity is also included in this chapter.

In Chapter 7, the mathematical topics regarding Petrov's algebraic classification of the Riemann and the conformal tensor are studied. Moreover, the Newman–Penrose versions of Einstein's field equations, incorporating Petrov's classification, are explored. This is done in great detail, as it is a difficult topic and we feel that detailed derivations of some of the equations are useful.

In Chapter 8, we introduce the coupled Einstein–Maxwell–Klein–Gordon field equations. This complicated system of equations classically describes the self-gravitation of charged scalar wave fields. In the special arena of spherically symmetric, static space–time, these field equations, with suitable boundary conditions, yield a nonlinear eigenvalue problem for the allowed theoretical charges of gravitationally bound wave-mechanical condensates.

Eight appendices are also provided that deal with special topics in classical general relativity as well as some necessary background mathematics.

The notation used in this book is as follows: The Roman letters i, j, k, l, m, n, etc. are used to denote subscripts and superscripts (i.e., covariant and contravariant indices) of a tensor field's components relative to a coordinate basis and span the full dimensionality of the manifold. However, we employ parentheses around the letters (a), (b), (c), (d), (e), (f), etc. to indicate components of a tensor field relative to an orthonormal basis. Greek indices are used to denote components that only span the dimensionality of a hypersurface. In our discussions of space–time, these Greek indices indicate spatial components only. The flat Minkowskian metric tensor components are denoted by d_{ij} or $d_{(a)(b)}$. Numerically they are the same, but conceptually there is a subtle difference. The signature of the space–time metric is

$+2$ and the conventions for the definitions of the Riemann, Ricci, and conformal tensors follow the classic book of Eisenhart.

We would like to thank many people for various reasons. As there are so many who we are indebted to, we can only explicitly thank a few here, in the hope that it is understood that there are many others who have indirectly contributed to this book in many, sometimes subtle, ways.

I (A. Das) learned much of general relativity from the late Professors J. L. Synge and C. Lanczos during my stay at the Dublin Institute for Advanced Studies. Before that period, I had as mentors in relativity theory Professors S. N. Bose (of Bose–Einstein statistics), S. D. Majumdar, and A. K. Raychaudhuri in Kolkata. During my stay in Pittsburgh, I regularly participated in, and benefited from, seminars organized by Professor E. Newman. In Canada, I had informal discussions with Professors F. Cooperstock, J. Gegenberg, W. Israel, and E. Pechlaner and Drs. P Agrawal, S. Kloster, M. M. Som, M. Suvegas, and N. Tariq. Moreover, in many international conferences on general relativity and gravitation, I had informal discussions with many adept participants through the years.

I taught the theory of relativity at University College of Dublin, Jadavpur University (Kolkata), Carnegie-Mellon University, and mostly at Simon Fraser University (Canada). Stimulations received from the inquiring minds of students, both graduate and undergraduate, certainly consolidated my understanding of this subject.

Finally, I thank my wife, Mrs. Purabi Das. I am very grateful for her constant encouragement and patience.

I (A. DeBenedictis) would like to thank all of the professors, colleagues, and students who have taught and influenced me. As mentioned previously, there are far too many to name them all individually. I would like to thank Professor E. N. Glass of the University of Michigan-Ann Arbor and the University of Windsor, who gave me my first proper introduction to this fascinating field of physics and mathematics. I would like to thank Professor K. S. Viswanathan of Simon Fraser University, from whom I learned, among the many things he taught me, that this field has consequences in theoretical physics far beyond what I originally had thought.

I would also like to thank my colleagues whom I have met over the years at various institutions and conferences. All of them have helped me, even if they do not know it. Discussions with them, and their hospitality during my visits, are worthy of great thanks. During the production of this work, I was especially indebted to my colleagues in quantum gravity. They have given me the appreciation of how difficult it is to turn the subject matter of this book into a quantum theory, and opened up a fascinating new area of research to me. The quantization of the gravitational field is likely to be one of the deepest, difficult, and most interesting puzzles in theoretical physics for some time. I hope that this text will provide a solid background for half of that puzzle to those who choose to tread down this path.

I would also like to thank the students whom I have taught, or perhaps they have taught me. Whether it be freshman level or advanced graduate level, I can honestly say that I have learned something from every class that I have taught.

Not least, I extend my deepest thanks and appreciation to my wife Jennifer for her encouragement throughout this project. I do not know how she did it.

We both extend great thanks to Mrs. Sabine Lebhart for her excellent and timely typesetting of a very difficult manuscript.

Finally, we wish the best to all students, researchers, and curious minds who will each in their own way advance the field of gravitation and convey this beautiful subject to future generations. We hope that this book will prove useful to them.

Vancouver, Canada

Anadijiban Das
Andrew DeBenedictis

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Symbols[1], [2]

$\square$	d'Alembertian operator, completion of an example
$\blacksquare$	Q.E.D., completion of proof
$\cdot$	Central dot denotes multiplication (used to make crowded equations more readable)
$\equiv$	Identity
$:=$, $=:$	Definition, which is an identity involving new notation
$*$	Hodge star operation, tortoise coordinate designation
$\vert_{..}$	Constrained to a curve or surface
$\in$	Belongs to
${}^{r}_{s}\mathbf{0}$, $\mathbf{O}^{\cdot}_{\cdot\cdot\cdot}$	$(r+s)$th order zero tensor. (In the latter the number of dots indicate r and s.)
$\vec{\mathbf{O}}$	Zero vector
$[jk, i]$	Christoffel symbol of the 1st kind
$\left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\}$	Christoffel symbol of the 2nd kind
$[c \leftrightarrow d]$, $\{\mu \leftrightarrow \nu\}$, etc.	Represents the previous term in brackets of an expression but with the given indices interchanged
a	Angular momentum parameter, expansion factor in F–L–R–W metric
$\Vert\vec{\mathbf{A}}\Vert$	Norm or length of a vector
$A \cup B$	Union of two sets
$A \cap B$	Intersection of two sets
$A \times B$	Cartesian product of two sets

[1]For common tensors, only the coordinate component form is shown in this list.

[2]Occasionally the symbols listed here will also have other definitions in the text. We tabulate the most common definitions here as it should be clear in the text where the meanings differ from those in this list.

$A \subset B$	A is a subset of B
$(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})_g \equiv \mathbf{g}..(\vec{\mathbf{A}}, \vec{\mathbf{B}})$	Inner product between two vectors
${}_p\mathbf{A} \wedge {}_q\mathbf{B}$	Wedge product between a p-form and a q-form
$[\vec{\mathbf{A}}, \vec{\mathbf{B}}] := \vec{\mathbf{A}}\,\vec{\mathbf{B}} - \vec{\mathbf{B}}\,\vec{\mathbf{A}}$	Lie bracket or commutator
$\mathcal{A}$	Electric potential, (also a function used in five-dimensional cosmologies)
A^i	Components of the electromagnetic 4-potential
α	Affine parameter for a null geodesic, Newman–Penrose spin-coefficient
$\mathcal{B}$	Magnetic potential, bivector set
β	Expansion coefficient for 5th dimension, Newman–Penrose spin-coefficient
c	Speed of light (usually set to 1)
$\mathcal{C}$	Conformal group, causality violating region
C^r	Differentiability class r
$\mathcal{C}^i$	Coordinate conditions
${}^u_v\mathcal{C}[{}^r_s\mathbf{T}]$	Contraction operation of a tensor field ${}^r_s\mathbf{T}$
$C^i{}_{jkl}$	Components of Weyl's conformal tensor
$\mathbb{C}$	The set of all complex numbers
(χ, U)	Coordinate chart for a differentiable manifold
$\chi(p) = x$	Local coordinates of a point p in a manifold.
$\equiv (x^1, x^2, \ldots, x^N)$	In some places $x \in \mathbb{R}$.
χ_{ij}	Extended extrinsic curvature components
D	A domain in $\mathbb{R}^N$ (open and connected)
D_i	Gauge covariant derivative
∂D	$(N-1)$-dimensional boundary of D
∇_i	Covariant derivatives
$\dfrac{D}{\partial t}$	Covariant derivative along a curve
Δ	Laplacian in a manifold with metric, determinant of Γ_{ij}
∇^2	Laplacian in a Euclidean space
δ^i_j	Components of Kronecker delta (or identity matrix)
${}^p_p\delta,\ \delta^{i_1,\ldots,i_p}{}_{j_1,\ldots,j_p}$	Generalized Kronecker delta
$\mathrm{d}f$, $\mathrm{d}[{}_p\mathbf{W}]$	Exterior derivative of f or ${}_p\mathbf{W}$
d_{ij}	Components of flat space metric
$\dfrac{\partial(\widehat{x}^1, \ldots, \widehat{x}^N)}{\partial(x^1, \ldots, x^N)}$	Jacobian of a coordinate transformation
e	Electric charge, exponential
$\mathcal{E}_{ij}$, $\tilde{\mathcal{E}}_{ij}$, $\mathcal{E}^l{}_{ijk}$	Components of Einstein equations (in various forms)
$\vec{\mathbf{E}}$, E_α	Electric field and its components
$\mathbb{E}_N$	N-dimensional Euclidean space
$\{\vec{\mathbf{e}}_{(a)}\}_1^N := \{\vec{\mathbf{e}}_{(1)}, \ldots, \vec{\mathbf{e}}_{(N)}\}$	A basis set for a vector space

$\{\vec{\mathbf{E}}_{(a)}\}_1^4 := \left\{\vec{\mathbf{m}}, \vec{\bar{\mathbf{m}}}, \vec{\mathbf{l}}, \vec{\mathbf{k}}\right\}$	A complex null basis set for Newman–Penrose formalism
ε	A small number, Newman–Penrose spin coefficient, coefficient of a perturbation
$\varepsilon_{i_1 i_2,\ldots,i_N}$	Totally antisymmetric permutation symbol (Levi-Civita)
η^i	Components of the geodesic deviation vector
$\eta^{(a)(b)}$	Components of metric tensor relative to a complex null tetrad
$\eta_{i_1,i_2,\ldots,i_N}$	Totally antisymmetric pseudo (or oriented) tensor (Levi-Civita)
f^α	Newtonian force
$f_{ij}(x,u)$	Finsler metric components
F^i	4-force components
F_{ij}	Electromagnetic tensor field tensor components
g, $\|g\|$	Metric tensor determinant and its absolute value
G	Gravitational constant (usually set to 1)
g_{ij}	Metric tensor components
G^i_j	Einstein tensor components
γ	A parametrized curve into a manifold, Newman–Penrose spin coefficient
$\mathcal{X} := \chi \circ \gamma$	A parametrized curve into $\mathbb{R}^N$
Γ	The image of a parametrized curve into $\mathbb{R}^N$, characteristic matrix
Γ_{ij}	Characteristic matrix components
$\mathbb{\Gamma}_{(a)(b)(c)}$	Complex Ricci rotation coefficients
$\gamma_{(a)(b)(c)}$	Ricci rotation coefficients
$\gamma^k{}_{ij}$	Independent connection components in Hilbert-Palatini variational approach
$\hbar$	Reduced Planck's constant (usually set to 1)
h^i, h^{ij}, h^k_{ij}	Variations of vector, second-rank tensor, Christoffel connection respectively
$\vec{\mathbf{H}}$, H_α	Magnetic field and its components
$\mathcal{H}$	Relativistic Hamiltonian
$\mathbf{I}^{\cdot}_{\cdot}$	Identity tensor
J	Action functional or action integral
J^i	4-current components
J^{ik}	Total angular momentum components
$\vec{\mathbf{k}}$	Real null tetrad vector
k^i	Wave vector (or number) components
k_0	Curvature of spatial sections of F–L–R–W metric
$K(u)$	Gaussian curvature
$\vec{\mathbf{K}}$, K_i	A Killing vector and corresponding components
$K_{\mu\nu}$	Extrinsic curvature components of a hypersurface

κ	Einstein equation constant ($= 8\pi G/c^4$ in common units)
$\kappa_{(A)}$, $\kappa_{(0)}$	A^{th} curvature, Newman–Penrose spin coefficient
$\vec{\mathbf{l}}$	Real null tetrad vector
$l^i{}_j$, $L^i{}_j$	Components of a generalized Lorentz transformation
$[L]^{\mathrm{T}}$	Transposed matrix
L	A Lagrangian function
$L_{\vec{\mathbf{v}}}$	Lie derivative
$\mathcal{L}$	Lagrangian density
$\mathcal{L}_{(\mathrm{I})}$	Lagrangian function from super-Hamiltonian
λ	Eigenvalue, Lagrange multiplier, electromagnetic gauge function, Newman–Penrose spin coefficient
$\lambda_{(i)}$	ith eigenvalue
$\vec{\lambda}_{(A)}(s)$	Ath normal vector to a curve
$\lambda^i{}_{(a)}$, $\mu^{(a)}{}_i$	Components of orthonormal basis
Λ	Cosmological constant
m, $M(s)$	Mass, mass function
$\vec{\mathbf{m}}$, $\vec{\overline{\mathbf{m}}}$	Complex null tetrad vectors
M, M_N	A differentiable manifold, N dimensional differentiable manifold
M	"Total mass" of the universe
$\mathcal{M}^i$, ${}^*\mathcal{M}^i$	Maxwell vector (and dual) components
μ	Mass density, Newman–Penrose spin coefficient
N	Dimension of tangent vector space, lapse function in A.D.M. formalism
n^i	Unit normal vector components
N^α	Shift vector in A.D.M. formalism
ν	Frequency, Newman–Penrose spin coefficient
$O(p, n; \mathbb{R})$	Generalized Lorentz group
$\mathcal{I}O(p, n; \mathbb{R})$	Generalized Poincaré group
p	Point in a manifold, polynomial equation, pressure
$p^\#$	Polynomial equation for invariant eigenvalues
$p_\parallel$, $p_\perp$	Parallel pressure and transverse pressure respectively
p^i, $\mathcal{P}^i$	4-momentum components
$\pi_{(0)}$	Newman–Penrose spin coefficient
π^k	Projection mapping
$\mathcal{P}^i{}_j$	Projection tensor field components
ϕ	Characteristic surface function of a p.d.e., scalar field
Φ	Born-Infeld (or tachyonic) scalar field, (also a function used in five-dimensional cosmologies)
$\phi^\alpha_{(\mathrm{ext})}$	External force density
φ^{ij}	Complex electromagnetic field tensor components
$\Phi_{(A)(B)}$	Complex Ricci components (A, $B \in \{0, 1, 2\}$)
$\Phi^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}$	Components of an oriented, relative tensor field of weight w

Ψ	Complex Klein-Gordon field
$\Psi_{(J)}$	Complex Jth Weyl components ($J \in \{0, \ldots, 4\}$)
$Q_{(a)(d)}$	Complex Weyl tensor with second and third index projected in a timelike direction
R	Ricci curvature scalar (or invariant)
R_{ij}	Components of Ricci tensor
$R^{i}{}_{jk}$	Components of Cotton–Schouten–York tensor
$R^{i}{}_{jkl}$	Components of Riemann-Christoffel tensor
$\mathbb{R}$	The set of real numbers, complex Ricci scalar
$\mathbb{R}^N := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{N}$	Cartesian product of N copies of the set $\mathbb{R}$
ρ	Mass density, proper energy density, Newman–Penrose spin coefficient
s^{ij}, S^{ij}	Components of relativistic stress tensor (special and general respectively)
S_{ijkl}	Components of symmetrized curvature tensor
s	Arc separation parameter
S^2	Two-dimensional spherical surface
σ	Electrical charge density, Newman–Penrose spin coefficient, separation of a vector field, Klein-Gordon equation
$\sigma^{\alpha\beta}, \sigma^{ij}$	Stress density, shear tensor components
$\sum$, Σ	Arc separation function, function in Kerr metric, summation
$\vec{\mathbf{t}}_x$	Tangent vector of the image Γ at the point x
T_x	Tangent vector space of a manifold
$\tilde{T}_x$	Cotangent (or dual) vector space of a manifold
T^{ij}	Components of energy–momentum–stress tensor
$\mathbf{T}^{\cdot}_{\cdot\cdot}$, $T^{i}{}_{jk}$	Torsion tensor and the corresponding components
${}^{r}_{s}\mathbf{T}$	Tensor field of order $(r+s)$
$T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}$	Coordinate components of the (same) tensor field
$T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}$	Orthonormal components of the (same) tensor field
$\mathbb{T}^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}$	Complex tensor field components
${}^{r}_{s}\mathbf{T} \otimes {}^{p}_{q}\mathbf{S}$	Tensor (or outer product) of two tensor fields
$\mathcal{T}^i$	Conservation law components
τ	Affine parameter along geodesic (usu. proper time), Newman–Penrose spin coefficient
${}^{r}_{s}\mathcal{T}(T_x(\mathbb{R}^N))$	Tensor bundle
Θ_{ij}	Expansion tensor components, T-domain energy–momentum–stress tensor
${}^{r}_{s}\Theta$	Components of a relative tensor field
U	an open subset of a manifold
$\mathsf{U}_{(a)(b)}$, $\mathsf{V}_{(a)(b)}$, $\mathsf{W}_{(a)(b)}$	Components of complex bivector fields (see definitions (7.48i–vi))

u^i, U^i, $\mathcal{U}^i$	4-velocity components
$V^\alpha(t)$, $\mathcal{V}^\alpha$	Newtonian or Galilean velocity
W, w	Effective Newtonian potential
W	Lambert's W-function, (symbol also used for other functions in axi-symmetric metrics)
$\mathcal{W}$	Work function
${}_p\mathbf{W}$, $W_{i_1,\ldots,i_p}$	p-form and its antisymmetric components
ω_{ij}	Vorticity tensor components
Ω	Synge's world function
$x = \mathcal{X}(t)$, $x^i = \mathcal{X}^i(t)$	A parametrized curve in $\mathbb{R}^N$
$x = \xi(u)$, $x^i = \xi^i(u^1, \ldots, u^D)$	A parametrized submanifold
Y	Coefficient of spherical line element in Tolman-Bondi coordinates
z, $\bar{z}$	A complex variable and its conjugate
$\mathbb{Z}$, $\mathbb{Z}^+$	The set of integers, the set of positive integers

Chapter 1
Tensor Analysis on Differentiable Manifolds

1.1 Differentiable Manifolds

We will begin by briefly defining an N-dimensional differentiable manifold M. (See [23, 38, 56, 130].) There are a few assumptions in this definition. A set with a *topology* is one in which open subsets are known. Furthermore, if for every two distinct elements (or points) p and q there exist open and disjoint subsets containing p and q, respectively, then the topology is called *Hausdorff*. A connected Hausdorff manifold is *paracompact* if and only if it has a countable basis of open sets. (See [1, 130, 132].)

1. The first assumption we make about an applicable differentiable manifold M is that it is endowed with a paracompact topology.

(*Remark*: This assumption is necessary for the purpose of integration in any domain.)

We also consider only *a connected set* M for physical reasons. Moreover, we mostly deal with situations where M is an open set.

Now we shall introduce local coordinates for M. A *chart* (χ, U) or a *local coordinate system* is a pair consisting of an open subset $U \subset M$ together with a continuous, one-to-one mapping (*homeomorphism*) χ from U into (codomain) $D \subset \mathbb{R}^N$. Here, D is an open subset of $\mathbb{R}^N$ with the usual Euclidean topology.[1] For a point $p \in M$, we have $x \equiv (x^1, x^2, \ldots, x^N) = \chi(p) \in D$. The coordinates $(x^1, x^2, \ldots, x^N)$ are the coordinates of the point p in the chart (χ, U).

Each of the N coordinates is obtained by the projection mappings $\pi^k : D \subset \mathbb{R}^N \longrightarrow \mathbb{R}$, $k \in \{1, \ldots, N\}$. These are defined by $\pi^k(x) \equiv \pi^k(x^1, \ldots, x^N) =: x^k \in \mathbb{R}$. (See Fig. 1.1.)

[1] We can visualize *the coordinate space* $\mathbb{R}^N$ as an N-dimensional Euclidean space.

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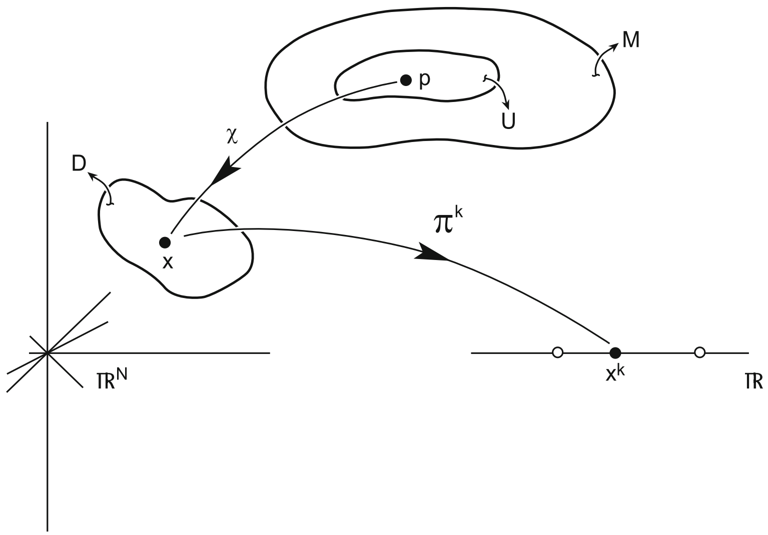

Fig. 1.1 A chart (χ, U) and projection mappings

Consider two coordinate systems or charts (χ, U) and $(\widehat{\chi}, \widehat{U})$ such that the point p is in the nonempty intersection of U and $\widehat{U}$. From Fig. 1.2, we conclude that

$$\begin{aligned} \widehat{x} &= (\widehat{\chi} \circ \chi^{-1})(x), \\ x &= (\chi \circ \widehat{\chi}^{-1})(\hat{x}), \end{aligned} \tag{1.1}$$

where $x \in D_{\mathrm{s}} \subset D$ and $\widehat{x} \in \widehat{D}_{\mathrm{s}} \subset \widehat{D}$.

From the preceding considerations, the mappings $\widehat{\chi} \circ \chi^{-1}$ and $\chi \circ \widehat{\chi}^{-1}$ are continuous and one-to-one. By projection of these points, we get

$$\begin{aligned} \widehat{x}^k &= [\pi^k \circ \widehat{\chi} \circ \chi^{-1}](x) =: \widehat{X}^k(x) \equiv \widehat{X}^k(x^1, \ldots, x^N), \\ x^k &= [\pi^k \circ \chi \circ \widehat{\chi}^{-1}](\widehat{x}) =: X^k(\widehat{x}) \equiv X^k(\widehat{x}^1, \ldots, \widehat{x}^N). \end{aligned} \tag{1.2}$$

The $2N$ functions $\widehat{X}^k$ and X^k are continuous. However, for a differentiable manifold, we assume that $\widehat{X}^k$ and X^k are differentiable functions of N real variables. In this context, we introduce some new notations. In case a function $f : D \subset \mathbb{R}^N \longrightarrow \mathbb{R}^M$ can be continuously differentiated r times with respect to every variable, we define the function f to belong to the class $C^r(D \subset \mathbb{R}^N; \mathbb{R}^M)$, $r \in \{0, 1, 2, \ldots\}$. In case the function is Taylor expandable (or a real-analytic function), the symbol $C^w(D \subset \mathbb{R}^N; \mathbb{R}^M)$ is used for the class.

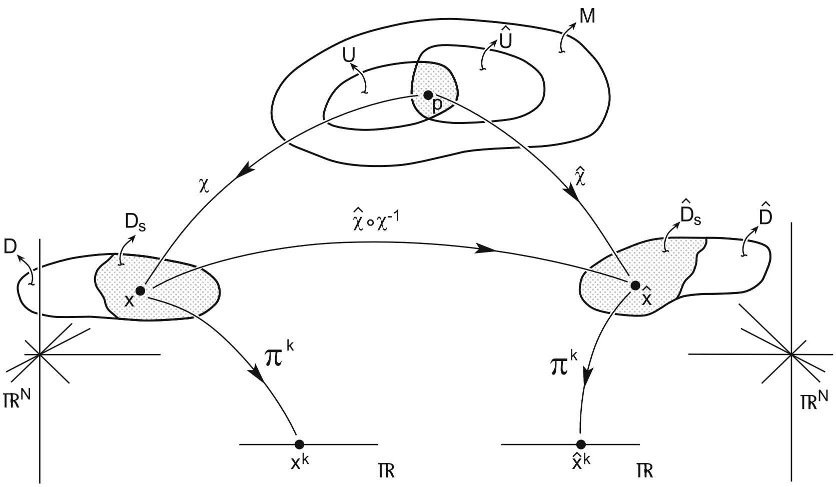

Fig. 1.2 Two charts in M and a coordinate transformation

By the first assumption of paracompactness, we can conclude [38, 130] that open subsets U_h exist[2] such that $M = \bigcup_h U_h$. The second assumption about M is the following:

2. There exist countable charts (χ_h, U_h) for M. Moreover, wherever there is a nonempty intersection between charts, coordinate transformations as in (1.2) of class C^r can be found.

Such a basis of charts for M is called a C^r-*Atlas*. A maximal collection of C^r-related atlases is called a *maximal* C^r-*Atlas*. (It is also called the complete atlas.) Finally, we are in a position to define a *differentiable manifold*.

3. An N-dimensional C^r-differentiable manifold is a set M with a maximal C^r-atlas.

(*Remark*: For $r = 0$, the set M is called a *topological manifold*.)

A differentiable manifold is said to be *orientable* if there exists an atlas (χ_h, U_h) such that the *Jacobian* det $\left[\frac{\partial \widehat{X}^k(x)}{\partial x^j}\right]$ is nonzero and of *one sign* everywhere.

Example 1.1.1. Consider the two-dimensional Euclidean manifold $\mathbb{E}_2$. One global chart $(\chi, \mathbb{E}_2)$ is furnished by $x = (x^1, x^2) = \chi(p)$, $p \in \mathbb{E}_2$, $D = \mathbb{R}^2$. Let $(\chi, \mathbb{E}_2)$ be one of infinitely many Cartesian coordinate systems.

Another chart, $(\hat{\chi}, \hat{U})$, is given by $\hat{x} \equiv (\hat{x}^1, \hat{x}^2) \equiv (r, \varphi) = \hat{\chi}(p)$; $\hat{D} := \left\{(\hat{x}^1, \hat{x}^2) \in \mathbb{R}^2 : \hat{x}^1 > 0, -\pi < \hat{x}^2 < \pi\right\}$.

[2]Some authors depict the boundary of an open subset by a dashed curve.

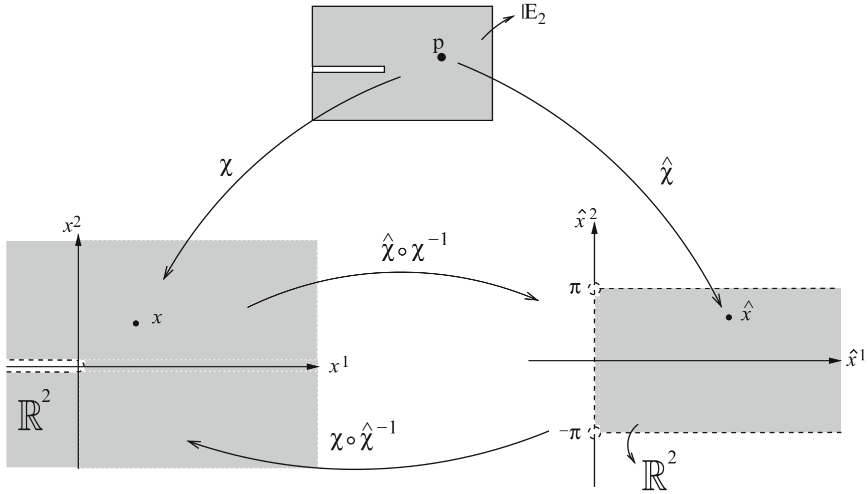

Fig. 1.3 The polar coordinate chart

The transformation to the polar coordinate chart is characterized by

$$\begin{aligned} x^1 &= X^1(\hat{x}) = \hat{x}^1 \cos \hat{x}^2, \\ x^2 &= X^2(\hat{x}) = \hat{x}^1 \sin \hat{x}^2, \end{aligned}$$

and the inverse transformation by

$$\hat{x}^1 = \hat{X}^1(x) = +\sqrt{(x^1)^2 + (x^2)^2} =: r,$$

$$\hat{x}^2 = \hat{X}^2(x) = \operatorname{arc}(x^1, x^2) \equiv \varphi := \begin{cases} \operatorname{Arctan}(x^2/x^1) \ \text{ for } \ x^1 > 0, \\ \frac{\pi}{2}\operatorname{sgn}(x^2) \ \text{ for } \ x^1 = 0 \ \text{ and } \ x^2 \neq 0, \\ \operatorname{Arctan}(x^2/x^1) + \pi \ \operatorname{sgn}(x^2) \ \text{ for } \ x^1 < 0 \\ \text{and } \ x^2 \neq 0. \end{cases}$$

Note that $-\pi/2 < \operatorname{Arctan}(x^2/x^1) < \pi/2$, so that $-\pi < \operatorname{arc}(x^1, x^2) < \pi$. (See Fig. 1.3.) □

Example 1.1.2. The two-dimensional (boundary) surface S^2 of the unit solid sphere can be constructed in the Euclidean space $\mathbb{E}_3$ by one constraint expressible in terms of the Cartesian coordinates as $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. Equivalently, the same constraint is expressible in the spherical polar coordinates as the radial coordinate $\widehat{x}^1 = 1$. The remaining spherical polar angular coordinates $(\widehat{x}^2, \widehat{x}^3) =: (\theta, \varphi)$ can be used as a possible coordinate system over an open subset of S^2, which is a

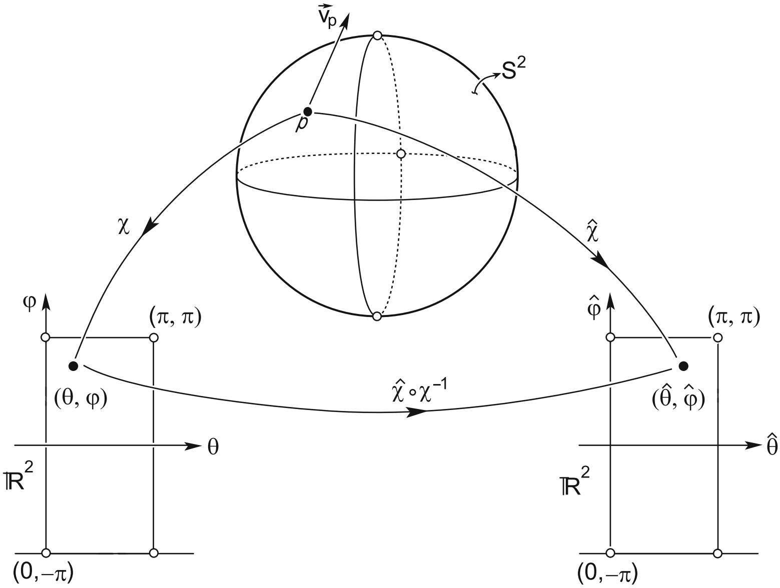

Fig. 1.4 Spherical polar coordinates

two-dimensional differentiable manifold in its own right. This coordinate chart is characterized by

$$x = (\theta, \varphi) = \chi(p), \quad p \in U \subset S^2;$$
$$D := \left\{(\theta, \varphi) \in \mathbb{R}^2 : 0 < \theta < \pi, -\pi < \varphi < \pi\right\} \subset \mathbb{R}^2.$$

Another *distinct* spherical polar chart is furnished by

$$\begin{aligned}
\widehat{\theta} \quad &= \widehat{\Theta}(\theta, \varphi) := \operatorname{Arc}\cos(-\sin\theta \sin\varphi), \\
\widehat{\varphi} \quad &= \widehat{\Phi}(\theta, \varphi) := \operatorname{arc}(-\sin\theta\cos\varphi, \cos\theta), \\
\widehat{D} :&= \left\{(\widehat{\theta}, \widehat{\varphi}) \in \mathbb{R}^2 : 0 < \widehat{\theta} < \pi, -\pi < \widehat{\varphi} < \pi\right\} \subset \mathbb{R}^2.
\end{aligned}$$

Neither of the two charts is global. However, the union $U \cup \widehat{U} = S^2$. Thus, the collection of two charts (χ, U) and $(\widehat{\chi}, \widehat{U})$ constitutes an atlas for S^2. (The minimum number of charts for an atlas of S^2 is two.) See Fig. 1.4 for the illustration. (For future use, a vector $\vec{\mathbf{v}}_p$ at p is shown.) □

Let us now consider the space in the arena of Newtonian physics. It is mathematically represented by the three-dimensional Euclidean space $\mathbb{E}_3$. This space admits infinitely many global Cartesian charts. Physically, vectors in the tangent space describe velocities, accelerations, forces, etc. Each of these vectors has a (homeomorphic) image in $\mathbb{R}^3$ of a Cartesian chart. To obtain an intuitive

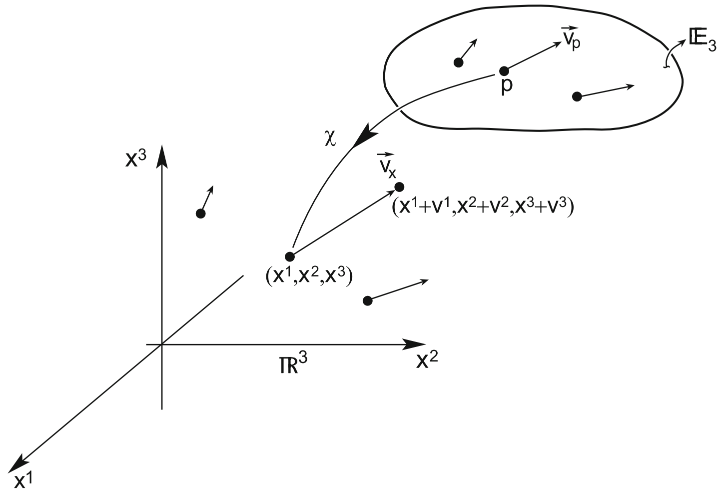

Fig. 1.5 Tangent vector in $\mathbb{E}_3$ and $\mathbb{R}^3$

definition of a *tangent vector* $\vec{\mathbf{v}}_p$ or its image $\vec{\mathbf{v}}_x$ in $\mathbb{R}^3$, we shall visualize an "arrow" in $\mathbb{R}^3$ with *the starting point* $x \equiv (x^1, x^2, x^3)$ in $\mathbb{R}^3$ and the directed displacement $\vec{\mathbf{v}} \equiv (v^1, v^2, v^3)$ necessary to reach the point $(x^1 + v^1, x^2 + v^2, x^3 + v^3)$ in $\mathbb{R}^3$. (See Fig. 1.5.)

Example 1.1.3. Let us choose the following as standard (Cartesian) basis vectors at x in $\mathbb{R}^3$:

$$\vec{\mathbf{i}}_x \equiv \vec{\mathbf{e}}_{1x} := (1,0,0)_x, \quad \vec{\mathbf{j}}_x \equiv \vec{\mathbf{e}}_{2x} := (0,1,0)_x, \quad \vec{\mathbf{k}}_x \equiv \vec{\mathbf{e}}_{3x} := (0,0,1)_x.$$

Let a three-dimensional vector be given by $\vec{\mathbf{v}} := 3\vec{\mathbf{e}}_1 + 2\vec{\mathbf{e}}_2 + \vec{\mathbf{e}}_3 = (3,2,1)$. Let us choose a point $x \equiv (x^1, x^2, x^3) = (1,2,3)$. Therefore, the tangent vector $\vec{\mathbf{v}}_x = (3,2,1)_{(1,2,3)}$ starts from the point $(1,2,3)$ and terminates at $(4,4,4)$. □

However, such a simple definition runs into problems in a curved manifold. An example of a simple curved manifold is the spherical surface S^2 we have previously discussed and as shown in Fig. 1.4. We have drawn an intuitive picture of a tangent vector $\vec{\mathbf{v}}_p$ on S^2. The starting point p of $\vec{\mathbf{v}}_p$ is on S^2. However, the end point of $\vec{\mathbf{v}}_p$ is *not* on S^2. The problem is how to define a tangent vector *intrinsically* on S^2, without going out of the spherical surface. One logical possibility is to introduce *directional derivatives* of a smooth function F defined at p in a subset of S^2. Such a definition involves only the point p and its neighboring points all on S^2. Thus, we shall represent tangent vectors by the directional derivatives. This concept appears to be very abstract at the beginning. (See [56, 121, 197].)

Before proceeding, it is useful to introduce the *Einstein summation convention*. In a mathematical expression, wherever two repeated indices are present, automatic sums over the repeated index is implied. For example, we denote

$$u^k v_k := \sum_{k=1}^{N} u^k v_k \equiv \sum_{j=1}^{N} u^j v_j = u^j v_j \ ,$$

$$g_{ij} a^i b^j := \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} a^i b^j \equiv \sum_{k=1}^{N} \sum_{l=1}^{N} g_{kl} a^k b^l = g_{kl} a^k b^l.$$

The summation indices are also called *dummy indices*, since they can be replaced by another set of repeated indices over the same range, as demonstrated in the above equations.

Now, let us define *generalized directional derivatives*. We shall use a coordinate chart (χ, U) instead of the abstract manifold M. Let $x \equiv (x^1, \ldots, x^N) = \chi(p)$. Let an N-tuple of vector components be given by $(v^1, \ldots, v^N)$. Then, the *tangent vector* $\vec{\mathbf{v}}_x$ in $D \subset \mathbb{R}^N$ is defined by the generalized directional derivative:

$$\vec{\mathbf{v}}_x := v^i \frac{\partial}{\partial x^i},$$
$$\vec{\mathbf{v}}_x[f] := v^i \frac{\partial f(x)}{\partial x^i}. \tag{1.3}$$

Here, f belongs to $C^1(D \subset \mathbb{R}^N; \mathbb{R})$. (Note that in the notation of the usual calculus, $\vec{\mathbf{v}}_x[f] = \vec{\mathbf{v}} \cdot \mathbf{grad}\, f$.)

The set of all tangent vectors $\vec{\mathbf{v}}_x$ constitutes the N-dimensional *tangent vector space* $T_x(\mathbb{R}^N)$ in $D \subset \mathbb{R}^N$. It is an isomorphic image of the tangent vector space $T_p(M)$. The coordinate basis set $\{\vec{\mathbf{e}}_{1x}, \ldots, \vec{\mathbf{e}}_{Nx}\}$ for $T_x(\mathbb{R}^N)$ is defined by the differential operators

$$\vec{\mathbf{e}}_{1x} := \frac{\partial}{\partial x^1}, \ldots, \vec{\mathbf{e}}_{Nx} := \frac{\partial}{\partial x^N};$$
$$\vec{\mathbf{e}}_{kx}[f] := \frac{\partial f(x)}{\partial x^k}. \tag{1.4}$$

(See [23, 197].)

Now, we shall define a tangent vector field $\vec{\mathbf{v}}(p)$ in $U \subset M$ or equivalently the tangent vector field $\vec{\mathbf{v}}(x)$ in $D \subset \mathbb{R}^N$. It involves N real-valued functions $v^1(x), \ldots, v^N(x)$. The tangent vector field is defined by the operator

$$\vec{\mathbf{v}}(x) := v^j(x) \frac{\partial}{\partial x^j},$$
$$\vec{\mathbf{v}}(x)[f] := v^j(x) \frac{\partial f(x)}{\partial x^j}. \tag{1.5}$$

Here, $f \in C^1(D \subset \mathbb{R}^N; \mathbb{R})$.

Example 1.1.4. Let $D := \{(x^1, x^2) \in \mathbb{R}^2 \ : \ x^1 \in \mathbb{R}, 1 < x^2\}$. Moreover, let $f(x) := x^1 \cdot [(x^2)^{x^2}]$ and

$$\vec{\mathbf{v}}(x^1, x^2) := \frac{e^{x^2}}{2} \frac{\partial}{\partial x^1} - (2\cosh x^1)\frac{\partial}{\partial x^2}.$$

Therefore, by (1.5), we get

$$\vec{\mathbf{v}}(x^1, x^2)[f] = [(x^2)^{x^2}]\left[\frac{e^{x^2}}{2} - 2(x^1\cosh x^1)(1 + \ln x^2)\right],$$

$$\lim_{x^1\to 1}\lim_{x^2\to 1_+}\left\{\vec{\mathbf{v}}(x^1, x^2)[f]\right\} = -\left(\frac{e}{2} + e^{-1}\right). \qquad \square$$

The *Kronecker delta* is defined by the (real) numbers:

$$\delta^i{}_j := \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{1.6}$$

The $N \times N$ matrix with entries $\delta^i{}_j$ is the *unit matrix* $[I] = [\delta^i{}_j]$.

Example 1.1.5. Consider a vector $\vec{\boldsymbol{\alpha}} = (\alpha^1, \alpha^2, \ldots, \alpha^N)$. Using the summation convention,

$$\begin{aligned} \delta^1{}_j\alpha^j &= \delta^1{}_1\alpha^1 + \delta^1{}_2\alpha^2 + \cdots + \delta^1{}_N\alpha^N = \alpha^1, \\ \delta^i{}_j\alpha^j &= \alpha^i, \\ \delta^i{}_j\delta^j{}_k &= \delta^i{}_k, \quad \delta^i{}_j\delta^j{}_k\delta^k{}_l\delta^l{}_i = N. \end{aligned} \qquad \square$$

Example 1.1.6. The coordinate basis vectors from (1.4) can be expressed as

$$\vec{\mathbf{e}}_j(x) = \frac{\partial}{\partial x^j}.$$

Therefore, for the function $f(x) := x^i$,

$$\vec{\mathbf{e}}_j(x)[x^i] = \frac{\partial(x^i)}{\partial x^j} = \delta^i{}_j,$$

$$\vec{\mathbf{v}}(x)[x^i] = v^j(x)\frac{\partial}{\partial x^j}(x^i) = v^j(x)\delta^i{}_j = v^i(x). \tag{1.7}$$

$\square$

The main properties of tangent vector fields can be summarized in the following theorem:

Theorem 1.1.7. *If $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are tangent vector fields in $D \subset \mathbb{R}^N$, and $f, g, h \in C^1(D \subset \mathbb{R}^N; \mathbb{R})$, then*

(*i*)

$$[f(x)\vec{\mathbf{v}}(x) + g(x)\vec{\mathbf{w}}(x)][h] = f(x)(\vec{\mathbf{v}}(x)[h]) + g(x)(\vec{\mathbf{w}}(x)[h]), \tag{1.8}$$

(*ii*)

$$\vec{\mathbf{v}}(x)[cf + kg] = c(\vec{\mathbf{v}}(x)[f]) + k(\vec{\mathbf{v}}(x)[g]), \tag{1.9}$$

for all constants c and k

(*iii*)

$$\vec{\mathbf{v}}(x)[fg] = (\vec{\mathbf{v}}(x)[f])g(x) + f(x)(\vec{\mathbf{v}}(x)[g])\,. \tag{1.10}$$

The proof is left as an exercise.

Now we shall define a *cotangent* (or *covariant*) *vector field*. Consider a function $\tilde{\mathbf{f}}(x)$ which maps the tangent vector space $T_x(\mathbb{R}^N)$ into $\mathbb{R}$ such that

$$\tilde{\mathbf{f}}(x)\left[g(x)\vec{\mathbf{v}}(x) + h(x)\vec{\mathbf{w}}(x)\right] = g(x)\left[\tilde{\mathbf{f}}(x)\left(\vec{\mathbf{v}}(x)\right)\right] + h(x)\left[\tilde{\mathbf{f}}(x)\left(\vec{\mathbf{w}}(x)\right)\right] \tag{1.11}$$

for all functions $g(x), h(x)$ and all tangent vector fields $\vec{\mathbf{v}}(x), \vec{\mathbf{w}}(x)$ in $T_x(\mathbb{R}^N)$. Such a function is called a cotangent (or covariant) vector field.

Example 1.1.8. The (unique) zero covariant vector field $\tilde{\mathbf{0}}(x)$ is defined by

$$\tilde{\mathbf{0}}(x)\left(\vec{\mathbf{v}}(x)\right) := 0$$

for all $\vec{\mathbf{v}}(x)$ in $T_x(\mathbb{R}^N)$. □

(*Remark*: In Newtonian physics, the gradient of the gravitational potential is a covariant vector field.)

We define the linear combinations of covariant vector fields as

$$\left[\Lambda(x)\tilde{\mathbf{f}}(x) + \Omega(x)\tilde{\mathbf{g}}(x)\right]\left[\vec{\mathbf{v}}(x)\right] := \Lambda(x)\left[\tilde{\mathbf{f}}(x)(\vec{\mathbf{v}}(x))\right] + \Omega(x)\left[\tilde{\mathbf{g}}(x)(\vec{\mathbf{v}}(x))\right] \tag{1.12}$$

for all functions $\Lambda(x)$, $\Omega(x)$ and all tangent vectors $\vec{\mathbf{v}}(x)$ in $T_x(\mathbb{R}^N)$. Under such a rule, the set of all covariant vector fields constitutes an N-dimensional *cotangent* (or *dual*) *vector space* $\tilde{T}_x(\mathbb{R}^N)$.

Now we shall introduce the notion of a *1-form* which will be identified with a covariant vector field. We need to define a (totally) differentiable function f over $D \subset \mathbb{R}^N$. In case f satisfies the following criterion,

$$\lim_{(h^1,\ldots,h^N)\to(0,\ldots,0)} \left\{ \frac{\left[f(x^1+h^1,\ldots,x^N+h^N) - f(x^1,\ldots,x^N) - h^j \frac{\partial f(x)}{\partial x^j} \right]}{\sqrt{(h^1)^2 + \cdots + (h^N)^2}} \right\} = 0, \tag{1.13}$$

for arbitrary $(h^1, \ldots, h^N)$, we call the function f a *totally differentiable function* at x. The usual condensed form for denoting (1.13) is to write

$$\mathrm{d}f(x) = \frac{\partial f(x)}{\partial x^j}\,\mathrm{d}x^j. \tag{1.14}$$

Each side of (1.14) is called a 1-form. It is customary to identify a 1-form, $\mathrm{d}f(x)$, with a covariant vector field, $\tilde{\mathbf{f}}(x)$, with the following rule of operation:

$$\mathrm{d}f(x)[\vec{\mathbf{v}}(x)] \equiv \tilde{\mathbf{f}}(x)\left[\vec{\mathbf{v}}(x)\right] := v^j(x)\frac{\partial f(x)}{\partial x^j}. \tag{1.15}$$

Here, $\vec{\mathbf{v}}(x)$ is an arbitrary vector field.

Example 1.1.9. Let $f(x) \equiv f(x^1, x^2) := (1/3)[(x^1)^3 - 3e^{x^2}]$, $(x^1, x^2) \in \mathbb{R}^2$. Therefore, by (1.14) and (1.15),

$$\begin{aligned}
\mathrm{d}f(x) &= (x^1)^2\mathrm{d}x^1 - e^{x^2}\mathrm{d}x^2,\\
\mathrm{d}f(x)[\vec{\mathbf{v}}(x)] &= (x^1)^2 v^1(x) - e^{x^2} v^2(x),\\
\mathrm{d}f(0,0)[\vec{\mathbf{v}}(0,0)] &= -v^2(0,0).
\end{aligned}$$

□

Example 1.1.10. Consider the function $f(x) := x^k$. Then

$$\begin{aligned}
\mathrm{d}f(x) &= \mathrm{d}x^k\ ,\\
\frac{\partial}{\partial x^i} &= \delta^j_{\ i}\frac{\partial}{\partial x^j}.
\end{aligned}$$

Furthermore, by (1.15),

$$\mathrm{d}x^k\left[\frac{\partial}{\partial x^i}\right] = \delta^j_{\ i}\frac{\partial\left(x^k\right)}{\partial x^j} = \delta^j_{\ i}\delta^k_{\ j} = \delta^k_{\ i}.$$

□

Therefore, we identify the coordinate covariant basis field for $\tilde{T}_x(\mathbb{R}^N)$ as

$$\{\tilde{\mathbf{e}}_1(x), \ldots, \tilde{\mathbf{e}}_N(x)\} =: \left\{\mathrm{d}x^1, \ldots, \mathrm{d}x^N\right\}, \tag{1.16}$$

$$\tilde{\mathbf{e}}^k(x)\left[\vec{\mathbf{e}}_i(x)\right] = \delta^k_{\ i}. \tag{1.17}$$

Thus, every covariant vector (or 1-form) admits the linear combination

$$\tilde{\mathbf{W}}(x) = W_j(x)\mathrm{d}x^j \tag{1.18}$$

in terms of the basis covariant vectors $\mathrm{d}x^j$'s. The (unique) functions, $W_j(x)$, are called *covariant components*.

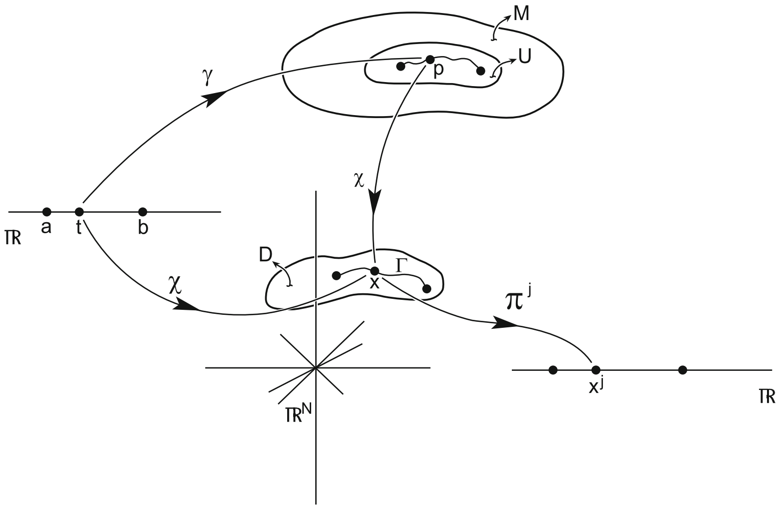

Fig. 1.6 A parametrized curve γ into M

Example 1.1.11. Consider the two-dimensional Euclidean manifold and a Cartesian coordinate chart. Let another chart be given by

$$\hat{x}^1 = \hat{X}^1(x) := (x^1)^3,$$

$$\hat{x}^2 = \hat{X}^2(x) := (x^2)^3, \ (x^1, x^2) \in \mathbb{R}^2 - \{(0, 0)\}; \ (\hat{x}^1, \hat{x}^2) \in \mathbb{R}^2 - \{(0, 0)\}.$$

Then, by direct computations, we deduce that

$$\frac{\partial}{\partial \hat{x}^1} = \frac{1}{3(x^1)^2} \frac{\partial}{\partial x^1}, \quad \frac{\partial}{\partial \hat{x}^2} = \frac{1}{3(x^2)^2} \frac{\partial}{\partial x^2},$$

$$\mathrm{d}\hat{x}^1 = 3(x^1)^2 \,\mathrm{d}x^1, \quad \mathrm{d}\hat{x}^2 = 3(x^2)^2 \,\mathrm{d}x^2.$$

It can now be verified that

$$\mathrm{d}\hat{x}^i \left(\frac{\partial}{\partial \hat{x}^j} \right) = \mathrm{d}x^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i{}_j. \qquad \square$$

Now we shall discuss a different topic, namely, a *parametrized curve* γ. Consider an interval $[a, b] \subset \mathbb{R}$.

(*Remark*: Open or semiopen intervals are also allowed. Moreover, unbounded intervals are permitted too.)

The parametrized curve γ is a function from the interval $[a, b]$ into a differentiable manifold. (See Fig. 1.6.)

(Note that the *function* γ is called the parametrized curve.) The image of the composite function $\mathcal{X} := \chi \circ \gamma$ in $D \subset \mathbb{R}^N$ is denoted by the symbol Γ. The coordinates on Γ are furnished by

$$\begin{aligned} x &= [\chi \circ \gamma](t) = \mathcal{X}(t), \\ x^j &= [\pi^j \circ \chi \circ \gamma](t) = \left[\pi^j \circ \mathcal{X}\right](t) =: \mathcal{X}^j(t). \end{aligned} \tag{1.19}$$

Here, $t \in [a, b] \subset \mathbb{R}$. The functions χ^j are usually *assumed* to be differentiable and piecewise twice-differentiable. The condition of the *nondegeneracy* (or *regularity*) is

$$\sum_{j=1}^{N} \left[\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\right]^2 > 0. \tag{1.20}$$

In Newtonian physics, t is taken to be the time variable and $M = \mathbb{E}_3$, the physical space. Moreover, Γ is a particle trajectory relative to a Cartesian coordinate system. The nondegeneracy condition (1.20) implies that the speed of the motion is strictly positive.

Example 1.1.12. Let the image Γ in $\mathbb{R}^3$ be given by

$$x = \mathcal{X}(t) := (2\cos^2 t,\ \ \sin 2t,\ \ 2\sin t); \quad 0 < t < \pi/2.$$

The curve is nondegenerate and the coordinate functions are real-analytic. Consider a circular cylinder in $\mathbb{R}^3$ such that it intersects the $x^1 - x^2$ plane on the unit circle with the center at $(1, 0, 0)$. Now, consider a spherical surface in $\mathbb{R}^3$ given by the equation $(x^1)^2 + (x^2)^2 + (x^3)^2 = 4$. The image Γ lies in the intersection of the cylinder and the sphere. □

Let us consider the tangent vector $\vec{\mathbf{t}}_x$ of the image Γ at the point x in $\mathbb{R}^N$ (see Fig. 1.7). In calculus, the components of the tangent vector $\vec{\mathbf{t}}_{\mathcal{X}(t)}$ are taken to be $\left(\frac{\mathrm{d}\mathcal{X}^1(t)}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}\mathcal{X}^N(t)}{\mathrm{d}t}\right)$. Therefore, the tangent vector $\vec{\mathbf{t}}_{\mathcal{X}(t)} \equiv \vec{\boldsymbol{\mathcal{X}}}'(t)$ along Γ (according to (1.3)) must be defined as the generalized directional derivative:

$$\vec{\mathbf{t}}_{\mathcal{X}(t)} \equiv \vec{\boldsymbol{\mathcal{X}}}'(t) := \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\left[\frac{\partial}{\partial x^j}\right]_{|\mathcal{X}(t)}. \tag{1.21}$$

The above tangent vector field belongs to the tangent vector space $T_{\mathcal{X}(t)}(\mathbb{R}^3)$ along Γ.

Example 1.1.13. Let a real-analytic, nondegenerate curve $\mathcal{X}$ into $\mathbb{R}^4$ be defined by

$$\begin{aligned} x &= \mathcal{X}(t) := ((t)^3, t, e^t, \sinh t), \quad -\infty < t < \infty; \\ \mathcal{X}(0) &= (0, 0, 1, 0). \end{aligned}$$

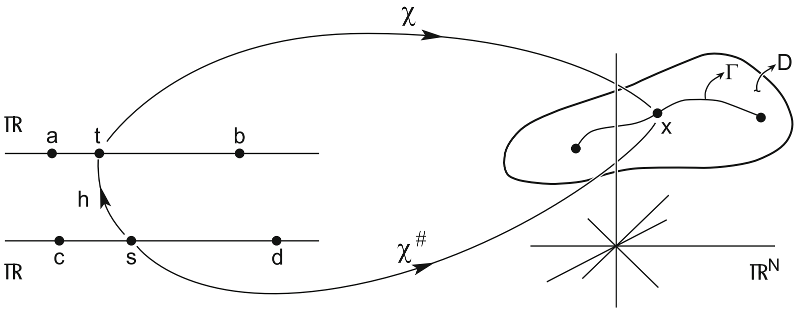

Fig. 1.7 Reparametrization of a curve

The corresponding tangent vector field along Γ is furnished by the generalized directional derivative:

$$\vec{\boldsymbol{\mathcal{X}}}'(t) = (3t^2)\left[\frac{\partial}{\partial x^1}\right]_{|\mathcal{X}(t)} + \left[\frac{\partial}{\partial x^2}\right]_{|\mathcal{X}(t)} + (e^t)\left[\frac{\partial}{\partial x^3}\right]_{|\mathcal{X}(t)} + (\cosh t)\left[\frac{\partial}{\partial x^4}\right]_{|\mathcal{X}(t)},$$

$$\vec{\boldsymbol{\mathcal{X}}}'(0) = \left[\frac{\partial}{\partial x^2}\right]_{|\mathcal{X}(0)} + \left[\frac{\partial}{\partial x^3}\right]_{|\mathcal{X}(0)} + \left[\frac{\partial}{\partial x^4}\right]_{|\mathcal{X}(0)}. \qquad \square$$

The tangent vector $\vec{\boldsymbol{\mathcal{X}}}'(t)$ can act on a differentiable function f (restricted to Γ) by (1.22) to follow. On this topic, we state and prove the following theorem.

Theorem 1.1.14. *Let a parametrized curve $\mathcal{X} : [a, b] \subset \mathbb{R} \to \mathbb{R}^N$ be differentiable and nondegenerate. Let $f : D \subset \mathbb{R}^N \to \mathbb{R}$ be a (totally) differentiable function. Then*

$$\vec{\boldsymbol{\mathcal{X}}}'(t)[f(\mathcal{X}(t))] = \frac{\mathrm{d}}{\mathrm{d}t}[f(\mathcal{X}(t))]. \tag{1.22}$$

Proof. By (1.15) and the chain rule of differentiation, the left-hand side of the above equation yields

$$\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\left[\frac{\partial}{\partial x^j}\right]_{|\mathcal{X}(t)}[f(\mathcal{X}(t))] \equiv \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\left[\frac{\partial f(x)}{\partial x^j}\right]_{|\mathcal{X}(t)} = \frac{\mathrm{d}}{\mathrm{d}t}[f(\mathcal{X}(t))]. \qquad \blacksquare$$

Now, we shall discuss the *reparametrization* of a curve. Let h be a differentiable and one-to-one mapping from $[c, d] \subset \mathbb{R}$ into $\mathbb{R}$. (See Fig. 1.7.).

The image Γ in $\mathbb{R}^N$ is given by the points

$$\begin{aligned} &t = h(s), \\ &x = \mathcal{X}(t), \quad a \le t \le b, \quad a = h(c), \quad b = h(d), \\ &x = \mathcal{X}^{\#}(s), \; c \le s \le d. \end{aligned}$$

Here,

$$\mathcal{X}^{\#} := \mathcal{X} \circ h \tag{1.23}$$

is the *reparametrized curve* into $\mathbb{R}^N$.

Theorem 1.1.15. *If $\mathcal{X}^{\#}$ is the reparametrization of the differentiable curve $\mathcal{X}$ by the function h, then the tangent vector*

$$\vec{\boldsymbol{\mathcal{X}}}^{\#\prime}(s) = \frac{\mathrm{d}h(s)}{\mathrm{d}s} \vec{\boldsymbol{\mathcal{X}}}'(h(s)). \tag{1.24}$$

Proof. By (1.23), we get

$$\mathcal{X}^{\# j}(s) = \mathcal{X}^j(h(s)).$$

By the assumption of differentiabilities and the chain rule, we obtain

$$\frac{\mathrm{d}\mathcal{X}^{\# j}(s)}{\mathrm{d}s} = \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\Big|_{t=h(s)} \cdot \frac{\mathrm{d}h(s)}{\mathrm{d}s}\,.$$

Therefore, the tangent vector

$$\begin{aligned}\vec{\boldsymbol{\mathcal{X}}}^{\#\prime}(s) &= \frac{\mathrm{d}\mathcal{X}^{\# j}(s)}{\mathrm{d}s}\left[\frac{\partial}{\partial x^j}\right]_{|\mathcal{X}^{\#}(s)} = \frac{\mathrm{d}h(s)}{\mathrm{d}s}\left[\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\right]_{|t=h(s)}\left[\frac{\partial}{\partial x^j}\right]_{|\mathcal{X}(h(s))} \\ &\equiv \frac{\mathrm{d}h(s)}{\mathrm{d}s}\vec{\boldsymbol{\mathcal{X}}}'(h(s)).\end{aligned}$$

■

Example 1.1.16. Consider the Euclidean plane $\mathbb{E}_2$ and a parametrized (real-analytic) curve $\mathcal{X} := \chi \circ \gamma$ given by

$$\mathcal{X}(t) := (\cos t, \sin t), \quad -2\pi \leq t \leq 2\pi.$$

The image γ is a unit circle in $\mathbb{R}^2$ with self-intersections. The *winding number* of this circle is exactly two.

Let a reparametrization be defined by

$$\begin{aligned} t &= h(s) := 2s, \quad -\pi \leq s \leq \pi\,; \\ \frac{\mathrm{d}h(s)}{\mathrm{d}s} &\equiv 2 > 0. \end{aligned}$$

The reparametrized curve $\mathcal{X}^{\#}$ is furnished by

$$\mathcal{X}^{\#}(s) = \mathcal{X}(2s) = (\cos(2s), \sin(2s)).$$

The new tangent vector is given by

$$\begin{aligned}\vec{\mathcal{X}}^{\#\prime}(s) &= 2\left[-(\sin 2s)\frac{\partial}{\partial x^1} + (\cos 2s)\frac{\partial}{\partial x^2}\right]_{|\mathcal{X}^{\#}(s)} \\ &= 2\vec{\mathcal{X}}'(2s).\end{aligned}$$

□

Exercises 1.1

1. Consider the three-dimensional Euclidean manifold and a global Cartesian chart $(\chi, \mathbb{E}_3)$ given by

$$x = (x^1,\, x^2,\, x^3) = \chi(p),\; x \text{ in } D = \mathbb{R}^3.$$

   The spherical polar chart $(\hat{\chi},\, \hat{U})$ is given by

$$\begin{aligned}\hat{x} &= (\hat{x}^1,\, \hat{x}^2,\, \hat{x}^3) \equiv (r,\, \theta,\, \varphi) = \hat{\chi}(p), \\ \hat{D} &:= \left\{(\hat{x}^1,\, \hat{x}^2,\, \hat{x}^3) \in \mathbb{R}^3 :\; \hat{x}^1 > 0,\; 0 < \hat{x}^2 < \pi,\; -\pi < \hat{x}^3 < \pi\right\}.\end{aligned}$$

   The coordinate transformation is characterized by

$$\begin{aligned}x^1 &= X^1(\hat{x}) := \hat{x}^1 \sin\hat{x}^2 \cos\hat{x}^3, \\ x^2 &= X^2(\hat{x}) := \hat{x}^1 \sin\hat{x}^2 \sin\hat{x}^3, \\ x^3 &= X^3(\hat{x}) := \hat{x}^1 \cos\hat{x}^2.\end{aligned}$$

   Obtain the inverse functions $\hat{X}^1(x),\; \hat{X}^2(x),\; \hat{X}^3(x)$ explicitly.
2. Prove (1.10) in the text.
3. Consider a function of two variables given by

$$f(x^1,\, x^2) := \begin{cases} (x^1)^2\sin(1/x^1) + (x^2)^2\sin(1/x^2) & \text{for } \; x^1x^2 \neq 0, \\ (x^1)^2\sin(1/x^1) & \text{for } \; x^1 \neq 0 \; \text{ and } \; x^2 = 0, \\ (x^2)^2\sin(1/x^2) & \text{for } \; x^1 = 0 \; \text{ and } \; x^2 \neq 0, \\ 0 & \text{for } \; x^1 = x^2 = 0. \end{cases}$$

   (i) Prove that f is totally differentiable at the origin $(0,\, 0)$.
   (ii) Prove that $\frac{\partial f(x^1,x^2)}{\partial x^1}$ and $\frac{\partial f(x^1,x^2)}{\partial x^2}$ exist but are discontinuous at the origin.

4. Evaluate the 1-form $\tilde{\mathbf{w}}(x) := \delta_{ij} x^i \mathrm{d}x^j$ on the tangent vector field $\vec{\mathbf{V}}(x) := \sum_{j=1}^{N} (x^j)^5 \frac{\partial}{\partial x^j}$. Prove that $\tilde{\mathbf{w}}(x)[\vec{\mathbf{V}}(x)] \geq 0$.
5. Consider the *semicubical parabola* in $\mathbb{R}^2$ given by

$$\mathcal{X}(t) := (t^2, t^3), \quad t \in \mathbb{R}.$$

The curve is degenerate at $\mathcal{X}(0) = (0, 0)$. Prove that reparametrization of this curve *cannot remove the degeneracy*.

(*Remark*: The degenerate point is called a *cusp*.)

Answers and Hints to Selected Exercises

1.

$$\hat{x}^1 = \hat{X}^1(x) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2},$$

$$\hat{x}^2 = \hat{X}^2(x) = \mathrm{Arccos}\left[x^3 / \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right],$$

$$\hat{x}^3 = \hat{X}^3(x) = \mathrm{arc}(x^1,\ x^2).$$

3.

$$\frac{\partial f(x^1,\ x^2)}{\partial x^1} = \begin{cases} 2x^1 \sin\left(\dfrac{1}{x^1}\right) - \cos\left(\dfrac{1}{x^1}\right) & \text{for} \quad x^1 \neq 0, \\ 0 \quad \text{for} \quad x^1 = 0\,; \end{cases}$$

$$\frac{\partial f(x^1,\ x^2)}{\partial x^2} = \begin{cases} 2x^2 \sin\left(\dfrac{1}{x^2}\right) - \cos\left(\dfrac{1}{x^2}\right) & \text{for} \quad x^2 \neq 0, \\ 0 \quad \text{for} \quad x^2 = 0. \end{cases}$$

(See [112].)

1.2 Tensor Fields Over Differentiable Manifolds

Consider a point $x_0 \equiv (x_0^1, \ldots, x_0^N) = \chi(p_0) \in \mathbb{R}^N$. The corresponding tangent vector space and the cotangent (or covariant) vector space are denoted by $T_{x_0}\big(\mathbb{R}^N\big)$ and $\tilde{T}_{x_0}\big(\mathbb{R}^N\big)$, respectively. An $\boldsymbol{(r+s)}$th *order* (or *rank*) *mixed tensor* ${}^r_s\mathbf{T}_{x_0}$ is a function[3] from the Cartesian product $\underbrace{\tilde{T}_{x_0} \times \cdots \times \tilde{T}_{x_0}}_{r} \times \underbrace{T_{x_0} \times \cdots \times T_{x_0}}_{s}$ into $\mathbb{R}$. Moreover, it has to satisfy the following *multilinearity conditions*:

$$\begin{aligned} &{}^r_s\mathbf{T}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \lambda\tilde{\mathbf{u}}^k_{x_0} + \mu\tilde{\mathbf{v}}^k_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}; \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right) \\ &\quad = \lambda\left[{}^r_s\mathbf{T}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^k_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}; \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right)\right] \\ &\qquad + \mu\left[{}^r_s\mathbf{T}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{v}}^k_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}; \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right)\right], \end{aligned} \tag{1.25i}$$

$$\begin{aligned} &{}^r_s\mathbf{T}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}; \vec{\mathbf{a}}_{1\,x_0}, \ldots, \lambda\vec{\mathbf{a}}_{j\ x_0} + \mu\vec{\mathbf{b}}_{j\ x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right) \\ &\quad = \lambda\left[\mathbf{T}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}; \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{j\ x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right)\right] \\ &\qquad + \mu\left[{}^r_s\mathbf{T}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}; \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{b}}_{j\ x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right)\right], \end{aligned} \tag{1.25ii}$$

for all λ, μ in $\mathbb{R}$, all $\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}; \tilde{\mathbf{v}}^k_{x_0}$ in $\tilde{T}_{x_0}\left(\mathbb{R}^N\right)$, all $\vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}; \vec{\mathbf{b}}^j_{x_0}$ in $T_{x_0}\left(\mathbb{R}^N\right)$, all k in $\{1, \ldots, r\}$, and all j in $\{1, \ldots, s\}$. Here, the nonnegative integer r represents the *contravariant order* and the nonnegative integer s represents the *covariant order*. Note that the function ${}^r_s\mathbf{T}_{x_0}$ is *linear in each of the $r + s$ slots* in the argument. (See [23, 56, 195, 197].)

Example 1.2.1. The (unique) $(r + s)$th order zero tensor is defined by

$${}^r_s\mathbf{0}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}; \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right) := 0 \tag{1.26}$$

for all $\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}$ in $\tilde{T}_{x_0}\left(\mathbb{R}^N\right)$ and all $\vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}$ in $T_{x_0}\left(\mathbb{R}^N\right)$. □

Example 1.2.2. Let $\tilde{\mathbf{b}}^k_{x_0} := \beta^k{}_1\tilde{\mathbf{e}}^1_{x_0} + \cdots + \beta^k_N\tilde{\mathbf{e}}^N_{x_0}$ and $\vec{\mathbf{a}}_{j\ x_0} = \alpha^1{}_j\vec{\mathbf{e}}_{1\,x_0} + \cdots + \alpha^N{}_j\vec{\mathbf{e}}_{N\,x_0}$. (Here, $\left\{\vec{\mathbf{e}}_{1\,x_0}, \ldots, \vec{\mathbf{e}}_{N\,x_0}\right\}$ and $\left\{\tilde{\mathbf{e}}^1_{x_0}, \ldots, \tilde{\mathbf{e}}^N_{x_0}\right\}$ are basis sets for $T_{x_0}(\mathbb{R}^N)$ and $\tilde{T}_{x_0}(\mathbb{R}^N)$, respectively.) Let a function be defined by

$${}^r_s\mathbf{F}_{x_0}\left(\tilde{\mathbf{b}}^1_{x_0}, \ldots, \tilde{\mathbf{b}}^r_{x_0}; \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right) := \left(\beta^1{}_1 \cdot \beta^2{}_2 \cdots \beta^r{}_r\right)\left(\alpha^1{}_1 \cdot \alpha^2{}_2 \cdots \alpha^s{}_s\right).$$

The function ${}^r_s\mathbf{F}_{x_0}$ can be proved to be a mixed tensor of order $(r + s)$. □

[3] The rank of a tensor is *different* from the rank of a matrix. The rank of a matrix is defined as the number of linearly independent rows or columns of a matrix. (These two numbers are the same if the elements are commutative.) Because of this, one sometimes uses the term "tensor order" or "degree of the tensor" when referring to tensor rank to avoid confusion. The tensor order or degree is the dimension of the minimum array required to represent the tensor. In index notation, this is equivalent to the number of indices required to represent the tensor. (See [8].)

The linear combination of two mixed tensors are defined by:

$$\begin{aligned}
&\left[\lambda\ {}^r_s\mathbf{T}_{x_0} + \mu\ {}^r_s\mathbf{W}_{x_0}\right]\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0};\ \vec{\mathbf{a}}_{1\,x_0}, \ldots,\ \vec{\mathbf{a}}_{s\,x_0}\right) \\
&:= \lambda\left[{}^r_s\mathbf{T}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0};\ \vec{\mathbf{a}}_{1\,x_0}, \ldots,\ \vec{\mathbf{a}}_{s\,x_0}\right)\right] \\
&\quad + \mu\left[{}^r_s\mathbf{W}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0};\ \vec{\mathbf{a}}_{1\,x_0}, \ldots,\ \vec{\mathbf{a}}_{s\,x_0}\right)\right]
\end{aligned} \tag{1.27}$$

for all λ, μ in $\mathbb{R}$, all $\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0}$ in $\tilde{\mathbf{T}}_{x_0}(\mathbb{R}^N)$ and all $\vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}$ in $\mathbf{T}_{x_0}(\mathbb{R}^N)$. (Note that addition between two tensors of different order is *not permitted*.)

It can be proved via (1.27) that the set of all mixed tensors of order $(r+s)$ constitutes a (real) vector space of dimension N^{r+s}.

The *tensor product* (or *outer product*) between two tensors ${}^r_s\mathbf{T}_{x_0}$ and ${}^p_q\mathbf{W}_{x_0}$ is defined by the function ${}^r_s\mathbf{T}_{x_0} \otimes\ {}^p_q\mathbf{W}_{x_0}$ from $\underbrace{\tilde{T}_{x_0}(\mathbb{R}^N) \times \cdots \times \tilde{T}_{x_0}(\mathbb{R}^N)}_{r+p}$ $\times \underbrace{T_{x_0}(\mathbb{R}^N) \times \cdots \times T_{x_0}(\mathbb{R}^{\mathbb{N}})}_{s+q}$ into $\mathbb{R}$ such that

$$\begin{aligned}
&\left[{}^r_s\mathbf{T}_{x_0} \otimes\ {}^p_q\mathbf{W}_{x_0}\right]\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0};\ \tilde{\mathbf{v}}^1_{x_0}, \ldots, \tilde{\mathbf{v}}^p_{x_0};\ \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0};\ \vec{\mathbf{b}}_{1\,x_0}, \ldots, \vec{\mathbf{b}}_{q\,x_0}\right) \\
&\quad =: \left[{}^r_s\mathbf{T}_{x_0}\left(\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0};\ \vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0}\right)\right] \times \left[{}^p_q\mathbf{W}_{x_0}\left(\tilde{\mathbf{v}}^1_{x_0}, \ldots, \tilde{\mathbf{v}}^p_{x_0};\ \vec{\mathbf{b}}_{1\,x_0}, \ldots, \vec{\mathbf{b}}_{q\,x_0}\right)\right]
\end{aligned} \tag{1.28}$$

for all $\vec{\mathbf{a}}_{1\,x_0}, \ldots, \vec{\mathbf{a}}_{s\,x_0};\ \vec{\mathbf{b}}_{1\,x_0}, \ldots, \vec{\mathbf{b}}_{q\,x_0}$ in $T_{x_0}(\mathbb{R}^N)$ and all $\tilde{\mathbf{u}}^1_{x_0}, \ldots, \tilde{\mathbf{u}}^r_{x_0};\ \tilde{\mathbf{v}}^1_{x_0}, \ldots, \tilde{\mathbf{v}}^p_{x_0}$ in $\tilde{T}_{x_0}(\mathbb{R}^N)$. Note that ${}^r_s\mathbf{T}_{x_0} \otimes\ {}^p_q\mathbf{W}_{x_0}$ is *a tensor* of order $(r+p) + (s+q)$.

Example 1.2.3. Let $N = 2$ and $\left\{\vec{\mathbf{e}}_{1\,x_0},\ \vec{\mathbf{e}}_{2\,x_0}\right\}$ be a basis set for $T_{x_0}(\mathbb{R}^N)$. Let two covariant tensors, ${}^0_1\mathbf{T}_{x_0}$ and ${}^0_2\mathbf{W}_{x_0}$, be defined by

$$\begin{aligned}
&\vec{\mathbf{a}}_{x_0} = \alpha^1\vec{\mathbf{e}}_{1\,x_0} + \alpha^2\vec{\mathbf{e}}_{2\,x_0},\ \vec{\mathbf{b}}_{1\,x_0} = \beta^1{}_1\vec{\mathbf{e}}_{1\,x_0} + \beta^2{}_1\vec{\mathbf{e}}_{2\,x_0}\ ,\ \vec{\mathbf{b}}_{2\,x_0} = \beta^1{}_2\vec{\mathbf{e}}_{1\,x_0} + \beta^2{}_2\vec{\mathbf{e}}_{2\,x_0}, \\
&\vec{\mathbf{c}}_{x_0} = \gamma^1\vec{\mathbf{e}}_{1\,x_0} + \gamma^2\vec{\mathbf{e}}_{2\,x_0}\ ; \\
&{}^0_1\mathbf{T}_{x_0}(\vec{\mathbf{a}}_{x_0}) := \alpha^1,\ \ {}^0_2\mathbf{W}_{x_0}(\vec{\mathbf{b}}_{1\,x_0},\ \vec{\mathbf{b}}_{2\,x_0}) := \beta^1{}_1\beta^2{}_2.
\end{aligned}$$

Then, by the tensor product rule (1.28), we derive that

$$\left[{}^0_1\mathbf{T}_{x_0} \otimes\ {}^0_2\mathbf{W}_{x_0}\right]\left(\vec{\mathbf{a}}_{x_0};\ \vec{\mathbf{b}}_{1\,x_0},\ \vec{\mathbf{b}}_{2\,x_0}\right) = \alpha^1\left(\beta^1{}_1\beta^2{}_2\right)\ \in\ \mathbb{R}.$$

Moreover,

$$\begin{aligned}
&\left[{}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right]\left(\lambda\vec{\mathbf{a}}_{x_0} + \mu\vec{\mathbf{c}}_{x_0};\ \vec{\mathbf{b}}_{1\,x_0},\ \vec{\mathbf{b}}_{2\,x_0}\right)\\
&\quad = \left(\lambda\alpha^1 + \mu\gamma^1\right)\left(\beta^1_1\beta^2_2\right) = \lambda\left[\alpha^1\left(\beta^1_1\beta^2_2\right)\right] + \mu\left[\gamma^1\left(\beta^1_1\beta^2_2\right)\right]\\
&\quad = \lambda\left[\left({}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right)\left(\vec{\mathbf{a}}_{x_0};\ \vec{\mathbf{b}}_{1\,x_0},\ \vec{\mathbf{b}}_{2\,x_0}\right)\right]\\
&\qquad + \mu\left[\left({}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right)\left(\vec{\mathbf{c}}_{x_0};\ \vec{\mathbf{b}}_{1\,x_0},\ \vec{\mathbf{b}}_{2\,x_0}\right)\right].
\end{aligned}$$

Similarly, we can deduce that

$$\begin{aligned}
&\left[{}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right]\left(\vec{\mathbf{a}}_{x_0};\ \lambda\vec{\mathbf{b}}_{1\,x_0} + \mu\vec{\mathbf{c}}_{x_0},\ \vec{\mathbf{b}}_{2\,x_0}\right)\\
&\quad = \lambda\left[\left({}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right)\left(\vec{\mathbf{a}}_{x_0};\ \vec{\mathbf{b}}_{1\,x_0},\ \vec{\mathbf{b}}_{2\,x_0}\right)\right]\\
&\qquad + \mu\left[\left({}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right)\left(\vec{\mathbf{a}}_{x_0};\ \vec{\mathbf{c}}_{x_0},\ \vec{\mathbf{b}}_{2\,x_0}\right)\right],\\
&\left[{}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right]\left(\vec{\mathbf{a}}_{x_0},\ \vec{\mathbf{b}}_{1\,x_0},\ \lambda\vec{\mathbf{b}}_{2\,x_0} + \mu\vec{\mathbf{c}}_{x_0}\right)\\
&\quad = \lambda\left[\left({}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right)\left(\vec{\mathbf{a}}_{x_0},\ \vec{\mathbf{b}}_{1\,x_0},\ \vec{\mathbf{b}}_{2\,x_0}\right)\right]\\
&\qquad + \mu\left[\left({}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}\right)\left(\vec{\mathbf{a}}_{x_0},\ \vec{\mathbf{b}}_{1\,x_0},\ \vec{\mathbf{c}}_{x_0}\right)\right].
\end{aligned}$$

Thus, by (1.28), the tensor product ${}^0_1\mathbf{T}_{x_0} \otimes {}^0_2\mathbf{W}_{x_0}$ is a tensor of order $[0 + (1 + 2)]$. □

Now we shall introduce the concept of a *tensor field* in $D \subset \mathbb{R}^N$. A (tangent) tensor field ${}^r_s\mathbf{T}$ is a function that assigns a tensor ${}^r_s\mathbf{T}(x)$ for each point x in $D \subset \mathbb{R}^N$. (A more rigorous definition of a tensor field is as a section of the *tensor (fiber) bundle*. See [38, 56, 267].)

Lower order tensors have special names and abbreviated notations. For example, ${}^0_0\mathbf{T}(x) \in \mathbb{R}$ is called a *scalar field*. ${}^1_0\mathbf{T}(x)$ is a tangent vector field (or contravariant vector field), whereas ${}^0_1\mathbf{T}(x)$ is a cotangent (or covariant) vector field. We denote lower order tensors by the following simplified notations:

$$\begin{aligned}
&{}^0_0\mathbf{T}(x) =: T(x),\\
&{}^0_2\mathbf{g}(x) =: \mathbf{g}_{..}(x),\\
&{}^1_3\mathbf{R}(x) =: \mathbf{R}^{\boldsymbol{\cdot}}_{\ \ldots}(x), \quad \text{etc.}
\end{aligned} \tag{1.29}$$

Let $\{\vec{\mathbf{e}}_i(x)\}_1^N$ and $\{\tilde{\mathbf{e}}^i(x)\}_1^N$ be basis sets for $T_x(\mathbb{R}^{\mathbb{N}})$ and $\tilde{T}_x(\mathbb{R}^{\mathbb{N}})$, respectively. It can be proved that $\{\vec{\mathbf{e}}_{i1}(x) \otimes \vec{\mathbf{e}}_{i2}(x) \otimes \cdots \otimes \vec{\mathbf{e}}_{ir}(x) \otimes \tilde{\mathbf{e}}^{j1}(x) \otimes \tilde{\mathbf{e}}^{j2}(x) \otimes \cdots \otimes \tilde{\mathbf{e}}^{js}(x)\}_1^N$ is a basis set for the set of all $(r+s)$th order tensor fields. Let us choose coordinate basis sets $\vec{\mathbf{e}}_i(x) = \frac{\partial}{\partial x^i}$ and $\tilde{\mathbf{e}}^j(x) = \mathrm{d}x^j$ as in (1.7) and (1.16). Therefore, a tensor field ${}^r_s\mathbf{T}(x)$ can be expressed as a linear combination

$$ {}^r_s\mathbf{T}(x) = T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s} \,. \tag{1.30} $$

The N^{r+s} real-valued functions

$$ T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x) := {}^r_s\mathbf{T}\left(\mathrm{d}x^{i_1}, \ldots, \mathrm{d}x^{i_r}; \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}\right) \tag{1.31} $$

are called the *coordinate components* of the tensor field ${}^r_s\mathbf{T}(x)$ (relative to the chosen coordinate chart).

There exist other distinct but isomorphic (exactly similar) tensors of the same order ($r' = r$, $s' = s$). For example, we can define an isomorphic tensor by

$$ \begin{aligned} &T'_{j_1,\ldots,j_s}{}^{i_1,\ldots,i_r}(x)\, \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \\ &= {}^r_s\mathcal{I}(x)\left[{}^r_s\mathbf{T}(x)\right] \\ &:= T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)\, \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}}. \end{aligned} $$

The linear combination in (1.27) and the tensor product in (1.28) yield, respectively,

$$ \begin{aligned} &\lambda(x)\, {}^r_s\mathbf{T}(x) + \mu(x)\, {}^r_s\mathbf{W}(x) \\ &\left[\lambda(x)T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x) + \mu(x)W^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)\right]\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}, \end{aligned} \tag{1.32i} $$

$$ \begin{aligned} &{}^r_s\mathbf{T}(x) \otimes {}^p_q\mathbf{B}(x) \\ &= \left[T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x) B^{u_1,\ldots,u_p}{}_{v_1,\ldots,v_q}(x)\right]\left[\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes \frac{\partial}{\partial x^{u_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \right. \\ &\qquad \left. \otimes\, \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s} \otimes \mathrm{d}x^{v_1} \otimes \cdots \otimes \mathrm{d}x^{v_q}\right]. \end{aligned} \tag{1.32ii} $$

Now, we shall define the *contraction operation*. It is defined by

$$\begin{aligned}
{}^{u}_{v}\mathcal{C}\left[{}^{r}_{s}\mathbf{T}(x)\right] := \Big[& T^{i_1,\ldots,i_{u-1}ci_{u+1},\ldots,i_r}{}_{j_1,\ldots,j_{v-1}cj_{v+1},\ldots,j_s}(x)\Big] \\
& \times \left[\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_{u-1}}} \otimes \frac{\partial}{\partial x^{i_{u+1}}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes \mathrm{d}x^{j_1} \otimes \right. \\
& \left. \cdots \otimes \mathrm{d}x^{j_{v-1}} \otimes \mathrm{d}x^{j_{v+1}} \otimes \cdots \otimes \mathrm{d}x^{j_s}\right].
\end{aligned} \tag{1.33}$$

It can be proved that ${}^{u}_{v}\mathcal{C}\left[{}^{r}_{s}\mathbf{T}(x)\right]$ is a tensor field of order $(r-1)+(s-1)$.

Example 1.2.4. Let us choose a two-dimensional manifold and a coordinate chart. Let a basis field and its conjugate covariant field be expressed as

$$\vec{\mathbf{e}}_p(x) := E^i{}_p(x)\frac{\partial}{\partial x^i}, \quad \tilde{\mathbf{e}}^q(x) = \tilde{E}^q{}_j(x)\,\mathrm{d}x^j;$$
$$\tilde{E}^q{}_i(x)E^i{}_p(x) = \delta^p_q, \; i, \; j \in \{1, \; 2\}; \; x \in D := \left\{x \in \mathbb{R}^2 : \; |x^1| < 1, \; |x^2| < 1\right\}.$$

Let a scalar field be defined by

$$\lambda(x) := \exp\left(x^1 - x^2\right).$$

Furthermore, let two tensor fields be furnished by (*suspending the summation convention temporarily in this example*)

$$\mathbf{T}^{\cdot}{}_{\cdot}(x) := \left[\sum_{t=1}^{2}\delta_{qt}x^p x^t\right]\vec{\mathbf{e}}_p(x) \otimes \tilde{\mathbf{e}}^q(x),$$
$$\mathbf{W}^{\cdot}{}_{\cdot}(x) := \sum_{u=1}^{2}\sum_{s=1}^{2}\left[\delta_{vs}(x^4)^3(x^s)^3\right]\vec{\mathbf{e}}_u(x) \otimes \tilde{\mathbf{e}}^v(x).$$

Then, by (1.32i), (1.32ii), and (1.33), we get

$$\lambda(x)T^p_q(x) + W^p_q(x) = \sum_{t=1}^{2}\delta_{qt}\left[\exp(x^1 - x^2)x^p x^t + (x^p)^3(x^t)^3\right],$$
$$\left[\mathbf{T}^{\cdot}{}_{\cdot}(x) \otimes \mathbf{W}^{\cdot}{}_{\cdot}(x)\right]^{pu}{}_{qv} = T^p_q(x)W^u_v(x) = \sum_{s=1}^{2}\sum_{t=1}^{2}\left[\delta_{qt}\delta_{vs}x^p x^t(x^u)^3(x^s)^3\right],$$
$${}^{1}_{1}\mathcal{C}\left(\mathbf{T}^{\cdot}{}_{\cdot}(x)\right) = T^c_c(x) = \delta_{cq}x^c x^q \geq 0,$$
$${}^{1}_{1}\mathcal{C}\left(\mathbf{T}^{\cdot}{}_{\cdot}\left(0, \frac{1}{2}\right)\right) = \frac{1}{4}.$$

□

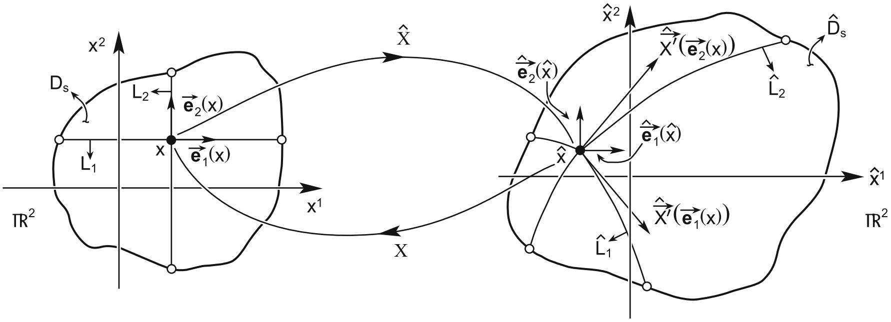

Fig. 1.8 The Jacobian mapping of tangent vectors

Now we shall discuss the transformation of tensor components under a general coordinate transformation discussed in (1.2). To reiterate briefly,

$$\hat{x}^k = \hat{X}^k(x) \equiv \hat{X}^k(x^1, \ldots, x^N),$$

$$x^k = X^k(\widehat{x}) \equiv X^k(\hat{x}^1, \ldots, \hat{x}^N),$$

$$x \in D_s \subset \mathbb{R}^N;\ \hat{x} \in \hat{D}_s \subset \mathbb{R}^N.$$

The functions $\hat{X}^k$ and X^k are assumed to be differentiable. The above coordinate transformation induces a one-to-one mapping $\widehat{\vec{\mathbf{X}}}'$ from $T_x\left(\mathbb{R}^N\right)$ into $T_{\hat{x}}\left(\mathbb{R}^N\right)$ (see Fig. 1.8). This mapping is called the *Jacobian mapping* or *derivative mapping*. It is furnished by

$$\left[\hat{\vec{\mathbf{X}}}'(\vec{\mathbf{t}}(x))\right][f(\hat{x})] := \vec{\mathbf{t}}(x)\left[f \circ \hat{X}(x)\right],$$

$$\vec{\mathbf{X}}' = \left(\hat{\vec{\mathbf{X}}}'\right)^{-1}. \tag{1.34}$$

Here, $\vec{\mathbf{t}}(x)$ is in $T_x(\mathbb{R}^N)$, $\hat{\vec{\mathbf{X}}}'\left(\vec{\mathbf{t}}(x)\right)$ is in $T_{\hat{x}}\left(\mathbb{R}^N\right)$ and f is a real-valued differentiable function from $\hat{D}s \subset \mathbb{R}^N$.

The coordinate transformations $\hat{X}$ and $X = \hat{X}^{-1}$ induce the mapping $\hat{\tilde{\mathbf{X}}}'$ from $\tilde{T}_x(\mathbb{R}^N)$ into $\tilde{T}_{\hat{x}}(\mathbb{R}^N)$ by the following rule:

$$\left[\hat{\tilde{\mathbf{X}}}'(\tilde{\boldsymbol{\omega}}(x))\right]\left[\hat{\vec{\mathbf{V}}}(\hat{x})\right] := \tilde{\boldsymbol{\omega}}(x)\left[\vec{\mathbf{X}}'\left(\hat{\vec{\mathbf{V}}}(\hat{x})\right)\right]. \tag{1.35}$$

Example 1.2.5. Let us choose $\vec{\mathbf{t}}(x) = \frac{\partial}{\partial x^i}$ in $T_x(\mathbb{R}^N)$. Moreover, we choose $f = \pi^j$, the projection mapping in (1.2). By (1.34), we obtain

$$\left[\hat{\vec{\mathbf{X}}}'\left(\frac{\partial}{\partial x^i}\right)\right]\left[\pi^j(\hat{x})\right] = \frac{\partial}{\partial x^i}\left[\pi^j \circ \hat{X}(x)\right] = \frac{\partial \hat{X}^j(x)}{\partial x^i}.$$

Now, choosing $\tilde{\boldsymbol{\omega}}(x) = \mathrm{d}x^i$ and $\hat{\vec{V}}(\hat{x}) = \frac{\partial}{\partial \hat{x}^j}$, we derive, from (1.35), that

$$\begin{aligned}\left[\hat{\tilde{\mathbf{X}}}'(\mathrm{d}x^i)\right]\left[\frac{\partial}{\partial \hat{x}^j}\right] &= \mathrm{d}x^i\left[\vec{\mathbf{X}}'\left(\frac{\partial}{\partial \hat{x}^j}\right)\right] \\ &= \mathrm{d}x^i\left[\frac{\partial X^k(\hat{x})}{\partial \hat{x}^j}\,\frac{\partial}{\partial x^k}\right] = \frac{\partial X^k(\hat{x})}{\partial \hat{x}^j}\left\{\mathrm{d}x^i\left[\frac{\partial}{\partial x^k}\right]\right\} = \frac{\partial X^i(\hat{x})}{\partial \hat{x}^j}.\end{aligned}$$

But we have

$$\left[\frac{\partial X^i(\hat{x})}{\partial \hat{x}^k}\mathrm{d}\hat{x}^k\right]\left[\frac{\partial}{\partial \hat{x}^j}\right] = \frac{\partial X^i(\hat{x})}{\partial \hat{x}^k}\left[\mathrm{d}\hat{x}^k\,\frac{\partial}{\partial x^j}\right] = \frac{\partial X^i(\hat{x})}{\partial \hat{x}^j}.$$

Comparing both equations, we conclude that

$$\hat{\tilde{\mathbf{X}}}'(\mathrm{d}x^i) = \frac{\partial X^i(\hat{x})}{\partial \hat{x}^k}\,\mathrm{d}\hat{x}^k. \qquad \square$$

The coordinate transformations $\hat{X}$ and X induce transformations from ${}^r_s\mathbf{T}(x)$ into ${}^r_s\hat{\mathbf{T}}(\hat{x})$. They are explicitly defined as

$$\begin{aligned}&{}^r_s\hat{\mathbf{T}}(\hat{x}) := {}^r_s\hat{\mathbf{X}}'\left[{}^r_s\mathbf{T}(x)\right], \\ &{}^r_s\hat{\mathbf{X}}'\left[{}^r_s\mathbf{T}(x)\right]\left[\hat{\tilde{\mathbf{X}}}'\left(\tilde{\mathbf{w}}_1(x)\right),\ldots,\hat{\tilde{\mathbf{X}}}'\left(\tilde{\mathbf{w}}_r(x)\right);\hat{\vec{\mathbf{X}}}'\left(\vec{\mathbf{v}}_1(x)\right),\ldots,\hat{\vec{\mathbf{X}}}'\left(\vec{\mathbf{v}}_s(x)\right)\right] \\ &\quad := \left[{}^r_s\mathbf{T}(x)\right]\left(\tilde{\mathbf{w}}_1(x),\ \ldots,\ \tilde{\mathbf{w}}_r(x);\ \vec{\mathbf{v}}_1(x),\ \ldots,\ \vec{\mathbf{v}}_s(x)\right),\end{aligned} \tag{1.36}$$

for all $\vec{\mathbf{v}}_1(x),\ldots,\vec{\mathbf{v}}_s(x)$ in $T_x(\mathbb{R}^N)$ and all $\tilde{\mathbf{w}}_1(x),\ \ldots,\ \tilde{\mathbf{w}}_r(x)$ in $\tilde{T}_x(\mathbb{R}^N)$.

We shall now derive the transformation rules for the (coordinate) components of the tensor field in (1.30).

Theorem 1.2.6. *The transformation rules of the $(r+s)$th order tensor field components $T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)$ under a general coordinate transformation (1.2) are furnished by*

$$\hat{T}^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(\hat{x}) = \frac{\partial \hat{X}^{k_1}(x)}{\partial x^{i_1}},\ldots,\frac{\partial \hat{X}^{k_r}(x)}{\partial x^{i_r}}\frac{\partial X^{j_1}(\hat{x})}{\partial \hat{x}^{l_1}},\ldots,\frac{\partial X^{j_s}(\hat{x})}{\partial \hat{x}^{l_s}}\,T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x), \tag{1.37}$$

where $x \in D_s \subset D \subset \mathbb{R}^N$ and $\hat{x} \in \hat{D}_s \subset \hat{D} \subset \mathbb{R}^N$.

Proof. By Definition (1.36) and (1.34) and (1.35), we obtain

$$\begin{aligned}
&\left[{}^r_s\mathbf{T}(x)\right]\left(\mathrm{d}x^{i_1},\ldots,\mathrm{d}x^{i_r};\ \frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_s}}\right)\\
&\quad=\left[{}^r_s\hat{\mathbf{T}}(\hat{x})\right]\left(\frac{\partial X^{i_1}(\hat{x})}{\partial \hat{x}^{k_1}}\mathrm{d}\hat{x}^{k_1},\ \ldots,\ \frac{\partial \hat{X}^{l_s}(x)}{\partial x^{j_s}}\frac{\partial}{\partial \hat{x}^{l_s}}\right)\\
&\quad=\frac{\partial X^{i_1}(\hat{x})}{\partial \hat{x}^{k_1}},\ldots,\frac{\partial \hat{X}^{l_s}(x)}{\partial x^{j_s}}\left\{\left[{}^r_s\hat{\mathbf{T}}(\hat{x})\right]\left(\mathrm{d}\hat{x}^{k_1},\ \ldots,\ \frac{\partial}{\partial \hat{x}^{l_s}}\right)\right\}
\end{aligned}$$

or

$$T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)=\frac{\partial X^{i_1}(\hat{x})}{\partial \hat{x}^{k_1}},\ldots,\frac{\partial X^{i_r}(\hat{x})}{\partial \hat{x}^{k_r}}\,\frac{\partial \hat{X}^{l_1}(x)}{\partial x^{j_1}},\ldots,\frac{\partial \hat{X}^{l_s}(x)}{\partial x^{j_s}}\,\hat{T}^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(\hat{x}).$$

Inverting the above equation by using

$$\frac{\partial \hat{X}^{k}(x)}{\partial x^{m}}\,\frac{\partial X^{m}(\hat{x})}{\partial \hat{x}^{i}}=\delta^{k}_{\ i},\quad \frac{\partial X^{j}(\hat{x})}{\partial \hat{x}^{m}}\,\frac{\partial \hat{X}^{m}(x)}{\partial x^{l}}=\delta^{j}_{\ l}\,, \tag{1.38}$$

we obtain the transformation in (1.37). ■

Example 1.2.7. Consider the case of $N = 2$ and $D = D_s = \mathbb{R}^2$. Let a (real) analytic transformation be provided by

$$\begin{aligned}
\hat{x}^1 &= \hat{X}^1(x) := x^1 + x^2,\\
\hat{x}^2 &= \hat{X}^2(x) := x^1 - x^2,\\
\frac{\partial\left(\hat{x}^1,\ \hat{x}^2\right)}{\partial\left(x^1,\ x^2\right)} &\equiv \det\left[\frac{\partial \hat{X}^i(x)}{\partial x^j}\right] \equiv -2.
\end{aligned}$$

Let a $(2+0)$th order tensor field be furnished by

$$\begin{aligned}
\mathbf{T}^{\cdot\cdot}(x) &:= (x^i x^j)\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j},\\
T^{ij}(x) &= x^i x^j.
\end{aligned}$$

The transformed components $\hat{T}^{kl}(\hat{x})$ are given by (1.37) as

$$\begin{aligned}
\hat{T}^{11}(\hat{x}) &= \left[\frac{\partial \hat{X}^1(x)}{\partial x^1}\right]^2 T^{11}(x)+2\left[\frac{\partial \hat{X}^1(x)}{\partial x^1}\frac{\partial \hat{X}^1(x)}{\partial x^2}\right]T^{12}(x)+\left[\frac{\partial \hat{X}^1(x)}{\partial x^2}\right]^2 T^{22}(x)\\
&= (x^1 + x^2)^2 = (\hat{x}^1)^2.
\end{aligned}$$

Similarly, we can deduce that

$$\hat{T}^{kl}(\hat{x}) = \hat{x}^k \hat{x}^l \ .$$

In many physical applications, the components of a tensor *restricted to a special curve* are of interest. The values of $\hat{T}^{kl}(\hat{x})$ restricted to a special curve

$$\begin{aligned} \hat{\mathcal{X}}^1(t) &:= t, \\ \hat{\mathcal{X}}^2(t) &:= -t, \\ t &\in \mathbb{R}, \end{aligned}$$

are furnished by

$$\hat{T}^{11}(\hat{x})_{|\hat{\mathcal{X}}(t)} = (t)^2 \equiv \hat{T}^{22}(\hat{x})_{|\hat{\mathcal{X}}(t)} \equiv -\hat{T}^{12}(\hat{x})_{|\hat{\mathcal{X}}(t)} \equiv -\hat{T}^{21}(\hat{x})_{|\hat{\mathcal{X}}(t)}.$$

Evaluated at a particular point, corresponding to $t = 1/\sqrt{2}$ say, and the image point $\hat{\mathcal{X}}(1/\sqrt{2})$, the values of the tensor's components are

$$\begin{aligned} &\hat{T}^{11}\left(1/\sqrt{2},\, -1/\sqrt{2}\right) = \hat{T}^{22}\left(1/\sqrt{2},\, -1/\sqrt{2}\right) \\ &= -\hat{T}^{12}\left(1/\sqrt{2},\, -1/\sqrt{2}\right) = -\hat{T}^{21}\left(1/\sqrt{2},\, -1/\sqrt{2}\right) = 1/2. \end{aligned}$$

□

Now we shall define the transformation rules for the components of a *relative tensor field* ${}^r_s\Theta(x)$ of weight w (sometimes called a tensor density field of weight w). These are furnished by

$$\begin{aligned} \hat{\Theta}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\hat{x}) &= \left\{ \det\left[\frac{\partial X^m(\hat{x})}{\partial \hat{x}^n}\right]\right\}^w \frac{\partial \hat{X}^{i_1}(x)}{\partial x^{k_1}}, \ldots, \frac{\partial \hat{X}^{i_r}(x)}{\partial x^{k_r}} \\ &\times \frac{\partial X^{l_1}(\hat{x})}{\partial \hat{x}^{j_1}}, \ldots, \frac{\partial X^{l_s}(\hat{x})}{\partial \hat{x}^{j_s}}\, \Theta^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(x) \ . \end{aligned} \tag{1.39}$$

In case $w = 0$, (1.39) reduces to (1.37). In case $w = 1$, the relative tensor $\Theta^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)$ is called a *tensor density field*. (See [244].)

Another generalization of tensor fields is possible. An *oriented relative tensor field* of weight w is characterized by the transformation rules:

$$\begin{aligned} \hat{\Phi}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\hat{x}) &= \operatorname{sgn}\left\{ \det\left[\frac{\partial X^m(\hat{x})}{\partial \hat{x}^n}\right]\right\} \left\{ \det\left[\frac{\partial X^m(\hat{x})}{\partial \hat{x}^n}\right]\right\}^w \frac{\partial \hat{X}^{i_1}(x)}{\partial x^{k_1}}, \ldots, \frac{\partial \hat{X}^{i_r}(x)}{\partial x^{k_r}} \\ &\times \frac{\partial X^{l_1}(\hat{x})}{\partial \hat{x}^{j_1}}, \ldots, \frac{\partial X^{l_s}(\hat{x})}{\partial \hat{x}^{j_s}}\, \Phi^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(x). \end{aligned} \tag{1.40}$$

Example 1.2.8. Suppose that $T_{ij}(x)$ are coordinate components of the tensor field $\mathbf{T}_{..}(x)$. By the transformation rules (1.37), we deduce that

$$\hat{T}_{ij}(\hat{x}) = \frac{\partial X^k(\hat{x})}{\partial \hat{x}^i} T_{kl}(x) \frac{\partial X^l(\hat{x})}{\partial \hat{x}^j} \; .$$

Taking the determinant of both sides, we obtain

$$\begin{aligned}\det\left[\hat{T}_{ij}(\hat{x})\right] &= \det\left[\frac{\partial X^k(\hat{x})}{\partial \hat{x}^i}\right] \cdot \det\left[T_{kl}(x)\right] \cdot \det\left[\frac{\partial X^l(\hat{x})}{\partial \hat{x}^j}\right] \\ &= \left\{\det\left[\frac{\partial X^m(\hat{x})}{\partial \hat{x}^n}\right]\right\}^2 \cdot \det\left[T_{kl}(x)\right].\end{aligned}$$

Extracting the positive square root of both sides, we get

$$\begin{aligned}\sqrt{\left|\det\left[\hat{T}_{ij}(\hat{x})\right]\right|} &= \left|\det\left[\frac{\partial X^m(\hat{x})}{\partial \hat{x}^n}\right]\right| \cdot \sqrt{\left|\det\left[T_{kl}(x)\right]\right|} \\ &= \operatorname{sgn}\left\{\det\left[\frac{\partial X^m(\hat{x})}{\partial \hat{x}^n}\right]\right\} \cdot \det\left[\frac{\partial X^m(\hat{x})}{\partial \hat{x}^n}\right] \cdot \sqrt{\left|\det\left[T_{kl}(x)\right]\right|} \; .\end{aligned}$$

Comparing the above with (1.40), we conclude that $\sqrt{|\det[T_{kl}(x)]|}$ transforms as an oriented relative scalar field of weight 1. It is also called an oriented scalar density field. □

Now we shall generate some new tensor fields out of the old tensor fields ${}_p\mathbf{T}(x) \equiv {}^0_p\mathbf{T}(x)$. The *symmetrization operation* is defined by

$$\begin{aligned}&\operatorname{Symm}\left[{}_p\mathbf{T}(x)\right]\left(\vec{\mathbf{a}}_1(x), \ldots, \vec{\mathbf{a}}_p(x)\right) \\ &\quad := \left(\frac{1}{p!}\right) \sum_{\sigma \in S_p} \left[{}_p\mathbf{T}(x)\right]\left(\vec{\mathbf{a}}_{\sigma(1)}(x), \ldots, \vec{\mathbf{a}}_{\sigma(p)}(x)\right) \qquad (1.41)\end{aligned}$$

for all $\vec{\mathbf{a}}_1(x), \ldots, \vec{\mathbf{a}}_p(x)$ in $T_x\left(\mathbb{R}^N\right)$. Here, σ denotes *a permutation* of $(1, 2, \ldots, p)$ into $(\sigma(1), \sigma(2), \ldots, \sigma(p))$. The corresponding (coordinate) components must satisfy:

$$\left(\operatorname{Symm}\left[{}_p\mathbf{T}(x)\right]\right)\left(\vec{\mathbf{a}}_{\sigma(1)}(x), \ldots, \vec{\mathbf{a}}_{\sigma(p)}(x)\right) \equiv \left(\operatorname{Symm}\left[{}_p\mathbf{T}(x)\right]\right)\left(\vec{\mathbf{a}}_1(x), \ldots, \vec{\mathbf{a}}_p(x)\right). \qquad (1.42)$$

Example 1.2.9. Consider $p = 2, \; N \geq 2$. There exist two distinct permutations of $(1,2)$. These are given by $\sigma_1(1,2) := (1,2)$ and $\sigma_2(1,2) = (2, 1)$. Therefore, by (1.41), we obtain

$$\begin{aligned}
&\text{Symm}\,[\mathbf{T}_{..}(x)]\left(\vec{\mathbf{a}}_1(x),\ \vec{\mathbf{a}}_2(x)\right)\\
&\quad= \frac{1}{2}\left\{[\mathbf{T}_{..}(x)]\left(\vec{\mathbf{a}}_1(x),\ \vec{\mathbf{a}}_2(x)\right) + [\mathbf{T}_{..}(x)]\left(\vec{\mathbf{a}}_2(x),\ \vec{\mathbf{a}}_1(x)\right)\right\},\\
&\text{Symm}\,[\mathbf{T}_{..}(x)]\left(\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial x^j}\right) = \frac{1}{2}\left[T_{ij}(x) + T_{ji}(x)\right].
\end{aligned} \tag{1.43}$$

□

In case the tensor components satisfy $T_{i_1\ldots i_p\ldots i_q\ldots}(x) \equiv T_{i_1\ldots i_q\ldots i_p\ldots}(x)$ (in any dimension), the tensor is said to be symmetric in the pth and qth index. If a second rank tensor satisfies $T_{ij}(x) \equiv T_{ji}(x)$, the tensor $\mathbf{T}_{..}(x)$ is called a *symmetric tensor*. A symmetric second-order tensor field in an N-dimensional manifold has $N(N+1)/2$ linearly independent components.

Antisymmetrization is defined by the *alternating operation*

$$\begin{aligned}
&\left[\text{Alt}\ {}_p\mathbf{T}(x)\right]\left(\vec{\mathbf{a}}_1(x), \ldots, \vec{\mathbf{a}}_p(x)\right)\\
&:= \left(\frac{1}{p!}\right)\sum_{\sigma\in S_p}[\text{sgn}(\sigma)]\left[{}_p\mathbf{T}(x)\right]\left(\vec{\mathbf{a}}_{\sigma(1)}(x), \ldots, \vec{\mathbf{a}}_{\sigma(p)}(x)\right)
\end{aligned} \tag{1.44}$$

for all $\vec{\mathbf{a}}_1(x), \ldots, \vec{\mathbf{a}}_p(x)$ in $T_x\left(\mathbb{R}^N\right)$.

We can prove that

$$\begin{aligned}
&\left[\text{Alt}\ {}_p\mathbf{T}(x)\right]\left(\vec{\mathbf{a}}_{\sigma(1)}(x), \ldots, \vec{\mathbf{a}}_{\sigma(p)}(x)\right)\\
&\quad\equiv [\text{sgn}(\sigma)]\left[\text{Alt}\ {}_p\mathbf{T}(x)\right]\left(\vec{\mathbf{a}}_1(x), \ldots, \vec{\mathbf{a}}_p(x)\right).
\end{aligned} \tag{1.45}$$

The tensor $\text{Alt}\ {}_p\mathbf{T}(x)$ is called a *totally antisymmetric tensor*. (See [56, 195, 237].)

Example 1.2.10. Consider the case of $N = p = 2$. The permutations σ_1 and σ_2 have $\text{sgn}(\sigma_1) = 1$ and $\text{sgn}(\sigma_2) = -1$. Therefore, by (1.44),

$$\begin{aligned}
[\text{Alt}\ \mathbf{T}_{..}(x)]\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right) &= \frac{1}{2}\left[T_{12}(x) - T_{21}(x)\right],\\
[\text{Alt}\ \mathbf{T}_{..}(x)]_{ij} &= \frac{1}{2}\left[T_{ij}(x) - T_{ji}(x)\right] \equiv -[\text{Alt}\ \mathbf{T}_{..}(x)]_{ji},\\
[\text{Alt}\ \mathbf{T}_{..}(x)]_{11} &= [\text{Alt}\ \mathbf{T}_{..}(x)]_{22} \equiv 0.
\end{aligned}$$

□

In case the tensor components satisfy $T_{i_1\ldots i_p\ldots i_q\ldots}(x) \equiv -T_{i_1\ldots i_q\ldots i_p\ldots}(x)$ (in any dimension), the tensor is said to be antisymmetric in the pth and qth index. If a second rank tensor satisfies $T_{ij}(x) \equiv -T_{ji}(x)$, the tensor $\mathbf{T}_{..}(x)$ is called an *antisymmetric tensor*. The number of linearly independent components of such a tensor in N-dimensions is exactly $N(N-1)/2$.

Now we are ready to define the *exterior* or *wedge product*. Let ${}_p\mathbf{W}(x)$ and ${}_q\mathbf{U}(x)$ be two totally antisymmetric tensor fields (antisymmetric on the interchange of any two indices). Then, the wedge product is defined by

$$\begin{aligned}
&{}_p\mathbf{W}(x) \wedge {}_q\mathbf{U}(x) := \left[\frac{(p+q)!}{(p!)(q!)}\right] \mathrm{Alt}\left[{}_p\mathbf{W}(x) \otimes {}_q\mathbf{U}(x)\right],\\
&\left[{}_p\mathbf{W}(x) \wedge {}_q\mathbf{U}(x)\right]\left[\vec{\mathbf{a}}_1(x), \ldots, \vec{\mathbf{a}}_p(x); \vec{\mathbf{a}}_{p+1}(x), \ldots, \vec{\mathbf{a}}_{p+q}(x)\right]\\
&\quad = \left[\frac{1}{(p!)(q!)}\right] \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\left[{}_p\mathbf{W}(x) \otimes {}_q\mathbf{U}(x)\right]\\
&\qquad \cdot \left[\vec{\mathbf{a}}_{\sigma(1)}(x), \ldots, \vec{\mathbf{a}}_{\sigma(p)}(x); \vec{\mathbf{a}}_{\sigma(p+1)}(x), \ldots, \vec{\mathbf{a}}_{\sigma(p+q)}(x)\right]
\end{aligned} \tag{1.46}$$

for all $\vec{\mathbf{a}}_1(x), \ldots, \vec{\mathbf{a}}_{p+q}(x)$ in $T_x\left(\mathbb{R}^N\right)$.

Example 1.2.11. Let the dimension $N \geq 3$. There are six distinct permutations of $(1, 2, 3)$. These can be listed as follows: $\sigma_1(1, 2, 3) := (1, 2, 3)$, $\sigma_2(1, 2, 3) := (2, 1, 3)$, $\sigma_3(1, 2, 3) := (1, 3, 2)$, $\sigma_4(1, 2, 3) := (3, 2, 1)$, $\sigma_5(1, 2, 3) := (2, 3, 1)$, $\sigma_6(1, 2, 3) := (3, 1, 2)$. The permutations $\sigma_2, \sigma_3, \sigma_4$ are odd and the permutations $\sigma_1, \sigma_5, \sigma_6$ are even. Let us consider the wedge product between $\mathbf{w}_{..}(x)$ and $\tilde{\mathbf{u}}(x)$. By (1.46), it is given by

$$\begin{aligned}
&[\mathbf{w}_{..}(x) \wedge \tilde{\mathbf{u}}(x)]\left[\vec{\mathbf{a}}_1(x), \vec{\mathbf{a}}_2(x); \vec{\mathbf{a}}_3(x)\right]\\
&= \frac{1}{2}\Big\{\mathbf{w}_{..}(x)\left[\vec{\mathbf{a}}_1(x), \vec{\mathbf{a}}_2(x)\right] \cdot \tilde{\mathbf{u}}\left[\vec{\mathbf{a}}_3(x)\right] - \mathbf{w}_{..}(x)\left[\vec{\mathbf{a}}_2(x), \vec{\mathbf{a}}_1(x)\right] \cdot \tilde{\mathbf{u}}(x)\left[\vec{\mathbf{a}}_3(x)\right]\\
&\quad - \mathbf{w}_{..}(x)\left[\vec{\mathbf{a}}_1(x), \vec{\mathbf{a}}_3(x)\right] \cdot \tilde{\mathbf{u}}(x)\left[\vec{\mathbf{a}}_2(x)\right] - \mathbf{w}_{..}(x)\left[\vec{\mathbf{a}}_3(x), \vec{\mathbf{a}}_2(x)\right] \cdot \tilde{\mathbf{u}}(x)\left[\vec{\mathbf{a}}_1(x)\right]\\
&\quad + \mathbf{w}_{..}(x)\left[\vec{\mathbf{a}}_2(x), \vec{\mathbf{a}}_3(x)\right] \cdot \tilde{\mathbf{u}}(x)\left[\vec{\mathbf{a}}_1(x)\right] + \mathbf{w}_{..}(x)\left[\vec{\mathbf{a}}_3(x), \vec{\mathbf{a}}_1(x)\right] \cdot \tilde{\mathbf{u}}(x)\left[\vec{\mathbf{a}}_2(x)\right]\Big\}\\
&= (-1)^{(2\cdot 1)}\left[\tilde{\mathbf{u}}(x) \wedge \mathbf{w}_{..}(x)\right]\left[\vec{\mathbf{a}}_1(x), \vec{\mathbf{a}}_2(x); \vec{\mathbf{a}}_3(x)\right]. \qquad \square
\end{aligned}$$

Now we state the important properties of the wedge product in the following theorem.

Theorem 1.2.12. *The wedge product satisfies the following rules:*

$$\left[{}_p\mathbf{A}(x) + {}_p\mathbf{B}(x)\right] \wedge {}_q\mathbf{C}(x) = \left[{}_p\mathbf{A}(x) \wedge {}_q\mathbf{C}(x)\right] + \left[{}_p\mathbf{B}(x) \wedge {}_q\mathbf{C}(x)\right], \tag{1.47i}$$

$${}_p\mathbf{A}(x) \wedge \left[{}_q\mathbf{B}(x) + {}_q\mathbf{C}(x)\right] = \left[{}_p\mathbf{A}(x) \wedge {}_q\mathbf{B}(x)\right] + \left[{}_p\mathbf{A}(x) \wedge {}_q\mathbf{C}(x)\right], \tag{1.47ii}$$

$$\left[\lambda(x)\, {}_p\mathbf{U}(x)\right] \wedge \left[\mu(x)\, {}_q\mathbf{W}(x)\right] = \left[\lambda(x)\mu(x)\right]\left[{}_p\mathbf{U}(x) \wedge {}_q\mathbf{W}(x)\right], \tag{1.47iii}$$

$$\left[{}_p\mathbf{A}(x) \wedge {}_q\mathbf{B}(x)\right] \wedge [{}_r\mathbf{C}(x)] = {}_p\mathbf{A}(x) \wedge \left[{}_q\mathbf{B}(x) \wedge {}_r\mathbf{C}(x)\right]$$
$$=: {}_p\mathbf{A}(x) \wedge {}_q\mathbf{B}(x) \wedge {}_r\mathbf{C}(x), \qquad (1.47\text{iv})$$

$${}_q\mathbf{U}(x) \wedge {}_p\mathbf{W}(x) = (-1)^{pq}\left[{}_p\mathbf{W}(x) \wedge {}_q\mathbf{U}(x)\right]. \qquad (1.47\text{v})$$

The proof is left as an exercise for the reader.

Next we shall introduce a $(p + p)$th order $(2 \leq p \leq N)$ totally antisymmetric numerical tensor. It is called the *generalized Kronecker tensor*, ${}^p_p\boldsymbol{\delta}$. It is defined by

$${}^p_p\boldsymbol{\delta} := \delta^{a_1,\ldots,a_p}{}_{b_1,\ldots,b_p}\frac{\partial}{\partial x^{a_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{a_p}} \otimes \mathrm{d}x^{b_1} \otimes \cdots \otimes \mathrm{d}x^{b_p},$$

$$\delta^{a_1,\ldots,a_p}{}_{b_1,\ldots,b_p} := \begin{vmatrix} \delta^{a_1}_{b_1} & \delta^{a_1}_{b_2} & \ldots & \delta^{a_1}_{b_p} \\ \vdots & & & \\ \delta^{a_p}_{b_1} & \delta^{a_p}_{b_2} & \ldots & \delta^{a_p}_{b_p} \end{vmatrix}. \qquad (1.48)$$

Example 1.2.13. $\delta^{a_1a_2}_{b_1b_2} = \delta^{a_1}_{b_1}\delta^{a_2}_{b_2} - \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}$ are the components of ${}^2_2\boldsymbol{\delta} \equiv \boldsymbol{\delta}^{\cdot\cdot}_{\cdot\cdot}$. □

The pertinent properties of ${}^p_p\boldsymbol{\delta}$ are listed below.

Theorem 1.2.14. *The components of* ${}^p_p\boldsymbol{\delta}$ *satisfy the following rules:*

(*i*)

$$\delta^{a_1,\ldots,a_p}{}_{b_1,\ldots,b_p} = \begin{cases} 1 & \textit{if } (b_1,\ldots,b_p) \textit{ is an even permutation of } (a_1,\ldots,a_p), \\ -1 & \textit{if } (b_1,\ldots,b_p) \textit{ is an odd permutation of } (a_1,\ldots,a_p), \\ 0 & \textit{otherwise}. \end{cases} \qquad (1.49)$$

(*ii*)

$$\delta^{a_1,\ldots,a_q a_{q+1},\ldots,a_p}{}_{b_1,\ldots,b_q a_{q+1},\ldots,a_p} = [(N-q)!/(N-p)!]\,\delta^{a_1,\ldots,a_q}{}_{b_1,\ldots,b_q} \;\; \textit{for } \; 1 \leq q < p \leq N. \qquad (1.50)$$

The proof will be left as an exercise.

Another totally antisymmetric *permutation symbol* is that of Levi-Civita. The corresponding components are defined by:

$$\varepsilon_{a_1,\ldots,a_N} := \delta^{1,\ldots,N}{}_{a_1,\ldots,a_N}, \qquad (1.51)$$

$$\varepsilon^{b_1,\ldots,b_N} := \delta^{b_1,\ldots,b_N}{}_{1,\ldots,N}. \qquad (1.52)$$

Only $N!$ among N^N components are nonzero.

Example 1.2.15.

$$\varepsilon_{123\ldots N} = -\varepsilon_{213\ldots N} = \delta^{12\ldots N}{}_{12\ldots N} = 1,$$

$$\varepsilon_{113\ldots N} = 0, \text{ etc.} \tag{1.53}$$

□

The main properties of the permutation symbols are summarized in the following theorem.

Theorem 1.2.16. *The components of the permutation symbols satisfy the following rules:*

(*i*)

$$\varepsilon_{a_1,\ldots,a_N} = \varepsilon^{a_1,\ldots,a_N}$$

$$= \begin{cases} 1 & \textit{if } (a_1,\ldots,a_N) \textit{ is an even permutation of } (1,\ldots,N), \\ -1 & \textit{if } (a_1,\ldots,a_N) \textit{ is an odd permutation of } (1,\ldots,N), \\ 0 & \textit{otherwise}. \end{cases} \tag{1.54}$$

(*ii*)

$$\varepsilon^{a_1,\ldots,a_p a_{p+1},\ldots,a_N}\,\varepsilon_{b_1,\ldots,b_p a_{p+1},\ldots,a_N} = (N-p)!\,\delta^{a_1,\ldots,a_p}{}_{b_1,\ldots,b_p}. \tag{1.55}$$

(*iii*)

$$\lambda^{b_1}{}_{a_1},\ldots,\lambda^{b_N}{}_{a_N}\,\varepsilon_{b_1,\ldots,b_N} = \left\{det\left[\lambda^{c}{}_{d}\right]\right\}\varepsilon_{a_1,\ldots,a_N}. \tag{1.56}$$

The proof will be left as an exercise. (See [171, 195].)

Example 1.2.17. Consider a two-dimensional vector space. In this case, the permutation symbols satisfy

$$\varepsilon_{11} = \varepsilon^{11} = \varepsilon_{22} = \varepsilon^{22} = 0,\ \varepsilon_{12} = \varepsilon^{12} = -\varepsilon_{21} = -\varepsilon^{21} = 1\,;$$

$$\varepsilon^{ab}\varepsilon_{cd} = \delta^{ab}{}_{12}\cdot\delta^{12}{}_{cd},\ \varepsilon^{ab}\varepsilon_{ab} = 2\,;$$

$$\varepsilon_{ab}\psi^{a}\chi^{b} = \det\begin{bmatrix}\psi^1 & \psi^2 \\ \chi^1 & \chi^2\end{bmatrix}. \tag{1.57}$$

□

(This example is relevant in the theory of spin-$\frac{1}{2}$ particles in nature.)

Now we shall define a *differential form* of various orders. The simplest examples of differential forms are 0-forms and 1-forms. A 0-form is defined by a scalar field (or $(0+0)$th order tensor field) $f(x)$. A 1-form is defined by a covariant vector field $\tilde{\mathbf{T}}(x) = T_j(x)\mathrm{d}x^j$. Consider now a totally antisymmetric covariant tensor field, ${}_p\mathbf{W}(x)$. According to (1.46), we can express ${}_p\mathbf{W}(x)$ as

$$\begin{aligned}{}_p\mathbf{W}(x) &= W_{i_1,\ldots,i_p}(x)\,\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_p}\\ &= \left(\frac{1}{p!}\right)W_{i_1,\ldots,i_p}(x)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\\ &= \sum_{1\le i_1<\cdots<i_p}^{N} W_{i_1,\ldots,i_p}(x)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge d^{i_p}\,. \end{aligned} \tag{1.58}$$

The antisymmetric tensor $\frac{1}{p!}W_{i_1,\ldots,i_p}(x)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}$ is called a pth order differential form or a **p**-*form*. (See [23, 38, 56, 104, 237, 267].)

Example 1.2.18. Choosing $p = q = 1$ in N-dimensions, we obtain, from (1.47v), that the two-form $\mathrm{d}x^j\wedge\mathrm{d}x^i$ has the property

$$\mathrm{d}x^j\wedge\mathrm{d}x^i = -\mathrm{d}x^i\wedge\mathrm{d}x^j\,,$$

which implies that

$$\mathrm{d}x^1\wedge\mathrm{d}x^1 = \mathrm{d}x^2\wedge\mathrm{d}x^2 = \ldots = \mathrm{d}x^N\wedge\mathrm{d}x^N = \mathbf{0}_{..}(x). \tag{1.59}$$

(The components of $\mathbf{0}_{..}(x)$ are all identically zero.) □

Now we shall introduce the concept of an *exterior derivative*. In case f is a (totally) differentiable function from $D\subset\mathbb{R}^N$ into $\mathbb{R}$, the exterior derivative is defined by:

$$\begin{aligned}\mathrm{d}f(x) &:= \frac{\partial f(x)}{\partial x^j}\,\mathrm{d}x^j\,,\\ \mathrm{d}f(x)\left[\vec{\mathbf{V}}(x)\right] &= V^j(x)\frac{\partial f(x)}{\partial x^j}\,. \end{aligned} \tag{1.60}$$

(Compare with (1.14) and (1.15).) Therefore, $\mathrm{d}f(x)$ is a 1-form field. In case $\tilde{\mathbf{T}}(x) = T_j(x)\mathrm{d}x^j$ is a differentiable 1-form, its exterior derivative is defined as:

$$\begin{aligned}\mathrm{d}\tilde{\mathbf{T}}(x) &:= [\mathrm{d}T_i(x)]\wedge\mathrm{d}x^i = \left[\frac{\partial T_i(x)}{\partial x^j}\right]\mathrm{d}x^j\wedge\mathrm{d}x^i\\ &= \frac{1}{2}\left[\frac{\partial T_i(x)}{\partial x^j} - \frac{\partial T_j(x)}{\partial x^i}\right]\mathrm{d}x^j\wedge\mathrm{d}x^i\,. \end{aligned} \tag{1.61}$$

The above is obviously a 2-form.

In general, the exterior derivative of a p-form is furnished by:

$$\begin{aligned}\mathrm{d}\left[{}_p\mathbf{W}(x)\right] &:= \left(\frac{1}{p!}\right)\left[\mathrm{d}W_{j_1,\dots,j_p}(x)\right]\wedge \mathrm{d}x^{j_1}\wedge\ldots\wedge \mathrm{d}x^{j_p}\\ &= \left(\frac{1}{p!}\right)\left[\frac{\partial W_{j_1,\dots,j_p}(x)}{\partial x^k}\right]\mathrm{d}x^k\wedge \mathrm{d}x^{j_1}\wedge\ldots\wedge \mathrm{d}x^{j_p}\,. \end{aligned}\tag{1.62}$$

The exterior derivative possesses the following wedge product rule:

$$\mathrm{d}\left[{}_p\mathbf{A}(x)\wedge {}_q\mathbf{B}(x)\right] = \mathrm{d}\left[{}_p\mathbf{A}(x)\right]\wedge {}_q\mathbf{B}(x) + (-1)^p\cdot {}_p\mathbf{A}(x)\wedge \mathrm{d}\left[{}_q\mathbf{B}(x)\right]. \tag{1.63}$$

Example 1.2.19. Consider a differentiable 2-form

$$\mathbf{F}_{\cdot\cdot}(x) \equiv {}_2\mathbf{F}(x) = \frac{1}{2}F_{ij}(x)\,\mathrm{d}x^i\wedge \mathrm{d}x^j\,. \tag{1.64}$$

The exterior derivative of ${}_2\mathbf{F}(x)$ is the 3-form

$$\begin{aligned}\mathrm{d}\left[{}_2\mathbf{F}(x)\right] &= \left(\frac{1}{2}\right)\left[\frac{\partial F_{ij}(x)}{\partial x^k}\right]\mathrm{d}x^k\wedge \mathrm{d}x^i\wedge \mathrm{d}x^j\\ &= \frac{1}{6}\left[\frac{\partial F_{ij}(x)}{\partial x^k}+\frac{\partial F_{jk}(x)}{\partial x^i}+\frac{\partial F_{ki}(x)}{\partial x^j}\right]\mathrm{d}x^k\wedge \mathrm{d}x^i\wedge \mathrm{d}x^j\,. \end{aligned}\tag{1.65}$$

□

(*Remark*: The above example is relevant in electromagnetic field theory.)

Now we shall state and prove a useful *lemma of Poincaré*.

Lemma 1.2.20. *Let ${}_p\mathbf{W}(x)$ be a (continuously) twice-differentiable p-form in $D\subset\mathbb{R}^N$. Then*

$$\mathrm{d}^2\left[{}_p\mathbf{W}(x)\right] := \mathrm{d}\left\{\mathrm{d}\left[{}_p\mathbf{W}(x)\right]\right\} \equiv {}_{p+2}\mathbf{O}(x). \tag{1.66}$$

Proof. By (1.62),

$$\begin{aligned}&\mathrm{d}^2\left[{}_p\mathbf{W}(x)\right]\\ &= \sum_{1\le j_1<\cdots<j_p}^{N}\left[\frac{\partial W_{j_1,\dots,j_p}(x)}{\partial x^l\,\partial x^k}\right]\mathrm{d}x^l\wedge \mathrm{d}x^k\wedge \mathrm{d}x^{j_1}\wedge\ldots\wedge \mathrm{d}x^{j_p}\\ &= \frac{1}{2}\sum_{1\le j_1<\cdots<j_p}^{N}\left[\frac{\partial W_{j_1,\dots,j_p}(x)}{\partial x^l\,\partial x^k}-\frac{\partial W_{j_1,\dots,j_p}(x)}{\partial x^k\,\partial x^l}\right]\mathrm{d}x^l\wedge \mathrm{d}x^k\wedge \mathrm{d}x^{j_1}\wedge\ldots\wedge \mathrm{d}x^{j_p}\\ &\equiv {}_{p+2}\mathbf{O}(x)\,. \end{aligned}$$

■

The exterior derivative operator, acting on a p-form, is therefore a *nilpotent operator* .

A differential p-form ${}_p\mathbf{W}(x)$ is called a *closed form* in case $d\left[{}_p\mathbf{W}(x)\right] \equiv {}_{p+1}\mathbf{O}(x)$. A p-form ${}_p\mathbf{A}(x)$ is called an *exact form* provided there exists a form ${}_{p-1}\mathbf{B}(x)$ such that ${}_p\mathbf{A}(x) = \mathrm{d}\left[{}_{p-1}\mathbf{B}(x)\right]$. *Poincare's lemma* shows that every exact form is closed. We naturally ask whether or not the converse statement is true. That is, are all closed forms exterior derivatives of lower rank forms? In the following theorem, we answer that question.

Theorem 1.2.21. *Let D^* be an open domain which is star-shaped with respect to an interior point $x_0 \in D^*$. Moreover, let ${}_p\mathbf{W}(x)$ be a differentiable p-form satisfying $\mathrm{d}\left[{}_p\mathbf{W}(x)\right] \equiv {}_{p+1}\mathbf{O}(x)$ in D^*. Then, there exists a continuously twice-differentiable $(p-1)$-form ${}_{p-1}\mathbf{A}(x)$ in D^* such that*

$$ {}_p\mathbf{W}(x) = \mathrm{d}\left[{}_{p-1}\mathbf{A}(x)\right]. \tag{1.67}$$

(For the proof, the reader is asked to consult Flander's book [104].)

Remark: The $(p-1)$th form ${}_{p-1}\mathbf{A}(x)$ in (1.67) is *not unique*. We can replace it by the $(p-1)$th form

$$ {}_{p-1}\hat{\mathbf{A}}(x) = {}_{p-1}\mathbf{A}(x) - \mathrm{d}\left[{}_{p-2}\boldsymbol{\lambda}(x)\right]. \tag{1.68}$$

Here ${}_{p-2}\boldsymbol{\lambda}(x)$ is an arbitrary $(p-2)$th form of differentiability class C^3.

Example 1.2.22. Let us choose $N = 4$ and the *electromagnetic field tensor* $\mathbf{F}_{..}(x)$ identified with the 2-form:

$$ {}_2\mathbf{F}(x) = \frac{1}{2} F_{ij}(x)\mathrm{d}x^i \wedge \mathrm{d}x^j. $$

Four out of the eight *Maxwell's equations* can be cast into tensor field equations:

$$ \begin{gathered} \mathrm{d}\left[{}_2\mathbf{F}(x)\right] = \mathbf{0}_{...}(x), \\ \frac{\partial F_{ij}(x)}{\partial x^k} + \frac{\partial F_{jk}(x)}{\partial x^i} + \frac{\partial F_{ki}(x)}{\partial x^j} = 0. \end{gathered} \tag{1.69}$$

In a star-shaped domain $D^* \subset \mathbb{R}^4$ (corresponding to space–time), the above equations imply, by (1.67), the existence of a one-form $\tilde{\mathbf{A}}(x)$ such that

$$ \begin{gathered} {}_2\mathbf{F}(x) = \mathrm{d}\left[\tilde{\mathbf{A}}(x)\right], \\ F_{ij}(x) = \frac{\partial A_j(x)}{\partial x^i} - \frac{\partial A_i(x)}{\partial x^j}. \end{gathered} \tag{1.70}$$

The 1-form $\tilde{\mathbf{A}}(x)$ is called the *electromagnetic 4-potential*. By (1.68), we can make the following transformation:

$$\hat{\tilde{\mathbf{A}}}(x) = \tilde{\mathbf{A}}(x) - \mathrm{d}\lambda(x), \tag{1.71i}$$

$$\hat{A}_i(x) = A_i(x) - \frac{\partial\lambda(x)}{\partial x^i}, \tag{1.71ii}$$

such that the electromagnetic field tensor's components remain unchanged. Equations (1.71i) and (1.71ii) represent what are known as *gauge transformations* of the 4-potential. □

Now we shall briefly discuss the integration of a p-form in (an open, connected) domain D of $\mathbb{R}^N$. Suppose that f is a Riemann (or Lebesgue) integrable function over $D \cup \partial D$. Then, we define

$$\int_D f(x)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^N := \int_D f(x^1, \ldots, x^N)\,\mathrm{d}x^1, \ldots, \mathrm{d}x^N\,. \tag{1.72}$$

The right-hand side of (1.72) indicates a standard multiple integral.

For a general N-form ${}_N\mathbf{A}(x)$, we define

$$\begin{aligned}\int_D {}_N\mathbf{A}(x) &:= \frac{1}{N!}\int_D A_{i_1,\ldots,i_N}(x)\,\mathrm{d}x^{i_1}, \ldots, \mathrm{d}x^{i_N}\\ &= \sum_{1\le i_1<\cdots<i_N} \int_D A_{i_1,\ldots,i_N}(x)\,\mathrm{d}x^{i_1}, \ldots, \mathrm{d}x^{i_N}.\end{aligned} \tag{1.73}$$

This allows us to state the *generalized Stokes' theorem*:

Theorem 1.2.23. *Let D^*_{p+1} be an open, star-shaped $(p+1)$th dimensional domain of $\mathbb{R}^N$ $(p+1 \le N)$ with a continuous, piecewise differentiable, orientable p-dimensional boundary ∂D^*_{p+1}. Then, for any continuously differentiable p-form ${}_p\mathbf{W}(x)$,*

$$\int_{D^*_{p+1}} \mathrm{d}\left[{}_p\mathbf{W}(x)\right] = \int_{\partial D^*_{p+1}} {}_p\mathbf{W}(x). \tag{1.74}$$

(For the proof of this theorem, we refer to the book by Spivak [237].)

Example 1.2.24. Let us choose $N = 2$ and $p = 1$. Moreover, we choose $D^*_2 \subset \mathbb{R}^2$ to be the rectangle $(a,b) \times (c,d)$. The boundary ∂D^*_2 of the rectangle is to be traversed counterclockwise in manner (to maintain the right orientation). For a continuously differentiable 1-form $\tilde{\mathbf{A}}(x)$, (1.74) yields

$$\int_{(a,b)\times(c,d)} \mathrm{d}\left[\tilde{\mathbf{A}}(x)\right] = \int_{\partial[(a,b)\times(c,d)]} \tilde{\mathbf{A}}(x),$$

$$\text{or,} \quad \int_{(a,b)\times(c,d)} \left[\frac{\partial A_2(x)}{\partial x^1} - \frac{\partial A_1(x)}{\partial x^2}\right] \mathrm{d}x^1 \mathrm{d}x^2$$

$$= \int_{\partial[(a,b)\times(c,d)]} \left[A_1(x)\,\mathrm{d}x^1 + A_2(x)\,\mathrm{d}x^2\right].$$

The above is just Stokes' theorem in two-dimensional vector calculus. □

Definition (1.62) shows that the exterior derivative produces a $(p+1)$th order antisymmetric tensor field out of a pth order antisymmetric tensor. There exists another derivative of tensor fields, generating higher order tensor fields. This is called the *Lie derivative*.

Consider a continuous vector field $\vec{\mathbf{V}}(x)$ in $D \subset \mathbb{R}^N$ and a system of ordinary differential equations:

$$\vec{\boldsymbol{\mathcal{X}}}'(t) = \vec{\mathbf{V}}\left[\mathcal{X}(t)\right],$$

$$\text{or,} \quad \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} = V^j\left[\mathcal{X}^1(t), \ldots, \mathcal{X}^N(t)\right];$$

$$t \in [-a, a] \subset \mathbb{R}. \tag{1.75}$$

In case the *Lipschitz condition* is satisfied [38], the system of (1.75) admits *a unique solution* of the initial value problem $\mathcal{X}(0) = x_0$. The solution is called the *integral curve* of the system (1.75) passing through x_0. We now consider *the family* of integral curves passing through various initial points by putting

$$x = \xi(t, x_0), \tag{1.76i}$$

$$x_0 \equiv \xi(0, x_0). \tag{1.76ii}$$

Changing the notation in (1.76i) and (1.76ii), we write

$$\hat{x} = \xi(t, x), \tag{1.77i}$$

$$x \equiv \xi(0, x). \tag{1.77ii}$$

The original differential equations yield

$$\frac{\partial \xi(t, x)}{\partial t} = \vec{\mathbf{V}}\left(\xi(x, t)\right). \tag{1.78}$$

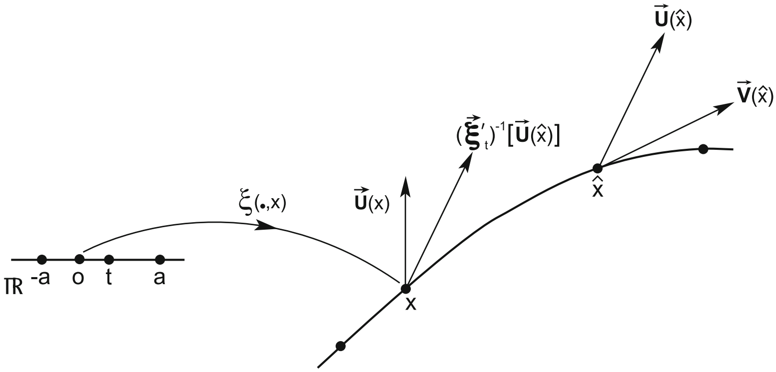

Fig. 1.9 A vector field $\vec{\mathbf{U}}(x)$ along an integral curve $\xi(\cdot, x)$

Now we introduce the related mapping:

$$\begin{aligned} \xi_t &:= \xi(t, \cdot) : x \mapsto \hat{x}, \\ \xi_0 &\equiv \xi(0, \cdot) = \text{Identity}. \end{aligned} \tag{1.79}$$

This mapping takes the point x in $\hat{x}$ along an integral curve. Moreover, the mapping ξ_t takes the neighborhood $N_\delta(x)$ into the neighborhood $N_{\hat{\delta}}(\hat{x})$ along integral curves. Furthermore, the mapping ξ_t induces a derivative mapping $\boldsymbol{\xi}'_t$ from the tangent space ${}_xT(\mathbb{R}^N)$ into $T_{\hat{x}}(\mathbb{R}^N) \equiv T_{\xi_t(x)}(\mathbb{R}^{\mathbb{N}})$. (See (1.34).)

We define the Lie derivative of a vector field $\vec{\mathbf{U}}(x)$ by the following rule:

$$L_{\vec{\mathbf{V}}}\left[\vec{\mathbf{U}}(x)\right] := \lim_{t \to 0} \left\{ \frac{(\vec{\boldsymbol{\xi}}'_t)^{-1}\left[\vec{\mathbf{U}}(\hat{x})\right] - \vec{\mathbf{U}}(x)}{t} \right\}. \tag{1.80}$$

(See Fig. 1.9.)

Example 1.2.25. Let us work out the Lie derivative of the coordinate basis field $\frac{\partial}{\partial x^i}$. By Example 1.2.5 and (1.77i) and (1.78), we obtain

$$\left[\vec{\boldsymbol{\xi}}'_t\right]^{-1}\left(\frac{\partial}{\partial \hat{x}^i}\right) = \frac{\xi^j(-t, \hat{x})}{\partial \hat{x}^i} \frac{\partial}{\partial x^j}.$$

Therefore, by (1.80), we get

$$\begin{aligned} L_{\vec{\mathbf{V}}}\left[\frac{\partial}{\partial x^i}\right] &= \lim_{t \to 0} \left\{ \frac{\left[\vec{\boldsymbol{\xi}}'_t\right]^{-1}\left(\frac{\partial}{\partial \hat{x}^i}\right) - \frac{\partial}{\partial x^i}}{t} \right\} \\ &= \lim_{t \to 0} \left\{ t^{-1}\left[\frac{\partial \xi^j(-t, \hat{x})}{\partial \hat{x}^i} - \delta^j_i\right] \right\} \frac{\partial}{\partial x^j} \end{aligned}$$

$$= \lim_{t \to 0} \left\{ t^{-1} \left[\frac{\partial \hat{x}^j}{\partial \hat{x}^i} - t \left(\frac{\partial}{\partial(-t)} \frac{\partial \xi^j(-t, \hat{x})}{\partial \hat{x}^i} \right) + \mathcal{O}(t^2) - \delta^j_i \right] \right\} \frac{\partial}{\partial x^j}$$

$$= - \frac{\partial}{\partial \hat{x}^i} \left[\frac{\partial \xi^j(-t, \hat{x})}{\partial(-t)} \right]_{|t=0} \frac{\partial}{\partial x^j}$$

$$= - \left\{ \frac{\partial V^j\left[\xi(-t, \hat{x})\right]}{\partial \hat{x}^i} \right\}_{|t=0} \frac{\partial}{\partial x^j} = - \frac{\partial V^j(x)}{\partial x^i} \frac{\partial}{\partial x^j}. \qquad \square$$

Using the above example and the definition

$$L_{\vec{\mathbf{V}}}\left[f(x)\right] := \vec{\mathbf{V}}(x)\left[f\right], \tag{1.81}$$

we can derive the vector field component

$$\left\{ L_{\vec{\mathbf{V}}} \left[\vec{\mathbf{U}}(x) \right] \right\}^i = \frac{\partial U^i(x)}{\partial x^j} V^j(x) - \frac{\partial V^i(x)}{\partial x^j} U^j(x). \tag{1.82}$$

We can generalize definition (1.80) for a general tensor field ${}^r_s\mathbf{T}(x)$ by the following:

$$L_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x)\right] := \lim_{t \to 0} \left\{ \frac{\left[{}^r_s\boldsymbol{\xi}'_t\right]^{-1} \left[{}^r_s\mathbf{T}(\hat{x})\right] - {}^r_s\mathbf{T}(x)}{t} \right\}. \tag{1.83}$$

(Here, we assumed (1.36).)

We can work out the components of $L_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x)\right]$ which are furnished by

$$\begin{aligned}
\left\{L_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x)\right]\right\}^{i_1,\ldots,i_r}_{\;j_1,\ldots,j_s} = {} & \left[\frac{\partial}{\partial x^k} T^{i_1,\ldots,i_r}_{\;j_1,\ldots,j_s}(x) \right] V^k(x) \\
& - \sum_{\alpha=1}^{r} \left[T^{i_1,\ldots,i_{\alpha-1} k\, i_{\alpha+1},\ldots,i_r}_{\;j_1,\ldots,j_s}(x) \right] \frac{\partial V^{i_\alpha}(x)}{\partial x^k} \\
& + \sum_{\beta=1}^{s} \left[T^{i_1,\ldots,i_r}_{\;j_1,\ldots,j_{\beta-1} k\, j_{\beta+1},\ldots,j_s}(x) \right] \frac{\partial V^k(x)}{\partial x^{j_\beta}}.
\end{aligned} \tag{1.84}$$

Example 1.2.26. Consider a $(1+1)$th order mixed tensor field $\mathbf{T}^{\cdot}_{\cdot}(x)$. From (1.84),

$$\left\{L_{\vec{\mathbf{V}}}\left[\mathbf{T}^{\cdot}_{\cdot}(x)\right]\right\}^i_{\;j} = \left[\frac{\partial T^i_j(x)}{\partial x^k} \right] \cdot V^k(x) - T^k_j(x) \frac{\partial V^i(x)}{\partial x^k} + T^i_k(x) \frac{\partial V^k(x)}{\partial x^j}.$$

The contraction of the above equation yields

$$\left\{L_{\vec{\mathbf{V}}}\left[\mathbf{T}^{\cdot}_{\cdot}(x)\right]\right\}^{i}_{\ i} = \frac{\partial T^{i}_{\ i}(x)}{\partial x^k} V^k(x) = L_{\vec{V}}\left[T^{i}_{\ i}(x)\right],$$

$$\mathcal{C}^1_1\left\{L_{\vec{\mathbf{V}}}\left[\mathbf{T}^{\cdot}_{\cdot}(x)\right]\right\} = L_{\vec{\mathbf{V}}}\left\{\mathcal{C}^1_1\left[\mathbf{T}^{\cdot}_{\cdot}(x)\right]\right\}.$$

(Note that *the contraction operation commutes with the Lie derivative*.) □

Exercises 1.2

1. Let $T^{i_1,\ldots,i_r}_{\ \ \ \ \ \ \ \ j_1,\ldots,j_s}(x)$ be the differentiable coordinate components of an $(r+s)$th tensor field ${}^r_s\mathbf{T}(x)$ in $D \subset \mathbb{R}^N$. Prove that the partial derivatives $\frac{\partial}{\partial x^k} T^{i_1,\ldots,i_r}_{\ \ \ \ \ \ \ \ j_1,\ldots,j_s}(x)$ *do not transform as tensor components* under a general twice-differentiable coordinate transformation.
2. Let $T_{jk}(x)$ be the components of a $(0+2)$th order tensor field in $D \subset \mathbb{R}^N$. Show that the cofactors $C^{jk}(x)$ of the entries $T_{jk}(x)$ in $\det\left[T_{jk}(x)\right]$ transform as components of a relative, second-order contravariant tensor field of weight $+2$.
3. Consider the numerical identity tensor field in $D \subset \mathbb{R}^N$ defined by

   $$\mathbf{I}^{\cdot}_{\cdot}(x) := \delta^i_{\ j} \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j.$$

   Prove that

   $$\left\{{}^1_1\mathcal{C}\left[\mathbf{I}^{\cdot}_{\cdot}(x) \otimes \mathbf{I}^{\cdot}_{\cdot}(x)\right]\right\}\left(\tilde{\mathbf{w}}(x), \vec{\mathbf{v}}(x)\right) \equiv N\left[w_j(x) v^j(x)\right]$$

   for every pair of fields $\tilde{\mathbf{w}}(x)$ and $\vec{\mathbf{v}}(x)$ in $D \subset \mathbb{R}^N$.
4. Prove the following equations:

   (i) $\varepsilon^{j_1,\ldots,j_r j_{r+1},\ldots,j_N}\delta^{h_{r+1},\ldots,h_N}_{\ \ \ \ \ \ \ \ \ \ \ j_{r+1},\ldots,j_N} = (N-r)!\varepsilon^{j_1,\ldots,j_r h_{r+1},\ldots,h_N}$.

   (ii) $\det\left[a_{ij}\right] = \varepsilon^{j_1,\ldots,j_N} a_{1\,j_1}, \ldots, a_{N\,j_N}$.
5. Let n be a nonnegative integer such that $2n+1 \le N$. Show that every $(2n+1)$-form ${}_{2n+1}\mathbf{W}(x)$ satisfies

   $${}_{2n+1}\mathbf{W}(x) \wedge {}_{2n+1}\mathbf{W}(x) \equiv {}_{4n+2}\mathbf{0}(x).$$

6. Consider a 1-form defined by

$$\tilde{\mathbf{W}}(x) := -\left[\frac{x^2}{(x^1)^2 + (x^2)^2}\right] \mathrm{d}x^1 + \left[\frac{x^1}{(x^1)^2 + (x^2)^2}\right] \mathrm{d}x^2,$$

$$(x^1, x^2) \in \mathbb{R}^2 - \{(0,0)\}.$$

   (i) Prove that $\mathrm{d}\tilde{\mathbf{W}}(x) \equiv \mathbf{0}_{..}(x)$ in $\mathbb{R}^2 - \{(0,0)\}$.
   (ii) Show the nonexistence of a differentiable function, $f(x)$, in $\mathbb{R}^2 - \{(0,0)\}$ such that $\tilde{\mathbf{W}}(x) = \mathrm{d}f(x)$. Explain.

7. Consider the differentiable symmetric tensor field $\mathbf{S}_{..}(x)$ in $D \subset \mathbb{R}^N$ with $\det\left[S_{ij}(x)\right] \neq 0$.

   (i) Prove that

$$\left\{L_{\vec{\mathbf{V}}}\left[\mathbf{S}_{..}(x)\right]\right\}_{ij} = \left[\frac{\partial}{\partial x^k} S_{ij}(x)\right] V^k(x) + S_{kj}(x)\frac{\partial V^k(x)}{\partial x^i} + S_{ik}(x)\frac{\partial V^k(x)}{\partial x^j}.$$

   (ii) In case $S_{ij}(x) = c_{ij}$, a constant-valued tensor, prove that the partial differential equations

$$\left\{L_{\vec{\mathbf{V}}}\left[\mathbf{S}_{..}(x)\right]\right\}_{ij} = 0$$

   are solved by

$$c_{jk} V^k(x) = t_j + w_{jk} x^k, \quad w_{kj} \equiv -w_{jk}.$$

   Here, the t_j's and w_{jk}'s represent $N + N(N-1)/2$ arbitrary constants of integration.

   (*Remark:* The right-hand side in the last equation constitutes a *rigid motion*.)

8. Let a *Lie bracket* (or *commutator*) for differentiable vector fields $\vec{\mathbf{U}}(x)$, $\vec{\mathbf{V}}(x)$ be defined by $\left[\vec{\mathbf{U}}(x), \vec{\mathbf{V}}(x)\right] := \vec{\mathbf{U}}(x)\vec{\mathbf{V}}(x) - \vec{\mathbf{V}}(x)\vec{\mathbf{U}}(x)$. Prove that for twice-differentiable functions $f(x)$ and $h(x)$,

   (i) $\left[\vec{\mathbf{V}}(x), \vec{\mathbf{U}}(x)\right][f] \equiv -\left[\vec{\mathbf{U}}(x), \vec{\mathbf{V}}(x)\right][f]$.
   (ii) $L_{\vec{\mathbf{V}}}\left[\vec{\mathbf{U}}(x)\right] = \left[\vec{\mathbf{V}}(x), \vec{\mathbf{U}}(x)\right]$.
   (iii) $\left[\vec{\mathbf{U}}(x), [\vec{\mathbf{V}}(x), \vec{\mathbf{W}}(x)]\right] + \left[\vec{\mathbf{V}}(x), [\vec{\mathbf{W}}(x), \vec{\mathbf{U}}(x)]\right]$
   $+ \left[\vec{\mathbf{W}}(x), [\vec{\mathbf{U}}(x), \vec{\mathbf{V}}(x)]\right] \equiv \vec{\mathbf{0}}(x)$.
   (iv) $\left[f\vec{\mathbf{V}}(x), h\vec{\mathbf{W}}(x)\right] = \left\{f(x)\vec{\mathbf{V}}(x)[h]\right\}\vec{\mathbf{W}}(x) - \left\{h(x)\vec{\mathbf{W}}(x)[f]\right\}\vec{\mathbf{V}}(x)$
   $- f(x)h(x)\left[\vec{\mathbf{V}}(x), \vec{\mathbf{W}}(x)\right]$.
   (v) $\left[\vec{\mathbf{e}}_i, \vec{\mathbf{e}}_j\right]\left[\tilde{\mathbf{e}}^k\right] =: \chi^k{}_{ij}(x) \equiv -\chi^k{}_{ji}(x)$.
   (vi) $\left[\vec{\mathbf{e}}_{(a)}, \vec{\mathbf{e}}_{(b)}\right]\left[\tilde{\mathbf{e}}^{(c)}\right] =: \chi^{(c)}{}_{(a)(b)}(x) \equiv -\chi^{(c)}{}_{(b)(a)}(x)$.

Answers and Hints to Selected Exercises

4. (ii)

$$\varepsilon^{j_1,\ldots,j_N} a_{1\,j_1},\ldots,a_{N\,j_N} = \sum_{\sigma\in S_N} [\mathrm{sgn}(\sigma)]\, a_{1\sigma_{(1)}}\cdot a_{2\sigma_{(2)}},\ldots,a_{N\sigma_{(N)}} = \quad \det\left[a_{ij}\right].$$

5. Use (1.47v).
7. (ii) The partial differential equations reduce to

$$\frac{\partial}{\partial x^i}\left[c_{kj} V^k(x)\right] + \frac{\partial}{\partial x^j}\left[c_{ik} V^k(x)\right] = w_{ij} + w_{ji} \equiv 0.$$

8. (i)

$$\vec{\mathbf{U}}(x) = U^i(x)\,\partial_i\,,\ \vec{\mathbf{V}}(x) = V^j(x)\,\partial_j\,, \left[\vec{\mathbf{V}}(x),\, \vec{\mathbf{U}}(x)\right][f] = 0$$
$$+\left[V^j(\partial_j U^i)\,\partial_i f - U^i(\partial_i V^j)\,\partial_j f\right].$$

(ii) Using (1.84),

$$L_{\vec{\mathbf{V}}}\left[\vec{\mathbf{U}}\right] = \left[V^j\cdot\partial_j U^i - U^j\cdot\partial_j V^i\right]\partial_i$$
$$= \vec{\mathbf{V}}(x)\vec{\mathbf{U}}(x) - \vec{\mathbf{U}}(x)\vec{\mathbf{V}}(x)$$
$$= \left[\vec{\mathbf{V}}(x),\, \vec{\mathbf{U}}(x)\right].$$

1.3 Riemannian and Pseudo-Riemannian Manifolds

In the preceding section, for the real tangent vector space $T_x(\mathbb{R}^N)$, no other additional structure was assumed. Now we shall impose an interesting structure, namely, the *inner-product* (or *dot product*) to $T_x(\mathbb{R}^N)$. This is accomplished by assuming the existence of a second-order tensor field $\mathbf{g}_{..}(x)$ in $D \subset \mathbb{R}^N$. The corresponding axioms are as follows:

I1 $\mathbf{g}_{..}(x)(\vec{\mathbf{A}}(x), \vec{\mathbf{B}}(x)) \in \mathbb{R}$ for all $\vec{\mathbf{A}}(x), \vec{\mathbf{B}}(x) \in T_x(\mathbb{R}^N)$. (1.85i)

I2 $\mathbf{g}_{..}(x)(\vec{\mathbf{B}}(x), \vec{\mathbf{A}}(x)) \equiv \mathbf{g}_{..}(x)(\vec{\mathbf{A}}(x), \vec{\mathbf{B}}(x))$
for all $\vec{\mathbf{A}}(x), \vec{\mathbf{B}}(x) \in T_x(\mathbb{R}^N)$. (1.85ii)

I3 $$\mathbf{g}_{..}(x)(\lambda(x)\vec{\mathbf{A}}(x) + \mu(x)\vec{\mathbf{B}}(x),\ \vec{\mathbf{C}}(x))$$
$$= \lambda(x)\mathbf{g}_{..}(x)(\vec{\mathbf{A}}(x),\ \vec{\mathbf{C}}(x)) + \mu(x)\mathbf{g}_{..}(x)(\vec{\mathbf{B}}(x),\ \vec{\mathbf{C}}(x)). \tag{1.85iii}$$

I4 $$\mathbf{g}_{..}(x)(\vec{\mathbf{A}}(x),\ \vec{\mathbf{V}}(x)) \equiv 0 \ \text{ for all } \ \vec{\mathbf{V}}(x) \in T_x(\mathbb{R}^N) \ \text{ iff } \ \vec{\mathbf{A}}(x) = \vec{\mathbf{0}}(x). \tag{1.85iv}$$

Axioms **I**1, **I**2, and **I**3 imply that $\mathbf{g}_{..}(x)$ is *a symmetric second-order tensor field*. Axiom **I**4 is the condition of *nondegeneracy*. The tensor $\mathbf{g}_{..}(x)$ is known as the *metric tensor*. (See [56, 195].)

The *separation of a vector field* $\vec{\mathbf{V}}(x)$ is given by

$$\sigma(\vec{\mathbf{V}}(x)) := +\sqrt{\left|\mathbf{g}_{..}(x)(\vec{\mathbf{V}}(x),\ \vec{\mathbf{V}}(x))\right|}\,. \tag{1.86}$$

It satisfies the rules

$$\begin{aligned} \sigma(\vec{\mathbf{V}}(x)) &\geq 0, \\ \sigma(\lambda(x)\vec{\mathbf{V}}(x)) &= |\lambda(x)|\,\sigma(\vec{\mathbf{V}}(x)). \end{aligned} \tag{1.87}$$

A *unit vector field* $\vec{\mathbf{U}}(x)$ satisfies

$$\sigma(\vec{\mathbf{U}}(x)) \equiv 1. \tag{1.88}$$

Two *orthogonal vector fields*, $\vec{\mathbf{A}}(x)$ and $\vec{\mathbf{B}}(x)$, satisfy

$$\mathbf{g}_{..}(x)(\vec{\mathbf{A}}(x),\ \vec{\mathbf{B}}(x)) \equiv 0. \tag{1.89}$$

An *orthonormal basis set* consists[4] of vectors $\vec{\mathbf{e}}_{(a)}(x)$ and is denoted by $\{\vec{\mathbf{e}}_{(1)}(x), \ldots, \vec{\mathbf{e}}_{(N)}(x)\} =: \{\vec{\mathbf{e}}_{(a)}(x)\}_1^N$. These vectors must satisfy

$$\begin{aligned} \mathbf{g}_{..}(x)(\vec{\mathbf{e}}_{(a)}(x),\ \vec{\mathbf{e}}_{(b)}(x)) =: d_{(a)(b)} &:= \operatorname{diag}\Bigl(\underbrace{1,\ 1}_{p},\ \underbrace{-1, \ldots, -1}_{n}\Bigr), \\ p + n &= N, \\ \left|d_{(a)(b)}\right| &= \delta_{(a)(b)}, \\ \operatorname{sgn}(\mathbf{g}_{..}(x)) &:= p - n \geq 0. \end{aligned} \tag{1.90}$$

[4] Some authors call it a pseudoorthonormal basis set.

The integer $p-n$ of the metric tensor is known as the *signature of the metric*. In the conventions of this book, an N-dimensional manifold endowed with metric whose signature is $N-2$ ($p=N-1$, $n=1$) is said to possess *Lorentz signature*.[5]

We use lowercase Latin letters from the beginning of the alphabet, and in parentheses, to denote the indices of a tensor relative to an orthonormal frame ((a), (b), (c), (d), (e), (f), etc.). *We use lowercase Latin letters from the middle of the alphabet (i, j, k, l, m, n, etc.) to denote the indices of a tensor relative to a coordinate basis set.*

In cases where the metric is *positive-definite* ($n=0$), axiom **I**4 is replaced by the stronger axiom:

$$\begin{aligned} \mathbf{I}4+ \qquad & \mathbf{g}_{..}(x)(\vec{\mathbf{V}}(x), \vec{\mathbf{V}}(x)) \geq 0\,; \\ & \mathbf{g}_{..}(x)(\vec{\mathbf{V}}(x), \vec{\mathbf{V}}(x)) \equiv 0 \;\text{ iff }\; \vec{\mathbf{V}}(x) = \vec{\mathbf{0}}(x). \end{aligned} \tag{1.91}$$

For such a case, the *norm* or *length* associated with a vector field is furnished by:

$$\parallel \vec{\mathbf{V}}(x) \parallel := +\sqrt{\mathbf{g}_{..}(x)(\vec{\mathbf{V}}(x), \vec{\mathbf{V}}(x))}. \tag{1.92}$$

The norm satisfies the rules:

$$\begin{aligned} \parallel \vec{\mathbf{V}}(x) \parallel &\geq 0, \\ \parallel \lambda(x)\vec{\mathbf{V}}(x) \parallel &= |\lambda(x)| \parallel \vec{\mathbf{V}}(x) \parallel, \\ \parallel \vec{\mathbf{A}}(x) + \vec{\mathbf{B}}(x) \parallel &\leq \; \parallel \vec{\mathbf{A}}(x) \parallel + \parallel \vec{\mathbf{B}}(x) \parallel . \end{aligned} \tag{1.93}$$

Comparing (1.87) and (1.93), we conclude that separation and norm (or length) are *not equivalent*.

In the case of a positive-definite metric, the *angle field* between two non-zero vector fields, $\vec{\mathbf{U}}(x)$ and $\vec{\mathbf{V}}(x)$, can be defined by

$$\cos[\theta(x)] := \frac{\left[\mathbf{g}_{..}(\vec{\mathbf{U}}(x), \vec{\mathbf{V}}(x))\right]}{\parallel \vec{\mathbf{U}}(x) \parallel \cdot \parallel \vec{\mathbf{V}}(x) \parallel}. \tag{1.94}$$

By the Cauchy–Schwarz inequality, we must have

$$-1 \leq \cos[\theta(x)] \leq 1. \tag{1.95}$$

[5]Some relativity texts use a metric which is the negative of the one used here, so that a Lorentz signature metric has a signature of $2-N$. Caution must be applied when comparing the equations in this text with those in texts utilizing the opposite sign convention.

However, we can prove that for the separation $\sigma(\vec{\mathbf{U}}(x)) > 0$ and $\sigma(\vec{\mathbf{V}}(x)) > 0$, the *pseudoangle* is *unbounded*, that is,

$$-\infty < \frac{\left[\mathbf{g}_{..}(\vec{\mathbf{U}}(x), \vec{\mathbf{V}}(x))\right]}{\sigma(\vec{\mathbf{U}}(x)) \cdot \sigma(\vec{\mathbf{V}}(x))} < \infty. \tag{1.96}$$

Comparing (1.95) and (1.96), we conclude that *for a non-positive-definite metric, the usual angle between two vectors is not defined!*

Relative to a coordinate basis set $\left\{\frac{\partial}{\partial x^i}\right\}_1^N$, the *metric tensor components* are

$$g_{ij}(x) = \mathbf{g}_{..}(x)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \equiv g_{ji}(x). \tag{1.97}$$

The nondegeneracy axiom (1.85iv) implies that

$$g = g(x) := \det[g_{ij}(x)] \neq 0, \quad |g| = |g(x)| > 0. \tag{1.98}$$

The inverse matrix of $\left[g_{ij}(x)\right]$ yields the contravariant second-order tensor components

$$\left[g^{ij}(x)\right] := \left[g_{ij}(x)\right]^{-1}, \quad g^{ij}(x) \equiv g^{ji}(x),$$
$$\mathbf{g}^{..}(x) = g^{ij}(x)\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}. \tag{1.99}$$

The above tensor is called the *conjugate metric tensor* or *contravariant metric tensor* (sometimes called the *inverse metric tensor*).

Example 1.3.1. Let $\mathbb{E}_N$ be the N-dimensional Euclidean manifold. Let $(\chi, \mathbb{E}_N)$ be one of the global *Cartesian charts*. Relative to the corresponding basis $\left\{\frac{\partial}{\partial x^i}\right\}_1^N$, the corresponding components of the metric tensor are:

$$g_{ij}(x) \equiv \delta_{ij}, \quad g^{ij}(x) \equiv \delta^{ij},$$
$$\| \vec{\mathbf{V}}(x) \|^2 = \delta_{ij} V^i(x) V^j(x) = \sum_{i=1}^{N} \left[V^i(x)\right]^2 \geq 0. \qquad \square$$

Example 1.3.2. Let M be an N-dimensional (flat) manifold with the *Lorentz metric* $\mathbf{g}_{..}(x)$. (We choose $N \geq 2$.) In a global *pseudo-Cartesian chart* (χ, M), this metric field is furnished by

$$\mathbf{g}_{..}(x) = d_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j := \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots + \mathrm{d}x^{N-1} \otimes \mathrm{d}x^{N-1} - \mathrm{d}x^N \otimes \mathrm{d}x^N.$$
$$d^{ij} = d_{ij}, \quad \det[d_{ij}] = -1.$$
$$\operatorname{sgn}(\mathbf{g}_{..}(x)) = N - 2. \qquad \square$$

In the sequel, we shall deal primarily with two types of basis sets. These are the coordinate basis set $\left\{\frac{\partial}{\partial x^i}\right\}_1^N$ and the orthonormal basis set $\left\{\vec{\mathbf{e}}_{(a)}(x)\right\}_1^N$. By (1.30), (1.92), and (1.99), we can express

$$\mathbf{g}_{..}(x) = g_{ij}(x)\,\mathrm{d}x^i \otimes \mathrm{d}x^j = d_{(a)(b)}\tilde{\mathbf{e}}^{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b)}(x). \tag{1.100}$$

The components of the conjugate (contravariant) metric tensor field $\mathbf{g}^{..}(x)$ are given by

$$\begin{aligned}\mathbf{g}^{..}(x) &= g^{ij}(x)\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} = d^{(a)(b)}\,\vec{\mathbf{e}}_{(a)}(x) \otimes \vec{\mathbf{e}}_{(b)}(x),\\ g^{ik}(x)g_{kj}(x) &\equiv \delta^i{}_j,\ d^{(a)(c)}d_{(c)(b)} \equiv \delta^{(a)}_{(b)}.\end{aligned} \tag{1.101}$$

Now we will consider the transformation from one basis set to another. For two coordinate basis sets, transformations have been furnished in (1.34) and (1.35) which we repeat succinctly as

$$\hat{\mathbf{X}}'\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial \hat{X}^k(x)}{\partial x^i}\frac{\partial}{\partial \hat{x}^k}, \hat{\mathbf{X}}'\left(\mathrm{d}x^i\right) = \frac{\partial X^i(\hat{x})}{\partial \hat{x}^k}\mathrm{d}\hat{x}^k. \tag{1.102}$$

The transformation from one orthonormal basis set to another can be summarized as:

$$\begin{aligned}\hat{\vec{\mathbf{e}}}_{(a)}(x) &= L^{(b)}_{(a)}(x)\vec{\mathbf{e}}_{(b)}(x),\\ \hat{\tilde{\mathbf{e}}}^{(a)}(x) &= A^{(a)}_{(b)}(x)\tilde{\mathbf{e}}^{(b)}(x),\\ \left[A^{(b)}_{(a)}(x)\right] &:= \left[L^{(a)}_{(b)}(x)\right]^{-1},\\ L^{(a)}_{(b)}(x)d_{(a)(c)}L^{(c)}_{(e)}(x) &= d_{(b)(e)}.\end{aligned} \tag{1.103}$$

This last equation defines a *generalized Lorentz transformation* (at $x \in \mathbb{R}^N$). The set of all generalized Lorentz transformations, at a particular point $x \in \mathbb{R}^N$, constitutes a *Lie group* denoted by $O(p, n; \mathbb{R})$, $(p + n = N)$. (See [114, 123].)

The (nonsingular) transformation from one coordinate basis to an orthonormal basis is furnished by

$$\begin{aligned}\vec{\mathbf{e}}_{(a)}(x) &= \lambda^i_{(a)}(x)\frac{\partial}{\partial x^i},\\ \frac{\partial}{\partial x^i} &= \mu^{(a)}_i(x)\vec{\mathbf{e}}_{(a)}(x),\\ \tilde{\mathbf{e}}^{(a)}(x) &= \mu^{(a)}_i(x)\,\mathrm{d}x^i,\end{aligned}$$

$$\begin{aligned} dx^i &= \lambda^i{}_{(a)}(x)\tilde{\mathbf{e}}^{(a)}(x), \\ \left[\mu^{(a)}_i(x)\right] &:= \left[\lambda^i{}_{(a)}(x)\right]^{-1}. \end{aligned} \tag{1.104}$$

It follows from (1.25i), (1.25ii), (1.100), and (1.104) that

$$\begin{aligned} d_{(a)(b)} &= \mathbf{g}_{..}(x)(\vec{\mathbf{e}}_{(a)}(x), \vec{\mathbf{e}}_{(b)}(x)) = g_{ij}(x)\lambda^i{}_{(a)}(x)\lambda^j{}_{(b)}(x), \\ g_{ij}(x) &= \mathbf{g}_{..}(x)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = d_{(a)(b)}\mu^{(a)}_i(x)\mu^{(b)}_j(x), \\ d^{(a)(b)} &= \mathbf{g}^{..}(x)(\tilde{\mathbf{e}}^{(a)}(x), \tilde{\mathbf{e}}^{(b)}(x)) = g^{ij}(x)\mu^{(a)}_i(x)\mu^{(b)}_j(x), \\ g^{ij}(x) &= \mathbf{g}^{..}(x)(dx^i, dx^j) = d^{(a)(b)}\lambda^i{}_{(a)}(x)\lambda^j{}_{(b)}(x), \\ \delta^i{}_j &= \mathbf{g}^{.}{}_{.}(x)\left(dx^i, \frac{\partial}{\partial x^j}\right) := \mathbf{I}^{.}{}_{.}(x)\left(dx^i, \frac{\partial}{\partial x^j}\right) = \lambda^i_{(a)}(x)\mu^{(a)}_j(x), \\ \delta^{(a)}_{(b)} &= \mathbf{g}^{.}{}_{.}(x)(\tilde{\mathbf{e}}^{(a)}(x), \vec{\mathbf{e}}_{(b)}(x)) = \mathbf{I}^{.}{}_{.}(x)(\tilde{\mathbf{e}}^{(a)}(x), \vec{\mathbf{e}}_{(b)}(x)) = \mu^{(a)}_i(x)\lambda^i{}_{(b)}(x). \end{aligned} \tag{1.105}$$

Here, $\mathbf{I}^{.}{}_{.}(x)$ is the *identity tensor*.

Example 1.3.3. Consider the two-dimensional spherical surface S^2 of unit radius and the usual spherical polar coordinate chart. (See Example 1.1.2.) The metric tensor field is characterized by

$$\begin{aligned} \mathbf{g}_{..}(x) &= dx^1 \otimes dx^1 + \sin^2(x^1)\, dx^2 \otimes dx^2 \\ &= \tilde{\mathbf{e}}^{(1)}(x) \otimes \tilde{\mathbf{e}}^{(1)}(x) + \tilde{\mathbf{e}}^{(2)}(x) \otimes \tilde{\mathbf{e}}^{(2)}(x) = \delta_{(a)(b)}\tilde{\mathbf{e}}^{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b)}(x), \end{aligned}$$

$$\operatorname{sgn}[\mathbf{g}_{..}(x)] = 2, \quad g = \det[g_{ij}(x)] = \sin^2(x^1) > 0,$$

$$\begin{aligned} \mathbf{g}^{..}(x) &= \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1} + \left(\sin(x^1)\right)^{-2} \frac{\partial}{\partial x^2} \otimes \frac{\partial}{\partial x^2} \\ &= \vec{\mathbf{e}}_{(1)}(x) \otimes \vec{\mathbf{e}}_{(1)}(x) + \vec{\mathbf{e}}_{(2)}(x) \otimes \vec{\mathbf{e}}_{(2)}(x), \\ D &:= \left\{x \in \mathbb{R}^2 : 0 < x^1 < \pi, -\pi < x^2 < \pi\right\}. \end{aligned}$$

Here, the coordinate basis set is $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$, whereas the natural orthonormal basis set is $\left\{\vec{\mathbf{e}}_{(1)}(x), \vec{\mathbf{e}}_{(2)}(x)\right\}$. The transformation between the two basis sets is governed by

$$
\begin{aligned}
\vec{\mathbf{e}}_{(1)}(x) &= \frac{\partial}{\partial x^1}, \quad \vec{\mathbf{e}}_{(2)}(x) = \left(\sin(x^1)\right)^{-1} \frac{\partial}{\partial x^2},\\
\tilde{\mathbf{e}}^{(1)}(x) &= \mathrm{d}x^1, \quad \tilde{\mathbf{e}}^{(2)}(x) = \sin(x^1)\, \mathrm{d}x^2,\\
\lambda^i{}_{(1)}(x) &= \delta^i{}_{(1)}, \quad \lambda^i{}_{(2)}(x) = \left(\sin(x^1)\right)^{-1} \delta^i{}_{(2)},\\
\mu^{(1)}_{\ i}(x) &= \delta^{(1)}_i, \quad \mu^{(2)}_{\ i}(x) = \sin(x^1)\, \delta^{(2)}_i.
\end{aligned}
\qquad \square
$$

A tensor field ${}^r_s\mathbf{T}(x)$ can, by (1.30) and (1.36), be expressed in terms of various basis sets as:

$$
\begin{aligned}
{}^r_s\mathbf{T}(x) &= T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}\\
&= \overline{T}^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}(x)\, \vec{\mathbf{e}}_{(a_1)}(x) \otimes \cdots \otimes \vec{\mathbf{e}}_{(a_r)}(x) \otimes \tilde{\mathbf{e}}^{(b_1)}(x) \otimes \cdots \otimes \tilde{\mathbf{e}}^{(b_s)}(x).
\end{aligned}
\tag{1.106}
$$

By (1.37), (1.103), (1.104), and (1.106), we have the transformation rules for the tensor field components:

$$
\hat{T}^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(\hat{x}) = \frac{\partial \hat{X}^{k_1}(x)}{\partial x^{i_1}}, \ldots, \frac{\partial \hat{X}^{k_r}(x)}{\partial x^{i_r}} \left[\frac{\partial X^{j_1}(\hat{x})}{\partial \hat{x}^{l_1}}, \ldots, \frac{\partial X^{j_s}(\hat{x})}{\partial \hat{x}^{l_s}}\right]_{|\hat{x}=\hat{X}(x)}
\times T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x), \tag{1.107i}
$$

$$
\hat{\overline{T}}^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}(x) = A^{(a_1)}{}_{(c_1)}(x), \ldots, A^{(a_r)}{}_{(c_r)}(x) L^{(d_1)}{}_{(b_1)}(x), \ldots, L^{(d_s)}{}_{(b_s)}(x)
\times \overline{T}^{(c_1),\ldots,(c_r)}{}_{(d_1),\ldots,(d_s)}(x), \tag{1.107ii}
$$

$$
\overline{T}^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}(x) = \mu^{(a_1)}{}_{i_1}(x), \ldots, \mu^{(a_r)}{}_{i_r}(x) \lambda^{j_1}{}_{(b_1)}(x), \ldots, \lambda^{j_s}{}_{(b_s)}(x) T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x). \tag{1.107iii}
$$

$$
T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x) = \lambda^{i_1}{}_{(a_1)}(x), \ldots, \lambda^{i_r}{}_{(a_r)}(x) \cdot \mu^{(b_1)}{}_{j_1}(x), \ldots, \mu^{(b_s)}{}_{j_s}(x) \cdot \overline{T}^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}(x). \tag{1.107iv}
$$

The components $\overline{T}^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}(x)$ are called the *orthonormal components of the tensor* ${}^r_s\mathbf{T}(x)$.

Example 1.3.4. Let us consider the spherical surface S^2 of unit radius and the spherical coordinate chart of the preceding example. Let a second-order symmetric tensor field defined by

$$T^{ij}(x)\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} := (x^i x^j)\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \ .$$

The corresponding orthonormal components, from (1.107iii) are furnished by

$$\begin{aligned}\overline{T}^{(a)(b)}(x) &= \mu_i^{(a)}(x)\,\mu_j^{(b)}(x) T^{ij}(x),\\ &= (x^i x^j)\,\mu_i^{(a)}(x)\mu_j^{(b)}(x).\end{aligned}$$

From the preceding example,

$$\mu_i^{(1)}(x) = \delta_i^{(1)},\ \ \mu_i^{(2)}(x) = \sin(x^1)\,\delta_i^{(2)},$$

we obtain

$$\begin{aligned}&\overline{T}^{(1)(1)}(x) = (x^1)^2,\ \ \overline{T}^{(1)(2)}(x) \equiv \overline{T}^{(2)(1)}(x) = \sin(x^1)(x^1 x^2),\\ &\overline{T}^{(2)(2)}(x) = \left(\sin(x^1)\right)^2\ (x^2)^2.\end{aligned}$$

Evaluated at the "equator" (except one point), the above yields:

$$\begin{aligned}&\overline{T}^{(1)(1)}\left(\pi/2,\ x^2\right) \equiv (\pi/2)^2\,,\ \ \overline{T}^{(1)(2)}\left(\pi/2,\ x^2\right) = (\pi/2)\,x^2,\\ &\overline{T}^{(2)(2)}\left(\pi/2,\ x^2\right) = (x^2)^2 \quad \text{for all} \quad x^2 \in (-\pi,\ \pi).\end{aligned}$$

Thus, for a sphere of unit radius, the coordinate and corresponding orthonormal components of $\mathbf{T}^{\cdot\cdot}(x)$ coincide on the equator. □

For the interested reader, tensor bundles ${}^r_s\boldsymbol{\mathcal{T}}(T_x(\mathbb{R}^N))$ are discussed in [38, 56, 267].

The metric tensor $\mathbf{g}_{..}(x)$ and the conjugate metric tensor $\mathbf{g}^{\cdot\cdot}(x)$ induce a *tensor space isomorphism* $\mathcal{I}_g(x)$ from the tangent tensor bundle ${}^r_s\mathcal{T}\left(T_x(\mathbb{R}^N)\right)$ into ${}^{r'}_{s'}\mathcal{T}\left(T_x(\mathbb{R}^N)\right)$ for $r+s = r'+s'$. This isomorphism is brought about by the *raising and lowering of indices* of tensor components. The rules are elaborated for two different tensors in the following:

$$\begin{aligned}&\mathcal{I}_g\left[T^i{}_{j_1,\ldots,j_s}(x)\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}\right] = T_{kj_1,\ldots,j_s}(x)\,\mathrm{d}x^k \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}\\ &\quad =: g_{ki}(x)T^i{}_{j_1,\ldots,j_s}(x)\,\mathrm{d}x^k \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s},\end{aligned} \tag{1.108i}$$

$$\begin{aligned}&\mathcal{I}_g\left[T_{j_1,\ldots,j_s}(x)\,\mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}\right] = T^k{}_{j_2,\ldots,j_s}(x)\frac{\partial}{\partial x^k} \otimes \mathrm{d}x^{j_2} \otimes \cdots \otimes \mathrm{d}x^{j_s}\\ &\quad =: g^{kj_1}(x)T_{j_1,j_2,\ldots,j_s}(x)\frac{\partial}{\partial x^k} \otimes \mathrm{d}x^{j_2} \otimes \cdots \otimes \mathrm{d}x^{j_s},\end{aligned} \tag{1.108ii}$$

$$\begin{aligned}
&\mathcal{I}_g \left[\overline{T}^{(a)}{}_{(b_1),\ldots,(b_s)}(x)\, \vec{\mathbf{e}}_{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b_1)}(x) \otimes \cdots \otimes \tilde{\mathbf{e}}^{(b_s)}(x) \right] \\
&\quad = \overline{T}_{(c)(b_1),\ldots,(b_s)}(x)\, \tilde{\mathbf{e}}^{(c)}(x) \otimes \tilde{\mathbf{e}}^{(b_1)}(x) \otimes \cdots \otimes \tilde{\mathbf{e}}^{(b_s)}(x) \\
&\quad =: d_{(c)(a)} \overline{T}^{(a)}{}_{(b_1),\ldots,(b_s)}(x)\, \tilde{\mathbf{e}}^{(c)}(x) \otimes \tilde{\mathbf{e}}^{(b_1)}(x) \otimes \cdots \otimes \tilde{\mathbf{e}}^{(b_s)}(x),
\end{aligned} \tag{1.108iii}$$

$$\begin{aligned}
&\mathcal{I}_g \left[\overline{T}_{(b_1),\ldots,(b_s)}(x)\, \tilde{\mathbf{e}}^{(b_1)}(x) \otimes \cdots \otimes \tilde{\mathbf{e}}^{(b_s)}(x) \right] \\
&\quad = \overline{T}^{(a)}{}_{(b_2),\ldots,(b_s)}(x)\, \vec{\mathbf{e}}_{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b_2)}(x) \otimes \tilde{\mathbf{e}}^{(b_s)}(x) \\
&\quad =: d^{(a)(b_1)} \overline{T}_{(b_1)(b_2),\ldots,(b_s)}(x)\, \vec{\mathbf{e}}_{(a)}(x) \tilde{\mathbf{e}}^{(b_2)}(x) \otimes \cdots \otimes \tilde{\mathbf{e}}^{(b_s)}(x).
\end{aligned} \tag{1.108iv}$$

Example 1.3.5. Let us work out the raising and lowering of a second-order tensor field from (1.108i)–(1.108iv):

$$T^{ij}(x) = g^{jk}(x) T^i{}_k(x) = g^{ik}(x) g^{il}(x) T_{kl}(x) = g^{ik}(x) T_k{}^j(x),$$

$$T_{ij}(x) = g_{ik}(x) T^k{}_j(x) = g_{ik}(x) g_{jl}(x) T^{kl}(x) = g_{jk}(x) T_i{}^k(x),$$

$$\overline{T}^{(a)(b)}(x) = d^{(b)(c)} \overline{T}^{(a)}{}_{(c)}(x) = d^{(a)(c)} d^{(b)(f)} \overline{T}_{(c)(f)}(x) = d^{(a)(c)} \overline{T}_{(c)}{}^{(b)}(x),$$

$$\overline{T}_{(a)(b)}(x) = d_{(a)(c)} \overline{T}^{(c)}{}_{(b)}(x) = d_{(a)(e)} d_{(b)(f)} \overline{T}^{(e)(f)}(x) = d_{(b)(f)} \overline{T}_{(a)}{}^{(f)}(x). \tag{1.109}$$

In the sequel, we shall drop the bar from the orthonormal components.

For the metric tensor components, we obtain from (1.101),

$$\begin{aligned}
g^i{}_j(x) &= g^{ik}(x) g_{kj}(x) \equiv \delta^i{}_j, \\
d^{(a)}{}_{(b)} &= d^{(a)(c)} d_{(c)(b)} \equiv \delta^{(a)}{}_{(b)}.
\end{aligned}$$

□

Now we will generalize the totally antisymmetric permutation symbols of (1.51) and (1.52). We define the *totally antisymmetric (oriented) tensor components of Levi-Civita* by:

$$\eta_{i_1,\ldots,i_N}(x) := \sqrt{|g(x)|}\, \varepsilon_{i_1,\ldots,i_N}, \tag{1.110i}$$

$$\eta^{i_1,\ldots,i_N}(x) := \frac{\operatorname{sgn}[g(x)]}{\sqrt{|g(x)|}} \varepsilon^{i_1,\ldots,i_N}, \tag{1.110ii}$$

$$\eta_{(a_1),\ldots,(a_N)}(x) := \varepsilon_{(a_1),\ldots,(a_N)}, \tag{1.110iii}$$

$$\eta^{(a_1),\ldots,(a_N)}(x) := \operatorname{sgn}\left\{\det\left[d_{(a)(b)}\right]\right\} \varepsilon^{(a_1),\ldots,(a_N)}. \tag{1.110iv}$$

Note that $\eta_{(a_1),\ldots,(a_N)}(x)$ and $\eta^{(a_1),\ldots,(a_N)}(x)$ *are constant-valued*, whereas $\eta_{i_1,\ldots,i_N}(x)$ and $\eta^{i_1,\ldots,i_N}(x)$ are not necessarily constant-valued.

The choice in (1.110ii) is reasonable since it can be proved that

$$\eta^{i_1,\ldots,i_N}(x) = g^{i_1 j_1}(x), \ldots, g^{i_N j_N}(x)\, \eta_{j_1,\ldots,j_N}(x). \tag{1.111}$$

The transformation properties of the components $\eta_{i_1,\ldots,i_N}(x)$ and $\eta^{i_1,\ldots,i_N}(x)$ under coordinate transformations (1.2) are furnished by

$$\hat{\eta}_{i_1,\ldots,i_N}(\hat{x}) = \operatorname{sgn}\left\{\det\left[\frac{\partial X^k(\hat{x})}{\partial \hat{x}^l}\right]\right\} \frac{\partial X^{j_1}(\hat{x})}{\partial \hat{x}^{i_1}}, \ldots, \frac{\partial X^{j_N}(\hat{x})}{\partial \hat{x}^{i_N}} \eta_{j_1,\ldots,j_N}(x), \tag{1.112i}$$

$$\hat{\eta}^{i_1,\ldots,i_N}(\hat{x}) = \operatorname{sgn}\left\{\det\left[\frac{\partial X^k(\hat{x})}{\partial \hat{x}^l}\right]\right\} \frac{\partial \hat{X}^{i_1}(x)}{\partial x^{j_1}}, \ldots, \frac{\partial \hat{X}^{i_N}(x)}{\partial x^{j_N}} \eta^{j_1,\ldots,j_N}(x). \tag{1.112ii}$$

These are components of *oriented tensor fields*.

We shall define now the *Hodge star-operation* (or Hodge-$*$ operation). The Hodge-$*$ function maps an antisymmetric tensor field ${}_p\mathbf{W}(x)$ into an antisymmetric oriented tensor field ${}_{N-p}{}^{*}\mathbf{W}(x)$ by the following rules:

$$\begin{aligned} {}^{*}W_{i_1,\ldots,i_{N-p}}(x) &:= \frac{1}{p!}\, \eta^{j_1,\ldots,j_p}{}_{i_1,\ldots,i_{N-p}}(x) W_{j_1,\ldots,j_p}(x), \\ \eta^{j_1,\ldots,j_p}{}_{i_1,\ldots,i_{N-p}}(x) &:= g^{j_1 k_1}(x), \ldots, g^{j_p k_p}(x) \eta_{k_1,\ldots,k_p i_1,\ldots,i_{N-p}}(x). \end{aligned} \tag{1.113}$$

(See [38, 56, 104].)

Example 1.3.6. Consider a four-dimensional manifold of Lorentz signature $+2$. Let an antisymmetric tensor field be given by

$$\mathbf{F}_{..}(x) = \frac{1}{2} F_{ij}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

The Hodge star mapping will generate the antisymmetric components

$${}^{*}F^{kl}(x) = \frac{1}{2}\eta^{ijkl}(x) F_{ij}(x).$$

Thus, for the Lorentz metric, we have

$$\begin{aligned} {}^{*}F^{12}(x) &= -\frac{1}{\sqrt{|g|}} F_{34}(x), & {}^{*}F^{23}(x) &= -\frac{1}{\sqrt{|g|}} F_{14}(x), \\ {}^{*}F^{31}(x) &= -\frac{1}{\sqrt{|g|}} F_{24}(x), & {}^{*}F^{14}(x) &= -\frac{1}{\sqrt{|g|}} F_{23}(x), \\ {}^{*}F^{24}(x) &= -\frac{1}{\sqrt{|g|}} F_{31}(x), & {}^{*}F^{34}(x) &= -\frac{1}{\sqrt{|g|}} F_{12}(x). \end{aligned}$$

This example is relevant in electromagnetic field theory, where $F_{ij}(x)$ represents the components of what is known as the *electromagnetic field strength tensor*. □

We have noticed in Problem # 1 of Exercises 1.2 that partial derivatives of a differentiable tensor field *do not transform in general as tensor fields*. We can remedy that situation in a well-known manner. First we will need to introduce some new notations and definitions.

Partial derivatives of a differentiable function will often be denoted in the sequel by

$$\partial_j f := \frac{\partial}{\partial x^j} f(x) = \vec{\mathbf{e}}_j(x)[f], \tag{1.114i}$$

$$\partial_{(a)} f := \lambda^j{}_{(a)}(x) \frac{\partial}{\partial x^j} f(x) = \lambda^j{}_{(a)}(x) \partial_j f \equiv \vec{\mathbf{e}}_{(a)}(x)\,[f]\,. \tag{1.114ii}$$

Now we shall introduce a new concept of differentiations of tensor fields. It is called *connection* (sometimes written as *connexion*) or *covariant*[6] *derivative* and will be denoted by ∇. The operator ∇ assigns to each $\vec{\mathbf{V}}(x) \in T_x(\mathbb{R}^N)$ of class C^r $(r \geq 1)$ an $(1+1)$th order tensor field $\nabla \vec{\mathbf{V}}(x)$ of class C^{r-1}. The covariant derivative satisfies the following two rules:

$$\text{I.} \quad \nabla \left[\vec{\mathbf{V}}(x) + \vec{\mathbf{W}}(x) \right] = \nabla \left[\vec{\mathbf{V}}(x) \right] + \nabla \left[\vec{\mathbf{W}}(x) \right], \tag{1.115i}$$

$$\text{II.} \quad \nabla \left[f(x) \vec{\mathbf{V}}(x) \right] = \vec{\mathbf{V}}(x) \otimes \mathrm{d} f(x) + f(x) \left[\nabla \vec{\mathbf{V}}(x) \right]. \tag{1.115ii}$$

Let us consider basis fields $\{\partial_i\}_1^N \equiv \left\{\vec{\mathbf{e}}_i(x)\right\}_1^N$ and $\left\{\partial_{(a)}\right\}_1^N \equiv \left\{\vec{\mathbf{e}}_{(a)}(x)\right\}_1^N$. Moreover, the corresponding covariant basis fields are $\left\{\mathrm{d}x^i\right\}_1^N$ and $\left\{\tilde{\mathbf{e}}^{(a)}(x)\right\}_1^N \equiv \left\{\mu^{(a)}{}_i(x)\mathrm{d}x^i\right\}_1^N$ in (1.104). The corresponding basis fields for $(1+1)$-order tensors are $\left\{\partial_i \otimes \mathrm{d}x^j\right\}_1^N$ and $\left\{\partial_{(a)} \otimes \tilde{\mathbf{e}}^{(b)}\right\}_1^N$. Therefore, we can express the mixed tensors $\nabla \vec{\mathbf{e}}_i$ and $\nabla \vec{\mathbf{e}}_{(a)}$ by the linear combinations

$$\nabla\left(\partial_i\right) = \left\{ \begin{matrix} j \\ k\,i \end{matrix} \right\} (x) \cdot \partial_j \otimes \mathrm{d}x^k, \tag{1.116i}$$

$$\nabla\left(\partial_{(a)}\right) = -\,\gamma^{(b)}{}_{(c)(a)}(x)\; \partial_{(b)} \otimes \tilde{\mathbf{e}}^{(c)}(x). \tag{1.116ii}$$

[6]The term *covariant* has several meanings in tensor analysis. One meaning relates to the transformation properties of a tensor and, in terms of component indices, refers to an index written as a subscript (vs. superscript, which is *contravariant*). Another meaning refers to independence from any coordinate system. In the latter sense, all tensor equations are covariant. Up until this point in the book, we have only used the former meaning (index-down). Here, we refer to the latter meaning (coordinate independence).

Here, $\left\{ {j \atop k\ i} \right\}(x)$ and $\gamma^{(b)}_{\ (c)(a)}(x)$ are some suitable coefficient functions. The following should be noted: (1) The components $\left\{ {j \atop k\ i} \right\}(x)$ and $\gamma^{(b)}_{\ (c)(a)}(x)$ are *not tensor field components in general*. (2) The negative sign in (1.116ii) is a historical convention. (3) In the sequel, we shall often *omit* (x) from these symbols. (4) *At this stage, the symbols* $\left\{ {j \atop k\ i} \right\}$ *are more general than the Christoffel symbols of (1.134).*

By rules (1.115i), (1.115ii), (1.116i), and (1.116ii), we obtain

$$
\begin{aligned}
\nabla\vec{\mathbf{V}}(x) &= \nabla\left[V^i(x)\,\vec{\mathbf{e}}_i(x)\right] = \left[\vec{\mathbf{e}}_i(x)\otimes \mathrm{d}V^i(x)\right] + V^i(x)\left[\nabla\vec{\mathbf{e}}_i(x)\right] \\
&= \partial_i \otimes \left[\mathrm{d}V^i(x) + \left\{ {i \atop j\ k} \right\} V^k(x)\mathrm{d}x^j\right] \\
&= \left[\partial_j V^i + \left\{ {i \atop j\ k} \right\} V^k(x)\right]\left[\partial_i \otimes \mathrm{d}x^j\right] =: \nabla_j V^i \left[\partial_i \otimes \mathrm{d}x^j\right].
\end{aligned}
\tag{1.117i}
$$

$$
\begin{aligned}
\nabla\vec{\mathbf{V}}(x) &= \left[\nabla_{(b)}V^{(a)}\right]\left[\partial_{(a)}\otimes\tilde{\mathbf{e}}^{(b)}(x)\right] \\
&:= \left[\partial_{(b)}V^{(a)} - \gamma^{(a)}_{\ (c)(b)}V^{(c)}(x)\right]\left[\partial_{(a)}\otimes\tilde{\mathbf{e}}^{(b)}(x)\right].
\end{aligned}
\tag{1.117ii}
$$

Note that $\nabla_j V^i$ and $\nabla_{(b)}V^{(a)}$ *are components of the tensor field* $\nabla\vec{\mathbf{V}}(x)$ of order $(1+1)$.

The *directional covariant derivative* of a vector field $\vec{\mathbf{V}}(x)$ in the direction of the vector field $\vec{\mathbf{U}}(x)$ is defined by projecting the covariant derivative in the direction of $\vec{\mathbf{U}}(x)$. The directional derivative is denoted by $\nabla_{\vec{\mathbf{U}}}\left[\vec{\mathbf{V}}(x)\right]$. The operator $\nabla_{\vec{\mathbf{U}}}$ maps a vector field $\vec{\mathbf{V}}(x)$ of class C^r into another vector field of class C^{r-1}. The definition is furnished by:

$$
\nabla_{\vec{\mathbf{U}}}\left[\vec{\mathbf{V}}(x)\right]\left(\tilde{\mathbf{W}}(x)\right) := \left[\nabla\vec{\mathbf{V}}(x)\right]\left(\tilde{\mathbf{W}}(x),\ \vec{\mathbf{U}}(x)\right)
\tag{1.118}
$$

for all $\tilde{\mathbf{W}}(x) \in \tilde{T}_x(\mathbb{R}^N)$. (See [38, 56, 267].)

The main properties of the operator $\nabla_{\vec{\mathbf{U}}}$ are summarized in the following theorem.

Theorem 1.3.7. *The covariant derivatives along directions of vector fields under suitable differentiability conditions must satisfy:*

$$
(i)\quad \nabla_{\vec{\mathbf{V}}}\left[\vec{\mathbf{Y}}(x) + \vec{\mathbf{Z}}(x)\right] = \nabla_{\vec{\mathbf{V}}}\left[\vec{\mathbf{Y}}(x)\right] + \nabla_{\vec{\mathbf{V}}}\left[\vec{\mathbf{Z}}(x)\right];
\tag{1.119i}
$$

$$
(ii)\quad \nabla_{\vec{\mathbf{Y}}+\vec{\mathbf{Z}}}\left[\vec{\mathbf{V}}(x)\right] = \nabla_{\vec{\mathbf{Y}}}\left[\vec{\mathbf{V}}(x)\right] + \nabla_{\vec{\mathbf{Z}}}\left[\vec{\mathbf{V}}(x)\right];
\tag{1.119ii}
$$

$$(iii) \quad \nabla_{f\vec{\mathbf{V}}}\left[\vec{\mathbf{Y}}(x)\right] = f(x)\nabla_{\vec{\mathbf{V}}}\left[\vec{\mathbf{Y}}(x)\right]; \tag{1.119iii}$$

$$(iv) \quad \nabla_{\vec{\mathbf{V}}}\left[h(x)\vec{\mathbf{Y}}(x)\right] = \left\{\vec{\mathbf{V}}(x)[h]\right\}\vec{\mathbf{Y}}(x) + h(x)\left\{\nabla_{\vec{\mathbf{V}}}\left[\vec{\mathbf{Y}}(x)\right]\right\}. \tag{1.119iv}$$

Proof is left to the reader.

Example 1.3.8. From (1.117i), (1.117ii), and (1.118), we derive that

$$\left\{\nabla_{\partial_i}\left[\vec{\mathbf{V}}(x)\right]\right\}(\mathrm{d}x^i) = \nabla_i V^j,$$
$$\left\{\nabla_{\partial_{(a)}}\left[\vec{\mathbf{V}}(x)\right]\right\}\left[\tilde{\mathbf{e}}^{(b)}(x)\right] = \nabla_{(a)}V^{(b)}.$$
□

The definition of the covariant derivative along a vector field in (1.119i) can be extended to the covariant derivative of a tensor field ${}^r_s\mathbf{T}(x)$. The corresponding properties are the following:

I. In case ${}^r_s\mathbf{T}(x)$ is of class C^r, the tensor $\nabla_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x)\right]$ is of the same order and of class C^{r-1} $(r \geq 1)$. (1.120i)

II. $\nabla_{\vec{\mathbf{V}}}\left[f(x)\right] := \vec{\mathbf{V}}(x)[f],$ (1.120ii)

III. $\nabla_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x) + {}^r_s\mathbf{B}(x)\right] = \nabla_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x)\right] + \nabla_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{B}(x)\right],$ (1.120iii)

IV. $\nabla_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x) \otimes {}^m_n\mathbf{G}(x)\right] = \nabla_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x)\right] \otimes {}^m_n\mathbf{G}(x) + {}^r_s\mathbf{T}(x) \otimes \nabla_{\vec{\mathbf{V}}}\left[{}^m_n\mathbf{G}(x)\right],$ (1.120iv)

V. $\nabla_{\vec{\mathbf{V}}}\left\{{}^p_q\mathcal{C}\left[{}^r_s\mathbf{T}(x)\right]\right\} = {}^p_q\mathcal{C}\left\{\nabla_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x)\right]\right\}.$ (1.120v)

Example 1.3.9. Now we shall work out the covariant derivative of a dual vector field:

$$\nabla_{\partial_i}\left[\tilde{\mathbf{W}}(x)\right](\partial_j) = \partial_i W_j - \begin{Bmatrix} k \\ i\ j \end{Bmatrix} W_k(x), \tag{1.121}$$

$$\left[\nabla_{\partial_{(a)}}\tilde{\mathbf{W}}(x)\right]\left[\tilde{\mathbf{e}}_{(b)}(x)\right] = \partial_{(a)}W_{(b)} + \gamma^{(c)}{}_{(b)(a)}W_{(c)}(x). \tag{1.122}$$
□

We can generalize the definition of the covariant derivative in (1.117i) and (1.117ii) for a general tensor field ${}^r_s\mathbf{T}(x)$. It is furnished by

$$\left\{\nabla\left[{}^r_s\mathbf{T}(x)\right]\right\}\left(\tilde{\mathbf{W}}^1(x), \ldots, \tilde{\mathbf{W}}^r(x),\ \vec{\mathbf{U}}_1(x), \ldots, \vec{\mathbf{U}}_s(x);\ \vec{\mathbf{V}}(x)\right)$$

$$:= \left\{\nabla_{\vec{\mathbf{V}}}\left[{}^r_s\mathbf{T}(x)\right]\right\}\left(\tilde{\mathbf{W}}^1(x), \ldots, \tilde{\mathbf{W}}^r(x),\ \vec{\mathbf{U}}_1(x), \ldots, \vec{\mathbf{U}}_s(x)\right) \tag{1.123}$$

for all $\tilde{\mathbf{W}}^1(x), \ldots, \tilde{\mathbf{W}}^r(x),\ \vec{\mathbf{U}}_1(x), \ldots, \vec{\mathbf{U}}_s(x)$ fields.

In terms of the explicit components of ${}^r_s\mathbf{T}(x)$, the covariant derivative is expressed below.

Theorem 1.3.10. *Let ${}^r_s\mathbf{T}(x)$ be a differentiable tensor field of class C^r $(r \geq 1)$ in $D \in \mathbb{R}^N$. The covariant derivative is furnished by*

$$\left\{\nabla_{\partial_k}\left[{}^r_s\mathbf{T}(x)\right]\right\}\left(\mathrm{d}x^{i_1}, \ldots, \mathrm{d}x^{i_r}, \partial_{j_1}, \ldots, \partial_{j_s}\right) = \nabla_k T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}$$
$$:= \partial_k T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} + \sum_{\alpha=1}^{r}\begin{Bmatrix} i_\alpha \\ k\ l \end{Bmatrix} T^{i_1,\ldots,i_{\alpha-1} l i_{\alpha+1},\ldots,i_r}{}_{j_1,\ldots,j_s}$$
$$-\sum_{\beta=1}^{s}\begin{Bmatrix} l \\ k\ j_\beta \end{Bmatrix} T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_{\beta-1} l j_{\beta+1},\ldots,j_s}, \tag{1.124i}$$

$$\left\{\nabla_{\partial_{(c)}}\left[{}^r_s\mathbf{T}(x)\right]\right\}\left(\tilde{\mathbf{e}}^{(a_1)}, \ldots, \tilde{\mathbf{e}}^{(a_r)},\ \partial_{(b_1)}, \ldots, \partial_{(b_s)}\right) = \nabla_{(c)} T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}$$
$$:= \partial_{(c)} T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)} - \sum_{\alpha=1}^{r}\gamma^{(a_\alpha)}{}_{(d)(c)} T^{(a_1),\ldots,(a_{\alpha-1})(d)(a_{\alpha+1}),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)}$$
$$+\sum_{\beta=1}^{s}\gamma^{(d)}{}_{(b_\beta)(c)} T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_{\beta-1})(d)(b_{\beta+1}),\ldots,(b_s)}. \tag{1.124ii}$$

The proof follows from (1.117i), (1.117ii), (1.120ii), (1.120iii), (1.120iv), (1.121)–(1.123). (See [56, 90, 266].)

Example 1.3.11. Let the coordinate components of a constant-valued tangent vector field $\vec{\mathbf{V}}(x)$ be given by $V^1(x) \equiv 1$, $V^2(x) = \cdots = V^N(x) \equiv 0$ in a non-Cartesian basis:

$$\nabla_i V^k = \begin{Bmatrix} k \\ i\ 1 \end{Bmatrix} \not\equiv 0. \qquad \square$$

Example 1.3.12. Consider the identity tensor

$$\mathbf{I}^{\cdot}{}_{\cdot}(x) = \delta^i{}_j\, \partial_i \otimes \mathrm{d}x^j = \delta^{(a)}{}_{(b)}\, \partial_{(a)} \otimes \tilde{\mathbf{e}}^{(b)}.$$
$$\nabla_k \delta^i{}_j = \partial_k\left(\delta^i{}_j\right) + \begin{Bmatrix} i \\ k\ l \end{Bmatrix}\delta^l{}_j - \begin{Bmatrix} l \\ k\ j \end{Bmatrix}\delta^i{}_l \equiv 0,$$

$$\nabla_{(c)}\delta^{(a)}_{\ \ (b)} = \partial_{(c)}\left(\delta^{(a)}_{\ \ (b)}\right) - \gamma^{(a)}_{\ \ (d)(c)}\delta^{(d)}_{\ \ (b)} + \gamma^{(d)}_{\ \ (b)(c)}\delta^{(a)}_{\ \ (d)} \equiv 0,$$

$$\nabla_{\vec{\mathbf{V}}}\left[\mathbf{I}^{\cdot}{}_{\cdot}(x)\right] \equiv \mathbf{0}^{\cdot}{}_{\cdot}(x). \qquad \square$$

We have defined the Lie bracket $\left[\vec{\mathbf{U}}, \vec{\mathbf{V}}\right]$ and its properties in Problem # 8 of Exercise 1.2. With the help of the Lie bracket, we can define the *torsion tensor* field in the following:

$$\begin{aligned}
&\left[\mathbf{T}^{\cdot}{}_{\cdot\cdot}(x)\right]\left(\tilde{\mathbf{W}}(x),\, \vec{\mathbf{U}}(x),\, \vec{\mathbf{V}}(x)\right) \\
&\quad := \tilde{\mathbf{W}}(x)\left\{\nabla_{\vec{\mathbf{U}}}\left[\vec{\mathbf{V}}(x)\right] - \nabla_{\vec{\mathbf{V}}}\left[\vec{\mathbf{U}}(x)\right] - \left[\vec{\mathbf{U}}(x),\, \vec{\mathbf{V}}(x)\right]\right\} \\
&\quad \equiv -\left[\mathbf{T}^{\cdot}{}_{\cdot\cdot}(x)\right]\left(\tilde{\mathbf{W}}(x),\, \vec{\mathbf{V}}(x),\, \vec{\mathbf{U}}(x)\right).
\end{aligned} \tag{1.125}$$

Example 1.3.13. Now, $\mathbf{T}^{\cdot}{}_{\cdot\cdot}(x) = T^{(a)}_{\ \ (b)(c)}(x)\, \partial_{(a)} \otimes \tilde{\mathbf{e}}^{(b)}(x) \otimes \tilde{\mathbf{e}}^{(c)}(x) = T^{i}_{\ jk}(x)$ $\partial_i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$. Using (1.124i), (1.124ii), and (1.125), we also deduce that

$$\begin{aligned}
T^{(a)}_{\ \ (b)(c)}(x) &= \left[\mathbf{T}^{\cdot}{}_{\cdot\cdot}(x)\right]\left(\tilde{\mathbf{e}}^{(a)}(x),\, \partial_{(b)},\, \partial_{(c)}\right) \\
&= \tilde{\mathbf{e}}^{(a)}(x)\left\{\nabla_{\partial_{(b)}}\left[\partial_{(c)}\right]\right\} - \tilde{\mathbf{e}}^{(a)}(x)\left\{\nabla_{\partial_{(c)}}\left[\partial_{(b)}\right]\right\} \\
&\quad - \tilde{\mathbf{e}}^{(a)}(x)\left[\partial_{(b)},\, \partial_{(c)}\right] = \gamma^{(a)}_{\ \ (b)(c)}(x) - \gamma^{(a)}_{\ \ (c)(b)}(x) - \chi^{(a)}_{\ \ (b)(c)}(x).
\end{aligned} \tag{1.126}$$

Using $[\partial_i,\, \partial_j] \equiv \vec{\mathbf{0}}(x)$, we derive that

$$T^{i}_{\ jk}(x) = \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\} - \left\{\begin{matrix} i \\ k\ j \end{matrix}\right\} \equiv -T^{i}_{\ kj}(x). \tag{1.127}$$

The above equations reveal the antisymmetry properties of the torsion tensor. $\square$

Next we shall introduce an important tensor field. Namely, the $(1 + 3)$th order *curvature tensor* $\mathbf{R}^{\cdot}{}_{\cdot\cdot\cdot}(x)$ which is defined by

$$\begin{aligned}
\left[\mathbf{R}^{\cdot}{}_{\cdot\cdot\cdot}(x)\right]\left(\tilde{\mathbf{W}}(x),\, \vec{\mathbf{Y}}(x),\, \vec{\mathbf{U}}(x),\, \vec{\mathbf{V}}(x)\right) &:= \tilde{\mathbf{W}}(x)\left\{\nabla_{\vec{\mathbf{U}}}\left[\nabla_{\vec{\mathbf{V}}}\vec{\mathbf{Y}}(x)\right]\right. \\
&\left. -\nabla_{\vec{\mathbf{V}}}\left[\nabla_{\vec{\mathbf{U}}}\vec{\mathbf{Y}}(x)\right] - \nabla_{[\vec{\mathbf{U}},\, \vec{\mathbf{V}}]}\left[\vec{\mathbf{Y}}(x)\right]\right\}.
\end{aligned} \tag{1.128}$$

(See [38, 56, 90, 130, 171, 244, 266].)

Example 1.3.14. The components of $\mathbf{R}^{\cdot}{}_{\cdot\cdot\cdot}(x)$ will be computed relative to the basis sets $\{\partial_i \otimes dx^j \otimes dx^k \otimes dx^l\}_1^{N^4}$:

$$\begin{aligned} R^i{}_{jmn}(x) &= \mathbf{R}^{\cdot}{}_{\cdot\cdot\cdot}(x)\left(dx^i \otimes \partial_j \otimes \partial_m \otimes \partial_n\right) \\ &= dx^i\left\{\nabla_{\partial_m}\left[\nabla_n\left(\partial_j\right)\right] - \partial_n\left[\nabla_{\partial_m}\left(\partial_j\right)\right] - \nabla_{[\partial_m,\,\partial_n]}\left(\partial_j\right)\right\}. \end{aligned}$$

Using (1.123), (1.124i) and the fact $[\partial_m, \partial_n][f] \equiv 0$ (for any C^2-function f), we obtain that

$$\begin{aligned} R^i{}_{jmn}(x) &= \partial_m \begin{Bmatrix} i \\ n\ j \end{Bmatrix} - \partial_n \begin{Bmatrix} i \\ m\ j \end{Bmatrix} + \begin{Bmatrix} i \\ m\ h \end{Bmatrix}\begin{Bmatrix} h \\ n\ j \end{Bmatrix} - \begin{Bmatrix} i \\ n\ h \end{Bmatrix}\begin{Bmatrix} h \\ m\ j \end{Bmatrix} \\ &\equiv -R^i{}_{jnm}(x). \end{aligned} \tag{1.129}$$

(If the connection is known, the above formula is the most straightforward method to calculate the curvature tensor.) □

Example 1.3.15. Let us compute the orthonormal components $R^{(a)}{}_{(b)(c)(d)}(x)$ of the tensor $\mathbf{R}^{\cdot}{}_{\cdot\cdot\cdot}(x)$ relative to the orthonormal basis set $\{\vec{\mathbf{e}}_{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b)}(x) \otimes \tilde{\mathbf{e}}^{(c)}(x) \otimes \tilde{\mathbf{e}}^{(d)}(x)\}$:

$$\begin{aligned} R^{(a)}{}_{(b)(c)(d)}(x) &= \mathbf{R}^{\cdot}{}_{\cdot\cdot\cdot}(x)\left(\tilde{\mathbf{e}}^{(a)}(x), \vec{\mathbf{e}}_{(b)}(x), \vec{\mathbf{e}}_{(c)}(x), \vec{\mathbf{e}}_{(d)}(x)\right) \\ &= \tilde{\mathbf{e}}^{(a)}(x)\left\{\nabla_{\partial_{(c)}}\left[\nabla_{\partial_d}\left(\vec{\mathbf{e}}_{(b)}(x)\right)\right] - \partial_{\partial_{(d)}}\left[\nabla_{\partial_{(c)}}\left(\vec{\mathbf{e}}_{(b)}(x)\right)\right]\right. \\ &\qquad \left. - \nabla_{[\partial_{(d)},\,\partial_{(c)}]}\left[\vec{\mathbf{e}}_{(b)}(x)\right]\right\}. \end{aligned}$$

Using (1.123), (1.124ii), and $[\partial_{(c)}, \partial_{(d)}] = \chi^{(f)}{}_{(c)(d)}(x)\,\partial_{(f)}$, we derive that

$$\begin{aligned} R^{(a)}{}_{(b)(c)(d)}(x) = &-\tilde{\mathbf{e}}^{(a)}(x)\left\{\vec{\mathbf{e}}_{(c)}\left[\gamma^{(f)}{}_{(b)(d)}\right]\vec{e}_{(f)} + \gamma^{(f)}{}_{(b)(c)}\nabla_{\partial_{(c)}}\left(\vec{\mathbf{e}}_{(f)}\right)\right. \\ &\left. - \vec{\mathbf{e}}_{(d)}\left[\gamma^{(f)}{}_{(b)(c)}\right]\vec{\mathbf{e}}_{(f)} - \gamma^{(f)}{}_{(b)(c)}\nabla_{\partial_{(d)}}\left(\vec{\mathbf{e}}_{(f)}\right) - \chi^{(f)}{}_{(c)(d)}\nabla_{\partial_{(f)}}\left(\vec{\mathbf{e}}_{(b)}\right)\right\} \\ \equiv &-R^{(a)}{}_{(b)(d)(c)}(x). \end{aligned} \tag{1.130}$$

(Note the antisymmetry with respect to the last two indices.) □

We shall discuss the geometrical significance of the curvature tensor later.

If $\left\{ {i \atop j\ k} \right\} \equiv 0$ for $x \in D \subset \mathbb{R}^N$, then $R^i{}_{jmn}(x) \equiv 0$. However, $R^i{}_{jmn}(x) \equiv 0$ does *not* imply $\left\{ {i \atop j\ k} \right\} \equiv 0$ as will be shown in a later example.

Note that (1.85i)–(1.113) deal with the metric and orthonormality. However, (1.115i)–(1.130) investigate various properties of connections or covariant derivatives. A closer inspection of (1.115i)–(1.130) will reveal that *nowhere* are the metric tensor or orthonormality explicitly used. (In fact, *there exist differential manifolds with connections devoid of a metric tensor employing similar equations!*) However, the definition of a *Riemannian* or *pseudo-Riemannian* manifold demands some relationships between the metric and the connection. In the following, we discuss such properties.

Theorem 1.3.16. *In the domain $D \subset \mathbb{R}^N$ corresponding to a domain of a manifold endowed with a differentiable metric, there exists a unique connection such that:*

(i) *The torsion tensor* $\mathbf{T}^{\cdot}{}_{\cdot\cdot}(x) \equiv \mathbf{0}^{\cdot}{}_{\cdot\cdot}(x).$ (1.131i)

(ii) *The covariant derivative of the metric identically vanishes:*

$$\nabla \mathbf{g}_{\cdot\cdot}(x) \equiv \mathbf{0}_{\cdot\cdot\cdot}(x). \tag{1.131ii}$$

Proof. Let the coordinate basis $\{\partial_i\}_1^N$ be used to investigate (1.131i) and (1.131ii). By (1.131i) and (1.128), we deduce

$$T^i{}_{jk}(x) = \left\{ {i \atop j\ k} \right\} - \left\{ {i \atop k\ j} \right\} \equiv 0, \tag{1.132}$$

and that

$$\left\{ {i \atop j\ k} \right\} \equiv \left\{ {i \atop k\ j} \right\}.$$

Using (1.131ii) and (1.124i), we derive

$$\nabla_i g_{jk}(x) = \partial_i g_{jk} - \left\{ {l \atop i\ j} \right\} g_{lk} - \left\{ {l \atop i\ k} \right\} g_{jl} \equiv 0. \tag{1.133}$$

With cyclic permutations of the indices i, j, and k in (1.133), we obtain

$$\begin{aligned} \partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk} &= g_{il} \left[\left\{ {l \atop j\ k} \right\} + \left\{ {l \atop k\ j} \right\} \right] + 0 + 0 \\ &= 2 g_{il} \left\{ {l \atop j\ k} \right\}. \end{aligned}$$

Therefore,

$$\left\{ {l \atop j\,k} \right\} = \frac{1}{2} g^{li} \left[\partial_j \, g_{ki} + \partial_k \, g_{ij} - \partial_i \, g_{jk} \right]. \tag{1.134}$$

By this *unique expression* for $\left\{ {l \atop j\,k} \right\}$, the proof is established. ■

We define a related entity by

$$[jk, \, i] := \frac{1}{2} \left[\partial_j \, g_{ki} + \partial_k \, g_{ij} - \partial_i \, g_{jk} \right] \equiv [kj, \, i] \, ,$$

$$\left\{ {l \atop j\,k} \right\} = g^{li} \, [jk, \, i] \, . \tag{1.135}$$

The symbols $[jk, \, i]$ and $\left\{ {l \atop j\,k} \right\} \equiv 0$ are called *Christoffel symbols of the first and second kind*, respectively. These connections possess $N^2(N+1)/2$ linearly independent components. They are *not components of a tensor field under general coordinate transformations*. The salient properties of these symbols are listed here:

$$g_{il} \left\{ {l \atop j\,k} \right\} = [jk, \, i] \, , \tag{1.136i}$$

$$\partial_j \, g_{ik} = [ij, \, k] + [jk, \, i] \, , \tag{1.136ii}$$

$$\partial_k g^{ij} = -\, g^{ib} \cdot g^{lj} \cdot \partial_k g_{bl} = - \left[g^{il} \left\{ {j \atop k\,l} \right\} + g^{jl} \left\{ {i \atop k\,l} \right\} \right], \tag{1.136iii}$$

$$\left(g \cdot g^{il} \right) \cdot g_{ij} = g \cdot \delta^l{}_j , \;\; \partial_k g = \left(g \cdot g^{ij} \right) \cdot \partial_k g_{ij} \, , \tag{1.136iv}$$

$$\left\{ {j \atop i\,j} \right\} = \partial_i \left[\ln \left(\sqrt{|g|} \right) \right]. \tag{1.136v}$$

Example 1.3.17. Consider the polar chart for a subset of the Euclidean plane $\mathbb{E}_2$. (See Example 1.1.1). The metric tensor is furnished by

$$\mathbf{g}_{..}(x) = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + (x^1)^2 \, \mathrm{d}x^2 \otimes \mathrm{d}x^2 ,$$

$$D := \left\{ x \in \mathbb{R}^2 : \; x^1 > 0, \; -\pi < x^2 < \pi \right\} ,$$

$$\left[g_{ij}(x) \right] = \begin{bmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{bmatrix}, \quad g = (x^1)^2 > 0.$$

The nonvanishing Christoffel symbols from (1.134) and (1.135) are given by

$$[12,\,2] = -[22,\,1] = x^1,$$

$$\begin{Bmatrix} 2 \\ 1\,2 \end{Bmatrix} = (x^1)^{-1}, \begin{Bmatrix} 1 \\ 2\,2 \end{Bmatrix} = -x^1,$$

$$\begin{Bmatrix} j \\ i\,j \end{Bmatrix} = \begin{Bmatrix} 2 \\ i\,2 \end{Bmatrix} = \partial_i\left(\ln x^1\right) = \partial_i\left[\ln\left(\sqrt{|g|}\right)\right].$$

In this example, one can show that $R^i{}_{jmn}(x) \equiv 0$ in spite of $\begin{Bmatrix} i \\ j\,k \end{Bmatrix} \not\equiv 0$. □

The investigations of (1.131i) and (1.131ii) in an orthonormal basis set $\{\vec{\mathbf{e}}_{(a)}\}_1^N$ lead to the unique expression for connection coefficients as

$$\gamma^{(c)}{}_{(b)(a)}(x) = \frac{1}{2}\left[\chi^{(c)}{}_{(b)(a)} + d^{(c)(e)}\left(d_{(b)(f)}\chi^{(f)}{}_{(a)(e)} + d_{(a)(f)}\chi^{(f)}{}_{(b)(e)}\right)\right],$$
$$\gamma^{(b)}{}_{(c)(a)}(x) - \gamma^{(b)}{}_{(a)(c)}(x) = \chi^{(b)}{}_{(c)(a)}(x). \tag{1.137}$$

The *Ricci rotation coefficients* are defined by[7]

$$\gamma_{(a)(b)(c)}(x) := d_{(a)(e)}\gamma^{(e)}{}_{(b)(c)}(x),$$
$$\gamma_{(b)(a)(c)}(x) \equiv -\gamma_{(a)(b)(c)}(x). \tag{1.138}$$

The number of linearly independent components of $\gamma_{(a)(b)(c)}(x)$ is $N^2(N-1)/2$. (See [56, 90].)

The important properties of the Ricci rotation coefficients are listed here:

$$\gamma_{(a)(b)(c)}(x) = \mathbf{g}_{..}(x)\left(\partial_{(a)},\, \nabla_{\partial_{(c)}}\left(\partial_{(a)}\right)\right), \tag{1.139i}$$

$$\gamma_{(a)(b)(c)}(x) = g_{jl}(x)\left(\nabla_k \lambda^l{}_{(a)}\right)\lambda^j{}_{(b)}(x)\lambda^k{}_{(c)}(x), \tag{1.139ii}$$

$$g_{jl}(x)\left(\nabla_i \lambda^l{}_{(a)}\right) = \gamma_{(a)(b)(c)}(x)\mu^{(b)}{}_j(x)\mu^{(c)}{}_i(x), \tag{1.139iii}$$

$$\left[\partial_{(a)},\, \partial_{(b)}\right][f] = d^{(c)(e)}\left[\gamma_{(e)(a)(b)} - \gamma_{(e)(b)(a)}\right]\partial_{(c)} f. \tag{1.139iv}$$

Example 1.3.18. Consider the spherical surface of unit radius, S^2. (See Example 1.1.2.) In the spherical polar coordinate chart, the metric is provided by

---

[7]The Ricci rotation coefficients, analogous to the Christoffel symbols $[ij,\,k]$, *do not transform tensorially in general.*

$$
\begin{aligned}
\mathbf{g}_{..}(x) &= \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left(\sin x^1\right)^2 \mathrm{d}x^2 \otimes \mathrm{d}x^2 \\
&= \tilde{\mathbf{e}}^{(1)}(x) \otimes \tilde{\mathbf{e}}^{(1)}(x) + \tilde{\mathbf{e}}^{(2)}(x) \otimes \tilde{\mathbf{e}}^{(2)}(x), \\
D &:= \left\{x \in \mathbb{R}^2 : \ 0 < x^1 < \pi, \ -\pi < x^2 < \pi\right\}. \qquad (1.140)
\end{aligned}
$$

The natural orthonormal basis is furnished by

$$
\begin{aligned}
&\tilde{\mathbf{e}}^{(1)}(x) = \mathrm{d}x^1, \ \ \tilde{\mathbf{e}}^{(2)}(x) = \left(\sin x^1\right) \mathrm{d}x^2, \\
&\vec{\mathbf{e}}_{(1)}(x) \equiv \partial_{(1)} = \partial_1, \ \ \vec{\mathbf{e}}_{(2)}(x) \equiv \partial_{(2)} = \left(\sin x^1\right)^{-1} \partial_2, \\
&\lambda^i{}_{(1)}(x) = \delta^i{}_{(1)}, \ \ \lambda^i{}_{(2)}(x) = \left(\sin x^1\right)^{-1} \delta^i{}_{(2)}, \\
&\mu^{(1)}{}_i(x) = \delta^{(1)}{}_i, \ \ \mu^{(2)}{}_i(x) = \left(\sin x^1\right) \delta^{(2)}{}_i.
\end{aligned}
$$

Using (1.139ii), we obtain the nonzero components of the Ricci rotation coefficients as

$$
\gamma_{(1)(2)(2)}(x) = \cot x^1 \equiv -\gamma_{(2)(1)(2)}(x). \qquad \square
$$

The curvature tensor $\mathbf{R}^{\cdot}{}_{...}(x)$ in (1.128), (1.129), and (1.130) is called the *Riemann–Christoffel curvature tensor*, provided the connection satisfies (1.131i,ii), (1.134), and (1.137). The components of the Riemann–Christoffel curvature tensor satisfy, from (1.129), (1.130), (1.135), and (1.137):

$$
R^i{}_{jkl}(x) = \partial_k \begin{Bmatrix} i \\ l\ j \end{Bmatrix} - \partial_l \begin{Bmatrix} i \\ k\ j \end{Bmatrix} + \begin{Bmatrix} i \\ k\ h \end{Bmatrix} \begin{Bmatrix} h \\ l\ j \end{Bmatrix} - \begin{Bmatrix} i \\ l\ h \end{Bmatrix} \begin{Bmatrix} h \\ k\ j \end{Bmatrix}, \qquad (1.141\text{i})
$$

$$
R_{ijkl}(x) = \partial_k\,[lj,\, i] - \partial_l\,[kj,\, i] + [il,\, h] \begin{Bmatrix} h \\ j\ k \end{Bmatrix} - [ik,\, h] \begin{Bmatrix} h \\ j\ l \end{Bmatrix}, \qquad (1.141\text{ii})
$$

$$
\begin{aligned}
R^{(a)}{}_{(b)(c)(d)}(x) &= \partial_{(d)}\gamma^{(a)}{}_{(b)(c)} - \partial_{(c)}\gamma^{(a)}{}_{(b)(d)} + \gamma^{(a)}{}_{(h)(c)}\gamma^{(h)}{}_{(b)(d)} - \gamma^{(a)}{}_{(h)(d)}\gamma^{(h)}{}_{(b)(c)} \\
&\quad + \gamma^{(a)}{}_{(b)(h)}\left(\gamma^{(h)}{}_{(c)(d)} - \gamma^{(h)}{}_{(d)(c)}\right), \qquad (1.141\text{iii})
\end{aligned}
$$

$$
\begin{aligned}
R_{(a)(b)(c)(d)}(x) &= \partial_{(d)}\gamma_{(a)(b)(c)} - \partial_{(c)}\gamma_{(a)(b)(d)} + d^{(h)(e)}\left[\gamma_{(h)(a)(d)}\gamma_{(e)(b)(c)}\right. \\
&\quad \left. -\gamma_{(h)(a)(c)}\gamma_{(e)(b)(d)} + \gamma_{(a)(b)(h)}\left(\gamma_{(e)(c)(d)} - \gamma_{(e)(d)(c)}\right)\right].
\end{aligned}
\qquad (1.141\text{iv})
$$

The algebraic identities of the Riemann–Christoffel tensor are as follows:

Table 1.1 The number of independent components of the Riemann–Christoffel tensor for various dimensions N

N	Number of independent components
2	1
3	6
4	20
5	50
10	825

Theorem 1.3.19. *The algebraic identities of the Riemann–Christoffel tensor components are furnished by*

$$R_{jikl}(x) \equiv -R_{ijkl}(x), \tag{1.142i}$$

$$R_{ijlk}(x) \equiv -R_{ijkl}(x), \tag{1.142ii}$$

$$R_{klij}(x) \equiv R_{ijkl}(x), \tag{1.142iii}$$

$$R_{ijkl}(x) + R_{iklj}(x) + R_{iljk}(x) \equiv 0. \tag{1.142iv}$$

$$R_{(a)(b)(c)(d)}(x) \equiv -R_{(b)(a)(c)(d)}(x), \tag{1.142v}$$

$$R_{(a)(b)(c)(d)}(x) \equiv -R_{(a)(b)(d)(c)}(x), \tag{1.142vi}$$

$$R_{(c)(d)(a)(b)}(x) \equiv R_{(a)(b)(c)(d)}(x), \tag{1.142vii}$$

$$R_{(a)(b)(c)(d)}(x) + R_{(a)(c)(d)(b)}(x) + R_{(a)(d)(b)(c)}(x) \equiv 0. \tag{1.142viii}$$

The proof may be found in many of the textbooks in [56, 244, 267].

The algebraic identities in the previous theorem imply that the number of linearly independent components of the Riemann–Christoffel curvature tensor for a manifold of dimension N is $N^2(N^2-1)/12$. The number of independent components for various dimensions of importance in physics is listed in Table 1.1. (Of particular importance to general relativity is the case $N = 4$.)

Now, we shall state *Bianchi's differential identities* for the curvature components.

Theorem 1.3.20. *Let the Riemann–Christoffel curvature tensor* $\mathbf{R}^{\cdot}{}_{\cdot\cdot\cdot}(x)$ *be of class* $C^1(D \subset \mathbb{R}^N)$*. Then the corresponding components satisfy the following differential identities:*

$$\nabla_j R^h{}_{ikl} + \nabla_k R^h{}_{ilj} + \nabla_l R^h{}_{ijk} \equiv 0, \tag{1.143i}$$

$$\nabla_{(d)} R^{(e)}{}_{(a)(b)(c)} + \nabla_{(b)} R^{(e)}{}_{(a)(c)(d)} + \nabla_{(c)} R^{(e)}{}_{(a)(d)(b)} \equiv 0. \tag{1.143ii}$$

The proof may be found in many of the textbooks in the references.[8]

Example 1.3.21. Consider a two-dimensional manifold M with positive-definite metric. In an *orthogonal coordinate chart*, the metric is expressible as:

$$\begin{aligned}\mathbf{g}_{..}(x) &= g_{11}(x)\,\mathrm{d}x^1 \otimes \mathrm{d}x^1 + g_{22}(x)\,\mathrm{d}x^2 \otimes \mathrm{d}x^2 \\ &= \tilde{\mathbf{e}}^{(1)}(x) \otimes \tilde{\mathbf{e}}^{(1)}(x) + \tilde{\mathbf{e}}^{(2)}(x) \otimes \tilde{\mathbf{e}}^{(2)}(x).\end{aligned}$$

We can derive that

$$\begin{aligned}g(x) &= g_{11}(x) \cdot g_{22}(x) > 0, \\ \tilde{\mathbf{e}}^{(1)}(x) &= \sqrt{g_{11}}\,\mathrm{d}x^1, \quad \tilde{\mathbf{e}}^{(2)}(x) = \sqrt{g_{22}}\,\mathrm{d}x^2, \\ \lambda^i{}_{(1)}(x) &= \frac{1}{\sqrt{g_{11}}}\delta^i{}_{(1)}, \quad \lambda^i{}_{(2)}(x) = \frac{1}{\sqrt{g_{22}}}\delta^i{}_{(2)}, \\ \partial_{(1)} &= \frac{1}{\sqrt{g_{11}}}\partial_1, \quad \partial_{(2)} = \frac{1}{\sqrt{g_{22}}}\partial_2.\end{aligned} \tag{1.144}$$

Using (1.141i) and (1.141ii), we obtain for one independent component:

$$\begin{aligned}R_{1212}(x) &= -\frac{\sqrt{g}}{2}\left[\partial_1\left(\frac{1}{\sqrt{g}}\partial_1 g_{22}\right) + \partial_2\left(\frac{1}{\sqrt{g}}\partial_2 g_{11}\right)\right], \\ R_{(1)(2)(1)(2)}(x) &= -\frac{\sqrt{g}}{2}\left[\sqrt{g_{11}}\partial^2_{(1)}\left(\sqrt{g}\right) + \sqrt{g_{22}}\partial^2_{(2)}\left(\sqrt{g_{11}}\right)\right].\end{aligned}$$

□

Now we shall state the general rule for commutators or Lie brackets of two covariant derivatives. These are known as the *Ricci identities* .

Theorem 1.3.22. *Let ${}^r_s\mathbf{T}(x)$ be a tensor field of class $C^2(D \subset \mathbb{R}^N)$. Then, the following commutation rules hold:*

$$\begin{aligned}(\nabla_k\nabla_l - \nabla_l\nabla_k)\, T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} &= \sum_{\alpha=1}^{r}\left[T^{i_1,\ldots,i_{\alpha-1}hi_{\alpha+1},\ldots,i_r}{}_{j_1,\ldots,j_s} R^{i_\alpha}{}_{hkl}\right] \\ &\quad - \sum_{\beta=1}^{s}\left[T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_{\beta-1}hj_{\beta+1},\ldots,j_s} R^h{}_{j_\beta kl}\right],\end{aligned} \tag{1.145i}$$

[8] The number of linearly independent Bianchi's identities is furnished by $1/24 \cdot N^2 \cdot (N^2 - 1) \cdot (N - 2)$. In case $N = 4$, this number is exactly 20. (See [229].)

$$
\begin{aligned}
&\left(\nabla_{(c)}\nabla_{(d)} - \nabla_{(d)}\nabla_{(c)}\right) T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)} \\
&\quad = \sum_{\alpha=1}^{r} \left[T^{(a_1),\ldots,(a_{\alpha-1})(e)(a_{\alpha+1}),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)} R^{(a_\alpha)}{}_{(e)(c)(d)} \right] \\
&\qquad - \sum_{\beta=1}^{s} \left[T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_{\beta-1})(e)(b_{\beta+1}),\ldots,(b_s)} R^{(e)}{}_{(b_\beta)(c)(d)} \right]. \qquad (1.145\text{ii})
\end{aligned}
$$

The above can be proved by induction.

Example 1.3.23. Consider a twice-differentiable tensor field $\mathbf{T}_{..}(x)$ in $D \subset \mathbb{R}^N$. Equation (1.145i) yields:

$$(\nabla_k \nabla_l - \nabla_l \nabla_k)\, T_{ij} = -T_{hj} R^h{}_{ikl} - T_{ih} R^h{}_{jkl}.$$

In case $\mathbf{T}_{..}(x) = \mathbf{g}_{..}(x)$, the left-hand side of the above equations is zero by (1.133), whereas the right-hand side vanishes by (1.142i). □

Now we shall discuss contractions of the Riemann–Christoffel curvature tensor. The *Ricci tensor* $\mathbf{R}_{..}(x) \equiv \mathbf{Ric}(x)$ is defined by the single contraction[9]

$$\mathbf{Ric}(x) \equiv \mathbf{R}_{..}(x) := \frac{1}{3}\mathcal{C}\left[\mathbf{R}^{\cdot}{}_{...}(x)\right] = R^k{}_{ijk}(x)\, dx^i \otimes dx^j =: R_{ij}(x)\, dx^i \otimes dx^j, \qquad (1.146\text{i})$$

$$\mathbf{R}_{..}(x) = R^{(c)}{}_{(a)(b)(c)}(x)\tilde{\mathbf{e}}^{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b)}(x) =: R_{(a)(b)}(x)\tilde{\mathbf{e}}^{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b)}(x). \qquad (1.146\text{ii})$$

By (1.141i,iii) and (1.146i,ii), we deduce

$$R_{ij}(x) = \partial_j \begin{Bmatrix} k \\ k\, i \end{Bmatrix} - \partial_k \begin{Bmatrix} k \\ i\, j \end{Bmatrix} + \begin{Bmatrix} k \\ i\, l \end{Bmatrix} \begin{Bmatrix} l \\ k\, j \end{Bmatrix} - \begin{Bmatrix} l \\ l\, k \end{Bmatrix} \begin{Bmatrix} k \\ i\, j \end{Bmatrix}, \qquad (1.147\text{i})$$

$$\text{or,}\quad R_{ij}(x) = \partial_i \partial_j \ln\sqrt{|g|} - \frac{1}{\sqrt{|g|}}\partial_k\left[\sqrt{|g|}\begin{Bmatrix} k \\ i\, j \end{Bmatrix}\right] + \begin{Bmatrix} k \\ i\, l \end{Bmatrix} \begin{Bmatrix} l \\ k\, j \end{Bmatrix} \equiv R_{ji}(x), \qquad (1.147\text{ii})$$

$$
\begin{aligned}
R_{(a)(b)}(x) &= \partial_{(c)}\gamma^{(c)}{}_{(a)(b)} - \partial_{(b)}\gamma^{(c)}{}_{(a)(c)} + \gamma^{(c)}{}_{(a)(d)}\gamma^{(d)}{}_{(b)(c)} - \gamma^{(c)}{}_{(d)(c)}\gamma^{(d)}{}_{(a)(b)} \\
&\equiv R_{(b)(a)}(x). \qquad (1.147\text{iii})
\end{aligned}
$$

Thus, the Ricci tensor $\mathbf{R}_{..}(x)$ is symmetric and has $N(N+1)/2$ linearly independent components.

[9] Some texts define the Ricci tensor as the contraction of the Riemann tensor on *the first and third indices*. In texts using that convention, the Ricci tensor is opposite in sign to what is presented here.

The *curvature scalar* (or *curvature invariant*) is defined by the contraction

$$R(x) := R^i{}_i(x) = g^{ij}(x)\, R_{ij}(x), \tag{1.148i}$$

$$R(x) = R^{(a)}{}_{(a)}(x) = d^{(a)(b)}\, R_{(a)(b)}(x). \tag{1.148ii}$$

Of particular importance to general relativity theory is the *Einstein tensor*, $\mathbf{G}_{..}(x)$. It is defined by

$$\mathbf{G}_{..}(x) := \mathbf{R}_{..}(x) - \frac{1}{2} R(x)\, \mathbf{g}_{..}(x), \tag{1.149i}$$

$$G_{ij}(x) = R_{ij}(x) - \frac{1}{2} R(x)\, g_{ij}(x), \tag{1.149ii}$$

$$G_{(a)(b)}(x) = R_{(a)(b)}(x) - \frac{1}{2} R(x)\, d_{(a)(b)}. \tag{1.149iii}$$

The Einstein tensor is a (0+2)th order *symmetric tensor* and possesses $N(N+1)/2$ linearly independent components.

There exist more differential identities involving the tensor fields $\mathbf{R}^{\cdot}{}_{...}(x)$, $\mathbf{R}_{..}(x)$, and $\mathbf{G}_{..}(x)$.

Theorem 1.3.24. *Let a metric field* $\mathbf{g}_{..}(x)$ *be of class* C^3 *in* $D \subset \mathbb{R}^N$. *Then the following differential identities hold:*

$$\nabla_j R^j{}_{ikl} + \nabla_k R_{il} - \nabla_l R_{ik} \equiv 0, \tag{1.150i}$$

$$\nabla_j G^j{}_i \equiv 0, \tag{1.150ii}$$

with similar identities in the orthonormal frame.

Proof. The identities (1.150i) follow from Bianchi's identities (1.143i,ii) by a single contraction. The second set of identities follows by a further contraction. ■

Remarks: (i) Identities (1.150i) are known as the *first contracted Bianchi's identities*. (ii) Identities (1.150ii) are called the *second contracted Bianchi's identities*. We will see that these identities yield differential conservation laws in Einstein's theory of gravitation.

Now we shall define the *covariant differentiation of a relative and oriented relative tensor field of weight* **w**. (See (1.39) and (1.40).) It is defined (in both cases) by:

$$\nabla_k \Phi^{i_1,\ldots,i_r}{}_{j_i,\ldots,j_s} := \partial_k \Phi^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} + \sum_{\alpha=1}^{r} \begin{Bmatrix} i_\alpha \\ k\, l \end{Bmatrix} \Phi^{i_1,\ldots,i_{\alpha-1} l i_{\alpha+1},\ldots,i_r}{}_{j_1,\ldots,j_s}$$

$$-\sum_{\beta=1}^{s}\begin{Bmatrix} l \\ k\ j_\beta \end{Bmatrix} \Phi^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_{\beta-1} l j_{\beta+1},\ldots,j_s} - w \begin{Bmatrix} l \\ k\ l \end{Bmatrix} \Phi^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}, \tag{1.151i}$$

$$\nabla_{(c)}\Phi^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)} = \partial_{(c)}\Phi^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_s)} - \sum_{\alpha=1}^{r}\gamma^{(a_\alpha)}{}_{(d)(c)}\Phi^{(a_1),\ldots,(a_{\alpha-1})(d)(a_{\alpha+1}),\ldots,(a_r)}{}_{(b_1),\ldots,(b)s)}$$
$$+\sum_{\beta=1}^{s}\gamma^{(d)}{}_{(b_\beta)(c)}\Phi^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_{\beta-1})(d)(b_{\beta+1}),\ldots,(b_s)}. \tag{1.151ii}$$

(See [56, 90, 171, 244].)

Example 1.3.25. The transformation property of $\sqrt{|g|}$ is that of an oriented scalar density field (see Example 1.2.8). Therefore, by (1.151i,ii), and $w = 1$, we obtain:

$$\nabla_k\left(\sqrt{|g|}\right) = \partial_k\left(\sqrt{|g|}\right) - \begin{Bmatrix} l \\ k\ l \end{Bmatrix}\sqrt{|g|} \equiv 0. \tag{1.152i}$$

$$\nabla_{(c)}\left\{\sqrt{|\det[d_{(a)(b)}]|}\right\} = \partial_{(c)}(1) \equiv 0. \tag{1.152ii}$$

□

A manifold M with metric $\mathbf{g}_{..}$ satisfying (1.131i,ii) is called a *Riemannian manifold* in case the metric is positive-definite.[10] If (1.131i,ii) hold, and the metric is not positive-definite, the manifold is called a *pseudo-Riemannian manifold*.

We shall now classify various manifolds. In case $\mathbf{R}^{\cdot}{}_{...}(x) \equiv \mathbf{0}^{\cdot}{}_{...}(x)$ in $D \subset \mathbb{R}^N$, the domain D (and corresponding $U \subset M$) is called a *flat domain*. In case $\mathbf{R}^{\cdot}{}_{...}(x) \not\equiv \mathbf{0}^{\cdot}{}_{...}(x)$ in $D \subset \mathbb{R}^N$, the domain D (and corresponding $U \subset M$) is called a *curved domain* in some sense. For a positive-definite metric, $\text{sgn}[\mathbf{g}_{..}(x)] = N$. For a Lorentz metric, $\text{sgn}[\mathbf{g}_{..}(x)] = N-2$. For an indefinite, non-Lorentz metric, $0 \le \text{sgn}[\mathbf{g}_{..}(x)] \le N-4$. We represent the classification of manifolds with metrics in Fig. 1.10.

Remarks: (i) In Newtonian physics, a three-dimensional Euclidean manifold represents space. (ii) In the special theory of relativity, a four-dimensional flat manifold with Lorentz metric represents the space–time continuum. (iii) In the general theory of relativity, a four-dimensional curved manifold with Lorentz metric represents the space–time continuum.

We shall briefly discuss some integral theorems. Firstly, we define the *oriented volume* of a domain $D \subset \mathbb{R}^N$ by the multiple integral

[10] It should be noted that in much of the literature involving relativistic physics, the term Euclidean is often synonymous with Riemannian. In this text, Euclidean and Riemannian refer to two distinct properties.

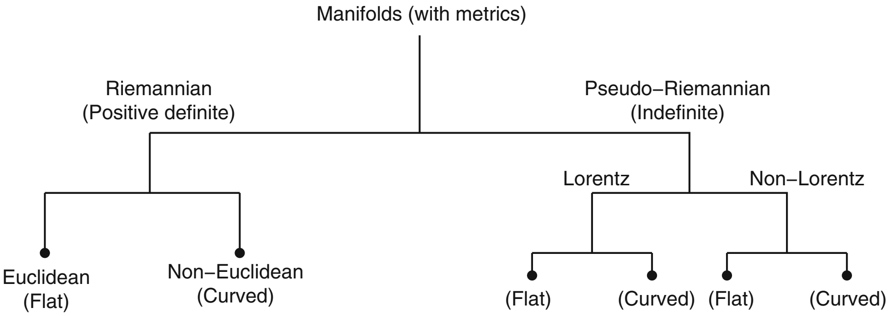

Fig. 1.10 A classification chart for manifolds endowed with metric

$$V(D) = \int_D \mathrm{d}^N V := \frac{1}{N!}\int_D \eta_{i_1,\ldots,i_N}(x)\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_N} \equiv \int_D \sqrt{|g|}\,\mathrm{d}x^1, \ldots, \mathrm{d}x^N. \tag{1.153}$$

Example 1.3.26. Consider a sphere in three-dimensional Euclidean space $\mathbb{E}_3$ and the spherical polar coordinate chart (see Example 1.1.2). The metric tensor is provided by:

$$\begin{aligned} \mathbf{g}_{..}(x) &= \mathrm{d}x^1 \otimes \mathrm{d}x^1 + (x^1)^2\left[\mathrm{d}x^2 \otimes \mathrm{d}x^2 + \sin^2(x^2)\mathrm{d}x^3 \otimes \mathrm{d}x^3\right], \\ D &:= \left\{x \in \mathbb{R}^3 :\ 0 < x^1 < a,\ 0 < x^2 < \pi,\ -\pi < x^3 < \pi\right\}. \end{aligned} \tag{1.154}$$

Therefore, the volume of a sphere, as given by (1.153), is

$$V(D) = \int_{0+}^{a-}\int_{0+}^{\pi-}\int_{-\pi+}^{\pi-}(x^1)^2\sin(x^2)\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \frac{4\pi(a)^3}{3}. \qquad \square$$

Now we shall state the *generalized Gauss' theorem.*

Theorem 1.3.27. *Let D^* be a star-shaped domain in $\mathbb{R}^N$ with a continuous, piecewise-differentiable, orientable, closed, nonnull boundary, ∂D^*, with unit outward normal n_j. Moreover, let $\vec{\mathbf{A}}(x)$ be a differentiable vector field in $D^* \cup \partial D^*$. Then, the following integral relation holds:*

$$\int_{D^*}\left(\nabla_j A^j\right)\mathrm{d}^N v = \int_{\partial D^*} A^j n_j\,\mathrm{d}^{N-1}v. \tag{1.155}$$

For proof, consult textbooks [38, 56, 130, 171, 237, 244, 266] in the references.

Example 1.3.28. Consider the Newtonian theory of gravitation in the Euclidean space $\mathbb{E}_3$. The gravitational potential $W(x)$, which is of class piecewise-C^2, satisfies:

$$\nabla_i \nabla^i W = \begin{cases} 4\pi G\, \rho(x) & \text{in} \quad D^* \in \mathbb{R}^3, \\ 0 & \text{in} \quad \mathbb{R}^3 - D^*. \end{cases} \tag{1.156}$$

Here, G is the *Newtonian constant of gravitation* ($G = 6.67 \times 10^{-8}\text{cm}^3 \cdot \text{gm}^{-1} \cdot \text{s}^{-2}$ in the c.g.s. system of units), $\rho(x) > 0$ is the mass density (which is assumed to be a piecewise-continuous function). By Gauss' theorem, we obtain:

$$\int_{D^*} \left(\nabla_i \nabla^i W(x)\right) \mathrm{d}^3 v = \int_{\partial D^*} \left(\nabla^i W(x)\right) n_i \, \mathrm{d}^2 v,$$

$$4\pi G M := 4\pi G \int_{D^*} \rho(x)\, \mathrm{d}^3 v = \int_{\partial D^*} \left(\nabla^i W(x)\right) n_i \, \mathrm{d}^2 v. \tag{1.157}$$

Here, M is *the total mass* of the body inside D^*. Moreover, the integral on the right-hand side is called the total normal flux of the gravitational field across the boundary surface ∂D^*. □

(*Remark:* At a distance r from the center of a spherically symmetric source (such as the Earth), the Newtonian gravitational potential is given by $W(r) = -GM/r$, where M is the total mass located inside radius r.)

Now we shall discuss briefly special coordinate charts. The metric field at a particular point x_0 of a chart is given by:

$$\mathbf{g}_{..}(x_0) = g_{ij}(x_0)\, \mathrm{d}x^i \otimes \mathrm{d}x^j =: s_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Here, the symmetric matrix $[S] := \left[s_{ij}\right]$ has p positive-definite eigenvalues and n negative-definite eigenvalues. In a different coordinate chart, by (1.37), we derive

$$\hat{s}_{ij} := \hat{g}_{ij}(\hat{x}_0) = \frac{\partial X^k(\hat{x})}{\partial \hat{x}^i}_{|\hat{x}_0} \frac{\partial X^l(\hat{x})}{\partial \hat{x}^j}_{|\hat{x}_0} g_{kl}(x_0),$$

or, in the language of matrices,

$$\left[\hat{S}\right] = [\Lambda]^T [S] [\Lambda],$$

$$[\Lambda] := \left[\frac{\partial X^l(\hat{x})}{\partial \hat{x}^j}\right]_{|\hat{x}_0}.$$

In linear algebra, there is a theorem [113] asserting the existence of a matrix $[\Lambda]$ such that

$$\left[\hat{S}\right] = [D] =: \left[d_{ij}\right].$$

Thus, there are coordinate charts for which

$$\hat{g}_{ij}(\hat{x}_0)\,\mathrm{d}\hat{x}^i \otimes \mathrm{d}\hat{x}^j = d_{ij}\,\mathrm{d}\hat{x}^i \otimes \mathrm{d}\hat{x}^j \,.$$

Now we mention another important coordinate chart. At the point $x_0 \in D \subset \mathbb{R}^N$ of a chart, there exists a *Riemann normal coordinate chart* such that

$$\hat{x}_0 = \hat{X}(x_0) = (0, 0, \ldots, 0)\,,$$
$$\hat{\partial}_k \hat{g}_{ij\,|(0,0,\ldots,0)} = 0, \quad \widehat{\left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\}}_{|(0,0,\ldots,0)} = 0. \tag{1.158}$$

(For a proof, see textbooks [56, 90] in the references.)

There exists another coordinate chart such that the metric field is expressible as

$$\hat{\mathbf{g}}_{..}(\hat{x}) = \hat{g}_{\alpha\beta}(\hat{x})\,\mathrm{d}\hat{x}^\alpha \otimes \mathrm{d}\hat{x}^\beta + \hat{g}_{NN}(\hat{x})\,\mathrm{d}\hat{x}^N \otimes \mathrm{d}\hat{x}^N,$$
$$\hat{g}_{NN}(\hat{x}) \neq 0, \quad \alpha, \beta \in \{1, 2, \ldots, N-1\}\,. \tag{1.159}$$

Here, the coordinate $\hat{x}^N$ is called a *normal* or *hypersurface orthogonal coordinate*. (For the proof of (1.159), see textbooks [56, 90, 244] in the references.) The normal coordinate curve x^N intersects orthogonally with each of $x^1, \ldots, x^{N-1}$-coordinate curves. (We have *assumed that $\hat{x}^N$ is a nonnull coordinate*.)

There exists a special type of normal coordinate chart, namely, a *Gaussian normal* or *geodesic normal coordinate chart*. In this chart, the metric field is furnished by

$$\begin{aligned}\mathbf{g}_{..}(x) &= g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + d_{NN}\,\mathrm{d}x^N \otimes \mathrm{d}x^N \\ &=: g_{\alpha\beta}(x)\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta + \varepsilon_N\,\mathrm{d}x^N \otimes \mathrm{d}x^N, \\ \varepsilon_N &= \pm 1. \end{aligned} \tag{1.160}$$

Now we shall touch upon briefly some special Riemannian and pseudo-Riemannian manifolds.

A domain $D \subset \mathbb{R}^N$ (corresponding to the open subset $U \subset M$) is a *flat domain* provided

$$\mathbf{R}^{\cdot}{}_{\cdots}(x) \equiv \mathbf{0}^{\cdot}{}_{\cdots}(x),$$

or, equivalently:

$$R^i{}_{jkl}(x) \equiv 0,$$
$$\text{or,}\;\; R^{(a)}{}_{(b)(c)(d)}(x) \equiv 0. \tag{1.161}$$

Eisenhart, in his book on Riemannian geometry [90], proves that the metric tensor components $g_{ij}(x)$ of a prescribed signature $p - n$ solves the system of nonlinear (1.161) if and only if

$$g_{ij}(x) = d_{kl}\,\partial_i f^k(x)\,\partial_j f^l(x). \tag{1.162}$$

Here, N functions $f^k(x)$ are of class C^3 in D and otherwise *arbitrary*.

If we make a C^3-coordinate transformation

$$\hat{x}^i = f^i(x),$$

then (1.162) yields

$$\hat{\mathbf{g}}_{..}(\hat{x}) = d_{kl}\,\mathrm{d}\hat{x}^k \otimes \mathrm{d}\hat{x}^l. \tag{1.163}$$

In the above equation, the metric tensor components are diagonal as well as constant-valued. The corresponding coordinate chart is often denoted by the following nomenclature: In case $\mathbf{g}_{..}(x)$ is positive-definite (thus $d_{ij} = \delta_{ij}$), the special coordinate system is called a *Cartesian coordinate chart*. In case $\mathbf{g}_{..}(x)$ is *not positive-definite*, the chart is called a *pseudo-Cartesian chart*.

Note that, relative to a Cartesian or pseudo-Cartesian chart, the covariant derivative of an arbitrary differentiable tensor field simplifies to

$$\hat{\nabla}_k \hat{T}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\hat{x}) \equiv \hat{\partial}_k \hat{T}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\hat{x}).$$

A Riemannian or pseudo-Riemannian manifold is called a *space of constant curvature* K_0, provided:

$$R_{lijk}(x) = K_0\left[g_{lj}(x)g_{ik}(x) - g_{lk}(x)g_{ij}(x)\right], \tag{1.164i}$$

$$R_{(a)(b)(c)(d)}(x) = K_0\left[d_{(a)(c)}d_{(b)(d)} - d_{(a)(d)}d_{(b)(c)}\right]. \tag{1.164ii}$$

Note that in a space of constant curvature, the orthonormal components of the Riemann tensor are *constant-valued*.

Example 1.3.29. In a two-dimensional Riemannian surface of constant curvature, we consider a Gaussian normal coordinate chart

$$\mathbf{g}_{..}(x) = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left[h(x^1, x^2)\right]^2 \mathrm{d}x^2 \otimes \mathrm{d}x^2.$$

Equation (1.164i) reduces to the partial differential equation

$$\frac{\partial^2}{(\partial x^1)^2} h(x^1, x^2) + K_0 h(x^1, x^2) = 0,$$

$$(x^1, x^2) \in D \subset \mathbb{R}^2.$$

For the flat case, $K_0 = 0$, and the general solution of the partial differential equation is furnished by

$$h(x^1, x^2) = f(x^2)x^1 + g(x^2).$$

Here, f and g are arbitrary C^2-functions of integration. For the sake of this example, let us solve the initial value problem:

$$h(0, x^2) \equiv 1, \quad \frac{\partial}{\partial x^1} h(x^1, x^2)_{|x^1=0} \equiv 0.$$

Thus, we finally obtain

$$\mathbf{g}_{..}(x) = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2.$$

That is, the domain is locally isometric to a Euclidean plane.

In case $K_0 > 0$ (constant positive curvature), we can prove that the domain is locally isometric to a spherical surface (with radius of curvature $1/\sqrt{K_0}$). Furthermore, for $K_0 < 0$ (constant negative curvature), the domain is locally isometric to the surface of revolution in the shape of a bugle. □

Now we shall furnish the canonical form of the metric tensor for a space of constant curvature.

Theorem 1.3.30. *Let M be a manifold of space of constant curvature with $N \geq 2$ and of differentiability class C^2. Then, there exists locally a coordinate chart such that*

$$\mathbf{g}_{..}(x) = \left[1 + \frac{K_0}{4}\left(d_{kl}\, x^k x^l\right)\right]^{-2} \left[d_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j\right],$$
$$x \in D \subset \mathbb{R}^N.$$

(For the proof, consult textbooks [56, 90] in the references.)

Now we shall define an *Einstein space*. It is a C^3-manifold with the following conditions:

$$\mathbf{R}_{..}(x) = \left[\frac{R(x)}{N}\right] \mathbf{g}_{..}(x), \tag{1.165i}$$

$$R_{ij}(x) = \left[\frac{R(x)}{N}\right] g_{ij}(x), \tag{1.165ii}$$

$$R_{(a)(b)}(x) = \left[\frac{R(x)}{N}\right] d_{(a)(b)}. \tag{1.165iii}$$

Example 1.3.31. Consider the following Gaussian normal metric for $N \geq 2$:

$$\mathbf{g}_{..}(x) := \sum_{\alpha=1}^{N-1} \left[h_{(\alpha)}(x^N)\right]^2 \mathrm{d}x^\alpha \otimes \mathrm{d}x^\alpha - \mathrm{d}x^N \otimes \mathrm{d}x^N,$$

$$h_{(\alpha)}(x^N) := \left(\sin(kx^N)\right)^{\frac{1}{N-1}} \left[\tan\left(kx^N/2\right)\right]^{\beta_\alpha},$$

$$k = \text{const.}, \;\; \beta_\alpha = \text{const. such that:} \;\; \sum_{\alpha=1}^{N-1} \beta_\alpha = 0 \;\; \text{and} \;\; \sum_{\alpha=1}^{N-1} \beta_\alpha^2 = \frac{N-2}{N-1}.$$

The above metric locally represents an Einstein space, as can be verified by calculating the Ricci tensor and Ricci scalar ((1.147ii) and (1.148i)) and showing that (1.165i) holds. (See [210].) □

A *Ricci flat space* is one for which the components of the Ricci tensor vanish. Note that for Einstein spaces with vanishing curvature scalar, the Ricci tensor components also vanish via (1.165i–iii), and hence, Einstein spaces with vanishing curvature scalar are Ricci flat. In the general theory of relativity, a four-dimensional Ricci flat domain indicates a space–time devoid of material sources (although "gravitational fields" may be present). We shall provide plenty of examples of such spaces later on!

We shall now investigate *conformal mappings*. This term is well known in complex analysis. We generalize this concept to the mapping of a regular domain M into $\overline{M}$. It is assumed that M and $\overline{M}$ have *identical atlases*. The conformal mapping is defined by

$$\overline{\mathbf{g}}_{..}(x) := \exp\left[2\mu(x)\right] \mathbf{g}_{..}(x), \tag{1.166i}$$

$$\overline{\tilde{\mathbf{e}}}^{(a)}(x) = \exp\left[\mu(x)\right] \tilde{\mathbf{e}}^{(a)}(x). \tag{1.166ii}$$

Here, μ is assumed to be a C^3-function into $\mathbb{R}^N$. We note that by (1.166i,ii), we can obtain

$$\overline{\mathbf{g}}_{..}(x)\left(\vec{\mathbf{v}}(x), \vec{\mathbf{w}}(x)\right) = \exp\left[2\mu(x)\right] \mathbf{g}_{..}(x)\left(\vec{\mathbf{v}}(x), \vec{\mathbf{w}}(x)\right) \tag{1.167}$$

for an arbitrary pair of vector fields $\vec{\mathbf{v}}(x)$ and $\vec{\mathbf{w}}(x)$. The above equation clearly shows that *for a positive-definite metric, the angle (see (1.94)) between two nonzero vectors is exactly preserved.*

By (1.166i,ii), we also derive (after a long calculation) the transformed curvature components as

$$\begin{aligned}
\overline{R}^{i}{}_{jkl}(x) = R^{i}{}_{jkl}(x) &+ \delta^{i}{}_{l}\left[\nabla_k \nabla_j \mu - (\partial_k \mu)(\partial_j \mu)\right] - \delta^{i}{}_{k}\left[\nabla_l \nabla_j \mu - (\partial_l \mu)(\partial_j \mu)\right] \\
&+ \left(\delta^{i}{}_{l} g_{jk} - \delta^{i}{}_{k} g_{jl}\right)\left[g^{mn}(\partial_m \mu)(\partial_n \mu)\right] \\
&+ g^{im}\left\{g_{kj}\left[\nabla_l \nabla_m \mu - (\partial_l \mu)(\partial_m \mu)\right] - g_{lj}\left[\nabla_k \nabla_m \mu - (\partial_k \mu)(\partial_m \mu)\right]\right\},
\end{aligned} \tag{1.168i}$$

$$
\begin{aligned}
e^{2\mu(x)}\overline{R}^{(a)}{}_{(b)(c)(d)}(x) = {} & R^{(a)}{}_{(b)(c)(d)}(x) + \delta^{(a)}{}_{(d)}\left[\nabla_{(c)}\nabla_{(b)}\mu - (\partial_{(c)}\mu)(\partial_{(b)}\mu)\right] \\
& - \delta^{(a)}{}_{(c)}\left[\nabla_{(d)}\nabla_{(b)}\mu - (\partial_{(d)}\mu)(\partial_{(b)}\mu)\right] \\
& + \left(\delta^{(a)}{}_{(d)} d_{(b)(c)} - \delta^{(a)}{}_{(c)} d_{(b)(d)}\right)\left[d^{(e)(f)}(\partial_{(e)}\mu)(\partial_{(f)}\mu)\right] \\
& + d^{(a)(e)}\big\{d_{(b)(c)}\left[\nabla_{(d)}\nabla_{(e)}\mu - (\partial_{(d)}\mu)(\partial_{(e)}\mu)\right] \\
& - d_{(b)(d)}\left[\nabla_{(c)}\nabla_{(e)}\mu - (\partial_{(c)}\mu)(\partial_{(e)}\mu)\right]\big\}. \qquad (1.168\text{ii})
\end{aligned}
$$

(See [56, 90, 171, 266].)

Now we shall define *Weyl's conformal tensor*.[11] For $N \geq 3$, the Weyl conformal tensor's components are defined by:

$$
\begin{aligned}
C^{l}{}_{ijk}(x) := {} & R^{l}{}_{ijk}(x) + \frac{1}{(N-2)}\left[\delta^{l}{}_{j} R_{ik} - \delta^{l}{}_{k} R_{ij} + g_{ik} R^{l}{}_{j} - g_{ij} R^{l}{}_{k}\right] \\
& + \frac{R(x)}{(N-1)(N-2)}\left[\delta^{l}{}_{k} g_{ij} - \delta^{l}{}_{j} g_{ik}\right], \qquad (1.169\text{i})
\end{aligned}
$$

$$
\begin{aligned}
C^{(d)}{}_{(a)(b)(c)}(x) := {} & R^{(d)}{}_{(a)(b)(c)}(x) + \frac{1}{(N-2)} \\
& \cdot \left[\delta^{(d)}{}_{(b)} R_{(a)(c)} - \delta^{(d)}{}_{(c)} R_{(a)(b)} + d_{(a)(c)} R^{(d)}{}_{(b)} - d_{(a)(b)} R^{(d)}{}_{(c)}\right] \\
& + \frac{R(x)}{(N-1)(N-2)}\left[\delta^{(d)}{}_{(c)} d_{(a)(b)} - \delta^{(d)}{}_{(b)} d_{(a)(c)}\right]. \qquad (1.169\text{ii})
\end{aligned}
$$

An important theorem regarding the conformal tensor is the following:

Theorem 1.3.32. *Consider a conformal mapping* $\overline{\mathbf{g}}_{..}(x) = \exp[2\mu(x)]\mathbf{g}_{..}(x)$ *of class* C^3 *in a domain* $D \subset \mathbb{R}^N$ $(N \geq 3)$*. The components of the conformal tensor remain invariant under the conformal transformations, that is,*

$$
\overline{C}^{l}{}_{ijk}(x) \equiv C^{l}{}_{ijk}(x). \qquad (1.170)
$$

Proof. The proof follows from calculating all parts of the conformal tensor in (1.169i), via (1.166i), (1.168i), (1.147ii), and (1.148i) using both $\mathbf{g}_{..}(x)$ and $\overline{\mathbf{g}}_{..}(x)$. ■

A domain of M is *conformally flat* provided the corresponding chart satisfies

$$
\mathbf{g}_{..}(x) = \exp[-2\mu(x)] \cdot \overline{\mathbf{g}}_{..}(x),
$$

[11] There exists another tensor called *Weyl's projective tensor*. It is defined in [90].

with

$$\overline{R}^{i}{}_{jkl}(x) \equiv 0.$$

We shall now state an important theorem regarding conformal flatness.

Theorem 1.3.33. *A domain of M for $N \geq 4$ is conformally flat if and only if $C^{i}{}_{jkl}(x) \equiv 0$ in the corresponding domain $D \subset \mathbb{R}^N$.*

(For a proof, consult textbooks [56, 90, 171].)

The symmetries or isometries of a Riemannian or pseudo-Riemannian manifold are obtained from the existence of *Killing vectors*, $\vec{\mathbf{K}}(x)$ satisfying

$$L_{\vec{\mathbf{K}}}\left[\mathbf{g}_{..}(x)\right] = \mathbf{0}_{..}(x), \tag{1.171i}$$

$$\left(\partial_l g_{ij}\right) K^l(x) + g_{lj}(x)\partial_i K^l + g_{il}(x)\partial_j K^l = 0, \tag{1.171ii}$$

$$\nabla_i K_j + \nabla_j K_i = 0. \tag{1.171iii}$$

(Compare with Problem # 7 of Exercises 1.2.)

Example 1.3.34. Consider the surface S^2 of the unit sphere. (See Examples 1.1.2 and 1.3.18.) Recall (1.155) which states

$$\mathbf{g}_{..}(x) = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left(\sin x^1\right)^2 \mathrm{d}x^2 \otimes \mathrm{d}x^2. \tag{1.172}$$

The corresponding Killing equation (1.171ii) yields

$$\partial_1 K^1 = 0, \tag{1.173i}$$

$$\partial_2 K^2 + \left(\cot x^1\right) K^1(x) = 0, \tag{1.173ii}$$

$$\partial_2 K^1 + \left(\sin x^1\right)^2 \partial_1 K^2 = 0. \tag{1.173iii}$$

The general solutions of the above system of partial differential equations are furnished by

$$\begin{aligned}
K^1(x) &= -A\sin x^2 + B\cos x^2,\\
K^2(x) &= C - \cot x^1\left(A\cos x^2 + B\sin x^2\right),\\
\vec{\mathbf{K}}(x) &= -A\left(\sin x^2 \frac{\partial}{\partial x^1} + \cot x^1 \cos x^2 \frac{\partial}{\partial x^2}\right)\\
&\quad + B\left(\cos x^2 \frac{\partial}{\partial x^1} - \cot x^1 \sin x^2 \frac{\partial}{\partial x^2}\right) + C\frac{\partial}{\partial x^2}.
\end{aligned}$$

Here, A, B, and C are arbitrary constants of integration. Thus, there exist three linearly independent Killing vectors:

$$\vec{\mathbf{K}}_{\langle 1\rangle}(x) := \sin x^2 \frac{\partial}{\partial x^1} + \cot x^1 \cos x^2 \frac{\partial}{\partial x^2}\,,$$
$$\vec{\mathbf{K}}_{\langle 2\rangle}(x) := -\cos x^2 \frac{\partial}{\partial x^1} + \cot x^1 \sin x^2 \frac{\partial}{\partial x^2}\,,$$
$$\vec{\mathbf{K}}_{\langle 3\rangle}(x) := -\frac{\partial}{\partial x^2}\,.$$

(Multiplying the above by the imaginary number i yields the orbital angular momentum operators of quantum mechanics!)

In other words, it is said that S^2 admits a three-parameter (rotation) *group of motion*. □

Remarks: (i) Suppose that $\vec{\mathbf{K}}(x)$ is a Killing vector field in $D_s \subset D \subset \mathbb{R}^N$. Then, there exists a coordinate chart such that

$$\hat{K}^i(\hat{x}) = \delta^i{}_{(N)}, \quad \frac{\partial \hat{g}_{ij}(\hat{x})}{\partial \hat{x}^N} \equiv 0 \quad \text{for all} \quad \hat{x} \in \hat{D}_s \subset \hat{D} \subset \mathbb{R}^N. \tag{1.174}$$

(ii) An N-dimensional Riemannian or pseudo-Riemannian manifold can admit *at most* an $N(N+1)/2$-parameter group of motion. Such a space must be a space of constant curvature given by (1.164i).

(Proofs of the above remarks are available in the textbook by Eisenhart [90].)

We have discussed a parametrized curve $\mathcal{X}(t)$ in (1.19). Furthermore, we have defined a tensor field ${}^r_s\mathbf{T}(x)$ and its components in (1.30) and (1.31). Now, we shall investigate a tensor field *restricted* to the curve. We denote this by ${}^r_s\mathbf{T}(x)_{|\mathcal{X}(t)} \equiv {}^r_s\mathbf{T}(\mathcal{X}(t))$. The components of the $(r+s)$th order derived tensor field $\frac{D\,{}^r_s\mathbf{T}(\mathcal{X}(t))}{\mathrm{d}t}$ restricted to the curve are furnished by:

$$\begin{aligned}
\frac{D\,T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\mathcal{X}(t))}{\mathrm{d}t} &:= \left[\nabla_k T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}\right]_{|\mathcal{X}(t)} \frac{\mathrm{d}\mathcal{X}^k(t)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\,T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}}{\mathrm{d}t} + \left[\sum_{\alpha=1}^{r} \begin{Bmatrix} i_\alpha \\ k\ l \end{Bmatrix} T^{i_1,\ldots,i_{\alpha-1} l i_{\alpha+1},\ldots,i_r}{}_{j_1,\ldots,j_s}\right. \\
&\qquad \left. - \sum_{\beta=1}^{s} \begin{Bmatrix} l \\ k\ j_\beta \end{Bmatrix} T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_{\beta-1} l j_{\beta+1},\ldots,j_s}\right]_{|\mathcal{X}(t)} \frac{\mathrm{d}\mathcal{X}^k(t)}{\mathrm{d}t}.
\end{aligned} \tag{1.175}$$

(See [56, 90, 171, 244].)

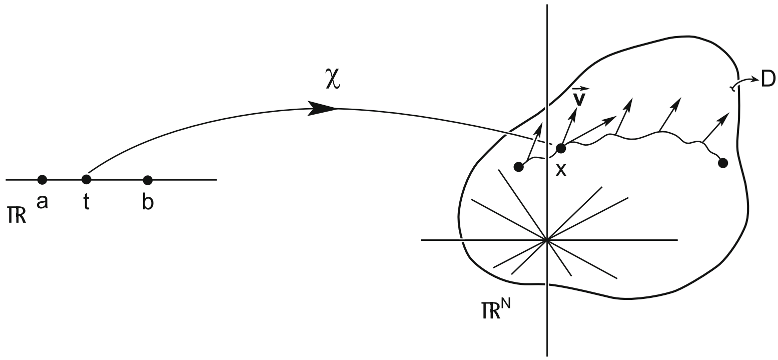

Fig. 1.11 Parallel propagation of a vector along a curve

Example 1.3.35. Equation (1.175) for a vector field restricted to a curve yields

$$\nabla_{\vec{\mathcal{X}}'}\left[\vec{\mathbf{V}}\,(\mathcal{X}(t))\right] = \frac{DV^i\,(\mathcal{X}(t))}{\mathrm{d}t}\left[\frac{\partial}{\partial x^i}\right]_{|\mathcal{X}(t)} = \frac{DV^{(a)}\,(\mathcal{X}(t))}{\mathrm{d}t}\vec{\mathbf{e}}_{(a)}\,(\mathcal{X}(t))\,, \tag{1.176i}$$

$$\frac{D\,V^i\,(\mathcal{X}(t))}{\mathrm{d}t} = \frac{\mathrm{d}\,V^i\,(\mathcal{X}(t))}{\mathrm{d}t} + \left[\begin{Bmatrix} i \\ j\;k \end{Bmatrix}\,V^k(x)\right]_{|\mathcal{X}(t)}\frac{\mathrm{d}\,\mathcal{X}^j(t)}{\mathrm{d}t}, \tag{1.176ii}$$

$$\frac{D\,V^{(a)}\,(\mathcal{X}(t))}{\mathrm{d}t} = \left[\partial_{(b)}V^{(a)}(x) - \gamma^{(a)}_{\ \ (c)(b)}(x)V^{(c)}(x)\right]_{|\mathcal{X}(t)}\mathcal{X}'^{(b)}(t). \tag{1.176iii}$$

If we choose the vector field as the tangent vector to the twice-differentiable parametrized curve, then $V^i\,(\mathcal{X}(t)) = \frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}$. Also, (1.176ii) implies

$$\frac{D\,V^i\,(\mathcal{X}(t))}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathcal{X}^i(t)}{\mathrm{d}t^2} + \begin{Bmatrix} i \\ j\;k \end{Bmatrix}_{|\mathcal{X}(t)}\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{X}^k(t)}{\mathrm{d}t}. \tag{1.177}$$

In case the manifold is the three-dimensional Euclidean space $\mathbb{E}_3$, and t is the time parameter, (1.177) denotes the *Newtonian acceleration* of a particle in a general coordinate chart. □

A vector field $\mathbf{V}(\mathcal{X}(t))$ along a curve $\mathcal{X}(t)$ is said to be *parallelly propagated* provided

$$\nabla_{\vec{\mathcal{X}}'}\left[\vec{\mathbf{V}}(\mathcal{X}(t))\right] = \vec{\mathbf{O}}(\mathcal{X}(t)), \tag{1.178i}$$

$$\frac{\mathrm{d}V^i(\mathcal{X}(t))}{\mathrm{d}t} + \begin{Bmatrix} i \\ j\;k \end{Bmatrix}_{|\mathcal{X}(t)}V^j(\mathcal{X}(t))\frac{\mathrm{d}\mathcal{X}^k(t)}{\mathrm{d}t} = 0. \tag{1.178ii}$$

(See Fig. 1.11.)

(*Remark:* The averaging of tensorial quantities at different space–time points requires the concept of parallel propagation between the points. (See [29].))

Suppose that a differentiable vector field $\vec{\mathbf{V}}(x)$ is defined in $D \subset \mathbb{R}^N$. Let there be a differentiable curve $\mathcal{X}$ with image Γ in D. Moreover, let the vector field $\vec{\mathbf{V}}(\mathcal{X}(t))$ restricted to the curve, be parallelly propagated over Γ. Then, by (1.178ii), we obtain the difference

$$\begin{aligned} V^i(\mathcal{X}(b)) - V^i(\mathcal{X}(a)) = & \int_a^b \frac{\mathrm{d}V^i(\mathcal{X}(t))}{\mathrm{d}t}\,\mathrm{d}t = \int_\Gamma \left[\partial_k V^i\right] \mathrm{d}x^k \\ & = -\int_\Gamma \begin{Bmatrix} i \\ k\ j \end{Bmatrix} V^j(x)\mathrm{d}x^k. \end{aligned} \tag{1.179}$$

In case Γ is a simple closed curve and it is the boundary of a star-shaped surface D_2^* interior to D, we can apply Stokes' Theorem 1.2.23. By (1.74), (1.179), and (1.141i), we obtain that the jump discontinuity

$$\begin{aligned} \left[\Delta V^i\right] &:= -\oint_\Gamma \begin{Bmatrix} i \\ k\ j \end{Bmatrix} V^j(x)\,\mathrm{d}x^k \\ &= (1/2)\int_{D_2^*} \left[\partial_k \left(\begin{Bmatrix} i \\ l\ j \end{Bmatrix} V^j\right) - \partial_l \left(\begin{Bmatrix} i \\ k\ j \end{Bmatrix} V^j\right)\right] \mathrm{d}x^l \wedge \mathrm{d}x^k \\ &= (1/2)\int_{D_2^*} \left[R^i{}_{jkl}(x) V^j(x)\right] \mathrm{d}x^l \wedge \mathrm{d}x^k. \end{aligned} \tag{1.180}$$

(See Fig. 1.12.)

Note that for a flat manifold, (1.180) yields $\left[\Delta V^i\right] \equiv 0$, or parallel transportation is *integrable*.

Figure 1.13 illustrates a subtle issue between parallel transport around a closed curve and flatness. Figure 1.13a represents a flat sheet, and transporting the vector around the closed curve shown yields trivial parallel transport. In a similar manner, if the sheet is folded into a cylinder as in Fig. 1.13b, transport around the cylinder (or any closed path) also yields trivial parallel transport, indicating that the cylinder is also intrinsically flat. If the sheet is folded into a cone as in Fig. 1.13c, the sheet is everywhere intrinsically flat, save for the apex. However, there are closed paths which will yield nontrivial parallel transport of the vector. Specifically, these are paths which enclose the apex. If we excise a section near the top of the cone (creating a truncated cone), we now have an everywhere flat manifold but vectors transported around the truncated region will still possess nontrivial parallel transport. This is

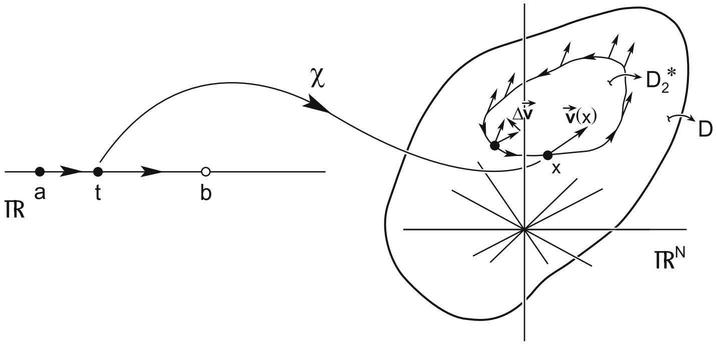

Fig. 1.12 Parallel transport along a closed curve

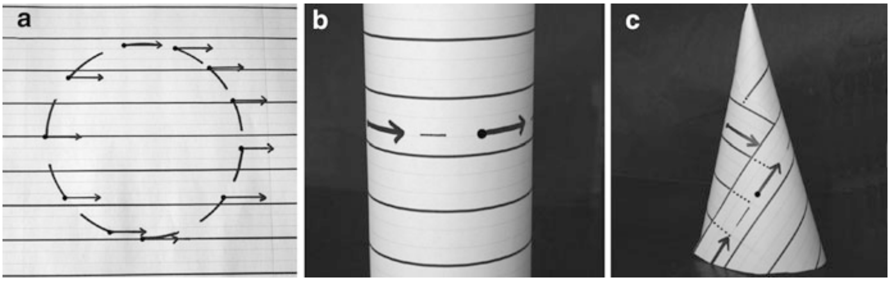

Fig. 1.13 Parallel transport along closed curves on several manifolds. Although all manifolds here are intrinsically flat, except for the apex of (**c**), the cone yields nontrivial parallel transport of the vector when it is transported around the curve shown, which encompasses the apex. The domain enclosed by a *curve* encircling the apex is non-star-shaped, and therefore, nontrivial parallel transport may be obtained even though the entire curve is located in regions where the manifold is flat

not in contradiction with the above arguments for flatness since the surface which is bound by this curve is not star-shaped, and therefore, the above argument does not apply. Paths on the cone which do not enclose the apex still yield trivial parallel transport.

A *geodesic* $\mathcal{X}$ is a nondegenerate, twice-differentiable curve into $D \subset \mathbb{R}^N$ such that

$$\nabla_{\vec{\mathcal{X}}'}\left[\vec{\boldsymbol{\mathcal{X}}}'(t)\right] = \Lambda(t)\vec{\boldsymbol{\mathcal{X}}}'(t), \tag{1.181i}$$

$$\frac{\mathrm{d}^2\mathcal{X}^i(t)}{\mathrm{d}t^2} + \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\}_{\big|\mathcal{X}(t)} \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^k(t)}{\mathrm{d}t} = \Lambda(t)\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t} \tag{1.181ii}$$

for some continuous function Λ over $[a,b] \subset \mathbb{R}$. The geometrical implication of (1.181i) is that the tangent vector along the curve is parallelly transported.

Equation (1.181ii) can be simplified by a reparametrization discussed in (1.23). Changing the notation, we express the reparametrization as

$$\tau = \mathcal{T}(t) := k_1 \left\{ \int \exp\left[\int \Lambda(t)\mathrm{d}t \right] \mathrm{d}t \right\} + e^{k_2},$$
$$x = \mathcal{X}(t) = \widehat{\mathcal{X}}(\tau). \tag{1.182}$$

Here, $k_1 \neq 0$ and k_2 are arbitrary constants. It can be proved that under the reparametrization, (1.181ii) reduces to

$$\frac{\mathrm{d}^2\widehat{\mathcal{X}}^i(\tau)}{\mathrm{d}\tau^2} + \left\{ {i \atop j\ k} \right\}_{|\hat{\mathcal{X}}(\tau)} \frac{\mathrm{d}\widehat{\mathcal{X}}^j(\tau)}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\widehat{\mathcal{X}}^k(\tau)}{\mathrm{d}\tau} = 0. \tag{1.183}$$

These are the *geodesic equations*.

Remark: The parameter τ in (1.182) is called an *affine parameter* along a geodesic. A nonhomogeneous linear transformation

$$\hat{\tau} = c_1\tau + c_2, \qquad c_1 \neq 0$$

produces another affine parameter.

Equation (1.183) admits the first integral

$$g_{ij}(\widehat{\mathcal{X}}(\tau)) \frac{\mathrm{d}\widehat{\mathcal{X}}^i(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\widehat{\mathcal{X}}^j(\tau)}{\mathrm{d}\tau} = k = \text{ const.} \tag{1.184}$$

In case $k = 0$, we call the curve a *null geodesic*. For a *nonnull geodesic*, we introduce another (affine) reparametrization by

$$s = \mathcal{S}(\tau) := \sqrt{|k|}\,(\tau),$$
$$x = \widehat{\mathcal{X}}(\tau) = \mathcal{X}^{\#}(s), \tag{1.185}$$

$$g_{ij}(\mathcal{X}^{\#}(s)) \frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s} \frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s} \equiv \text{ sgn}\,(k) = \pm 1. \tag{1.186}$$

The parameter s is called an *arc separation parameter*.

Example 1.3.36. Consider the Euclidean plane $\mathbb{E}_2$ and the polar coordinate chart. (See Example 1.1.1.) We have

$$\mathbf{g}_{..}(x) = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + (x^1)^2 \mathrm{d}x^2 \otimes \mathrm{d}x^2,$$

$$\left\{ {1 \atop 11} \right\} = \left\{ {1 \atop 12} \right\} = \left\{ {2 \atop 11} \right\} = \left\{ {2 \atop 22} \right\} = 0,$$

$$\left\{ {1 \atop 22} \right\} = -x^1, \quad \left\{ {2 \atop 12} \right\} = (x^1)^{-1}.$$

The geodesic equations (1.183) and (1.186) in arc separation parameter s yield

$$\frac{d^2\mathcal{X}^{\#1}(s)}{ds^2} - \mathcal{X}^{\#1}(s)\left[\frac{d\mathcal{X}^{\#2}(s)}{ds}\right]^2 = 0,$$

$$\frac{d}{ds}\left\{\left[\mathcal{X}^{\#1}(s)\right]^2 \frac{d\mathcal{X}^{\#2}(s)}{ds}\right\} = 0,$$

$$\left[\frac{d\mathcal{X}^{\#1}(s)}{ds}\right]^2 + \left[\mathcal{X}^{\#1}(s)\right]^2\left[\frac{d\mathcal{X}^{\#2}(s)}{ds}\right]^2 = 1.$$

The second of the above equations yields

$$\left[\mathcal{X}^{\#1}(s)\right]^2 \frac{d\mathcal{X}^{\#2}(s)}{ds} = h,$$

where h is an arbitrary constant of integration. (The above equation implies Kepler's law of the constancy of areal velocity). In case $h = 0$, the general solutions of the equations are furnished by

$$r \equiv x^1 = \mathcal{X}^{\#1}(s) = s + c_1,$$

$$\varphi \equiv x^2 = \mathcal{X}^{\#2}(s) = c_2.$$

Here, c_1 and c_2 are two arbitrary constants of integration. Note that the above geodesic, in the polar coordinate chart of $\mathbb{E}_2$, represents a portion of a radial straight line. In case $h \neq 0$, we can prove that geodesics are portions of straight lines *not passing through origin*. □

The geodesic equations (1.183) for a nonnull curve can be derived from the *Euler–Lagrange variational equations* [171]:

$$\left.\frac{\partial L(x,u)}{\partial x^k}\right|_{\substack{x=\mathcal{X}^{\#}(s)\\ u=\frac{d\mathcal{X}^{\#}(s)}{ds}}} - \frac{d}{ds}\left\{\left[\frac{\partial L(..)}{\partial u^k}\right]_{|..}\right\} = 0,$$

$$L(x,u) := \sqrt{|g_{ij}(x)u^i u^j|} > 0. \tag{1.187}$$

Therefore, along a geodesic curve, the total arc separation attains any of the minimum, or maximum, or stationary value. (See Appendix 1.)

Now we shall derive the equations for the geodesic deviation. Consider a two-dimensional parametric surface ξ into $D \subset \mathbb{R}^N$ furnished by

$$x = \xi(\tau, v),$$

$$\xi \in C^3(\mathcal{D} \in \mathbb{R}^2; \mathbb{R}^N), \quad x^i = \xi^i(\tau, v),$$

$$\text{Rank}\left[\frac{\partial \xi^i}{\partial \tau}, \frac{\partial \xi^i}{\partial v}\right] = 2. \tag{1.188}$$

(See Fig. 1.14.)

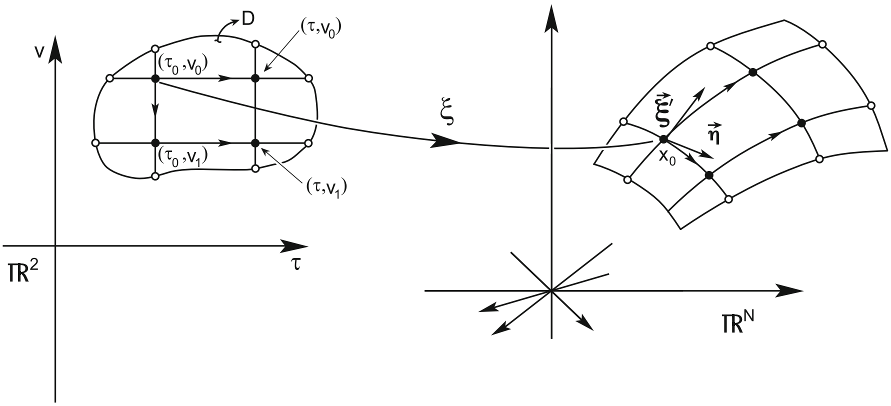

Fig. 1.14 Two-dimensional surface generated by geodesics

Let a typical parametric curve $\xi(\tau, v_0)$ be *a geodesic* in $D \subset \mathbb{R}^N$, with τ as an *affine parameter*. Therefore, we have

$$\xi'^i(\tau, v) := \frac{\partial \xi^i(\tau, v)}{\partial \tau},$$
$$\frac{D\xi'^i(\tau, v)}{\partial \tau} = \frac{\partial}{\partial \tau}\, \xi'^i(\tau, v) + \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\}_{\big|\xi(\tau,v)} \xi'^j(\tau, v)\, \xi'^k(\tau, v) = 0. \tag{1.189}$$

Let us consider the other coordinate curve $x = \xi(\tau_0, v)$. This curve is non-degenerate, differentiable, and *need not be a geodesic*. The tangent vector components along $\xi(\tau_0, v)$ is given by

$$\eta^i(\tau, v) := \frac{\partial \xi^i(\tau, v)}{\partial v}, \tag{1.190i}$$

$$\xi^i(\tau_o, v_1) - \xi^i(\tau_0, v_0) = (v_1 - v_0)\, \eta^i(\tau_0, v_0) + \frac{1}{2}\left[\frac{\partial \eta^i}{\partial v}\right]_{\big|v^\#} (v_1 - v_0)^2,$$
$$v^\# := v_0 + \theta(v_1 - v_0), \quad 0 < \theta < 1. \tag{1.190ii}$$

Therefore, $|v_1 - v_0| \cdot \sigma\left(\vec{\boldsymbol{\eta}}(\tau_0, v_0)\right)$ is the first approximation of the separation between two neighboring geodesics $\xi(\tau, v_0)$ and $\xi(\tau, v_1)$ at τ_0. The vector field $(v_1 - v_0)\, \frac{D\vec{\boldsymbol{\eta}}(\tau,v_0)}{\partial \tau}$ is the approximate *relative velocity* between two neighboring geodesics. Similarly, the approximate *relative acceleration* between two geodesics is given by $(v_1 - v_0)\, \frac{D^2\vec{\boldsymbol{\eta}}(\tau,v)}{\partial \tau^2}$. Now, we shall furnish *an exact equation* for $\frac{D^2\eta^i(\tau,v)}{\partial \tau^2}$ and it is given by

$$\frac{D^2 \eta^i(\tau, v)}{\partial \tau^2}\bigg|_{\xi(\tau,v)} + R^i{}_{ljk}(\xi(\tau, v))\, \xi'^l(\tau, v)\, \eta^j(\tau, v)\, \xi'^k(\tau, v) = 0. \qquad (1.191)$$

Remark: The above equations are called the *geodesic deviation equations*. (For the proof of (1.191), see textbooks [56, 244] in the references.)

Example 1.3.37. Consider the surface S^2 of the unit sphere embedded in $\mathbb{R}^3$. (See Examples 1.1.2, 1.3.3, 1.3.18, and 1.3.34.)

The metric tensor, the nonzero Christoffel symbols, and the nonzero Riemann–Christoffel tensor components are provided by:

$$\begin{aligned}
\mathbf{g}_{..}(x) &= \mathrm{d}x^1 \otimes \mathrm{d}x^1 + (\sin x^1)^2\, \mathrm{d}x^2 \otimes \mathrm{d}x^2 \\
&\equiv \mathrm{d}\theta \otimes \mathrm{d}\theta + (\sin\theta)^2\, \mathrm{d}\varphi \otimes \mathrm{d}\varphi, \\
\left\{\begin{matrix} 1 \\ 22 \end{matrix}\right\} &= \sin\theta \cdot \cos\theta, \quad \left\{\begin{matrix} 2 \\ 12 \end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 21 \end{matrix}\right\} = \cot\theta, \\
R^1{}_{212} &\equiv -R^1{}_{221} = \sin^2\theta, \\
R^2{}_{112} &\equiv -R^2{}_{121} \equiv -1.
\end{aligned}$$

Consider the one-parameter family of geodesic congruence (or great circles) provided by the longitudes as

$$\begin{aligned}
\xi(\theta, \varphi) &:= (\theta, \varphi), \quad (\theta, \varphi) \in (0, \pi) \times (-\pi, \pi) \subset \mathbb{R}^2; \\
\xi'^i(\theta, \varphi) &= \delta^i{}_{(1)}, \quad \eta^i(\theta, \varphi) = \delta^i{}_{(2)},
\end{aligned}$$

$$\begin{aligned}
g_{ij}(\xi(\theta,\varphi))\xi'^i(\theta,\varphi)\xi'^j(\theta,\varphi) &\equiv 1, \quad g_{ij}(\xi(\theta,\varphi))\eta^i(\theta,\varphi)\eta^j(\theta,\varphi) \equiv \sin^2\theta > 0, \\
g_{ij}(\xi(\theta,\varphi))\xi'^i(\theta,\varphi)\eta^j(\theta,\varphi) &\equiv 0.
\end{aligned}$$

(See Fig. 1.15.)

The geodesic deviation (1.191) reduces to

$$\begin{aligned}
\frac{D^2 \delta^i{}_{(2)}}{\partial\theta^2} + R^i{}_{121}(\theta, \varphi) &= [\partial_1(\cot\theta)]\, \delta^i{}_{(2)} + \left\{\begin{matrix} i \\ 1\,k \end{matrix}\right\} \delta^k{}_{(2)} \cot\theta + R^2{}_{121}\delta^i{}_{(2)} \\
&= \delta^i{}_{(2)} \left[-\mathrm{cosec}^2\theta + \cot^2\theta + 1\right] \equiv 0.
\end{aligned}$$

Thus, the geodesic deviation equations are identically satisfied. Note that in this example,

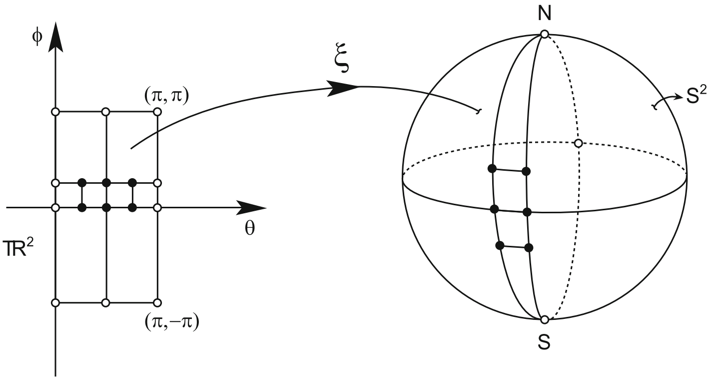

Fig. 1.15 Geodesic deviation between two neighboring longitudes

$$\| \vec{\boldsymbol{\eta}}(\theta, \varphi) \| := \sigma(\vec{\boldsymbol{\eta}}(\theta, \varphi)) = \sin\theta,$$

$$\lim_{\theta \to 0_+} \| \vec{\boldsymbol{\eta}}(\theta, \varphi) \| = 0 = \lim_{\theta \to \pi_-} \| \vec{\boldsymbol{\eta}}(\theta, \varphi) \| .$$

Longitudinal geodesics *do meet* at the north and south poles (which are *not* covered by the coordinate chart). These two points are called *conjugate points* on a geodesic $\xi(\theta, \varphi_0)$. □

Geodesics are "straightest curves" in a differentiable manifold. Now we shall deal with differentiable curves which are not necessarily geodesic. A *nonnull differentiable curve* is characterized by:

$$\sigma\left(\vec{\mathcal{X}}'(t)\right) = \sqrt{\left|\mathbf{g}_{..}(\mathcal{X}(t))\left[\vec{\mathcal{X}}'(t), \vec{\mathcal{X}}'(t)\right]\right|} > 0. \tag{1.192}$$

We shall reparametrize the curve according to (1.23) and (1.24) as

$$\begin{aligned}
s = \mathcal{S}(t) &:= \int_a^t \sigma\left(\vec{\mathcal{X}}'(u)\right) \mathrm{d}u, \\
\frac{\mathrm{d}\mathcal{S}(t)}{\mathrm{d}t} &= \sqrt{\left| g_{ij}(\mathcal{X}(t)) \frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t} \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} \right|}, \\
\sigma\left(\vec{\mathcal{X}}^{\#\prime}(s)\right) &= \sqrt{\left| g_{ij}(\mathcal{X}^{\#}(s)) \frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s} \frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s} \right|} \\
&\equiv 1.
\end{aligned} \tag{1.193}$$

By (1.186), it is clear that s is an arc separation parameter. (In case of a positive-definite metric, s is called an *arc length parameter*.)

Also, by (1.185) and (1.186), we can express

$$\left[\frac{\mathrm{d}\mathcal{S}(t)}{\mathrm{d}t}\right]^2 = \left|g_{ij}(\mathcal{X}(t))\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\right| > 0. \tag{1.194}$$

A popular way to write the above equation is

$$\mathrm{d}s^2 = g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j. \tag{1.195}$$

Mathematically, (1.195) is somewhat *dubious*. However, for popular demand, we shall keep on using (1.195) in the sequel.

In this chapter, we shall usually use the parameter s for a nonnull curve and drop # from $\mathcal{X}^{\#}$. (However, for a *null curve*, we use an affine parameter α and drop ^ from $\hat{\mathcal{X}}(\alpha)$.)

Now we shall state an important theorem regarding a nonnull differentiable curve.

Theorem 1.3.38. *Let $\mathcal{X}$ be a parametrized, nonnull curve of differentiability class $C^{N+1}([s_1, s_2], \mathbb{R}^N)$. Moreover, let the following finite sequence of N non-null vector fields, recursively defined, exist along the curve:*

$$\begin{aligned}
\lambda^i{}_{(1)}(s) &:= \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s},\\
\kappa_{(A)}(s)\lambda^i{}_{(A+1)}(s) &:= \frac{D\lambda^i{}_{(A)}(s)}{\mathrm{d}s} + d_{(A-1)(A-1)}\cdot d_{(A)(A)}\kappa_{(A-1)}(s)\lambda^i{}_{(A-1)}(s),\\
\kappa_{(0)}(s) = \kappa_{(N)}(s) &\equiv 0,\\
d_{(A)(A)} &= \pm 1, \quad A \in \{1, 2, \ldots, N\}.
\end{aligned} \tag{1.196}$$

Then, the set of vector fields $\left\{\vec{\boldsymbol{\lambda}}_{(A)}(s)\right\}_1^N$ is an orthonormal basis set along the curve. (For the proof, consult textbooks [56, 244] in the references.)

Remarks:

(i) The equations (1.196) are called the *generalized Frenet-Serret formulas*.

(ii) In case any of the vectors $\frac{D\vec{\boldsymbol{\lambda}}_{(A)}(s)}{\mathrm{d}s} + d_{(A-1)(A-1)}d_{(A)(A)}\kappa_{(A)}\vec{\boldsymbol{\lambda}}_{(A-1)}(s)$ becomes null, the procedure breaks down.
(For a geodesic curve, the above expression is zero for $A = 1$, implying $\kappa_{(1)}(s) \equiv 0$.))

(iii) The subscript (A) is not summed.

Example 1.3.39. Let us choose the Euclidean space $\mathbb{E}_3$ and a Cartesian coordinate chart. We use the alternate notations

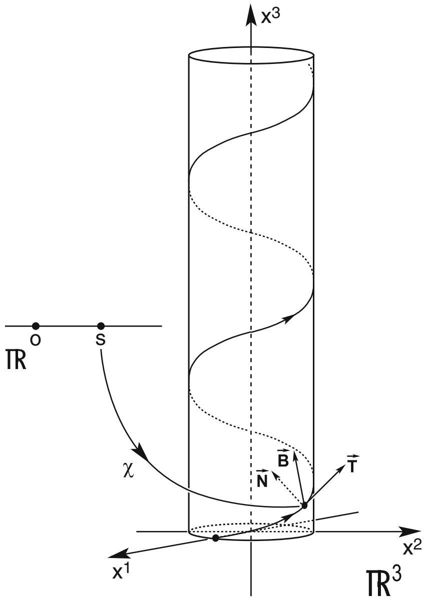

Fig. 1.16 A circular helix in $\mathbb{R}^3$

$$\begin{aligned} T^i(s) &:= \lambda^i{}_{(1)}(s), \quad N^i(s) := \lambda^i{}_{(2)}(s), \\ B^i(s) &:= \lambda^i{}_{(3)}(s), \\ \kappa(s) &:= \kappa_{(1)}(s), \quad \tau(s) := \kappa_{(2)}(s). \end{aligned}$$

The generalized Frenet-Serret formulas (1.196) yield

$$\begin{aligned} \frac{\mathrm{d}T^i(s)}{\mathrm{d}s} &= \kappa(s)\, N^i(s), \\ \frac{\mathrm{d}N^i(s)}{\mathrm{d}s} &= -\,\kappa(s)\, T^i(s) + \tau(s)\, B^i(s), \\ \frac{\mathrm{d}B^i(s)}{\mathrm{d}s} &= -\,\tau(s)\, N^i(s). \end{aligned} \tag{1.197}$$

(These are the usual *Frenet-Serret formulas*.)

Let us consider a circular helix in space furnished by

$$\begin{aligned} \mathcal{X}(s) &:= \left(\cos\left(s/\sqrt{2}\right), \sin\left(s/\sqrt{2}\right), s/\sqrt{2}\right), \\ & s \in [0, \infty). \end{aligned}$$

(See Fig. 1.16.)

In this case, we compute explicitly as

$$\vec{\mathbf{T}}(s) = -\frac{1}{\sqrt{2}}\sin(s/\sqrt{2})\left(\frac{\partial}{\partial x^1}\right)_{|\chi(s)} + \frac{1}{\sqrt{2}}\cos(s/\sqrt{2})\left(\frac{\partial}{\partial x^2}\right)_{|..} + \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^3}\right)_{|..},$$

$$\vec{\mathbf{N}}(s) = -\cos(s/\sqrt{2})\left(\frac{\partial}{\partial x^1}\right)_{|..} - \sin(s/\sqrt{2})\left(\frac{\partial}{\partial x^2}\right)_{|..},$$

$$\vec{\mathbf{B}}(s) = \frac{1}{\sqrt{2}}\sin(s/\sqrt{2})\left(\frac{\partial}{\partial x^1}\right)_{|\chi(s)} - \frac{1}{\sqrt{2}}\cos(s/\sqrt{2})\left(\frac{\partial}{\partial x^2}\right)_{|..} + \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^3}\right)_{|..},$$

$$\kappa(s) \equiv \left(\frac{1}{2}\right), \quad \tau(s) \equiv \left(\frac{1}{2}\right). \tag{1.198}$$

□

(See [95, 197, 266].)

Exercises 1.3

1. Compute the Christoffel symbols of the second kind and Ricci rotation coefficients for the two-dimensional metric of constant curvature

$$\mathbf{g}_{..}(x) = (x^1 + x^2)^{-2}\,\mathrm{d}x^1 \otimes \mathrm{d}x^2,$$

where the domain of validity $D := \{(x^1, x^2) : x^1 + x^2 > 0\}$.

2. Let the components $R^{(a)}{}_{ijk}(x) := \mathbf{R}^{\cdot}{}_{...}(x)\left[\tilde{\mathbf{e}}^{(a)}(x), \partial_i, \partial_j, \partial_k\right]$. Express $R^{(a)}{}_{ijk}(x)$ in terms of Christoffel symbols, Ricci rotation coefficients, and their first derivatives.

3. The *Laplacian operator* is defined by the second-order differential operator

$$\Delta := g^{ij}(x)\nabla_i\nabla_j =: \nabla^i\nabla_i = \nabla^{(a)}\nabla_{(a)}.$$

Prove that for a twice-differentiable scalar field $\phi(x)$,

$$\begin{aligned}\Delta\phi &= \left(1/\sqrt{|g|}\right)\partial_i\left[\sqrt{|g|}\,g^{ij}\,\partial_j\phi\right] \\ &= d^{(a)(b)}\left[\partial_{(b)}\partial_{(a)}\phi + \gamma^{(c)}{}_{(a)(b)}(x)\partial_{(c)}\phi\right].\end{aligned}$$

Remark: In the case the manifold is Riemannian, $\Delta =: \nabla^2$. Moreover, in the case where the metric is Lorentzian (possesses Lorentz signature), $\Delta =: \Box$.)

4. Prove that in a regular domain of a two-dimensional pseudo-Riemannian manifold

$$G_{ij}(x) \equiv 0.$$

5. The *double Hodge-dual* of Riemann–Christoffel curvature tensor in four dimensions is defined by

$$**R^{ijkl}(x) := (1/4)\,\eta^{ijmn}(x) R_{mnpq}(x)\,\eta^{pqkl}(x).$$

Prove that

$$**R^{i}{}_{jki}(x) = G_{jk}(x).$$

6. Suppose that the metric tensor field $\mathbf{g}_{..}(x)$ is of class C^4 in a domain $D \subset \mathbb{R}^N$. Show that the following second-order differential identities hold in D:

$$\begin{aligned}\nabla_i \nabla_j R_{klmn} - \nabla_j \nabla_i R_{klmn} + \nabla_l \nabla_k R_{mnji} - \nabla_k \nabla_l R_{mnji}\\ + \nabla_m \nabla_n R_{ijkl} - \nabla_n \nabla_m R_{ijkl} \equiv 0.\end{aligned}$$

7. Prove that for differentiable Ricci coefficients, the following differential identities hold:

$$\begin{aligned}&\partial_{(c)}\left(\gamma^{(c)}{}_{(a)(b)} - \gamma^{(c)}{}_{(b)(a)}\right) + \partial_{(a)}\,\gamma^{(c)}{}_{(b)(c)}\\ &- \partial_{(b)}\,\gamma^{(c)}{}_{(a)(c)} + \gamma^{(c)}{}_{(d)(c)}\left(\gamma^{(d)}{}_{(b)(a)} - \gamma^{(d)}{}_{(a)(b)}\right)\\ &+ \gamma^{(c)}{}_{(a)(d)}\,\gamma^{(d)}{}_{(b)(c)} - \gamma^{(c)}{}_{(b)(d)}\,\gamma^{(d)}{}_{(a)(c)} \equiv 0.\end{aligned}$$

8. Show that the *arc separation function*

$$\sum(\mathcal{X}) := \int_{t_1}^{t_2} \sqrt{\left| g_{ij}(\mathcal{X}(t)) \frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t} \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} \right|}\, \mathrm{d}t$$

for a differentiable curve $\mathcal{X}$ is *invariant under a reparametrization*.

9. Consider the Euclidean space $\mathbb{E}_3$ and a Cartesian coordinate chart. Let $\mathcal{X}$ be a twice-differentiable curve from the arc length parameter s into $\mathbb{R}^3$. Assuming $\kappa(s) := \kappa_{(1)}(s) > 0$, prove that $\mathcal{X}$ is a plane curve if and only if $\tau(s) := \kappa_{(2)}(s) \equiv 0$.

10. Consider a locally flat manifold and a Cartesian or a pseudo-Cartesian chart with $\mathbf{R}^{\cdot}{}_{\cdots}(x) \equiv \mathbf{0}^{\cdot}{}_{\cdots}(x)$. A one-parameter family of geodesics congruence spanning a *ruled 2-surface* is provided by

$$x^i = \xi^i(\tau, v) = \tau F^i(v) + G^i(v).$$

Here, $F^i(v)$ and $G^i(v)$ are differentiable. Prove that the geodesic deviation equations (1.191) are identically satisfied.

11. Let a twice-differentiable geodesic $\mathcal{X}$ in terms of an affine parameter τ be expressed as

$$x = \mathcal{X}(\tau) \in D \subset \mathbb{R}^N,$$
$$x^i = \mathcal{X}^i(\tau),$$
$$\tau \in [\tau_1, \tau_2] \subset \mathbb{R}.$$

Synge's world function Ω is defined by

$$\Omega(x_2, x_1) \equiv \Omega\,(\mathcal{X}(\tau_2), \mathcal{X}(\tau_1))$$
$$:= (1/2)(\tau_2 - \tau_1) \int_{\tau_1}^{\tau_2} g_{ij}(\mathcal{X}(\tau)) \frac{\mathrm{d}\mathcal{X}^i(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\mathcal{X}^j(\tau)}{\mathrm{d}\tau}\, \mathrm{d}\tau.$$

Prove that for a flat manifold with the metric $g_{ij}(x) = d_{ij}$,

$$\frac{\partial \Omega(x_2, x_1)}{\partial x_1^i} = -d_{ij}\left(x_2^j - x_1^j\right).$$

12. Show that the *Kretschmann invariant* for a space of constant curvature K_0 is given by

$$R_{ijkl}(x)\, R^{ijkl}(x) = R_{(a)(b)(c)(d)}(x)\, R^{(a)(b)(c)(d)}(x)$$
$$= 2N(N-1)(K_0)^2.$$

13. Prove that in a regular domain of a three-dimensional manifold, the conformal tensor $\mathbf{C}^{\cdot}_{\ \cdots}(x) \equiv \mathbf{0}^{\cdot}_{\ \cdots}(x)$.
14. Show that the number of linearly independent components of the conformal tensor $C^i{}_{jkl}(x)$ for $N \geq 3$ is $\frac{N^2(N^2-1)}{12} - \frac{N(N+1)}{2}$.
15. (i) Show that

$$\nabla_{(e)}\left[d_{(a)(b)} \cdot d_{(c)(d)}\right] \equiv 0.$$

(ii) Prove by explicit computation that

$$\partial_{(a)}\partial_{(b)} f = \nabla_{(a)}\nabla_{(b)} f - \gamma^{(d)}{}_{(b)(a)}(\cdot\cdot) \cdot \partial_{(d)} f.$$

(iii) Show that under a general Lorentz transformation (1.103), the Ricci rotation coefficients undergo the following transformations:

$$\widehat{\gamma}^{(c)}{}_{(a)(b)}(x) = A^{(c)}{}_{(r)}(x) \cdot L^{(p)}{}_{(a)}(x) \cdot L^{(q)}{}_{(b)}(x) \cdot \gamma^{(r)}{}_{(p)(q)}(x)$$
$$+ L^{(r)}{}_{(b)}(x) \cdot \widehat{\vec{\mathbf{e}}}_{(a)}(x)\left[A^{(c)}{}_{(r)}(x)\right].$$

(iv) Prove the particular Ricci identity:

$$\left[\nabla_{(c)}\nabla_{(d)} - \nabla_{(d)}\nabla_{(c)}\right] T_{(b)} = -R^{(a)}{}_{(b)(c)(d)} \cdot T_{(a)},$$

for an arbitrary differentiable covector field $T_{(a)}$.

Answers and Hints to Selected Exercises

5. Use (1.145i).
6. Use Ricci identities (1.145i,ii).
7. Identities follow from (1.147iii).
8. Use (1.23) and (1.24).
11. $\Omega(x_2, x_1) = (1/2)\, d_{ij}(x_2^i - x_1^i)(x_2^j - x_1^j)$.
(For more applications of the world function, consult [243].)
13. Use an orthogonal coordinate chart (*suspending summation convention*) to express

$$\mathbf{g}_{..}(x) = g_{11}(x)\, \mathrm{d}x^1 \otimes \mathrm{d}x^1 + g_{22}(x)\, \mathrm{d}x^2 \otimes \mathrm{d}x^2 + g_{33}(x)\, \mathrm{d}x^3 \otimes \mathrm{d}x^3.$$

Obtain for *distinct indices* i, j, k,

$$R_{ii} = \frac{1}{g_{jj}} R_{ijji} + \frac{1}{g_{kk}} R_{ikki},$$

$$R_{ij} = \frac{1}{g_{kk}} R_{ikkj},$$

$$R = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{g_{ii}\, g_{jj}} R_{ijji}.$$

Use the definition in (1.169i) for $N = 3$.
15. (ii) By (1.124ii), it follows that

$$\nabla_{(a)}\left[\nabla_{(b)} f\right] = \partial_{(a)}\left[\nabla_{(b)} f\right] + \gamma^{(d)}{}_{(b)(a)}(\cdot\cdot) \cdot \left[\nabla_{(d)} f\right].$$

(iii) Use (1.104), (1.107ii), and 1.116ii to obtain

$$\widehat{\gamma}^{(c)}{}_{(a)(b)(c)}(x) = -\left\{\nabla_{\widehat{\vec{\mathbf{e}}}_{(c)}}(x)\left[\widehat{\vec{\mathbf{e}}}_{(b)}(x)\right]\right\} \cdot \left[\widehat{\vec{\mathbf{e}}}_{(a)}(x)\right].$$

(*Remark*: Note that in general, the transformation under consideration is not tensorial.)

(iv) Using (1.124ii), (1.139iv), and (1.141iii), the L.H.S. of the equation yields:

$$\begin{aligned}
&\left\{\partial_{(c)}\left[\nabla_{(d)}T_{(b)}\right] + \gamma^{(a)}{}_{(d)(c)}\cdot\left[\nabla_{(a)}T_{(b)}\right] + \gamma^{(a)}{}_{(b)(c)}\cdot\left[\nabla_{(d)}T_{(a)}\right]\right\}\\
&\quad - \left\{\partial_{(d)}\left[\nabla_{(c)}T_{(b)}\right] + \gamma^{(a)}{}_{(c)(d)}\cdot\left[\nabla_{(a)}T_{(b)}\right] + \gamma^{(a)}{}_{(b)(d)}\cdot\left[\nabla_{(c)}T_{(a)}\right]\right\}\\
&= \left\{\left[\partial_{(c)}\partial_{(d)}T_{(b)} + \left(\partial_{(c)}\gamma^{(e)}{}_{(b)(d)}\right)\cdot T_{(e)} + \gamma^{(a)}{}_{(b)(d)}\cdot\partial_{(c)}T_{(a)}\right]\right.\\
&\qquad + \left[\gamma^{(a)}{}_{(d)(c)}\cdot\partial_{(a)}T_{(b)} + \gamma^{(a)}{}_{(d)(c)}\gamma^{(e)}{}_{(b)(a)}\cdot T_{(e)}\right]\\
&\qquad \left. + \left[\gamma^{(a)}{}_{(b)(c)}\cdot\partial_{(d)}T_{(a)} + \gamma^{(a)}{}_{(b)(c)}\gamma^{(e)}{}_{(a)(d)}\cdot T_{(e)}\right]\right\}\\
&\qquad - [c \leftrightarrow d]\,.\\
&= \left\{\partial_{(c)}\gamma^{(e)}{}_{(b)(d)} - \partial_{(d)}\gamma^{(e)}{}_{(b)(c)} + \left(\gamma^{(a)}{}_{(d)(c)} - \gamma^{(a)}{}_{(c)(d)}\right)\cdot\gamma^{(e)}{}_{(b)(a)}\right.\\
&\qquad \left. + \gamma^{(a)}{}_{(b)(c)}\cdot\gamma^{(e)}{}_{(a)(d)} - \gamma^{(a)}{}_{(b)(d)}\cdot\gamma^{(e)}{}_{(a)(c)}\right\}\cdot T_{(e)} + 0\\
&= -R^{(e)}{}_{(b)(c)(d)}\cdot T_{(e)}.
\end{aligned}$$

(Here, and throughout this book, the notation $[c \leftrightarrow d]$ or $\{c \leftrightarrow d\}$ represents the previous term but with the indices c and d interchanged.)

1.4 Extrinsic Curvature

Consider a two-dimensional surface $\sum_2$ embedded in the three-dimensional Euclidean space $\mathbb{E}_3$. We denote the nondegenerate mapping ξ of class C^3 which has the image $\sum_2$. Let ξ be furnished in a Cartesian coordinate chart by the equations

$$\begin{aligned}
&x = \xi(u) \in \mathbb{R}^3,\\
&x^i = \xi^i(u) \equiv \xi^i(u^1, u^2), \quad (u^1, u^2) \in \mathcal{D}_2 \subset \mathbb{R}^2\,;\\
&\partial_\mu \xi^i := \frac{\partial \xi^i(u^1, u^2)}{\partial u^\mu}\,,\\
&\text{Rank}\left[\partial_\mu \xi^i\right] = 2\,;\\
&\mu \in \{1, 2\}, \quad i \in \{1, 2, 3\}.
\end{aligned} \tag{1.199}$$

The mapping ξ is called the *parametric surface*. (See [56, 95].)

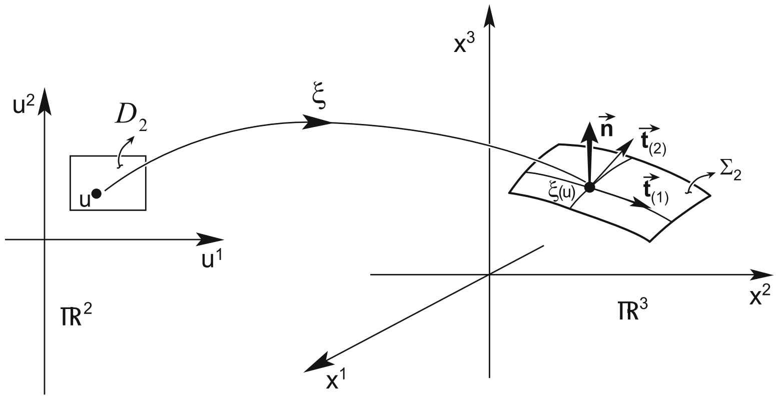

Fig. 1.17 A two-dimensional surface $\sum_2$ embedded in $\mathbb{R}^3$

The three-dimensional vectors

$$\vec{\mathbf{t}}_{(\mu)}(\xi(u)) := \partial_\mu \xi^i \frac{\partial}{\partial x^i}_{|\xi(u)} \tag{1.200}$$

are *tangential to the coordinate curves* on $\sum_2$. (See Fig. 1.17.)

These two vectors span a two-dimensional vector *subspace* of $T_{\xi(u)}(\mathbb{R}^3)$. It is isomorphic to the intrinsic tangent planes $T_{\xi(u)}\left(\sum_2\right)$ and $T_u(\mathbb{R}^2)$. Any two vectors $\vec{\mathbf{V}}(\xi(u))$ and $\vec{\mathbf{W}}(\xi(u))$ in the two-dimensional vector subspace can be expressed by the linear combinations

$$\vec{\mathbf{V}}(\xi(u)) = \left[v^\mu(u)\partial_\mu \xi^i\right] \frac{\partial}{\partial x^i}_{|\xi(u)},$$

$$\vec{\mathbf{W}}(\xi(u)) = \left[w^\mu(u)\partial_\mu \xi^i\right] \frac{\partial}{\partial x^i}_{|\xi(u)}. \tag{1.201}$$

Now, the inner product for three-dimensional vectors is provided by

$$\mathbf{g}_{..}(x)\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] \equiv \mathbf{I}_{..}(x)\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] := \delta_{ij}. \tag{1.202}$$

By (1.200), (1.201), and (1.202), we derive that

$$\mathbf{I}_{..}(\xi(u))\left(\vec{\mathbf{t}}_{(\mu)}(\xi(u)), \vec{\mathbf{t}}_{(\nu)}(\xi(u))\right) = \delta_{ij}(\partial_\mu \xi^i)(\partial_\nu \xi^j) =: \overline{g}_{\mu\nu}(u), \tag{1.203}$$

$$\mathbf{I}_{..}(\xi(u))\left(\vec{\mathbf{V}}(\xi(u)), \vec{\mathbf{W}}(\xi(u))\right) = \overline{g}_{\mu\nu}(u)\, v^\mu(u)\, w^\nu(u). \tag{1.204}$$

From the above equations, we identify

$$\overline{\mathbf{g}}_{..}(u) := \overline{g}_{\mu\nu}(u)\, \mathrm{d}u^{\mu} \otimes \mathrm{d}u^{\nu} \tag{1.205}$$

as the *intrinsic metric* for the (image) surface $\sum_2$.

We can express the unit outer normal vector $\vec{\mathbf{n}}(\xi(u))$ (using the cross product) as

$$\begin{aligned}
\vec{\mathbf{n}}(\xi(u)) &= \left[\vec{\mathbf{t}}_{(1)}(\xi(u)) \times \vec{\mathbf{t}}_{(2)}(\xi(u))\right] \Big/ \parallel \vec{\mathbf{t}}_{(1)}(\xi(u)) \times \vec{\mathbf{t}}_{(2)}(\xi(u)) \parallel \\
&= \left[\vec{\mathbf{t}}_{(1)}(\xi(u)) \times \vec{\mathbf{t}}_{(2)}(\xi(u))\right] \Big/ \sqrt{\overline{g}(u)}, \\
\overline{g}(u) &:= \det\left[\overline{g}_{\mu\nu}(u)\right] > 0, \\
\left[\vec{\mathbf{t}}_{(1)}(\cdot\cdot) \times \vec{\mathbf{t}}_{(2)}(\cdot\cdot)\right]^{k} &= \delta^{kj} \cdot \varepsilon_{jil} \cdot \partial_1 \xi^{i} \cdot \partial_2 \xi^{l}.
\end{aligned} \tag{1.206}$$

(See [95].)

Now, consider the rate of change of vector fields $\vec{\mathbf{t}}_{(\mu)}(\xi(u))$ along the coordinate curves on $\sum_2$. For that purpose, we define vector fields

$$\vec{\mathbf{t}}_{(\mu\nu)}(\xi(u)) := (\partial_\mu \partial_\nu \xi^{i})\, \frac{\partial}{\partial x^{i}}_{|\xi(u)} \equiv \vec{\mathbf{t}}_{(\nu\mu)}(\xi(u)). \tag{1.207}$$

Since $\left\{\vec{\mathbf{t}}_{(1)}(\xi(u)), \vec{\mathbf{t}}_{(2)}(\xi(u)), \vec{\mathbf{n}}(\xi(u))\right\}$ is a basis set for the vector space $T_{\xi(u)}(\mathbb{R}^3)$, we can express the vector field $\vec{\mathbf{t}}_{(\mu\nu)}(\xi(u))$ as a linear combination:

$$\vec{\mathbf{t}}_{(\mu\nu)}(\xi(u)) = C^{\lambda}{}_{\mu\nu}(u)\, \vec{\mathbf{t}}_{(\lambda)}(\xi(u)) + K_{\mu\nu}(u)\, \vec{\mathbf{n}}(\xi(u)). \tag{1.208}$$

Here, $C^{\lambda}{}_{\mu\nu}(u)$ and $K_{\mu\nu}(u)$ are some suitable coefficients.

By (1.207), we can prove the symmetries

$$\begin{aligned}
C^{\lambda}{}_{\nu\mu}(u) &\equiv C^{\lambda}{}_{\mu\nu}(u), \\
K_{\nu\mu}(u) &\equiv K_{\mu\nu}(u).
\end{aligned} \tag{1.209}$$

Let us try to determine coefficients $C^{\lambda}{}_{\mu\nu}(u)$ and $K_{\mu\nu}(u)$ in terms of known quantities of $\sum_2$. It can be proved that

$$C^{\lambda}{}_{\mu\nu}(u) = \overline{\begin{Bmatrix} \lambda \\ \mu\ \nu \end{Bmatrix}} = (1/2)\, \overline{g}^{\lambda\sigma} \left[\partial_\mu \overline{g}_{\nu\sigma} + \partial_\nu \overline{g}_{\sigma\mu} - \partial_\sigma \overline{g}_{\mu\nu}\right]. \tag{1.210}$$

(See [56, 95].)

Moreover, (1.207) and (1.208) yield

$$\begin{aligned} K_{\mu\nu}(u) &= \mathbf{I}_{..}(\xi(u))\big(K_{\mu\nu}(u)\,\vec{\mathbf{n}}(\xi(u)),\ \vec{\mathbf{n}}(\xi(u))\big) \\ &= \mathbf{I}_{..}(\xi(u))\left(\vec{\mathbf{t}}_{(\mu\nu)}(\xi(u)) - \overline{\left\{ \begin{matrix}\lambda \\ \mu\ \nu\end{matrix}\right\}}\,\vec{\mathbf{t}}_{(\lambda)}(\xi(u)),\ \vec{\mathbf{n}}(\xi(u))\right) \\ &= \delta_{ij}\,(\partial_\mu\partial_\nu\xi^i)\,n^j\,(\xi(u)) - 0. \end{aligned} \tag{1.211}$$

The symmetric tensor

$$\mathbf{K}_{..}(u) := K_{\mu\nu}(u)\,\mathrm{d}u^\mu \otimes \mathrm{d}u^\nu \tag{1.212}$$

is called the *extrinsic curvature tensor*. The *first fundamental form* $\Phi_{\mathrm{I}} \equiv \mathrm{d}s^2 = \overline{g}_{\mu\nu}(u)\,\mathrm{d}u^\mu \mathrm{d}u^\nu$ determines *the intrinsic geometry* of the surface. The *second fundamental form* $\Phi_{\mathrm{II}} := K_{\mu\nu}(u)\,\mathrm{d}u^\mu\mathrm{d}u^\nu$ reveals how the *surface curves in the external three-dimensional embedding space*.

Now we shall consider the *invariant eigenvalue problem* posed in the following equations:

$$\begin{aligned} &\det\left[K_{\mu\nu}(u) - k(u)\,\overline{g}_{\mu\nu}(u)\right] = 0, \\ &\overline{g}\,k^2 - \left(\overline{g}^{\mu\nu}K_{\mu\nu}\right)k + \det\left[K_{\mu\nu}\right] = 0. \end{aligned} \tag{1.213}$$

Since $\overline{\mathbf{g}}_{..}(u)$ is positive-definite, (1.213) always yield two real roots $k_{(1)}(u)$ and $k_{(2)}(u)$. These roots are *the invariant eigenvalues*. The *Gaussian curvature* and the *mean curvature* of the surface are furnished by

$$K(u) := k_{(1)}(u)k_{(2)}(u) = \det\left[K_{\mu\nu}(u)\right] / \,\overline{g}(u), \tag{1.214}$$

$$\mu(u) := (1/2)\left[k_{(1)}(u) + k_{(2)}(u)\right] = (1/2)\,\overline{g}^{\mu\nu}(u)K_{\mu\nu}(u). \tag{1.215}$$

Example 1.4.1. We shall investigate a smooth *surface of revolution*. Consider a nondegenerate, parametrized curve of class C^3 given by

$$\begin{aligned} &x = \mathcal{X}(u^1) := \left(f(u^1), 0, h(u^1)\right), \quad u^1 \in (a,b) \subset \mathbb{R}, \\ &f(u^1) > 0, \\ &\left[f'(u^1)\right]^2 + \left[h'(u^1)\right]^2 > 0. \end{aligned} \tag{1.216}$$

The image of $\mathcal{X}$ is called the *profile curve* and it is totally contained in the $x^1 - x^3$ plane of $\mathbb{R}^3$. In case the profile curve is revolved around the x^3-axis, the resulting surface of revolution is furnished by

$$\begin{aligned} &x = \xi(u^1, u^2) = \left(f(u^1)\cos u^2, f(u^1)\sin u^2, h(u^1)\right), \\ &\mathcal{D}_2 := \left\{u \in \mathbb{R}^2 : a < u^1 < b,\ -\pi < u^2 < \pi\right\}. \end{aligned} \tag{1.217}$$

(See Fig. 1.18.)

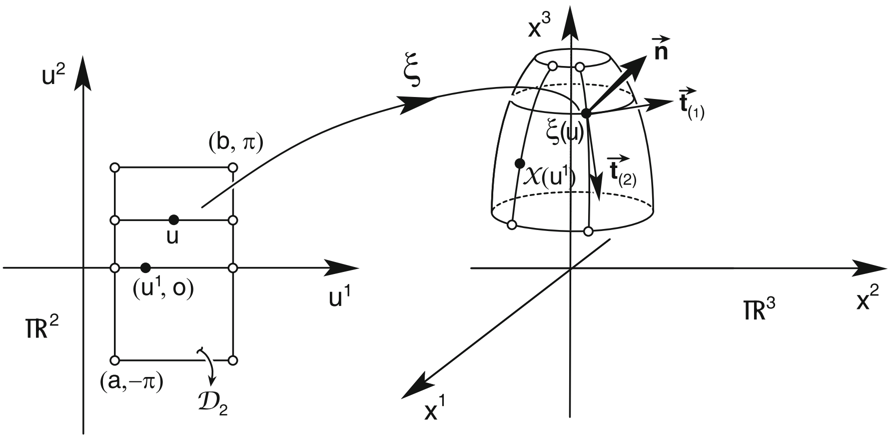

Fig. 1.18 A smooth surface of revolution

By (1.217), (1.200), and (1.203), we obtain

$$\vec{\mathbf{t}}_{(1)}(\xi(u)) = \left[f'(u^1)\left(\cos u^2 \frac{\partial}{\partial x^1} + \sin u^2 \frac{\partial}{\partial x^2} \right) + h'(u^1)\frac{\partial}{\partial x^3} \right]_{|\xi(u)},$$

$$\vec{\mathbf{t}}_{(2)}(\xi(u)) = \left[f(u^1)\left(-\sin u^2 \frac{\partial}{\partial x^1} + \cos u^2 \frac{\partial}{\partial x^2} \right) \right]_{|\xi(u)},$$

$$\overline{g}_{11}(u) = \left[f'(u^1)\right]^2 + \left[h'(u^1)\right]^2 > 0,$$

$$\overline{g}_{22}(u) = \left[f(u^1)\right]^2 > 0,$$

$$\overline{g}_{12}(u) = \overline{g}_{21}(u) \equiv 0,$$

$$\overline{g}(u) = \left[f(u^1)\right]^2 \left\{ \left[f'(u^1)\right]^2 + \left[h'(u^1)\right]^2 \right\} > 0. \tag{1.218}$$

The unit normal from (1.206) is given by

$$\vec{\mathbf{n}}(\xi(u)) = \left[(f')^2 + (h')^2\right]^{-1/2} \times \left\{ -h'(u^1)\left[\cos u^2 \frac{\partial}{\partial x^1} + \sin u^2 \frac{\partial}{\partial x^2} \right] + f'(u^1)\frac{\partial}{\partial x^3} \right\}_{|\xi(u)}. \tag{1.219}$$

Using (1.217), (1.218), and (1.207), we deduce that

$$\vec{\mathbf{t}}_{(11)}(\xi(u)) = \left\{ f''(u^1)\left[\cos u^2 \frac{\partial}{\partial x^1} + \sin u^2 \frac{\partial}{\partial x^2}\right] + h''(u^1)\frac{\partial}{\partial x^3}\right\}_{|\xi(u)},$$

$$\vec{\mathbf{t}}_{(22)}(\xi(u)) = -f(u^1)\left[\cos u^2 \frac{\partial}{\partial x^1} + \sin u^2 \frac{\partial}{\partial x^2}\right]_{|\xi(u)},$$

$$\vec{\mathbf{t}}_{(12)}(\xi(u)) \equiv \vec{\mathbf{t}}_{(21)}(\xi(u)) = f'(u^1)\left[-\sin u^2 \frac{\partial}{\partial x^1} + \cos u^2 \frac{\partial}{\partial x^2}\right]_{|\xi(u)}. \quad (1.220)$$

Now, by (1.220), (1.219), and (1.211), the extrinsic curvature components are

$$K_{11}(u) = \left[(f')^2 + (h')^2\right]^{-1/2}\left[f'(u^1)\,h''(u^1) - f''(u^1)\,h'(u^1)\right],$$

$$K_{22}(u) = \left[(f')^2 + (h')^2\right]^{-1/2} f(u^1)\,h'(u^1),$$

$$K_{12}(u) \equiv K_{21}(u) \equiv 0. \quad (1.221)$$

(In many problems of physics or engineering, one deals with metallic or ceramic surfaces of revolution.) □

Now we shall discuss the embedding of an $(N{-}1)$th dimensional manifold M_{N-1} into a manifold M_N. The differentiable manifold M_{N-1} is said to be embedded in M_N provided there exists *a one-to-one* mapping $F : M_{N-1} \longrightarrow M_N$ such that F is of class C^r $(r \in \mathbb{Z}^+)$. (In the case when mapping F is *not* assumed to be globally one-to-one, it is defined as an *immersion*.) The image $F(M_{N-1})$ is called the *embedded manifold*. By restricting the coordinate charts, we define a *parametrized hypersurface* ξ with the image $\sum_{N-1}$ in $D \subset \mathbb{R}^N$. (See Fig. 1.19.)

We explain various mappings in Fig. 1.19 by the following equations:

$$q = F(p),$$

$$\chi_{|..} := \chi_{|U\cap F(M_{N-1})},$$

$$x = \chi_{|..}(q) = \left[\chi_{|..} \circ F \circ \phi^{-1}\right](u) =: \xi(u),$$

$$x^i = \xi^i(u) \equiv \xi^i(u^1, \ldots, u^{N-1}),$$

$$\text{Rank}\left[\partial_\mu \xi^i\right] = N - 1 > 0,$$

$$\xi \in C^r(\mathcal{D}_{N-1} \subset \mathbb{R}^{N-1}; \mathbb{R}^N)\,; \quad r \geq 3. \quad (1.222)$$

Here, Roman indices are taken from $\{1, \ldots, N\}$ and the Greek indices are taken from $\{1, \ldots, N-1\}$. *The summation convention is followed for both sets of indices.* (See [56, 90].)

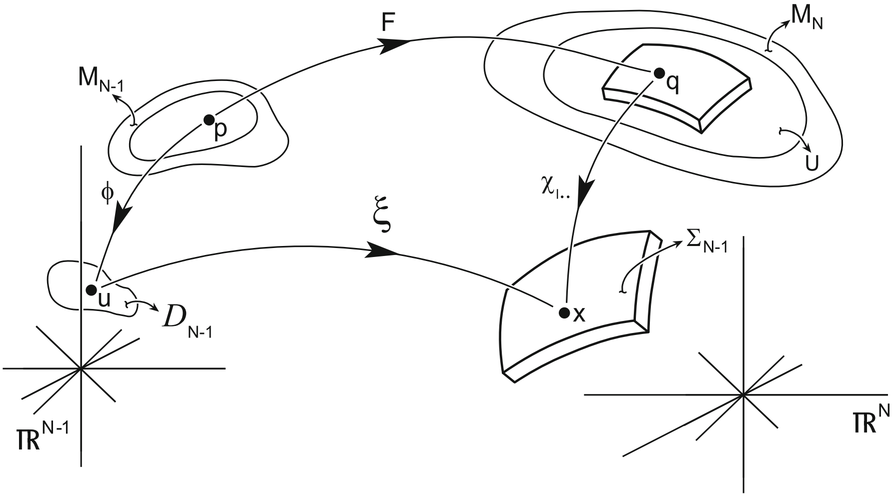

Fig. 1.19 The image $\sum_{N-1}$ of a parametrized hypersurface ξ

The metric $\mathbf{g}_{..}(x)$ and the induced metric $\overline{\mathbf{g}}_{..}(u)$ are furnished (as in (1.203)) respectively by:

$$\begin{aligned} \mathbf{g}_{..}(x) &= g_{ij}(x)\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \\ \overline{\mathbf{g}}_{..}(u) &= \overline{g}_{\mu\nu}(u)\,\mathrm{d}u^\mu \otimes \mathrm{d}u^\nu =: \left[g_{ij}(\xi(u))\,\partial_\mu \xi^i\,\partial_\nu \xi^j\right]\mathrm{d}u^\mu \otimes \mathrm{d}u^\nu. \end{aligned} \tag{1.223}$$

An outer normal to $\sum_{N-1}$ is provided by

$$\nu^k(\xi(u)) := \frac{1}{(N-1)!}\, g^{kj}(\xi(u))\,\varepsilon_{jj_2\ldots j_N}\,\frac{\partial(x^{j_2},\ldots,x^{j_N})}{\partial(u^1,\ldots,u^{N-1})}\,. \tag{1.224}$$

Here, indices $j_2, \ldots, j_N$ are summed. (See [56, 171].)

In the sequel, we assume that $g_{kj}(\cdot\cdot)\,\nu^k(\cdot\cdot)\,\nu^j(\cdot\cdot) \neq 0$. Thus, we can define the unit normal as

$$\begin{aligned} &n^k(\xi(u)) := \nu^k(\xi(u)) \Big/ \sqrt{|g_{ij}(\xi(u))\,\nu^i(\xi(u))\,\nu^j(\xi(u))|}\,, \\ &g_{kl}(\cdot\cdot)\,n^k(\cdot\cdot)\,n^l(\cdot\cdot) \equiv \pm 1 =: \varepsilon(n), \quad \varepsilon(n) = \operatorname{sgn}\left[\mathbf{g}_{..}(\vec{\mathbf{n}}, \vec{\mathbf{n}})\right], \\ &g_{ij}(\cdot\cdot)\,n^i(\cdot\cdot)\,\partial_\mu \xi^j \equiv 0. \end{aligned} \tag{1.225}$$

The last equation in (1.225) follows from the fact that $\frac{\partial(x^j, x^{j_2}, \ldots, x^{j_N})}{\partial(u^\mu, u^1, \ldots, u^{N-1})} \equiv 0$ for $\mu \in \{1, \ldots, N-1\}$ because the Jacobian matrix rank is $N-1$.

Recall the coordinate transformations in (1.2) for M_N. We can have a similar coordinate transformation, called a *reparametrization*, in M_{N-1}. (See Fig. 1.20.)

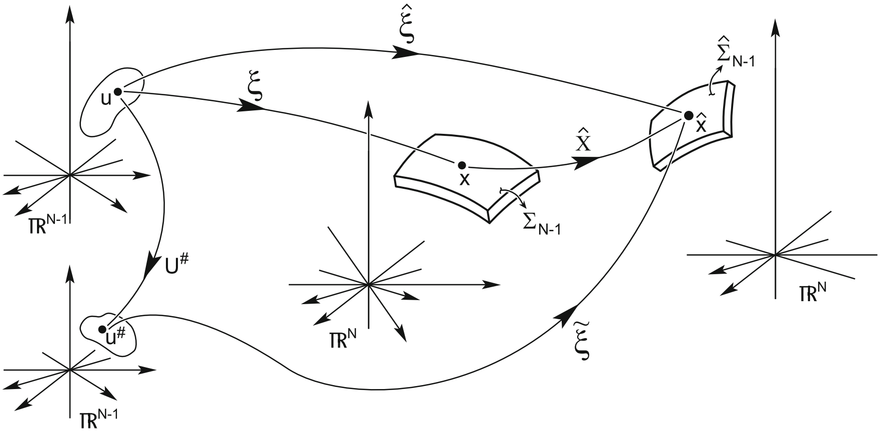

Fig. 1.20 Coordinate transformation and reparametrization of hypersurface ξ

The mappings occurring in Fig. 1.20 are elaborated below:

$$u^{\#} = U^{\#}(u), \quad \text{Rank}\left[\frac{\partial u^{\#\alpha}}{\partial u^{\beta}}\right] = N - 1, \tag{1.226i}$$

$$U := \left[U^{\#}\right]^{-1}, \tag{1.226ii}$$

$$x = \xi(u) = \xi^{\#}(u^{\#}) := \left[\xi \circ (U^{\#})^{-1}\right](u^{\#}), \tag{1.226iii}$$

$$\widehat{x} = \widehat{X}_{|\Sigma_{N-1}}(x) = \left[\widehat{X} \circ \xi\right](u) =: \widehat{\xi}(u), \tag{1.226iv}$$

$$\widehat{x} = \left[\widehat{X} \circ \xi \circ (U^{\#})^{-1}\right](u^{\#}) =: \tilde{\xi}(u^{\#}). \tag{1.226v}$$

Let us consider the transformation properties of $\partial_\mu \xi^i$ under each of (1.226iii), (1.226iv), and (1.226v). By the chain rules of differentiation, we obtain

$$\frac{\partial \xi^{\#i}(u^{\#})}{\partial u^{\#\mu}} = \frac{\partial U^{\nu}(u^{\#})}{\partial u^{\#\mu}} \cdot \frac{\partial \xi^i(u)}{\partial u^{\nu}}, \tag{1.227i}$$

$$\frac{\partial \widehat{\xi}^i(u)}{\partial u^{\mu}} = \frac{\partial \widehat{X}^i(x)}{\partial x^k}_{|\xi(u)} \cdot \frac{\partial \xi^k(u)}{\partial u^{\mu}}, \tag{1.227ii}$$

$$\frac{\partial \tilde{\xi}^i(u^{\#})}{\partial u^{\#\mu}} = \frac{\partial \widehat{X}^i(x)}{\partial x^k}_{|..} \cdot \frac{\partial \xi^k(u)}{\partial u^{\nu}} \cdot \frac{\partial U^{\nu}(u^{\#})}{\partial u^{\#\mu}}. \tag{1.227iii}$$

Therefore, we conclude that: (1) under a reparametrization, $\partial_\mu \xi^i$ behaves like components of a covariant vector field; (2) under a coordinate transformation, $\partial_\mu \xi^i$ behaves as components of a contravariant vector field, and (3) under a combined transformation, $\partial_\mu \xi^i$ behaves like components of *a mixed second-order (hybrid) tensor field*. These transformation rules prompt us to define formally *hybrid tensor field* [56] components by the following transformation rules:

$$\begin{aligned}
&\tilde{T}^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q}{}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}\,(u^\#,\,\tilde{\xi}(u^\#)) \\
&\quad = \frac{\partial u^{\#\mu_1}}{\partial u^{\lambda_1}} \cdots \frac{\partial u^{\#\mu_p}}{\partial u^{\lambda_p}} \cdot \frac{\partial u^{\rho_1}}{\partial u^{\#\nu_1}} \cdots \frac{\partial u^{\rho_q}}{\partial u^{\#\nu_q}} \cdot \frac{\partial \widehat{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \widehat{x}^{i_r}}{\partial x^{k_r}} \cdot \frac{\partial x^{l_1}}{\partial \widehat{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial \widehat{x}^{j_s}} \\
&\qquad \times\; T^{\lambda_1,\ldots,\lambda_p}{}_{\rho_1,\ldots,\rho_q}{}^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}\,(u,\,\xi(u)).
\end{aligned} \tag{1.228}$$

Example 1.4.2. Consider $N = 3$ and $M_N = \mathbb{E}_3$, the Euclidean space. A smooth, nondegenerate parametrized surface is given by

$$\begin{aligned}
&\xi(u) := \left(\alpha^1(u^1) + u^2\mathcal{X}^1(u^1),\, \alpha^2(u^1) + u^2\mathcal{X}^2(u^1),\, \alpha^3(u^1) + u^2\mathcal{X}^3(u^1)\right), \\
&(u^1, u^2) \in (a, b) \times (c, d) \subset \mathbb{R}^2; \\
&\text{Rank } \left[\partial_\mu \xi^i\right] = 2.
\end{aligned}$$

At each point $\xi(u_0)$ on $\sum_2$, a line segment or ruling passes through $\sum_2$. These rulings are furnished by

$$\begin{aligned}
&\mathcal{X}_{(0)}(u^2) := \left(\alpha^1(u_0^1) + u^2\mathcal{X}^1(u_0^1),\, \alpha^2(u_0^1) + u^2\mathcal{X}^2(u_0^1),\, \alpha^3(u_0^1) + u^2\mathcal{X}^3(u_0^1)\right), \\
&\qquad u^2 \in (c, d).
\end{aligned}$$

Such a surface is called a *ruled 2-surface*. Let us consider a reparametrization

$$u^{\#1} = u^1, \quad u^{\#2} = 2u^2.$$

For the purpose of this example, we introduce a translation of Cartesian axes given by

$$\widehat{x}^1 = x^1 + 1, \quad \widehat{x}^2 = x^2 + 2, \quad \widehat{x}^3 = x^3 + 3.$$

Transformation (1.227iii) can be obtained explicitly by the matrix equation

$$\underset{(3\times 2)}{\left[\frac{\partial \tilde{\xi}^i(u^\#)}{\partial u^{\#\mu}}\right]} = \begin{bmatrix} \alpha^{1\prime}(u^{\#1}) + (1/2)u^{\#2}\mathcal{X}^{1\prime}(u^{\#1}),\ (1/2)\mathcal{X}^1(u^{\#1)} \\ \alpha^{2\prime}(u^{\#1}) + (1/2)u^{\#2}\mathcal{X}^{2\prime}(u^{\#1}),\ (1/2)\mathcal{X}^2(u^{\#2)} \\ \alpha^{3\prime}(u^{\#1}) + (1/2)u^{\#2}\mathcal{X}^{3\prime}(u^{\#1}),\ (1/2)\mathcal{X}^3(u^{\#3)} \end{bmatrix}.$$

□

Now we shall define *a suitable covariant derivative for a hybrid tensor field*. The covariant derivative is given by the components

$$\tilde{\nabla}_\lambda T^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q}{}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}\,(u,\,\xi(u)) := \Big[\overline{\nabla}_\lambda T^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q}{}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}\,(u,x)$$
$$+\,\partial_\lambda \xi^k\,\nabla_k T^{\mu_1,\ldots,\mu_p}{}_{\nu_1,\ldots,\nu_q}{}^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}\,(u,x)\Big]_{|x=\xi(u)}\,. \tag{1.229}$$

Note that on the right-hand side, *the covariant derivative acts on the Greek indices in the first term and only on the Roman indices in the second term*. Moreover, the left-hand side constitutes components of a higher order hybrid tensor. To clarify, we now provide some examples of covariant differentiations.

Example 1.4.3. We shall discuss here covariant derivatives of hybrid scalar fields, vector fields, and tensor fields:

$$\tilde{\nabla}_\lambda \phi(u,\xi(u)) = \Big[\overline{\nabla}_\lambda \phi(u,x) + \partial_\lambda \xi^k \cdot \nabla_k \phi(u,x)\Big]_{|x=\xi(u)}$$
$$= \left[\frac{\partial}{\partial u^\lambda}\phi(u,x) + \partial_\lambda \xi^k \cdot \frac{\partial \phi(u,x)}{\partial x^k}\right]_{|x=\xi(u)}; \tag{1.230i}$$

$$\tilde{\nabla}_\lambda V_\mu(u) = \overline{\nabla}_\lambda V_\mu(u) = \frac{\partial V_\mu(u)}{\partial u^\lambda} - \overline{\begin{Bmatrix} \nu \\ \mu\ \lambda \end{Bmatrix}}\, V_\nu(u)\,; \tag{1.230ii}$$

$$\tilde{\nabla}_\lambda T_j(\xi(u)) = \partial_\lambda \xi^k \Big[\nabla_k T_j(x)\Big]_{|x=\xi(u)}$$
$$= \partial_\lambda \xi^k(u) \left[\frac{\partial T_j(x)}{\partial x^k} - \begin{Bmatrix} i \\ k\ j \end{Bmatrix}\, T_i(x)\right]_{|x=\xi(u)}; \tag{1.230iii}$$

$$\tilde{\nabla}_\lambda \left[\partial_\mu \xi^i(u)\right] = \overline{\nabla}_\lambda \left(\partial_\mu \xi^i\right) + \begin{Bmatrix} i \\ k\ l \end{Bmatrix}_{|\xi(u)} \partial_\lambda \xi^k \cdot \partial_\mu \xi^l\,; \tag{1.230iv}$$

$$\tilde{\nabla}_\lambda\, \overline{g}_{\mu\nu}(u) = \overline{\nabla}_\lambda\, \overline{g}_{\mu\nu}(u) \equiv 0\,; \tag{1.230v}$$

$$\tilde{\nabla}_\lambda\, g_{ij}(\xi(u)) = \partial_\lambda \xi^k \Big[\nabla_k g_{ij}(x)\Big]_{|x=\xi(u)} \equiv 0. \tag{1.230vi}$$

□

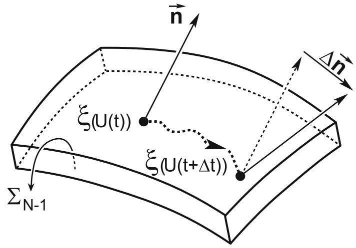

Fig. 1.21 Change of normal vector due to the extrinsic curvature

Now we generalize (1.211) to define the extrinsic curvature as

$$K_{\mu\nu}(u) := g_{ij}\,(\xi(u))\; n^i\,(\xi(u))\,\tilde{\nabla}_\mu\left(\partial_\nu \xi^j\right). \tag{1.231}$$

By (1.225), (1.231), and (1.230iv) we have

$$\tilde{\nabla}_\mu\left(\partial_\nu \xi^j\right) = \varepsilon(n)\cdot n^j\,(\xi(u))\cdot K_{\mu\nu}(u), \tag{1.232i}$$

$$\partial_\mu\partial_\nu\xi^j = \overline{\left\{\begin{matrix}\lambda\\ \mu\;\nu\end{matrix}\right\}}\,\partial_\lambda\xi^j - \left\{\begin{matrix} j\\ i\;k\end{matrix}\right\}_{|\xi(u)}\left(\partial_\mu\xi^i\right)\left(\partial_\nu\xi^k\right)$$
$$+\,\varepsilon(n)\cdot n^j\,(\xi(u))\cdot K_{\mu\nu}(u), \tag{1.232ii}$$

$$K_{\nu\mu}(u) \equiv K_{\mu\nu}(u). \tag{1.232iii}$$

Since by (1.225)

$$g_{ij}\,(\cdot\cdot)\left[\left(\tilde{\nabla}_\lambda\partial_\mu\xi^i\right)n^j + \partial_\mu\xi^i\left(\tilde{\nabla}_\lambda n^i\right)\right] \equiv 0, \tag{1.233}$$

we obtain from (1.231)

$$K_{\mu\nu}(u) = -g_{ij}\,(\xi(u))\left(\tilde{\nabla}_\mu n^i\right)\left(\partial_\nu\xi^j\right) = -\left(\nabla_i n_j\right)_{|\xi(u)}\left(\partial_\mu\xi^i\right)\left(\partial_\nu\xi^j\right). \tag{1.234}$$

Using (1.234) and (1.230iii), we can prove that

$$\tilde{\nabla}_\mu n^i = \partial_\mu\xi^k\left[\nabla_k n^i\right]_{|\xi(u)} = -\overline{g}^{\rho\mu}(u)\,K_{\mu\nu}(u)\,\partial_\rho\xi^i. \tag{1.235}$$

The above equations are the generalizations of *Weingarten's equations*. These equations allow the following geometrical interpretation: In Fig. 1.21, the dotted vector at $\xi\;(\mathcal{U}(t+\Delta t))$ is the parallelly transported normal vector. The difference vector $\Delta\vec{\mathbf{n}}$ indicates the change due to the extrinsic curvature.

Now, we need to digress and discuss briefly *integrability conditions* of partial differential equations. Consider the system of $N-1$ partial differential equations

$$\partial_\mu V(u) = P_\mu(u), \tag{1.236}$$

in a star-shaped domain $\mathcal{D}^*_{N-1} \subset \mathbb{R}^{N-1}$. In (1.236), the prescribed functions $P_\mu(u)$ are of class C^1. We define a one-form

$$\tilde{\mathbf{w}}(u) := P_\mu(u)\,\mathrm{d}u^\mu.$$

By (1.236),

$$\tilde{\mathbf{w}}(u) = \left[\partial_\mu V(u)\right]\mathrm{d}u^\mu = \mathrm{d}V(u).$$

Using Poincaré's lemma (1.66), we deduce that

$$\mathrm{d}\left[P_\mu(u)\,\mathrm{d}u^\mu\right] = \mathrm{d}\tilde{\mathbf{w}}(u) = \mathrm{d}^2 V(u) \equiv \mathbf{0}_{..}(u). \tag{1.237}$$

Therefore, by Theorem 1.2.21, there exists a C^1-function $F(u)$ such that

$$\mathrm{d}V(u) = \tilde{\mathbf{w}}(u) = \mathrm{d}\left(F(u)\right).$$

Integrating along a continuous and piecewise differentiable curve of image $\Gamma \subset \mathcal{D}^*_{N-1}$, we obtain the solution of (1.236) as

$$V(u) = \int_{u_{(0)}[\Gamma]}^{u} P_\mu(y)\,\mathrm{d}y^\mu = F(u) - F\left(u_{(0)}\right). \tag{1.238}$$

The conditions (1.237), yielding

$$\partial_\nu P_\mu(u) - \partial_\mu P_\nu(u) = 0, \tag{1.239}$$

are called *the integrability conditions* for (1.236).

Suppose that we are confronted with the question of existence of a hypersurface ξ of class C^r $(r \geq 3)$ for a prescribed extrinsic curvature $K_{\mu\nu}(u)$. In that case, we have to investigate the integrability conditions of partial differential equations (1.232i) or (1.232ii). These conditions can be summarized in the following lemma.

Lemma 1.4.4. *Suppose that ξ is an $(N-1)$th dimensional parametrized hypersurface embedded in $\mathbb{R}^N$ according to (1.222). Then, the integrability conditions of partial differential equations (1.232i) or (1.232ii) are furnished by the following hybrid tensor equation:*

$$\begin{aligned}\Theta^i{}_{\mu\nu\lambda}\left(u, \xi(u)\right) &:= \left[\overline{R}^\rho{}_{\mu\nu\lambda}(u) + \varepsilon(n)\left(K_{\mu\nu}\overline{K}^\rho{}_\lambda - K_{\mu\lambda}\overline{K}^\rho{}_\nu\right)\right]\partial_\rho \xi^i \\ &\quad + \left[R^i{}_{jlk}\left(\xi(u)\right)\partial_\mu\xi^j \cdot \partial_\lambda \xi^l \cdot \partial_\nu \xi^k + \varepsilon(n)\left[\tilde{\nabla}_\nu K_{\mu\lambda} - \tilde{\nabla}_\lambda K_{\mu\nu}\right] n^i\left(\xi(u)\right)\right] \\ &= 0.\end{aligned} \tag{1.240}$$

Proof. The integrability of partial differential equations (1.232i) or (1.232ii) are

$$\tilde{\nabla}_\nu \tilde{\nabla}_\lambda \left(\partial_\mu \xi^i\right) - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu \left(\partial_\mu \xi^i\right) = \varepsilon(n) \left[\tilde{\nabla}_\nu \left(n^i K_{\mu\lambda}\right) - \tilde{\nabla}_\lambda \left(n^i K_{\mu\nu}\right)\right].$$

By the Leibniz rule of differentiation and the Weingarten equations (1.235), we derive that

$$\tilde{\nabla}_\nu \left(K_{\mu\lambda} n^i\right) - \tilde{\nabla}_\lambda \left(n^i K_{\mu\nu}\right) = \left(\tilde{\nabla}_\nu K_{\mu\lambda} - \tilde{\nabla}_\lambda K_{\mu\nu}\right) n^i + \left(K_{\mu\nu} \overline{K}^{\rho}{}_{\lambda} - K_{\mu\lambda} \overline{K}^{\rho}{}_{\nu}\right) \partial_\rho \xi^i.$$

With the above equations and the following:

$$\tilde{\nabla}_\nu \tilde{\nabla}_\lambda \left(\partial_\mu \xi^i\right) - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu \left(\partial_\mu \xi^i\right) = \overline{R}^{\sigma}{}_{\mu\nu\lambda}(u) \cdot \partial_\sigma \xi^i + R^i{}_{jkl}\left(\xi(u)\right) \cdot \partial_\mu \xi^j \cdot \partial_\nu \xi^k \cdot \partial_\lambda \xi^l, \tag{1.241}$$

Equation (1.240) is established. (See Problem # 3 of Exercises 1.4.) ■

The above lemma leads to the following generalizations of *Gauss' equations* and the *Codazzi-Mainardi equations*.

Theorem 1.4.5. *Let ξ be a nondegenerate, $(N-1)$-dimensional parametric hypersurface of class C^r $(r \geq 3)$ into $\mathbb{R}^N$. Then, the integrability conditions of partial differential equations* (1.232i) *or* (1.232ii) *imply that*

$$(i) \qquad \overline{R}_{\sigma\mu\nu\lambda}(u) = \varepsilon(n)\left(K_{\mu\lambda} K_{\sigma\nu} - K_{\mu\nu} K_{\sigma\lambda}\right) + R_{ijkl}\left(\xi(u)\right) \partial_\sigma \xi^i \cdot \partial_\mu \xi^j \cdot \partial_\nu \xi^k \cdot \partial_\lambda \xi^l\,; \tag{1.242i}$$

$$(ii) \qquad \tilde{\nabla}_\nu K_{\mu\lambda} - \tilde{\nabla}_\lambda K_{\mu\nu} = R_{ijkl}\left(\xi(u)\right) \partial_\mu \xi^j \cdot \partial_\nu \xi^k \cdot \partial_\lambda \xi^l \cdot n^i\left(\xi(u)\right). \tag{1.242ii}$$

Proof. By (1.240), (1.223), and (1.225), we deduce that the "tangential component" yields

$$0 = \left(g_{ij}\, \partial_\sigma \xi^j\right) \Theta^i{}_{\mu\nu\lambda} = \overline{g}_{\sigma\rho}\, \overline{g}^{\rho\gamma}\left[\overline{R}_{\gamma\mu\nu\lambda} + \varepsilon(n)\left(K_{\mu\nu} K_{\lambda\gamma} - K_{\mu\lambda} K_{\nu\gamma}\right)\right] - \left(g_{ih}\, \partial_\sigma \xi^h\right) R^i{}_{jkl}\, \partial_\mu \xi^j \cdot \partial_\lambda \xi^l \cdot \partial_\nu \xi^k + 0.$$

Thus, (1.242i) follows.

By (1.240) and (1.225), we derive that the "normal component" implies

$$0 = \left(g_{ih}\, n^h\right) \Theta^i{}_{\mu\nu\lambda} = 0 - \left(g_{ih}\, n^h\right) R^i{}_{jkl}\, \partial_\mu \xi^j \cdot \partial_\nu \xi^k \cdot \partial_\lambda \xi^l + \left[\tilde{\nabla}_\nu K_{\mu\lambda} - \tilde{\nabla}_\lambda K_{\mu\nu}\right].$$

Therefore, (1.242ii) follows. ■

Now we shall provide some examples.

Example 1.4.6. Consider the three-dimensional Euclidean space, a Cartesian coordinate chart, and an embedded surface $\sum_2$. In this case, $R_{ijkl}\left(\xi(u)\right) \equiv 0$.

Equations (1.242i,b) reduce respectively to

$$\overline{R}_{1212}(u) = K_{11}K_{22} - (K_{12})^2 = \det\left[K_{\mu\nu}(u)\right], \tag{1.243i}$$

$$\overline{\nabla}_\lambda K_{\mu\nu} - \overline{\nabla}_\nu K_{\mu\lambda} \equiv 0. \tag{1.243ii}$$

(Equations (1.243i) and (1.243ii) were first discovered by Gauss in 1828 and Codazzi-Mainardi (in 1868 and 1856), respectively.)

Using (1.243i) and (1.214), it follows that the Gaussian curvature

$$K(u) = \overline{R}_{1212}(u) \big/ \overline{g}(u). \tag{1.244}$$

The left-hand side of (1.244) is related to *the extrinsic curvature*, whereas the right-hand side is given by *the intrinsic curvature* of the surface. The surprising equality of these two was called by Gauss as *Theorema Egregium*. □

Example 1.4.7. Consider an N-dimensional differential manifold with a local coordinate chart such that

$$\begin{aligned} \mathbf{g}_{..}(x) &= g_{ij}(x)\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \\ g^{NN}(x) &\neq 0. \end{aligned} \tag{1.245}$$

(In this example, the index N is *not* summed.)

Let an $(N-1)$-dimensional hypersurface be furnished by

$$\begin{aligned} x^\mu &= \xi^\mu(u) := u^\mu, \\ x^N &= \xi^N(u) := c = \text{const.}, \\ (u) &\equiv (\mathbf{x}) := \left(x^1, \ldots, x^{N-1}\right), \\ \partial_\mu \xi^\nu &= \delta^\nu_{\ \mu}, \quad \partial_\mu \xi^N \equiv 0. \end{aligned} \tag{1.246}$$

The intrinsic metric is provided by

$$\overline{g}_{\mu\nu}(u) \equiv \overline{g}_{\mu\nu}(\mathbf{x}) := g_{\mu\nu}(x)_{|x^N=c}, \tag{1.247}$$

and the unit normal to the hypersurface is furnished by

$$n^i(x)_{|x^N=c} = \left[g^{iN}(x) \Big/ \sqrt{|g^{NN}(x)|}\right]_{|x^N=c}. \tag{1.248}$$

Equation (1.231), by (1.246)–(1.248), shows that

$$K_{\mu\nu}(u) \equiv K_{\mu\nu}(\mathbf{x}) = \left[\frac{1}{\sqrt{|g^{NN}(x)|}} \begin{Bmatrix} N \\ \mu\ \nu \end{Bmatrix} \right]_{|x^N=c} . \tag{1.249}$$

In the case of a *geodesic normal coordinate chart*, it is assumed that

$$g^{NN}(x) \equiv \varepsilon_N, \quad g^{N\mu}(x) \equiv 0. \tag{1.250}$$

The extrinsic curvature components in (1.249) reduce by (1.250) to

$$K_{\mu\nu}(\mathbf{x}) = -\frac{\varepsilon_N}{2} \left.\frac{\partial g_{\mu\nu}(x)}{\partial x^N}\right|_{x^N=c} . \tag{1.251}$$

(The above equations will be applied in the next chapter.) □

Finally, we shall now define a *parameterized submanifold* by a mapping $\xi \in C^r(\mathcal{D}_D \subset \mathbb{R}^D; \mathbb{R}^N)$. We generalize (1.222) by the following:

$$\begin{aligned} x &= \xi(u) \in \mathbb{R}^N, \\ x^i &= \xi^i(u) \equiv \xi^i(u^1, \ldots, u^D), \\ \text{Rank}\left[\partial_\mu \xi^i\right] &= D, \\ 1 &< D < N, \\ \overline{g}_{\mu\nu}(u)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu &:= \left[g_{ij}\left(\xi(u)\right) \partial_\mu \xi^i\, \partial_\nu \xi^j\right] \mathrm{d}u^\mu \otimes \mathrm{d}u^\nu, \\ \mu, \nu &\in \{1, ., D\}. \end{aligned} \tag{1.252}$$

(For a detailed treatment, see [56, 90, 121].)

Exercises 1.4

1. Consider *a nondegenerate Monge surface* embedded in a three-dimensional Euclidean space $\mathbb{E}_3$. Let it be parametrically specified by

$$x = \xi(u) := \left(u^1, u^2, f(u^1, u^2)\right), \quad u \equiv (u^1, u^2) \in \mathcal{D}_2 \subset \mathbb{R}^2.$$

   (Assume that f is of class C^4.)

   (i) Prove that the extrinsic curvature components are furnished by

$$K_{\mu\nu}(u) = \left[1 + \delta^{\lambda\rho}(\partial_\lambda f)(\partial_\rho f)\right]^{-1/2} \cdot (\partial_\mu \partial_\nu f).$$

   (ii) Show that the vanishing of the Gaussian curvature $K(u)$ in a domain implies the existence of a nonconstant differentiable function F such that $F(\partial_1 f, \partial_2 f) \equiv 0$.

2. Suppose that a nondegenerate, twice-differentiable parametrized surface ξ is given by (1.199). Prove that the extrinsic curvature is provided by

$$\sqrt{g(u)} \cdot K_{\mu\nu}(u) = \det \begin{bmatrix} \partial_\mu \partial_\nu \xi^1 & \partial_\mu \partial_\nu \xi^2 & \partial_\mu \partial_\nu \xi^3 \\ \partial_1 \xi^1 & \partial_1 \xi^2 & \partial_1 \xi^3 \\ \partial_2 \xi^1 & \partial_2 \xi^2 & \partial_2 \xi^3 \end{bmatrix}.$$

3. Prove (1.241) of the text.
4. Suppose that M_N is a space of constant curvature. (See (1.164i).) Prove that the Gauss equation (1.242i) and the Codazzi-Mainardi equation (1.242ii) reduce respectively to

$$\overline{R}_{\sigma\mu\nu\lambda}(u) = \varepsilon(n)\left(K_{\mu\lambda} K_{\sigma\nu} - K_{\mu\nu} K_{\sigma\lambda}\right) + K_0 \left(\overline{g}_{\mu\lambda}\, \overline{g}_{\sigma\nu} - \overline{g}_{\mu\nu}\, \overline{g}_{\sigma\lambda}\right)$$

and

$$\overline{\nabla}_\nu K_{\mu\lambda} - \overline{\nabla}_\lambda K_{\mu\nu} = 0.$$

Answers and Hints to Selected Exercises

1. (ii) The condition $K(u) = 0$ is satisfied if and only if the determinant

$$\det(\partial_\mu \partial_\nu f) = 0.$$

Denoting by $g(u) := \partial_1 f$ and $h(u) := \partial_2 f$, the preceding condition yields the Jacobian

$$\frac{\partial(g, h)}{\partial(u^1, u^2)} := \begin{vmatrix} \partial_1 g & \partial_2 g \\ \partial_1 h & \partial_2 h \end{vmatrix} = 0.$$

Thus, the functions g and h are functionally related. Therefore, there exists a nonconstant differentiable function F such that $F(g, h) \equiv 0$.
3. Using (1.228), (1.141i), and (1.230i–vi), we derive that

$$\begin{aligned}
&\left[\tilde{\nabla}_\nu \tilde{\nabla}_\lambda - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu\right]\left(\partial_\mu \xi^i\right) \\
&= \left\{\overline{\nabla}_\nu\left[\tilde{\nabla}_\lambda \partial_\mu \xi^i\right] + \partial_\nu \xi^k \cdot \nabla_k \left[\tilde{\nabla}_\lambda \partial_\mu \xi^i(u)\right]_{|..}\right\} - \{\nu \leftrightarrow \lambda\} \\
&= \left\{\overline{\nabla}_\nu \left[\overline{\nabla}_\lambda \overline{\nabla}_\mu \xi^i + \left.\begin{Bmatrix} i \\ k\ l \end{Bmatrix}\right|_{..} \partial_\lambda \xi^k \cdot \partial_\mu \xi^l\right] + \partial_\nu \xi^k \cdot \left[0 + \left.\begin{Bmatrix} i \\ k\ j \end{Bmatrix}\right|_{..} \cdot \tilde{\nabla}_\lambda \partial_\mu \xi^j\right]\right\} \\
&\qquad - \{\nu \leftrightarrow \lambda\}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\overline{\nabla}_\nu \overline{\nabla}_\lambda - \overline{\nabla}_\lambda \overline{\nabla}_\nu\right) \overline{\nabla}_\mu \xi^i + \left[\partial_j \begin{Bmatrix} i \\ k\, l \end{Bmatrix}\right]\Big|_{\xi(u)} \cdot \left[\partial_\nu \xi^j \cdot \partial_\lambda \xi^k - \partial_\lambda \xi^j \cdot \partial_\nu \xi^k\right] \cdot \partial_\mu \xi^l \\
&\quad + \begin{Bmatrix} i \\ j\, k \end{Bmatrix} \begin{Bmatrix} j \\ l\, h \end{Bmatrix}\Big|_{..} \cdot \left[\partial_\lambda \xi^l \cdot \partial_\nu \xi^k - \partial_\nu \xi^l \cdot \partial_\lambda \xi^k\right] \cdot \partial_\mu \xi^h \\
&= \overline{R}^{\sigma}{}_{\mu\nu\lambda}(u) \cdot \partial_\sigma \xi^i(u) + R^i{}_{jkl}\left(\xi(u)\right) \cdot \partial_\mu \xi^j \cdot \partial_\nu \xi^k \cdot \partial_\lambda \xi^l .
\end{aligned}
$$

(Here, and throughout this book, the notation $\{\nu \leftrightarrow \lambda\}$ or $[\nu \leftrightarrow \lambda]$ represents the previous term but with the indices ν and λ interchanged.)

Chapter 2
The Pseudo-Riemannian Space–Time Manifold M_4

2.1 Review of the Special Theory of Relativity

We shall provide here a brief review of the special theory of relativity. Since the emphasis of this book is general relativity, we will state some of the results in this section without proof, for future application. For more details, the reader is referred to the references provided within this chapter.

The arena of this theory is *the four-dimensional flat pseudo-Riemannian space–time manifold* M_4 endowed with a Lorentz metric. (See [55,189,242].) Let us choose a global pseudo-Cartesian or *Minkowskian chart* for the flat manifold M_4. A typical tangent vector is depicted in Fig. 2.1.

In Fig. 2.1, the vector $\vec{\mathbf{v}}_{x_0}$ is drawn on a sheet of paper with *inherent Euclidean geometry*. However, in the pseudo-Riemannian manifold M_4 with Lorentz metric, we obtain from Examples 1.2.1 and 1.3.2 (also from (1.85i–iv)) that

$$\vec{\mathbf{v}}_{x_0} = v^i \frac{\partial}{\partial x^i}\Big|_{x_0},$$

$$\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{v}}_{x_0}, \vec{\mathbf{v}}_{x_0}\right) = d_{ij} v^i v^j := \left(v^1\right)^2 + \left(v^2\right)^2 + \left(v^3\right)^2 - \left(v^4\right)^2,$$

$$\sigma\left(\vec{\mathbf{v}}_{x_0}\right) = \sqrt{|d_{ij} v^i v^j|}. \tag{2.1}$$

Note that the components v^i of a tangent vector $\vec{\mathbf{v}}_{x_0}$ are *the same either for a positive-definite metric or for a Lorentz metric*.

The length $\sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2}$ of the vector $\vec{\mathbf{v}}_{x_0}$, as drawn in Fig. 2.1, is *different from the separation* $\sigma\left(\vec{\mathbf{v}}_{x_0}\right)$ in (2.1). Moreover, from the discussions after (1.95), it is clear that the angular inclination of the vector $\vec{\mathbf{v}}_{x_0}$ in Fig. 2.1 is *not meaningful*. Therefore, it is judicious to *exercise caution in interpreting lengths and angles of space–time vectors drawn on a piece of paper!*

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4_2,

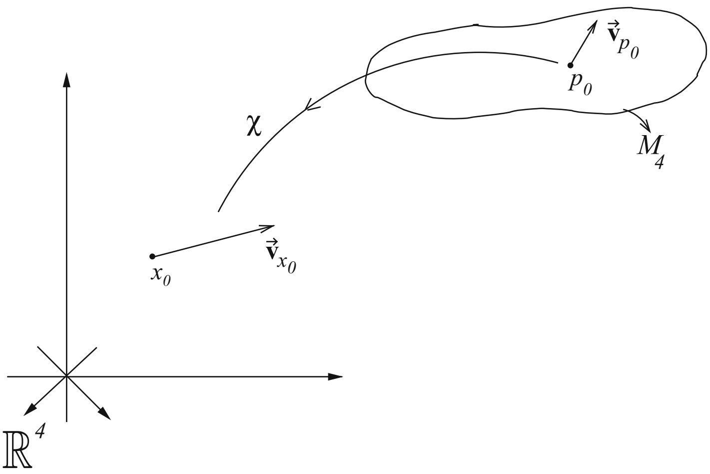

Fig. 2.1 A tangent vector $\vec{\mathbf{v}}_{p_0}$ in M_4 and its image $\vec{\mathbf{v}}_{x_0}$ in $\mathbb{R}^4$

The tangent vector space $T_{x_0}(\mathbb{R}^4)$ splits into *three distinct proper subsets*. The subset of vectors

$$W_S(x_0) := \left\{\vec{\mathbf{s}}_{x_0} : \mathbf{g}_{..}(x_0)\left(\vec{\mathbf{s}}_{x_0}, \vec{\mathbf{s}}_{x_0}\right) > 0\right\} \tag{2.2i}$$

is called the subset of *spacelike vectors*. The subset

$$W_T(x_0) := \left\{\vec{\mathbf{t}}_{x_0} : \mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \vec{\mathbf{t}}_{x_0}\right) < 0\right\} \tag{2.2ii}$$

is called the subset of *timelike vectors*. Finally, the subset

$$W_N(x_0) := \left\{\vec{\mathbf{n}}_{x_0} : \mathbf{g}_{..}(x_0)\left(\vec{\mathbf{n}}_{x_0}, \vec{\mathbf{n}}_{x_0}\right) = 0\right\} \tag{2.2iii}$$

is the subset of *null vectors*.

Example 2.1.1. Let $\left\{\vec{\mathbf{e}}_{(a)x_0}\right\}_1^4$ be an orthonormal basis set or *tetrad*. (See (1.87) and (1.88).) Therefore,

$$\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{e}}_{(a)x_0}, \vec{\mathbf{e}}_{(b)x_0}\right) = d_{(a)(b)}. \tag{2.3}$$

Let a tangent vector be given by the equation $\vec{\mathbf{s}}_{x_0} := 2\,\vec{\mathbf{e}}_{(1)x_0}$. Therefore, $\mathbf{g}_{..}(x_0)$ $\left(\vec{\mathbf{s}}_{x_0}, \vec{\mathbf{s}}_{x_0}\right) = 4$ and $\vec{\mathbf{s}}_{x_0}$ are spacelike. Let another vector be furnished by $\vec{\mathbf{t}}_{x_0} := \sqrt{3}\,\vec{\mathbf{e}}_{(4)x_0}$. In this case, $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \vec{\mathbf{t}}_{x_0}\right) = -3$. Thus, $\vec{\mathbf{t}}_{x_0}$ is timelike. Finally, let $\vec{\mathbf{n}}_{x_0} := \vec{\mathbf{e}}_{(2)x_0} - \vec{\mathbf{e}}_{(4)x_0}$. Thus, $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{n}}_{x_0}, \vec{\mathbf{n}}_{x_0}\right) = 0$. Therefore, $\vec{\mathbf{n}}_{x_0}$ is *a nonzero null vector*. □

Now, we shall state several theorems and a corollary regarding various tangent vectors.

Theorem 2.1.2. *Let* $\vec{\mathbf{t}}_{x_0}$ *and* $\widehat{\vec{\mathbf{t}}}_{x_0}$ *be two timelike vectors in* $T_{x_0}(\mathbb{R}^4)$. *Then* $\mathbf{g}_{..}(x_0)$ $(\vec{\mathbf{t}}_{x_0}, \widehat{\vec{\mathbf{t}}}_{x_0}) \neq 0$.

Corollary 2.1.3. *Let two timelike tangent vectors* $\vec{\mathbf{t}}_{x_0}$ *and* $\widehat{\vec{\mathbf{t}}}_{x_0}$ *have for their components* $t^4 > 0$ *and* $\hat{t}^4 > 0$. *Then,* $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \widehat{\vec{\mathbf{t}}}_{x_0}\right) < 0$.

Theorem 2.1.4. *Let* $\vec{\mathbf{t}}_{x_0}$ *be a timelike vector and* $\vec{\mathbf{n}}_{x_0}$ *be a nonzero null vector in* $T_{x_0}(\mathbb{R}^4)$. *Then,* $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \vec{\mathbf{n}}_{x_0}\right) \neq 0$.

Next, we state a very counterintuitive theorem.

Theorem 2.1.5. *Two nonzero null vectors* $\vec{\mathbf{n}}_{x_0}$ *and* $\widehat{\vec{\mathbf{n}}}_{x_0}$ *are orthogonal* (*i.e.*, $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{n}}_{x_0}, \widehat{\vec{\mathbf{n}}}_{x_0}\right) = 0$), *if and only if they are scalar multiples of each other.*

Proofs of above theorems and corollary are available in books [55, 243].

Now, we shall briefly discuss the physical significance of the flat manifold M_4 in special relativity. Firstly, we choose physical units so that *the speed of light* $c = 1$. (In the popular c.g.s. units, $c = 2.998 \times 10^{10}$ cm.s^{-1}) *Roman indices* take from $\{1, 2, 3, 4\}$, whereas *Greek indices* take from $\{1, 2, 3\}$. The summation convention is followed for *both sets of indices*. An element $p \in M_4$ represents *an idealized point event* in the space–time continuum. In a global Minkowskian coordinate chart, $x = \chi(p) \equiv (x^1, x^2, x^3, x^4)$ stands for the coordinates of an event. The coordinates x^1, x^2, x^3 represent the usual Cartesian coordinates of *the spatial point* associated with the event. The fourth coordinate x^4 stands for the time coordinate of the event. (*Remark*: The time coordinate x^4 *need not provide the actual temporal separation!*) In units where the speed of light is not set to unity, x^4 is the time coordinate multiplied by c. Therefore, in the Minkowskian chart, the time coordinate x^4 possesses *the same units as the spatial coordinates*. (On the other hand, a spatial coordinate x^α divided by c possesses the same unit as time.)

The three-dimensional hypersurface given by

$$\mathcal{N}_{x_0} := \left\{x : d_{ij}\left(x^i - x_0^i\right)\left(x^j - x_0^j\right) = 0\right\} \tag{2.4}$$

is called the *null cone* (or "light cone") with vertex at x_0. (See Fig. 2.2.)

The points inside and on the upper half of the null cone represent *future events* relative to x_0. Similarly, points inside and on the lower half of the cone represent *past events* relative to x_0. The events outside the cone are the (relativistic) *present events* relative to x_0.

The *causal cone* relative to x_0 is represented by the proper subset

$$\mathcal{C}_{x_0} := \left\{x : d_{ij}\left(x^i - x_0^i\right)\left(x^j - x_0^j\right) \leq 0\right\}. \tag{2.5}$$

Events in $\mathcal{C}_{x_0}$ can either causally affect or be affected by the event in x_0. This statement incorporates one of the physical postulates of the special relativity, namely, *physical actions cannot propagate faster than light*. We shall motivate this statement shortly.

The global Minkowskian charts are very useful in the special theory of relativity. The coordinate transformation (see Fig. 1.2) from one such chart to another is furnished by the equations

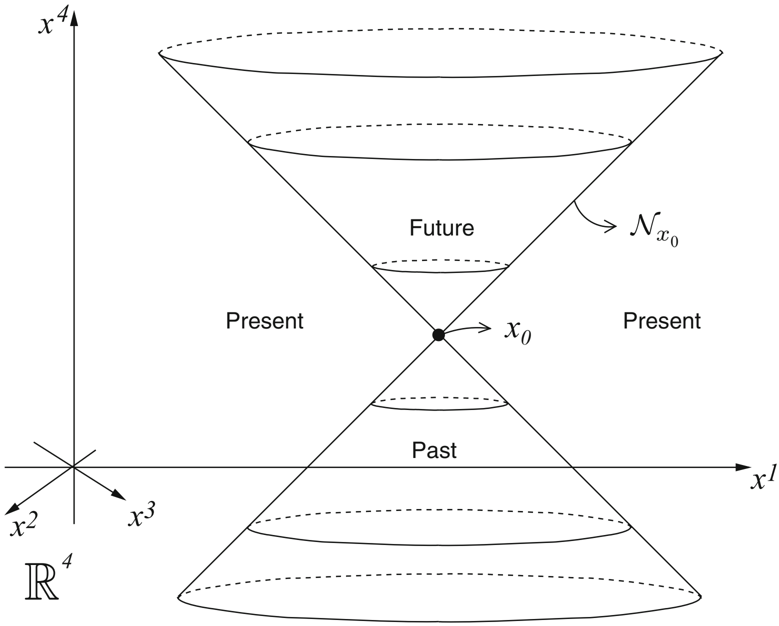

Fig. 2.2 Null cone $\mathcal{N}_{x_0}$ with vertex at x_0 (*circles* represent suppressed spheres)

$$\hat{x}^i = \widehat{X}^i(x) = c^i + l^i{}_j \, x^j, \tag{2.6i}$$

$$l^k{}_i \, d_{km} \, l^m{}_j = d_{ij}, \tag{2.6ii}$$

$$x^i = a^i{}_j \left(\hat{x}^j - c^j\right), \tag{2.6iii}$$

$$l^m{}_i \, a^i{}_k = \delta^m{}_k. \tag{2.6iv}$$

Here, c^i and $l^i{}_j$ are ten independent constants, or, parameters. The set of all transformations in (2.6i–iv) constitutes a Lie group denoted by $\mathcal{I}O(3,1;\mathbb{R})$. It is usually called the *Poincaré group*. The subgroup characterized by $c^1 = c^2 = c^3 = c^4 = 0$, is denoted by $O(3,1;\mathbb{R})$. It is usually known as the six-parameter *Lorentz group* [55, 114].

Example 2.1.6. Assuming $|v| < 1$, a *Lorentz transformation* is provided by

$$\begin{aligned} \hat{x}^1 &= x^1, \quad \hat{x}^2 = x^2, \\ \hat{x}^3 &= \frac{x^3 - vx^4}{\sqrt{1-(v)^2}}, \\ \hat{x}^4 &= \frac{-vx^3 + x^4}{\sqrt{1-(v)^2}}. \end{aligned} \tag{2.7}$$

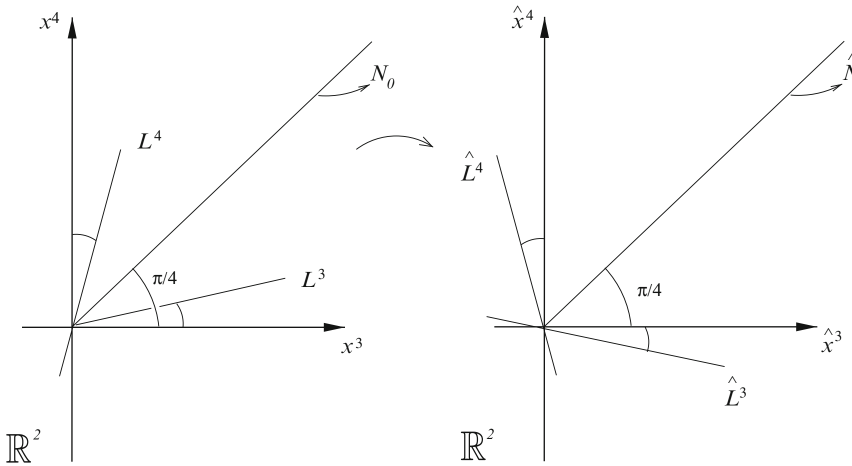

Fig. 2.3 A Lorentz transformation inducing a mapping between two coordinate planes

Suppressing two dimensions characterized by the x^1-axis and x^2-axis, we shall exhibit qualitatively the transformation (2.7) in Fig. 2.3. (We assume $0 < v < 1$.)

In Fig. 2.3, the x^3-axis and x^4-axis are mapped into two straight lines $\hat{L}^3$ and $\hat{L}^4$ respectively. The line N_0 is mapped into $\widehat{N}_0$. Thus, N_0 remains *invariant* under the mapping. (This line physically represents the trajectory of a photon, or other massless particle, in space–time!) The preimages of $\hat{x}^3$-axis and $\hat{x}^4$-axis are lines L^3 and L^4, respectively. In the inherent Euclidean geometry of the plane of Fig. 2.3, two lines L^3 and L^4 do *not intersect orthogonally!* However, in the relativistic Lorentz metric, two lines L^3 and L^4 *do* intersect orthogonally! The Lorentz transformation in (2.7) mediates, physically speaking, the transformation from one (inertial) observer to another (inertial) observer moving with velocity v along the x^3-axis. (A popular name for the Lorentz transformation in (2.7) is a *boost*.) This mapping induces the phenomena of length contraction and time dilation between observers in relative motion [55, 89]. □

Now, we shall discuss (Minkowskian) tensor fields in the flat space–time manifold. We use a global Minkowskian coordinate chart so that

$$\begin{aligned}\mathbf{g}_{..}(x) &= d_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \mathrm{d}x^4 \otimes \mathrm{d}x^4 \\ &= \delta_{\alpha\beta}\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta - \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\ \mathbf{g}^{..}(x) &= d^{ij}\, \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}\,, \\ \operatorname{sgn}(\mathbf{g}_{..}(x)) &= +2. \end{aligned} \tag{2.8}$$

The natural orthonormal basis set or tetrad is provided by

$$\vec{\mathbf{e}}_{(a)}(x) = \lambda^i_{\ (a)}(x)\,\frac{\partial}{\partial x^i} := \delta^i_{\ (a)}\,\frac{\partial}{\partial x^i}\,. \tag{2.9}$$

(See (1.104).)

A tensor field ${}^r_s\mathbf{T}(x)$ and its Minkowskian components satisfy

$${}^r_s\mathbf{T}(x) = T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)\,\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}. \tag{2.10}$$

(Compare the above with (1.30).)

The transformations of tensor field components under a Poincaré transformation in (2.6i–iv) are provided by

$$\widehat{T}^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(\hat{x}) = l^{k_1}{}_{i_1},\ldots,\, l^{k_r}{}_{i_r}\, a^{j_1}{}_{l_1},\ldots,\, a^{j_s}{}_{l_s}\, T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x). \tag{2.11}$$

(Compare the above with (1.37).)

It should be noted that covariant derivatives in Minkowskian charts are *equivalent to partial derivatives*, as the Christoffel symbols are identically zero in Minkowskian charts in flat space–time.

Example 2.1.7. Consider the Lorentz metric tensor in (2.8). Under a Poincaré transformation, using (2.11), we deduce that

$$\hat{d}_{ij} = a^k{}_i\, a^l{}_j\, d_{kl} = d_{ij}\,.$$

(For the derivation of the last step, we have used (2.6ii, iv).) Therefore, the Lorentz metric tensor behaves as a *numerical tensor* under Poincaré transformations. (Consult [56] for discussions on numerical tensors.) □

One of the mathematical axioms of the special relativity is that *every natural law must be expressible as a tensor field equation*

$${}^r_s\mathbf{T}(x) = {}^r_s\mathbf{O}(x),$$

$$\text{or}\quad T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x) = 0. \tag{2.12}$$

(Sometimes, such tensor equations are restricted on a curve.)

By a Poincaré transformation, (2.12) simply imply that

$$\widehat{T}^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(\hat{x}) = 0. \tag{2.13}$$

Therefore, a natural law, as formulated by one inertial (i.e., experiencing no net force) observer, must be *exactly similar* to that of another moving (inertial) observer. This is the principle of special relativistic "covariance." (Later, other types of objects, *spinor fields*, had to be added on.)

Now, we shall deal with differentiable, parametric curves in space–time. We shall consider a nondegenerate parametric curve $\mathcal{X}$ of class C^2. (See Fig. 1.6.) We reiterate (1.19) and (1.20) as

$$\begin{aligned} x &= \mathcal{X}(t), \\ x^i &= \mathcal{X}^i(t), \\ \sum_{j=1}^{4} &\left[\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\right]^2 > 0, \\ t &\in [a,b] \subset \mathbb{R}. \end{aligned} \tag{2.14}$$

Now, we explore the invariant values of

$$\mathbf{g}_{..}(\mathcal{X}(t))\left[\vec{\boldsymbol{\mathcal{X}}}'(t), \vec{\boldsymbol{\mathcal{X}}}'(t)\right] = d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}. \tag{2.15}$$

A *spacelike curve* satisfies

$$d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} > 0. \tag{2.16i}$$

A *timelike curve* is characterized by

$$d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} < 0. \tag{2.16ii}$$

Moreover, a (nondegenerate) *null curve* satisfies

$$d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} = 0. \tag{2.16iii}$$

Qualitative pictures, together with an upper null cone, are exhibited in Fig. 2.4. Note that the slope of timelike curves is everywhere steeper than that of the null cone, the slope of null curves is coincident with the slope of the null cone, and the slope of spacelike curves is shallower than the null cone.

From relativistic physics, it is known that (1) an (idealized) massive point particle travels along a timelike curve; (2) a massless point particle (like a photon) traverses along a null curve; (3) and moreover, hypothetical superluminal particles called tachyons possess trajectories which are represented by spacelike curves.

Now, we shall introduce the arc separation along *a timelike curve*. (See (1.193).) It is furnished by

$$s = \mathcal{S}(t) := \int_a^t \sqrt{-d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(u)}{\mathrm{d}u}\,\frac{\mathrm{d}\mathcal{X}^j(u)}{\mathrm{d}u}} \cdot \mathrm{d}u. \tag{2.17}$$

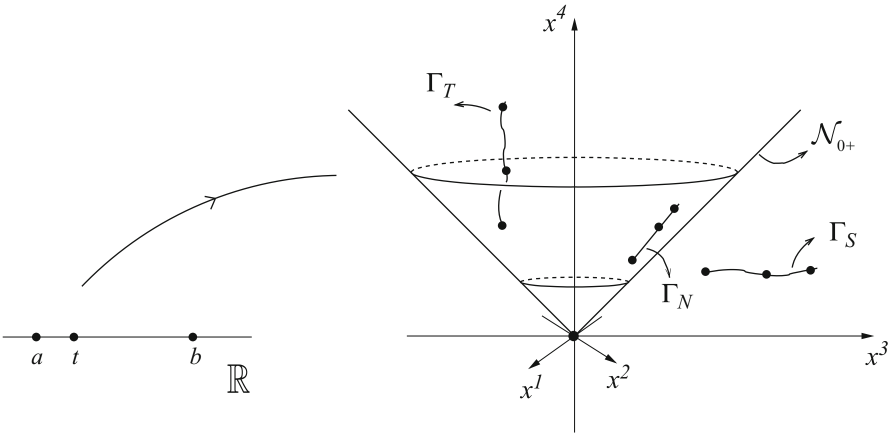

Fig. 2.4 Images Γ_S, Γ_T, and Γ_N of a spacelike, timelike, and a null curve

The above arc separation s (sometimes popularly denoted by τ also) is called the *proper time* along the timelike curve. It is *assumed*[1] to provide *the actual time separation* between the events $\mathcal{X}(a)$ and $\mathcal{X}(t)$ as an idealized, standard point clock traverses along the timelike curve $\mathcal{X}$ (or, the *world line*). The arc separation along a null curve is *exactly zero*.

An *inertial observer* in special relativity theory traverses a timelike, straight line satisfying

$$\frac{\mathrm{d}^2\mathcal{X}^i(t)}{\mathrm{d}t^2} = 0. \tag{2.18}$$

The above equation is that of *a geodesic* in flat space–time with a Minkowskian coordinate chart. (Compare with (1.183).)

An inertial observer is an idealized, massive point being carrying a (point) standard clock and four orthogonal directions inherent in a tetrad. He or she is also capable of sending photons outward and to receive them at later times after reflections from other specular point objects travelling on timelike world lines. In this process, the observer can assign *operational Minkowskian coordinates* to the events around him/her. Such a method of measurements is called *Minkowskian chronometry*. (Consult the book by Synge [242].) *A non-inertial, idealized observer* follows a timelike world line with $\frac{\mathrm{d}^2\mathcal{X}^i(t)}{\mathrm{d}t^2} \neq 0$.

Remark: It should be mentioned that the world line of an inertial (or else non-inertial) observer violates the uncertainty principle of quantum mechanics, which prohibits the simultaneous existence of exact position and velocity. (The arguments here are for classical idealized objects where quantum corrections are negligible for these purposes.)

[1]There are strong physical arguments for this assumption which are beyond the scope of this brief review of special relativity. The interested reader is referred to [55].

Now, we shall study the equations of motion of a massive particle of *constant mass*. In prerelativistic physics, Newton's equations of motion are furnished by

$$m\frac{\mathrm{d}^2\mathcal{X}^\alpha(t)}{\mathrm{d}t^2} =: m\frac{\mathrm{d}V^\alpha(t)}{\mathrm{d}t} = f^\alpha(\mathbf{x}, t, \mathbf{v})_{|x^\alpha=\mathcal{X}^\alpha(t)}, \quad V^\alpha(t) = \frac{\mathrm{d}\mathcal{X}^\alpha(t)}{\mathrm{d}t}. \tag{2.19i}$$

Here, $m > 0$ is the mass parameter and t is the (absolute) time. Moreover, $\mathbf{x} = (x^1, x^2, x^3)$ are Cartesian coordinates; $V^\alpha(t)$ and $f^\alpha(\mathbf{x}, t, \mathbf{v})_{|..}$ are the Cartesian components of *instantaneous velocity* and the net external force vector, respectively. One consequence of (2.19i) is that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[(m/2)\delta_{\alpha\beta}V^\alpha(t)V^\beta(t)\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[k + (m/2)\delta_{\alpha\beta}V^\alpha(t)V^\beta(t)\right] \\ &= \delta_{\alpha\beta}V^\alpha(t)\left[f^\beta(\cdot\cdot)_{|..}\right].\end{aligned} \tag{2.19ii}$$

The above equation can be physically interpreted as "the rate of increase of the kinetic energy plus an undetermined energy constant k is equal to the rate of work performed by the external force."

In special relativity, using a Minkowskian chart, we represent the motion curve by

$$\begin{aligned}x^\alpha &= \mathcal{X}^{\#\alpha}(s), \\ t \equiv x^4 &= \mathcal{X}^{\#4}(s).\end{aligned} \tag{2.20}$$

Here, $s \in [0, s_1]$ and $\mathcal{X}^{\#k}$ are C^2-functions. Furthermore, we have chosen s to be the arc separation parameter or proper time. (See (1.193).) It is also assumed that the motion curve is timelike and future pointing, that is,

$$d_{kl}\,\frac{\mathrm{d}\mathcal{X}^{\#k}(s)}{\mathrm{d}s}\,\frac{\mathrm{d}\mathcal{X}^{\#l}(s)}{\mathrm{d}s} \equiv -1, \tag{2.21i}$$

$$\frac{\mathrm{d}\mathcal{X}^{\#4}(s)}{\mathrm{d}s} > 0, \tag{2.21ii}$$

$$d_{kl}\,\frac{\mathrm{d}\mathcal{X}^{\#k}(s)}{\mathrm{d}s}\,\frac{\mathrm{d}^2\mathcal{X}^{\#l}(s)}{\mathrm{d}s^2} \equiv 0. \tag{2.21iii}$$

We denote the *four-velocity* components by

$$u^i = \mathcal{U}^i(s) := \frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s}\,, \tag{2.22i}$$

$$d_{ij}\,u^i u^j = -1, \tag{2.22ii}$$

$$|u^4| \geq 1. \tag{2.22iii}$$

In the four-dimensional space of four-velocity components, the quadratic constraint (2.22ii) yields a three-dimensional hyperhyperboloid of two disconnected subsets. (See Fig. 2.5.)

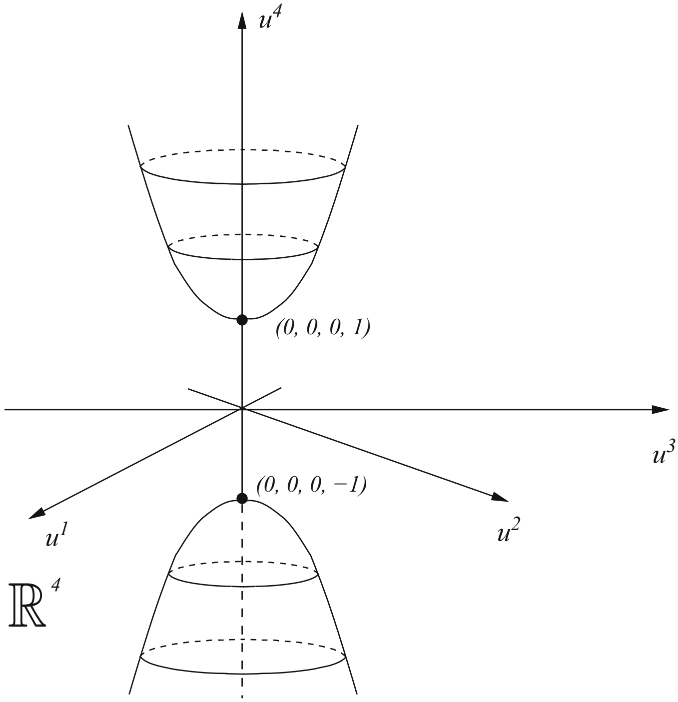

Fig. 2.5 The three-dimensional hyperhyperboloid representing the 4-velocity constraint

To compare relativistic velocity components with nonrelativistic velocity components, we need to reparametrize the motion curve by (2.17). Thus, we obtain

$$\begin{aligned} x^\alpha &= \mathcal{X}^{\#\alpha}(\mathcal{S}(t)) = \mathcal{X}^\alpha(t), \\ x^4 &= \mathcal{X}^{\#4}(s) = \mathcal{X}^4(t) = t, \\ V^\alpha(t) &:= \frac{\mathrm{d}\mathcal{X}^\alpha(t)}{\mathrm{d}t}, \\ \|\vec{\mathbf{V}}(t)\| &:= +\sqrt{\delta_{\mu\nu}V^\mu(t)V^\nu(t)}\,. \end{aligned} \tag{2.23}$$

(Note that $V^\alpha(t)$ are *not spatial components of a relativistic 4-vector*.)

Using (1.24), (2.17), (2.22i), and (2.23), we derive that

$$\begin{aligned} \mathcal{U}^\alpha(s) &= \frac{\mathrm{d}\mathcal{X}^{\#\alpha}(s)}{\mathrm{d}s} = \left[\frac{\mathrm{d}\mathcal{S}(t)}{\mathrm{d}t}\right]^{-1}\frac{\mathrm{d}\mathcal{X}^\alpha(t)}{\mathrm{d}t} = \frac{V^\alpha(t)}{\sqrt{1-\|\vec{\mathbf{V}}(t)\|^2}}, \\ \mathcal{U}^4(s) &= \frac{\mathrm{d}\mathcal{X}^{\#4}(s)}{\mathrm{d}s} = \left[\frac{\mathrm{d}\mathcal{S}(t)}{\mathrm{d}t}\right]^{-1}\frac{\mathrm{d}\mathcal{X}^4(t)}{\mathrm{d}t} = \frac{1}{\sqrt{1-\|\vec{\mathbf{V}}(t)\|^2}}. \end{aligned} \tag{2.24}$$

The above equations are mathematically valid *only for* $\| \vec{\mathbf{V}}(t) \| < 1$. Or, in other words, the nonrelativistic speed of a massive material particle *must be strictly less than the speed of light*.

Now, we shall generalize Newton's equations of motion (2.19i, ii) to the appropriate relativistic equations. For a particle with *constant mass* $m > 0$, we *assume that the relativistic equations are furnished by*:

$$m \frac{\mathrm{d}^2 \mathcal{X}^{\# i}(s)}{\mathrm{d}s^2} = F^i(x,u)_{\big| x=\mathcal{X}^{\#}(s),\, u=\frac{\mathrm{d}\mathcal{X}^{\#}(s)}{\mathrm{d}s}}. \tag{2.25}$$

Here, the *4-force vector* components $F^i(\cdot\cdot)$ are continuous functions of *eight variables*. (See [55] and [243].) Note that from (2.21iii), we obtain

$$\begin{aligned} &d_{ij} \frac{\mathrm{d}\mathcal{X}^{\# i}(s)}{\mathrm{d}s} F^j(\cdot\cdot)_{|\cdot\cdot} \equiv 0, \\ &\frac{\mathrm{d}\mathcal{X}^{\# 4}(s)}{\mathrm{d}s} F^4(\cdot\cdot)_{|\cdot\cdot} = \delta_{\alpha\beta} \frac{\mathrm{d}\mathcal{X}^{\#\alpha}(s)}{\mathrm{d}s} F^\beta(\cdot\cdot)_{|\cdot\cdot}. \end{aligned} \tag{2.26}$$

Thus, by Theorems 2.1.2 and 2.1.4, *a nonzero 4-force vector must be spacelike*.

Relativistic equations (2.25) yield, using (2.23) and (2.24), the equations

$$m \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{V^\alpha(t)}{\sqrt{1 - \| \vec{\mathbf{V}}(t) \|^2}} \right] = \sqrt{1 - \| \vec{\mathbf{V}}(t) \|^2} \cdot F^\alpha(\cdot\cdot)_{|x=\mathcal{X}(t)}, \tag{2.27i}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{m}{\sqrt{1 - \| \vec{\mathbf{V}}(t) \|^2}} \right] = \sqrt{1 - \| \vec{\mathbf{V}}(t) \|^2} \cdot F^4(\cdot\cdot)_{|\cdot\cdot}. \tag{2.27ii}$$

In the low speed $\left(\| \vec{\mathbf{V}}(t) \| \ll 1 \right)$ regime, the equations in (2.27i) reduce to Newton's equations of motion (2.19i), provided we equate

$$f^\alpha(\cdot\cdot)_{|\cdot\cdot} = \sqrt{1 - \| \vec{\mathbf{V}}(t) \|^2} \cdot F^\alpha(\cdot\cdot)_{|\cdot\cdot}. \tag{2.28}$$

From (2.28) and (2.26), we derive that

$$\sqrt{1 - \| \vec{\mathbf{V}}(t) \|^2} \cdot F^4(\cdot\cdot)_{|\cdot\cdot} = \delta_{\alpha\beta} V^\alpha(t) f^\beta(\cdot\cdot)_{|\cdot\cdot}. \tag{2.29}$$

Substituting (2.29) into (2.27ii), we deduce that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{m}{\sqrt{1 - \| \vec{\mathbf{V}}(t) \|^2}} \right] &= \frac{\mathrm{d}}{\mathrm{d}t} \left[m + (m/2)\delta_{\alpha\beta} V^\alpha(t) V^\beta(t) + 0\left(\| \vec{\mathbf{V}}(t) \|^4 \right) \right] \\ &= \delta_{\alpha\beta} V^\alpha(t) f^\beta(\cdot\cdot)_{|\cdot\cdot}. \end{aligned} \tag{2.30}$$

Comparing (2.30) with (2.19ii), we are prompted to express the energy of the particle as

$$E\left(\| \vec{\mathbf{v}} \|\right) := \frac{m}{\sqrt{1- \| \vec{\mathbf{v}} \|^2}}, \tag{2.31i}$$

$$\lim_{\|\vec{\mathbf{v}}\| \to 0_+} E\left(\| \vec{\mathbf{v}} \|\right) = E(0_+) = m. \tag{2.31ii}$$

Explicitly reinstating (temporarily) the speed of light c, (2.31ii) reveals the rest energy as $E(0_+) = mc^2$. (*This is arguably the most celebrated equation* in modern science! [87].) Since energy $E(\cdot\cdot)$ was derived from the expression $m\frac{d\mathcal{X}^{\#4}(s)}{ds}$, it is logical to define the (kinetic) *4-momentum* of a massive particle as

$$\begin{aligned} p^i &= \mathcal{P}^i(s) := m\frac{d\mathcal{X}^{\#i}(s)}{ds}, \\ p_i &= \mathcal{P}_i(s) = d_{ij}\,\mathcal{P}^j(s), \\ p_\alpha &= p^\alpha, \quad p_4 = -p^4. \end{aligned} \tag{2.32}$$

The constraint (2.22ii) yields a similar constraint on 4-momentum components in (2.32), and it is furnished by

$$d_{ij}\,p^i p^j = d^{ij}\,p_i p_j = -(m)^2. \tag{2.33}$$

The above yields a three-dimensional hyperhyperboloid in the four-dimensional momentum space. (Compare with the corresponding Fig. 2.5.) This hypersurface is known as the *mass shell* in relativistic physics.

Example 2.1.8. Let us investigate the timelike motion of a constant mass particle under a nonzero, constant-valued 4-force vector. By (2.25), we write

$$m\frac{d^2\mathcal{X}^{\#i}(s)}{ds^2} = K^i = \text{const.}, \tag{2.34i}$$

$$\frac{d^2\mathcal{X}^{\#i}(s)}{ds^2} = \frac{K^i}{m} =: C^i = \text{const.} \tag{2.34ii}$$

By the Theorems 2.1.2 and 2.1.4, we conclude that

$$d_{ij}\,C^i C^j > 0. \tag{2.35}$$

Solving differential equations (2.34ii) under the initial values $\mathcal{X}^{\#i}(0) = x_0^i$, $\frac{d\mathcal{X}^{\#i}(s)}{ds}_{|s=0} = u_0^i$, we obtain the solution as

$$\mathcal{X}^{\#i}(s) = x_0^i + u_0^i \cdot s + (1/2)C^i \cdot (s)^2. \tag{2.36}$$

Using the identity (2.21i), we deduce from (2.36) that

$$0 \equiv \frac{\mathrm{d}}{\mathrm{d}s}\left[d_{ij}\,\frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s}\,\frac{\mathrm{d}^2\mathcal{X}^{\#j}(s)}{\mathrm{d}s^2}\right] = d_{ij}\,C^i C^j. \tag{2.37}$$

Equation (2.37) yielding $d_{ij}C^iC^j = 0$ obviously *violates the strict inequality (2.35)*. Therefore, we conclude that such *a particle motion with constant 4-acceleration is impossible*.[2] □

Now, we shall investigate the motions of particles inside *an extended body*. It is the theme of the usual nonrelativistic *continuum mechanics*. The scope of this branch of applied mathematics encompasses fluids, elastic materials, visco-elastic materials, plasmas, and so on. (Usually, the relativistic effects, which are dominant at high velocities, are neglected in these disciplines.)

Let us consider a nonrelativistic fluid motion in Euler's approach. There is a three-velocity field $\mathcal{V}^\alpha(\mathbf{x}, t)$ which is a measure of the instantaneous fluid velocity of a particle at the spatial point $\mathbf{x} = (x^1, x^2, x^3)$ and time t. The nonlinear equations of motion of streamlines are governed by

$$\frac{\partial\mathcal{V}^\alpha(\cdot\cdot)}{\partial t} + \mathcal{V}^\beta(\mathbf{x}, t)\,\partial_\beta\mathcal{V}^\alpha(\cdot\cdot) = [\rho(\mathbf{x}, t)]^{-1}\,\partial_\beta\sigma^{\alpha\beta} + \phi^\alpha_{\text{(ext)}}(\mathbf{x}, t). \tag{2.38}$$

Here, $\rho(\mathbf{x}, t) > 0$ is the *mass density*, $\sigma^{\beta\alpha}(\mathbf{x}, t) \equiv \sigma^{\alpha\beta}(\mathbf{x}, t)$ is the *stress density*, and $\phi^\alpha_{\text{(ext)}}(\mathbf{x}, t)$ is the force density due to external influences. (See [244].) The streamlines are furnished by solutions or integral curves of the differential equations

$$\frac{\mathrm{d}\mathcal{X}^\alpha(t)}{\mathrm{d}t} = \mathcal{V}^\alpha(\mathbf{x}, t)_{|..}, \tag{2.39}$$

where $\mathcal{V}^\alpha(\mathbf{x},\ t)$ satisfy (2.38). (Compare with (1.75).)

Restricting the vector field equations (2.38) on a streamline $x^\alpha = \mathcal{X}^\alpha(t)$, we obtain

$$\rho(\mathbf{x}, t)_{|..}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{V}^\alpha(\cdot\cdot)\right]_{|..} = \left[\partial_\beta\sigma^{\alpha\beta} + \rho\phi^\alpha_{\text{(ext)}}(\cdot\cdot)\right]_{|..}. \tag{2.40}$$

It is obvious that Newton's equations of motion (2.19i) have motivated (2.40). *In the absence of an external force*, (2.38) reduce to

$$\rho(\cdot\cdot)\left[\frac{\partial\mathcal{V}^\alpha(\cdot\cdot)}{\partial t} + \mathcal{V}^\beta(\cdot\cdot)\,\partial_\beta\mathcal{V}^\alpha(\cdot\cdot)\right] - \partial_\beta\sigma^{\alpha\beta} = 0. \tag{2.41}$$

Now, in the relativistic theory, we need to visualize a (bounded) compact fluid body as a *world tube* in space–time. (See the Fig. 2.6.)

[2]Often, the term "constant acceleration" in the relativity literature refers to the condition $d_{ij}\,\frac{\mathrm{d}\mathcal{U}^i(s)}{\mathrm{d}s}\,\frac{\mathrm{d}\mathcal{U}^j(s)}{\mathrm{d}s} = \text{const.}$, which may be satisfied with nonconstant $\frac{\mathrm{d}\mathcal{U}^i(s)}{\mathrm{d}s} = \frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}$.

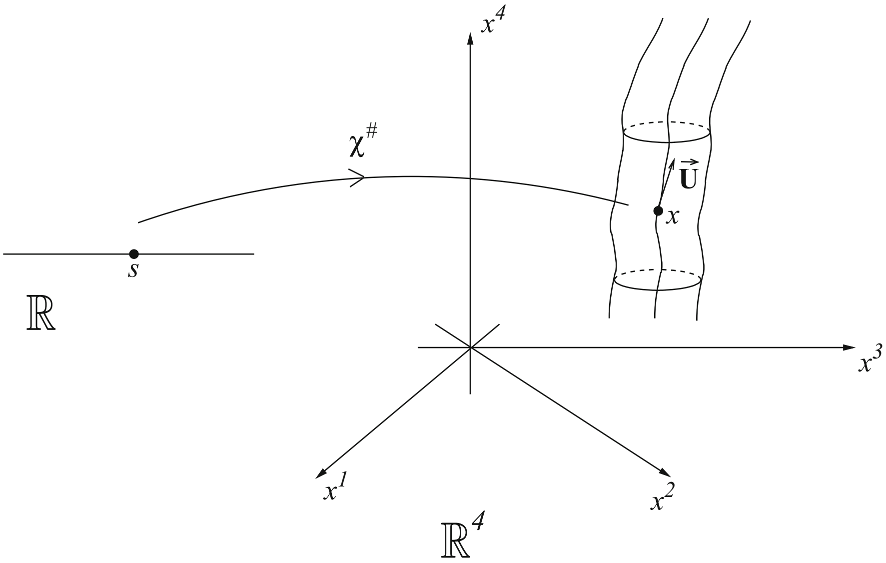

Fig. 2.6 A world tube and a curve representing a fluid streamline

The timelike, future-pointing 4-velocity vector field $\vec{\mathbf{U}}(x)$ satisfies

$$\begin{aligned} U_i(x)U^i(x) = d_{ij}U^i(x)U^j(x) &\equiv -1, \\ U^4(x) &\geq 1. \end{aligned} \tag{2.42}$$

A fluid streamline or curve of class C^2 is characterized by integral curves of the differential equations

$$\frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s} = U^i(x)_{|x=\mathcal{X}^{\#}(s)}. \tag{2.43}$$

(Compare with (2.39).)

The appropriate relativistic generalizations of the equations of motion in (2.41) are furnished by

$$\rho(x)\left[U^j(x)\,\partial_j U^i\right] - \left[\delta^i{}_k + U^i(x)U_k(x)\right]\partial_j s^{kj} = 0. \tag{2.44}$$

Here, $\rho(x) > 0$ is the proper energy density (including mass energy density). Moreover, $U^i(x)$ are components of the 4-velocity field of streamlines, and $s^{ji}(x) \equiv s^{ij}(x)$ are components of *the relativistic symmetric stress tensor field* of differentiability class C^1.

Now, we are in a position to define the relativistic, symmetric *energy–momentum–stress tensor* field by

$$\mathbf{T}^{\cdot\cdot}(x) := T^{ij}(x)\,\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$
$$T^{ij}(x) := \rho(x)U^i(x)U^j(x) - s^{ij}(x)$$
$$\equiv T^{ji}(x). \tag{2.45}$$

Remarks: (i) Some authors call the above tensor the stress–momentum–energy tensor, or stress–energy tensor, or simply the stress tensor.

(ii) In relativistic quantum field theory, $T^{ij}(x)$ sometimes represents components of *the canonical energy–momentum–stress tensor* for which $T^{ji}(x) \not\equiv T^{ij}(x)$. In this text, *we will not be utilizing the canonical variant of this tensor*.

Note that $T^{ij}(x)$ in (2.45) possesses ten linearly independent components.

Now, we shall state and prove a theorem about the energy–momentum–stress tensor field.

Theorem 2.1.9. *Let a symmetric energy–momentum–stress tensor field* $\mathbf{T}^{\cdot\cdot}(x)$ *be defined by* (2.45) *in* $D \subset \mathbb{R}^4$ *representing a fluid world tube. Furthermore, let the tensor field components* $T^{ij}(x)$ *be of differentiability class* C^1*. Then, the four partial differential equations*

$$\partial_j\, T^{ij} = 0 \tag{2.46}$$

imply the relativistic equations of streamline motion in (2.44)*. Moreover,* (2.46) *implies one additional differential equation*

$$\partial_j\left[\rho U^j\right] + U_i(x)\,\partial_j s^{ij} = 0. \tag{2.47}$$

Proof. By (2.45) and (2.46), we obtain

$$\partial_j\left[\rho U^i U^j - s^{ij}\right] = \rho(x)\,U^j(x)\,\partial_j\,U^i + U^i(x)\,\partial_j\left[\rho U^j\right] - \partial_j s^{ij} = 0. \tag{2.48}$$

Multiplying the above by $U_i(x)$ and contracting, we derive that

$$(1/2)\rho U^j \partial_j\left[U_i U^i\right] + \left[U_i U^i\right]\partial_j\left[\rho U^j\right] - U_i\,\partial_j s^{ij} = 0.$$

Now, substituting $U_i(x)U^i(x) \equiv -1$ from (2.42) into the preceding equation, we deduce (2.47). Putting (2.47) into (2.48), we derive the equations of motion for the streamlines in (2.44). ■

Remark: The equation $\partial_j T^{ij} = 0$ in (2.46) is called the *differential conservation of energy–momentum*.

Now, we shall try to understand physically the energy–momentum–stress tensor in (2.45). Suppose that an idealized inertial observer passes through the fluid and intersects a streamline at the event x in Fig. 2.6. Assume that *this test observer does not disturb any of the streamlines*. He carries an orthonormal basis $\left\{\vec{\mathbf{e}}_{(a)}(x)\right\}_1^4$ or a tetrad. Imagine that this observer manages to measure numbers associated with

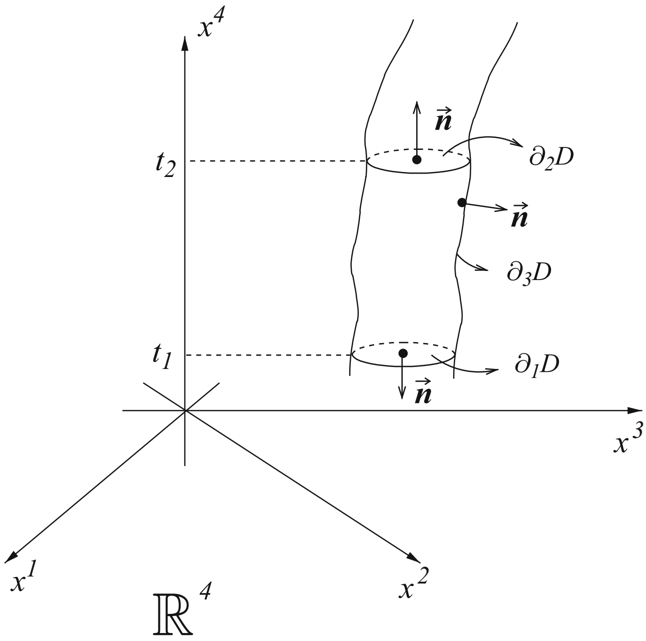

Fig. 2.7 A doubly sliced world tube of an extended body

the orthonormal components (or *physical components*) $T^{(a)(b)}(x)$ of the energy–momentum–stress tensor $\mathbf{T}^{\cdot\cdot}(x)$. In such a scenario, physical interpretations can be furnished. $T^{(4)(4)}(x)$ is the sum of the material energy density $\rho(x)\left[U^4(x)\right]^2$ and the internal energy density $-s^{(4)(4)}(x)$. $T^{(\alpha)(4)}(x)$ are the sums of kinetic momentum density $\rho(x)U^\alpha(x)U^4(x)$ and internal momentum density $-s^{(\alpha)(4)}(x)$. ($T^{(\alpha)(4)}(x)$ is often called the *energy flux* or *energy flux vector*, although it is not truly a vector.) Finally, $T^{(\alpha)(\beta)}(x)$ are sums of material stress components $\rho(x)U^\alpha(x)U^\beta(x)$ and the *negative of the usual internal stress components* $-s^{(\alpha)(\beta)}(x)$. A thorough discussion of the physical interpretation of the energy–momentum–stress tensor may be found in [230, 243].

Now, we are in a position to investigate the *integral conservation* or *total conservation of energy–momentum* of a fluid or other extended bodies. We provide another figure of a material world tube with two "horizontal caps" in Fig. 2.7.

Now, we shall state and prove the integral conservation laws. (See Fig. 2.7.)

Theorem 2.1.10. *Let $\mathbf{T}^{\cdot\cdot}(x) \not\equiv \mathbf{0}^{\cdot\cdot}(x)$ and be of class C^1 inside a material world tube. Furthermore, let $\mathbf{T}^{\cdot\cdot}(x) \equiv \mathbf{0}^{\cdot\cdot}(x)$ outside the world tube. Let $x^4 = t_1$ and $x^4 = t_2$ yield three-dimensional cross sections of the world tube denoted by $\partial_1 D$ and $\partial_2 D$, respectively. Moreover, let $\partial_3 D$ be the three-dimensional boundary "wall" around the domain D of the tube. Furthermore, let boundary conditions $T^{ij}(x)n_j(x)_{|\partial_3 D} = 0$, where n^i denotes components of unit outer* *normal, be satisfied. Let the overall boundary* $\partial D := \partial_1 D \cup \partial_2 D \cup \partial_3 D$ *be non-null, continuous, piecewise differentiable, and orientable. Then, the differential conservation equations* (2.46) *imply the integral conservation equations*

$$
\begin{aligned}
P^i &:= \int\limits_{\partial_1 D} T^{4i}(x^1, x^2, x^3, t_1)\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \\
&= \int\limits_{\partial_2 D} T^{4i}(x^1, x^2, x^3, t_2)\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \\
&= \int\limits_{\partial D_{(t)}} T^{4i}(x^1, x^2, x^3, t)\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \\
&= \text{const.}
\end{aligned}
\tag{2.49}
$$

Proof. Applying Gauss' Theorem 1.3.27 and differential conservation equations (2.46), we obtain

$$
\begin{aligned}
0 &= \int\limits_{D} \left[\partial_j T^{ij}\right] \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \mathrm{d}x^4 \\
&= \int\limits_{\partial_1 D} T^{ij}\, n_j\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 + \int\limits_{\partial_2 D} T^{ij}\, n_j\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 + \int\limits_{\partial_3 D} T^{ij}\, n_j\, \mathrm{d}^3 v \\
&= -\int\limits_{\partial_1 D} T^{i4}\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 + \int\limits_{\partial_2 D} T^{i4} \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 + 0.
\end{aligned}
$$

Thus, (2.49) is validated. ■

The constant-valued components P^i represent the conserved *total 4-momentum* of the extended body.

We can define the *relativistic total angular momentum* of an extended body *relative to the event* x_0, by the following equations:

$$
\begin{aligned}
J^{ik} &:= \int\limits_{\partial D_{(t)}} \big[(x^i - x_0^i)\, T^{k4}(x^1, x^2, x^3, t) \\
&\qquad - (x^k - x_0^k)\, T^{i4}(x^1, x^2, x^3, t)\big] \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3, \\
J^{ki} &= -J^{ik} = \text{const.}
\end{aligned}
\tag{2.50}
$$

The proof for the constancies of J^{ik} will be left as Problem #7 in Exercise 2.1.

Example 2.1.11. Consider the case of *the incoherent dust* (pressureless fluid) which happens to be one of the simplest of all materials. The definition is summarized in the energy–momentum–stress tensor

$$T^{ij}(x) := \rho(x)U^i(x)U^j(x). \tag{2.51}$$

Equation (2.47) yields

$$\partial_j \left[\rho U^j\right] = 0 \tag{2.52}$$

and is known as the *relativistic continuity equation* for the dust.

The equations for streamline motion (2.44) reduce to

$$\begin{aligned} \rho(x)\left[U^j(x)\partial_j U^i\right] &= 0, \\ \text{or,} \quad \left[U^j(x)\partial_j U^i\right]_{|x=\mathcal{X}^{\#}(s)} &= \frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2} = 0. \end{aligned} \tag{2.53}$$

Therefore, it is evident that the dust particles, in the absence of external forces, move along timelike straight lines (or geodesics) in the flat space–time. (It is an expected result, as the absence of pressure and shears implies that adjacent elements of the dust cannot exert any forces on each other.) □

The special theory of relativity emerged from investigations of the speed of light, which is also the speed of electromagnetic wave propagation. Therefore, the electromagnetic field equations are of particular interest in special relativity theory.

The classical electromagnetic field is governed by *Maxwell's equations*.[3] As three-dimensional vector field equations, they are (in the absence of sources):

$$\nabla \times \vec{\mathbf{H}}(\mathbf{x}, t) = \frac{\partial \vec{\mathbf{E}}(\mathbf{x}, t)}{\partial t}, \tag{2.54i}$$

$$\nabla \cdot \vec{\mathbf{E}}(\cdot\cdot) = 0, \tag{2.54ii}$$

$$\nabla \cdot \vec{\mathbf{H}}(\cdot\cdot) = 0, \tag{2.54iii}$$

$$\nabla \times \vec{\mathbf{E}}(\cdot\cdot) = -\frac{\partial \vec{\mathbf{H}}(\cdot\cdot)}{\partial t}. \tag{2.54iv}$$

Here, t is the time variable, $\mathbf{x} = (x^1, x^2, x^3)$ represents Cartesian coordinates, and ∇ is the (spatial) gradient operator. Moreover, $\vec{\mathbf{E}}(\cdot\cdot)$ and $\vec{\mathbf{H}}(\cdot\cdot)$ are electric and magnetic fields, respectively. The above equations can be cast neatly into Minkowskian tensor

[3] We are using *Lorentz–Heaviside units*. Especially, we have set c, the speed of light, to be *unity* to be consistent with units used throughout most of the book.

field equations. To achieve this, we have to identify the electric and magnetic three-vectors with an antisymmetric, second-order Minkowskian tensor as shown below:

$$\begin{aligned}
\left[F_{ij}(x)\right]_{4\times 4} &:= \begin{bmatrix} 0 & H_3(x) & -H_2(x) & E_1(x) \\ -H_3(x) & 0 & H_1(x) & E_2(x) \\ H_2(x) & -H_1(x) & 0 & E_3(x) \\ -E_1(x) & -E_2(x) & -E_3(x) & 0 \end{bmatrix} \\
&\equiv -\left[F_{ji}(x)\right]. \qquad (2.55)
\end{aligned}$$

Maxwell's equations (2.54i–iv) are (exactly) equivalent to the four-dimensional tensor field equations:

$$\partial_j F^{ij} = 0, \qquad (2.56\text{i})$$

$$\text{and} \quad \partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki} = 0. \qquad (2.56\text{ii})$$

Thus, Maxwell's equations (2.54i–iv), which were discovered approximately 40 years before the advent of the special theory of relativity, were *already relativistic!*

Example 2.1.12. We assume here the tensor field $\mathbf{F}_{..}(x)$ is of class C^2. Differentiating the mixed tensor form of (2.56ii), we obtain

$$\begin{aligned}
0 &= \partial_i \partial_k F^i_{\ j} + d^{il} \partial_i \partial_l F_{jk} - \partial_i \partial_j F^i_{\ k} \\
&= \partial_k \left[\partial_i F^i_{\ j}\right] + \Box F_{jk} - \partial_j \left[\partial_i F^i_{\ k}\right] \\
&= 0 + \Box F_{jk} - 0 \\
&= \left[\frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2} - \frac{\partial^2}{(\partial x^4)^2}\right] F_{jk}(x).
\end{aligned}$$

Note that these equations are wave equations with a speed of unity. Therefore, the above equations show that electromagnetic wave fields travel with the speed of light. The wave operator $\Box$ remains invariant under Poincaré transformations (2.6i), particularly under the Lorentz transformation (2.7). Therefore, *the speed of light (or a photon) remains unchanged for a moving inertial observer.* □

Equations (2.56i,ii) can be expressed succinctly in terms of differential 2-forms as introduced in (1.64). Let us repeat the 2-form here as

$$\mathbf{F}_{..}(x) = (1/2) F_{ij}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j .$$

By (1.69), we can express Maxwell's equations (2.56ii) as

$$\mathrm{d}\,[\mathbf{F}_{..}(x)] = \mathbf{0}_{...}(x). \tag{2.57}$$

(Glance through (1.70) for the electromagnetic 4-potential $A_i(x)\,\mathrm{d}x^i$ and for gauge transformations.) We repeat the Hodge-star operation in (1.113). (See also Example 1.3.6.) Thus, we express

$$*\,\mathbf{F}_{..}(x) = (1/2)\left[\eta^{ij}{}_{kl}\,F_{ij}(x)\right]\mathrm{d}x^k \wedge \mathrm{d}x^l. \tag{2.58}$$

Therefore, Maxwell's equations (2.56i), by the use of Example 1.3.6, reduce to

$$\mathrm{d}\,[*\mathbf{F}_{..}(x)] = *\mathbf{0}_{...}. \tag{2.59}$$

In the presence of electrically charged matter, the *Maxwell–Lorentz equations* (using Lorentz–Heaviside units) are

$$\partial_k F^{ik} = J^i(x), \tag{2.60i}$$

$$\partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki} = 0, \tag{2.60ii}$$

$$\partial_i J^i = \partial_i \partial_k F^{ik} \equiv 0. \tag{2.60iii}$$

In physical terms, $J^4(x)$ and $J^\alpha(x)$ represent the electrical charge density and current density, respectively.

We introduce the Hodge-star operation (1.113) on the 4-*charge-current* vector $J^i(x)$ to express

$$*\,\mathbf{J}_{...}(x) = \left[\eta^l{}_{ijk}\,J_l(x)\right]\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k. \tag{2.61}$$

Therefore, the electromagnetic equations (2.60i–iii) reduce to

$$\mathrm{d}\,[*\mathbf{F}_{..}(x)] = \ *\,\mathbf{J}_{...}(x), \tag{2.62i}$$

$$\mathrm{d}\,[\mathbf{F}_{..}(x)] = \mathbf{0}_{...}(x), \tag{2.62ii}$$

$$\mathrm{d}\,[*\mathbf{J}_{...}(x)] = \mathrm{d}^2\,[*\mathbf{F}_{..}(x)] \equiv *\mathbf{0}_{....}(x). \tag{2.62iii}$$

(The Poincaré lemma in (1.66) is evident for (2.62iii).)

Now, we shall introduce the electromagnetic energy–momentum–stress tensor as the following:

$$\begin{aligned} {}_{(\mathrm{em})}T^{ij}(x) &:= F^{ik}(x)F^{j}{}_{k}(x) - (1/4)d^{ij}\,F_{kl}(x)F^{kl}(x) \\ &\equiv {}_{(\mathrm{em})}T^{ji}(x). \end{aligned} \tag{2.63}$$

Example 2.1.13. It will be instructive to work out the divergence $\partial_j \left[{}_{(\text{em})}T^{ij}\right]$, which represents the force density experienced by the charged material due to the electromagnetic field. By (2.63), (2.60i), and (2.60ii), we derive that

$$\begin{aligned}
\partial_j\, {}_{(\text{em})}T^{ij} &= F^{ik}(x)\partial_j F^{j}_{\ k} + (1/2)\left[\partial_j F^{ik} \cdot F^{j}_{\ k}(x) + \partial_j F^{ik} \cdot F^{j}_{\ k}(x)\right] \\
&\quad -(1/2)\cdot d^{ij} \cdot \partial_j F_{kl} \cdot F^{kl}(x) \\
&= -F^{ik}(x)\partial_j\, F_k^{\ j} + (1/2)\, d^{il} F^{jk}(x)\left[\partial_j F_{lk} + \partial_k F_{jl} + \partial_l F_{kj}\right] \\
&= -F^{ik}(x) J_k(x) + 0.
\end{aligned}$$ □

Now, we shall explore properties of a *charged dust* with assumptions

$$J^i(x) := \sigma(x)U^i(x), \tag{2.64i}$$

$$T^{ij}(x) = \rho(x)U^i(x)U^j(x) + {}_{(\text{em})}T^{ij}(x). \tag{2.64ii}$$

Here, $\sigma(x)$ is the (proper) *electrical charge density* and the 4-velocity field components $U^i(x)$ satisfy (2.42). A pertinent consequence for a charged dust model will be furnished now.

Theorem 2.1.14. *Let the functions $\rho(x)$, $\sigma(x)$, and $U^i(x)$ be of class C^1 and $F_{ij}(x)$ be of class C^2 in a domain $D \subset \mathbb{R}^4$. Then, the differential conservation equations, $\partial_j T^{ij} = 0$, imply the equations:*

$$\rho(x)U^j(x)\partial_j U^i = \sigma(x)F^{ik}(x)U_k(x). \tag{2.65}$$

Proof. Using (2.64ii), (2.60i,ii), (2.42), and Example 2.1.13, we obtain that

$$\begin{aligned}
0 = \partial_j T^{ij} &= \partial_j\left[\rho U^i U^j\right] + \partial_j\left[{}_{(\text{em})}T^{ij}\right] \\
&= \rho(x)U^j(x)\partial_j U^i + U^i(x)\partial_j\left[\rho U^j\right] - \sigma(x)F^{ik}(x)U_k(x).
\end{aligned}$$

Multiplying the above by $U_i(x)$ and using $U_i(x)U^i(x) \equiv -1$, $F^{ik}(x)U_i(x)U_k(x) \equiv 0$, we deduce the continuity equation

$$\partial_j\left[\rho U^j\right] = 0.$$

Therefore, (2.65) follows. ■

Restricting on a charged stream line, we derive the equations of motion as

$$\rho\left(\mathcal{X}^{\#}(s)\right)\frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2} = \sigma\left(\mathcal{X}^{\#}(s)\right)\left[F^{i}_{\ k}(x)\right]_{|..}\frac{\mathrm{d}\mathcal{X}^{\#k}(s)}{\mathrm{d}s}. \tag{2.66}$$

For comparison, we cite here that the *relativistic Lorentz equations of motion* of a charged particle of mass m and charge e are known to be

$$m\frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2} = e\left[F^i_{\ k}(x)\right]_{|..}\frac{\mathrm{d}\mathcal{X}^{\#k}(s)}{\mathrm{d}s}. \tag{2.67}$$

The similarity between (2.66) and (2.67) is unmistakable.

Example 2.1.15. The three spatial components of the equations of motion (2.67) yield, by (2.24):

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{V^\alpha(t)}{\sqrt{1-\parallel\vec{\mathbf{V}}(t)\parallel^2}}\right] = e\delta^{\alpha\beta}\left[F_{\beta\gamma}(\mathbf{x},t)_{|\mathcal{X}^\alpha(t)}V^\gamma(t) + F_{\beta 4}(\cdot\cdot)_{|..}\right].$$

Using (2.54i–iv) and the antisymmetric permutation symbol in (1.54), the preceding equations imply that

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{V^\alpha(t)}{\sqrt{1-\parallel\vec{\mathbf{V}}(t)\parallel^2}}\right] = e\left[E^\alpha(\mathbf{x},t)_{|..} + \delta^{\alpha\beta}\varepsilon_{\beta\gamma\mu}V^\gamma(t)H^\mu(\cdot\cdot)_{|..}\right]. \tag{2.68}$$

Similarly, the fourth equation in (2.67) yields

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\sqrt{1-\parallel\vec{\mathbf{V}}(t)\parallel^2}}\right] = e\delta_{\alpha\beta}V^\alpha(t)E^\beta(\cdot\cdot)_{|..}. \tag{2.69}$$

The right-hand side of the above equation represents the rate of work done by the external electric field. *The rate of work done by the magnetic field is exactly zero.* □

We shall now summarize the main theoretical accomplishments of the special theory of relativity in the following:

1. The special theory of relativity explains the puzzle about the speed of light arising out of the Michelson-Morley experiments [181], which indicated that the speed of light is invariant. As a consequence, it is established that the maximal speed of propagation of physical actions is the speed of light.
2. There is *an equivalence of all inertial observers or frames* in regards to the formulation of natural laws.
3. Length contraction of a moving rod and time dilation of a moving clock are predicted (see [55, 89, 230]).
4. The rest mass $m > 0$ of a body possesses potential energy $E = mc^2$. This enormously potent equation was first discovered by Einstein [87].

In Table 2.1, we elaborate on some aesthetic aspects of the special theory of relativity in regards to the unification of distinct physical entities occurring in the nonrelativistic regime.

Table 2.1 Correspondence between relativistic and non-relativistic physical quantities

Physical entities in nonrelativistic physics	Corresponding unification in special relativity
Absolute space $\mathbb{E}_3$ and absolute time $\mathbb{R}$	Flat space–time manifold M_4
3-momentum p_α and energy E	4-momentum p_i with $p_i p^i = -m^2$
For electromagnetic wave, 3-wave numbers k_α and the frequency ν	4-wave numbers k_i with $k_i k^i = 0$
Electric field vector $\vec{\mathbf{E}}$ and magnetic field vector $\vec{\mathbf{H}}$	Second-order, antisymmetric electromagnetic field tensor F_{ij}
For a fluid or a deformable body: energy density ρ, momentum density ρV^α, and symmetric stress tensor $\sigma^{\alpha\beta}$	Symmetric energy–momentum–stress tensor $T^{ij} := \rho U^i U^j - s^{ij}$

So far, we have used only pseudo-Cartesian or Minkowskian coordinate charts for the flat space–time manifold M_4. Sometimes, more general coordinate charts are necessary. For example, to derive the energy levels of a hydrogen atom, the relativistic Dirac equation is investigated in the spherical polar coordinates for spatial dimensions. Let us go back to the equations pertaining to a general coordinate chart in (1.1) and Figs. 1.1 and 1.2. We shall apply such mappings to the flat space–time manifold M_4. Consider the mappings $\widehat{\chi} : \widehat{U} \to \mathbb{R}^4$ and $\widehat{\chi}^{-1} : \widehat{D} \to M_4$. If we restrict the mapping $\widehat{\chi}^{-1}$ to the two-dimensional domain $\widehat{D}_{(2)} := \{\widehat{x} \in \widehat{D} \subset \mathbb{R}^4 : \widehat{x}^1 = \widehat{x}^2 = 0\}$, then the corresponding restricted mapping $\widehat{\chi}^{-1}{}_{|\widehat{D}_2}$ can be qualitatively exhibited as in Fig. 2.8.

By (1.1) and (1.37), we conclude that

$$
{}_2\widehat{\mathbf{X}}'\,[\mathbf{g}_{..}(x)] = \widehat{\mathbf{g}}_{..}(\widehat{x}) = \left[\frac{\partial X^k(\widehat{x})}{\partial \widehat{x}^i}\,\frac{\partial X^l(\widehat{x})}{\partial x^j}\,d_{kl}\right] \mathrm{d}\widehat{x}^i \otimes \mathrm{d}\widehat{x}^j, \tag{2.70i}
$$

$$
\widehat{\mathbf{e}}^{(a)}(\widehat{x}) = \delta^{(a)}{}_k\,\frac{\partial X^k(\widehat{x})}{\partial \widehat{x}^i}\,\mathrm{d}\widehat{x}^i, \tag{2.70ii}
$$

$$
\widehat{\left\{ {i \atop j\;k} \right\}} \not\equiv 0, \tag{2.70iii}
$$

$$
\widehat{\gamma}_{(a)(b)(c)}(\widehat{x}) \not\equiv 0. \tag{2.70iv}
$$

Equations (1.163), (1.161), and (2.70i–iv) imply that

$$
\widehat{\mathbf{R}}^{\boldsymbol{\cdot}}{}_{...}(\widehat{x}) \equiv \widehat{\mathbf{0}}^{\boldsymbol{\cdot}}{}_{...}\,(\widehat{x}). \tag{2.71}
$$

The vanishing of the curvature tensor in the domain $\widehat{D} \subset \mathbb{R}^4$ indicates *the flatness of the domain $\widehat{U} \subset M_4$ in the curvilinear coordinate chart* $(\widehat{\chi}, \widehat{U})$ (in spite of $\widehat{\left\{ {i \atop j\;k} \right\}} \not\equiv 0$ and $\widehat{\gamma}_{(a)(b)(c)}(\widehat{x}) \not\equiv 0$).

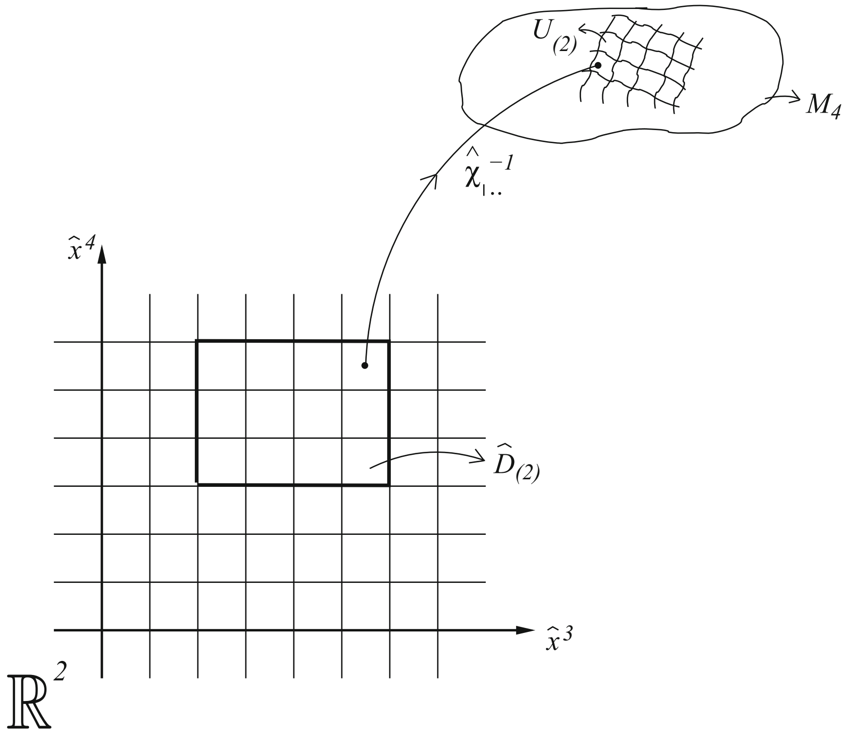

Fig. 2.8 Mapping of a rectangular coordinate grid into a curvilinear grid in the space–time manifold

Example 2.1.16. Consider the flat space–time manifold and a global Minkowskian chart (χ, M_4) given by

$$x \equiv \left(x^1, x^2, x^3, x^4\right) = \chi(p) \in D = \mathbb{R}^4.$$

Consider the coordinate transformation to spherical polar and time coordinates which is furnished by

$$\begin{aligned}
\widehat{x}^1 &= \widehat{X}^1(x) := +\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \equiv r,\\
\widehat{x}^2 &= \widehat{X}^2(x) := \mathrm{Arccos}\left[x^3 \Big/ \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right] \equiv \theta,\\
\widehat{x}^3 &= \widehat{X}^3(x) := \mathrm{arc}(x^1, x^2) \equiv \varphi,\\
\widehat{x}^4 &= x^4 \equiv t,
\end{aligned}$$

$$\widehat{D}_s = \widehat{D} := \left\{\widehat{x} \in \mathbb{R}^4 : \widehat{x}^1 > 0,\ 0 < \widehat{x}^2 < \pi,\ -\pi < \widehat{x}^3 < \pi,\ -\infty < \widehat{x}^4 < \infty\right\}.$$

(See Problem # 1 in Exercise 1.1.) It can be noted that the above chart is *not global*.

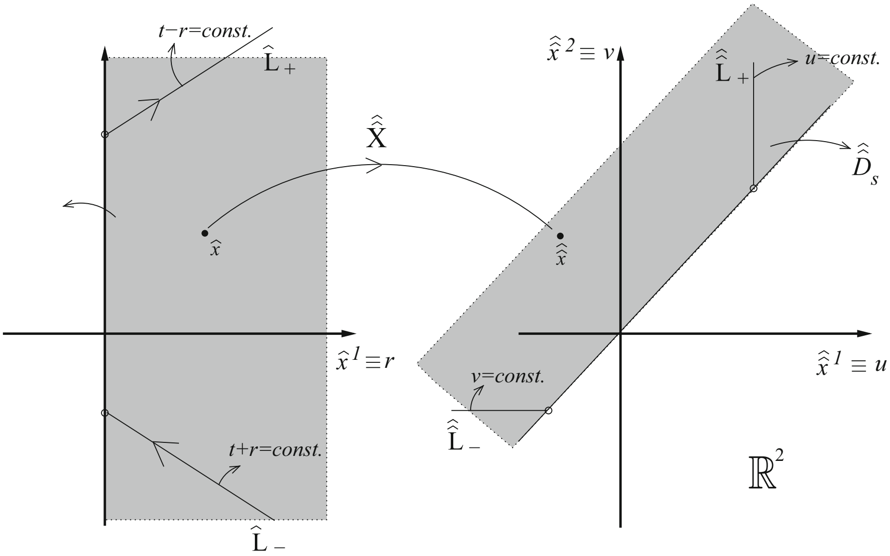

Fig. 2.9 A coordinate transformation mapping half lines $\hat{L}_+$ and $\hat{L}_-$ into half lines $\hat{\hat{L}}_+$ and $\hat{\hat{L}}_-$

The metric tensor in (2.70i) reduces to the orthogonal form

$$
\begin{aligned}
{}_2\widehat{\mathbf{X}}'\,[\mathbf{g}_{..}(x)] = \widehat{\mathbf{g}}_{..}(\widehat{x}) &= \mathrm{d}\widehat{x}^1 \otimes \mathrm{d}\widehat{x}^1 + (\widehat{x}^1)^2 \mathrm{d}\widehat{x}^2 \otimes \mathrm{d}\widehat{x}^2 + (\widehat{x}^1 \sin\widehat{x}^2)^2 \mathrm{d}\widehat{x}^3 \otimes \mathrm{d}\widehat{x}^3 \\
&\quad - \mathrm{d}\widehat{x}^4 \otimes \mathrm{d}\widehat{x}^4, \\
\mathrm{d}s^2 &= (\mathrm{d}\widehat{x}^1)^2 + (\widehat{x}^1)^2 \left[(\mathrm{d}\widehat{x}^2)^2 + (\sin\widehat{x}^2)^2(\mathrm{d}\widehat{x}^3)^2\right] - (\mathrm{d}\widehat{x}^4)^2 \\
&\equiv \mathrm{d}r^2 + r^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta\,(\mathrm{d}\varphi^2)\right] - \mathrm{d}t^2.
\end{aligned}
$$

(See (1.195).) □

Example 2.1.17. Let us make another coordinate transformation from the preceding chart in Example 2.1.16. Let it be furnished by equations

$$
\begin{aligned}
&\widehat{\widehat{x}}^1 = \widehat{x}^4 - \widehat{x}^1 \equiv u, \\
&\widehat{\widehat{x}}^2 = \widehat{x}^4 + \widehat{x}^1 \equiv v, \\
&\widehat{\widehat{x}}^3 = \widehat{x}^2 \equiv \theta, \\
&\widehat{\widehat{x}}^4 = \widehat{x}^3 \equiv \varphi, \\
&\widehat{\widehat{D}}_s := \left\{\widehat{\widehat{x}} \in \mathbb{R}^4 : -\infty < \widehat{\widehat{x}}^2 < \infty,\ 0 < \widehat{\widehat{x}}^2 - \widehat{\widehat{x}}^1,\ 0 < \widehat{\widehat{x}}^3 < \pi,\ -\pi < \widehat{\widehat{x}}^4 < \pi\right\}.
\end{aligned}
$$

See Fig. 2.9 depicting the transformation (*suppressing angular coordinates*).

The oriented half lines $\widehat{L}_+$ and $\widehat{L}_-$ represent possible trajectories of outgoing and incoming photons for an observer pursuing the $\widehat{x}^4$-coordinate line, respectively. Since the null lines (or photon paths) are expressed by $u =$ const. and $v =$ const., these pair of coordinates are called *double-null coordinates*, as u and v coordinate lines are coincident with the paths of null (light) rays.

The metric in the preceding Example 2.1.17 is transformed into

$$
\begin{aligned}
{}_2\widehat{\widehat{\mathbf{X}}}'\left[\widehat{\mathbf{g}}_{..}(\widehat{x})\right] = \widehat{\widehat{\mathbf{g}}}_{..}\left(\widehat{\widehat{x}}\right) &= -\mathrm{d}\widehat{\widehat{x}}^1 \otimes \mathrm{d}\widehat{\widehat{x}}^2 + (1/4)\left[\widehat{\widehat{x}}^2 - \widehat{\widehat{x}}^1\right]^2 \\
&\quad \times \left[\mathrm{d}\widehat{\widehat{x}}^3 \otimes \mathrm{d}\widehat{\widehat{x}}^3 + \left(\sin\widehat{\widehat{x}}^3\right)^2 \mathrm{d}\widehat{\widehat{x}}^4 \otimes \mathrm{d}\widehat{\widehat{x}}^4\right], \\
\mathrm{d}s^2 &= -\,\mathrm{d}u\,\mathrm{d}v + (1/4)(v-u)^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right].
\end{aligned}
$$

Consider a three-dimensional *null hypersurface* specified by

$$
\mathcal{N}_3 := \left\{\widehat{\widehat{x}} \in \mathbb{R}^4 : \widehat{\widehat{x}}^2 = k,\ 0 < k - \widehat{\widehat{x}}^1 < \infty,\ 0 < \widehat{\widehat{x}}^3 < \pi,\ -\pi < \widehat{\widehat{x}}^4 < \pi\right\}.
$$

The induced metric on $\mathcal{N}_3$, by (1.224), is furnished as

$$
\begin{aligned}
\widehat{\widehat{\mathbf{g}}}_{..}(\widehat{\widehat{x}})_{|\mathcal{N}_3} &= (1/4)\left(k - \widehat{\widehat{x}}^1\right)^2\left[\mathrm{d}\widehat{\widehat{x}}^3 \otimes \mathrm{d}\widehat{\widehat{x}}^3 + \left(\sin\widehat{\widehat{x}}^3\right)^2 \mathrm{d}\widehat{\widehat{x}}^4 \otimes \mathrm{d}\widehat{\widehat{x}}^4\right], \\
\mathrm{d}s^2_{|..} &= (1/4)(k-u)^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right].
\end{aligned}
$$

(Although (1.223) was pursued later for nonnull hypersurfaces, it is still valid for a null hypersurface.) The preceding equation yields *a two-dimensional metric for a three-dimensional hypersurface* $\mathcal{N}_3$! There is a loss of one dimension because a null coordinate line has a zero separation. □

A tensor field equation

$$
{}^r_s\mathbf{T}(x) = {}^r_s\mathbf{0}(x), \quad x \in D_{(e)} \subset \mathbb{R}^4
$$

in a Minkowskian chart implies that

$$
{}^r_s\widehat{\mathbf{T}}(\widehat{x}) = {}^r_s\widehat{\mathbf{0}}(\widehat{x}), \quad \widehat{x} \in \widehat{D}_{(e)} \subset \widehat{D} \subset \mathbb{R}^4 \tag{2.72}
$$

in a general coordinate chart and vice-versa. Under a successive general transformation, (2.72) yields an equivalent tensor field equation

$$
{}^r_s\widehat{\widehat{\mathbf{T}}}\left(\widehat{\widehat{x}}\right) = {}^r_s\widehat{\widehat{\mathbf{0}}}\left(\widehat{\widehat{x}}\right), \quad \widehat{\widehat{x}} \in \widehat{\widehat{D}}_{(e)} \subset \widehat{\widehat{D}}_s \subset \widehat{\widehat{D}} \subset \mathbb{R}^4. \tag{2.73}
$$

The equivalence of tensor field equations (2.72) and (2.73) is called the *general covariance* of the tensor field equations. The special theory of relativity therefore, by demanding that physically relevant formulae be expressible as tensor equations, is covariant under the group of general coordinate transformations. This reflects the fact that physical effects must be independent of the coordinate system with which they are measured.

Let us cast some of the tensor field equations in a general coordinate chart characterized by (2.70i,ii). The metric in (2.70i) and the orthonormal 1-forms in (2.70ii) will yield Christoffel symbols and Ricci rotation coefficients which are *not necessarily zero*. We can define covariant derivatives from the definitions in (1.124i), (1.124ii), (1.134), and (1.139ii).

It is useful to transform some of the special relativistic equations we have studied into general coordinate systems. Equation (2.42) implies that

$$\begin{aligned}\widehat{g}_{ij}(\widehat{x})\,\widehat{U}^i(\widehat{x})\,\widehat{U}^j(\widehat{x}) &= d_{(a)(b)}\,\widehat{U}^{(a)}(\widehat{x})\,\widehat{U}^{(b)}(\widehat{x})\\ &\equiv -1.\end{aligned} \tag{2.74}$$

The equations of motion for stream lines in (2.44) go over into

$$\widehat{\rho}(\widehat{x})\left[\widehat{U}^j(\widehat{x})\widehat{\nabla}_j\widehat{U}^i\right]-\left[\delta^i{}_k+\widehat{U}^i(\widehat{x})\,\widehat{U}_k(\widehat{x})\right]\widehat{\nabla}_j\,\widehat{s}^{kj}=0. \tag{2.75}$$

The energy–momentum–stress tensor in (2.45) and the differential conservation equation (2.46) transform into

$$\widehat{T}^{ij}(\widehat{x}) = \widehat{\rho}(\widehat{x})\,\widehat{U}^i(\widehat{x})\,\widehat{U}^j(\widehat{x})-\widehat{s}^{ij}(\widehat{x})\equiv\widehat{T}^{ji}(\widehat{x}), \tag{2.76i}$$

$$\widehat{T}^{(a)(b)}(\widehat{x}) = \widehat{\rho}(\widehat{x})\,\widehat{U}^{(a)}(\widehat{x})\,\widehat{U}^{(b)}(\widehat{x})-\widehat{s}^{(a)(b)}(\widehat{x})\equiv\widehat{T}^{(a)(b)}(\widehat{x}), \tag{2.76ii}$$

$$\widehat{\nabla}_j\widehat{T}^{ij} = 0, \tag{2.76iii}$$

$$\widehat{\nabla}_{(b)}\widehat{T}^{(b)(a)} = 0. \tag{2.76iv}$$

Finally, the Maxwell–Lorentz (electromagnetic) field equations (2.60i–iii) transform into

$$\widehat{\nabla}_k\widehat{F}^{ik} = \left(1/\sqrt{|\widehat{g}|}\right)\frac{\partial}{\partial\widehat{x}^k}\left[\sqrt{|\widehat{g}|}\,\widehat{F}^{ik}\right]=\widehat{J}^i(\widehat{x}), \tag{2.77i}$$

$$\widehat{\partial}_k\widehat{F}_{ij}+\widehat{\partial}_i\widehat{F}_{jk}+\widehat{\partial}_j\widehat{F}_{ki}=0, \tag{2.77ii}$$

$$\widehat{\nabla}_i\widehat{J}^i=0, \tag{2.77iii}$$

$$\widehat{\nabla}_{(c)}\widehat{F}^{(a)(c)}=\widehat{J}^{(a)}(\widehat{x}), \tag{2.78i}$$

$$\widehat{\nabla}_{(c)}\widehat{F}_{(a)(b)}+\widehat{\nabla}_{(a)}\widehat{F}_{(b)(c)}+\widehat{\nabla}_{(b)}\widehat{F}_{(c)(a)}=0, \tag{2.78ii}$$

$$\widehat{\nabla}_{(a)}\widehat{J}^{(a)}=0. \tag{2.78iii}$$

Remarks: (i) Equations (2.75), (2.76i–iv), (2.77i–iii), and (2.78i–iii) *all possess "general covariance" under the set of general coordinate transformations*.

(ii) *Special relativistic equations, expressed as tensor equations in general coordinates, are already "general relativistic"!* In the sequel, the space–time continuum will be treated as *a curved manifold*, so that $\widehat{\mathbf{R}}^{\cdot}{}_{\cdots}(\widehat{x}) \not\equiv \widehat{\mathbf{0}}^{\cdot}{}_{\cdots}(\widehat{x})$. Fortunately, (2.75), (2.76i–iv), (2.77i–iii), (2.78i–iii), and others, pass over smoothly into the arena of curved space–time.

Exercises 2.1

1. Determine whether or not any of the subset of vectors among $W_S(x_0)$, $W_T(x_0)$, or $W_N(x_0)$ defined in (2.2i–iii) is a vector subspace.
2. Let $\vec{\mathbf{t}}_{x_0}$ and $\widehat{\vec{\mathbf{t}}}_{x_0}$ be two future-pointing timelike vectors. Prove *the reversed Schwarz inequality*

$$\sigma\left(\vec{\mathbf{t}}_{x_0}\right) \cdot \sigma\left(\widehat{\vec{\mathbf{t}}}_{x_0}\right) \leq \left|\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \widehat{\vec{\mathbf{t}}}_{x_0}\right)\right|.$$

3. Define a 4×4 *Lorentz matrix* by $[L] := [l^i{}_j]$, where entries $l^i{}_j$ satisfy (2.66).
   (i) Prove that (2.66) can be expressed as the matrix equation

$$[L]^{\mathrm{T}}[D]\,[L] = [D]. \quad (\text{Here, } [D] := [d_{ij}].)$$

   (ii) Deduce that $\det[L] = \pm 1$.
   (iii) Prove that under the composition rule as the matrix multiplication, the set of all (4×4) Lorentz matrices constitutes *a group*.

Remark: This group, which is called *the Lorentz group*, is denoted by $O(3, 1; \mathbb{R})$.)

4. Consider a special relativistic tensor wave equation

$$\Box\phi_{l_1,\ldots,l_s} := d^{ij}\,\partial_i\,\partial_j\,\phi_{l_1,\ldots,l_s} = 0.$$

   Let $2 \times (4)^s$ functions $f_{(l_1,\ldots,l_s)},\ g_{(l_1,\ldots,l_s)} \in C^2(D_{(e)} \subset \mathbb{R}^4; \mathbb{R})$. These functions are *otherwise arbitrary*. Prove that

$$\begin{aligned}\phi_{l_1,\ldots,l_s}(x) := {} & f_{(l_1,\ldots,l_s)}\left(k_1x^1 + k_2x^2 + k_3x^3 - \nu(\mathbf{k})\right) \\ & + g_{(l_1,\ldots,l_s)}\left(k_1x^1 + k_2x^2 + k_3x^3 + \nu(\mathbf{k})\right),\end{aligned}$$

   where $\nu(\mathbf{k}) := \sqrt{\delta^{\alpha\beta}k_\alpha k_\beta}$, solves the wave equation.

5. Consider the upper branch of the three-dimensional hyperhyperbola $\sum_3$ in Fig. 2.5. A parametric representation is furnished by

$$
\begin{aligned}
u^1 &= \xi^1 \left(w^1, w^2, w^3\right) := \sinh w^1 \cdot \sin w^2 \cdot \cos w^3, \\
u^2 &= \xi^2 \left(w^1, w^2, w^3\right) := \sinh w^1 \cdot \sin w^2 \cdot \sin w^3, \\
u^3 &= \xi^3 \left(w^1, w^2, w^3\right) := \sinh w^1 \cdot \cos w^2, \\
u^4 &= \cosh w^1, \\
\mathcal{D}_3 &:= \left\{\left(w^1, w^2, w^3\right) \in \mathbb{R}^3 : -\infty < w^1 < \infty,\ 0 < w^2 < \pi,\ -\pi < w^3 < \pi\right\}.
\end{aligned}
$$

Prove that the hypersurface $\sum_3$ is a three-dimensional space of constant curvature.

6. Suppose that a particle with possibly variable mass (like a radioactive particle or exhaust-emitting rocket) is moving under an external force. The relativistic equations of motion (2.25) are generalized to

$$
\frac{\mathrm{d}}{\mathrm{d}s}\left[M(s)\frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s}\right] = F^i(x,u)\Big|_{x=\mathcal{X}^{\#}(s),\, u=\frac{\mathrm{d}\mathcal{X}^{\#}(s)}{\mathrm{d}s}}, \qquad M(s) > 0.
$$

   (i) Using the equations above, prove that $M(s)$ is constant-valued if and only if the orthogonality $d_{ij}\, F^i(\cdot\cdot)_{|..}\, \frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s} \equiv 0$ holds.
   (ii) Prove that the separation $\sqrt{d_{ij}\, \frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}\, \frac{\mathrm{d}^2\mathcal{X}^{\#j}(s)}{\mathrm{d}s^2}}$ of the 4-acceleration vector $\frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}\, \frac{\partial}{\partial x^i}{}_{|..}$ is constant-valued if and only if $[M(s)]^{-1}\, d_{ij}\, \frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}\, F^j(\cdot\cdot)_{|..}$ is constant-valued.

7. Prove the integral conservation laws for the relativistic total angular momentum given in (2.50).
8. Consider the electrically charged dust model discussed in (2.64i,ii) and Theorem 2.1.14.

   (i) Assuming the junction conditions $\rho U^i n_{i\,|...} = 0 = \sigma U^i n_{i\,|...}$, prove the following integral conservation laws regarding the *total mass* and the *total charge*:

$$
M := \int\limits_{D(t)} \rho\,(\mathbf{x},t)\, U^4(\mathbf{x},t)\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 = \text{const.},
$$

$$
Q := \int\limits_{D(t)} \sigma\,(\mathbf{x},t)\, U^4(\mathbf{x},t)\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 = \text{const.}
$$

(ii) Show the existence of two fixed spatial points $\mathbf{x}_{(1)}, \mathbf{x}_{(2)} \in D_{(t)}$ such that

$$\frac{Q}{M} = \frac{\sigma\,(\mathbf{x}_{(2)}, t)\, U^4(\mathbf{x}_{(2)}, t)}{\rho\,(\mathbf{x}_{(1)}, t)\, U^4(\mathbf{x}_{(1)}, t)} = \text{const.}$$

9. Consider a coordinate chart $(\widehat{\chi}, \widehat{U})$ for M_4 such that

$$\widehat{\mathbf{g}}_{..}(\widehat{x}) = \delta_{\alpha\beta}\, \mathrm{d}\hat{x}^{\alpha} \otimes \mathrm{d}\hat{x}^{\beta} - \left(\widehat{x}^1 + \widehat{x}^2 + \widehat{x}^3\right)^2 \mathrm{d}\widehat{x}^4 \otimes \mathrm{d}\widehat{x}^4,$$
$$\widehat{D} := \left\{\widehat{x} \in \mathbb{R}^4 : \widehat{x}^1 + \widehat{x}^2 + \widehat{x}^3 \geq 1,\; \widehat{x}^4 \in \mathbb{R}\right\}.$$

Prove by explicit computations that $\widehat{\mathbf{R}}^{\cdot}{}_{...}(\widehat{x}) \equiv \widehat{\mathbf{0}}^{\cdot}{}_{...}(\widehat{x})$ for $\widehat{x} \in \widehat{D} \subset \mathbb{R}^4$.

10. Consider the spherical polar coordinate chart dealt with in Example 2.1.16. In this chart, check Maxwell's equation (2.57) for the following two cases:

(i) The electromagnetic 2-form field for an electric dipole of strength $p_{(1)}$ given by

$$\widehat{\mathbf{F}}_{..}(\widehat{x}) = p_{(1)} \cdot \left[\frac{2\cos\widehat{x}^2}{(\widehat{x}^1)^3}\, \mathrm{d}\widehat{x}^1 \wedge \mathrm{d}\widehat{x}^4 + \frac{\sin\widehat{x}^2}{(\widehat{x}^1)^2}\, \mathrm{d}\widehat{x}^2 \wedge \mathrm{d}\widehat{x}^4\right]$$

(ii) The electromagnetic 2-form field for a vibrating electric dipole with frequency ν, furnished by

$$\begin{aligned}\widehat{\mathbf{F}}_{..}(\widehat{x}) = \text{Real part of }\; p_{(1)} \cdot \Bigg\{ \mathrm{e}^{i\nu(\widehat{x}^1 - \widehat{x}^4)} \Bigg[& 2\cos\widehat{x}^2 \left(\frac{1}{(\widehat{x}^1)^3} - \frac{i\nu}{(\widehat{x}^1)^2}\right) \mathrm{d}\widehat{x}^1 \wedge \mathrm{d}\widehat{x}^4 \\ & + \sin\widehat{x}^2 \left(\frac{1}{(\widehat{x}^1)^2} - \frac{i\nu}{\widehat{x}^1} - \nu^2\right) \mathrm{d}\widehat{x}^2 \wedge \mathrm{d}\widehat{x}^4 \\ & - \sin\widehat{x}^2 \left(\frac{i\nu}{\widehat{x}^1} + \nu^2\right) \mathrm{d}\widehat{x}^1 \wedge \mathrm{d}\widehat{x}^2 \Bigg]\Bigg\}.\end{aligned}$$

11. Using a Minkowskian coordinate chart, solve for the Killing vector fields of flat M_4 with help of (1.171ii).

Answers and Hints to Selected Exercises

1. None of the subsets of $W_S(x_0)$, $W_T(x_0)$, or $W_N(x_0)$ is a vector subspace.
3. (ii) Taking determinants of both sides of the matrix equation, it can be shown that

$$\left\{\det[L]^{\mathrm{T}}\right\} \cdot (-1) \cdot \{\det[L]\} = -1,$$
$$\text{or}\quad \{\det[L]\}^2 = 1.$$

5. The intrinsic metric of the hypersurface $\sum_3$ is given by

$$\overline{\mathbf{g}}_{..}(w) = \mathrm{d}w^1 \otimes \mathrm{d}w^1 + (\sinh w^1)^2\, \mathrm{d}w^2 \otimes \mathrm{d}w^2 + \left[(\sinh w^1)(\sin w^2)\right]^2 \mathrm{d}w^3 \otimes \mathrm{d}w^3,$$
$$\overline{R}_{\alpha\beta\gamma\delta}(w) = (-1) \cdot \left[\overline{g}_{\alpha\gamma}(w) \cdot \overline{g}_{\beta\delta}(w) - \overline{g}_{\alpha\delta}(w) \cdot \overline{g}_{\beta\gamma}(w)\right].$$

6. (i) Multiplying equations of motion by $d_{ij} \frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s}$ and contracting, it can be proved that

$$-\frac{\mathrm{d}M(s)}{\mathrm{d}s} + M(s) \underbrace{\left[d_{ij} \frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s} \frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}\right]}_{\equiv 0} = d_{ij}\, F^i(\cdot\cdot)_{|..} \frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s}.$$

7. Introduce a $(3+0)$th order tensor field by

$$\mathcal{T}^{ijk}(x) := \left(x^i - x_0^i\right) T^{jk}(x) - \left(x^j - x_0^j\right) T^{ik}(x).$$

Derive, using $T^{ji}(x) \equiv T^{ij}(x)$ and $\partial_j T^{ij} = 0$, that $\partial_k \mathcal{T}^{ijk} = 0$.

8. (ii) Use the mean value theorem of integrals. (See [32].)
9. Often, for such calculations, it is time-saving to use computational symbolic algebra programs as in Appendix 8.
10. (i)

$$\mathrm{d}\,\widehat{\mathbf{F}}_{..}(\widehat{x}) = p_{(1)} \cdot \left[-\frac{2\sin\widehat{x}^2}{(\widehat{x}^1)^3}\, \mathrm{d}\widehat{x}^2 \wedge \mathrm{d}\widehat{x}^1 \wedge \mathrm{d}\widehat{x}^4 - \frac{2\sin\widehat{x}^2}{(\widehat{x}^1)^3}\, \mathrm{d}\widehat{x}^1 \wedge \mathrm{d}\widehat{x}^2 \wedge \mathrm{d}\widehat{x}^4\right]$$
$$\equiv \widehat{\mathbf{0}}_{...}(\widehat{x}).$$

11. Ten Killing vector fields are:

$$\vec{\mathbf{K}}_{(r)}(x) := \delta^l{}_{(r)} \frac{\partial}{\partial x^l},$$
$$\vec{\mathbf{K}}_{(r)(s)}(x) := \left(d_{(r)i}\delta^j{}_{(s)} - d_{(s)i}\delta^j{}_{(r)}\right) x^i \frac{\partial}{\partial x^j} \equiv -\vec{\mathbf{K}}_{(s)(r)}(x),$$
$$r, s \in \{1, 2, 3, 4\}.$$

(*Remarks*: (i) Compare the above with the answers for Problem #7(ii) in Exercise 1.2. (ii) These Killing vectors are also called *generators for the Poincaré group* $\mathcal{IO}(3,1;\mathbb{R})$.)

2.2 Curved Space–Time and Gravitation

In the preceding section on special relativity theory, the set of all inertial observers (observers moving relative to each other with constant three-velocities) forms a privileged class of observers. In Minkowskian coordinate charts, natural laws are expressed by each of the inertial observers with *exactly similar* (Minkowskian) tensor field equations. Naturally, the next investigations focus on natural laws expressible by an observer moving with *constant three-acceleration* (i.e., $\frac{\mathrm{d}^2\mathcal{X}^\alpha(t)}{\mathrm{d}t^2} =$ const.) relative to an inertial observer. Our familiar experiences of objects moving with constant three-accelerations include apples (or other massive objects) falling under Earth's gravity. (We are neglecting air resistance due to Earth's atmosphere, and we are assuming the fall is taking place over distances where changes in the strength of the gravitational field can be neglected.)

Recall that Newton's equations of motion in an external gravitational field are provided by

$$m\frac{\mathrm{d}^2\mathcal{X}^\alpha(t)}{\mathrm{d}t^2} = f^\alpha(..)_{|..} =: -m\delta^{\alpha\beta}\frac{\partial W(\mathbf{x})}{\partial x^\beta}_{|x^\alpha=\mathcal{X}^\alpha(t)}, \tag{2.79i}$$

$$\text{or,}\quad \frac{\mathrm{d}^2\mathcal{X}^\alpha(t)}{\mathrm{d}t^2} = -\,\delta^{\alpha\beta}\partial_\beta W_{|..}. \tag{2.79ii}$$

(See (2.19i).) In (2.79i, ii), $W(\mathbf{x})$ represents the *Newtonian gravitational potential*. From (2.79ii), it is clear that the motion curve in a gravitational field is *independent of the mass* $m > 0$ *of the particle*.[4]

Very near the Earth's surface, the potential may be well approximated by $W(\mathbf{x}) = gx^3 + \text{const.}$, where the x^3-axis is coincident with the vertical direction. (Here, g denotes the constant gravitational acceleration *which must not be confused with* $\det\left[g_{ij}\right]$.) Integrating the equations of motion (2.79ii) in Earth's gravity yields

$$\begin{aligned}
\mathcal{X}^1(t) &= {}_{(i)}x^1 + {}_{(i)}v^1 t,\\
\mathcal{X}^2(t) &= {}_{(i)}x^2 + {}_{(i)}v^2 t,\\
\mathcal{X}^3(t) &= {}_{(i)}x^3 + {}_{(i)}v^3 t - \frac{1}{2}\,g\,(t)^2.
\end{aligned} \tag{2.80}$$

Here, ${}_{(i)}x^1$, ${}_{(i)}x^2$, ${}_{(i)}x^3$ represent the initial position of the particle whereas ${}_{(i)}v^1$, ${}_{(i)}v^2$, ${}_{(i)}v^3$ represent the initial three-velocity components. Let a second idealized

[4]The equivalency between *gravitational mass* and *inertial mass* is assumed. Current experimental limits place the equivalency of these two types of mass to within 10^{-12} [226], and it is widely believed that they are equivalent.

(point) observer start initially from $({}_{(0)}x^1, {}_{(0)}x^2, {}_{(0)}x^3)$ and with zero initial three-velocity. Her trajectory, falling freely under gravity, will be given by

$$\begin{aligned}\hat{\mathcal{X}}^1(t) &= {}_{(0)}x^1,\\ \hat{\mathcal{X}}^2(t) &= {}_{(0)}x^2,\\ \hat{\mathcal{X}}^3(t) &= {}_{(0)}x^3 - \frac{1}{2}g\,(t)^2.\end{aligned} \tag{2.81}$$

The relative three-velocity components, between two falling particles, are furnished by

$$\frac{\mathrm{d}\mathcal{X}^\alpha(t)}{\mathrm{d}t} - \frac{\mathrm{d}\hat{\mathcal{X}}^\alpha(t)}{\mathrm{d}t} = {}_{(i)}v^\alpha = \text{const.},$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta^\alpha(t) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\mathcal{X}^\alpha(t) - \hat{\mathcal{X}}^\alpha(t)\right] \equiv 0. \tag{2.82}$$

Or, in other words, *nearby objects free-falling near the surface of the Earth move from each other with constant 3-velocities*. However, *the relative 3-acceleration* $\frac{\mathrm{d}^2\eta^\alpha(t)}{\mathrm{d}t^2} \equiv 0$.

Now, consider an idealized observer inside a freely falling elevator in Earth's gravitational field. Let him throw some massive particles inside the falling elevator. In free fall, he experiences no effects of gravity; moreover, he sees the particles move around him with constant 3-velocities (ignoring bounces off the walls). He can well conclude that gravitational forces have *disappeared*. Consider another example: Astronauts inside the international space station work under conditions similar to the observer in the elevator.[5] Therefore, these astronauts are experiencing conditions valid in the special theory of relativity in inertial frames. On the other hand, it is well known that in instants after a rocket launching, the astronauts inside the rocket experience *an apparent enhancing* of gravity. Thus, one may conclude that 3-accelerations of an observer in space can *apparently generate or annihilate gravitation* (at least in the local sense). This surprising fact is known as the *principle of equivalence*.

Let us investigate more critically the scenarios involving constant 3-accelerations on *a nonlocal scale*. Consider three massive point objects falling freely under the influence of gravity near the surface of the Earth where the acceleration is

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[5] The apparent zero-gravity effects experienced by astronauts in orbit are exactly analogous to the elevator example. The "zero-gravity" effects are due to the fact that the astronauts and the orbiting vehicle are in free fall and *not* due to the fact that gravity is too weak to have any appreciable effects. A quick calculation, using (2.79ii) and the remark after Example 1.3.28, reveals that the gravitational acceleration at 320 km above the Earth's surface (low Earth orbit) is approximately 91% of its value at the surface of the Earth.

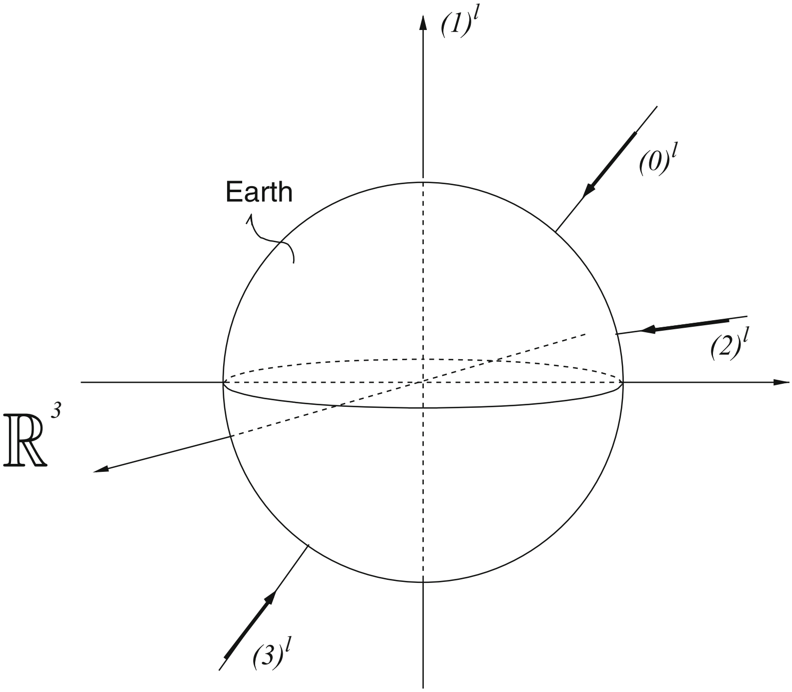


Fig. 2.10 Three massive particles falling freely in space under Earth's gravity

approximately constant (with no angular motion). The particles are initially located at three different locations. Each of the particles follows trajectories governed by (2.81). (See Fig. 2.10 where the 3-acceleration vectors are *highlighted*.)

It is obvious from Fig. 2.10 that the three 3-acceleration vectors have distinct directions. Therefore, the observers falling freely along the lines ${}_{(0)}l$, ${}_{(2)}l$ and ${}_{(3)}l$ possess nonzero relative 3-accelerations. Thus, gravitational effects *cannot be eliminated (or enhanced) nonlocally!* That is to say, there is no global accelerating reference frame which will yield apparent weightlessness for all observers. Let us quote Synge's comments on this topic [242]:

The principle of equivalence performed the essential office of midwife at the birth of general relativity, but, as Einstein remarked, the infant would never have got beyond its long-clothes had it not been for Minkowski's concept. I suggest that the midwife now be buried with appropriate honours and the facts of absolute space–time faced.

Let us go back to Fig. 2.10 and furnish a space and time version of the same phenomenon. (Recall that (2.81) yields a parabola in space and time.) Thus, we put forward Fig. 2.11, ignoring the third particle pursuing ${}_{(3)}l$. In Fig. 2.11, parabolic world lines ${}_{(0)}L$ and ${}_{(2)}L$ have constant 3-accelerations and *a nonzero relative 3-acceleration*. Moreover, we included a vertical world line ${}_{(e)}L := {}_{(1)}L$ representing a static idealized observer on the Earth's surface. However, each of the two

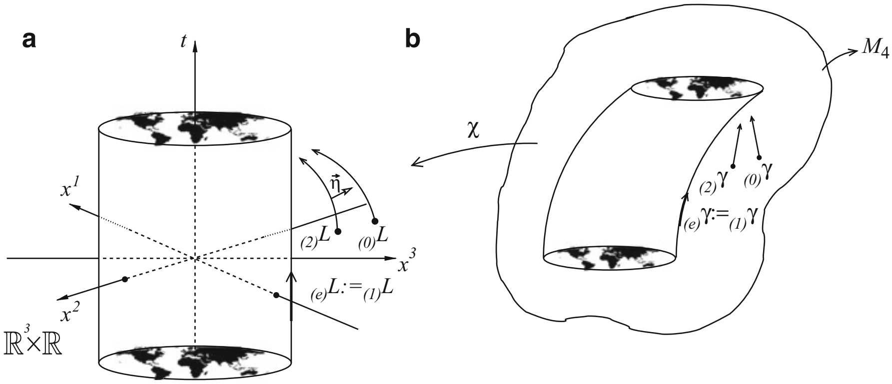


Fig. 2.11 (**a**) Space and time trajectories of two geodesic particles freely falling towards the Earth. (**b**) A similar figure but adapted to the geodesic motion of the two freely falling observers in curved space–time M_4

observers pursuing trajectories $_{(0)}L$ and $_{(2)}L$ experience no apparent gravitational force. Thus, they rightfully consider themselves as *inertial observers* following geodesic world lines. (Such world lines have zero 4-accelerations.) However, the Earth-bound observer, following the straight world line (in the sense depicted in the figure) $_{(e)}L$, experiences his own weight under gravity. Therefore, he concludes that he is *not an inertial observer* and is pursuing *a nongeodesic trajectory in space and time*. Now, in a Minkowskian coordinate chart of *a flat space–time*, neither of the parabolic curves $_{(0)}L$ and $_{(2)}L$ could be considered geodesics. As well, neither could the straight line $_{(e)}L$ be nongeodesic. Hence, we are confronted with a dilemma. The only logical way out is to recognize the Fig. 2.11a as the arena of *a coordinate chart for a curved pseudo-Riemannian space–time manifold*. (See the Fig. 2.11b where we *qualitatively represent* a similar diagram but with the two geodesic observers pursuing straight lines.) Furthermore, the fact that relative 4-acceleration components $\frac{D^2\eta^i(\tau)}{d\tau^2}$ between two geodesics $_{(0)}L$ and $_{(2)}L$ *are not all zero*, implies, by the geodesic deviation equations (1.191), that $R^i{}_{ljk} \not\equiv 0$.

In Einstein's theory of gravitation, either there *is an intrinsic gravitational field or there is none*, according to whether the Riemann curvature tensor of space–time vanishes or not. This is an absolute property; *it has nothing to do with apparent gravitational effects* such as those due to nongeodesic motion of observers. In other words, both the astronauts in the international space station, experiencing apparent weightlessness, and Einstein, standing in the Prussian Academy of Sciences experiencing his weight, are immersed in a nonzero gravitational field since $R^i{}_{ljk} \neq 0$ in their vicinities. However, very far away from gravitating bodies, straight lines in a Minkowskian coordinate chart are geodesics, $R^i{}_{ljk} \to 0$, and gravitational effects are vanishingly small. So we conclude, symbolically speaking, *intrinsic gravitation* $\equiv$ *nonzero Riemann curvature tensor*.

It is well known that the Newtonian theory of gravitation explains usual phenomena involving terrestrial and planetary motions under gravitation. We naturally

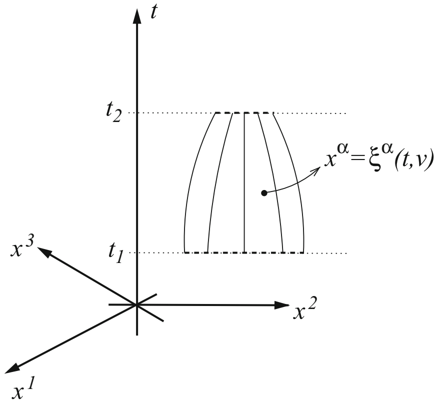


Fig. 2.12 Qualitative representation of a swarm of particles moving under the influence of a gravitational field

ask: "What is the significance of the nonzero curvature tensor of Einstein's theory of gravitation in terms of the Newtonian gravitational potential?" We answer this legitimate question in the following example.

Example 2.2.1. Consider a swarm of massive particles in a gravitational field following Newtonian equations of motion (2.79ii) and subject only to gravity. Let these particles span a very smooth world surface in space and time. Moreover, let this surface be parametrically represented by the equations

$$x^{\alpha} = \xi^{\alpha}(t, v),$$
$$t \in [t_1, t_2], \quad v \in [v_1, v_2],$$

(see Fig. 2.12).

We assume that the functions ξ^{α} are of class C^3 and that the potential W is of class C^2. By Newton's equations of motion (2.79ii), we have

$$\frac{\partial^2 \xi^{\alpha}(t, v)}{\partial t^2} + \delta^{\alpha\beta} \left.\frac{\partial W(\mathbf{x})}{\partial x^{\beta}}\right|_{x^{\mu}=\xi^{\mu}(\cdot\cdot)} = 0. \tag{2.83}$$

Differentiating the above equation with respect to v and commuting the order of differentiations, we obtain

$$\frac{\partial^3 \xi^{\alpha}(t, v)}{\partial t^2 \partial v} + \delta^{\alpha\beta} \left[\frac{\partial^2 W(\cdot\cdot)}{\partial x^{\beta} \partial x^{\gamma}}\right]_{|\cdot\cdot} \cdot \frac{\partial \xi^{\gamma}(t, v)}{\partial v} = 0. \tag{2.84}$$

Denoting *the relative separation components* by $\eta^{\alpha}(t, v) := \frac{\partial \xi^{\alpha}(t,v)}{\partial v}$, (2.84), goes over into

$$\frac{\partial^2 \eta^{\alpha}(t, v)}{\partial t^2} + \delta^{\alpha\beta} \left[\frac{\partial^2 W(\cdot\cdot)}{\partial x^{\beta} \partial x^{\gamma}}\right]_{|\cdot\cdot} \cdot \eta^{\gamma}(t, v) = 0. \tag{2.85}$$

The second term in the equation above gives rise to the usual *tidal forces* caused by gravitation. Now, going back to the pseudo-Riemannian space–time M_4, the spatial components of the geodesic deviation equation (1.192) yield

$$\frac{D^2\eta^\alpha(\tau,v)}{\partial\tau^2} + \left[R^\alpha{}_{ljk}(\cdot\cdot)\right]_{|\cdot\cdot} \cdot \frac{\partial\xi^l(\cdot\cdot)}{\partial\tau} \cdot \eta^j(\cdot\cdot) \cdot \frac{\partial\xi^k(\cdot\cdot)}{\partial\tau} = 0. \tag{2.86}$$

Comparing (2.85) and (2.86), we conclude that nonzero curvature components represent *relativistic tidal forces of the intrinsic gravitation*.

Now, let us explore another line of reasoning. In case tidal forces vanish, (2.84) implies that $\frac{\partial^2 W(\mathbf{x})}{\partial x^\beta \partial x^\gamma} = 0$. Solving the partial differential equations, we obtain the general solution as $W(\mathbf{x}) = c_{(0)} + \sum_{\mu=1}^{3} c_{(\mu)} x^\mu$. (Here, $c_{(0)}, c_{(\mu)}$'s are arbitrary constants of integration.) Thus, equations of motion (2.79ii) yields $\frac{\mathrm{d}^2\mathcal{X}^\alpha(t)}{\mathrm{d}t^2} = -\delta^{\alpha\beta}c_{(\beta)} = \text{const}$. Therefore, we are *back to the scenario of constant 3-acceleration!* By free fall, the apparent gravity can be eliminated. Thus, as expected, the intrinsic gravitation, equivalently the curvature tensor, must vanish. □

We shall now elaborate on the curved pseudo-Riemannian space–time M_4. (It is suggested to glance through the definition of a differentiable manifold in p. 1–3.) *The first assumption* on the pseudo-Riemannian manifold M_4 is that it is a connected, four-dimensional, Hausdorff, C^4-manifold with a Lorentz metric (of signature $+2$). The assumptions of Hausdorff topology and Lorentz metric imply that M_4 is paracompact. Thus, M_4 has a countable basis of open sets [150] providing a C^4-atlas.

In mathematical physics, the mathematical treatments should be rigorous and the mathematical symbols should be ultimately related to experimental observations. Consider an idealized event $p \in U \subset M_4$. The corresponding coordinates in the chart (χ, U) are provided by $x^i = \chi^i(p) := \left[\pi^i \circ \chi\right](p) = \pi^i(x) \in \mathbb{R}$ for $x \in D \subset \mathbb{R}^4$. We have to construct devices and methods to measure these four numbers x^i. For that purpose, we shall first investigate idealized histories of particles in $U \subset M_4$, equivalently in $D \subset \mathbb{R}^4$, by parametrized curves called world lines. In the sequel, we shall use arc separation parameters in (1.186) and (1.187) for timelike curves, which are furnished by $x^i = \mathcal{X}^i(s)$. (We *drop the symbol* #.) For a null curve, we express $x^i = \mathcal{X}^i(\alpha)$, where we use an affine parameter α. We assume the parametrized curve $\mathcal{X}$ to be of class C^3. Therefore, we can express from (1.186) that

$$\begin{aligned} g_{ij}\left(\mathcal{X}(s)\right)\frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s}\,\frac{\mathrm{d}\mathcal{X}^j(s)}{\mathrm{d}s} &\equiv -1, \\ g_{ij}\left(\mathcal{X}(\alpha)\right)\frac{\mathrm{d}\mathcal{X}^i(\alpha)}{\mathrm{d}\alpha}\,\frac{\mathrm{d}\mathcal{X}^j(\alpha)}{\mathrm{d}\alpha} &\equiv 0. \end{aligned} \tag{2.87}$$

As a consequence of the above equation, we derive that

$$g_{ij}\left(\mathcal{X}(s)\right)\frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s}\left[\frac{\mathrm{d}^2\mathcal{X}^j(s)}{\mathrm{d}s^2}+\left\{\begin{matrix} j \\ k\ l \end{matrix}\right\}\Big|_{..}\frac{\mathrm{d}\mathcal{X}^k(s)}{\mathrm{d}s}\frac{\mathrm{d}\mathcal{X}^l(s)}{\mathrm{d}s}\right]\equiv 0. \tag{2.88}$$

Therefore, *4-velocity is always orthogonal to 4-acceleration, whether or not the curve is a geodesic*.

Now, we investigate *the proper time and proper length* along a nonnull differentiable curve. By (1.186) and (2.17), the proper time along a timelike curve is furnished by

$$s=\mathcal{S}(t):=\int_{t_0}^{t}\sqrt{-g_{ij}\left(\mathcal{X}(u)\right)\frac{\mathrm{d}\mathcal{X}^i(u)}{\mathrm{d}u}\frac{\mathrm{d}\mathcal{X}^j(u)}{\mathrm{d}u}}\,\mathrm{d}u. \tag{2.89}$$

(The proper time is also denoted by τ by some authors.)

The proper length along a spacelike curve is given by

$$s=\mathcal{S}(l):=\int_{l_0}^{l}\sqrt{g_{ij}\left(\mathcal{X}(w)\right)\frac{\mathrm{d}\mathcal{X}^i(w)}{\mathrm{d}w}\frac{\mathrm{d}\mathcal{X}^j(w)}{\mathrm{d}w}}\,\mathrm{d}w. \tag{2.90}$$

(The arc separation along a null curve is exactly zero.)

Example 2.2.2. Consider the pseudo-Riemannian space–time manifold M_4 and an orthogonal coordinate chart (χ, U). Let the corresponding domain $D \subset \mathbb{R}^4$ contain the origin $(0,0,0,0)$. Moreover, let the metric tensor components satisfy $g_{11}(x) > 0$, $g_{22}(x) > 0$, $g_{33}(x) > 0$, and $g_{44}(x) < 0$. Then, the proper time along the x^4-axis is given by

$$\tau \equiv s=\int_{0}^{x^4}\sqrt{-g_{44}(0,0,0,t)}\,\mathrm{d}t. \tag{2.91}$$

The proper length along the x^1-axis is furnished by

$$l^1 \equiv s=\int_{0}^{x^1}\sqrt{g_{11}(w,0,0,0)}\,\mathrm{d}w. \tag{2.92}$$

Similar equations hold for the x^2-axis and x^3-axis.

The proper two-dimensional area of the coordinate rectangle $(0, x^1) \times (0, x^2)$ is given by

$$A := \int_0^{x^1} \int_0^{x^2} \sqrt{\gamma(u^1, u^2)}\, \mathrm{d}u^1\, \mathrm{d}u^2,$$
$$\gamma(u^1, u^2) := \det \begin{bmatrix} g_{11}(u^1, u^2, 0, 0) & g_{12}(u^1, u^2, 0, 0) \\ g_{12}(u^1, u^2, 0, 0) & g_{22}(u^1, u^2, 0, 0) \end{bmatrix}. \tag{2.93}$$

(We assume that $\gamma(u^1, u^2) > 0$.)

The proper three-dimensional volume of the "coordinate rectangular solid" $(0, x^1) \times (0, x^2) \times (0, x^3)$ on the spacelike three-dimensional hypersurface $x^4 = 0$ is furnished by

$$V := \int_0^{x^1} \int_0^{x^2} \int_0^{x^3} \sqrt{\overline{g}(u^1, u^2, u^3)}\, \mathrm{d}u^1\, \mathrm{d}u^2\, \mathrm{d}u^3,$$
$$\overline{g}(u^1, u^2, u^3) := \det \left[g_{\alpha\beta}(u^1, u^2, u^3, 0) \right]. \tag{2.94}$$

(We assume that $\overline{g}(u^1, u^2, u^3) > 0$.)

In a flat space–time, relative to a Minkowskian coordinate chart, the right-hand sides of (2.91)–(2.94) will reduce to x^4, x^1, $x^1 x^2$, and $x^1 x^2 x^3$, respectively. (We have tacitly assumed in this example that the matrix $[g_{ij}(x)]$ has four real eigenvalues $\lambda_{(1)}(x) > 0$, $\lambda_{(2)}(x) > 0$, $\lambda_{(3)}(x) > 0$, and $\lambda_{(4)}(x) < 0$.) □

Now, we discuss an idealized observer following a future-pointing timelike world line. He or she carries a regular (point) clock to measure the proper time along the world line. An orthonormal basis set or a tetrad is transported along with the observer. He or she may be an inertial or noninertial observer. The inertial observer obviously follows a *timelike geodesic* furnished by

$$\mathcal{U}^i(s) := \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s}, \tag{2.95i}$$

$$\frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = 0, \tag{2.95ii}$$

$$g_{ij}\,(\mathcal{X}(s))\,\mathcal{U}^i(s)\,\mathcal{U}^j(s) \equiv -1, \tag{2.95iii}$$

$$g_{ij}\,(\mathcal{X}(s))\,\mathcal{U}^i(s)\frac{D\mathcal{U}^j(s)}{\mathrm{d}s} \equiv 0. \tag{2.95iv}$$

Equation (2.95i) defines *the 4-velocity components* $\mathcal{U}^i(s)$. The geodesic equation (2.95ii) indicates that 4-velocity components $\mathcal{U}^i(s)$ undergo *parallel transportations*. (See (1.175) and (1.178i,ii).)

It is convenient to choose the timelike unit vector $\vec{\mathbf{e}}_{(4)}(\mathcal{X}(s))$ of the orthonormal tetrad as $e^i_{(4)}(\mathcal{X}(s)) = \mathcal{U}^i(s)$. The choice of the other three spacelike unit vectors of the tetrad is arbitrary up to a rotation or a reflection induced by an element of the orthogonal group $O(3, \mathbb{R})$. Let the choice of orthonormality be made at the initial event $\mathcal{X}(0)$. Therefore,

$$g_{ij}(\mathcal{X}(0))\, e^i_{(a)}(\mathcal{X}(0))\, e^j_{(b)}(\mathcal{X}(0)) = d_{(a)(b)}. \tag{2.96}$$

Assume now that each of the four 4-vectors is *transported parallelly* along the geodesic, that is,

$$\frac{De^i_{(a)}(\mathcal{X}(s))}{\mathrm{d}s} = 0. \tag{2.97}$$

By the Leibniz rule for the derivative $\frac{D}{\mathrm{d}s}$ in (1.175), we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}(\mathcal{X}(s))\, e^i_{(a)}(\mathcal{X}(s))\, e^j_{(b)}(\mathcal{X}(s))\right] \\
&\quad = \frac{D}{\mathrm{d}s}\left[g_{ij}(\cdot\cdot)\, e^i_{(a)}(\cdot\cdot)\, e^j_{(b)}(\cdot\cdot)\right] \\
&\quad = g_{ij}(\cdot\cdot)\left\{\left[\frac{D}{\mathrm{d}s}\, e^i_{(a)}(\cdot\cdot)\right]\cdot e^j_{(b)}(\cdot\cdot) + e^i_{(a)}(\cdot\cdot)\cdot\left[\frac{D}{\mathrm{d}s}\, e^j_{(b)}(\cdot\cdot)\right]\right\} \\
&\quad = 0,
\end{aligned}$$

or,

$$g_{ij}(\mathcal{X}(s))\, e^i_{(a)}(\mathcal{X}(s))\, e^j_{(b)}(\mathcal{X}(s)) = \text{const.}$$

By the initial conditions (2.96), the constants must be $d_{(a)(b)}$. Thus, the orthonormality of the tetrad is preserved under parallel transport along a geodesic. Such a tetrad is essential for the inertial observer to measure physical components like $T_{(a)(b)}(\mathcal{X}(s))$ and physical (or orthonormal) components of other tensor fields.

Now, let us investigate *a noninertial observer pursuing a nongeodesic timelike world line*. An obvious example is an idealized (point) scientist working in a fixed location of a laboratory on Earth's surface. What kind of orthonormal tetrad can this observer carry along their world line? We can explore the Frenet-Serret orthonormal tetrad introduced in (1.196). Recapitulating the formulas, we provide the following equations:

$$\begin{aligned}
\lambda^i_{(1)}(s) &:= \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} \equiv \mathcal{U}^i(s), \\
\frac{D\lambda^i_{(1)}(s)}{\mathrm{d}s} &= \kappa_{(1)}(s)\lambda^i_{(2)}(s),
\end{aligned}$$

$$\frac{D\lambda^i_{(2)}(s)}{\mathrm{d}s} = \kappa_{(2)}(s)\lambda^i_{(3)}(s) + \kappa_{(1)}(s)\lambda^i_{(1)}(s),$$
$$\frac{D\lambda^i_{(3)}(s)}{\mathrm{d}s} = \kappa_{(3)}(s)\lambda^i_{(4)}(s) - \kappa_{(2)}(s)\lambda^i_{(2)}(s),$$
$$\frac{D\lambda^i_{(4)}(s)}{\mathrm{d}s} = -\kappa_{(3)}(s)\lambda^i_{(3)}(s). \tag{2.98}$$

Here, $\kappa_{(1)}(s)$ is the *principal* or *first curvature.* Moreover, $\kappa_{(2)}(s)$ and $\kappa_{(3)}(s)$ are *the second and third curvature*, respectively. The spacelike unit vectors $\vec{\boldsymbol{\lambda}}_{(2)}(s), \vec{\boldsymbol{\lambda}}_{(3)}(s)$, and $\vec{\boldsymbol{\lambda}}_{(4)}(s)$ are *the first, second, and third normal* to the curve. For a nongeodesic curve, $\kappa_{(1)}(s) \neq 0$. Therefore, by (2.98), the orthonormal Frenet-Serret tetrad $\{\vec{\boldsymbol{\lambda}}_{(a)}(s)\}_1^4$ is *not parallelly transported*. Thus, we look for another mode of transportation of an orthonormal tetrad or frame along a *nongeodesic* curve (or an accelerating observer). We define the *Fermi derivative* of a vector field along a curve by the following string of equations:

$$\mathcal{U}^i(s) = \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s},$$
$$\kappa_{(1)}(s)N^i(s) := \frac{D\mathcal{U}^i(s)}{\mathrm{d}s},$$
$$\frac{D_F\vec{\mathbf{V}}(\mathcal{X}(s))}{\mathrm{d}s} := \frac{D\vec{\mathbf{V}}(\mathcal{X}(s))}{\mathrm{d}s} - \kappa_{(1)}(s)\left[\mathbf{g}_{..}(\mathcal{X}(s))\left(\vec{\mathbf{N}},\vec{\mathbf{V}}\right)\right]\vec{\mathbf{U}}(s)$$
$$+\kappa_{(1)}(s)\left[\mathbf{g}_{..}(\mathcal{X}(s))\left(\vec{\mathbf{U}},\vec{\mathbf{V}}\right)\right]\vec{\mathbf{N}}(s). \tag{2.99}$$

(Consult [126, 243].)

The *Fermi-Walker transport* (*F–W transport* in short) is defined by

$$\frac{D_F\vec{\mathbf{V}}(\mathcal{X}(s))}{\mathrm{d}s} = \vec{\mathbf{O}}(\mathcal{X}(s))$$

or $$\frac{DV^i(\mathcal{X}(s))}{\mathrm{d}s} = \kappa_{(1)}(s)\left[\mathcal{U}^i(s)N^j(s) - \mathcal{U}^j(s)N^i(s)\right]V_j(\mathcal{X}(s))$$
$$= \left[\mathcal{U}^i(s)\cdot\frac{D\mathcal{U}^j(s)}{\mathrm{d}s} - \mathcal{U}^j(s)\frac{D\mathcal{U}^i(s)}{\mathrm{d}s}\right]V_j(\mathcal{X}(s)). \tag{2.100}$$

An important property of F–W transport is discussed next.

Theorem 2.2.3. *Let $\mathcal{X}$ be a parametrized timelike curve of class C^2 with the image $\Gamma \subset D \subset \mathbb{R}^4$. Let a differentiable tetrad $\left\{\vec{\mathbf{e}}_{(a)}(\mathcal{X}(s))\right\}_1^4$ be orthonormal at the initial point $\mathcal{X}(0)$. Furthermore, let each of the vectors $\vec{\mathbf{e}}_{(a)}(\mathcal{X}(s))$ be subjected to F–W transport along the curve. Then, the orthonormality of the tetrad $\left\{\vec{\mathbf{e}}_{(a)}(\mathcal{X}(s))\right\}_1^4$ is preserved for all $s \in [0, s_1]$.*

Proof. By the assumptions made and (2.100), it follows that

$$g_{ij}\left(\mathcal{X}(0)\right) e^{i}_{(a)}\left(\mathcal{X}(0)\right) e^{j}_{(b)}\left(\mathcal{X}(0)\right) = d_{(a)(b)},$$
$$\frac{De^{i}_{(a)}(\mathcal{X}(s))}{\mathrm{d}s} = \kappa_{(1)}(s)\, g_{jk}(\cdot\cdot)\left[\mathcal{U}^{i}(s)N^{j}(s) - \mathcal{U}^{j}(s)N^{i}(s)\right] e^{k}_{(a)}\left(\mathcal{X}(s)\right). \tag{2.101}$$

Therefore, we obtain by (1.175) and the Leibniz property, that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\left(\mathcal{X}(s)\right) e^{i}_{(a)}\left(\mathcal{X}(s)\right) e^{j}_{(b)}\left(\mathcal{X}(s)\right)\right] \\
&= \frac{D}{ds}\left[g_{ij}(\cdot\cdot)\, e^{i}_{(a)}(\cdot\cdot)\, e^{j}_{(b)}(\cdot\cdot)\right] \\
&= g_{ik}(\cdot\cdot)\left\{\left[\frac{De^{i}_{(a)}(\cdot\cdot)}{\mathrm{d}s}\right]\cdot e^{k}_{(b)}(\cdot\cdot) + e^{i}_{(a)}(\cdot\cdot)\cdot\left[\frac{De^{k}_{(b)}(\cdot\cdot)}{\mathrm{d}s}\right]\right\} \\
&= \kappa_{(1)}(s)\, g_{ik}(\cdot\cdot)\, g_{jl}(\cdot\cdot)\left[e^{k}_{(b)}(\cdot\cdot)\, e^{l}_{(a)}(\cdot\cdot) + e^{k}_{(a)}(\cdot\cdot)\, e^{l}_{(b)}(\cdot\cdot)\right] \\
&\quad\times\left[\mathcal{U}^{i}(\cdot\cdot)N^{j}(\cdot\cdot) - \mathcal{U}^{j}(\cdot\cdot)N^{i}(\cdot\cdot)\right] \equiv 0.
\end{aligned}$$

The zero on the right-hand side resulted from the double contraction of a symmetric tensor with an antisymmetric one. Thus, we conclude that

$$g_{ij}\left(\mathcal{X}(s)\right) e^{i}_{(a)}\left(\mathcal{X}(s)\right) e^{j}_{(b)}\left(\mathcal{X}(s)\right) = \text{ const. } = d_{(a)(b)}.$$ ■

It can be noted by (2.100), (2.95iv), and (2.98) that the tangent vector field $\mathcal{U}^{i}(s)$ undergoes F–W transport *automatically*. Moreover, in the limit $\kappa_{(1)}(s) \to 0$, the F–W transport → parallel transport. In Fig. 2.13, we compare and contrast parallel transport versus F–W transport.

In Fig. 2.13b, the tangent vector $\mathcal{U}^{i}(s) = e^{i}_{(4)}\left(\mathcal{X}(s)\right)$ *remains a tangent vector to the nongeodesic curve* under F–W transport. Thus, F–W transport provides a noninertial observer with an orthonormal tetrad. Furthermore, it also provides an orthonormal (spatial) triad orthogonal to the 4-velocity vector $\mathcal{U}^{i}(s)$. This triad forms a spatial frame of reference for the observer. It is this frame of reference which provides us a correct generalization of the Newtonian concept of *a nonrotating moving frame*.

Example 2.2.4. Consider a *conformally flat space–time* M_4. (Consult Theorem 1.3.33 and Appendix 4.) The metric is furnished by

$$\mathrm{d}s^2 = \left[\phi(x)\right]^2 d_{ij}\,\mathrm{d}x^{i}\,\mathrm{d}x^{j},$$
$$\phi \in C^3(D \subset \mathbb{R}^4; \mathbb{R}), \quad \phi(x) > 0, \quad \partial_4\phi \neq 0. \tag{2.102i}$$

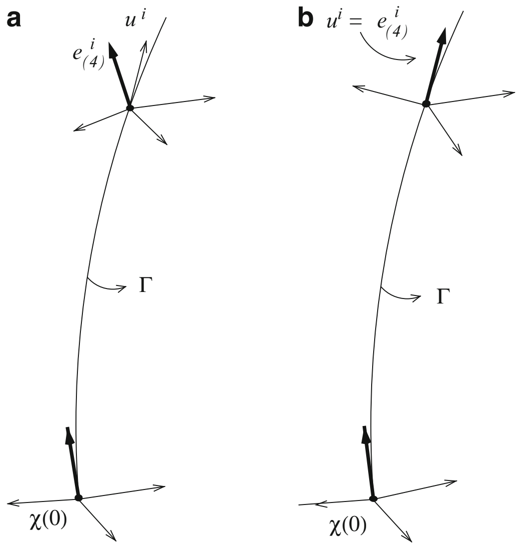


Fig. 2.13 (**a**) shows the parallel transport along a nongeodesic curve. (**b**) depicts the F–W transport along the same curve

$$\mathbf{g}_{..}(x) = [\phi(x)]^2 \, d_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{2.102ii}$$

$$\vec{\mathbf{e}}_{(a)}(x) = [\phi(x)]^{-1} \, \delta^i_{(a)} \frac{\partial}{\partial x^i}, \tag{2.102iii}$$

$$g_{ij}(x) \, e^i_{(a)}(x) \, e^j_{(b)}(x) = d_{(a)(b)}. \tag{2.102iv}$$

Consider the image Γ of a differentiable curve given by

$$\begin{aligned} x^\alpha &= \mathcal{X}^\alpha(s) \equiv 0, \\ x^4 &= \mathcal{X}^4(s) = \mathcal{S}^{-1}(s), \\ s = \mathcal{S}(x^4) &:= \int_0^{x^4} \phi(0,0,0,t)\, \mathrm{d}t. \end{aligned} \tag{2.103}$$

(The existence of $\mathcal{S}^{-1}$ is assured by the fact that $\frac{\mathrm{d}\mathcal{S}(x^4)}{\mathrm{d}x^4} = \phi(0,0,0,x^4) > 0$.) Therefore, Γ is a portion of the x^4-axis. The unit tangential vector along Γ is given by (2.103) and (2.102iii) as

$$\begin{aligned} \mathcal{U}^i(s) &= \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} = [\phi(\mathcal{X}(s))]^{-1} \, \delta^i_{(4)} = e^i_{(4)}(\mathcal{X}(s)), \\ \mathcal{U}^\alpha(s) &\equiv 0, \quad \mathcal{U}^4(s) > 0. \end{aligned} \tag{2.104}$$

Now, from the metric (2.102ii), the Christoffel symbols can be computed as

$$\left\{ {i \atop j\,k} \right\} = [\phi(x)]^{-1} \cdot d^{\,il} \cdot \left[d_{jl} \partial_k \phi + d_{lk} \partial_j \phi - d_{kj} \partial_l \phi \right]. \tag{2.105}$$

Thus, by (2.104) and (2.105), we obtain

$$\frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = \left\{[\phi(x)]^{-3}\left(\delta^i_{(4)}\partial_4\phi + d^{\,ij}\partial_j\phi\right)\right\}_{|x=\mathcal{X}(s)}. \tag{2.106}$$

The above equation demonstrates that the principle curvature $\kappa_{(1)}(s)$ (and therefore the 4-acceleration) is nonzero, unless $\partial_\alpha\phi \equiv 0$.

Equation (2.100) implies that the tangential vector $\mathcal{U}^i(s)\left[\frac{\partial}{\partial x^i}\right]_{|..}$ *automatically undergoes F–W transport along Γ.*

For the other three spacelike vector fields $e^i_{(\alpha)}(\mathcal{X}(s))$, the left-hand side of (2.100), by consequences of (2.104), (2.102iii), and (2.105), yields

$$\begin{aligned}\frac{De^i_{(\alpha)}(\cdot\cdot)}{\mathrm{d}s} &= \frac{de^i_{(\alpha)}(\cdot\cdot)}{\mathrm{d}s} + \left\{\begin{matrix} i \\ j\ k\end{matrix}\right\}_{|..}\mathcal{U}^j(s)\,e^k_{(\alpha)}(\cdot\cdot) \\ &= \left\{[\phi(x)]^{-3}\cdot\delta^i_{(4)}\cdot\partial_\alpha\phi\right\}_{|..}. \end{aligned}\tag{2.107}$$

On the other hand, the right-hand side of (2.100), by use of (2.2.4,ii), (2.104), and (2.106), implies that

$$\begin{aligned}&g_{jk}(\cdot\cdot)\left\{\mathcal{U}^i(s)\,e^j_{(\alpha)}(\cdot\cdot)\frac{D\mathcal{U}^k(s)}{\mathrm{d}s} - \mathcal{U}^j(s)\,e^k_{(\alpha)}(\cdot\cdot)\frac{D\mathcal{U}^i(s)}{\mathrm{d}s}\right\} \\ &= \left\{[\phi(x)]^{-3}\cdot\delta^i_{(4)}\cdot\partial_\alpha\phi\right\}_{|..}. \end{aligned}\tag{2.108}$$

Comparing (2.107) and (2.108), we conclude that the orthonormal tetrad $\left\{\vec{\mathbf{e}}_{(a)}(\mathcal{X}(s))\right\}_1^4$ indeed undergoes the F–W transport along Γ. □

(Conformally flat space–times are discussed in detail in Appendix 4.)

We shall now consider an operational method for measurement of the proper length of a spacelike curve. Recall that *the arc separation function* $\sum$ of a differentiable curve $\mathcal{X}$ was discussed in # 8 of Exercise 1.3. It is defined by

$$\sum(\mathcal{X}) := \int_{t_{(1)}}^{t_{(2)}}\sqrt{\left|g_{ij}(\mathcal{X}(t))\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\right|}\,\mathrm{d}t. \tag{2.109}$$

(The integral above is *invariant under reparametrization as well as under a general coordinate transformation*.)

Consider an observer with an orthonormal tetrad, clock, and extremely small devices to emit and receive photons. (See the Fig. 2.14.)

In Fig. 2.14, $\Gamma_{(1)}$ and $\Gamma_{(2)}$ are images of curves $\mathcal{X}_{(1)}$ and $\mathcal{X}_{(2)}$ representing the same observer. Null curves (or photon trajectories) are represented by $\mathcal{X}_{(3)}$ and $\mathcal{X}_{(4)}$ with images $\Gamma_{(3)}$ and $\Gamma_{(4)}$ respectively. Moreover, Γ is the image of a spacelike

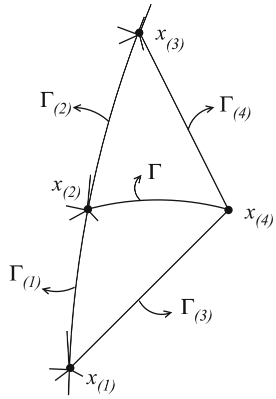


Fig. 2.14 Measurement of a spacelike separation along the image Γ

curve $\mathcal{X}$ under scrutiny. In special relativity, the measurement of a spacelike distance is much simpler to analyze than in the case of curved space–time. The procedure was briefly mentioned after (2.18). Now, we shall deal with the problem in *flat space–time* in greater detail. (This will allow us to have better insight into the corresponding analysis when the space–time is curved.) Let us choose Minkowskian coordinates and each of the five curves in Fig. 2.14 as geodesic or *straight lines*. We choose proper time parameter $s \equiv \tau$ for the inertial observer, proper length parameter l for the spacelike straight line $\mathcal{X}$, and an affine parameter α for the null straight lines $\mathcal{X}_{(3)}$ and $\mathcal{X}_{(4)}$. Therefore, we can express

$$
\begin{aligned}
\mathcal{X}^i_{(1)}(s) &= t^i_{(1)}s + c^i_{(1)}, \quad s \in [s_1, s_2), d_{ij}\, t^i_{(1)}t^j_{(1)} = -1\,;\\
\mathcal{X}^i_{(2)}(s) &= t^i_{(2)}s + c^i_{(2)}, \quad s \in [s_2, s_3], d_{ij}\, t^i_{(2)}t^j_{(2)} = -1, \quad t^i_{(2)} = t^i_{(1)}\,;\\
\mathcal{X}^i(l) &= t^i l + c^i, \quad l \in [l_1, l_2], d_{ij}\, t^i t^j = 1\,;\\
\mathcal{X}^i_{(3)}(\alpha) &= t^i_{(3)}\alpha + c^i_{(3)}, \quad \alpha \in [\alpha_1, \alpha_2), d_{ij}\, t^i_{(3)}t^j_{(3)} = 0\,;\\
\mathcal{X}_{(4)}(\alpha) &= t^i_{(4)}\alpha + c^i_{(4)}, \quad \alpha \in (\alpha_2, \alpha_3], d_{ij}\, t^i_{(4)}t^j_{(4)} = 0.
\end{aligned} \tag{2.110}
$$

Here, $t^i_{(\cdot\cdot)}$'s and $c^i_{(\cdot\cdot)}$'s are const. By (2.109), (2.110) yields

$$
\sum(\mathcal{X}_{(1)}) = \sqrt{-d_{ij}\, t^i_{(1)}t^j_{(1)}} \cdot (s_2 - s_1) = \sqrt{-d_{ij}\left(x^i_{(2)} - x^i_{(1)}\right)\left(x^j_{(2)} - x^j_{(1)}\right)},
$$

$$
\sum(\mathcal{X}_{(2)}) = \sqrt{-d_{ij}\left(x^i_{(3)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)},
$$

$$
\begin{aligned}
\sum(\mathcal{X}) &= \sqrt{d_{ij}\, t^i t^j} \cdot (l_2 - l_1) = \sqrt{d_{ij}\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(4)} - x^j_{(2)}\right)},\\
\sum(\mathcal{X}_{(3)}) &= \sqrt{d_{ij}\left(x^i_{(4)} - x^i_{(1)}\right)\left(x^j_{(4)} - x^j_{(1)}\right)} = 0,\\
\sum(\mathcal{X}_{(4)}) &= \sqrt{d_{ij}\left(x^i_{(3)} - x^i_{(4)}\right)\left(x^j_{(3)} - x^j_{(4)}\right)} = 0.
\end{aligned} \tag{2.111}
$$

Now, we can express

$$
\begin{aligned}
x^i_{(4)} - x^i_{(3)} &= \left(x^i_{(4)} - x^i_{(2)}\right) - \left(x^i_{(3)} - x^i_{(2)}\right),\\
x^i_{(4)} - x^i_{(1)} &= \left(x^i_{(4)} - x^i_{(2)}\right) + \left(x^i_{(2)} - x^i_{(1)}\right),\\
x^i_{(2)} - x^i_{(1)} &= \lambda\left(x^i_{(3)} - x^i_{(2)}\right),\\
\lambda &:= (s_2 - s_1)\big/(s_3 - s_2) > 0.
\end{aligned} \tag{2.112}
$$

Putting the above equations into the null separations of (2.111), we obtain

$$
\begin{aligned}
&\lambda\Big\{d_{ij}\Big[\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(4)} - x^j_{(2)}\right) + \left(x^i_{(3)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)\\
&\quad - 2\left(x^i_{4)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)\Big]\Big\} = 0,\\
&d_{ij}\Big[\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(4)} - x^j_{(2)}\right) + \lambda^2\left(x^i_{(3)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)\\
&\quad + 2\lambda\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)\Big] = 0.
\end{aligned} \tag{2.113}
$$

Adding the above equations and cancelling the factor $(1 + \lambda) > 0$, we deduce that

$$
d_{ij}\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(4)} - x^j_{(2)}\right) = -\lambda d_{ij}\left(x^i_{(3)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right), \tag{2.114}
$$

or,

$$\left[\sum(\mathcal{X})\right]^2 = \lambda\left[\sum\left(\mathcal{X}_{(2)}\right)\right]^2 = \sum\left(\mathcal{X}_{(1)}\right)\cdot\sum\left(\mathcal{X}_{(2)}\right).$$

Therefore, *the (inertial) observer can measure spatial distances by the readings of his clock!* [55, 242].

Now, let us try to compute the spacelike separation along the image Γ (Fig. 2.14), in the case when the curvature tensor is nonzero. We employ Riemann normal coordinates for the point $x_{(2)}$ in Fig. 2.14. (See p. 67 for the definition.) Let the domain of validity $D \subset \mathbb{R}^4$ encompass the space–time diagram in Fig. 2.14. So, we assume that

$$\begin{aligned} x_{(2)} &= (0,0,0,0), \\ g_{ij}(0,0,0,0) &= d_{ij}, \\ \partial_k g_{ij}(x)_{|(0,0,0,0)} &= 0. \end{aligned} \tag{2.115}$$

From the point of view of physics, we refer to this coordinate chart as *local Minkowskian coordinates*.

Let us assume that the observer is inertial. In such a case, geodesics originating at $x_{(2)}$ are furnished by straight lines [55, 242]

$$\begin{aligned} &\mathcal{X}^i_{(1)}(s) = t^i_{(1)}\cdot s, \quad s\in[-s_1,0), \\ &d_{ij}\,t^i_{(1)}\,t^j_{(1)} = -1, \quad \mathcal{X}^i_{(1)}(0) = (0,0,0,0) = x^i_{(2)}\,; \\ &\mathcal{X}^i_{(2)}(s) = t^i_{(2)}\cdot s = t^i_{(1)}\cdot s, \quad s\in[0,s_3]\,; \\ &\mathcal{X}^i(s) = t^i\cdot l, \;\; l\in[0,l_2]. \end{aligned} \tag{2.116}$$

In the first of the above equations, we have admitted negative values of the parameters, indicating that it is *a signed arc separation parameter*. The null geodesics with images $\Gamma_{(3)}$ and $\Gamma_{(4)}$ are *not* necessarily straight lines. They are governed by equations

$$\begin{aligned} &\frac{\mathrm{d}^2\mathcal{X}^i_{(3)}(\alpha)}{\mathrm{d}\alpha^2} + \left\{\begin{matrix} i \\ j\;k \end{matrix}\right\}_{|\mathcal{X}_{(3)}(\alpha)}\cdot\frac{\mathrm{d}\mathcal{X}^j_{(3)}(\alpha)}{\mathrm{d}\alpha}\,\frac{\mathrm{d}\mathcal{X}^k_{(3)}(\alpha)}{\mathrm{d}\alpha} = 0, \quad \alpha\in[\alpha_1,\alpha_2)\,; \\ &\frac{\mathrm{d}^2\mathcal{X}^i_{(4)}(\alpha)}{\mathrm{d}\alpha^2} + \left\{\begin{matrix} i \\ j\;k \end{matrix}\right\}_{|\mathcal{X}_{(4)}(\alpha)}\cdot\frac{\mathrm{d}\mathcal{X}^j_{(4)}(\alpha)}{\mathrm{d}\alpha}\,\frac{\mathrm{d}\mathcal{X}^k_{(4)}(\alpha)}{\mathrm{d}\alpha} = 0, \quad \alpha\in(\alpha_2,\alpha_3]. \end{aligned} \tag{2.117}$$

(Recall that α is an affine parameter.) Now, computing arc separations along various curves, we obtain

$$\begin{aligned}
\sum(\mathcal{X}_{(1)}) &= \int_{-s_1}^{0} \sqrt{-g_{ij}(\mathcal{X}(t))\frac{\mathrm{d}\mathcal{X}^i_{(1)}(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}}\,\mathrm{d}t \\
&= \sqrt{-g_{ij}\left(\mathcal{X}_{(1)}(0)\right)t^i_{(1)}\,t^j_{(1)}}\cdot(s_1) \\
&= \sqrt{-d_{ij}\left(x^i_{(2)}-x^i_{(1)}\right)\left(x^j_{(2)}-x^j_{(1)}\right)}, \\
\sum(\mathcal{X}_{(2)}) &= \sqrt{-d_{ij}\left(x^i_{(3)}-x^i_{(2)}\right)\left(x^j_{(3}-x^j_{(2)}\right)}, \\
\sum(\mathcal{X}) \quad &= \sqrt{d_{ij}\left(x^i_{(4)}-x^i_{(2)}\right)\left(x^j_{(4)}-x^j_{(2)}\right)}.
\end{aligned} \tag{2.118}$$

Comparing the above equations with the corresponding equations in (2.111), we conclude that the agreements are *exact*. However, null curves, which are assumed to be of class C^3, pose some complications. Expressing the 4-velocity components $\mathcal{U}^i_{(3)}(\alpha) = \frac{\mathrm{d}\mathcal{X}^i_{(3)}(\alpha)}{\mathrm{d}\alpha}$ along the null curve $\mathcal{X}_{(3)}$, we derive from (2.117) that

$$\frac{\mathrm{d}\mathcal{U}^i_{(3)}(\alpha)}{\mathrm{d}\alpha} = -\left\{\begin{matrix} i \\ j\;k \end{matrix}\right\}_{\big|\mathcal{X}_{(3)}(\alpha)} \cdot \mathcal{U}^j_{(3)}(\alpha)\,\mathcal{U}^k_{(3)}(\alpha), \quad \alpha \in [\alpha_1, \alpha_2). \tag{2.119}$$

By the mean value theorem [32], we can deduce from (2.119) that

$$\begin{aligned}
\mathcal{X}^i_{(3)}(\alpha) = \;& x^i_{(1)} + (\alpha-\alpha_1)\,\mathcal{U}^i_{(3)}(\alpha_1) + \frac{1}{2}(\alpha-\alpha_1)^2\left[\frac{\mathrm{d}\mathcal{U}^i_{(3)}(\alpha)}{\mathrm{d}\alpha}\right]_{\big|\beta}, \\
&\beta := \alpha_1 + \theta(\alpha-\alpha_1), \quad 0 < \theta < 1.
\end{aligned} \tag{2.120}$$

Using (2.119) in (2.120), we conclude that

$$\begin{aligned}
(\alpha_2-\alpha_1)\,\mathcal{U}^i_{(3)}(\alpha_1) &= x^i_{(3)} - x^i_{(1)} - r^i\left(|\alpha_2-\alpha_1|\,,\partial_k\, g_{ij\;|\beta_2}\right), \\
r^i(\cdot\cdot) &:= \frac{(\alpha_2-\alpha_1)^2}{2}\left[\left\{\begin{matrix} i \\ j\;k \end{matrix}\right\}\mathcal{U}^j_{(3)}\,\mathcal{U}^k_{(3)}\right]_{\big|\beta}.
\end{aligned} \tag{2.121}$$

For a "small" domain and "weak" intrinsic gravitation, the remainder term $\left|r^i(\cdot\cdot)\right|$ is very small compared to $\left|x^i_{(3)} - x^i_{(1)}\right|$. It should be remarked, however, that (2.121) is *exact*. Since we are dealing with geodesic triangles in a curved manifold [266], we do expect some deviations from the conclusions in the flat manifold. (We have tacitly assumed here that there are *no conjugate points for geodesics* in the domain of consideration.)

Example 2.2.5. Consider a Riemannian normal or local Minkowskian coordinate chart. By the conditions

$$\begin{aligned} g_{ij}(0,0,0,0) &= d_{ij}, \\ \partial_k\, g_{ij}(x)_{|(0,0,0,0)} &= 0, \end{aligned}$$

we can derive from (1.141ii)

$$R_{ijkl}(0,0,0,0) = (1/2)\big[\partial_j\,\partial_k g_{il} + \partial_i\,\partial_l g_{jk} - \partial_j\,\partial_l g_{ik} - \partial_i\,\partial_k g_{jl}\big]_{|(0,0,0,0)}. \quad (2.122)$$

With the above equations, we can easily prove the algebraic identities of the Riemann–Christoffel tensor stated in Theorem 1.3.19.

We can furthermore prove that

$$\begin{aligned} \big\{\partial_l\,\partial_k g_{ij} - \partial_i[kl, j] - \partial_j[kl, i]\big\}_{|(0,0,0,0)} &= -\big[R_{ikjl} + R_{iljk}\big]_{|(0,0,0,0)} \\ &=: 3S_{ijkl}(0,0,0,0). \end{aligned} \quad (2.123)$$

(See Problem # 4 of Exercise 2.2.) □

Synge, in his book [243], dealt extensively with the problem of physical measurements with help of the world function, $\Omega(x_2, x_1)$ of p. 86, the *symmetrized curvature tensor* (in (2.123)), the principal curvature of observer's world line, etc. The book [184] by Misner et al. also deals with such topics from slightly different perspectives.

We shall now turn our attention to the equations of motion of a particle in curved space–time. We consider the general case of a possibly variable mass $M(s) > 0$ for the particle. We generalize the special relativistic equations given in # 6 of Exercise 2.1. From the definition of force as the rate of change of momentum, *we postulate the equations of motion* along a future-pointing timelike curve as

$$\begin{aligned} & M(s)\left[\frac{\mathrm{d}^2\mathcal{X}^i(s)}{\mathrm{d}s^2} + \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\}_{|..} \frac{\mathrm{d}\mathcal{X}^j(s)}{\mathrm{d}s}\,\frac{\mathrm{d}\mathcal{X}^k(s)}{\mathrm{d}s}\right] + \frac{\mathrm{d}M(s)}{\mathrm{d}s}\,\frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} \\ & \quad = F^i(x,u)\Big|_{\substack{x^k=\mathcal{X}^k(s) \\ u^k=\frac{\mathrm{d}\mathcal{X}^k(s)}{\mathrm{d}s}}}, \end{aligned} \quad (2.124)$$

$$\text{or} \quad \frac{D}{\mathrm{d}s}\big[M(s)\mathcal{U}^i(s)\big] = F^i(x,u)_{|..}.$$

On the right-hand sides of (2.124), the 4-force vector components $F^i(\cdot\cdot)_{|..}$ are solely due to *nongravitational effects*.

In terms of *physical* (*or orthonormal*) *components*, (2.124) yields

$$M(s)\,\mathcal{U}^{(b)}(s)\left[\partial_{(b)}u^{(a)}(x) - \gamma^{(a)}{}_{(d)(b)}(x)\cdot u^{(d)}(x)\right]\Big|_{\substack{x^k=\mathcal{X}^k(s)\\ u^{(e)}=\mathcal{U}^{(e)}(s)}}$$

$$+\frac{\mathrm{d}M(s)}{\mathrm{d}s}\cdot\mathcal{U}^{(a)}(s) = F^{(a)}(x,u)_{|..}, \tag{2.125i}$$

$$\frac{\mathrm{d}M(s)}{\mathrm{d}s} = -d_{(a)(b)}\left[F^{(a)}(\cdot\cdot)\right]_{|..}\cdot\mathcal{U}^{(b)}(s). \tag{2.125ii}$$

Here, $\gamma^{(a)}{}_{(d)(b)}(x)$ are components of *the Ricci rotation coefficients* in (1.124ii). Although (2.124) and (2.125i) are mathematically equivalent, (2.125i) are preferable from the point of view of observation, as they correspond to the physical frame. Equation (2.125ii) is the relativistic generalization of the Newtonian power law (2.19ii). With help from (2.125ii), we can prove that *the mass function $M(s)$ is constant-valued if and only if vector fields $\mathcal{U}^i(s)$ and $F^i(\cdot\cdot)_{|..}$ are orthogonal.* (Compare with Problem # 6(i) of Exercise 2.1.) In such a scenario, (2.124), (2.98), and (2.99) yield

$$M(s) = m = \text{positive const.}, \tag{2.126}$$

$$\frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = (m)^{-1}\,F^i(\cdot\cdot)_{|..} = \kappa_{(1)}(s)\,N^i(s). \tag{2.127}$$

Therefore, *nongeodesic or 4-accelerated motions* of particles and observers, with $\kappa_{(1)}(s)\neq 0$, must be caused by *nongravitational forces*.

There is a special class of nongravitational forces called *monogenic forces* [159]. A monogenic force is derivable from *a single work function* $\mathcal{W}(x,u)$ of class C^2 such that

$$F^i(\cdot\cdot)_{|..} := g^{ij}(\mathcal{X}(s))\left\{\nabla_j\mathcal{W}(\cdot\cdot)_{|..} - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial\mathcal{W}(\cdot\cdot)}{\partial u^j}\right]_{|..}\right\}. \tag{2.128}$$

We elaborate on the consequences of a monogenic force in the following example.

Example 2.2.6. We can derive equations of motion (2.127) with (2.128) *from a variational principle*. (See (1.187) and Appendix 1.) Consider the Lagrangian function L of eight variables in the following:

$$L(x,u) := -m\sqrt{-g_{ij}(x)\,u^i u^j} + \mathcal{W}(x,u). \tag{2.129}$$

Therefore, taking partial derivatives, we obtain

$$\frac{\partial L(\cdot\cdot)}{\partial x^i} = \frac{m}{2}\,\frac{(\partial_i g_{jk})u^j u^k}{\sqrt{-g_{kl}u^k u^l}} + \frac{\partial \mathcal{W}(\cdot\cdot)}{\partial x^i},$$

$$\frac{\partial L(\cdot\cdot)}{\partial u^i} = \frac{m g_{ij}(\cdot\cdot)u^j}{\sqrt{-g_{kl}u^k u^l}} + \frac{\partial \mathcal{W}(\cdot\cdot)}{\partial u^i}. \tag{2.130}$$

The Euler–Lagrange equations, derived from (2.130), yield

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial L(\cdot\cdot)}{\partial u^i}\right]_{|..} - \frac{\partial L(\cdot\cdot)}{\partial x^i}_{|..} \\
&= m\left[g_{ij}(\mathcal{X}(s))\frac{\mathrm{d}\mathcal{U}^j(s)}{\mathrm{d}s} + \partial_k g_{ij\,|..}\cdot\mathcal{U}^j(s)\,\mathcal{U}^k(s)\right] + \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{W}(\cdot\cdot)_{|..} \\
&\quad - \frac{m}{2}\cdot\partial_i g_{jk\,|..}\cdot\mathcal{U}^j(s)\,\mathcal{U}^k(s) - \partial_i\mathcal{W}(\cdot\cdot)_{|..},
\end{aligned}$$

or, $$m\left[g_{ij}(\cdot\cdot)\frac{\mathrm{d}\mathcal{U}^j(s)}{\mathrm{d}s} + [jk,i]_{|..}\,\mathcal{U}^j(s)\,\mathcal{U}^k(s)\right] = \partial_i\mathcal{W}_{|..} - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial\mathcal{W}}{\partial u^i}\right]_{|..},$$

or, $$m\frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = g^{ij}(\cdot\cdot)_{|..}\left\{\nabla_j\mathcal{W}_{|..} - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial\mathcal{W}}{\partial u^j}\right]_{|..}\right\}. \tag{2.131}$$

□

We shall now consider a useful class of monogenic forces by requiring that the work function $\mathcal{W}(x,u)$ is a *positive, homogeneous function of degree one*. That is, for an arbitrary positive number $\lambda > 0$, we must have from (2.129),

$$\mathcal{W}(x,\lambda u) = \lambda\mathcal{W}(x,u)\,;$$

$$L(x,\lambda u) = \lambda L(x,u). \tag{2.132}$$

(Consult [55, 171, 242].)

By Euler's theorem on homogeneous functions [176], we derive from (2.132) that

$$u^i\frac{\partial\mathcal{W}(x,u)}{\partial u^i} = \mathcal{W}(x,u)\,;$$

$$u^i\frac{\partial L(x,u)}{\partial u^i} = L(x,u). \tag{2.133}$$

We may assume another condition on the above Lagrangian, namely,

$$\det \left\{ \frac{\partial^2 [L(x,u)]^2}{\partial u^i \, \partial u^j} \right\} \neq 0. \tag{2.134}$$

Equation (2.133), along with the inequality (2.134) and the differentiability $L \in C^2(D \subset \mathbb{R}^8; \mathbb{R})$, leads to a *Finsler metric* [2],

$$\begin{aligned} f_{ij}(x,u) &:= -(1/2)\frac{\partial^2 [L(x,u)]^2}{\partial u^i \, \partial u^j} \equiv f_{ji}(x,u), \\ \det\left[f_{ij}(\cdot\cdot)\right] &\neq 0. \end{aligned} \tag{2.135}$$

Now, we shall define conjugate 4-momentum vector components from (2.130) and (2.132) as

$$p_i = P_i(x,u) := \frac{\partial L(x,u)}{\partial u^i} = \frac{m g_{ij} u^j}{\sqrt{-g_{kl} u^k u^l}} + \frac{\partial \mathcal{W}(x,u)}{\partial u^i}. \tag{2.136}$$

We can deduce from the equations above that

$$g^{ij}(x)\left[p_i - \frac{\partial \mathcal{W}(x,u)}{\partial u^i}\right]\left[p_j - \frac{\partial \mathcal{W}(x,u)}{\partial u^j}\right] = -m^2. \tag{2.137}$$

This equation is the generalization of the special relativistic quadratic constraint (2.33). It is *the generalized mass-shell condition*. For a massless particle like the photon, we can put $m = 0$ on the right-hand side of (2.137).

We shall now investigate the arena of *relativistic Hamiltonian mechanics*. First, we will follow nonrelativistic Hamiltonian mechanics in order to introduce the relativistic version of the *Legendre transformation*. Thus, we define *the relativistic Hamiltonian* as

$$\widehat{\mathcal{H}}(x,p) := p_i u^i - L(x,u) = u^i \frac{\partial L(\cdot\cdot)}{\partial u^i} - L(\cdot\cdot). \tag{2.138}$$

Let us *assume* now (2.128) and (2.132). Therefore, by the conditions (2.136) and (2.133), we conclude that $\widehat{\mathcal{H}}(x,p) \equiv 0$. Thus, the obvious approach fails! So, we look for *another definition* of a relativistic Hamiltonian. We notice that by (2.136) and (2.133), the 4-momentum function $P_i(x,u)$ is a homogeneous polynomial of *degree zero* in the variables u^i. Therefore, for the positive number $\lambda := (u^4)^{-1}$, and the variables $v^\alpha := u^\alpha (u^4)^{-1}$, we obtain

$$p_i = P_i(x,u) = P_i(x,(u^4)^{-1}u) = P_i(x; v^1, v^2, v^3, 1)). \tag{2.139}$$

The above equations locally yield a parametrized seven-dimensional hyperspace in the eight-dimensional "covariant phase space" [55] coordinatized by the (x, u)-chart. Alternatively, the elimination of three variables v^α in (2.139) will result locally in a *holonomic constraint*

$$\mathcal{H}(x, p) = 0. \tag{2.140}$$

Since we have already derived such a constraint in (2.137), we identify *the relativistic Hamiltonian, or super-Hamiltonian*, as

$$\mathcal{H}(x, p) := \frac{1}{2m}\left\{ g^{ij}(x)\left[p_i - \frac{\partial \mathcal{W}(x,u)}{\partial u^i}\right]\left[p_j - \frac{\partial \mathcal{W}(x,u)}{\partial u^j}\right] + m^2\right\}, \tag{2.141i}$$

$$\frac{\partial}{\partial u^k}\left\{ g^{ij}(x)\left[p_i - \frac{\partial \mathcal{W}(x,u)}{\partial u^i}\right]\left[p_j - \frac{\partial \mathcal{W}(x,u)}{\partial u^j}\right]\right\} \equiv 0. \tag{2.141ii}$$

Remark: The super-Hamiltonian above is, of course, *different from that in (2.138)*.

The variationally derived equations of motion in (2.131) are obtainable by an alternative Lagrangian that involves the super-Hamiltonian $\mathcal{H}(x, p)$.

Let a parametrized curve of class C^2 in a domain of *a 13-dimensional space* be specified by:

$$x^i = \overline{\mathcal{X}}^i(t), \quad u^i = \frac{\mathrm{d}\overline{\mathcal{X}}^i(t)}{\mathrm{d}t}, \quad p_i = \overline{\mathcal{P}}_i(t), \quad \lambda = \overline{\Lambda}(t), \quad t \in [t_1, t_2]. \tag{2.142}$$

(Here, t is just a parameter.)

Let a Lagrangian of class C^2 be furnished by

$$\mathcal{L}_{(\mathrm{I})}(x, u; p; \lambda) := p_i u^i - \lambda \mathcal{H}(x, p). \tag{2.143}$$

Here, λ is the *Lagrange multiplier*. The corresponding nine Euler–Lagrange equations are given by

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}_{(\mathrm{I})}(\cdot\cdot)}{\partial u^i}\right]_{|..} - \left[\frac{\partial \mathcal{L}_{(\mathrm{I})}(\cdot\cdot)}{\partial x^i}\right]_{|..} \\ &= \frac{\mathrm{d}\mathcal{P}_i(t)}{\mathrm{d}t} - \left[\lambda \frac{\partial \mathcal{H}(\cdot\cdot)}{\partial x^i}\right]_{|..}, \end{aligned} \tag{2.144i}$$

$$0 = 0 - \left[\frac{\partial \mathcal{L}_{(\mathrm{I})}(\cdot\cdot)}{\partial p_i}\right]_{|..} = -\frac{\mathrm{d}\overline{\mathcal{X}}^i(t)}{\mathrm{d}t} + \left[\lambda \frac{\partial \mathcal{H}(\cdot\cdot)}{\partial p_i}\right]_{|..}, \tag{2.144ii}$$

$$0 = 0 - \left[\frac{\partial \mathcal{L}_{(\mathrm{I})}(\cdot\cdot)}{\partial \lambda}\right]_{|..} = \mathcal{H}(x, p)_{|..}. \tag{2.144iii}$$

We reparametrize the curve by setting

$$s = \mathcal{S}(t) := \int_{t_1}^{t} \overline{\Lambda}(w)\, \mathrm{d}w,$$
$$\mathcal{X}^i(s) = \overline{\mathcal{X}}^i(t), \quad \mathcal{P}_i(s) = \overline{\mathcal{P}}_i(t). \tag{2.145}$$

Equations (2.144i–iii) go over into

$$\frac{\mathrm{d}\mathcal{P}_i(s)}{\mathrm{d}s} = -\frac{\partial\mathcal{H}(\cdot\cdot)}{\partial x^i}\Big|_{\cdot\cdot}, \tag{2.146i}$$

$$\frac{\partial\mathcal{X}^i(s)}{\mathrm{d}s} = \frac{\partial\mathcal{H}(\cdot\cdot)}{\partial p_i}\Big|_{\cdot\cdot}, \tag{2.146ii}$$

$$\text{and} \quad \mathcal{H}(x, p)_{|\cdot\cdot} = 0\,. \tag{2.146iii}$$

The equations above are called the *relativistic Hamiltonian equations of motion* or *relativistic canonical equations of motion*. (See [243].)

We assert that the equations of motion in (2.146i–iii) are *equivalent to those in* (2.131).

Example 2.2.7. We explore the motion of a massive, charged particle in an external electromagnetic field. (See Example 1.2.22 and (2.67).) We choose for the work function as

$$\mathcal{W}(x, u) := e\, A_k(x)\, u^k, \quad \mathcal{W}(x, \lambda u) = \lambda\mathcal{W}(x, u),$$
$$\frac{\partial\mathcal{W}(x, u)}{\partial u^k} = eA_k(x). \tag{2.147}$$

Equations (2.129), (2.130), and (2.136) yield

$$L(x, u) = -m\sqrt{-g_{ij}(x)u^i u^j} + e\, A_k(x)\, u^k,$$
$$p_i = \frac{\partial L(x,\, u)}{\partial u^i} = \frac{m\, g_{ij}(x)\, u^j}{\sqrt{-g_{kl}\, u^k u^l}} + e\, A_i(x),$$
$$\left[p_i - e\, A_i(x)\right]_{|\cdot\cdot} = m\, g_{ij}\,(\mathcal{X}(s))\, \mathcal{U}^j(s). \tag{2.148}$$

The relativistic super-Hamiltonian function from (2.141i) is furnished by

$$\mathcal{H}(x, p) = (2m)^{-1}\{g^{kl}(x)\left[p_k - e\, A_k(x)\right]\left[p_l - e\, A_l(x)\right] + m^2\}. \tag{2.149}$$

Hamilton's canonical equations of motion (2.146i,ii) yield

$$\frac{d\mathcal{X}^i(s)}{ds} = \frac{\partial \mathcal{H}(\cdot\cdot)}{\partial p_i}_{|..} = m^{-1}\{g^{ij}(x)[p_j - e\,A_j(x)]\}_{|..} = \mathcal{U}^i(s)\,;$$

$$\frac{d\mathcal{P}_i(s)}{ds} = -\frac{\partial \mathcal{H}(\cdot\cdot)}{\partial x^i}_{|..} = -(2m)^{-1}\{\partial_i g^{kl}[p_k - eA_k][p_l - eA_l]$$

$$- 2e\, g^{kl}\cdot \partial_i A_k[p_l - eA_l]\}_{|..}$$

$$= (m)^{-1}\left\{(g^{lm}g^{kn}\partial_i g_{mn})\left(\frac{m^2}{2}u_k u_l\right) + emg^{kl}\cdot\partial_i A_k \cdot u_l\right\}_{|..},$$

$$\text{or,}\qquad \frac{d}{ds}\left[m\, g_{ik}u^i + eA_i\right]_{|..} = \left[\frac{m}{2}\partial_i g_{kl}\cdot u^k u^l + e\partial_i A_k \cdot u^k\right]_{|..},$$

$$\text{or,}\qquad m\left[\frac{d^2\mathcal{X}^i(s)}{ds^2} + \left\{\begin{matrix} i \\ j\ k\end{matrix}\right\}_{|..}\frac{d\mathcal{X}^j(s)}{ds}\frac{d\mathcal{X}^k(s)}{ds}\right]$$

$$= e\left\{g^{ij}(x)\left[\partial_j A_k - \partial_k A_j\right]\right\}_{|..}\frac{d\mathcal{X}^k(s)}{ds}$$

$$= e\left[g^{ij}(x)\,F_{jk}(s)\right]_{|..}\frac{d\mathcal{X}^k(s)}{ds}. \tag{2.150}$$

The above equations are the correct generalization of the relativistic Lorentz equations of motion in (2.67) to curved space–time. □

There exist alternative Lagrangians for the variational derivation of Hamilton's canonical equations (2.146i,ii). We cite two such Lagrangians in the following:

$$L_{(\mathrm{II})}(x; p, p'; \lambda) := x^i\, p_i' + \lambda\mathcal{H}(x, p), \tag{2.151i}$$

$$L_{(\mathrm{III})}(x, u; p, p'; \lambda) := (1/2)(p_i u^i - x^i\, p_i') - \lambda\mathcal{H}(x, p). \tag{2.151ii}$$

(Here, the prime denotes the derivative $p'_{i|...} := \frac{d\mathcal{P}(t)}{dt}$.) The Lagrangian function $\mathcal{L}_{(\mathrm{III})}$ in the equation above explicitly reveals the symplectic structure of the canonical equations (2.146i–iii). (See Problem # 8(ii) of Exercise 2.2.)

So far, in this chapter, we have assumed that the space–time is a pseudo-Riemannian manifold M_4 and has, in general terms, attributed its curvature to the intrinsic gravitation. However, we have set up no field equations by which the curvature of space–time is made to depend on material sources. That is, we do not yet have the relativistic analog of (1.156). Now, we shall pursue this project.

Recall the Newtonian equations of motion in an external gravitational field furnished by (2.79ii). In a curvilinear coordinate chart of the Euclidean space $\mathbb{E}_3$, these equations go over into

$$\overline{\mathbf{g}}_{..}(\mathbf{x}) = \overline{g}_{\alpha\beta}(\mathbf{x})\, dx^{\alpha} \otimes dx^{\beta}, \tag{2.152i}$$

$$\overline{\mathbf{R}}^{\cdot}_{...}(\mathbf{x}) \equiv \overline{\mathbf{O}}^{\cdot}_{...}(\mathbf{x}), \tag{2.152ii}$$

$$\frac{d^2\mathcal{X}^{\alpha}(t)}{dt^2} + \overline{\left\{\begin{matrix}\alpha \\ \beta\,\gamma\end{matrix}\right\}}_{|..} \frac{d\mathcal{X}^{\beta}(t)}{dt}\,\frac{d\mathcal{X}^{\gamma}(t)}{dt} = -\overline{g}^{\alpha\beta}(\mathbf{x})\,\partial_{\beta}W(\mathbf{x})_{|..}$$
$$= -\overline{\nabla}^{\alpha} W(\mathbf{x})_{|..}. \tag{2.152iii}$$

The equations of motion (2.152iii) is variationally derivable from the Lagrangian

$$L_{(N)}(\mathbf{x}, \mathbf{v}) := (m/2)\left[\overline{g}_{\alpha\beta}(\mathbf{x})\, v^{\alpha} v^{\beta} - 2W(\mathbf{x})\right]. \tag{2.153}$$

On the other hand, in the absence of *nongravitational forces*, a particle follows a geodesic world line in *the curved space–time*. Such a geodesic is variationally derivable from the Lagrangian in (2.129) as

$$L(x, u) = -m\sqrt{-g_{ij}(x)u^i u^j}. \tag{2.154}$$

If we use *any affine parameter*, including s, *an alternative Lagrangian* for the geodesic curve is provided by

$$L_{(2)}(x, u) := (m/2)\, g_{ij}(x)\, u^i u^j$$
$$= (m/2)\left[g_{\alpha\beta}(x)\, u^{\alpha} u^{\beta} + 2g_{\alpha 4}(x)\, u^{\alpha} u^4 + g_{44}(x)(u^4)^2\right]. \tag{2.155}$$

Defining *coordinate 3-velocity components* by $v^{\alpha} := u^{\alpha}/u^4$, the Lagrangian above can be expressed as

$$L_{(2)}(\cdot\cdot) = (m/2)\left[g_{\alpha\beta}(x)v^{\alpha}v^{\beta} + 2g_{\alpha 4}(x)v^{\alpha} + g_{44}(x)\right](u^4)^2. \tag{2.156}$$

Comparing Lagrangians $L_{(N)}(\cdot\cdot)$ in (2.153) and $L_{(2)}(\cdot\cdot)$ in (2.156), we conclude that in the low velocity regime, the metric tensor component $-g_{44}(x)$ corresponds to twice the Newtonian potential $W(\mathbf{x})$ plus an undetermined constant. In a weak gravitational field, this relationship is (approximately) given by $g_{44}(x) = -[1 + 2W(\mathbf{x})] < 0$.

Now, we shall explore the field equation for the Newtonian potential inside material sources. We choose physical units such that *the Newtonian constant of gravitation* $G = 1$. (Recall that we have already chosen units so that the speed of light $c = 1$.) In such units, physical quantities are expressible as powers of the spatial unit of "length" or else "time.")

The Newtonian potential $W(\mathbf{x})$, in curvilinear coordinates (2.152i,ii), is governed by *Poisson's equation*

$$\overline{\nabla}^2 W(\mathbf{x}) := \overline{g}^{\alpha\beta}(\mathbf{x})\overline{\nabla}_\alpha \overline{\nabla}_\beta W = 4\pi\rho(\mathbf{x}) \tag{2.157i}$$

$$\text{or,} \quad \overline{\nabla}^2 [-(1 + 2W(\mathbf{x}))] = -8\pi\rho(\mathbf{x}). \tag{2.157ii}$$

Here, $\rho(\mathbf{x})$ is the nonrelativistic *mass density*. In the last row of the table in p. 127, the nonrelativistic mass density, momentum density, and stress density are *all unified in relativity theory by the symmetric energy–momentum–stress tensor* $T_{ij}(x)$. Therefore, in the right-hand side of the relativistic gravitational field equations, we would like to have $-8\pi\, T_{ij}(x)$, instead of *just* $-8\pi\rho(\mathbf{x})$ as in (2.157ii). This choice forces us to consider *a symmetric tensor* $\theta_{ij}(\cdot\cdot)$ for the left-hand side of the field equations. Since the left-hand side of (2.157ii) is (approximately) given by $\overline{\nabla}^2 g_{44}(\cdot\cdot)$, our tensor $\theta_{ij}(\cdot\cdot)$ should contain up to and including second partial derivatives of $g_{ij}(x)$. Moreover, the curvature tensor plays an important role in the intrinsic gravitation. Therefore, we would like $\theta_{ij}(\cdot\cdot)$ to be expressible by *various contractions of the curvature tensor*. Moreover, the energy–momentum–stress tensor satisfies the useful differential conservation equations $\nabla_j T^{ij} = 0$. Therefore, we shall also require $\theta_{ij}(\cdot\cdot)$ to satisfy $\nabla_j \theta^{ij} = 0$. Now, we have accumulated enough criteria to derive $\theta_{ij}(\cdot\cdot)$ explicitly in the following theorem [90].

Theorem 2.2.8. *Let the metric field components $g_{ij}(x)$ be of class C^3 in a domain $D_{(e)} \subset D \subset \mathbb{R}^4$. Let a symmetric, differentiable tensor field $\theta^{ij}(x)$ in $D_{(e)}$ be defined by*

$$\theta^{ij}(x) := k\left\{R^{ij}(x) + g^{ij}(x) \cdot [\phi(x)\, R(x) + \psi(x)]\right\}.$$

Here, k is a nonzero constant, $R_{ij}(\cdot\cdot)$ are components of the Ricci tensor, $R(\cdot\cdot)$ is the curvature invariant, $\phi(x)$ and $\psi(x)$ are two differentiable scalar fields. Then, $\nabla_j \theta^{ij} = 0$, if and only if $\theta^{ij}(x) = k\left[G^{ij}(x) - \Lambda\, g^{ij}(x)\right]$, where $G^{ij}(x)$ are components of the Einstein tensor and Λ is a constant.

Proof. (i) Assume that $\theta^{ij}(x) = k\left[G^{ij}(x) - \Lambda\, g^{ij}(x)\right]$. Then, $\nabla_j \theta^{ij} = k \times \left[\nabla_j G^{ij} - \Lambda \nabla_j\, g^{ij}\right] \equiv 0$ by (1.150ii) and (1.131ii).

(ii) Now, assume that $\nabla_j \theta^{ij} = 0$. Then, from the definition of $\theta^{ij}(x)$, we conclude that

$$0 = k\left\{\nabla_j R^{ij} + g^{ij}(x) \cdot \partial_j\, [\phi(x)\, R(x) + \psi(x)]\right\}.$$

Using $\nabla_j R^{ij} = (1/2)\, g^{ij}(x)\, \partial_j R$, we obtain that

$$0 = g^{ij}(x) \cdot \partial_j\, [(1/2)\, R(x) + \phi(x)\, R(x) + \psi(x)]\,.$$

Therefore,

$$(1/2)\, R(x) + \phi(x)\, R(x) + \psi(x) = -\Lambda = \text{const.}$$

Thus,

$$\theta^{ij}(x) = k\left[G^{ij}(x) - \Lambda\, g^{ij}(x)\right]. \qquad \blacksquare$$

Choosing the constant $k = 1$ for physical reasons (e.g., an acceptable Newtonian law for weak fields), we *assume* that the *gravitational field equations* are furnished by

$$\theta_{ij}(x) = G_{ij}(x) - \Lambda\, g_{ij}(x) = -8\pi\, T_{ij}(x),$$
$$x \in D_{(e)} \subset D \subset \mathbb{R}^4. \tag{2.158}$$

These field equations constitute the second assumption of the gravitational theory of this book. The constant Λ is called the *cosmological constant*, due to the fact that it was useful to Einstein in yielding a static universe when applying his field equations to the universe as a whole.

The right-hand side of (2.158) will read, in the common c.g.s. units, as $-\kappa\, T_{ij}(x)$, where $\kappa := 8\pi G/c^4 = 2.07 \times 10^{-48}\ \mathrm{gm}^{-1}\,\mathrm{cm}^{-1}\,\mathrm{s}^2$. In the sequel, we shall *continue to use the constant* κ, although we shall *still employ geometrized units* where $c = G = 1$. (Thus, in these units, $\kappa = 8\pi$.)

Einstein did not consider the cosmological constant Λ in his original papers. He published as field equations the following (or equivalent):

$$G_{ij}(x) = \begin{cases} -\kappa\, T_{ij}(x) & \text{inside material sources,} \\ 0 & \text{outside material sources} \end{cases} \tag{2.159i}$$

or
$$G_{(a)(b)}(x) = \begin{cases} -\kappa\, T_{(a)(b)}(x) & \text{inside material sources,} \\ 0 & \text{outside material sources.} \end{cases} \tag{2.159ii}$$

The *Einstein field equations* above are *great intellectual achievements* of the twentieth century! Outside material sources, the *vacuum field equations* reduce to

$$R_{ij}(x) = 0, \tag{2.160i}$$

$$\text{or,} \qquad R_{(a)(b)}(x) = 0. \tag{2.160ii}$$

(Recall that the above are the conditions for *Ricci flatness* as discussed in p. 70.)

Now, alternative versions of field equations (2.159i,ii) will be provided. Following Lichnerowicz [166], we shall assign "names" to the field equations. Denoting by $T(x) := g^{ij}(x)\, T_{ij}(x)$, we furnish the alternative versions as

$$\mathcal{E}_{ij}(x) := G_{ij}(x) + \kappa\, T_{ij}(x) = 0, \tag{2.161i}$$

$$\mathcal{E}_{(a)(b)}(x) := G_{(a)(b)}(x) + \kappa\, T_{(a)(b)}(x) = 0\,; \tag{2.161ii}$$

$$\begin{aligned}\tilde{\mathcal{E}}_{ij}(x) &:= R_{ij}(x) + \kappa\left[T_{ij}(x) - (1/2)\, g_{ij}(x)\, T(x)\right] \\ &= R^{k}{}_{ijk}(x) + \kappa\left[T_{ij}(x) - (1/2)\, g_{ij}(x)\, T(x)\right] = 0,\end{aligned} \tag{2.162i}$$

$$\tilde{\mathcal{E}}_{(a)(b)}(x) := R_{(a)(b)}(x) + \kappa\left[T_{(a)(b)}(x) - (1/2)\, d_{(a)(b)}\, T(x)\right]$$
$$= R^{(c)}{}_{(a)(b)(c)}(x) + \kappa\left[T_{(a)(b)}(x) - (1/2)\, d_{(a)(b)}\, T(x)\right] = 0\,; \quad (2.162\text{ii})$$

$$\begin{aligned}\mathcal{E}^{l}{}_{ijk}(x) := {} & R^{l}{}_{ijk}(x) - C^{l}{}_{ijk}(x)\\ & + (\kappa/2)\Big\{\delta^{l}{}_{k}\, T_{ij}(x) - \delta^{l}{}_{j}\, T_{ik}(x) + g_{ij}(x)\, T^{l}{}_{k}(x) - g_{ik}(x)\, T^{l}{}_{j}(x)\\ & + (2/3)\cdot\left[\delta^{l}{}_{j}\, g_{ik}(x) - \delta^{l}{}_{k}\, g_{ij}(x)\right]\cdot T(x)\Big\} = 0,\end{aligned} \quad (2.163\text{i})$$

$$\begin{aligned}\mathcal{E}^{(d)}{}_{(a)(b)(c)}(x) := {} & R^{(d)}{}_{(a)(b)(c)}(x) - C^{(d)}{}_{(a)(b)(c)}(x)\\ & + (\kappa/2)\Big\{\delta^{(d)}{}_{(c)}\, T_{(a)(b)}(x) - \delta^{(d)}{}_{(b)}\, T_{(a)(c)}(x)\\ & + d_{(a)(b)} T^{(d)}{}_{(c)}(x) - d_{(a)(c)}\, T^{(d)}{}_{(b)}(x)\\ & + (2/3)\cdot\left[\delta^{(d)}{}_{(b)}\, d_{(a)(c)} - \delta^{(d)}{}_{(c)}\, d_{(a)(b)}\right]\cdot T(x)\Big\} = 0.\end{aligned} \quad (2.163\text{ii})$$

Here, $C^{l}{}_{ijk}(x)$ (and $C^{(d)}{}_{(a)(b)(c)}(x)$) are components of Weyl's conformal tensor, as defined in (1.169i, ii).

In the material "vacuum", $T_{ij}(x) \equiv 0$ (or, $T_{(a)(b)}(x) \equiv 0$). Therefore, the field equations in (2.161i)–(2.163ii) reduce to their vacuum (sourceless) versions

$$\tilde{\mathcal{E}}^{(0)}{}_{ij}(x) := R_{ij}(x) = 0, \quad (2.164\text{i})$$

$$\tilde{\mathcal{E}}^{(0)}{}_{(a)(b)}(x) := R_{(a)(b)}(x) = 0\,; \quad (2.164\text{ii})$$

$$\tilde{\mathcal{E}}^{(0)l}{}_{ijk}(x) := R^{l}{}_{ijk}(x) - C^{l}{}_{ijk}(x) = 0, \quad (2.165\text{i})$$

$$\tilde{\mathcal{E}}^{(0)(d)}{}_{(a)(b)(c)}(x) := R^{(d)}{}_{(a)(b)(c)}(x) - C^{(d)}{}_{(a)(b)(c)}(x) = 0. \quad (2.165\text{ii})$$

The system of field equations (2.161i)–(2.163ii) must be further augmented by the following equations:

$$\mathcal{T}^{i}(x) := \nabla_j\, T^{ij} = 0, \quad (2.166\text{i})$$

$$\mathcal{T}^{(a)}(x) := \nabla_{(b)}\, T^{(a)(b)} = 0\,; \quad (2.166\text{ii})$$

$$\nabla_j\, \mathcal{E}^{ij} \equiv \kappa\, \mathcal{T}^{i}(x), \quad (2.167\text{i})$$

$$\nabla_{(b)} \mathcal{E}^{(a)(b)} \equiv \kappa\, \mathcal{T}^{(a)}(x)\,; \quad (2.167\text{ii})$$

$$\mathcal{C}^i\left(g_{jk}, \partial_l g_{jk}\right) = 0, \tag{2.168i}$$

$$\mathcal{C}^{(a)}\left(e^i{}_{(b)}, \partial_j e^i{}_{(b)}\right) = 0. \tag{2.168ii}$$

The equations $\mathcal{C}^i(\cdot\cdot) = 0$ (or $\mathcal{C}^{(a)}(\cdot\cdot) = 0$) stand for four possible *coordinate conditions*. (e.g., four equations $g_{\alpha 4}(x) \equiv 0$ and $g_{44}(x) \equiv -1$ can be *locally imposed* to obtain a geodesic normal (or Gaussian normal) coordinate chart, (as discussed in (1.160)).)

The field equations (2.161i), (2.166i), (2.167i), and (2.168i) (or their orthonormal counterparts) constitute a system of *semilinear*, *second-order*, *coupled*, *partial differential equations* in the domain $D_{(e)} \subset D \subset \mathbb{R}^4$. (See Appendix 2.) It is useful to take a count of the number of unknown functions versus the number of (algebraically and differentially) *independent equations*. The counting process reveals whether the system is *underdetermined*, *overdetermined*, or *determinate*. (In an underdetermined system, we can *prescribe suitably some unknown functions*.) In the most general case, we assume that the ten functions $T_{ij}(x)$ are *unknown*, although often physical considerations may place constraints on them. In the scenario where the $T_{ij}(x)$ are unknown, the counting will be symbolically expressed as the following:

No. of unknown functions: $10(g_{ij}) + 10(T_{ij}) = 20.$

No. of equations: $10(\mathcal{E}_{ij} = 0) + 4(\mathcal{T}^i = 0) + 4(\mathcal{C}^i = 0) = 18.$

No. of differential identities: $4(\nabla_j \mathcal{E}^{ij} \equiv \kappa \mathcal{T}^i) = 4.$

No. of independent equations: $18 - 4 = 14.$

Therefore, *the most general system is underdetermined* and *six out of the twenty unknown functions can at most be prescribed*.

Now, we shall explore an isolated material body surrounded by the vacuum. Let it be represented by the world tube $D_{(b)} \subset D_{(e)} \subset D \subset \mathbb{R}^4$. (See Fig. 2.15.)

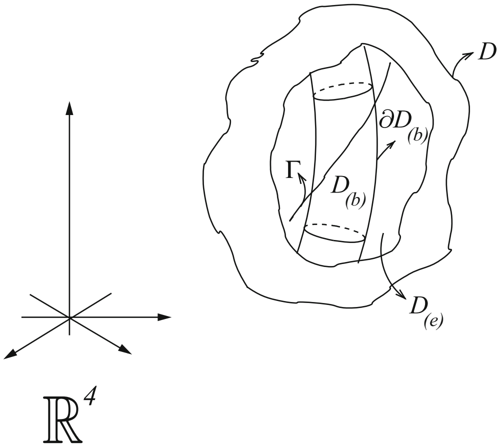
The relevant field equations from (2.159i,ii) are

$$G_{ij}(x) = \begin{cases} -\kappa\, T_{ij}(x) & \text{for } x \in D_{(b)} \subset D_{(e)} \subset \mathbb{R}^4, \\ 0 & \text{for } x \in D_{(e)} - D_{(b)}. \end{cases} \tag{2.169}$$

We assume certain jump discontinuities $\left[T_{ij}(x)\right] \not\equiv 0$ are allowed across the timelike three-dimensional boundary hypersurface $\partial D_{(b)}$. (*We do not deal with infinite discontinuities.*) There are two common jump conditions available in the literature. *Synge's junction conditions* [243] are the following:

$$\begin{aligned} \left[g_{ij}(x)\right]_{|\partial D_{(b)}} &= \left[\partial_k\, g_{ij}\right]_{|\partial D_{(b)}} \equiv 0, \\ \left[T_{ij}(x)\, n^j\right]_{|\partial D_{(b)}} &= 0. \end{aligned} \tag{2.170}$$

Fig. 2.15 A material world tube in the domain $D_{(b)}$



Here, $n^i \frac{\partial}{\partial x^i}{}_{|\partial D_{(b)}}$ denotes the unit, spacelike normal vector to the hypersurface $\partial D_{(b)}$. Moreover, the notation $[\ldots]$ indicates the measure of the jump discontinuity. Physically, this condition may be interpreted as implying that there should be no flux of matter or energy off of the junction surface (which, however, may not be static).

The other popular junction conditions are due to Israel, Sen, Lanczos, and Darmois [45, 140, 156, 234]. We shall call these conditions the *I–S–L–D jump conditions* to match the nomenclature often used in the literature. Suppose that the intrinsic metric components of $\partial D_{(b)}$ are given by $\overline{g}_{\mu\nu}(u)$ of (1.223). Moreover, *the extrinsic curvature* is furnished by the components $K_{\mu\nu}(u) = -(1/2)\left[\nabla_i n_j + \nabla_j n_i\right]\left(\partial_\mu \xi^i\right)\left(\partial_\nu \xi^j\right)$ in (1.234). Then, the I–S–L–D junction conditions are characterized by

$$\begin{aligned} \left[g_{ij}(x)\right]_{|\partial D_{(b)}} &\equiv 0, \\ \left[\nabla_i n_j + \nabla_j n_i\right]_{|\partial D_{(b)}} &\equiv 0. \end{aligned} \tag{2.171}$$

This junction condition implies the absence of surface layers (thin "δ-function" shells) of matter and will be further motivated when we consider the variational principle applied to gravitation. (See Appendix 1.)

Remark: The conditions (2.170) and (2.171) also apply to junctions separating different types of material (*not just matter-vacuum boundaries*).

Now, let us consider an (not necessarily inertial) observer with world line Γ as in Fig. 2.15. This observer is penetrating the material world tube $D_{(b)}$ with a parallelly transported (or F–W transported) orthonormal tetrad. The observer has devices to measure the physical components $G_{(a)(b)}(\mathcal{X}(s))$ (via measurements of $R_{(a)(b)(c)(d)}$)

and $T_{(a)(b)}(\mathcal{X}(s))$ along the world line $x^i = \mathcal{X}^i(s)$. Thus, he or she can validate field equations (2.169) along the world line by actual experiments. Therefore, from the point of view of physics, the field equations (2.161ii), (2.164ii), (2.166ii), etc., in terms of *physical components* are more relevant than the corresponding coordinate components!

Next, consider a special coordinate chart, namely, Riemann normal coordinates or local Minkowskian coordinates in Example 2.2.5. The vacuum equations (2.164i), in this chart, yields

$$\begin{aligned} R_{jk}(0,0,0,0) &= \left[g^{il}(x)\, R_{ijkl}(x)\right]_{|(0,0,0,0)} \\ &= (1/2)\left\{\Box\, g_{jk}(x) + \partial_j \partial_k \left(d^{il}\, g_{il}\right) - \partial^i \left(\partial_j\, g_{ik}\right) - \partial^l \left(\partial_k\, g_{jl}\right)\right\}_{|(0,0,0,0)} = 0. \end{aligned} \tag{2.172}$$

Here, *the wave operator (or D'Alembertian)* is defined as $\Box := d^{il}\partial_i\partial_l$. This linearized reduction is at *the single point* $(0,0,0,0)$. However, (2.172) is *exact*. Moreover, there is a glimpse of *gravitational waves* in (2.172). (Gravitational waves shall be briefly discussed in Appendix 5.)

The vacuum equations $G_{ij}(x) = 0$ (or, $R_{ij}(x) = 0$) are also derivable from a variational principle involving the *Lagrangian density* $\mathcal{L}\left(g_{ij}, \partial_k g_{ij}, \partial_l \partial_k g_{ij}\right) := \sqrt{-g(x)} \cdot R(x)$, known as the *Einstein-Hilbert Lagrangian density* (see the Appendix 1). (Hilbert discovered the vacuum equations *independently* [131], after attending Einstein's earlier seminars on the problem of gravitation.)

Tensorial field equations (2.161i), (2.162i), (2.163i), (2.166i), etc., *retain their forms intact* under a general coordinate transformation in (1.37). Similarly, field equations (2.161ii), (2.162ii), (2.163ii), (2.166ii), etc., preserve their forms under a variable Lorentz transformation of the tetrad fields. Both of these "covariances" are called "general covariances" of the field equations. We have already mentioned general covariances of (2.74)–(2.78iii) in *the theory of special relativity*. However, historically, the notion of general covariance or *general relativity* has been used *only in the context of the curved space–time*. We shall also continue to use "general relativity" only in curved space–time (by popular demand)!

Example 2.2.9. Consider a (static) metric of class C^{ω} in the following [243]:

$$\begin{aligned} \mathbf{g}_{..}(x) &:= [1 - 2W(\mathbf{x})]\, \delta_{\alpha\beta}\, \mathrm{d}x^{\alpha} \otimes \mathrm{d}x^{\beta} - [1 + 2W(\mathbf{x})]\, \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\ \mathrm{d}s^2 &= [1 - 2W(\mathbf{x})]\, \delta_{\alpha\beta}\, \mathrm{d}x^{\alpha}\mathrm{d}x^{\beta} - [1 + 2W(\mathbf{x})]\, (\mathrm{d}x^4)^2, \\ (x) &\equiv (\mathbf{x}, x^4) \in \mathbf{D} \times \mathbb{R} =: D \subset \mathbb{R}^4, \\ & |W(\mathbf{x})| < (1/2). \end{aligned} \tag{2.173}$$

Computation of the Einstein tensor components from (2.173), by (1.141i), (1.147ii), (1.148i), and (1.149i), leads to

$$
\begin{aligned}
G_{\alpha\beta}(\cdot\cdot) = {} & \frac{4W(\cdot\cdot)}{[1-4W^2]}\left[\delta_{\alpha\beta}\nabla^2 W - \partial_\alpha \partial_\beta W\right] \\
& - 2\left[\frac{1+4W+12W^2}{(1-4W^2)^2}\right]\partial_\alpha W \cdot \partial_\beta W \\
& + \left[\frac{3+4W+12W^2}{(1-4W^2)^2}\right]\delta_{\alpha\beta}\delta^{\mu\nu}\partial_\mu W \cdot \partial_\nu W \\
=: {} & -\kappa\, T_{\alpha\beta}(\mathbf{x}), \qquad (2.174\text{i})\\
G_{\alpha 4}(\cdot\cdot) \equiv {} & 0 =: -\kappa\, T_{\alpha 4}(\mathbf{x}), \qquad (2.174\text{ii})\\
G_{44}(\cdot\cdot) = {} & -2\left[\frac{1+2W}{1-2W)^2}\right]\cdot \nabla^2 W - 3\left[\frac{1+2W}{(1-2W)^3}\right]\cdot \delta^{\mu\nu}\partial_\mu W \cdot \partial_\nu W \\
=: {} & -\kappa\, T_{44}(\mathbf{x}), \qquad (2.174\text{iii})\\
\nabla^2 W := {} & \delta^{\mu\nu}\partial_\mu \partial_\nu W. \qquad (2.174\text{iv})
\end{aligned}
$$

It turns out that the above energy–momentum–stress components satisfy

$$
\begin{aligned}
& T_{ij}(\cdot\cdot) = \rho(\cdot\cdot)\, u_i(\cdot\cdot)\, u_j(\cdot\cdot) + p_{ij}(\cdot\cdot), \\
& u^i(\cdot\cdot) := [1+2W]^{-1/2}\cdot \delta^i{}_{(4)}, \\
& g_{ij}(\cdot\cdot)\, u^i(\cdot\cdot)\, u^j(\cdot\cdot) \equiv -1, \\
& p_{ij}(\cdot\cdot)\, u^j(\cdot\cdot) \equiv 0, \\
& \rho(\mathbf{x}) := \frac{\kappa^{-1}}{(1-2W)^2}\left[2\nabla^2 W + 3\frac{\delta^{\mu\nu}\partial_\mu W \cdot \partial_\nu W}{(1-2W)}\right]. \qquad (2.175)
\end{aligned}
$$

The above energy–momentum–stress tensor represents a complicated visco-anisotropic fluid, which may or may not exist in nature. (We are just exploring a mathematical model.)

Further, let $W(\mathbf{x})$ in the metric (2.173) satisfy the Newtonian gravitational equations

$$
\begin{aligned}
& \nabla^2 W = \begin{cases} (\kappa/2)\,\mu(\mathbf{x}) > 0 & \text{for } \mathbf{x} \in \mathbf{D}_{(1)} \cup \cdots \cup \mathbf{D}_{(n)}, \\ 0 & \text{for } \mathbf{x} \in \mathbf{D} - \left\{\mathbf{D}_{(1)} \cup \cdots \cup \mathbf{D}_{(n)}\right\}; \end{cases} \\
& W(\mathbf{x}) = -\int_{\mathbf{D}_{(1)}\cup\cdots\cup\mathbf{D}_{(n)}} \frac{\mu(\mathbf{y})\, \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}y^3}{\|\, \mathbf{x} - \mathbf{y} \,\|}. \qquad (2.176)
\end{aligned}
$$

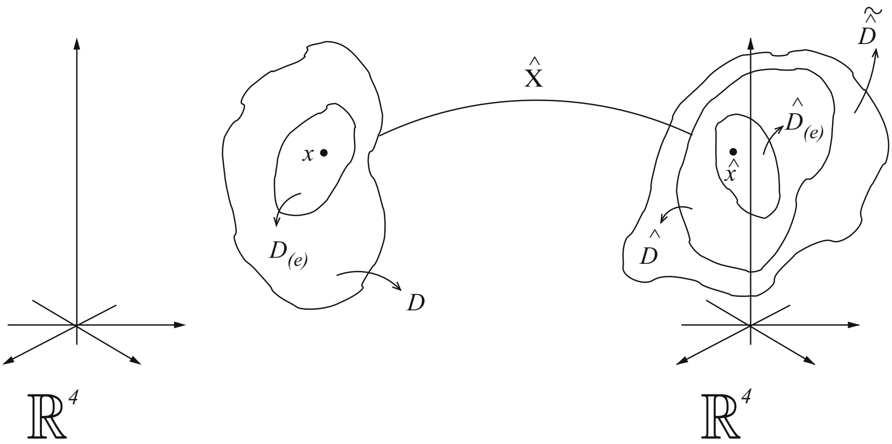


Fig. 2.16 Analytic extension of solutions from the original domain $D_{(e)}$ into $\tilde{\hat{D}}$

Therefore, the Newtonian potential $W(\mathbf{x})$ is due to superposition of gravitational potentials of n static, massive, extended bodies. These bodies are in equilibrium by virtue of mutual cancellations of gravitational attractions and visco-anisotropic repulsions. The metric (2.173), with help of (2.176), depicts *the exact general relativistic generalization of this Newtonian phenomenon*. □

The field equations (2.161i)–(2.163ii), (2.164i)–(2.165ii), and (2.166i,ii), which are partial differential equations involving metric tensor components or tetrad components, are posed in a suitable domain $D_{(e)} \subset D \subset \mathbb{R}^4$. If we can solve these equations in the domain $D_{(e)}$, then we can try to extend the solutions analytically to the whole coordinate domain $D \subset \mathbb{R}^4$. In cases where we succeed in this endeavor, we make a suitable coordinate transformation to the domain $\widehat{D} \subset \mathbb{R}^4$, where the solutions are automatically known by virtue of the transformation laws in (1.107i–iii). In the hatted coordinate chart, we try to analytically extend the solutions into a "larger" domain $\tilde{\hat{D}} \subset \mathbb{R}^4$. (See Fig. 2.16.)

We can try to extend solutions for the metric components into more and more coordinate charts and hope to finally construct an atlas for the differentiable pseudo-Riemannian space–time manifold M_4. *We may or may not succeed* in this endeavor. Global extensions and global analysis of the space–time continuum are quite involved as they require knowledge of the entire history and future of all gravitating bodies involved. (See the book by Hawking and Ellis [126] on this subject.) Here, we merely cite some toy models of two-dimensional curved surfaces, embedded in three-dimensional Euclidean space, to illustrate myriads of *strange possibilities* that global structures of space–time may acquire. (See Fig. 2.17.)

In Fig. 2.17a, the surface is that of a "bugle." It is a surface of negative Gaussian curvature, and it is *geodesically incomplete*. (For *a geodesically complete manifold*, geodesics can be extended for all (real) values of affine parameters.

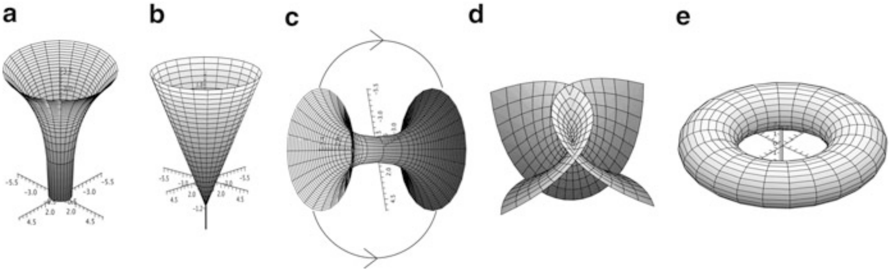


Fig. 2.17 Five two-dimensional surfaces with some peculiarities

(See [56, 126, 266].)) The surface in Fig. 2.17b represents a circular cone with a "hair" or *singularity* on the bottom which is *not a differentiable manifold*. Figure 2.17c represents a manifold with a "throat" and identifications. Figure 2.17d depicts a surface with *self-intersection*. It is an example of *a Riemannian variety* [90], which is *not a Riemannian manifold*. The closed surface in Fig. 2.17e has one handle, or equivalently, it is a surface of *genus* 1 [266]. In general relativity, 2.17a can qualitatively represent a submanifold of the space outside of a spherical star. Figure 2.17b can qualitatively represent the gravitational collapse of a spherical matter distribution to form a singularity, and Fig. 2.17c can be a spatial representation of an exotic object known as a *wormhole*. (See Appendix 6.)

Finally, let us be reminded again of two aspects of gravitational forces. Firstly, *apparent gravity* is always due to the 4-acceleration of the observer. Secondly, *intrinsic gravity* (the actual gravitational field) is always caused by tidal forces or the (nonzero) curvature of the space–time. To be honest, we have to admit that in the general theory of relativity, the notion of "gravitational forces" is just a myth! The curvature effects of the space–time are *erroneously misinterpreted* as "gravitational forces." (In the sequel, *whenever we use the word gravitation, we really imply curvature effects*.) We conclude this section with Wheeler's quotation: "Matter tells space how to curve, space tells matter how to move."

(Note that in the quotation above, the word "gravitation" is conspicuously *absent*!)

Exercises 2.2

1. Consider the Newtonian potential

$$W(\mathbf{x}) := (1/2)\left[(a+b)(x^1)^2 - a(x^2)^2 - b(x^3)^2\right],$$

where $a > 0,\ b > 0$ are constants.

Integrate differential equations (2.85) for the components $\eta^\alpha(t, v)$ of the relative separation vector field.

2. Consider the generalized Frenet-Serret formula (2.98). In the case of *a hyperbola of constant curvature*, we put $\kappa_{(1)}(s) = b = \text{const.} \neq 0$, $\kappa_{(2)}(s) \equiv 0$, $\kappa_{(3)}(s) \equiv 0$. In the case where the space–time is *flat*, obtain a class of general solutions to differential equations (2.98).
3. Let a differential tensor field ${}^{r}_{s}\mathbf{T}(x)$ be restricted to a differentiable timelike curve $x^i = \mathcal{X}^i(s)$. *The Fermi derivative* of the tensor field is defined by

$$\frac{D_F T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\mathcal{X}(s))}{\mathrm{d}s} := \frac{D T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\cdot\cdot)}{\mathrm{d}s} + \sum_{\alpha=1}^{r}\left[\mathcal{U}_k \cdot \frac{D\mathcal{U}^{i_\alpha}}{\mathrm{d}s} - \mathcal{U}^{i_\alpha} \cdot \frac{\mathrm{d}\mathcal{U}_k}{\mathrm{d}s}\right] \cdot T^{i_1,\ldots,k,\ldots,i_r}{}_{j_1,\ldots,j_s} + \sum_{\beta=1}^{s}\left[\mathcal{U}_{j_\beta} \cdot \frac{D\mathcal{U}^{l}}{\mathrm{d}s} - \mathcal{U}^{l} \cdot \frac{D\mathcal{U}_{j_\beta}}{\mathrm{d}s}\right] \cdot T^{i_1,\ldots,i_r}{}_{j_1,\ldots,l,\ldots,j_s}.$$

Prove the Leibniz property:

$$\frac{D_F\left[\left(A^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}\right)\left(B^{k_1,\ldots,k_p}{}_{l_1,\ldots,l_q}\right)\right]}{\mathrm{d}s} = \left[\frac{D_F A^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}}{\mathrm{d}s}\right] \cdot B^{k_1,\ldots,k_p}{}_{l_1,\ldots,l_q} + A^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} \cdot \left[\frac{D_F B^{k_1,\ldots,k_p}{}_{l_1,\ldots,l_q}}{\mathrm{d}s}\right].$$

4. The *symmetrized curvature tensor* $\mathbf{S}_{\ldots.}(x)$ is defined by the components

$$S_{ijkl}(x) := -(1/3)\left[R_{ikjl}(x) + R_{iljk}(x)\right].$$

(i) Prove that

$$S_{ijlk}(x) \equiv S_{ijkl}(x) \equiv S_{jikl}(x) \equiv S_{klij}(x);$$
$$S_{ijkl}(x) + S_{iklj}(x) + S_{iljk}(x) \equiv 0.$$

(ii) Compute the number of linearly independent components of $\mathbf{S}_{\ldots.}(x)$.

5. Show that the relativistic equation of motion (2.124) implies

$$M(s) = M(0) - \int_0^s g_{ij}(\mathcal{X}(t))\, F^i(\cdot\cdot)_{|\cdot\cdot} \cdot \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\,\mathrm{d}t.$$

(*Remarks*: The equation above proves that external gravitational forces, or combined external gravitational and electromagnetic forces alone, *cannot* alter the proper mass or rest energy of a (classical) particle.)

6. Consider the following Lagrangian function [55]:

$$L(x,u) := \left[1 + c_{(0)} V(x)\right] \Big\{ e A_i(x) u^i - m \sqrt{|g_{ij}(x) u^i u^j|}$$
$$+ c_{(2)} \sqrt[3]{|\phi_{ijk}(x) u^i u^j u^k|} + c_{(3)} \sqrt[4]{|\psi_{ijkl}(x) u^i u^j u^k u^l|} \Big\}.$$

   Here, $c_{(0)}, e, m > 0, c_{(2)}, c_{(3)}$, are constants. The functions $V(x)$, $A_i(x)$, $g_{ij}(x)$, $\phi_{ijk}(x)$, $\psi_{ijkl}(x)$ are of class C^3 in $D \subset \mathbb{R}^4$. Moreover, tensor fields $g_{ij}(x)$, $\phi_{ijk}(x)$, and $\psi_{ijkl}(x)$ are symmetric and totally symmetric fields. Using the fact that $L(x, \lambda u) = \lambda L(x,u)$ for $\lambda > 0$, construct the corresponding Finsler metric $f_{ij}(x,u)$, according to (2.135).
7. Prove the differential identities (2.141ii) involving the super-Hamiltonian function $\mathcal{H}(\cdot\cdot)$.
8. Consider the 4×4 identity matrix $[\,I\,] := [\delta^i{}_j]$. Define an antisymmetric, numerical 8×8 matrix by $[A] := \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$.

   Consider a general, differentiable coordinate transformation in *the extended phase space* (*or cotangent bundle* [38, 55, 56]) as

$$\hat{x}^i = \widehat{\xi}^i(x;p),$$
$$\hat{p}_i = \widehat{\eta}_i(x;p),$$
$$\underset{8\times 8}{[J]} := \left[\begin{array}{c|c} \partial\widehat{x}^i/\partial x^j & \partial\widehat{x}^i/\partial p_j \\ \hline \partial\widehat{p}_i/\partial x^j & \partial\widehat{p}_i/\partial p_j \end{array}\right], \quad \det[J] \neq 0.$$

   Let the (extended) Jacobian matrix satisfy $[J][A][J]^{\mathrm{T}} = [A]$. Prove that under such a transformation, relativistic canonical equation (2.146i,ii) remains intact (or "covariant").

(*Remarks*: (i) Transformations satisfying $[J][A][J]^{\mathrm{T}} = [A]$ are called *relativistic canonical transformations*.
(ii) The subset of linear, homogeneous, canonical transformations constitutes a Lie group called the *symplectic group* $S_p(8;\mathbb{R})$. (See [123, 160].))

9. Consider gravitational field equations (2.162ii), (2.164ii), and (2.166ii). Show that in terms of directional derivatives, Ricci rotation coefficients, and *the physical* (*or tetrad*) *components* $T_{(a)(b)}(x)$, the field equations are equivalent to the following:

$$\partial_{(c)}\gamma^{(c)}{}_{(a)(b)} - \partial_{(b)}\gamma^{(c)}{}_{(a)(c)} + \gamma^{(c)}{}_{(a)(d)}\cdot\gamma^{(d)}{}_{(b)(c)} - \gamma^{(c)}{}_{(d)(c)}\cdot\gamma^{(d)}{}_{(a)(b)}$$

$$= \begin{cases} -\kappa\left[T_{(a)(b)}(x) - (1/2)\, d_{(a)(b)}\cdot d^{(c)(d)}T_{(c)(d)}(x)\right], & \text{inside matter} \\ 0, \quad \text{outside matter}; \end{cases}$$

$$\partial_{(b)}T^{(a)(b)} - \gamma^{(a)}{}_{(d)(b)}T^{(d)(b)}(x) - \gamma^{(b)}{}_{(d)(b)}T^{(a)(b)}(x) = 0, \quad \text{inside matter.}$$

10. *Harmonic coordinate conditions* (or the *harmonic gauge*) are characterized by four differential equations $\partial_j\left(\sqrt{|g|}\, g^{ij}\right) = 0$. (See [155].)

   (i) Consider a differentiable scalar field defined by $\phi(x) := x^i$. (Here, the index i takes *one of the values* in $\{1, 2, 3, 4\}$.) Prove that in the harmonic gauge,

$$\Box\phi := g^{ij}(x)\nabla_i\nabla_j\phi = 0.$$

   (ii) Prove that in the harmonic gauge, gravitational field equations reduce to

$$\frac{1}{2}\, g^{kl}(x)\,\partial_k\partial_l\, g^{ij} - \begin{Bmatrix} i \\ k\ p \end{Bmatrix} \begin{Bmatrix} j \\ l\ q \end{Bmatrix}\, g^{kl}(x)\, g^{pq}(x)$$

$$= \begin{cases} -\kappa\left[T^{ij}(x) - \frac{1}{2}\, g^{ij}(x)\, T^k{}_k(x)\right] & \text{inside material sources,} \\ \quad 0 & \text{outside material sources.} \end{cases}$$

(*Remarks*: The harmonic gauge conditions are analogous to the *Lorentz gauge condition* $\nabla_i A^i = \frac{1}{\sqrt{|g|}}\,\partial_i[\sqrt{|g|}\, A^i] = 0$ in electromagnetic field theory.)

Answers and Hints to Selected Exercises

1.

$$\eta^1(t, v) = A^1(v)\cos\left(\sqrt{a+b}\cdot t\right) + B^1(v)\sin\left(\sqrt{a+b}\cdot t\right),$$
$$\eta^2(t, v) = A^2(v)\,\mathrm{e}^{\sqrt{a}\,t} + B^2(v)\,\mathrm{e}^{-\sqrt{a}\,t},$$
$$\eta^3(t, v) = A^3(v)\,\mathrm{e}^{\sqrt{b}\,t} + B^3(v)\,\mathrm{e}^{-\sqrt{b}\,t}.$$

Six arbitrary functions $A^i(v)$, $B^i(v)$ resulted from integration.

2.

$$
\begin{aligned}
\mathcal{X}^1(s) &= (b)^{-1}\,\cosh(bs) + c^1,\\
\mathcal{X}^2(s) &= c^2, \qquad \mathcal{X}^3(s) = c^3,\\
\mathcal{X}^4(s) &= b^{-1}\,\sinh(bs) + c^4.
\end{aligned}
$$

There are four arbitrary constants c^i's of integration.

4. (ii) The number is 20, which is the same number of independent components for $R_{ijkl}(x)$.
7. Using (2.133), we get

$$
\begin{aligned}
& u^i\,\frac{\partial \mathcal{W}(\cdot\cdot)}{\partial u^i} = \mathcal{W}(\cdot\cdot),\\
\text{or}\qquad & u^i\,\frac{\partial^2 \mathcal{W}(\cdot\cdot)}{\partial u^k\,\partial u^i} \equiv 0.
\end{aligned}
$$

Now, employing (2.136), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial u^k}\left\{g^{ij}(x)\left[p_i - \frac{\partial \mathcal{W}(\cdot\cdot)}{\partial u^i}\right]\left[p_j - \frac{\partial \mathcal{W}(\cdot\cdot)}{\partial u^j}\right]\right\}\\
&= -2g^{ij}(x)\cdot\frac{\partial^2 \mathcal{W}(\cdot\cdot)}{\partial u^k\,\partial u^i}\cdot\left[p_j - \frac{\partial \mathcal{W}(\cdot\cdot)}{\partial u^j}\right]\\
&= -\frac{2m}{\sqrt{-g_{mn}u^m u^n}}\cdot\left[u^i\,\frac{\partial^2 \mathcal{W}(\cdot\cdot)}{\partial u^k\,\partial u^i}\right] \equiv 0.
\end{aligned}
$$

8. Write canonical equations (2.146i,ii) as block-matrix equations [243]:

$$
\underset{8\times 1}{\begin{bmatrix} \dfrac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} \\ \hline \dfrac{\mathrm{d}\mathcal{P}_i(s)}{\mathrm{d}s} \end{bmatrix}} = \underset{8\times 8}{\left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]} \underset{8\times 1}{\begin{bmatrix} \dfrac{\partial \mathcal{H}(\cdot\cdot)}{\partial x^i}\Big|_{\cdot\cdot} \\ \hline \dfrac{\partial \mathcal{H}(\cdot\cdot)}{\partial p_i}\Big|_{\cdot\cdot} \end{bmatrix}} = \underset{8\times 8}{[A]} \underset{8\times 1}{\begin{bmatrix} \dfrac{\partial \mathcal{H}(\cdot\cdot)}{\partial x^i} \\ \hline \dfrac{\partial \mathcal{H}(\cdot\cdot)}{\partial p_i} \end{bmatrix}}.
$$

Transforming into hatted coordinates by the chain rule, the equations above imply that

$$\left[\begin{array}{c|c} \dfrac{\partial x^i}{\partial \widehat{x}^j} & \dfrac{\partial x^i}{\partial \widehat{p}_j} \\ \hline \dfrac{\partial p_i}{\partial \widehat{x}^j} & \dfrac{\partial p_i}{\partial \widehat{p}_j} \end{array}\right] \left[\begin{array}{c} \dfrac{d\widehat{\mathcal{X}}^j(s)}{ds} \\ \hline \dfrac{d\widehat{\mathcal{P}}_j(s)}{ds} \end{array}\right] = [A] \left[\begin{array}{c|c} \dfrac{\partial \widehat{x}^j}{\partial x^i} & \dfrac{\partial \widehat{p}_j}{\partial x^i} \\ \hline \dfrac{\partial \widehat{x}^j}{\partial p_i} & \dfrac{\partial \widehat{p}_j}{\partial p_i} \end{array}\right] \left[\begin{array}{c} \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{x}^j}_{|..} \\ \hline \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{p}_j}_{|..} \end{array}\right],$$

$$\text{or,} \qquad [J]^{-1} \left[\begin{array}{c} \dfrac{d\widehat{\mathcal{X}}^j(s)}{ds} \\ \hline \dfrac{d\widehat{\mathcal{P}}_j(s)}{ds} \end{array}\right] = [A][J]^{\mathrm{T}} \left[\begin{array}{c} \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{x}^j}_{|..} \\ \hline \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{p}_j}_{|..} \end{array}\right],$$

$$\text{or,} \qquad \left[\begin{array}{c} \dfrac{d\widehat{\mathcal{X}}^j(s)}{ds} \\ \hline \dfrac{d\widehat{\mathcal{P}}_j(s)}{ds} \end{array}\right] = [A] \left[\begin{array}{c} \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{x}^j}_{|..} \\ \hline \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{p}_j}_{|..} \end{array}\right].$$

2.3 General Properties of T_{ij}

We shall now explore *the algebraic classification* of orthonormal components of the energy–momentum–stress tensor field $T_{(a)(b)}(x)$ at a particular event, corresponding to $x_0 \in D \subset \mathbb{R}^4$. The 4×4 real, symmetric matrix $\left[T_{(a)(b)}(x_0)\right]$ has the (usual) *characteristic polynomial equation*

$$p(\lambda) := \det\left[T_{(a)(b)}(x_0) - \lambda \cdot \delta_{(a)(b)}\right] = 0. \tag{2.177}$$

The corresponding roots, which are *the usual eigenvalues*, are *all real but non-relativistic*! However, physically relevant, relativistic *invariant eigenvalues* are furnished by another polynomial equation

$$\begin{aligned} p^{\#}(\lambda) &:= \det\left[T_{(a)(b)}(x_0) - \lambda \cdot d_{(a)(b)}\right] = 0. \\ \text{or} \quad &\det\left[T^{(a)}{}_{(b)}(x_0) - \lambda \cdot \delta^{(a)}{}_{(b)}\right] = 0. \end{aligned} \tag{2.178}$$

As discussed in Appendix 3, *the relativistic or Lorentz-invariant eigenvalues need neither be real nor the* 4×4 *matrix* $\left[T_{(a)(b)}(x_0)\right]$ *be diagonalizable*. In the special cases where $\left[T_{(a)(b)}(x_0)\right]$ admits only *real invariant eigenvalues*, the classification of the matrix is provided by the following types. (See Example A3.8 for more details.)

$$\text{Type-I:} \qquad \left[T_{(a)(b)}(x_0)\right]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)} & 0 \\ 0 & 0 & 0 & \lambda_{(4)} \end{bmatrix}. \tag{2.179}$$

This matrix is already diagonalized. The 4×1 column vectors representing the relativistic eigenvectors are (obviously) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. These vectors are isomorphic to the (natural) orthonormal basis vectors in $\left\{\vec{\mathbf{e}}_{(a)}(x_0)\right\}_1^4$. The energy–momentum–stress tensor is expressible as

$$\begin{aligned} \mathbf{T}^{\cdot\cdot}(x_0) &= \sum_{a=1}^{4} \lambda_{(a)} \left[\vec{\mathbf{e}}_{(a)}(x_0) \otimes \vec{\mathbf{e}}_{(a)}(x_0)\right], \\ T^{ij}(x_0) &= \lambda_{(1)} e^i_{(1)}(x_0) e^j_{(1)}(x_0) + \lambda_{(2)} e^i_{(2)}(x_0) e^j_{(2)}(x_0) \\ &\quad + \lambda_{(3)} e^i_{(3)}(x_0) e^j_{(3)}(x_0) + \lambda_{(4)} e^i_{(4)}(x_0) e^j_{(4)}(x_0). \end{aligned} \tag{2.180}$$

In case invariant eigenvalues $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}, -\lambda_{(4)}$ are all distinct, the Segre characteristic is[6] $[1, 1, 1, 1]$. (See (A3.4).) This type of the energy–momentum–stress tensor is known as Type-I$_{(a)}$.

In the case of Type-I$_{(b)}$, we assume that $\lambda_{(1)} = \lambda_{(2)}$ and $\lambda_{(1)}, \lambda_{(3)}, -\lambda_{(4)}$ are distinct. The Segre characteristic is $[(1, 1), 1, 1]$. The components $T^{ij}(x_0)$ are given by

$$\begin{aligned} T^{ij}(x_0) &= \lambda_{(1)} \left[e^i_{(1)}(x_0) e^j_{(1)}(x_0) + e^i_{(2)}(x_0) e^j_{(2)}(x_0)\right] \\ &\quad + \lambda_{(3)} e^i_{(3)}(x_0) e^j_{(3)}(x_0) + \lambda_{(4)} e^i_{(4)}(x_0) e^j_{(4)}(x_0) \\ &= \lambda_{(1)} g^{ij}(x_0) + \left[\lambda_{(3)} - \lambda_{(1)}\right] e^i_{(3)}(x_0) e^j_{(3)}(x_0) \\ &\quad + \left[\lambda_{(1)} + \lambda_{(4)}\right] e^i_{(4)}(x_0) e^j_{(4)}(x_0). \end{aligned} \tag{2.181}$$

Here, we have used equations (1.105) and *the notation* $e^i_{(a)}(x_0) \equiv \lambda^i_{(a)}(x_0)$.

The Type-I$_{(c)}$ is characterized by $\lambda_{(1)} = \lambda_{(2)}$, $\lambda_{(3)} = -\lambda_{(4)}$, and $\lambda_{(1)} \neq -\lambda_{(4)}$. The Segre characteristic is $[(1, 1), (1, 1)]$.

---

[6] Consult Appendix 3 for the definition of a Segre characteristic.

For the Type-I$_{(d)}$, $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)}$ and $\lambda_{(1)} \neq -\lambda_{(4)}$. The Segre characteristic is furnished by $[(1,1,1),1]$. The energy–momentum–stress components are given by

$$\begin{aligned} T^{ij}(x_0) &= \lambda_{(1)}\left[e^i_{(1)}(x_0)e^j_{(1)}(x_0) + e^i_{(2)}(x_0)e^j_{(2)}(x_0) + e^i_{(3)}(x_0)e^j_{(3)}(x_0)\right] \\ &\quad + \lambda_{(4)}e^i_{(4)}(x_0)e^j_{(4)}(x_0) \\ &= \lambda_{(1)}g^{ij}(x_0) + \left[\lambda_{(1)} + \lambda_{(4)}\right] e^i_{(4)}(x_0)e^j_{(4)}(x_0). \end{aligned} \tag{2.182}$$

Here, we have made use of (1.105).

The Type-I$_{(e)}$ is characterized by $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)} = -\lambda_{(4)}$. (See (A3.4).) The Segre characteristic is $[(1,1,1,1)]$. The energy–momentum–stress tensor field is provided by

$$\begin{aligned} T^{ij}(x) &= \lambda_{(1)}\Big[e^i_{(1)}(x)e^j_{(1)}(x) + e^i_{(2)}(x)e^j_{(2)}(x) \\ &\qquad + e^i_{(3)}(x)e^j_{(3)}(x) - e^i_{(4)}(x)e^j_{(4)}(x)\Big] \\ &= \lambda_{(1)}g^{ij}(x). \end{aligned} \tag{2.183}$$

The Type-II is characterized by

$$\left[T_{(a)(b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)}+1 & 1 \\ 0 & 0 & 1 & 1-\lambda_{(3)} \end{bmatrix},$$

$$\text{or,}\quad \left[T^{(a)}{}_{(b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)}+1 & 1 \\ 0 & 0 & -1 & \lambda_{(3)}-1 \end{bmatrix} \neq \left[T^{(a)}{}_{(b)}(x_0)\right]^{\mathrm{T}}. \tag{2.184}$$

The relativistic eigencolumn vectors for the above matrix are *isomorphic to spacelike vectors* $\vec{\mathbf{e}}_{(1)}(x_0), \vec{\mathbf{e}}_{(2)}(x_0)$, and *the null vector* $\vec{\mathbf{e}}_{(3)}(x_0) - \vec{\mathbf{e}}_{(4)}(x_0)$. In the case when $\lambda_{(1)}, \lambda_{(2)}$, and $\lambda_{(3)}$ are *distinct*, the Segre characteristic is furnished by $[1,1,2]$. (See (A3.5).)

For Type-II$_{(b)}$, assume that $\lambda_{(1)} = \lambda_{(2)}$, $\lambda_{(1)} \neq \lambda_{(3)}$. The Segre characteristic is provided by $[(1,1),2]$.

For Type-II$_{(c)}$, assume that $\lambda_{(1)} \neq \lambda_{(3)}$ and $\lambda_{(2)} = \lambda_{(3)}$. The Segre characteristic is furnished by $[1,(1,2)]$.

For Type-II$_{(d)}$, assume that $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)}$. The Segre characteristic is $[(1,1,2)]$.

The Type-III$_{(a)}$ is characterized by

$$\left[T^{(a)}_{\ (b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 1 & 0 & 0 \\ 0 & \lambda_{(1)} & 0 & 0 \\ 0 & 0 & \lambda_{(2)} & 1 \\ 0 & 0 & 0 & \lambda_{(2)} \end{bmatrix}. \tag{2.185}$$

Here, we assume that $\lambda_{(1)} \neq \lambda_{(2)}$ and the Segre characteristic is $[2, 2]$.

For the Type-III$_{(b)}$, we assume that $\lambda_{(1)} = \lambda_{(2)}$ and the Segre characteristic is $[(2, 2)]$.

The Type-IV$_{(a)}$ is characterized by

$$\left[T^{(a)}_{\ (b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 1 & 0 \\ 0 & 0 & \lambda_{(2)} & 1 \\ 0 & 0 & 0 & \lambda_{(2)} \end{bmatrix}. \tag{2.186}$$

In this case, $\lambda_{(1)} \neq \lambda_{(2)}$ and the corresponding Segre characteristic is $[1, 3]$.

For Type-IV$_{(b)}$, we assume that $\lambda_{(1)} = \lambda_{(2)}$. The Segre characteristic is $[(1, 3)]$.

Type-V is characterized by

$$\left[T^{(a)}_{\ (b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 1 & 0 & 0 \\ 0 & \lambda_{(1)} & 1 & 0 \\ 0 & 0 & \lambda_{(1)} & 1 \\ 0 & 0 & 0 & \lambda_{(1)} \end{bmatrix}. \tag{2.187}$$

The Segre characteristic is [4]. (Consult (A3.4)–(A3.10) of Appendix 3.)

Example 2.3.1. Consider the case of a *perfect fluid* defined by the energy–momentum–stress tensor field

$$\begin{aligned} T^{ij}(x) &:= p(x)g^{ij}(x) + [\rho(x) + p(x)]\, U^i(x)U^j(x) \\ &\equiv [\rho(x) + p(x)]\, U^i(x)U^j(x) + p(x)g^{ij}(x), \\ &\quad g_{ij}(x)U^i(x)U^j(x) \equiv -1. \end{aligned} \tag{2.188}$$

Here, $\rho(x)$ is *the proper mass density* and $p(x)$ is the *pressure*. (See (2.45).) By (2.182), it is clear that the Segre characteristic of $T_{ij}(x)$ in (2.188) throughout the space–time domain of consideration is $[(1, 1, 1), 1]$.

We can deduce from (2.188) that

$$T_{ij}(x)U^j(x) = -\rho(x)g_{ij}(x)U^j(x),$$
$$T_{ij}(x)V^j(x) = p(x)g_{ij}(x)V^j(x). \tag{2.189}$$

Here, $V^j(x)$ represents an arbitrary, nonzero spacelike vector field satisfying the orthogonality $U_i(x)V^i(x) \equiv 0$.

Therefore, we conclude that $p(x)$ and $-\rho(x)$ are *the invariant eigenvalues* of the energy–momentum–stress tensor field $T_{ij}(x)$. (Note that in this example, $T_{ij}(x)U^i(x)U^j(x) = \rho(x)$.) □

In a macroscopic domain of the space–time universe, we usually experience that the proper mass density is nonnegative, or $\rho(x) \geq 0$. To generalize this concept, invariant criteria on $T_{ij}(x)$, called *energy conditions*, are introduced. (See [126].)

I. The *Weak/Null Energy Condition*: This condition is furnished by the weak inequality

$$T_{ij}(x)W^i(x)W^j(x) \geq 0 \tag{2.190}$$

for *every timelike/null vector field* $W^i(x)\frac{\partial}{\partial x^i}$. Physically speaking, this inequality, for timelike $W^i(x)\frac{\partial}{\partial x^i}$, implies that the energy density of a material source as measured by any observer pursuing a timelike curve, must be nonnegative.

II. The *Dominant Energy Condition*: This condition is characterized by

(i) $$T_{ij}(x)W^i(x)W^j(x) \geq 0 \tag{2.191}$$

and (ii) $T^{ij}(x)W_j(x)\frac{\partial}{\partial x^i}$ is nonspacelike.
($W^i(x)\frac{\partial}{\partial x^i}$ is *an arbitrary timelike or null vector field*.)
The above conditions may be physically interpreted as the local energy density is always nonnegative and the local energy flux is always nonspacelike.

III. The *Strong Energy Condition*: This condition is governed by the weak inequality

$$T_{ij}(x)W^i(x)W^j(x) \geq (1/2)T^i{}_i(x)W^j(x)W_j(x) \tag{2.192}$$

for every timelike (or null) vector field $W^i(x)\frac{\partial}{\partial x^i}$.

The energy conditions are reasonable in a space–time domain containing regular, macroscopic, matter distribution. However, these conditions likely do not hold in the neighborhood of an extreme pressure and density, such as close to a singularity. Moreover, in the microcosm, the arena of quantum effects, these conditions are not meaningful. An investigation on the energy conditions may be found in [14].

In the case of a diagonalizable energy–momentum–stress tensor field given in (2.180), the energy conditions can be considerably sharpened.

Theorem 2.3.2. *Let the energy–momentum–stress tensor field $T^{ij}(x)$ be diagonalizable as*

$$
\begin{aligned}
&T^{ij}(x) = \rho(x)U^i(x)U^j(x) + \sum_{\mu=1}^{3} p_{(\mu)}(x)V^i_{(\mu)}(x)V^j_{(\mu)}(x),\\
&U^i(x)U_i(x) \equiv -1, \quad U_i(x)V^i_{(\mu)}(x) \equiv 0,\\
&g_{ij}(x)V^i_{(\mu)}(x)V^j_{(\nu)}(x) \equiv \delta_{(\mu)(\nu)}.
\end{aligned} \tag{2.193}
$$

Then, the weak energy condition (2.190) *is equivalent to*

$$\rho(x) \geq 0\,; \quad \rho(x) + p_{(\mu)}(x) \geq 0, \quad \mu \in \{1,2,3\}, \tag{2.194i}$$

and the null energy condition is equivalent to only the second condition in (2.194i).

The dominant energy condition (2.191) *is equivalent to*

$$\left| p_{(\mu)}(x) \right| \leq \rho(x), \quad \mu \in \{1,2,3\}. \tag{2.194ii}$$

Furthermore, the strong energy condition is equivalent to

$$
\begin{aligned}
&\rho(x) + p_{(\mu)}(x) \geq 0\\
&\textit{and}\\
&\rho(x) + \sum_{\mu=1}^{3} p_{(\mu)}(x) \geq 0.
\end{aligned} \tag{2.194iii}
$$

We shall skip the proof. (See [126, 257].)

Remarks: (i) The invariant eigenvalues $p_{(\mu)}(x)$ are called *principal pressures*.

(ii) The negative of principal pressures, namely, $\sigma_{(\mu)}(x) := -p_{(\mu)}(x)$ are called *principal tensions*.

(iii) A cosmological constant source can violate energy conditions.

Example 2.3.3. One can enforce the energy inequalities in (2.194i–iii) with help of *four*, *arbitrary*, *slack functions*.

Let five functions $q, f, h_{(1)}, h_{(2)}, h_{(3)}$ be arbitrary continuous functions in $x \in D \subset \mathbb{R}^4$. The weak/null energy inequality (2.194i) is solved by putting

$$\rho(x) := q(x), \quad p_{(\mu)}(x) := \left[h_{(\mu)}(x)\right]^2 - q(x).$$

In the case of the weak energy condition, the function $q(x)$ is subject to the extra restriction $q(x) = [f(x)]^2$ to ensure a nonnegative energy density.

The dominant energy condition (2.194ii) is solved by substituting

$$\rho(x) := [f(x)]^2, \quad p_{(\mu)}(x) := [f(x)]^2 \cdot \cos\left[\theta_{(\mu)}(x)\right].$$

Here, $f, \theta_{(1)}, \theta_{(2)}, \theta_{(3)}$ are four, arbitrary, continuous functions.

The strong energy condition (2.194iii) is solved by expressing

$$\begin{aligned}
\rho(x) &:= [f(x)\cdot\sinh\chi(x)]^2\,,\\
p_{(1)}(x) &:= 2\,[f(x)\cdot\cosh\chi(x)\cdot\sin\theta(x)\cdot\cos\phi(x)]^2\\
&\quad -[f(x)\cdot\sinh\chi(x)]^2\,,\\
p_{(2)}(x) &:= 2\,[f(x)\cdot\cosh\chi(x)\cdot\sin\theta(x)\cdot\sin\phi(x)]^2\\
&\quad -[f(x)\cdot\sinh\chi(x)]^2\,,\\
p_{(3)}(x) &:= 2\,[f(x)\cdot\cosh\chi(x)\cdot\cos\phi(x)]^2\\
&\quad -[f(x)\cdot\sinh\chi(x)]^2\,.
\end{aligned}$$

Here, four continuous functions f, χ, θ, ϕ are otherwise arbitrary. □

Now we shall investigate *macroscopic materials* in general. Constituent particles of such materials follow *a timelike* 4-*velocity field* $U^i(x)\frac{\partial}{\partial x^i}$ satisfying *the invariant eigenvalue equations*:

$$\begin{aligned}
T_{ij}(x)U^j(x) &= -\rho(x)U_i(x),\\
U_i(x)U^i(x) &\equiv -1.
\end{aligned}\tag{2.195}$$

The vector field $\vec{\mathbf{U}}(x) = U^i(x)\frac{\partial}{\partial x^i}$ is tangential to motion curves representing *stream lines*. (See (2.42) and Fig. 2.6.) It can be proved that the energy–momentum–stress tensor field $T_{ij}(x)$ in (2.195) must be of Type-I in (2.179). (See Problem # 2 of Exercise 2.3.)

We define a symmetric tensor field by

$$S_{ij}(x) := \rho(x)U_i(x)U_j(x) - T_{ij}(x).\tag{2.196}$$

It follows that

$$S_{ij}(x)U^j(x) = -\rho(x)U_i(x) - T_{ij}(x)U^j(x) \equiv 0.\tag{2.197}$$

We call $S_{ij}(x)$ *the general relativistic stress-tensor field*. (The special relativistic version was discussed in (2.45).)

We now define the *projection tensor* field

$$\begin{aligned}
\mathcal{P}^i{}_j(x) &:= \delta^i{}_j + U^i(x)U_j(x),\\
\mathcal{P}^i{}_j(x)U^j(x) &\equiv 0.
\end{aligned}\tag{2.198}$$

For *an arbitrary, nonzero vector field* $V^i(x)\frac{\partial}{\partial x^i}$, the projected vector $\mathcal{P}^i{}_j(x)$ $V^j(x)\frac{\partial}{\partial x^i}$ is *spacelike and orthogonal to* $U^i(x)\frac{\partial}{\partial x^i}$. (See Theorems 2.1.2 and 2.1.4.)

We shall now analyze *the relativistic kinematics* of material streamlines. We need to define several vector and tensor fields derived from the 4-velocity field $U^i(x)\frac{\partial}{\partial x^i}$. Assuming that streamlines are curves of class C^3, we define the *4-acceleration field*, *vorticity tensor*, *expansion tensor*, *expansion scalar*, and *shear tensor*, respectively, by

$$\dot{U}^i(x) := U^j \nabla_j U^i, \tag{2.199i}$$

$$2\omega_{ij}(x) := (\nabla_l U_k - \nabla_k U_l) \cdot \mathcal{P}^k{}_i(x) \cdot \mathcal{P}^l{}_j(x), \tag{2.199ii}$$

$$2\Theta_{ij}(x) := (\nabla_l U_k + \nabla_k U_l) \cdot \mathcal{P}^k{}_i(x) \cdot \mathcal{P}^l{}_j(x), \tag{2.199iii}$$

$$\Theta(x) := \Theta^k{}_k(x) = \nabla_k U^k, \tag{2.199iv}$$

$$\sigma_{ij}(x) := \Theta_{ij}(x) - (1/3)\Theta(x) \cdot \mathcal{P}_{ij}(x). \tag{2.199v}$$

Remarks: (i) The expansion scalar $\Theta(x)$ provides the expansion (or contraction) of a material domain (or body) containing streamlines.

(ii) *The symmetric shear tensor* $\sigma_{ij}(x)$ represents change of the shape of a material body, without any change of 3-volume content. (*Caution*: σ_{ij} is not to be confused with the nonrelativistic stress $\sigma_{\alpha\beta}$ of (2.38).)

(iii) The vanishing of *the antisymmetric vorticity tensor* $\omega_{ij}(x) \equiv 0$ represents *irrotational motions* of stream lines. (In such a case, we can prove that a family of three-dimensional hypersurfaces exists such that $U^i(x)\frac{\partial}{\partial x^i}$ are unit normals [86, 257].)

(iv) In the case where we have $\Theta(x) \equiv 0$, $\sigma_{ij}(x) \equiv 0$, and $\omega_{ij}(x) \neq 0$, the streamlines experience (relativistic) *rigid motions*. (See [86].)

Now, we shall state and prove a theorem about $\dot{U}^i(x)$, $\omega_{ij}(x)$, $\Theta_{ij}(x)$, $\Theta(x)$, and $\sigma_{ij}(x)$ fields.

Theorem 2.3.4. *Let the various fields* $\dot{U}^i(x)$, $\omega_{ij}(x)$, $\Theta_{ij}(x)$, $\Theta(x)$, *and* $\sigma_{ij}(x)$ *be defined according to* (2.199i–v) *for* $x \in D \subset \mathbb{R}^4$. *Then, the following identities hold:*

$$\dot{U}_j(x)U^j(x) = \omega_{ij}(x)U^j(x) = \Theta_{ij}(x)U^j(x) = \sigma_{ij}(x)U^j(x) \equiv 0; \tag{2.200}$$

$$\nabla_j U_i \equiv \omega_{ij}(x) + \sigma_{ij}(x) + (1/3)\Theta(x) \cdot \mathcal{P}_{ij}(x) - \dot{U}_i(x)U_j(x). \tag{2.201}$$

Proof. Equation (2.200) follows from definitions (2.199i–v) and the identity $\mathcal{P}^i{}_j(x)$ $U^j(x) \equiv 0$ in (2.198).

The right-hand side of (2.201) yields

$$\begin{aligned} \text{R.H.S.} = {} & (1/2)\left[\nabla_j U_i - \nabla_i U_j + \dot{U}_i U_j - \dot{U}_j U_i\right] \\ & +(1/2)\left[\nabla_j U_i + \nabla_i U_j + \dot{U}_i U_j + \dot{U}_j U_i\right] \\ & -\dot{U}_i U_j \equiv \nabla_j U_i. \end{aligned}$$

■

Now, we shall derive the *Raychaudhuri-Landau equation*. (See [214, 215, 257].)

Theorem 2.3.5. *Let the* 4*-velocity field* $U^i(x)\frac{\partial}{\partial x^i}$ *be of class* C^2 *in the domain* $D \subset \mathbb{R}^4$*. Then, the following differential equations hold:*

$$U^j(x)\nabla_j \Theta = \nabla_j \dot{U}^j + \omega^{jk}(x)\cdot\omega_{jk}(x) - \sigma^{jk}(x)\cdot\sigma_{jk}(x) \\ -(1/3)\cdot[\Theta(x)]^2 + R_{jk}(x)\cdot U^j(x)\cdot U^k(x), \tag{2.202i}$$

$$\frac{\mathrm{d}\Theta(\mathcal{X}(s))}{\mathrm{d}s} = \Big\{\nabla_j \dot{U}^j + \omega^{jk}(x)\cdot\omega_{jk}(x) - \sigma^{jk}(x)\cdot\sigma_{jk}(x) \\ -(1/3)\cdot[\Theta(x)]^2 + R_{jk}(x)\cdot U^j(x)\cdot U^k(x)\Big\}\Big|_{x^i=\mathcal{X}^i(s)}. \tag{2.202ii}$$

Proof. We start from the Ricci identity (1.145i) to express

$$(\nabla_k\nabla_l - \nabla_l\nabla_k)\,U_j = R^h{}_{jlk}(x)\cdot U_h(x).$$

Contracting with $g^{jl}(x)\cdot U^k(x)$, we deduce that

$$g^{jl}U^k\nabla_k\nabla_l U_j = g^{jl}\left\{\nabla_l\left[U^k\cdot\nabla_k U_j\right] - \left(\nabla_l U^k\right)\cdot\left(\nabla_k U_j\right)\right\} + R_{hk}U^hU^k \\ = \nabla_j\dot{U}^j - \left(\nabla^j U^k\right)\cdot\left(\nabla_k U_j\right) + R_{hk}U^hU^k.$$

Now, we use (2.201) for $\left(\nabla_k U_j\right)$. Substituting this expression in the middle term of the last equation and simplifying, we derive (2.202i). Restricting (2.202i) into the streamline given by $\frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} = U^i\left(\mathcal{X}(s)\right)$, we obtain the other equation (2.202ii). ■

Remark: Equation (2.202ii) is called the *Raychaudhuri-Landau equation*, and it is very relevant in proving the singularity theorems which will be discussed briefly later in the book.

Example 2.3.6. In this example, we deal with *incoherent dust* (pressureless fluid) characterized by

$$T^{ij}(x) := \rho(x)U^i(x)U^j(x), \\ U_i(x)U^i(x) \equiv -1. \tag{2.203}$$

(See Example 2.1.11.) The proper mass density $\rho(x)$ is assumed to be strictly positive.

By the conservation equations (2.166i), we obtain that

$$0 = \nabla_j T^{ij} = \rho(x)U^j(x)\nabla_j U^i + U^i(x)\nabla_j\left[\rho U^j\right]. \tag{2.204}$$

Multiplying the above by $U_i(x)$ and contracting, we get

$$(1/2)\rho U^j\nabla_j\left(U_iU^i\right) + \left[U_iU^i\right]\cdot\nabla_j\left[\rho U^j\right] \\ = 0 - \nabla_j\left[\rho U^j\right] = 0, \\ \text{or,}\quad \nabla_j\left[\rho U^j\right] = 0. \tag{2.205}$$

The above is *the general relativistic continuity equation*. Substituting (2.205) into (2.204) and dividing by $\rho(x) > 0$, we derive

$$U^j(x)\nabla_j U^i = \dot{U}^i(x) = 0, \tag{2.206i}$$

$$U^j(x)\nabla_j U^i{}_{|x^i=\mathcal{X}^i(s)} = \frac{D\mathcal{U}^i(s)}{ds} \tag{2.206ii}$$

$$= \frac{d^2\mathcal{X}^i(s)}{ds^2} + \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\}_{|..} \cdot \frac{d\mathcal{X}^j(s)}{ds} \cdot \frac{d\mathcal{X}^k(s)}{ds} = 0. \tag{2.206iii}$$

Therefore, by (2.95ii), the streamlines follow *timelike geodesics* (as expected for a free pressureless fluid). Thus, the geodesic equations of motion of dust particles emerged from *conservation equations* (*which are consequences of the gravitational field equations* (2.159i).)

Now, supplementing the above condition of $\dot{U}_i(x) = 0$, *we assume further that the dust particles are undergoing irrotational motion*, that is,

$$\omega_{ij}(x) \equiv 0. \tag{2.207}$$

The gravitational field equation (2.162i) yields, from (2.203),

$$R_{ij}(x) = -\kappa \left[\rho(x)U_i(x)U_j(x) + (1/2)g_{ij}(x) \cdot \rho(x)\right],$$
$$R_{ij}(x)U^i(x)U^j(x) = -(\kappa/2) \cdot \rho(x) < 0. \tag{2.208}$$

Substituting (2.206ii), (2.207), and (2.208) into (2.202ii), we derive, assuming $\sigma^{jk}(x)\sigma_{jk}(x) \geq 0$, that

$$\frac{d\Theta(\cdot\cdot)}{ds} = -\left\{\sigma^{jk}(x)\sigma_{jk}(x) + (1/3)[\Theta(\cdot\cdot)]^2 + (\kappa/2)\rho(x)\right\}_{|x^i=\mathcal{X}^i(s)} < 0. \tag{2.209}$$

The above inequality demonstrates that the rate of expansion of the dust body (with particles following timelike geodesics) slows down with (proper) time. That fact proves the attractive aspects of gravitational forces. □

(*Remarks*: Consult # 5(ii) of Exercise 2.3 for the proof of $\sigma^{jk}(x)\sigma_{jk}(x) \geq 0$.)

Now, we shall investigate the streamlines of *a general material continuum*. The energy–momentum–stress tensor is furnished by (2.196) and (2.197). The differential conservation equation (2.166i) yields

$$0 = \nabla_j T^{ij} = \nabla_j\left[\rho U^i U^j - S^{ij}\right]$$
$$= \rho U^j \nabla_j U^i + U^i\left[\nabla_j\left(\rho U^j\right)\right] - \nabla_j S^{ij}. \tag{2.210}$$

Multiplying the above by $U_i(x)$ and using $U_i(x)U^i(x) \equiv -1$, we derive that

$$\nabla_j\left[\rho U^j\right] + U_i \nabla_j S^{ij} = 0. \tag{2.211}$$

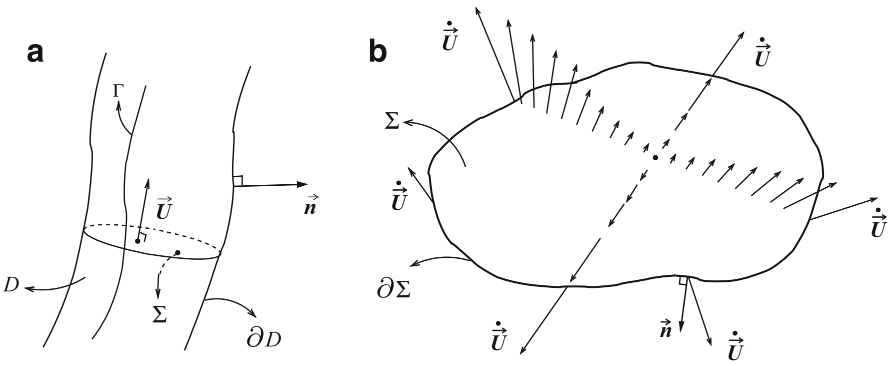


Fig. 2.18 (**a**) shows a material world tube. (**b**) shows the continuous $\dot{\vec{\mathbf{U}}}$ field over Σ

Substituting the equation above into (2.210), we deduce that

$$\rho U^j \nabla_j U^i = \left[\delta^i{}_k + U^i U_k\right] \nabla_j S^{kj}, \tag{2.212i}$$

$$\rho\,[\mathcal{X}(s)] \cdot \frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = \rho \dot{U}^i{}_{|..} = \left[\mathcal{P}^i{}_k(x) \cdot \nabla_j S^{kj}\right]_{|..} . \tag{2.212ii}$$

This expression provides the governing equations for the time evolution of streamlines.

In Example 2.3.6 involving *incoherent dust*, we concluded that streamlines pursue *timelike geodesics*. The proof for this fact emerged just from the gravitational field equations. Research has been pursued on the question of motion of an extended isolated material body [79].

We shall now provide some insight into such a problem under simplified assumptions. We assume the usual field equations and junction conditions:

$$G_{ij}(x) = \begin{cases} -\kappa\left[\rho(x)U_i(x)U_j(x) - S_{ij}(x)\right] & \text{inside } D, \\ 0 & \text{outside } D\,; \end{cases} \tag{2.213i}$$

$$G_{ij}(x)\, n^j(x)_{|\partial D} = 0. \tag{2.213ii}$$

(See Fig. 2.18a.)

We define another scalar field on $\partial\Sigma$ by

$$[\Phi(x)]_{|\partial\Sigma} := \left[\rho(x) n_i(x) \dot{U}^i(x)\right]_{|\partial\Sigma}. \tag{2.214}$$

Theorem 2.3.7. *Let the world tube of a material continuum be contained in the domain* $D \subset \mathbb{R}^4$. *Let the field equations and the junction conditions* (2.213i) *and* (2.213ii) *hold. Moreover, let the timelike* 4*-velocity tangent vector field* $U^i(x)\frac{\partial}{\partial x^i}$

be of class C^1 and $\dot{U}^i(x)\frac{\partial}{\partial x^i}|_{\partial\Sigma} \neq \vec{\mathbf{O}}(x)_{|\partial\Sigma}$. Furthermore, let the streamlines be irrotational so that $\omega_{ij}(x) \equiv 0$ in D. Then there exists at least one timelike continuous geodesic curve inside D.

Proof. Irrotational motion implies the existence of a one-parameter family of orthogonal, three-dimensional hypersurfaces [86, 126, 257]. One of these hypersurfaces, Σ, is shown in both of Figs. 2.18a, b. Since the 4-acceleration $\dot{U}^i(x)\frac{\partial}{\partial x^i}$ is orthogonal to timelike 4-velocity vector $U^i(x)\frac{\partial}{\partial x^i}$, it must be either the zero 4-vector or else a spacelike 4-vector inside Σ. Assuming $\rho(x) > 0$ in (2.213i), we conclude from (2.214) that $\text{sgn}\,[\Phi(x)]_{|\partial\Sigma} = \text{sgn}\big[n_i(x)\dot{U}^i(x)\big]_{|\partial\Sigma}$. By the assumption $\dot{U}^i(x)\frac{\partial}{\partial x^i}|_{\partial\Sigma} \neq \vec{\mathbf{O}}(x)_{|\partial\Sigma}$, we conclude that $\Phi(x)_{|\partial\Sigma} \neq 0$. Therefore, either $\Phi(x)_{|\partial\Sigma} > 0$ or else $\Phi(x)_{|\partial\Sigma} < 0$. Assume that $\Phi(x)_{|\partial\Sigma} > 0$. (The case of $\Phi(x)_{|\partial\Sigma} < 0$ can be treated in a similar fashion.) Now, the condition $\Phi(x)_{|\partial\Sigma} > 0$ implies that the 4-vector $\dot{U}^i(x)\frac{\partial}{\partial x^i}|_{\partial\Sigma}$ points outward everywhere on the continuous, piecewise-differentiable closed boundary curve *on* $\partial\Sigma$. (See Fig. 2.18b.) It is clear that *the winding number, or index*, of the continuous 4-vector field $\dot{U}^i(x)\frac{\partial}{\partial x^i}|_{\partial\Sigma}$ around the closed contour $\partial\Sigma$ is exactly one. Therefore, the fixed-point theorem[7] tells us that the 4-vector field $\dot{U}(x)\frac{\partial}{\partial x^i}|_{\Sigma}$ must be zero at *an interior point* of Σ. Since the world tube D contains a one-parameter family of spacelike hypersurfaces Σ, we have inside the world tube D, the image Γ of a continuous curve of zero 4-acceleration, or *a geodesic*. ■

Remarks: (i) The image of a curve Γ inside the material tube *need not be a stream line*.

(ii) A spinning body might not possess an interior geodesic. (See [46].)

Example 2.3.8. Consider the 1-form $\tilde{\mathbf{U}}(x) := U_i(x)\,\mathrm{d}x^i$. Assume that it is a *closed form*, or, $d\,\tilde{\mathbf{U}}(x) = \mathbf{O}_{..}(x)$. By (1.61), $\partial_i U_j - \partial_j U_i = 0$. Thus, the vorticity tensor components $\omega_{ij}(x) \equiv 0$. By Theorem 1.2.21, there exists a differentiable function ψ such that $U_i(x) = \partial_i\psi$. The function ψ must satisfy *the Hamilton–Jacobi equation* $g^{ij}(x)\cdot\partial_i\psi\cdot\partial_j\psi \equiv -1$. (See Example A2.3.) The one-parameter spacelike hypersurfaces Σ are provided by $\psi(x) = k =$ const. A typical two-dimensional boundary is $\partial\Sigma = \Sigma \cap \partial D$. Now, consider the equation $N^i(x)\partial_i\psi_{|\partial\Sigma} = 0$. Solving this underdetermined system, we arrive at a set of solutions:

$$-\vec{\mathbf{N}}_{(1)}(x) := (\partial_4\psi)\cdot\frac{\partial}{\partial x^1} - (\partial_1\psi)\cdot\frac{\partial}{\partial x^4},$$
$$-\vec{\mathbf{N}}_{(2)}(x) := (\partial_4\psi)\cdot\frac{\partial}{\partial x^2} - (\partial_2\psi)\cdot\frac{\partial}{\partial x^4},$$

[7] Let $\vec{\mathbf{V}}(\cdot\cdot)$ be a continuous vector field defined on Σ such that $\vec{\mathbf{V}}(\cdot\cdot) \neq \vec{\mathbf{0}}(\cdot\cdot)$ for any point x on $\partial\Sigma$. If the index or winding number of $\vec{\mathbf{V}}(\cdot\cdot)$ around $\partial\Sigma$ is not zero, then there exists at least one $x_0 \in \Sigma$ such that $\vec{\mathbf{V}}(x_0) = \vec{\mathbf{0}}(x_0)$. (See [37].)

$$
\begin{aligned}
&-\vec{\mathbf{N}}_{(3)}(x) := (\partial_4\psi)\cdot\frac{\partial}{\partial x^3} - (\partial_3\psi)\cdot\frac{\partial}{\partial x^4},\\
&\vec{\mathbf{n}}(x)_{|\partial\Sigma} = \sum_{\mu=1}^{3}\left[c^{(\mu)}(x)\vec{N}_{(\mu)}(x)\right]_{|\partial\Sigma},\\
&\sum_{\mu}\sum_{\nu}\left[c^{(\mu)}(x)\cdot c^{(\nu)}(x)\cdot g_{ij}(x)N^{i}_{(\mu)}(x)N^{j}_{(\nu)}(x)\right]_{|\partial\Sigma} \equiv 1\,.
\end{aligned}
$$

The last algebraic equation is underdetermined and solution functions $c^{(\mu)}(x)$ exist. By (2.214) and (2.199i),

$$
\Phi(x)_{|\partial\Sigma} = \left[\rho(x)\cdot\left(\Sigma\, c^{(\mu)}(x)N^{i}_{(\mu)}(x)\right)\cdot g^{jk}(x)\nabla_k\psi\cdot\nabla_j\nabla_i\psi\right]_{|\partial\Sigma}\,.
$$

In case $\Phi(x)_{|\partial\Sigma} \neq 0$, Theorem 2.3.7 asserts the existence of an interior geodesic. The complicated expression of $\Phi(x)$ can be considerably reduced for *a special case*. We *assume the metric* may be cast in the form

$$
\begin{aligned}
\mathbf{g}_{..}(x) &= g_{\alpha\beta}(x)\,\mathrm{d}x^{\alpha}\otimes\mathrm{d}x^{\beta} + g_{44}(x)\,\mathrm{d}x^4\otimes\mathrm{d}x^4,\\
\mathrm{d}s^2 &= g_{\alpha\beta}(x)\,\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta} - |g_{44}(x)|\,(\mathrm{d}x^4)^2.
\end{aligned}
$$

Moreover, we make a simple choice of Σ as

$$
\Sigma: \quad \psi(x) := -x^4 = k = \text{ const.}
$$

Therefore,

$$
\begin{aligned}
&U_{\alpha}(x) \equiv 0, \quad U_4(x) \equiv -1,\\
&g^{ij}(x)U_i(x)U_j(x) = g^{44}(x) \equiv -1,\\
\text{or,}\quad &g_{44}(x) \equiv -1.
\end{aligned}
$$

The vectors $\vec{\mathbf{N}}_{(\mu)}(x)$ and $\vec{\mathbf{n}}(x)_{|\partial\Sigma}$ satisfy

$$
\begin{aligned}
&\vec{\mathbf{N}}_{(\mu)}(x) = \frac{\partial}{\partial x^{\mu}}, \quad \vec{\mathbf{n}}(x)_{|\partial\Sigma} = \sum_{\mu} c^{(\mu)}(x)\frac{\partial}{\partial x^{\mu}},\\
&\sum_{\mu}\sum_{\nu}\left[g_{\mu\nu}(x)c^{(\mu)}(x)c^{(\nu)}(x)\right]_{|\partial\Sigma} \equiv 1.
\end{aligned}
$$

The function $\Phi(x)_{|\partial\Sigma}$ reduces to

$$\left[\rho(x)\sum_{\mu} c^{(\mu)}(x)\delta^{i}{}_{\mu}g^{jk}(x)\nabla_k\psi\cdot\nabla_j\nabla_i\psi\right]_{|\partial\Sigma}$$

$$= \rho(x)\sum_{\mu} c^{(\mu)}(x)g^{j4}(x)\left[0-\left\{\begin{matrix} i \\ j\ \mu\end{matrix}\right\}\partial_i\psi\right]_{|..}$$

$$= \left[\rho(x)\sum_{\mu} c^{(\mu)}(x)\left\{\begin{matrix} 4 \\ 4\ \mu\end{matrix}\right\}\right]_{|..} \equiv 0.$$

Therefore, streamlines on $\partial\Sigma$ are *all timelike geodesics*. If we analyze the metric under consideration

$$ds^2 = g_{\alpha\beta}(x)\,dx^{\alpha}dx^{\beta} - (dx^4)^2,$$

it turns out to be *a geodesic normal coordinate chart*. (Compare with (1.160).) Therefore, the streamlines *coincide* (inside and on the boundary of the whole domain D) with x^4-coordinate curves, which are all *timelike geodesics*. □

We have defined and discussed in Theorem 2.1.10, the total 4-momentum of an extended body in flat space–time. We would like to generalize those definitions for *curved space–time*. We can still use the Fig. 2.7 for the present purpose. The main difficulty in this endeavor is the problem of tensor transformations for a spatial integral under a general coordinate transformation in (1.2) and (1.37). The only logical choice is to express these integrals as *tensorially invariant entities*.

We start from four conservation equations (2.166i) explicitly stated as $\nabla_j T^{ij} = 0$. We introduce an additional differentiable vector field $\vec{\mathbf{V}}(x)$ satisfying

$$2\nabla_j\left[T^{ij}V_i\right] = T^{ij}(x)\left[\nabla_j V_i + \nabla_i V_j\right] = 0 \quad (2.215\text{i})$$

$$\text{or} \qquad G^{ij}(x)\left[\nabla_j V_i + \nabla_i V_j\right] = 0. \quad (2.215\text{ii})$$

The above equation (2.215ii) *generalizes the Killing vector equations* (1.171iii). The existence of each of the vector fields $\vec{\mathbf{V}}(x)$, satisfying (2.215i), gives rise to a conserved integral. Before we prove such a statement, consider the material world tube in Fig. 2.19. (Compare with Fig. 2.7.)

Now we shall state and prove the following theorem on integral conservations.

Theorem 2.3.9. *Let the components $T^{ij}(x)$ of the differentiable energy-momentum-stress tensor field be nonzero inside the domain D of the world tube and vanish outside. Let the nonnull boundary $\partial D := \partial D_{(3)} \cup \Sigma_{(0)} \cup \Sigma$ be continuous, piecewise-differentiable, orientable, and closed. Moreover, let the junction*

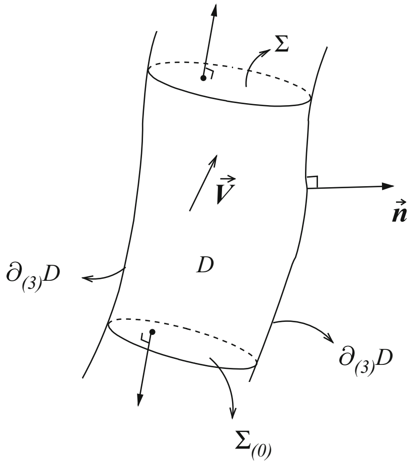


Fig. 2.19 A doubly sliced world tube of an isolated, extended material body

conditions $T^{ij}(x)n_j(x)_{|\partial D_{(3)}} = 0$ *hold. Furthermore, let a differentiable vector field* $\vec{\mathbf{V}}(x)$ *exist inside D satisfying* (2.215ii)*. Then, there exists an invariant, conserved integral:*

$$P := -\int\limits_{\Sigma} \left[T^{ij}(x)V_j(x)n_i(x)\right]_{|\Sigma} \cdot \mathrm{d}^3 v = \text{ const.} \tag{2.216}$$

Proof. Applying Gauss' Theorem 1.3.27, and the differential equation (2.215i), we obtain

$$0 = \int\limits_{D} \nabla_i \left[T^{ij} V_j\right] \cdot \mathrm{d}^4 v = \int\limits_{\partial D} \left[T^{ij} V_j n_i\right]_{|\partial D} \cdot \mathrm{d}^3 v$$

$$= \int\limits_{\Sigma_{(0)}} \left[T^{ij} V_j n_i\right]_{|\Sigma_{(0)}} \cdot \mathrm{d}^3 v + \int\limits_{\Sigma} \left[T^{ij} V_j n_i\right]_{|\Sigma} \cdot \mathrm{d}^3 v + \int\limits_{\partial D_{(3)}} \left[T^{ij} V_j n_i\right]_{|\partial D_{(3)}} \cdot \mathrm{d}^3 v.$$

or,

$$\int\limits_{\Sigma} \left[T^{ij} V_j n_i\right]_{|\Sigma} \cdot \mathrm{d}^3 v + 0 = -\int\limits_{\Sigma_{(0)}} \left[T^{ij} V_j n_i\right]_{|\Sigma_{(0)}} \cdot \mathrm{d}^3 v$$
$$= \text{const.}$$
■

Consider the scenario where the space–time domain D has some *symmetries*, or, admits groups of motion. Then, there will exist some Killing vector $\vec{\mathbf{K}}(x)$ satisfying $\nabla_j K_i + \nabla_i K_j = 0$. (See (1.171iii).) Therefore, in a domain with symmetry, *one possible solution* of (2.215i) is $\vec{\mathbf{V}}(x) = \vec{\mathbf{K}}(x)$.

Example 2.3.10. Consider *the flat space–time* and a global Minkowskian chart with $g_{ij}(x) = d_{ij}$. It is mentioned in p. 135 that there exist ten Killing vector fields furnished by

$$\vec{\mathbf{K}}_{(A)}(x) := \delta^i_{(A)} \frac{\partial}{\partial x^i}, \tag{2.217i}$$

$$\vec{\mathbf{K}}_{(A)(B)}(x) := d_{(A)i}\delta^j_{(B)}x^i \frac{\partial}{\partial x^j} - d_{(B)j}\delta^i_{(A)}x^j \frac{\partial}{\partial x^i}, \tag{2.217ii}$$

Here, $A, B \in \{1, 2, 3, 4\}$ are just *labels*. (We still use the summation convention for capital Roman indices.)

According to (2.216), there exist the following ten conserved invariant integrals:

$$P_{(A)} = -\int_{\Sigma} \left[d_{(A)i}T^{ij}(x)n_j(x)\right]_{|\Sigma} \cdot \mathrm{d}^3v, \tag{2.218i}$$

$$J_{(A)(B)} := \int_{\Sigma} d_{(A)k} \cdot d_{(B)j} \left\{\left[x^j T^{ki}(x) - x^k T^{ji}(x)\right] n_i(x)\right\}_{|\Sigma} \cdot \mathrm{d}^3v. \tag{2.218ii}$$

(Compare the equations above with (2.49) and (2.50).)

Consider *a constant-valued Lorentz transformation given by* [55]

$$\begin{aligned}
&\widehat{\vec{\mathbf{K}}}_{(A)} = l^{(B)}{}_{(A)}\vec{\mathbf{K}}_{(B)}(x),\\
&l^{(A)}{}_{(B)}\, d_{(A)(C)}\, l^{(C)}{}_{(E)} = d_{(B)(E)},\\
&l^{(4)}{}_{(4)} \geq 1.
\end{aligned} \tag{2.219}$$

(This class of transformation is known as *orthochronus*.) Note that for *constant-valued* $l^{(A)}{}_{(B)}$, the transformed vector $\widehat{\vec{\mathbf{K}}}_{(A)}(x)$ *is also a Killing vector field*. Therefore, by (2.217i), (2.218i), and (2.219), we derive transformation rules:

$$\widehat{P}_{(A)} = l^{(B)}{}_{(A)}P_{(B)}.$$

Thus, we identify *invariant constants* $P_{(A)}$'s and $J_{(A)(B)}$'s as components of the total 4-momentum and the total relativistic angular momentum of an extended material body. □

Example 2.3.11. Assume that $T^{ij}(x)$ has an invariant eigenvalue $\lambda(x) \neq 0$ satisfying $T^{ij}(x)e_j(x) = \lambda(x)e^i(x)$. Moreover, assume that

$$V_j(x) = \sigma(x)e_j(x), \quad \sigma(x) \neq 0.$$

Equation (2.215i) yields

$$\begin{aligned}0 &= T^{ij}\left[\sigma \cdot \nabla_i e_j + (\nabla_i \sigma) \cdot e_j\right]\\ &= \sigma\left[\nabla_i\left(T^{ij} e_j\right)\right] + (\nabla_i \sigma) \cdot (\lambda e^i)\\ &= \sigma\left[\lambda\left(\nabla_i e^i\right) + e^i \cdot \nabla_i \lambda\right] + (\lambda e^i) \cdot \nabla_i \sigma.\end{aligned}$$

or,

$$e^i \nabla_i (\ln|\sigma|) + \left[(\nabla_j e^j) + e^j \nabla_j (\ln|\lambda|)\right] = 0.$$

The above equation is a linear, first-order p.d.e. with the unknown function $\ln|\sigma(x)|$.

By the discussions in Appendix 2, especially by the equations in (A2.13), solutions of this equation exist in principle. Therefore, the corresponding conserved integral is

$$P := -\int_{\Sigma} \left[\sigma(x)\lambda(x)e^i(x)n_i(x)\right]_{|\Sigma} \cdot \mathrm{d}^3 v. \qquad \square$$

Now, we shall investigate the generalization (2.215i,ii) of Killing vector equation (1.171iii). A very general class of solutions of (2.215i,ii) is furnished by[8]

$$\begin{aligned}G^{ij}(x)V_j(x) = -\kappa T^{ij}(x)V_j(x) = \nabla^i h + \nabla_j A^{ij},\\ \Box h \equiv \nabla_i \nabla^i h = 0, \quad A^{ji}(x) \equiv -A^{ij}(x).\end{aligned} \tag{2.220}$$

Here, the scalar wave field $h(x)$ and the antisymmetric field $A^{ij}(x)$ are of class C^2 and otherwise *arbitrary*. There are *infinitely many solutions* in (2.220).

The physical implications of the solutions (2.220) are *not obvious*.

[8]1. The Helmotz theorem [159] on differentiable vector field $\vec{\mathbf{W}}(\mathbf{x})$ in a three-dimensional domain allows the decomposition

$$W^\alpha(\mathbf{x}) = \nabla^\alpha h + \eta^{\alpha\beta\gamma}(\mathbf{x})\left[\nabla_\gamma A_\beta - \nabla_\beta A_\alpha\right].$$

2. For *a closed, differential p-form* of (1.58), the Hodge decomposition theorem [104] asserts that

$$\begin{aligned}W_{i_1,\ldots,i_p}(x)\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} &= h_{i_1,\ldots,i_p}(x)\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p}\\ &\quad + \mathrm{d}\left[\alpha_{i_1,\ldots,i_{p-1}}(x)\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_{p-1}}\right],\\ \nabla_j \nabla^j h_{i_1,\ldots,i_p} &= 0.\end{aligned}$$

Exercises 2.3

1. Consider *the flat metric in double-null coordinates* of Example 2.1.17. In a similar fashion, the metric can be expressed as

$$\begin{aligned} ds^2 = d_{ij}\,dx^i dx^j &= (dx^1)^2 + (dx^2)^2 + (dx^3 - dx^4)(dx^3 + dx^4) \\ &=: \left(d\widehat{x}^1\right)^2 + \left(d\widehat{x}^2\right)^2 + 2\,d\widehat{x}^3 d\widehat{x}^4 =: \eta_{ij}\,d\widehat{x}^i d\widehat{x}^j. \end{aligned}$$

Consider an energy–momentum–stress tensor matrix $\left[T_{ij}(x_0)\right]$ of Segre class $[1, 3]$. Show that the transformed energy–momentum–stress tensor matrix

$$\left[\widehat{T}_{ij}(\widehat{x}_0)\right] := \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 1 & 0 \\ 0 & 1 & 0 & \lambda_{(2)} \\ 0 & 0 & \lambda_{(2)} & 0 \end{bmatrix}$$

remains of Segre class $[1, 3]$ (for $\lambda_{(1)} \neq \lambda_{(2)}$). Moreover, prove that the invariant triple eigenvector with respect to η^{ij} is along *the null direction* $\begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix}$, $t \neq 0$.

2. Let the matrix $\left[T_{(a)(b)}(x_0)\right]$ admit a timelike eigenvector $U^{(b)}(x_0)$ satisfying $T_{(a)(b)}(x_0)U^{(b)}(x_0) = -\rho(x_0)U_{(a)}(x_0)$, $d_{(a)(b)}U^{(a)}(x_0)U^{(b)}(x_0) = -1$. Consider the case that every invariant eigenvalue of $\left[T_{(a)(b)}(x_0)\right]$ is *real*. Prove that the matrix belongs to the Type-I of (2.179).
3. Prove Theorem 2.3.2.
4. (i) Let a projection tensor be defined as:

$$\mathcal{P}^i{}_j(x) := \delta^i{}_j - \varepsilon(v)V^i(x)V_j(x), \quad V_iV^j = \varepsilon(v) = \pm 1.$$

Show that $\mathcal{P}^i{}_k(x)\mathcal{P}^k{}_j(x) = \mathcal{P}^i{}_j(x)$.

(ii) Deduce that the invariant eigenvalues of $\mathcal{P}_{ij}(x)$ are exactly zero and one.

5. (i) Derive that $g^{ij}(x)\sigma_{ij}(x) = d^{(a)(b)}\sigma_{(a)(b)}(x) \equiv 0$.

(ii) Prove that $\omega_{ij}(x) \cdot \omega^{ij}(x) \geq 0$ and $\sigma_{ij}(x) \cdot \sigma^{ij}(x) \geq 0$.

6. Show that in the case of vanishing expansion, $\Theta(x) \equiv 0$, and shear tensor $\sigma_{ij}(x)\,dx^i \otimes dx^j \equiv \mathbf{O}_{..}(x)$, *the Lie derivative of the projection tensor* $\mathcal{P}_{..}(x) := \left[g_{ij}(x) + U_i(x)U_j(x)\right]dx^i \otimes dx^j$ reduces to $L_{\vec{U}}\left[\mathcal{P}_{..}(x)\right] \equiv \mathbf{O}_{..}(x)$.
7. Deduce that

$$\begin{aligned} U^k\nabla_k\left[\nabla_l U_j + \nabla_j U_l\right] &= \left[\nabla_l \dot{U}_j + \nabla_j \dot{U}_l\right] \\ &\quad - \left[\omega_j{}^k + \Theta_j{}^k - \dot{U}^k U_j\right] \cdot \left[\omega_{kl} + \Theta_{kl} - \dot{U}_k U_l\right] \end{aligned}$$

$$- \left[\omega_l{}^k + \Theta_l{}^k - \dot{U}^k U_l\right] \cdot \left[\omega_{kj} + \Theta_{kj} - \dot{U}_k U_j\right]$$
$$+ \left[R_{hjlk} + R_{hljk}\right] \cdot U^h U^k .$$

8. Consider a domain of material continuum with $T^{ij}(x) = \rho(x) U^i(x) U^j(x) - S^{ij}(x),\ U_i(x) U^i(x) \equiv -1$. Prove that along a streamline, the rate of change of the proper mass density is given by

$$\frac{\mathrm{d}\rho[\mathcal{X}(s)]}{\mathrm{d}s} = -\left[\rho(x) \cdot (\nabla_i U^i) + U_i(x) \cdot \nabla_j S^{ij}\right]_{|..} .$$

9. Consider the material world tube depicted in Fig. 2.18a. Let the 4-acceleration spacelike vector on the boundary be given by

$$\dot{U}^i(x)_{|\partial\Sigma} = C^i = \text{const.}, \quad |C^1| + |C^2| + |C^3| > 0.$$

In the case where the 4-acceleration vector field $\dot{\vec{\mathbf{U}}}$ is continuous in $\Sigma \cup \partial\Sigma$, determine whether or not there exists a geodesic inside the world tube.

10. An *anti-de Sitter space–time* domain (of constant negative curvature) is characterized by the metric (with cosmological constant set equal to -3 for notational convenience, since this corresponds to a radius of curvature of -1):

$$\mathbf{g}_{..}(x) := \left[1 + (x^1)^2\right]^{-1} \cdot \mathrm{d}x^1 \otimes \mathrm{d}x^1 + (x^1)^2 \cdot \mathrm{d}x^2 \otimes \mathrm{d}x^2$$
$$+ (x^1)^2 \cdot (\sin x^2)^2 \cdot \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \left[1 + (x^1)^2\right] \cdot \mathrm{d}x^4 \otimes \mathrm{d}x^4,$$

$$\mathrm{d}s^2 = (1 + r^2)^{-1} \mathrm{d}r^2 + r^2 \left[\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right] - (1 + r^2)\, \mathrm{d}t^2.$$

Prove that the following conserved integrals, representing the total energy and total "angular momentum components," respectively, exist:

$$P_{(t)} := - \int_{D_{(t)}} T^{44}(\cdot\cdot)(1 + r^2)\, r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi = \text{const.} ,$$

$$J_{(1)} := \int_{D_{(t)}} \left[\sin\varphi \cdot T^{24}(\cdot\cdot) + \sin\theta \cdot \cos\theta \cdot \cos\varphi \cdot T^{34}(\cdot\cdot)\right] r^4\, \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi,$$

$$J_{(2)} := \int_{D_{(t)}} \left[\cos\varphi \cdot T^{24}(\cdot\cdot) - \sin\theta \cdot \cos\theta \cdot \sin\varphi \cdot T^{34}(\cdot\cdot)\right] r^4\, \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi,$$

$$J_{(3)} := \int_{D_{(t)}} T^{34}(\cdot\cdot) \cdot r^4\, \sin^3\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi.$$

Answers and Hints to Selected Exercises

1. The 4×4 matrix to be investigated is given by:

$$\left[\widehat{T}_{ij}\,(\widehat{x}_0) - \lambda\eta_{ij}\right] := \begin{bmatrix} \lambda_{(1)} - \lambda & 0 & 0 & 0 \\ 0 & \lambda_{(2)} - \lambda & 1 & 0 \\ 0 & 1 & 0 & \lambda_{(2)} - \lambda \\ 0 & 0 & \lambda_{(2)} - \lambda & 0 \end{bmatrix}.$$

2. From (2.196) and (2.197), $S_{(a)(b)} := \rho(x_0) \cdot U_{(a)}(x_0) \cdot U_{(b)}(x_0) - T_{(a)(b)}(x_0)$, $S_{(a)(b)}(x_0) \cdot U^{(b)}(x_0) = 0$. Let $\lambda(x_0)$ be a real, nonzero invariant eigenvalue so that $S_{(a)(b)}(x_0) \cdot \mathrm{e}^{(b)}(x_0) = \lambda(x_0) \cdot e_{(a)}(x_0)$. Therefore, $\lambda(x_0) \cdot [e_{(a)}(x_0) \cdot U^{(a)}(x_0)] = [S_{(a)(b)}(x_0) \cdot U^{(a)}(x_0)] \cdot \mathrm{e}^{(b)}(x_0) = 0$. Thus, for $\lambda(x_0) \neq 0$, the corresponding eigenvector $\mathrm{e}^{(a)}(x_0)$ is *spacelike* and orthogonal to $U^{(a)}(x_0)$. In case $S_{(a)(b)}(x_0)$ has three nonzero invariant eigenvalues, there exist three spacelike eigenvectors orthogonal to $U^{(a)}(x_0)$. Thus,

$$T^{(a)(b)}(x_0) = \rho(x_0) \cdot U^{(a)}(x_0) \cdot U^{(b)}(x_0) + \sum_{\mu=1}^{3} \lambda_{(\mu)}(x_0) \cdot \mathrm{e}^{(a)}_{(\mu)} \cdot \mathrm{e}^{(b)}_{(\mu)}.$$

   Therefore, $[T_{ab}(x_0)]$ is of Type-I (even if some or all of $\lambda_{(\mu)}(x_0) = 0$.)

4. (i)

$$\left[\delta^i{}_k - \varepsilon(v)U^i U_k\right] \cdot \left[\delta^k{}_j - \varepsilon(v)U^k U_j\right]$$
$$= \delta^i{}_j + (-1 - 1 + 1) \cdot \varepsilon(v) \cdot U^i U_j.$$

5. (ii) Consider the vector field transformation equations

$$\widehat{U}^i\,(\widehat{x}) = \frac{\partial \widehat{X}^i(x)}{\partial x^j} \cdot U^j(x).$$

   Choosing three spatial equations from above, investigate

$$0 = \widehat{U}^\alpha\,(\widehat{x}) = U^j(x)\frac{\partial \widehat{X}^\alpha(x)}{\partial x^j}.$$

   Considering the equations above as linear, first-order, partial differential equations for *unknown functions*, $\widehat{X}^\alpha(x)$ conclude that solutions exist locally. (See Appendix 2.) In new coordinates, the condition $\widehat{U}_i\,(\widehat{x}) \cdot \widehat{U}^i\,(\widehat{x}) \equiv -1$ yields $\widehat{g}_{44}\,(\widehat{x})\left[\widehat{U}^4\,(\widehat{x})\right]^2 \equiv -1$. Therefore, $\widehat{g}_{44}\,(\widehat{x}) < 0$ and $\widehat{U}^4\,(\widehat{x}) \neq 0$. (The choice $\widehat{U}^4\,(\widehat{x}) > 0$ is the usual one.) Such coordinates constitute a *comoving coordinate chart*. Choosing the orthonormal tetrad

$\left\{\vec{e}_{(1)},\ \vec{e}_{(2)},\ \vec{e}_{(3)},\ [|g_{44}|]^{-\frac{1}{2}} \cdot \delta^i{}_{(4)}\partial_i\right\}$, the components $\widehat{\omega}_{(\alpha)(4)}(\widehat{x}) \equiv 0$. Thus,

$$\omega_{ij}(x) \cdot \omega^{ij}(x) = \widehat{\omega}_{(a)(b)}(\widehat{x}) \cdot \widehat{\omega}^{(a)(b)}(\widehat{x}) = \widehat{\omega}_{(\alpha)(\beta)}(\widehat{x}) \cdot \widehat{\omega}^{(\alpha)(\beta)}(\widehat{x}) + 0$$
$$= 2\left\{\left[\widehat{\omega}_{(1)(2)}(\widehat{x})\right]^2 + \left[\widehat{\omega}_{(2)(3)}(\widehat{x})\right]^2 + \left[\widehat{\omega}_{(3)(1)}(\widehat{x})\right]^2\right\} \geq 0.$$

7. By the Ricci identity (1.145i), obtain

$$(\nabla_k \nabla_l - \nabla_l \nabla_k)\, U_j = R_{hjlk} U^h,$$
$$U^k \left[\nabla_k \nabla_l U_j\right] = \nabla_l \dot{U}_j - \left(\nabla_l U^k\right) \cdot \left(\nabla_k U_j\right) + R_{hjlk} \cdot U^h U^k.$$

8. Use (2.211).
9. The winding number or the index of $\dot{\vec{\mathbf{U}}}(x)$ around $\partial\Sigma$ is exactly zero. There exist no $x_0 \in \Sigma$ such that $\dot{\vec{\mathbf{U}}}(x_0) = \vec{\mathbf{0}}(x_0)$.
10. There exist *ten generators* for the Killing vector fields [129]. Out of these, for the present problem, the following four are relevant (with cosmological constant set equal to -3):

$$\vec{\mathbf{K}}_{(t)}(\cdot\cdot) := \frac{\partial}{\partial t},$$
$$\vec{\mathbf{K}}_{(1)}(\cdot\cdot) := \sin\varphi \cdot \frac{\partial}{\partial\theta} + \cot\theta \cdot \cos\varphi \cdot \frac{\partial}{\partial\varphi},$$
$$\vec{\mathbf{K}}_{(2)}(\cdot\cdot) := \cos\varphi \cdot \frac{\partial}{\partial\theta} - \cot\theta \cdot \sin\varphi \cdot \frac{\partial}{\partial\varphi},$$
$$\vec{\mathbf{K}}_{(3)}(\cdot\cdot) := \frac{\partial}{\partial\varphi}.$$

Substitute the above vectors for $\vec{\mathbf{V}}(x) = \vec{\mathbf{K}}_{(\cdot\cdot)}(\cdot\cdot)$ in (2.216).

Remark: The remaining Killing vectors for the anti-de Sitter space–time (again with cosmological constant set equal to -3) are the following:

$$\vec{\mathbf{K}}_{(4)}(\cdot\cdot) := -r \cdot \sin(t) \cdot \sin\theta \cdot \cos\varphi \cdot (1 + r^2)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial t}$$
$$+ (1 + r^2)^{\frac{1}{2}} \cdot \cos(t) \cdot \sin\theta \cdot \cos\varphi \cdot \frac{\partial}{\partial r}$$
$$+ r^{-1} \cdot (1 + r^2)^{\frac{1}{2}} \cdot \cos(t) \cdot \left[\cos\theta \cdot \cos\varphi \cdot \frac{\partial}{\partial\theta} - \frac{\sin\varphi}{\sin\theta} \cdot \frac{\partial}{\partial\varphi}\right],$$

$$\begin{aligned}
\vec{\mathbf{K}}_{(5)}(\cdot\cdot) := & -r\cdot\sin(t)\cdot\sin\theta\cdot\sin\varphi\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& +(1+r^2)^{\frac{1}{2}}\cdot\cos(t)\cdot\sin\theta\cdot\sin\varphi\cdot\frac{\partial}{\partial r}\\
& +r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\cos(t)\cdot\left[\cos\theta\cdot\sin\varphi\cdot\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{\sin\theta}\cdot\frac{\partial}{\partial\varphi}\right],\\
\vec{\mathbf{K}}_{(6)}(\cdot\cdot) := & -r\cdot\sin(t)\cdot\cos\theta\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& +(1+r^2)^{\frac{1}{2}}\cdot\cos(t)\cdot\cos\theta\cdot\frac{\partial}{\partial r}-r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\cos(t)\cdot\sin\theta\cdot\frac{\partial}{\partial\theta},\\
\vec{\mathbf{K}}_{(7)}(\cdot\cdot) := & \; r\cdot\cos(t)\cdot\sin\theta\cdot\cos\varphi\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& +(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\sin\theta\cdot\cos\varphi\cdot\frac{\partial}{\partial r}\\
& +r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\left[\cos\theta\cdot\cos\varphi\cdot\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{\sin\theta}\cdot\frac{\partial}{\partial\varphi}\right],\\
\vec{\mathbf{K}}_{(8)}(\cdot\cdot) := & \; r\cdot\cos(t)\cdot\sin\theta\cdot\sin\varphi\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& +(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\sin\theta\cdot\sin\varphi\cdot\frac{\partial}{\partial r}\\
& +r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\left[\cos\theta\cdot\sin\varphi\cdot\frac{\partial}{\partial\theta}-\frac{\cos\varphi}{\sin\theta}\cdot\frac{\partial}{\partial\varphi}\right],\\
\vec{\mathbf{K}}_{(9)}(\cdot\cdot) := & \; r\cdot\cos(t)\cdot\cos\theta\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& +(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\cos\theta\cdot\frac{\partial}{\partial r}-r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\sin\theta\cdot\frac{\partial}{\partial\theta}.
\end{aligned}$$

2.4 Solution Strategies, Classification, and Initial-Value Problems

Let us consider a differentiable coordinate transformation (1.2) in a regular domain $D \subset \mathbb{R}^4$:

$$\begin{aligned}
\widehat{x}^i &= \widehat{X}^i(x),\\
\widehat{g}_{ij}(\widehat{x}) &= \frac{\partial X^k(\widehat{x})}{\partial\widehat{x}^i}\cdot\frac{\partial X^l(\widehat{x})}{\partial\widehat{x}^j}\cdot g_{kl}(x)\,.
\end{aligned} \tag{2.221}$$

We pose four reasonable coordinate conditions

$$\widehat{\mathcal{C}}^m \left(\widehat{g}_{ij}\,(\widehat{x})\right) = 0, \tag{2.222i}$$

or,
$$\widehat{\mathcal{C}}^m \left[\frac{\partial X^k\,(\widehat{x})}{\partial \widehat{x}^i} \cdot \frac{\partial X^l\,(\widehat{x})}{\partial \widehat{x}^j} \cdot g_{kl}(x)\right] = 0. \tag{2.222ii}$$

The four nonlinear partial differential equations in (2.222ii), for *the four unknown functions* $X^k\,(\widehat{x})$, *will locally admit solutions*. (See Appendix 2.) That is why, in a four-dimensional pseudo-Riemannian (or Riemannian) manifold, coordinate charts exist which admit *at most four (reasonable) coordinate conditions*.[9]

We recapitulate Einstein's interior field equations (2.161i), (2.166i), and differential identities (2.167i):

$$\mathcal{E}_{ij}\,(x) := G_{ij}\,(x) + \kappa\, T_{ij}\,(x) = 0, \tag{2.223i}$$

$$\mathcal{T}^i\,(x) := \nabla_j\, T^{ij} = 0, \tag{2.223ii}$$

$$\nabla_j\, \mathcal{E}^{ij} \equiv \kappa\, \mathcal{T}^i\,(x). \tag{2.223iii}$$

As in p. 164, we count the number of unknown functions versus the number of independent equations (without including coordinate conditions).

No. of unknown functions: $(10 g_{ij}) + 10(T_{ij}) = 20.$
No. of differential equations: $10(\mathcal{E}_{ij} = 0) + 4(\mathcal{T}^i = 0) = 14.$
No. of differential identities: $4(\nabla_j \mathcal{E}^{ij} - \kappa \mathcal{T}^i \equiv 0) = 4.$
No. of independent equations: $14 - 4 = 10.$

The system of semilinear, second- and first-order partial differential equations (2.223i,ii) is definitely *underdetermined*. We are allowed to prescribe *ten* out of *twenty* unknown functions to make the system *determinate*. We can make such choices in *11 distinct ways*.

Remarks: (i) In strategy-I, ten functions of $T_{ij}\,(x)$ are prescribed and ten functions of $g_{ij}\,(x)$ are treated as unknown. It is the most difficult strategy mathematically. However, from the perspective of physics, it is the most useful. This method is sometimes called the T-method [243].

(ii) In strategy-II, ten functions of $g_{ij}\,(x)$ are prescribed and ten functions of $T_{ij}\,(x) := -(\kappa)^{-1} G_{ij}\,(x)$ are treated as unknown. Thus, mathematically, it is

[9] In an N-dimensional domain, at most N (reasonable), coordinate conditions hold. Therefore, a two-dimensional metric can be locally reduced to *a conformally flat form*. A three-dimensional metric can be locally brought to *an orthogonal form*. However, orthogonal coordinates *may not exist* in dimensions $N > 3$. The coordinate conditions $\widehat{\mathcal{C}}^m \left(\widehat{g}_{ij}\,(\widehat{x})\right) = 0$ are *not* tensor field equations.

the simplest method. Although energy conditions (2.190) or (2.191) or (2.192) may be difficult to satisfy, there may arise occasion when this strategy is useful. This method is sometimes called the g-method [243].

(iii) In strategy-III there exist *nine mixed methods* [18] where $10 - s$ of functions among the $T_{ij}(x)$ and s of the metric functions are prescribed for $1 \leq s \leq 9$. Moreover, $10-s$ functions among the $g_{ij}(x)$ and s functions among the $T_{ij}(x)$ are treated as unknown.

(iv) In the case of *the vacuum field equations* (2.159i), the addition of the four coordinate conditions $\mathcal{C}^i(g_{kl}, \partial_l g_{jk}) = 0$, makes *the system determinate*.

(v) In the case when the space–time domain has some *symmetries* (or admits groups of motions), the counting has to be completely revised, as the symmetries impose extra conditions. The general scheme, however, remains the same. We will address some specific cases in later sections.

It should be noted that there is a relation among the coordinate conditions $\mathcal{C}^i(g_{kl}, \partial_l g_{jk}) = 0$ and the prescription of metric functions $g_{ij}(x)$. *In case four coordinate conditions are imposed, the above strategies undergo revisions.*

Example 2.4.1. We shall furnish an example of the g-method. Let the metric tensor components be *prescribed as*

$$g_{ij}(x) := d_{kl} \cdot \frac{\partial f^k(x)}{\partial x^i} \cdot \frac{\partial f^l(x)}{\partial x^j}. \tag{2.224}$$

Here, four prescribed functions $f^k(x)$ are of class C^4. By (1.161), the space–time domain is *flat*. Therefore,

$$T_{ij}(x) := -(\kappa)^{-1} G_{ij}(x) \equiv 0.$$

Thus, the choice (2.224) of $g_{ij}(x)$ has *annihilated the possibility of material sources completely*. □

Another example of the g-method has been given in Example 2.2.9.

Example 2.4.2. In the domain of consideration, we choose *four coordinate conditions* as $g_{\alpha 4}(x) \equiv 0$ and $g_{44}(x) \equiv -1$. Thus, the metric is expressible as

$$\mathbf{g}_{..}(x) = g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta - \mathrm{d}x^4 \otimes \mathrm{d}x^4,$$

$$\mathrm{d}s^2 = g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha \mathrm{d}x^\beta - (\mathrm{d}x^4)^2. \tag{2.225}$$

This is *the geodesic normal or Gaussian normal coordinate chart* of (1.160).

On the three-dimensional spacelike hypersurface characterized by $x^4 = T$, the intrinsic metric is furnished by (2.225) as

$$\begin{aligned} \overline{\mathbf{g}}_{..}(\mathbf{x}, T) &= g_{\alpha\beta}(\mathbf{x}, T)\,\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta \\ &=: \overline{g}_{\alpha\beta}(\mathbf{x}, T)\,\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta. \end{aligned} \tag{2.226}$$

Here, we have a slight difference of notation with (1.223). (Compare (A1.35i) and (A1.35ii).)

The Gauss' equations (1.242i) *in this example* (with notations $\overline{A}^{\alpha}(\cdot\cdot) := \overline{g}^{\alpha\beta}(\cdot\cdot)A_{\beta}(\cdot\cdot)$ and $\overline{\nabla}_{\beta}A_{\alpha} := \partial_{\beta}A_{\alpha} - \begin{Bmatrix}\gamma \\ \alpha\ \beta\end{Bmatrix} \cdot A_{\gamma}(\cdot\cdot)$) yield the following equations:

$$\begin{aligned} R_{\rho\mu\nu\sigma} &= \overline{R}_{\rho\mu\nu\sigma} + K_{\mu\sigma}\cdot K_{\rho\nu} - K_{\mu\nu}\cdot K_{\rho\sigma} \\ &= \overline{R}_{\rho\mu\nu\sigma} + \frac{1}{4}\left[\partial_4\overline{g}_{\mu\sigma}\cdot\partial_4\overline{g}_{\rho\nu} - \partial_4\overline{g}_{\mu\nu}\cdot\partial_4\overline{g}_{\rho\sigma}\right]; \end{aligned} \tag{2.227}$$

$$\begin{aligned} G_{\mu\nu} &= \overline{G}_{\mu\nu} - \frac{1}{2}\,\partial_4\partial_4\overline{g}_{\mu\nu} - \overline{K}^{\sigma}_{\ \sigma}\cdot K_{\mu\nu} + 2\overline{K}^{\alpha}_{\ \mu}\cdot K_{\alpha\nu} \\ &\quad + \frac{1}{2}\,\overline{g}_{\mu\nu}\left[\left(\overline{K}^{\sigma}_{\ \sigma}\right)^2 - 3\overline{K}^{\rho}_{\ \sigma}\cdot\overline{K}^{\sigma}_{\ \rho} + \overline{g}^{\alpha\beta}\cdot\partial_4\partial_4\overline{g}_{\alpha\beta}\right] \\ &= \overline{G}_{\mu\nu} - \frac{1}{2}\,\partial_4\partial_4\overline{g}_{\mu\nu} - \frac{1}{4}\,\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\cdot\partial_4\overline{g}_{\mu\nu} + \frac{1}{2}\,\overline{g}^{\alpha\beta}\cdot\partial_4\overline{g}_{\mu\beta}\cdot\overline{g}_{\nu\alpha} \\ &\quad + \overline{g}_{\mu\nu}\Big[\frac{1}{8}\left(\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\right)^2 - \frac{3}{8}\,\overline{g}^{\alpha\beta}\cdot\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\alpha\rho}\cdot\partial_4\overline{g}_{\beta\sigma} \\ &\qquad\qquad + \frac{1}{2}\,\overline{g}^{\rho\sigma}\cdot\partial_4\partial_4\overline{g}_{\rho\sigma}\Big] = -\kappa\, T_{\mu\nu}\,; \end{aligned} \tag{2.228}$$

$$\begin{aligned} G_{\mu 4} &= \partial_{\mu}\left[\overline{K}^{\sigma}_{\ \sigma}\right] - \overline{\nabla}^{\sigma}\left(K_{\mu\sigma}\right) \\ &= \frac{1}{2}\,\partial_{\mu}\left[\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\right] - \frac{1}{2}\,\overline{\nabla}^{\sigma}\left[\partial_4\overline{g}_{\mu\sigma}\right] = -\kappa\, T_{\mu 4}\,; \end{aligned} \tag{2.229}$$

$$\begin{aligned} G_{44} &= \frac{1}{2}\,\overline{R} - \frac{1}{2}\left(\overline{K}^{\sigma}_{\ \sigma}\right)^2 + \frac{1}{2}\,\overline{K}^{\rho}_{\ \sigma}\cdot\overline{K}^{\sigma}_{\ \rho} \\ &= \frac{1}{2}\,\overline{R} - \frac{1}{8}\left(\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\right)^2 + \frac{1}{8}\,\overline{g}^{\mu\nu}\cdot\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\mu\rho}\cdot\partial_4\overline{g}_{\nu\sigma} \\ &= -\kappa\, T_{44}\,. \end{aligned} \tag{2.230}$$

In this strategy, we *prescribe six* $T_{\alpha\beta}(\cdot\cdot)$ and *define* $T_{44} := -(\kappa)^{-1}G_{44}$ and $T_{\mu 4} := -(\kappa)^{-1}G_{\mu 4}$. Moreover, we solve *ten differential equations* (2.228) and $\mathcal{T}^i = 0$ for *six unknown functions* $\overline{g}_{\alpha\beta}(\cdot\cdot)$. (In these ten differential equations, there exist *four differential identities*.) Thus, this strategy is mathematically simpler than Strategy-I (the T-method).

In this method, the main energy condition $T_{44}(\mathbf{x}, T) \geq 0$ depends on the sign of the expression $\left[\frac{1}{2}\,\overline{R} - \frac{1}{2}\left(\overline{K}^{\sigma}_{\ \sigma}\right)^2 + \frac{1}{2}\overline{K}^{\rho}_{\ \sigma}\cdot\overline{K}^{\sigma}_{\ \rho}\right]$.

The field equation (2.230) is supposed to be a refinement on Poisson's equation (2.157i) of the Newtonian potential $W(\mathbf{x})$. Therefore, the Newtonian potential must be well *hidden in the six metric components* $\overline{g}_{\alpha\beta}(\mathbf{x}, T)$!

The *six field equations* (2.228) happen to be *subtensor field equations* [55, 244]. *These are covariant under the (restricted) transformations of spatial coordinates alone*. □

Let us go back to the system of the first- and second-order semilinear partial differential equations (2.223i, ii) representing field equations. *The most general solutions* of the system will involve *24 arbitrary functions of integration*! Out of these 24 functions, *four arbitrary functions* can be absorbed by coordinate functions. The most general solutions of *the vacuum field equations* contain *20 arbitrary functions of integration*. These arbitrary functions can be adjusted to match the prescribed initial-boundary value problems.

Example 2.4.3. Consider the space–time domain corresponding to

$$D := \left\{x : a < x^1 < b,\ x^2 \in \mathbb{R},\ x^3 > 2,\ x^4 \in \mathbb{R}\right\}.$$

We investigate the vacuum field equations

$$R_{ij}(x) = 0$$

in this domain.

A *class of* (*nonflat*) *general solutions*, [3, 205], is furnished by

$$\begin{aligned}
\mathbf{g}_{..}(x) := {} & \left(\ln|x^3|\right) \cdot \exp\left[F(x^1)\right] \cdot \mathrm{d}x^1 \otimes \mathrm{d}x^1 \\
& + \left(x^3\right)^2 \cdot \exp\left[2\alpha(x^2)\right] \cdot \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 \\
& + \exp\left[2\beta(x^4)\right] \cdot \left(\mathrm{d}x^1 \otimes \mathrm{d}x^4 + \mathrm{d}x^4 \otimes \mathrm{d}x^1\right),
\end{aligned}$$

$$\text{or,} \qquad \begin{aligned}
\mathrm{d}s^2 = {} & \left(\ln|x^3|\right) \cdot \exp\left[F(x^1)\right] \cdot \left(\mathrm{d}x^1\right)^2 + \left(x^3\right)^2 \cdot \exp\left[2\alpha(x^2)\right]\left(\mathrm{d}x^2\right)^2 \\
& + \left(\mathrm{d}x^3\right)^2 + 2\exp\left[2\beta(x^4)\right] \cdot \mathrm{d}x^1 \mathrm{d}x^4,
\end{aligned}$$

$$D := \left\{(x^1, x^2, x^3, x^4) : -\infty < x^1 < \infty, -\infty < x^2 < \infty, 0 < x^3, -\infty < x^4 < \infty\right\}.$$

Here, $F(x)$, $\alpha(x^2)$, and $\beta(x^4)$ are assumed to be *arbitrary functions* of class C^2. The arbitrary functions $\alpha(x^2)$ and $\beta(x^4)$ can be absorbed by the coordinate transformation:

$$\begin{aligned}
& \widehat{x}^1 = x^1, \quad \widehat{x}^2 = \int e^{\alpha(x^2)} \cdot \mathrm{d}x^2, \quad \widehat{x^3} = x^3, \\
& \widehat{x}^4 = \int e^{2\beta(x^4)} \cdot \mathrm{d}x^4.
\end{aligned}$$

Thus, the vacuum solution involving one arbitrary function is given by

$$ds^2 = \left(\ln|\widehat{x}^3|\right) \cdot \exp\left[F(\widehat{x}^1)\right] \cdot \left(d\widehat{x}^1\right)^2 + \left(\widehat{x}^3\right)^2 \cdot \left(d\widehat{x}^2\right)^2 + 2\, d\widehat{x}^1 d\widehat{x}^4.$$

The "initial-value problem"

$$\widehat{g}_{11}\left(0, \widehat{x}^2, \widehat{x}^3, \widehat{x}^4\right) = \frac{\partial \widehat{g}_{11}(\widehat{x})}{\partial \widehat{x}^1}\Big|_{(0,\widehat{x}^2,\widehat{x}^3,\widehat{x}^4)} = \ln|\widehat{x}^3|$$

yields one possible metric (not involving arbitrary functions) as

$$ds^2 = \left(\ln|\widehat{x}^3|\right) \cdot e^{\widehat{x}^1} \cdot \left(d\widehat{x}^1\right)^2 + \left(\widehat{x}^3\right)^2 \cdot \left(d\widehat{x}^2\right)^2 + \left(d\widehat{x}^3\right)^2 + 2\, d\widehat{x}^1 d\widehat{x}^4. \qquad \square$$

The classification of a semilinear (or a quasi-linear) second-order partial differential equation has been discussed in Appendix 2 after (A2.44). The classification of a system of first-order, semilinear (or quasi-linear) partial differential equations has been touched upon in (A2.51). (For a detailed treatment, we refer to the book by Courant and Hilbert [43].)

To illustrate briefly, we deal with *a couple of toy models* in the following:

Example 2.4.4. Consider a domain of *the flat space–time and the wave equation*

$$\square V = d^{ij}\,\partial_i \partial_j V = 0. \tag{2.231}$$

(See Problem # 4 of Exercise 2.1.) By the discussions after (A2.44), it is clear that (2.231) is *hyperbolic*. The second-order, linear p.d.e. (2.231) is equivalent to the following system of four first-order p.d.e.s:

$$\partial_4 \omega_\alpha - \partial_\alpha \omega_4 = 0, \tag{2.232i}$$

$$d^{ij}\,\partial_i \omega_j = 0, \tag{2.232ii}$$

$$\omega_j = \partial_j V. \tag{2.232iii}$$

(Compare with (A2.48).) We can combine (2.232ii) and (2.232i) into the following form:

$$\Gamma_{ij}{}^{k} \cdot \partial_k \omega_j = 0,$$

$$\Gamma_{1j}{}^{k} := d^{jk}, \quad \Gamma_{21}{}^{4} = \Gamma_{32}{}^{4} = \Gamma_{43}{}^{4} := 1,$$

$$\Gamma_{24}{}^{1} = \Gamma_{34}{}^{2} = \Gamma_{44}{}^{3} := -1\,;$$

$$\text{otherwise, } \Gamma_{ij}{}^{k} \equiv 0. \tag{2.233}$$

(Here, the index j is *also* summed.) Compare (2.233) with (A2.49). *Caution*: The components $\Gamma_{ij}{}^{k}$ are not tensorial.

The characteristic matrix of (2.233) is provided by

$$\underset{4\times 4}{\left[\Gamma_{ij}\right]} := \left[\Gamma_{ij}{}^{k} \cdot \partial_k \phi\right] = \begin{bmatrix} \partial_1\phi & \partial_2\phi & \partial_3\phi & -\partial_4\phi \\ \partial_4\phi & 0 & 0 & -\partial_1\phi \\ 0 & \partial_4\phi & 0 & -\partial_2\phi \\ 0 & 0 & \partial_4\phi & -\partial_3\phi \end{bmatrix}$$

$$\begin{aligned} \det[\Gamma_{ij}] &= -(\partial_4\phi)^2 \left[(\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 - (\partial_4\phi)^2\right] \\ &= -(\partial_4\phi)^2 \cdot \left[d^{ij}\, \partial_i\phi \cdot \partial_j\phi\right]. \end{aligned} \tag{2.234}$$

(See (2.51).)

Therefore, for the case $\partial_4\phi \neq 0$, the equation

$$\det[\Gamma_{ij}] = 0$$

yields *the characteristic* (*three-dimensional*) *hypersurfaces* governed by

$$d^{ij} \cdot \partial_i\phi \cdot \partial_j\phi = 0. \tag{2.235}$$

By the criteria in [43], *the system (2.233) is hyperbolic*. The p.d.e. (2.235) stands for *a null hypersurface* (like a null cone). (*Shock waves* of the wave equation (2.231) travel along such a hypersurface [43].) □

Example 2.4.5. Consider the analogous problem in a domain of the four-dimensional *Euclidean manifold. A harmonic function* $V(x)$ satisfies *the potential equation*:

$$\delta^{ij}\, \partial_i \partial_j V = 0. \tag{2.236}$$

The equivalent first-order system is furnished by

$$\Gamma_{ij}^{k}\, \partial_k \omega_j = 0. \tag{2.237}$$

(Here, we have set $\omega_j = \partial_j V$.) The characteristic matrix is provided by

$$\underset{4\times 4}{\left[\Gamma_{ij}\right]} = \begin{bmatrix} \partial_1\phi & \partial_2\phi & \partial_3\phi & \partial_4\phi \\ \partial_4\phi & 0 & 0 & -\partial_1\phi \\ 0 & \partial_4\phi & 0 & -\partial_2\phi \\ 0 & 0 & \partial_4\phi & -\partial_3\phi \end{bmatrix}. \tag{2.238}$$

The equation for the characteristic criterion is:

$$\det[\Gamma_{ij}] = -(\partial_4\phi)^2 \left[(\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 + (\partial_4\phi)^2\right] = 0. \tag{2.239}$$

Therefore, for the case $\partial_4\phi \neq 0$, the solutions are given by $\phi(x) = k = \text{const.}$

There exists *no nondegenerate characteristic hypersurface*. The system (2.237) is called *elliptic* [43]. (The harmonic function $V(x)$ must be *real-analytic*.) □

Example 2.4.6. Now, we shall classify gravitational field equations (2.223i). We use *harmonic coordinates* (or *the harmonic gauge*) to simplify calculations. (Recall the harmonic coordinate chart as introduced in Problem #10 of Exercise 2.2.) The corresponding field equations are given by

$$\begin{aligned} & g^{kl}(x)\,\partial_k\partial_l\,g^{ij} - 2\begin{Bmatrix} i \\ k\;p \end{Bmatrix}\begin{Bmatrix} j \\ l\;q \end{Bmatrix} g^{kl}(x)\cdot g^{pq}(x) \\ & = \begin{cases} -2\kappa\left[T^{ij}(x) - \frac{1}{2}g^{ij}(x)T^k_{\ k}(x)\right] & \text{inside material sources,} \\ 0 & \text{outside material sources.} \end{cases} \end{aligned} \tag{2.240}$$

Now, let us obtain a system of first-order quasi-linear p.d.e.s which is *equivalent to the second-order system in (2.240)*. Following Example 2.4.4, and putting $\omega^{ij}{}_k = \partial_k g^{ij}$, we arrive at the following system of *40 equations*:

$$\begin{aligned} \omega^{ij}{}_k(x) &\equiv \omega^{ji}{}_k(x), \\ \partial_\alpha\omega^{ij}{}_4 - \partial_4\omega^{ij}{}_\alpha &= 0, \\ g^{kl}\,\partial_k\omega^{ij}{}_l - h^{ij}(g^{kl},\omega^{kl}{}_p) &= \begin{cases} -2\kappa\left[T^{ij} - \frac{1}{2}g^{ij}T^k{}_k\right] & \text{inside,} \\ 0 & \text{outside.} \end{cases} \end{aligned} \tag{2.241}$$

We define the 4×4 submatrix as

$$\underset{4\times 4}{[\gamma]} = [\gamma_{ij}{}^k\cdot\partial_k\phi] := \begin{bmatrix} g^{1k}\partial_k\phi & g^{2k}\partial_k\phi & g^{3k}\partial_k\phi & g^{4k}\partial_k\phi \\ \partial_4\phi & 0 & 0 & -\partial_1\phi \\ 0 & \partial_4\phi & 0 & -\partial_2\phi \\ 0 & 0 & \partial_4\phi & -\partial_3\phi \end{bmatrix}. \tag{2.242}$$

The characteristic matrix and its determinant for the system (2.241) are furnished by

$$\underset{40\times 40}{[\Gamma]} := \begin{bmatrix} \boxed{\gamma} & & & & & & & & & \\ & \boxed{\gamma} & & & & & & & & \\ & & \boxed{\gamma} & & & & & & & \\ & & & \boxed{\gamma} & & & 0 & & & \\ & & & & \boxed{\gamma} & & & & & \\ & & & & & \boxed{\gamma} & & & & \\ & 0 & & & & & \boxed{\gamma} & & & \\ & & & & & & & \boxed{\gamma} & & \\ & & & & & & & & \boxed{\gamma} & \\ & & & & & & & & & \boxed{\gamma} \end{bmatrix}, \tag{2.243i}$$

$$\det[\Gamma] = \left[(\partial_4\phi)^2\cdot(g^{ij}\cdot\partial_i\phi\cdot\partial_j\phi)\right]^{10}. \tag{2.243ii}$$

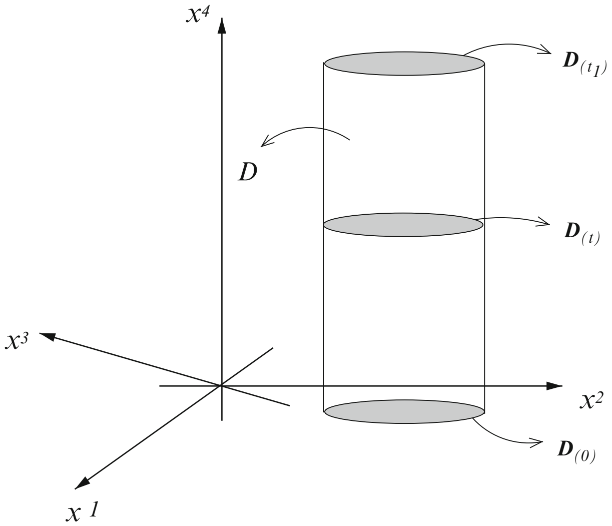


Fig. 2.20 Domain $D := \mathbf{D}_{(0)} \times (0, t_1) \subset \mathbb{R}^4$ for the initial-value problem

In the case $\partial_4\phi \neq 0$, the equation $\det[\Gamma] = 0$ yields

$$g^{ij}(x) \cdot \partial_i\phi \cdot \partial_j\phi = 0. \tag{2.244}$$

Above is the governing equation for *a three-dimensional null hypersurface* in the space–time metric of signature $+2$. Obviously, the system (2.241) is *hyperbolic*. □

We shall now explore the *initial-value problem* (or the *Cauchy problem*) of gravitational field equations (2.159i). However, we firstly examine a toy model of one hyperbolic p.d.e., namely, the usual wave equation:

$$\nabla_i \nabla^i V = 0. \tag{2.245}$$

Let the domain of validity $D := \mathbf{D}_{(0)} \times (0, t_1)$ have one boundary $\mathbf{D}_{(0)}$ as the initial hypersurface $x^4 = 0$. (See Fig. 2.20.)

The corresponding *Cauchy-Kowalewski theorem* is stated below:

Theorem 2.4.7. *Let metric components $g^{ij}(x)$ be real analytic with $g^{44}(x) < 0$ in a space–time domain $D := \mathbf{D}_{(0)} \times (0, t_1) \subset \mathbb{R}^4$ with one boundary at $x^4 = 0$. Moreover, let this boundary hypersurface contain the origin* $(0, 0, 0, 0)$. *Furthermore, let $\mathbf{D}_{(0)}$ be the projection of D onto the hypersurface $x^4 = 0$. Given real-analytic functions $f(\mathbf{x})$ and $h(\mathbf{x})$ in $\mathbf{x} \in \mathbf{D}_{(0)}$, there exists a half-neighborhood $\mathcal{N}_\delta^+(0, 0, 0, 0)$ with a unique solution $V(x)$ of the partial differential equation* (2.245) *such that* $\lim_{x^4 \to 0_+} V(\mathbf{x}, x^4) = f(\mathbf{x})$ *and* $\lim_{x^4 \to 0_+} \left[\frac{\partial V(\mathbf{x}, x^4)}{\partial x^4}\right] = h(\mathbf{x})$.

For the proof, consult [43].

As a preliminary to the investigation of initial-value problems for general relativity, we state the following *Lichnerowicz-Synge lemma* [166, 243]:

Lemma 2.4.8. *Consider gravitational field equations* (2.223i,ii) *in a domain* $D := \mathbf{D}_{(0)} \times (0, t_1)$. *Then, the following two statements are mathematically equivalent:*

$$\text{(A)} \qquad \mathcal{E}_{ij}(x) = 0 \quad in \quad x \in D. \tag{2.246i}$$

$$\text{(B)} \qquad \mathcal{E}_{\alpha\beta}(x) - \frac{1}{2}\, g_{\alpha\beta}(x)\, \mathcal{E}^i{}_i(x) = 0 \tag{2.246ii}$$

$$\textit{and} \qquad \nabla_j \mathcal{E}^{ij} = 0 \quad in \quad x \in D \tag{2.246iii}$$

$$\textit{with} \qquad \mathcal{E}^4{}_i(\mathbf{x}, 0) = 0 \quad for \quad \mathbf{x} \in \mathbf{D}_{(0)}. \tag{2.246iv}$$

For the proof, see [243].

Now, we shall state and prove a theorem on the solution of the initial-value problem in general relativity.

Theorem 2.4.9. *Let the energy–momentum–stress tensor components* $T_{ij}(x)$ *and metric tensor components* $g_{ij}(x)$ *be real analytic with* $g_{44}(x) < 0$ *in a space–time domain* $D := \mathbf{D}_{(0)} \times (0, t_1) \subset \mathbb{R}^4$ *with one boundary hypersurface at* $x^4 = 0$. *Given 30 real-analytic functions,* $g^{\#}_{ij}(\mathbf{x})$, $\psi_{ij}(\mathbf{x})$, $T^{\#}_{\alpha\beta}(\mathbf{x})$, *and* $\theta_i(\mathbf{x}) = T^{\#}_{4i}(\mathbf{x})$ *for* $\mathbf{x} \in \mathbf{D}_{(0)}$ *be prescribed. Moreover, let the initial constraints* $\left[G^4_{\;i}(x) + \kappa T^4_{\;i}(x)\right]_{|x^4=0} = 0$ *hold. Then, there exist 20 unique solutions* $g_{ij}(x)$ *and* $T_{ij}(x)$ *of the field equations* $\mathcal{E}_{ij}(x) = 0$ *in* $\mathbf{D}_{(0)} \times (0, t_1)$ *such that* $\lim_{x^4 \to 0_+} g_{ij}(x) = g^{\#}_{ij}(\mathbf{x})$, $\lim_{x^4 \to 0_+} \left[\partial_4 g_{ij}\right] = \psi_{ij}(\mathbf{x})$, $\lim_{x^4 \to 0_+} T_{\alpha\beta}(x) = T^{\#}_{\alpha\beta}(\mathbf{x})$, *and* $\lim_{x^4 \to 0_+} T_{i4}(x) = \theta_i(\mathbf{x})$.

Proof. Instead of using (2.246i), we use the equivalent equations (2.246ii), (2.246iii), and (2.246iv). Moreover, instead of using the T-method of p. 197, we use a mixed method of the p. 197 to solve the field equations. Therefore, by Example 2.4.2, we make use of geodesic normal coordinates (2.225) yielding

$$\mathbf{g}_{..}(x) = \overline{g}_{\alpha\beta}(x)\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta - \mathrm{d}x^4 \otimes \mathrm{d}x^4.$$

Moreover, we prescribe $T_{\alpha\beta}(x)$ in D and solve for $\overline{g}_{\alpha\beta}(x)$ and $T_{i4}(x)$ from (2.246ii–iv). The field equations (2.246ii) and (2.246iii) yield, respectively,

$$\overline{R}_{\mu\nu}(x) - \frac{1}{2}\left(\partial_4 \partial_4 \overline{g}_{\mu\nu}\right) - \frac{1}{4}\, \overline{g}^{\rho\sigma} \cdot \partial_4 \overline{g}_{\rho\sigma} \cdot \partial_4 \overline{g}_{\mu\nu} + \frac{1}{2}\, \overline{g}^{\alpha\beta} \cdot \partial_4 \overline{g}_{\mu\alpha} \cdot \partial_4 \overline{g}_{\nu\beta}$$
$$+ \kappa \left[T_{\mu\nu} - \frac{1}{2}\, \overline{g}_{\mu\nu} \left(\overline{g}^{\alpha\beta} T_{\alpha\beta} - T_{44} \right) \right] = 0, \tag{2.247i}$$

$$\partial_4 T_{j4}(x) = \partial_\alpha T^\alpha{}_j(x) + \begin{Bmatrix} i \\ i\ k \end{Bmatrix} T^k{}_j(x) - \begin{Bmatrix} k \\ i\ j \end{Bmatrix} T^i{}_k(x). \tag{2.247ii}$$

From (2.247i,ii), we derive the following string of partial differential equations by *successive differentiations*:

$$\partial_4\partial_4\overline{g}_{\mu\nu}(x) = 2\overline{R}_{\mu\nu}(x) - \frac{1}{2}\,\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\cdot\partial_4\overline{g}_{\mu\nu} + \overline{g}^{\alpha\beta}\cdot\partial_4\overline{g}_{\mu\alpha}\cdot\partial_4\overline{g}_{\nu\beta}$$
$$+\, 2\kappa\left[T_{\mu\nu}(x) - \frac{1}{2}\,\overline{g}_{\mu\nu}\cdot\left(\overline{g}^{\alpha\beta}T_{\alpha\beta} - T_{44}\right)\right],$$
$$\partial_4\partial_4\partial_4\overline{g}_{\mu\nu} = \partial_4\{\ldots\ldots\ldots\ldots\ldots\},\tag{2.248i}$$
$$\vdots \qquad\qquad \vdots$$

$$\partial_4 T_{j4}(x) = \partial_\alpha T^{\alpha}{}_{j}(x) + \begin{Bmatrix} i \\ i\ k \end{Bmatrix} T^{k}{}_{j}(x) - \begin{Bmatrix} k \\ i\ j \end{Bmatrix} T^{i}{}_{k}(x),$$
$$\partial_4\partial_4 T_{j4}(x) = \partial_4\{\ldots\ldots\ldots\ldots\ldots\},\tag{2.248ii}$$
$$\vdots \qquad\qquad \vdots$$

Now, *we prescribe 16 real-analytic initial values*

$$\overline{g}_{\alpha\beta}(\mathbf{x},0) = g^{\#}_{\alpha\beta}(\mathbf{x}),$$
$$\partial_4\overline{g}_{\alpha\beta}(x)_{|x^4=0} = \psi_{\alpha\beta}(\mathbf{x}) \equiv 2K^{\#}_{\alpha\beta}(\mathbf{x}),$$
$$T_{j4}(\mathbf{x},0) = \theta_j(\mathbf{x}).\tag{2.249}$$

The field equations (2.248i,ii) yield on the initial hypersurface $x^4 = 0$,

$$\partial_4\partial_4\overline{g}_{\mu\nu|x^4=0} = 2R^{\#}_{\mu\nu}(\mathbf{x}) - \frac{1}{2}\,g^{\#\rho\sigma}(\mathbf{x})\cdot\psi_{\rho\sigma}(\mathbf{x})\cdot\psi_{\mu\nu}(\mathbf{x})$$
$$+\, g^{\#\alpha\beta}(\mathbf{x})\cdot\psi_{\mu\alpha}(\mathbf{x})\cdot\psi_{\nu\beta}(\mathbf{x}) + 2\kappa\left[T^{\#}_{\mu\nu} - \frac{1}{2}\,g^{\#}_{\mu\nu}\left(g^{\#\alpha\beta}T^{\#}_{\alpha\beta} - T^{\#}_{44}\right)\right]_{|x^4=0}$$
$$= (\text{initial values}),$$
$$\partial_4\partial_4\partial_4\overline{g}_{\mu\nu|x^4=0} = (\text{initial values}),$$
$$\vdots \qquad\qquad \vdots\tag{2.250i}$$
$$\partial_4 T_{j4|x^4=0} = (\text{initial values})$$
$$\partial_4\partial_4 T_{j4|x^4=0} = (\text{initial values})$$
$$\vdots \qquad\qquad \vdots\tag{2.250ii}$$

The prescribed functions in (2.249) *have to satisfy partial differential equations* (2.246iv) on the initial hypersurface. These are explicitly provided by:

$$
\begin{aligned}
&\partial_\alpha \left[g^{\#\mu\nu}(\mathbf{x}) \cdot \psi_{\mu\nu}(\mathbf{x})\right] - \nabla^{\#\sigma}\left[\psi_{\alpha\sigma}(\mathbf{x})\right] + 2\kappa\,\theta_\alpha(\mathbf{x}) = 0,\\
\text{or,}\quad &\partial_\alpha \left[K^{\#\mu}{}_{\mu}(\mathbf{x})\right] - \nabla^{\#\sigma}\left[K_{\alpha\sigma}(\mathbf{x})\right] + \kappa\,\theta_\alpha(\mathbf{x}) = 0,\\
\text{and}\quad &R^{\#}(\mathbf{x}) - \frac{1}{4}\left[g^{\#\mu\nu}\cdot\psi_{\mu\nu}\right]^2 + \frac{1}{4}\left(g^{\#\mu\nu}\cdot g^{\#\rho\sigma}\cdot\psi_{\mu\rho}\cdot\psi_{\nu\sigma}\right) + 2\kappa\,\theta_4(\mathbf{x}) \quad = 0,\\
\text{or,}\quad &R^{\#}(\mathbf{x}) - \left[K^{\#\mu}{}_{\mu}(\mathbf{x})\right]^2 + K^{\#\nu}{}_{\rho}(\mathbf{x})\cdot K^{\#\rho}{}_{\nu}(\mathbf{x}) + 2\kappa\,\theta_4(\mathbf{x}) = 0.
\end{aligned}
\tag{2.251}
$$

(Here, $K^{\#}_{\mu\nu}(\mathbf{x})$ are components of *the extrinsic curvature* as given in (1.251).) Assuming that (2.251) is satisfied, the corresponding Taylor series are expressed as:

$$
\begin{aligned}
\overline{g}_{\alpha\beta}(x) &= g^{\#}_{\alpha\beta}(\mathbf{x}) + x^4\cdot\psi_{\alpha\beta}(\mathbf{x}) + \frac{1}{2}\left(x^4\right)^2\cdot\left(\partial_4\partial_4\overline{g}_{\alpha\beta}\right)_{|x^4=0} + \cdots,\\
T_{4j}(x) &= \theta_j(\mathbf{x}) + x^4\cdot\left(\partial_4 T_{4j}\right)_{|x^4=0} + \frac{1}{2}\left(x^4\right)^2\cdot\left(\partial_4\partial_4 T_{4j}\right)_{|x^4=0} + \cdots.
\end{aligned}
\tag{2.252}
$$

Using (2.250i,ii), one can explicitly construct the series in (2.252). The condition of real analyticity guarantees the absolute convergence of series in (2.252) for $0 \leq x^4 < t_1 := \min\left(\tau_{\alpha\beta}(\mathbf{x}), \tau_i(\mathbf{x})\right)$. Moreover, it is clear from (2.252) that $\lim_{x^4\to 0_+} \overline{g}_{\alpha\beta}(x) = g^{\#}_{\alpha\beta}(\mathbf{x})$, $\lim_{x^4\to 0_+} \partial_4 g_{\alpha\beta} = \psi_{\alpha\beta}(\mathbf{x})$, and $\lim_{x^4\to 0_+} T_{4j}(x) = \theta_j(\mathbf{x})$. Thus, the initial-value problem of general relativity is solved, and Theorem 2.4.9 is proved. ■

Example 2.4.10. Consider the initial-value problem for vacuum equations. These are summarized from (2.248i) as

$$
\partial_4\partial_4\overline{g}_{\mu\nu}(x) = 2\overline{R}_{\mu\nu}(x) - \frac{1}{2}\,\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\cdot\partial_4\overline{g}_{\mu\nu} + \overline{g}^{\alpha\beta}\cdot\partial_4\overline{g}_{\mu\alpha}\cdot\partial_4\overline{g}_{\nu\beta}. \tag{2.253}
$$

The initial data (*or Cauchy data*) $\overline{g}_{\mu\nu}(\mathbf{x}, 0) = g^{\#}_{\mu\nu}(\mathbf{x})$ and $\left[\partial_4\overline{g}_{\mu\nu}\right]_{|x^4=0} = \psi_{\mu\nu}(\mathbf{x})$ must satisfy, by (2.251), the following equations:

$$
\nabla^{\#}_{\alpha}\left[\psi^{\#\alpha}{}_{\beta} - \delta^{\alpha}{}_{\beta}\psi^{\#\mu}{}_{\mu}\right] = 0, \tag{2.254i}
$$

$$
4R^{\#}(\mathbf{x}) - \left[\psi^{\#\mu}{}_{\mu}\right]^2 + \left[\psi^{\#\nu}{}_{\rho}\cdot\psi^{\#\rho}{}_{\nu}\right] = 0. \tag{2.254ii}
$$

Equations (2.254i,ii) are physically important for *gravitational waves*. (See Appendix 5.) This system of four equations for 12 unknown functions is *undetermined*. It seems as if it would be easy to solve these, but in fact, it is not! □

Example 2.4.11. Consider a specific scenario for the initial-value problem for vacuum equations. We choose the initial data as

$$
\begin{aligned}
& g^{\#}_{\alpha\beta}(\mathbf{x}) = \delta_{\alpha\beta}, \quad R^{\#}_{\alpha\beta}(\mathbf{x}) \equiv 0, \quad R^{\#}(\mathbf{x}) \equiv 0\,; \\
& \left[\psi_{\alpha\beta}(\mathbf{x})\right] := \begin{bmatrix} 2c_{(1)} & 0 & 0 \\ 0 & 2c_{(2)} & 0 \\ 0 & 0 & 2c_{(3)} \end{bmatrix} = \left[2K^{\#}_{\mu\nu}(\mathbf{x})\right], \\
& \nabla^{\#}_{\alpha}\left[\psi^{\#\alpha}{}_{\beta}\right] \equiv 0, \quad \nabla^{\#}_{\beta}\left[\psi^{\#\mu}{}_{\mu}\right] \equiv 0.
\end{aligned}
$$

Here, *the constants* $c_{(1)}, c_{(2)}, c_{(3)}$ are assumed to satisfy the constraints:

$$c_{(1)} + c_{(2)} + c_{(3)} = 1 = \left[c_{(1)}\right]^2 + \left[c_{(2)}\right]^2 + \left[c_{(3)}\right]^2.$$

In this case, (2.254i,ii) are *identically satisfied*. Moreover, solving (2.253), we arrive at the special *Kasner metric* [147]

$$
\begin{aligned}
\mathbf{g}_{..}(x) = {} & \left(1 + x^4\right)^{2c_{(1)}} \cdot \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left(1 + x^4\right)^{2c_{(2)}} \cdot \mathrm{d}x^2 \otimes \mathrm{d}x^2 \\
& + \left(1 + x^4\right)^{2c_{(3)}} \cdot \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\
\mathrm{d}s^2 = {} & \left(1 + x^4\right)^{2c_{(1)}} \cdot \left(\mathrm{d}x^1\right)^2 + \left(1 + x^4\right)^{2c_{(2)}} \cdot \left(\mathrm{d}x^2\right)^2 \\
& + \left(1 + x^4\right)^{2c_{(3)}} \cdot \left(\mathrm{d}x^3\right)^2 - \left(\mathrm{d}x^4\right)^2.
\end{aligned}
\tag{2.255}
$$

□

We shall provide another example of the initial-value scheme in Example 6.2.3.

Exercises 2.4

1. Consider four harmonic coordinate conditions $\partial_j\left[\sqrt{|g|}\, g^{ij}\right] = 0$. Obtain a class of general solutions of these differential equations in terms of arbitrary functions.
2. Let six metric components be $g_{12}(x) = g_{13}(x) = g_{14}(x) = g_{23}(x) = g_{24}(x) = g_{34}(x) \equiv 0$ yielding an orthogonal coordinate chart:

$$
\begin{aligned}
\mathbf{g}_{..}(x) = {} & \left[h_{(1)}(x)\right]^2 \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left[h_{(2)}(x)\right]^2 \mathrm{d}x^2 \otimes \mathrm{d}x^2 \\
& + \left[h_{(3)}(x)\right]^2 \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \left[h_{(4)}(x)\right]^2 \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\
\mathrm{d}s^2 = {} & \sum_i \sum_j h_{(i)} \cdot h_{(j)} \cdot d_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j.
\end{aligned}
$$

(Summation convention is temporarily suspended.) Express a special class of vacuum field equations $R_{ij}(x) = 0$ in terms of four functions $h_{(i)}(x) > 0$ which are of class C^3.

3. Consider the conformal tensor components of definition (1.169i)

$$C^l{}_{ijk}(x) := R^l{}_{ijk}(x) + \frac{1}{2}\left[\delta^l{}_j R_{ik} - \delta^l{}_k R_{ij} + g_{ik} R^l{}_j - g_{ij} R^l{}_k\right] + \frac{R(x)}{6}\left[\delta^l{}_k\, g_{ij} - \delta^l{}_j\, g_{ik}\right]$$

in a space–time domain. Solve for ten functions $g_{ij}(x)$ in the system of ten independent quasi-linear second-order partial differential equations $C^l{}_{ijk}(x) = 0$.
4. Recall the vacuum field equations $R_{ij}(x) = 0$. Using 50 functions $g_{ij}(x)$ and $\left\{ {k \atop ij} \right\}$, express an equivalent system of first-order partial differential equations.
5. Let the metric be expressed as:

$$\begin{aligned} \mathbf{g}_{..}(x) &:= \left[a(x^4)\right]^2 \cdot \delta_{\alpha\beta}\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta - \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\ \mathrm{d}s^2 &= \left[a(x^4)\right]^2 \cdot \delta_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta - \left(\mathrm{d}x^4\right)^2, \\ x \in D &:= \mathbf{D} \times (t_0, t_1) \subset \mathbb{R}^4, \quad t_0 > 1. \end{aligned}$$

Moreover, the function $a(x^4) > 0$ is of class C^3. For the initial-value problem, the prescribed functions are assumed to be as follows: In $D \subset \mathbb{R}^4$, $T_{\alpha\beta}(x) \equiv 0$.

On the initial hypersurface at $x^4 = t_0$,

(i) $g^{\#}_{\alpha\beta}(\mathbf{x}) = (\alpha)^2 \cdot \delta_{\alpha\beta} =$ const.,
(ii) $\psi_{\alpha\beta}(\mathbf{x}) = \beta \cdot \delta_{\alpha\beta} =$ const.,
(iii) $\theta_\alpha(\mathbf{x}) \equiv 0$,
(iv) $\theta_4(\mathbf{x}) = \rho_0 =$ const.

Prove that, with the notation $\dot{a}(x^4) := \frac{\mathrm{d}a(x^4)}{\mathrm{d}x^4}$, (2.247i,ii) and (2.251) reduce to

$$-\left[a(x^4) \cdot \ddot{a}(x^4) + 2\left(\dot{a}(x^4)\right)^2\right] + \frac{\kappa}{2}\left[a(x^4)\right]^2 \cdot T_{44}(x^4) = 0,$$

$$\partial_4 T_{\alpha 4} \equiv 0, \quad \partial_4 T_{44} = -\frac{3[\dot{a}(x^4)]}{a(x^4)} \cdot T_{44}(x^4),$$

$$\text{and,} -\frac{3}{2}\left(\frac{\beta}{\alpha^2}\right)^2 + 2\kappa\, \rho_0 = 0.$$

Answers and Hints to Selected Exercises

1. Consider fields with the following symmetries:

$$\Gamma^{jikl}(x) \equiv \Gamma^{ijkl}(x) \equiv \Gamma^{ijlk}(x) \equiv -\Gamma^{iljk}(x).$$

The functions $\Gamma^{ijkl}(x)$ are of class C^4. A class of general solutions is furnished by

$$\sqrt{|g|}\, g^{ij}(x) = \partial_k \partial_l \Gamma^{ijkl}.$$

(*Remarks:* (i) Harmonic coordinate conditions and the class of general solutions are *not tensor field equations*.

(ii) In fact, *no coordinate condition is expressible as a tensor field equation*.)

2. *Suspend summation convention* in this answer [56, 90].
   For $l \neq k$, and $i \neq k$,

$$R_{lk}(x) = \sum_{i \neq k,l}{}' \left[h_{(i)}(x)\right]^{-1} \cdot \Big\{\partial_l \partial_k h_{(i)} - \left[\partial_l h_{(i)}\right] \cdot \left[\partial_k \ln h_{(l)}\right]$$

$$- \left[\partial_k h_{(i)}\right] \cdot \left[\partial_l \ln h_{(k)}\right]\Big\} = 0\,;$$

$$R_{kk}(x) = d_{kk} \cdot \left[h_{(k)}(x)\right] \cdot \sum_{i \neq k}{}' \left[h_{(i)}(x)\right]^{-1} \cdot \Big\{ d^{ii} \cdot \partial_i \left[h_{(i)}^{-1} \cdot \partial_i h_{(k)}\right]$$

$$+ d^{kk} \cdot \partial_k \left[h_{(k)}^{-1} \cdot \partial_k h_{(i)}\right] + \sum_{l \neq i,k}{}' d^{ll} \cdot \left[h_{(l)}^{-2} \cdot \partial_l h_{(i)} \cdot \partial_l h_{(k)}\right]\Big\} = 0.$$

(*Remarks*:

(i) There exist four differential identities among ten equations above. However, the system is still overdetermined and difficult to solve.
(ii) The above equations hold in an N-dimensional manifold.
(iii) Orthogonal coordinates may not exist for $N > 3$.)

3. The most general solution of the system of p.d.e.s is provided by:

$$g_{ij}(x) = \exp\left[-2\mu(x)\right] \cdot d_{kl} \cdot \partial_i f^k \cdot \partial_j f^l.$$

Here, $\mu(x)$ is of class C^3. Moreover, the four functions $f^k(x)$ are of class C^4, and they are functionally independent. These five functions are otherwise *arbitrary*. (Consult Theorem 1.3.32 and (1.162).)

(*Remark*: By the coordinate transformation $\widehat{x}^i = f^i(x)$, the metric components can be reduced to $\widehat{g}_{ij}(\widehat{x}) = \exp\left[-2\widehat{\mu}(\widehat{x})\right] \cdot d_{kl}$.)

4.

$$\partial_k g_{ij} = g_{jh}(x) \cdot \begin{Bmatrix} h \\ k\, i \end{Bmatrix} + g_{ih}(x) \cdot \begin{Bmatrix} h \\ k\, j \end{Bmatrix},$$

$$\partial_l \left[g_{jh} \cdot \begin{Bmatrix} h \\ k\, i \end{Bmatrix} + g_{ih} \cdot \begin{Bmatrix} h \\ k\, j \end{Bmatrix} \right] - [l \leftrightarrow k] = 0,$$

$$\partial_k \begin{Bmatrix} i \\ i\, j \end{Bmatrix} - \partial_i \begin{Bmatrix} i \\ k\, j \end{Bmatrix} + \begin{Bmatrix} i \\ k\, h \end{Bmatrix} \begin{Bmatrix} h \\ i\, j \end{Bmatrix} - \begin{Bmatrix} i \\ i\, h \end{Bmatrix} \begin{Bmatrix} h \\ k\, j \end{Bmatrix} = 0.$$

5. Use (2.247i,ii) and (2.252). Also, recall that the metric $\mathbf{g}^{\#}_{..} = (\alpha)^2 \cdot \mathbf{I}_{..}$ *is flat*.

(*Remarks*: Integration of the above equations leads to *the flat Friedmann model of cosmology* in Chap. 6.)

2.5 Fluids, Deformable Solids, and Electromagnetic Fields

We start with *a pressureless incoherent dust or a dust cloud*. In a flat space–time, we introduced the topic in Example 2.1.11. In a curved space–time, Example 2.3.6 dealt with such a material. The pertinent equations are the following:

$$
\begin{aligned}
T_{ij}(x) &:= \rho(x) \cdot U_i(x) \cdot U_j(x), \quad \rho(x) > 0, \\
U_i(x) \cdot U^i(x) &\equiv -1, \\
\nabla_i \left(\rho U^i\right) &= 0, \\
\dot{U}^{(a)}\left[\mathcal{X}(s)\right] &= \frac{\mathrm{d}\,\mathcal{U}^{(a)}(s)}{\mathrm{d}s} - \gamma^{(a)}{}_{(d)(b)} \cdot \mathcal{U}^{(d)}(s) \cdot \mathcal{U}^{(b)}(s) = 0.
\end{aligned} \tag{2.256}
$$

The last equation shows that streamlines follow timelike geodesics.

Now, we shall investigate a *perfect fluid* (or, *an ideal fluid*). This fluid was already introduced in Example 2.3.1. The field equations, from Equations (2.161i), (2.166i), and (2.168i), are given by the following:

$$T^{ij}(x) := \left[\rho(x) + p(x)\right] \cdot U^i(x) \cdot U^j(x) + p(x) \cdot g^{ij}(x), \tag{2.257i}$$

$$
\begin{aligned}
\mathcal{E}^{ij}(x) := G^{ij}(x) + \kappa\big\{&\left[\rho(x) + p(x)\right] \cdot U^i(x) \cdot U^j(x) \\
&+ p(x) \cdot g^{ij}(x)\big\} = 0,
\end{aligned} \tag{2.257ii}
$$

$$\mathcal{T}^i(x) := \nabla_j \left\{\left[\rho(x) + p(x)\right] \cdot U^i(x) \cdot U^j(x) + p(x) \cdot g^{ij}(x)\right\} = 0, \tag{2.257iii}$$

$$\mathcal{U}(x) := g_{ij}(x) \cdot U^i(x) \cdot U^j(x) + 1 = 0, \tag{2.257iv}$$

$$\mathcal{C}^i\left(g_{kj},\ \partial_l g_{kj}\right) = 0. \tag{2.257v}$$

The counting of the number of unknown functions versus the number of independent equations is provided by the following:

No. of unknown functions: $10(g_{ij}) + 4(U^i) + 1(\rho) + 1(p) = 16$.

No. of equations: $10(\mathcal{E}^{ij} = 0) + 4(\mathcal{T}^i = 0) + 1(\mathcal{U} = 0) + 4(\mathcal{C}^i = 0) = 19$.

No. of identities: $4(\nabla_j \mathcal{E}^{ij} \equiv \kappa \mathcal{T}^i) = 4$.

No. of independent equations: $19 - 4 = 15$.

Therefore, the system is *underdetermined*. To make the system *determinate*, it is permissible to impose an *equation of state*

$$F(\rho, p) = 0,$$
$$\left[\frac{\partial F(\cdot\cdot)}{\partial \rho}\right]^2 + \left[\frac{\partial F(\cdot\cdot)}{\partial p}\right]^2 > 0. \tag{2.258}$$

Remark: Choosing four coordinate conditions $\mathcal{C}^i(\cdot\cdot) = 0$ is a usual practice. However, in p. 197, it is mentioned that there exist *three other strategies of solutions*. In each strategy, the counting has to be done separately.

Let the world tube of the fluid body be exhibited in Fig. 2.19. Synge's junction conditions (2.170) across the hypersurface $\partial_{(3)}D$ of jump discontinuities, with $U^i(x)n_i(x)_{|\partial_{(3)}D} = 0$, reduce to

$$\left[g_{ij}(x)\right]_{|\partial_{(3)}D} = 0 = \left[\partial_k g_{ij}\right]_{|\partial_{(3)}D};$$
$$[p(x)]_{|\partial_{(3)}D} = 0. \tag{2.259}$$

Energy conditions from Theorem 2.3.2 reduce in this case to the following:

(i) Weak energy conditions:

$$\rho(x) \geq 0, \quad \rho(x) + p(x) \geq 0. \tag{2.260i}$$

(ii) Dominant energy conditions:

$$|p(x)| \leq \rho(x). \tag{2.260ii}$$

(iii) Strong energy conditions:

$$\rho(x) \geq 0, \quad \rho(x) + p(x) \geq 0, \quad \rho(x) + 3p(x) \geq 0. \tag{2.260iii}$$

The four differential conservation equations (2.257iii), by (2.211) and (2.212i,ii), yield the following constitutive equations:

$$\nabla_j \left[\rho(x) \cdot U^j(x)\right] = -p(x) \cdot \nabla_j U^j, \tag{2.261i}$$

$$\dot{\rho}(x) := U^j(x) \cdot \nabla_j \rho = -\left[\rho(x) + p(x)\right] \cdot \nabla_j U^j, \tag{2.261ii}$$

$$\begin{aligned}\left[\rho(x) + p(x)\right]\dot{U}^i(x) &= \left[\rho(x) + p(x)\right] \cdot U^j(x) \cdot \nabla_j U^i \\ &= -\left[g^{ij}(x) + U^i(x) \cdot U^j(x)\right] \cdot \nabla_j p,\end{aligned} \tag{2.261iii}$$

$$\left[\rho(x) + p(x)\right]_{|\mathcal{X}(s)} \cdot \frac{D\,\mathcal{U}^i(s)}{\mathrm{d}s} = -\left\{\left[g^{ij}(x) + U^i(x) \cdot U^j(x)\right] \cdot \nabla_j p\right\}_{|..}. \tag{2.261iv}$$

The last equations (2.261iv) govern streamlines of a perfect fluid flow.

Now, we shall introduce *physical (or orthonormal) components* of the nonrelativistic 3-velocity vector field analogous to those of (2.24). These are provided by

$$\begin{aligned}
\mathcal{V}^{(\alpha)}(x) &:= \frac{U^{(\alpha)}(x)}{U^{(4)}(x)},\\
[\mathcal{V}(x)]^2 &:= \delta_{(\alpha)(\beta)}\mathcal{V}^{(\alpha)}(\cdot\cdot)\cdot\mathcal{V}^{(\beta)}(\cdot\cdot),\\
U^{(\alpha)}(x) &= \mathcal{V}^{(\alpha)}(x)\Big/\sqrt{1-\mathcal{V}^2(\cdot\cdot)},\\
U^{(4)}(x) &= 1\Big/\sqrt{1-\mathcal{V}^2(\cdot\cdot)}.
\end{aligned} \tag{2.262}$$

Remark: Components $\mathcal{V}^{(\alpha)}(x)$'s *are not actual components of any four-dimensional vector or tensor*.

Example 2.5.1. Consider the relativistic equations of motion (2.261iii). In terms of physical (or orthonormal) components and Ricci rotation coefficients of (1.138), the spatial components of (2.261iii) can be expressed as

$$\begin{aligned}
&\left[\rho(x)+p(x)\right]\cdot\Bigg\{\sqrt{1-\mathcal{V}^2(\cdot\cdot)}\cdot\left[\partial_{(4)}\left(\frac{\mathcal{V}^{(\alpha)}}{\sqrt{\cdots}}\right)+\mathcal{V}^{(\beta)}\cdot\partial_{(\beta)}\left(\frac{\mathcal{V}^{(\alpha)}}{\sqrt{\cdots}}\right)\right]\\
&\quad-\gamma^{(\alpha)}{}_{(4)(4)}(\cdot\cdot)-\left(\gamma^{(\alpha)}{}_{(4)(\beta)}+\gamma^{(\alpha)}{}_{(\beta)(4)}\right)\cdot\mathcal{V}^{(\beta)}-\gamma^{(\alpha)}{}_{(\delta)(\beta)}\cdot\mathcal{V}^{(\delta)}\cdot\mathcal{V}^{(\beta)}\Bigg\}\\
&\quad=-\left[1-\mathcal{V}^2(\cdot\cdot)\right]\cdot\delta^{(\alpha)(\beta)}\cdot\partial_{(\beta)}p-\left(\partial_{(4)}p+\mathcal{V}^{(\beta)}\cdot\partial_{(\beta)}p\right)\cdot\mathcal{V}^{(\alpha)}.
\end{aligned} \tag{2.263}$$

The equation above is *the exact general relativistic version of Euler's equation of (perfect) fluid flow* [243]. *A simplistic interpretation* of (2.263) is that

$$\text{(mass)}\ \times\ \text{(acceleration)}\ =\ -\text{(gradient of pressure)}. \qquad \square$$

Now, we shall deal with a 4×4 matrix $\left[T_{ij}(x)\right]$ of Segre characteristic $[(1,1),1,1]$. Physically speaking, this class includes (1) an *anisotropic fluid,* (2) *a deformable solid with symmetry*, (3) *a perfect fluid plus a tachyonic dust* [62], and many other usual or exotic materials. By (2.181), we express

$$\begin{aligned}
T_{ij}(x) = &\left[\rho(x)+p_\perp(x)\right]\cdot U_i(x)\cdot U_j(x)+p_\perp(x)\cdot g_{ij}(x)\\
&+\left[p_\parallel(x)-p_\perp(x)\right]\cdot S_i(x)\cdot S_j(x).
\end{aligned} \tag{2.264}$$

The corresponding gravitational field equations are furnished by

$$\begin{aligned}
\mathcal{E}_{ij}(x) &:= G_{ij}(x)+\kappa\,T_{ij}(x)=0,\\
\mathcal{T}^i(x) &:= \nabla_j T^{ij}=0,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{U}(x) := U_i(x) \cdot U^i(x) + 1 = 0, \\
&\mathcal{S}(x) := S_i(x) \cdot S^i(x) - 1 = 0, \\
&\mathcal{P}(x) := U_i(x) \cdot S^i(x) = 0, \\
&\mathcal{C}^i \left(g_{jk}, \partial_l g_{jk}\right) = 0.
\end{aligned} \tag{2.265}$$

The above system is underdetermined and four subsidiary conditions can be imposed to make the system determinate.

The constitutive equations for the anisotropic fluid flow can be derived from $\mathcal{T}^i(x) = 0$, $U_i(x) \cdot \mathcal{T}^i(x) = 0$, and $S_i(x) \cdot \mathcal{T}^i(x) = 0$. These are explicitly given by

$$\begin{aligned}
&U^i \cdot \nabla_j \left[(\rho + p_\perp) \cdot U^j\right] + (\rho + p_\perp) \cdot U^j \cdot \nabla_j U^i + \nabla^i p_\perp \\
&\quad + S^i \cdot \nabla_j \left[(p_\parallel - p_\perp) S^j\right] + (p_\parallel - p_\perp)\, S^j \nabla_j S^i = 0,
\end{aligned} \tag{2.266i}$$

$$-\nabla_j \left[(\rho + p_\perp) U^j\right] + U_i \cdot \nabla^i p_\perp + (p_\parallel - p_\perp)\, U_i \cdot S^j \cdot \nabla_j S^i = 0, \tag{2.266ii}$$

$$(\rho + p_\perp) \cdot S_i U^j \cdot \nabla_j U^i + S_i \nabla^i p_\perp + \nabla_j \left[(p_\parallel - p_\perp) S^j\right] = 0. \tag{2.266iii}$$

Substituting (2.266ii,iii) into (2.266i), we finally deduce that

$$\begin{aligned}
&(\rho + p_\perp) \cdot (\delta^i{}_k - S^i \cdot S_k) \cdot U^j \cdot \nabla_j U^k \\
&\quad + (p_\parallel - p_\perp) \cdot (\delta^i{}_k + U^i \cdot U_k) \cdot S^j \cdot \nabla_j S^k \\
&\quad + (\delta^i{}_j + U^i \cdot U_j - S^i \cdot S_j) \cdot \nabla^j p_\perp = 0.
\end{aligned} \tag{2.267}$$

The equation above provides the streamlines of an anisotropic fluid flow.

The energy conditions of Theorem 2.3.2 reduce in this case to the following:

1. Weak energy conditions:
$$\rho \geq 0, \quad \rho + p_\perp \geq 0, \quad \rho + p_\parallel \geq 0. \tag{2.268i}$$
2. Dominant energy conditions:
$$\rho \geq |p_\perp|, \quad \rho \geq |p_\parallel|. \tag{2.268ii}$$
3. Strong energy conditions:
$$\rho \geq 0, \quad \rho + p_\perp \geq 0, \quad \rho + p_\parallel \geq 0, \quad \rho + 2p_\perp + p_\parallel \geq 0. \tag{2.268iii}$$

Synge's junction conditions (2.170) for jump discontinuities on the hypersurface $\partial D_{(3)}$, with $U^i(\cdot\cdot) n_i(\cdot\cdot)_{|..} = 0$, reduce to

$$\begin{aligned}
&\left[g_{ij}\right]_{|\partial_{(3)} D} = 0, \quad \left[\partial_k g_{ij}\right]_{|\partial_{(3)} D} = 0, \\
&\left[p_\perp \cdot \delta^i{}_j + (p_\parallel - p_\perp) \cdot S^i \cdot S_j\right] n^j{}_{|\partial_{(3)} D} = 0.
\end{aligned} \tag{2.269}$$

We shall provide some special examples of the anisotropic fluid in the next chapter. Also, see [63, 66] for a detailed treatment of fluids.

Now, we shall investigate the case of a *deformable solid* body. It is characterized by the energy–momentum–stress tensor components:

$$T^{ij}(x) := \rho(x) \cdot U^i(x) \cdot U^j(x) - S^{ij}(x),$$
$$S^{ij}(x) := \sum_{\alpha=1}^{3} \sigma_{(\alpha)}(x) \cdot e^i_{(\alpha)}(x) \cdot e^j_{(\alpha)}(x). \tag{2.270}$$

(We have already mentioned such equations in (2.183), (2.193), and (2.196).)

The gravitational field equations are furnished by

$$\mathcal{E}^{ij}(x) := G^{ij}(x) + \kappa \left[\rho \cdot U^i \cdot U^j - S^{ij}(x)\right] = 0, \tag{2.271i}$$

$$\mathcal{T}^i(x) := \nabla_j \left[\rho \cdot U^i \cdot U^j - S^{ij}\right] = 0, \tag{2.271ii}$$

$$\mathcal{U}(x) := U_i(x) U^i(x) + 1 = 0, \tag{2.271iii}$$

$$\mathcal{P}_{(\alpha)}(x) := U_i(x)\, e^i_{(\alpha)}(x) = 0, \tag{2.271iv}$$

$$\mathcal{N}_{(\alpha)(\beta)}(x) := g_{ij} \cdot e^i_{(\alpha)} \cdot e^j_{(\beta)} - \delta_{(\alpha)(\beta)} = 0, \tag{2.271v}$$

$$\mathcal{C}^i\left(g_{jk}, \partial_l g_{jk}\right) = 0. \tag{2.271vi}$$

The above system is underdetermined and five subsidiary conditions can be imposed.

The constitutive equations are

$$\nabla_j \left[\rho U^j\right] + U_i \cdot \nabla_j \left[\sum_{\alpha=1}^{3} \sigma_{(\alpha)} \cdot e^i_{(\alpha)} \cdot e^j_{(\alpha)}\right] = 0, \tag{2.272i}$$

$$\rho \cdot U^j \cdot \nabla_j U^i = \left(\delta^i{}_k + U^i \cdot U_k\right) \cdot \nabla_j \left[\sum_{\alpha=1}^{3} \sigma_{(\alpha)} \cdot \mathrm{e}^k_{(\alpha)} \cdot e^j_{(\alpha)}\right]. \tag{2.272ii}$$

(Compare the above equations with (2.211) and (2.212i).)

The energy conditions are provided by (2.194i–iii), and Synge's junction conditions (2.170), with $U^i(\cdot\cdot) n_i(\cdot\cdot)_{|\cdot\cdot} = 0$, in this case reduce to

$$\left[g_{ij}\right]_{|\partial_{(3)}D} = 0, \quad \left[\partial_k g_{ij}\right]_{|\partial_{(3)}D} = 0,$$
$$\left[\sum_{\alpha=1}^{3} \sigma_{(\alpha)} \cdot e^i_{(\alpha)} \cdot e^j_{(\alpha)} \cdot n_j\right]_{|\partial_{(3)}D} = 0. \tag{2.273}$$

Example 2.5.2. Consider the physical or orthonormal components of equations of motion (2.272ii). The spatial components, with help of (2.262) and (2.263), yield

$$\begin{aligned}
&\rho(x) \cdot \left\{ \frac{1}{\sqrt{1-\mathcal{V}^2(\cdot\cdot)}} \cdot \left[\partial_{(4)} \left(\frac{\mathcal{V}^{(\alpha)}}{\sqrt{\cdots}} \right) + \mathcal{V}^{(\beta)} \cdot \partial_{(\beta)} \left(\frac{\mathcal{V}^{(\alpha)}}{\sqrt{\cdots}} \right) \right.\right. \\
&\qquad - \left[1-\mathcal{V}^2(\cdot\cdot)\right]^{-1} \cdot \left[\gamma^{(\alpha)}{}_{(4)(4)}(\cdot\cdot) + \left(\gamma^{(\alpha)}{}_{(4)(\beta)} + \gamma^{(\alpha)}{}_{(\beta)(4)} \right) \cdot \mathcal{V}^{(\beta)} \right. \\
&\qquad \left.\left.\left. + \gamma^{(\alpha)}{}_{(\beta)(\delta)} \cdot \mathcal{V}^{(\beta)} \cdot \mathcal{V}^{(\delta)} \right]\right]\right\} \\
&= \left[\delta^{(\alpha)}{}_{(b)} + \left(1-\mathcal{V}^2\right)^{-1/2} \cdot \mathcal{V}^{(\alpha)} U_{(b)} \right] \cdot \nabla_{(c)} S^{(b)(c)}. \qquad (2.274)
\end{aligned}$$

(Compare with (2.41) and (2.125i).) For a class of *equilibrium of the deformable body*, we assume that $\mathcal{V}^{\alpha}(x) \equiv 0$. Thus, the conditions

$$0 = U_{(a)} S^{(a)(b)} = 0 - \frac{1}{\sqrt{\cdots}} \cdot S^{(4)(b)},$$

or,

$$S^{(4)(b)}(x) \equiv 0$$

holds.

Substituting the above into (2.274), we derive that

$$0 = \rho(x) \cdot \gamma^{(\alpha)}{}_{(4)(4)}(\cdot\cdot) + \delta^{(\alpha)}{}_{(\beta)} \cdot \nabla_{(\mu)} S^{(\beta)(\mu)}. \qquad (2.275)$$

The equation above can be physically interpreted as

"the gravitational forces exactly balance the elastic forces." □

Now, we shall explore electromagnetic fields in a curved space–time manifold. We have already touched upon electromagnetic fields in Examples 1.2.19, 1.2.22, 1.3.6, 2.1.12, and 2.1.13 and (2.56i,ii), (2.60i–iii), (2.63), (2.67), (2.77i–iii), and (2.78i–iii). All these equations allow for *a straight forward generalization* in a domain of curved space–time. Recall that the electromagnetic field is represented by *an antisymmetric tensor field*

$$\mathbf{F}_{\cdot\cdot}(x) = F_{ij}(x)\, dx^i \otimes dx^j = (1/2) F_{ij}(x)\, dx^i \wedge dx^j. \qquad (2.276)$$

Outside charged material sources, Maxwell's equations for an electromagnetic field in *a curved*, *background space–time*, are governed by

$$\nabla_j F^{ij} = 0, \qquad (2.277\text{i})$$

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} \equiv \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0. \qquad (2.277\text{ii})$$

By (1.69) and (1.70), we deduce from (2.277ii) and (1.145i) that

$$F_{ij}(x) = \partial_i A_j - \partial_j A_i \equiv \nabla_i A_j - \nabla_j A_i, \tag{2.278i}$$

$$\nabla_k \nabla^j A_j - \nabla^j \nabla_k A_j = R^h{}_k(x) \cdot A_h(x). \tag{2.278ii}$$

Denoting the *generalized wave operator* (or the *generalized D'Alembertian*) by $\Box := \nabla_j \nabla^j$, Maxwell's equations (2.277i,ii) reduce to

$$\nabla^i \left[\nabla_j A^j\right] - R^{ij}(x) \cdot A_j(x) - \Box A^i = 0. \tag{2.279}$$

Now under *a gauge transformation* of (1.71ii), namely,

$$\widehat{A}_i(x) = A_i(x) - \partial_i \lambda = A_i(x) - \nabla_i \lambda, \tag{2.280}$$

the electromagnetic field remains unchanged, or $\widehat{F}_{ij}(x) \equiv F_{ij}(x)$. Let us choose *a special class* of $\lambda(x)$ such that

$$\Box \lambda = \nabla_i \nabla^i \lambda = \nabla_i A^i. \tag{2.281}$$

By (2.280) and (2.281),

$$\nabla_i \widehat{A}^i = \nabla_i A^i - \Box \lambda = 0. \tag{2.282}$$

The above is called the *Lorentz gauge condition* on the vector field $\vec{\widehat{\mathbf{A}}}(x)$. Maxwell's equations (2.279) simplify into

$$\Box \widehat{A}^i + R^{ij}(x) \cdot \widehat{A}_j(x) = 0. \tag{2.283}$$

Electromagnetic field equations (2.277i,ii) possess *an additional covariance under the conformal transformation* (1.166i). We shall state and prove the following theorem on this topic.

Theorem 2.5.3. *Let a conformal transformation be furnished by*

$$\begin{aligned} \overline{\mathbf{g}}_{..}(x) &= \exp\left[2\mu(x)\right] \cdot \mathbf{g}_{..}(x), \\ \overline{\mathbf{F}}_{..}(x) &:= \mathbf{F}_{..}(x), \\ \overline{F}^{ij}(x) &:= \overline{g}^{ik}(x)\, \overline{g}^{jl}(x) F_{kl}(x). \end{aligned} \tag{2.284}$$

Then, the electromagnetic field equations (2.277i,ii) *remain unchanged.*

Proof. By (2.284), it follows that

$$\det\left[\overline{g}_{ij}\right] = \exp\left[8\mu\right] \cdot \det\left[g_{ij}\right],$$

$$\overline{F}^{ij}(x) = \exp\left[-4\mu\right] \cdot F^{ij}(x),$$

$$\overline{\nabla}_j \overline{F}^{ij} = \frac{1}{\sqrt{-\overline{g}}}\, \partial_j \left[\sqrt{-\overline{g}}\; \overline{F}^{ij}\right]$$
$$= \mathrm{e}^{-4\mu} \cdot \left[\nabla_j F^{ij}\right].$$

Therefore, (2.277i) remains intact. Since (2.277ii) can be expressed without the metric tensor, it remains valid automatically. ■

Example 2.5.4. Let us investigate electromagnetic field equations (2.277i,ii) in *a conformally flat*, background domain characterized by

$$\overline{g}_{ij}(x) = \exp\left[2\mu(x)\right] \cdot d_{ij}\,. \tag{2.285}$$

Equation (2.277i) still implies that $\overline{F}_{ij}(x) = \partial_i A_j - \partial_j A_i \equiv F_{ij}(x)$. The equations $\overline{\nabla}_j \overline{F}_i{}^j = 0$ yield

$$\nabla_j F_i{}^j = \partial_i \left[\partial_j A^j\right] - d^{kj}\partial_k \partial_j A_i = 0. \tag{2.286}$$

Assuming *the Lorentz gauge* condition:

$$\nabla_k A^k \equiv \partial_k A^k = 0, \tag{2.287}$$

the equations reduce to the wave equation

$$d^{kl}\partial_k \partial_l A^i = 0. \tag{2.288}$$

A general class of solutions of (2.287) and (2.288) is furnished (by the superposition of plane waves) as [55]:

$$\begin{aligned}
&A^j(x) = \int_{\mathbb{R}^3} \mathrm{Re}\left\{\alpha^j(\mathbf{k}) \cdot \exp\left[\mathrm{i}\left(k_l x^l - \theta(\mathbf{k})\right)\right]\right\} \cdot \mathrm{d}^3\mathbf{k},\\
&k_4 = \nu(\mathbf{k}) := \sqrt{\delta^{\alpha\beta} k_\alpha k_\beta},\\
&\alpha^4(\mathbf{k}) := -k_\mu \alpha^\mu(\mathbf{k}) \big/ \nu(\mathbf{k}) \quad \text{for} \quad \nu(\mathbf{k}) > 0,\\
&\alpha^\mu(\mathbf{k}) \neq \lambda(\mathbf{k})\, k^\mu.
\end{aligned} \tag{2.289}$$

Here, we have *assumed that the integrals in (2.289) converge absolutely and uniformly*. Moreover, *we assume that differentiations commute with integrals* [32]. The five functions $\alpha^j(\mathbf{k})$ and $\theta(\mathbf{k})$ are otherwise *arbitrary*. □

The classification of electromagnetic field equations (2.277i,ii) in flat space–time has been carried out in Appendix 2, Example A2.9. In case, $\partial_4 F_{ij} \neq 0$, the system is *hyperbolic*.

Now, we shall discuss the effects of electromagnetic energy–momentum–stress tensor on the curvature of the space–time manifold. The appropriate field equations for this investigation are the coupled *Einstein–Maxwell equations* as furnished in the following:

$$T^{ij}(x) := F^{ik}(x)F^{j}_{\ k}(x) - (1/4)g^{ij}(x)F_{kl}(x)F^{kl}(x), \tag{2.290i}$$

$$\mathcal{M}^{i}(x) := \nabla_j F^{ij} = 0, \tag{2.290ii}$$

$$^{*}\mathcal{M}^{i}(x) := \frac{1}{3!}\,\eta^{ijkl}(x)\big[\nabla_j F_{kl} + \nabla_k F_{lj} + \nabla_l F_{jk}\big] = 0, \tag{2.290iii}$$

$$\mathcal{E}^{ij}(x) := G^{ij}(x) + \kappa\, T^{ij}(x) = 0, \tag{2.290iv}$$

$$\mathcal{T}^{i}(x) := \nabla_j T^{ij} \equiv 0, \tag{2.290v}$$

$$\mathcal{C}^{i}\left(g_{jk}, \partial_l g_{jk}\right) = 0. \tag{2.290vi}$$

Remarks: (i) The special relativistic electromagnetic energy–momentum–stress was introduced in (2.63).

(ii) In (2.290iii), the definition of *the Hodge-star operation* in (1.113) is used.

(iii) A domain of space–time, in which (2.290i–vi) hold, is called an *electromagneto-vac domain*.

(iv) Equation (2.290i) implies that $T^{i}_{\ i}(x) \equiv 0$. Therefore, (2.290iv) yields that $R(x) \equiv 0$.

The number of unknown functions versus the number of independent equations in the system (2.290i–vi) is exhibited below:

(i) No. of unknown functions: $6(F_{ij}) + 10(g_{ij}) = 16$.
(ii) No. of equations: $4(\mathcal{M}_i = 0) + 4(^{*}\mathcal{M}_i = 0) + 10(\mathcal{E}^{ij} = 0) + 4(\mathcal{C}^{i} = 0) = 22$.
(iii) No. of identities: $1(\nabla_i \mathcal{M}^{i} \equiv 0) + 1(\nabla_i\, ^{*}\mathcal{M}^{i} \equiv 0) + 4(\nabla_j \mathcal{E}^{ij} \equiv 0) = 6$.
(iv) No. of independent equations: $22 - 6 = 16$.

Therefore, the system of (2.290i–vi) is *exactly determinate*!

Now, we shall discuss *the variational derivation* of the field equations (2.290i–vi). (Appendix 1 deals with variational derivation of differential equations.)

Example 2.5.5. Equations (2.290iii) yield (2.278i), which is

$$F_{ij}(x) = \partial_i A_j - \partial_j A_i \equiv \nabla_i A_j - \nabla_j A_i.$$

It turns out that the variational derivation demands the use of the *4-potential* $A^{i}(x)$, rather than $F_{ij}(x)$. Using (A1.25), we write the Lagrangian function for the coupled fields as

$$\begin{aligned}
&L\left(a_i, y^{ij}, \gamma^k{}_{ij}; a_{ij}, y^{ij}{}_k, \gamma^k{}_{ijl}\right) \\
&:= y^{ij}\left[\gamma^k{}_{kij} - \gamma^k{}_{ijk} - \gamma^l{}_{lk}\gamma^k{}_{ij} + \gamma^l{}_{ik}\gamma^k{}_{lj}\right] \\
&\quad + (\kappa/2)\cdot y^{ij}y^{kl}\left(a_{lj} - a_{jl}\right)(a_{ki} - a_{ik}) \\
&=: \rho_{ij}(\cdot\cdot) + (\kappa/2)\cdot y^{ij}y^{kl}\left(a_{lj} - a_{jl}\right)(a_{ki} - a_{ik}), \\
&L(\cdots)_{\left| \begin{array}{l} y^{ij}=g^{ij}(x),\, y^{ij}{}_k=\partial_k g^{ij}, \\ \gamma^k{}_{ij}=\left\{ \begin{array}{c} k \\ ij \end{array} \right\},\, \gamma^k{}_{ijl}=\partial_l\left\{ \begin{array}{c} k \\ ij \end{array} \right\}, \\ a_{lj}=\partial_j A_l. \end{array}\right.} = R(x) + (\kappa/2)\cdot F^{ij}(x)\cdot F_{ij}(x).
\end{aligned} \tag{2.291}$$

The corresponding action integral is provided by

$$J(F) := \int_{D_4} L(\cdots)_{|\ldots} \cdot \sqrt{-g(x)}\, \mathrm{d}^4x,$$
$$a_i = \pi_i \circ F(x) =: A_i(x). \tag{2.292}$$

(See Fig. A1.2.). We vary independently $\widehat{y}^{ij} = g^{ij}(x) + \varepsilon h^{ij}(x)$, $\widehat{y}^k{}_{ij} = \left\{\begin{array}{c} k \\ i\ j \end{array}\right\}$ $+\varepsilon h^k{}_{ij}(x)$, and $\widehat{a}_i = A_i(x) + \varepsilon h_i(x)$. Then, the vanishing of the variational derivative, that is,

$$\lim_{\varepsilon\to 0} \frac{\Delta J(F)}{\varepsilon} = 0,$$

yields, by (A1.26), etc., the field equations (2.290ii) and (2.290iv). Moreover, *the boundary terms* from (2.291), (A1.20ii), and (A1.29ii) imply that

$$\int_{\partial D_4} \left\{ g^{ij}(x)\cdot h^k{}_{ki}(x) - g^{ki}(x)\cdot h^j{}_{ki}(x) + 2\kappa\, F^{ij}(x) h_i(x) \right\} n_j(x)\cdot \mathrm{d}^3v = 0. \tag{2.293}$$

Therefore, variations on the boundary ∂D_4 satisfying

$$\left\{ \left[g^{ij}(x)\cdot h^k{}_{ki}(x) - g^{ki}(x)\cdot h^j{}_{ki}(x) + 2\kappa\, F^{ij}(x) h_i(x) \right] n_j(x) \right\}_{|\partial D_4} = 0, \tag{2.294}$$

imply (2.293). (But the converse is not true!). *A popular way* to write (2.293) is as follows:

$$\int_{\partial D_4} \left\{ \left[g^{ij}(x) \cdot \delta \begin{Bmatrix} k \\ k\,i \end{Bmatrix} - g^{ki} \cdot \delta \begin{Bmatrix} j \\ k\,i \end{Bmatrix} + 2\kappa\, F^{ij}(x) \cdot \delta A_i(x) \right] n_j(x) \right\} \mathrm{d}^3 v = 0. \tag{2.295}$$

Usually, boundary variations $\delta \begin{Bmatrix} k \\ j\,i \end{Bmatrix}_{|..} = 0$, $\delta A_i(x)_{|..} = 0$ are chosen (for the Dirichlet problem), but there exist *many more initial-boundary problems admitted by the variational principle*, as exhibited in (2.293) or (2.295). □

Now, we shall explore *coupled gravitational and electromagnetic fields*, generated by the distribution of a charged dust cloud (or a primitive plasma). In the flat space–time, we have already investigated *incoherent*, *charged dust* in (2.64i,ii). The pertinent field equations here are expressed in the following:

$$T^{ij}(x) := \rho(x)U^i(x)U^j(x) + F^{ik}(x)F_k^{\ j}(x) - (1/4)g^{ij}(x)F_{kl}(x)F^{kl}(x), \tag{2.296i}$$

$$\mathcal{M}^i(x) := \nabla_j F^{ij} - \sigma(x)U^i(x) = 0, \tag{2.296ii}$$

$$^*\mathcal{M}^i(x) := \frac{1}{3!}\eta^{ijkl}(x)\left[\nabla_j F_{kl} + \nabla_k F_{lj} + \nabla_l F_{jk}\right] = 0, \tag{2.296iii}$$

$$\mathcal{J}(x) := \nabla_i \left(\sigma U^i\right) = 0, \tag{2.296iv}$$

$$\mathcal{E}^{ij}(x) := G^{ij}(x) + \kappa\, T^{ij}(x) = 0, \tag{2.296v}$$

$$\mathcal{T}^i(x) := \nabla_j\, T^{ij} = 0, \tag{2.296vi}$$

$$\mathcal{U}(x) := U_i(x)U^i(x) + 1 = 0, \tag{2.296vii}$$

$$\mathcal{C}^i\left(g_{jk}, \partial_l g_{jk}\right) = 0. \tag{2.296viii}$$

We shall now count the number of unknown functions versus the number of independent equations.

No. of unknown functions: $6(F_{ij}) + 1(\sigma) + 1(\rho) + 4(U^i) + 10(g_{ij}) = 22.$

No. of equations: $4(\mathcal{M}^i = 0) + 4({}^*\mathcal{M}^i = 0) + 1(\mathcal{J} = 0) + 10(\mathcal{E}^{ij} = 0)$
$+ 4(\mathcal{T}^i = 0) + 1(\mathcal{U} = 0) + 4(\mathcal{C}^i = 0) = 28.$

No. of identities: $1(\nabla_i \mathcal{M}^i \equiv 0) + 1(\nabla_i\, {}^*\mathcal{M}^i \equiv 0) + 4(\nabla_j \mathcal{E}^{ij} \equiv \kappa \mathcal{T}^i) = 6.$

No. of independent equations: $28 - 6 = 22$.

Thus, *the system is determinate*.

Now, we shall derive the constitutive equations for the system in (2.296i–viii).

The differential conservation equations (2.296vi) yield

$$\begin{aligned} 0 &= \nabla_j \left[\rho U^i U^j + F^{ik} F^j{}_k - (1/4) g^{ij} F_{kl} F^{kl}\right] \\ &= U^i \nabla_j \left(\rho U^i\right) + \rho U^j \nabla_j U^i + F^{ik} \nabla_j F^j{}_k \\ &\quad + (1/2) g^{il} F^{jk} \left[\nabla_j F_{lk} + \nabla_l F_{kj} + \nabla_k F_{jl}\right] \\ \text{or} \qquad 0 &= U^i \nabla_j \left(\rho U^j\right) + \rho U^j \nabla_j U^i - \sigma F^{ik} U_k + 0. \end{aligned} \tag{2.297}$$

Multiplying the equation above by $U_i(x)$, we deduce that

$$\nabla_j \left(\rho U^j\right) = 0. \tag{2.298}$$

Thus, *the continuity of the material current still holds*. Substituting (2.298) into (2.297), we derive that

$$\begin{aligned} &\rho(x) \dot{U}^i(x) = \rho(x) U^j(x) \nabla_j U^i = \sigma(x) F^{ik}(x) U_k(x) \\ \text{or} \qquad &\rho\left[\mathcal{X}(s)\right] \cdot \frac{D\, \mathcal{U}^i(s)}{\mathrm{d}s} = \left[\sigma(x) F^i{}_k(x)\right]_{|..} \cdot \mathcal{U}^k(s). \end{aligned} \tag{2.299}$$

The above equation of a streamline is the appropriate (curved space–time) generalization of the Lorentz equation of motion (2.66) and (2.67) in the flat space–time scenario.

Here, we emphasize that the relativistic equations of motion for streamlines in (2.256)*,* (2.261iv)*, and* (2.267) *and equations of motion* (2.272ii) *for a solid particle are all consequences of Einstein's gravitational field equations. These equations are not added on as required in the corresponding non-general relativistic theories.*

Now, we shall touch upon briefly the algebraic properties of the antisymmetric electromagnetic tensor field

$$\mathbf{F}_{..}(x) = (1/2) F_{ij}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j = F_{(a)(b)}(x)\, \tilde{\mathbf{e}}^{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b)}(x).$$

The 4×4 antisymmetric matrix $\left[F_{(a)(b)}(x_0)\right]$ *does not possess any real, nonzero, usual eigenvalues*. However, the same matrix *has invariant, (real) eigenvalues*. The invariant eigenvalue probelm can be posed as:

$$F_{(a)(b)}(x_0) \cdot \nu^{(b)}(x_0) = \lambda\, d_{(a)(b)}\, \nu^{(b)}(x_0). \tag{2.300}$$

(Compare the above equation with equations in Example A3.5 of Appendix 3) It follows from (2.300) that

$$\lambda\, d_{(a)(b)}\, \nu^{(a)}(x_0)\, \nu^{(b)}(x_0) = 0. \tag{2.301}$$

Therefore, either the (invariant) eigenvalue $\lambda = 0$ or the "invariant eigenvector" $\nu^{(a)}(x_0)\, \vec{\mathbf{e}}_{(a)}(x_0)$ is *null or both*.

Now, consider two fields defined by

$$I_{(1)}(x) := F_{(a)(b)}(x) \cdot F^{(a)(b)}(x), \tag{2.302i}$$

$$I_{(2)}(x) := F_{(a)(b)}(x) \cdot {}^{*}F^{(a)(b)}(x). \tag{2.302ii}$$

(Here, the star stands for *the Hodge-star* operation in (1.113).) The field $I_{(1)}(x)$ is a scalar or invariant. However, the field $I_{(2)}(x)$ is *a pseudoscalar* and $\left[I_{(2)}(x)\right]^2$ is invariant.

We shall now introduce the physical or orthonormal components of the electric and magnetic field vectors. These are furnished by

$$E_{(\alpha)}(x) := F_{(\alpha)(4)}(x), \tag{2.303i}$$

$$H_{(\alpha)}(x) := \varepsilon_{(\alpha)(\beta)(\gamma)}\, F^{(\beta)(\gamma)}(x). \tag{2.303ii}$$

(Compare the equations above with (2.55).)

Remark: The components $E_{(\alpha)}(x)$ and $H_{(\alpha)}(x)$ are *not components of relativistic vector fields*.

The invariant in (2.302i) and the pseudoscalar in (2.302ii) can be expressed in terms of electric and magnetic vectors as

$$\begin{aligned} I_{(1)}(x) &= -\,2\delta^{(\alpha)(\beta)}\left[E_{(\alpha)}(x)E_{(\beta)}(x) - H_{(\alpha)}(x)H_{(\beta)}(x)\right] \\ &=: -\,2\left[\|\vec{\mathbf{E}}(x)\|^2 - \|\vec{\mathbf{H}}(x)\|^2\right], \end{aligned} \tag{2.304i}$$

$$I_{(2)}(x) = -\,4\delta^{(\alpha)(\beta)} E_{(\alpha)}(x) H_{(\beta)}(x) =: -4\left[\vec{\mathbf{E}}(x) \cdot \vec{\mathbf{H}}(x)\right]. \tag{2.304ii}$$

The energy–momentum–stress tensor of an electromagnetic field from (2.290i) is expressed as

$$T^{(a)(b)}(x) := F^{(a)(c)}(x) F^{(b)}{}_{(c)}(x) - (1/4)\, d^{(a)(b)} F_{(c)(d)}(x) F^{(c)(d)}(x). \tag{2.305}$$

We now choose the orthonormal tetrad $\left\{\vec{\mathbf{e}}_{(1)}(x), \vec{\mathbf{e}}_{(2)}(x), \vec{\mathbf{e}}_{(3)}(x), \vec{\mathbf{e}}_{(4)}(x)\right\}$ in a special way so that the vector $\vec{\mathbf{e}}_{(3)}(x)$ *is orthogonal to both* $\vec{\mathbf{E}}(x)$ *and* $\vec{\mathbf{H}}(x)$. (Rotation of spatial vectors $\{\vec{\mathbf{e}}_{(1)}(x), \vec{\mathbf{e}}_{(2)}(x), \vec{\mathbf{e}}_{(3)}(x)\}$ can always achieve this simplification.) Therefore, relative to this special class of frames,

$$E_{(3)}(x) = H_{(3)}(x) = 0. \tag{2.306}$$

Therefore, (2.304i,ii) simplify into

$$F_{(a)(b)}(x) F^{(a)(b)}(x) = -2\left[E_{(1)}^{2} + E_{(2)}^{2} - H_{(1)}^{2} - H_{(2)}^{2}\right], \tag{2.307i}$$

$$F_{(a)(b)}(x)\, {}^{*}F^{(a)(b)}(x) = -4\left[E_{(1)} H_{(1)} + E_{(2)} H_{(2)}\right]. \tag{2.307ii}$$

Moreover, the simplified version of (2.305) yields *the nonzero components* as

$$\begin{aligned}
T_{(1)(1)}(x) &= -T_{(2)(2)}(x) = (1/2)\left[H_{(2)}^{2} - H_{(1)}^{2} + E_{(2)}^{2} - E_{(1)}^{2}\right], \\
T_{(3)(3)}(x) &= T_{(4)(4)}(x) = (1/2)\left[H_{(1)}^{2} + H_{(2)}^{2} + E_{(1)}^{2} + E_{(2)}^{2}\right], \\
T_{(1)(2)}(x) &= -\left[E_{(1)} E_{(2)} + H_{(1)} H_{(2)}\right], \\
T_{(3)(4)}(x) &= H_{(1)} E_{(2)} - E_{(1)} H_{(2)}.
\end{aligned} \tag{2.308}$$

The 4×4 symmetric matrix $\left[T_{(a)(b)}(x_0)\right]$ becomes *block diagonal*.

Now, we shall explore the invariant eigenvalue problem for the matrix $\left[T_{(a)(b)}(x_0)\right]$.

Theorem 2.5.6. *The invariant eigenvalues from the equation* $\det\left[T_{(a)(b)}(x_0) - \lambda\, d_{(a)(b)}\right] = 0$ *are furnished by four real numbers:*

$$\begin{aligned}
&\lambda = \lambda_0, -\lambda_0, \lambda_0, -\lambda_0\,; \\
&\lambda_0 = (1/4)\left\{\left[F_{(a)(b)}(x_0) F^{(a)(b)}(x_0)\right]^2 + \left[F_{(a)(b)}(x_0)\, {}^{*}F^{(a)(b)}(x_0)\right]^2\right\}^{1/2}.
\end{aligned} \tag{2.309}$$

Proof. The determinantal equation for eigenvalues splits into two equations

$$\det\begin{bmatrix} T_{(1)(1)} - \lambda & T_{(1)(2)} \\ T_{(1)(2)} & T_{(2)(2)} - \lambda \end{bmatrix} = 0 \tag{2.310i}$$

and

$$\det\begin{bmatrix} T_{(3)(3)} - \lambda & T_{(3)(4)} \\ T_{(3)(4)} & T_{(4)(4)} + \lambda \end{bmatrix} = 0. \tag{2.310ii}$$

Equations (2.310i,ii) yield, respectively,

$$\lambda^2 = \left[T_{(1)(1)}\right]^2 + \left[T_{(1)(2)}\right]^2, \tag{2.311i}$$

$$\lambda^2 = \left[T_{(4)(4)}\right]^2 - \left[T_{(3)(4)}\right]^2. \tag{2.311ii}$$

By (2.308) and (2.307i,ii), the right-hand sides of (2.311i) and (2.311ii) *coincide* so that

$$\begin{aligned} 4\lambda^2 &= \left[\left(E_{(1)}\right)^2 + \left(E_{(2)}\right)^2 - \left(H_{(1)}\right)^2 - \left(H_{(2)}\right)^2\right]^2 + 4\left[E_{(1)}H_{(1)} + E_{(2)}H_{(2)}\right]^2 \\ &= (1/4)\cdot\left\{\left[F_{(a)(b)}(x_0)F^{(a)(b)}(x_0)\right]^2 + \left[F_{(a)(b)}(x_0)\,{}^{*}F^{(a)(b)}(x_0)\right]^2\right\}. \end{aligned}$$

Since the right-hand side of the equation is *an invariant*, the equation is valid in general for any other orthonormal tetrad. Thus, (2.309) is proved. ■

Remarks: (i) The proof for the similar theorem in *flat space–time* is exactly the same.
(ii) For $\lambda^2 > 0$, the Segre characteristic of $\left[T_{(a)(b)}(x_0)\right]$ is $[(1,1),(1,1)]$.

We define a *null electromagnetic field* by the condition $\lambda = 0$, which implies from (2.309) that

$$F_{(a)(b)}(x_0)F^{(a)(b)}(x_0) = 0,$$

and,

$$F_{(a)(b)}(x_0)\,{}^{*}F^{(a)(b)}(x_0) = 0. \tag{2.312}$$

(The Segre characteristic is $[(1,1,2)]$.)

In terms of electric and magnetic field vectors, (2.5) yields from (2.304i,ii)

$$\begin{aligned} \|\vec{\mathbf{E}}(x_0)\| &= \|\vec{\mathbf{H}}(x_0)\|, \\ \vec{\mathbf{E}}(x_0)\cdot\vec{\mathbf{H}}(x_0) &= 0. \end{aligned} \tag{2.313}$$

Therefore, a null electromagnetic field physically represents the analog of *a plane electromagnetic wave* in flat space–time.

Exercises 2.5

1. Using (2.256), (2.298), and the junction condition $\rho U^i n_i{}_{|..} = 0$, prove the existence of the conserved, total (proper) mass

$$M := -\int_{\Sigma} \left[\rho(x)\,U^i(x)\,n_i(x)\right]_{|..}\cdot \mathrm{d}^3 v.$$

2. Consider a perfect fluid with the equation of state: $\rho = \mathcal{R}(p)$, $\nu(p) := \int dp \,/\left[p + \mathcal{R}(p)\right]$, $\mu(x) := \nu\left[p(x)\right]$. Introduce a conformal transformation

$\overline{g}_{ij}(x) = e^{2\mu(x)} \cdot g_{ij}(x)$, $W_i(x) := e^{\mu} \cdot U_i(x)$, $\overline{W}^i(x) = e^{-\mu} \cdot U^i(x)$. Prove that the equations of motion (2.261iii) reduce to

$$(\rho + p) \cdot U^j \cdot \overline{\nabla}_j U^i = U^i \cdot U^j \cdot \overline{\nabla}_j p.$$

3. Consider a perfect fluid whose energy–momentum–stress tensor is given by (2.257i) which obeys the conservation law (2.257iii).
   (i) Consider the projection of the conservation law in the direction of the fluid 4-velocity (i.e., $\mathcal{T}^i U_i = 0$). Show that this equation yields the relativistic continuity equation for a perfect fluid:

   $$\nabla_i \left(\rho U^i\right) + p \nabla_i U^i = 0.$$

   (ii) Consider now a projection which is orthogonal to the fluid 4-velocity. By projecting the conservation law in an orthogonal direction (i.e., $\mathcal{T}^i \mathcal{P}_{ij} = 0$, with $\mathcal{P}_{ij}$ being the perpendicular projection tensor of (2.198)), show that this implies the relativistic Euler equations for a perfect fluid:

   $$(\rho + p) U^i \nabla_i U_j + \nabla_j p + U^i U_j \nabla_i p = 0.$$

4. A relativistic fluid with the *heat flow vector* $\vec{\mathbf{q}}(x)$ is characterized by:

   $$T^{ij}(\cdot\cdot) := (\rho + p) U^i U^j + p g^{ij} + U^i q^j + U^j q^i + \pi^{ij},$$
   $$q_i U^i = 0, \quad \pi_{ij} U^j = 0, \quad \pi^i{}_i = 0, \quad \pi_{ij} = \pi_{ji}.$$

   Prove that the constitutive equations are furnished by
   (i) $\dot{\rho} + (\rho + p)\,\Theta + \nabla_i q^i - U_i \dot{q}^i + \pi^{ij} \sigma_{ij} = 0,$
   (ii) $(\rho + p)\dot{U}^i + \left(\delta^i{}_j + U^i U_j\right) \cdot \left(\nabla^j p + \dot{q}^j + \nabla_k \pi^{kj}\right)$
   $+ q_j \left(\sigma^{ij} + \omega^{ij}\right) + (4/3)\,\Theta\, q^i = 0.$

5. Consider a deformable solid body in general relativity characterized by

   $$T^{ij}(\cdot\cdot) := \rho U^i U^j - S^{ij} \equiv \rho U^i U^j - \sum_{\mu} \sigma_{(\mu)}\, e^i_{(\mu)}\, e^j_{(\mu)},$$
   $$S_{ij} U^j = 0.$$

   Prove that the constitutive equations are provided by
   (i) $\dot{\rho} = -\rho\,\Theta + S^{ij}\left[\sigma_{ij} + (1/3)\,\Theta\, g_{ij}\right],$
   (ii) $\rho \dot{U}^i = \nabla_j S^{ij} - U^i S^{jk}\left[\sigma_{jk} + (1/3)\,\Theta\, g_{jk}\right].$

6. Maxwell's equations and the Lorentz gauge condition are given in (2.283) and (2.282). Let *the background metric* satisfy $G_{ij}(x) = \Lambda g_{ij}(x)$. (Here, Λ is the nonzero cosmological constant.) Show that the electromagnetic 4-potential $\vec{\mathcal{A}}(x)$ satisfies a massive vector-boson equation

$$\Box A^i - \Lambda A^i(x) = 0.$$

7. Consider electromagneto-vac equations (2.290i–iv) with the Lorentz gauge condition (2.282). Derive that the 4-potential vector $\vec{\mathcal{A}}(x)$ satisfies gauge-field type of equations

$$\Box A^i - \kappa \left[F^{ik} F_{jk} - (1/4)\, \delta^i{}_j\, F_{kl} F^{kl} \right] A^j = 0.$$

8. The Hodge-dual operation in Example 1.3.6 yielded

$${}^*F^{ij}(x) = (1/2)\, \eta^{ijkl}(x) \cdot F_{kl}(x).$$

Define a complex-valued, antisymmetric field by

$$\varphi^{kj}(x) := F^{kj}(x) - i\; {}^*F^{kj}(x).$$

(i) Deduce that Maxwell's equations (2.277i,ii) reduce to $\nabla_j \varphi^{kj} = 0$.
(ii) Prove that the energy–momentum–stress tensor (2.290i) yields $T^j_{\ k}(x) = (1/2) \cdot \mathrm{Re}[\overline{\varphi}^{jl}(x) \cdot \varphi_{kl}(x)]$. (Here, the bar denotes *complex-conjugation*.)

9. A global, *electromagnetic duality rotation* is furnished by

$$\widehat{\varphi}^{kj}(x) = \mathrm{e}^{i\alpha} \cdot \varphi^{kj}(x).$$

Here, α is *a real constant*. Show that electromagneto-vac equations (2.290ii,iii) are covariant and (2.290iv) are invariant under the duality rotation.

10. Prove that in terms of orthonormal or physical components, field equations (2.296v) for an incoherent charged dust are equivalent to

$$\begin{aligned} R^{(d)}{}_{(a)(b)(c)}(x) &= C^{(d)}{}_{(a)(b)(c)}(x) + (\kappa/2)\Big[\delta^{(d)}{}_{(b)} T_{(a)(c)} - \delta^{(d)}{}_{(c)} T_{(a)(b)} \\ &\quad + d_{(a)(c)} T^{(d)}{}_{(b)} - d_{(a)(b)} T^{(d)}{}_{(c)} + (2/3)\rho(x) \cdot \left(\delta^{(d)}{}_{(b)} d_{(a)(c)} - \delta^{(d)}{}_{(c)} d_{(a)(b)}\right)\Big]. \end{aligned}$$

11. Consider a null electromagnetic field given by (2.3.12).

(i) Prove that the corresponding stress–energy–momentum tensor can be written as

$$T_{(a)(b)}(x_0) = \mu(x_0)\, \nu_{(a)}(x_0)\, \nu_{(b)}(x_0),$$

$$\text{with:} \qquad \mu(x_0) > 0, \quad \nu^{(a)}(x_0)\, \nu_{(a)}(x_0) = 0.$$

(ii) Show that the 4×4 matrix $\left[T_{(a)(b)}(x_0)\right]$ belongs to the Segre characteristic $[(1, 1), 2]$.

Answers and Hints to Selected Exercises

1. See Fig. 2.19 and use (2.216).
2.
$$\partial_i \mu = \partial_i p \big/ [p + \mathcal{R}(p)]\,;$$
$$\overline{\begin{Bmatrix} k \\ i\ j \end{Bmatrix}} = \begin{Bmatrix} k \\ i\ j \end{Bmatrix} + \delta^k{}_i \cdot \partial_j \mu + \delta^k{}_j \cdot \partial_i \mu - g_{ij}\, g^{kl} \cdot \partial_l \mu.$$
3. Note that $U^i U_i = -1$ and that $U^i \nabla_j U_i = 0$ (show this).
4. (i) Use (2.199i–v) and the consequent equation (2.201) as

$$\nabla_k U_j = \omega_{jk} + \sigma_{jk} + \frac{1}{3} \Theta \mathcal{P}_{jk} - \dot{U}_j U_k, \ \Theta = \nabla_k U^k.$$

Moreover, apply the following equations:

$$\nabla_j \left(U_i\, \pi^{ij}\right) = 0,$$
$$\begin{aligned} \text{or,} \qquad U_i \nabla_j \pi^{ij} &= 0 - \pi^{ij} \cdot \nabla_j U_i \\ &= -\pi^{ij}\left[\sigma_{ij} + \omega_{ij} + (1/3)\, \Theta\, \mathcal{P}_{ij} - \dot{U}_i U_j\right] \\ &= -\pi^{ij} \sigma_{ij} - 0 - 0 - 0. \end{aligned}$$

5. (i) Use (2.211). Moreoever,

$$0 = \nabla_k \left[S^{jk} U_j\right],$$
$$\begin{aligned} \text{or,} \quad U_j \nabla_k S^{jk} &= -S^{jk} \cdot \nabla_k U_j = -S^{jk}\left[\omega_{jk} + \sigma_{jk} + (1/3)\, \Theta\, \mathcal{P}_{jk} - \dot{U}_j U_k\right] \\ &= -S^{jk}\left[0 + \sigma_{jk} + (1/3)\, \Theta\, g_{jk} - 0\right]. \end{aligned}$$

7. Use (2.283) and $R^i{}_j(x) = -\kappa\, T^i{}_j(x)$.

8. (ii)

$$
\begin{aligned}
&- \eta^{abjl}(x) \cdot \eta_{cdkl}(x) \\
&= \delta^a_{\ c} \left(\delta^b_{\ d}\, \delta^j_{\ k} - \delta^b_{\ k}\, \delta^j_{\ d}\right) + \delta^a_{\ d} \left(\delta^b_{\ k}\, \delta^j_{\ c} - \delta^b_{\ c}\, \delta^j_{\ k}\right) \\
&\quad + \delta^a_{\ k} \left(\delta^b_{\ c}\, \delta^j_{\ d} - \delta^b_{\ d}\, \delta^j_{\ c}\right).
\end{aligned}
$$

10. Use field equations (2.163ii) and (2.296v).
11. (i) Choose a special orthonormal tetrad such that

$$E_{(2)}(x_0) = E_{(3)}(x_0) = H_{(1)}(x_0) = H_{(3)}(x_0) = 0,$$

$$\mu(x) := (E_{(1)}(x_0))^2 \;=\; (H_{(2)}(x_0))^2 \;> 0.\text{Suppose} \quad E_{(1)}(x_0) = +H_{(2)}(x_0).$$

Eigenvalue equations $T_{(a)(b)}(x_0)\, \nu^{(b)}(x_0) = 0$ reduce, from (2.308), to *one independent equation* $\nu^{(3)} - \nu^{(4)} = 0$.

The null eigenvector can be chosen as $\vec{\boldsymbol{\nu}} = [\delta^i_{\ (3)} + \delta^i_{\ (4)}] \cdot \frac{\partial}{\partial x^i}$. In terms of its covariant components, $T_{(a)(b)}(x_0) = \mu(x_0)\, \nu_{(a)}(x_0)\, \nu_{(b)}(x_0)$.

Chapter 3
Spherically Symmetric Space–Time Domains

3.1 Schwarzschild Solution

The first exact nontrivial solution of Einstein's (vacuum) field equations (2.160i) was obtained by Schwarzschild [231]. This solution turned out to be very important in regard to experimental verifications of the theory of general relativity. Schwarzschild investigated a metric similar to

$$\begin{aligned}
\mathbf{g}_{..}(x) &= \mathrm{e}^{\alpha(x^1)}\,\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left(x^1\right)^2 \left[\mathrm{d}x^2 \otimes \mathrm{d}x^2 + \left(\sin x^2\right)^2 \mathrm{d}x^3 \otimes \mathrm{d}x^3\right] \\
&\quad - \mathrm{e}^{\gamma(x^1)}\,\mathrm{d}x^4 \otimes \mathrm{d}x^4, \\
\mathrm{d}s^2 &= \mathrm{e}^{\alpha(r)}\,(\mathrm{d}r)^2 + r^2\left[(\mathrm{d}\theta)^2 + (\sin\theta)^2(\mathrm{d}\varphi)^2\right] \\
&\quad - \mathrm{e}^{\gamma(r)}(\mathrm{d}t)^2.
\end{aligned} \tag{3.1}$$

The above metric incorporates *both spherical symmetry and the condition of staticity*. With this metric, two unknown functions $\alpha(r)$ and $\gamma(r)$ (of class C^3) are required to satisfy the vacuum equations $R_{ij}(x) = 0$.

The nonzero components of the Christoffel symbols $\left\{ {i \atop j\;k} \right\}$ are computed from (3.1) and (1.134) as:

$$\begin{aligned}
&\left\{ {1 \atop 11} \right\} = (1/2)\cdot\partial_1\alpha\,, \quad \left\{ {2 \atop 12} \right\} = (r)^{-1}, \quad \left\{ {1 \atop 22} \right\} = -r\mathrm{e}^{-\alpha}\,, \\
&\left\{ {3 \atop 13} \right\} = (r)^{-1}\,, \quad \left\{ {1 \atop 33} \right\} = -r\sin^2\theta\cdot\mathrm{e}^{-\alpha}\,, \\
&\left\{ {3 \atop 23} \right\} = \cot\theta, \quad \left\{ {2 \atop 33} \right\} = -\sin\theta\cdot\cos\theta, \\
&\left\{ {4 \atop 14} \right\} = (1/2)\cdot\partial_1\gamma, \quad \left\{ {1 \atop 44} \right\} = (1/2)\cdot\mathrm{e}^{\gamma-\alpha}\cdot\partial_1\gamma\,.
\end{aligned} \tag{3.2}$$

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The vacuum equations (2.16i), by (3.2) and (1.147ii) reduce to the following system of ordinary differential equations:

$$R_{11}(\cdot\cdot) = (1/2)\cdot\partial_1\partial_1\gamma - (1/4)\cdot\partial_1\alpha\cdot\partial_1\gamma + (1/4)\cdot(\partial_1\gamma)^2 - \big(\partial_1\alpha/r\big) = 0, \tag{3.3i}$$

$$R_{22}(\cdot\cdot) = \mathrm{e}^{-\alpha}\cdot\big[1 + (r/2)\cdot(\partial_1\gamma - \partial_1\alpha)\big] - 1 = 0, \tag{3.3ii}$$

$$R_{33}(\cdot\cdot) \equiv \sin^2\theta\cdot R_{22}(\cdot\cdot) = 0, \tag{3.3iii}$$

$$\begin{aligned} R_{44}(\cdot\cdot) = \mathrm{e}^{\gamma-\alpha}\cdot\big[&- (1/2)\cdot\partial_1\partial_1\gamma + (1/4)\cdot\partial_1\alpha\cdot\partial_1\gamma \\ &- (1/4)\cdot(\partial_1\gamma)^2 - \big(\partial_1\gamma/r\big)\big] = 0\,. \end{aligned} \tag{3.3iv}$$

Computing $R_{11}(\cdot\cdot) + \mathrm{e}^{\alpha-\gamma}\cdot R_{44}(\cdot\cdot) = 0$ from above, we obtain

$$(1/r)\cdot\big[\partial_1(\alpha+\gamma)\big] = 0\,,$$

$$\text{or} \qquad \alpha(r) + \gamma(r) = k = \text{const.} \tag{3.4}$$

Here, k is *the arbitrary constant of integration*. Substituting (3.4) into (3.3ii), we arrive at the nonlinear differential equation

$$\mathrm{e}^{-\alpha}\cdot[1 - r\cdot\partial_1\alpha] = 1\,. \tag{3.5}$$

The above equation is *a disguised linear equation* (as mentioned in Appendix 2). It can be expressed as *the linear differential equation*

$$r\cdot\partial_1(\mathrm{e}^{-\alpha}) + (\mathrm{e}^{-\alpha}) = 1\,. \tag{3.6}$$

Integrating the above, we obtain

$$\begin{aligned} \mathrm{e}^{-\alpha(r)} &= \left(1 - \frac{2m}{r}\right) > 0\,, \\ \mathrm{e}^{\gamma(r)} &= \left(1 - \frac{2m}{r}\right)\cdot e^k > 0\,. \end{aligned} \tag{3.7}$$

Here, m is *another arbitrary constant* of integration. (From physical considerations, we choose $m > 0$.) Equation (3.7) *automatically solves (3.3i) and (3.3iv)*. Thus, the metric (3.1) can be specified as

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1}(\mathrm{d}r)^2 + r^2\big[(\mathrm{d}\theta)^2 + (\sin\theta)^2(\mathrm{d}\varphi)^2\big] - \left(1 - \frac{2m}{r}\right)\cdot e^k\cdot(\mathrm{d}t)^2\,. \tag{3.8}$$

By a coordinate transformation

$$\widehat{r} = r\,, \quad \widehat{\theta} = \theta\,, \quad \widehat{\varphi} = \varphi\,, \quad \widehat{t} = e^{k/2} \cdot t,$$

and *dropping hats* in the sequel, we obtain the famous *Schwarzschild metric* as[1]

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)^{-1} (\mathrm{d}r)^2 + r^2\left[(\mathrm{d}\theta)^2 + (\sin\theta)^2(\mathrm{d}\varphi)^2\right] - \left(1 - \frac{2m}{r}\right)(\mathrm{d}t)^2. \quad (3.9)$$

The space–time domain of validity for (3.9) corresponds to

$$D := \left\{(r, \theta, \varphi, t) : 0 < 2m < r < \infty,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ -\infty < t < \infty\right\}. \quad (3.10)$$

Example 3.1.1. The natural orthonormal tetrad corresponding to (3.9) is furnished by

$$\begin{aligned}
\vec{\mathbf{e}}_{(1)}(\cdot\cdot) &= \left(1 - \frac{2m}{r}\right)^{1/2} \cdot \frac{\partial}{\partial r}\,, \\
\vec{\mathbf{e}}_{(2)}(\cdot\cdot) &= (r)^{-1} \cdot \frac{\partial}{\partial \theta}\,, \\
\vec{\mathbf{e}}_{(3)}(\cdot\cdot) &= (r\sin\theta)^{-1} \cdot \frac{\partial}{\partial \varphi}\,, \\
\vec{\mathbf{e}}_{(4)}(\cdot\cdot) &= \left(1 - \frac{2m}{r}\right)^{-1/2} \cdot \frac{\partial}{\partial t}\,. \quad (3.11)
\end{aligned}$$

The corresponding (nonzero) orthonormal or physical components of the curvature tensor of (1.141iv) are provided by

$$\begin{aligned}
R_{(2)(3)(2)(3)}(\cdot\cdot) &= -R_{(1)(4)(1)(4)}(\cdot\cdot) = \frac{2m}{r^3}\,, \\
R_{(2)(4)(2)(4)}(\cdot\cdot) &= R_{(3)(4)(3)(4)}(\cdot\cdot) = -R_{(1)(2)(1)(2)}(\cdot\cdot) = -R_{(1)(3)(1)(3)} = \frac{m}{r^3}\,. \quad (3.12)
\end{aligned}$$

We note here several properties from the above equations. First, that $\lim\limits_{m \to 0_+} R_{(a)(b)(c)(d)}(\cdot\cdot) \equiv 0$. Therefore, the mass, m, of the central body is solely responsible for the nonzero curvature of the surrounding space–time domain. Secondly, we conclude that $\lim\limits_{r \to \infty} R_{(a)(b)(c)(d)}(\cdot\cdot) = 0$. Thus, the Schwarzschild universe *is asymptotically flat* in some sense. □

Remark: The asymptotic symmetry group is not the Poincaré group of flat Minkowski space–time but instead a larger group known as the *Bondi–Metzner–Sachs group* which preserves *the asymptotic structure* of the metric [25, 223].

[1]The line element in this form is found in (14) of Schwarzschild's original paper [231].

We observe that the physical components $R_{(a)(b)(c)(d)}(\cdot\cdot)$ are also defined and real analytic in the interval $0 < r \leq 2m$. *This region is outside the boundary of* D in (3.10). (The boundary $r = 2m$ is called the *Schwarzschild radius*, a *coordinate singularity*.) This puzzling situation will be properly explained in Chap. 5 on black holes.

Now we shall elucidate the exact interpretation of the symbol $m > 0$ occurring in (3.9). We have *already* used the symbol $m > 0$ in (2.19i,ii), (2.25), (2.30), (2.31i,ii), (2.32), (2.33), (2.67), (2.131), (2.141i), etc. In all these equations, we have used $m > 0$ to denote *the mass of a test particle* following a certain space–time trajectory (or world line). However, in the Schwarzschild metric in (3.9), the symbol $m > 0$ stands for *the central mass* causing the gravitational field around it. In case we reinstate temporarily the speed of light c and the Newtonian constant of gravitation G, the Schwarzschild mass $m = GM/c^2$. Here, M is the total central mass in these units.

Example 3.1.2. In this example, we shall explore briefly *the non-Euclidean geometry of the two-dimensional spatial submanifold* M_2 of the Schwarzschild space–time. Let the submanifold be furnished by

$$\begin{aligned}
r &= x^1 = \xi^1(u) := u^1\,,\\
\theta &= x^2 = \xi^2(u) := \pi/2 = \text{const.}\,,\\
\varphi &= x^3 = \xi^3(u) := u^2\,,\\
t &= x^4 = \xi^4(u) := t_0 = \text{const.}
\end{aligned}$$

(Compare these with (1.252).) The corresponding metric from (3.9) is provided by

$$\begin{aligned}
\overline{\mathbf{g}}_{..}(u^1,u^2) &= \left(1-\frac{2m}{u^1}\right)^{-1}\cdot \mathrm{d}u^1\otimes \mathrm{d}u^1 + \left(u^1\right)^2\cdot \mathrm{d}u^2\otimes \mathrm{d}u^2\,,\\
\mathrm{d}l^2 &= \left(1-\frac{2m}{r}\right)^{-1}(\mathrm{d}r)^2 + r^2(\mathrm{d}\varphi)^2\,,\\
D_{(2)} &:= \left\{(r,\varphi)\,:\, 0<2m<r<\infty,\ -\pi<\varphi<\pi\right\}.
\end{aligned} \tag{3.13}$$

By (2.92) and (3.13), the ratio

$$\begin{aligned}
\frac{\text{“Circumference of a circle”}}{\text{“Radial distance”}} &= \left[\int_{-\pi}^{\pi} r\,\mathrm{d}\varphi\right]\Big/\left[\int_{(2m)_+}^{r}(1-2m/w)^{-1/2}\cdot \mathrm{d}w\right]\\
&= (2\pi)\Big/\left[\sqrt{1-(2m/r)} + (m/r)\cdot\ln|r/2m|\right].
\end{aligned} \tag{3.14}$$

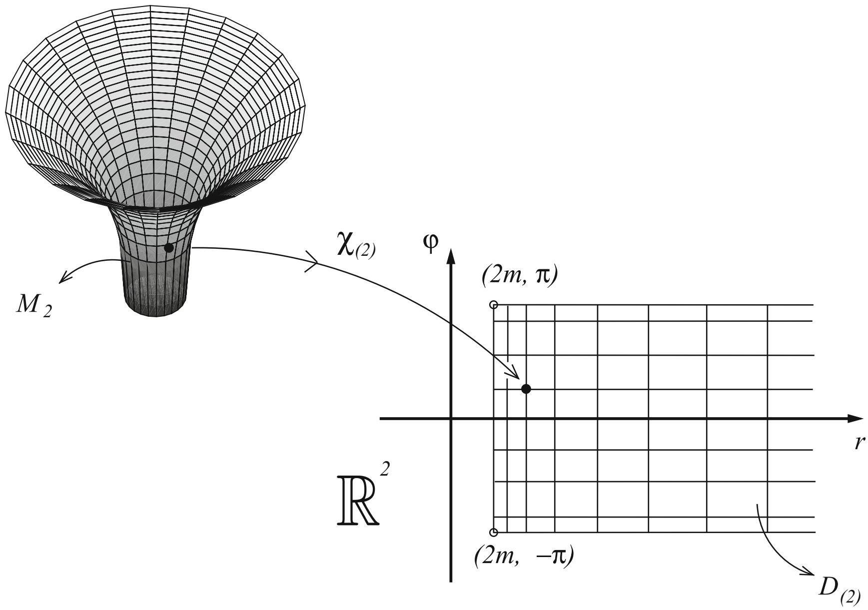


Fig. 3.1 Two-dimensional submanifold M_2 of the Schwarzschild space–time. The surface representing M_2 here is known as *Flamm's paraboloid* [102]

Note that the above ratio is a positive-valued, monotonically decreasing function of r. Moreover, *the ratio diverges in the limit* $r \to (2m)_+$. Furthermore, in the limit $r \to \infty$, the ratio goes to 2π, *the Euclidean value*.

Note that $\frac{\partial}{\partial\varphi}$ is *a Killing vector* for the metric in (3.13). With all this information at hand, we construct a qualitative picture for M_2 in Fig. 3.1.

(Compare Figs. 3.1 and 2.8.) In the metric in (3.13), radial coordinate lines are *geodesics*. Since these geodesics *cannot be continued* for the interval $0 < r \leq 2m$, we conclude that the submanifold M_2 and the Schwarzschild space–time are *geodesically incomplete*. □

Now we shall investigate timelike geodesics of the Schwarzschild space–time. These represent world lines of inertial test particles in the spherically symmetric gravitational field. We use *the proper time parameter* s as given in (1.186). We utilize the "squared Lagrangian" in (2.155) to express

$$\begin{aligned} L_{(2)}(\cdot\cdot) &= (1/2)\cdot g_{ij}(x)\, u^i u^j \\ &= (1/2)\left\{\left(1-\frac{2m}{r}\right)^{-1}\cdot (u^r)^2 + r^2\left[\left(u^\theta\right)^2 + \sin^2\theta\cdot(u^\varphi)^2\right]\right. \\ &\qquad \left. -\left(1-\frac{2m}{r}\right)\cdot\left(u^t\right)^2\right\}. \end{aligned} \tag{3.15}$$

The corresponding Euler–Lagrange equations in (1.187) and (A1.8) yield geodesic equations as

$$\frac{d^2\mathcal{R}(s)}{ds^2} - \frac{m}{\mathcal{R}(s)} \cdot \left[1 - \frac{2m}{\mathcal{R}(s)}\right]^{-1} \cdot \left(\frac{d\mathcal{R}(s)}{ds}\right)^2 - [\mathcal{R}(s) - 2m] \cdot \left\{ \left(\frac{d\Theta(s)}{ds}\right)^2 \right.$$

$$\left. + (\sin\Theta(s))^2 \cdot \left(\frac{d\Phi(s)}{ds}\right)^2 \right\} + \frac{m}{(\mathcal{R}(s))^2} \cdot \left[1 - \frac{2m}{\mathcal{R}(s)}\right] \left(\frac{d\mathcal{T}(s)}{ds}\right)^2 = 0\,, \tag{3.16i}$$

$$\frac{d^2\Theta(s)}{ds^2} + \frac{2}{\mathcal{R}(s)} \cdot \frac{d\mathcal{R}(s)}{ds} \cdot \frac{d\Theta(s)}{ds} - \sin\Theta(s) \cdot \cos\Theta(s) \cdot \left(\frac{d\Phi(s)}{ds}\right)^2 = 0\,, \tag{3.16ii}$$

$$\frac{d^2\Phi(s)}{ds^2} + \frac{2}{\mathcal{R}(s)} \cdot \frac{d\mathcal{R}(s)}{ds} \cdot \frac{d\Phi(s)}{ds} + 2\cot\Theta(s) \cdot \frac{d\Theta(s)}{ds} \cdot \frac{d\Phi(s)}{ds} = 0\,, \tag{3.16iii}$$

$$\frac{d^2\mathcal{T}(s)}{ds^2} + \frac{2m}{(\mathcal{R}(s))^2\,[1 - 2m/\mathcal{R}(s)]} \cdot \frac{d\mathcal{R}(s)}{ds} \cdot \frac{d\mathcal{T}(s)}{ds} = 0\,. \tag{3.16iv}$$

The above is a complicated system of semilinear, second-order, ordinary differential equations. Fortunately, the variables φ and t are *cyclic or ignorable*. Therefore, (3.16iii,iv) admit two first integrals:

$$(r\sin\theta)^2_{|..} \cdot \frac{d\Phi(s)}{ds} = h = \text{const.}\,, \tag{3.17i}$$

$$\left(1 - \frac{2m}{r}\right)_{|..} \cdot \frac{d\mathcal{T}(s)}{ds} = E = \text{const.} \tag{3.17ii}$$

The physical meaning of (3.17i) is that *the areal velocity h is conserved*. (See Example 1.3.36.) Equation (3.17ii) implies that *the total energy E of the test particle is conserved*. Such conservations of canonical momenta lead to a reduction of the original Lagrangian. We shall state *Routh's theorem* on this topic in an N-dimensional manifold.

Theorem 3.1.3. *Let the Lagrangian function* $L(x;u)$ *of* $2N$ *variables be of class* C^2 *and possess the property* $\det\left[\frac{\partial^2 L(\cdot\cdot)}{\partial u^i \partial u^j}\right] \neq 0$. *Moreover,* $L(\cdot\cdot)$ *be endowed with a cyclic (or an ignorable) variable* x^N *such that* $\frac{\partial L(\cdot\cdot)}{\partial x^N} \equiv 0$, $\frac{\partial L(\cdot\cdot)}{\partial u^N}{}_{|..} = c_{(N)} = \text{const.}$ *Then, the reduced Lagrangian* $\overline{L}(\cdot\cdot) := \left[L(\cdot\cdot) - c_{(N)}u^N\right]_{|..}$ *implies that* $\frac{\partial \overline{L}(\cdot\cdot)}{\partial x^N} = \frac{\partial \overline{L}(\cdot\cdot)}{\partial u^N} \equiv 0$. *Furthermore,* $N-1$ *Euler–Lagrange equations from* $\overline{L}(\cdot\cdot)$, *together with* $\frac{\partial L(\cdot\cdot)}{\partial u^N} = c_{(N)}$, *are implied by the original* N *Euler-Lagrangian equations.*

For the proof, we suggest [159]. As an example of Routh's procedure, we apply the theorem to the Lagrangian (3.15) and conservation equations (3.17i,ii). The corresponding twice reduced Lagrangian is furnished by

$$\overline{L}_{(2)}\left(r,\theta;u^r,u^\theta\right)=\frac{1}{2}\cdot\left\{\left(1-\frac{2m}{r}\right)^{-1}\cdot\ (u^r)^2+r^2\left(u^\theta\right)^2\right.$$
$$\left.-\frac{h^2}{(r\sin\theta)^2}+\frac{E^2}{\left(1-\frac{2m}{r}\right)}\right\}. \tag{3.18}$$

The Euler–Lagrange equation for the θ-coordinate from (3.18) yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[(\mathcal{R}(s))^2\cdot\frac{\mathrm{d}\Theta(s)}{\mathrm{d}s}\right]-\frac{h^2}{(\mathcal{R}(s))^2}\cdot\left[\frac{\cos\Theta(s)}{\sin^3\Theta(s)}\right]=0\,. \tag{3.19}$$

(The above equation is equivalent to (3.16ii).)

We notice that there exists *a special solution* of (3.19), namely,

$$\Theta(s)=\pi/2=\text{const.}\,,$$
$$\frac{\mathrm{d}\Theta(s)}{\mathrm{d}s}\equiv 0. \tag{3.20}$$

Physically, (3.20) means that the test particle is pursuing an "equatorial orbit." In such a case, the Euler–Lagrange equation for the r-coordinate from (3.18) implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left[\left(1-\frac{2m}{r}\right)^{-1}\cdot\ u^r\right]_{|..}\right\}-\frac{1}{2}\cdot\left\{\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(1-\frac{2m}{r}\right)^{-1}\right]\cdot(u^r)^2\right.$$
$$\left.+\frac{2h^2}{r^3}+E^2\cdot\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(1-\frac{2m}{r}\right)^{-1}\right]\right\}_{|..}=0. \tag{3.21}$$

(We have substituted (3.20) into (3.21).)

Equation (3.21) yields the first integral:

$$\left\{\left(1-\frac{2m}{r}\right)^{-1}\cdot(u^r)^2+\frac{h^2}{r^2}-E^2\left(1-\frac{2m}{r}\right)^{-1}\right\}_{|..}\equiv -1\,. \tag{3.22}$$

The above equation also follows from (1.186).

Now we assume that the constant h in (3.17i) is *nonzero*, so that $\frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}\neq 0$. Therefore, we can reparametrize the curve by

$$r=\mathcal{R}(s)=\widehat{\mathcal{R}}(\varphi)\,,$$

$$\frac{\mathrm{d}\mathcal{R}(s)}{\mathrm{d}s}=\frac{\mathrm{d}\widehat{\mathcal{R}}(\varphi)}{\mathrm{d}\varphi}\cdot\frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}=\frac{h}{\left[\widehat{\mathcal{R}}(\varphi)\right]^2}\cdot\frac{\mathrm{d}\widehat{\mathcal{R}}(\varphi)}{\mathrm{d}\varphi}\,. \tag{3.23}$$

Then, (3.22) and (3.23) yield

$$\left\{\left[\widehat{\mathcal{R}}(\varphi)\right]^{-2}\cdot\frac{\mathrm{d}\widehat{\mathcal{R}}(\varphi)}{\mathrm{d}\varphi}\right\}^{2}+\left[\widehat{\mathcal{R}}(\varphi)\right]^{-2}$$
$$=h^{-2}\cdot\left\{E^{2}-1+2m\left[\widehat{\mathcal{R}}(\varphi)\right]^{-1}\right\}+2m\cdot\left[\widehat{\mathcal{R}}(\varphi)\right]^{-3}. \tag{3.24}$$

Now we make another coordinate transformation:

$$y=(r)^{-1}, \quad r>2m>0,$$
$$y=Y(\varphi),\ \ \frac{\mathrm{d}Y(\varphi)}{\mathrm{d}\varphi}=-\left[\widehat{\mathcal{R}}(\varphi)\right]^{-2}\cdot\frac{\mathrm{d}\widehat{\mathcal{R}}(\varphi)}{\mathrm{d}\varphi}\not\equiv 0. \tag{3.25}$$

Equation (3.24) and its derivative imply that

$$\left[\frac{\mathrm{d}Y(\varphi)}{\mathrm{d}\varphi}\right]^{2}+\left[Y(\varphi)\right]^{2}=h^{-2}\left[E^{2}-1+2m\,Y(\varphi)\right]+2m\cdot\left[Y(\varphi)\right]^{3}, \tag{3.26i}$$

and,
$$\frac{\mathrm{d}^{2}Y(\varphi)}{\mathrm{d}\varphi^{2}}+Y(\varphi)=\left(m/h^{2}\right)+3m\left[Y(\varphi)\right]^{2}. \tag{3.26ii}$$

We shall try to investigate the above for a class of solutions given by perturbative techniques. We express the series and consequent equations as

$$Y(\varphi)=Y_{(0)}(\varphi)+Y_{(1)}(\varphi)+\cdots, \tag{3.27i}$$

$$\frac{\mathrm{d}^{2}Y_{(0)}(\varphi)}{\mathrm{d}\varphi^{2}}+Y_{(0)}(\varphi)=\frac{m}{h^{2}}, \tag{3.27ii}$$
$$\frac{\mathrm{d}^{2}Y_{(1)}(\varphi)}{\mathrm{d}\varphi^{2}}+Y_{(1)}(\varphi)=3m\left[Y_{(0)}(\varphi)\right]^{2}, \tag{3.27iii}$$
$$\cdots\cdots\cdots\ .$$

The general solution of (3.27ii), which is the *Newtonian approximation*, is given by

$$Y_{(0)}(\varphi)=\left(m/h^{2}\right)\cdot\left[1+e\,\cos\left(\varphi-\tilde{\omega}\right)\right]. \tag{3.28}$$

Here, e and $\tilde{\omega}$ are two constants of integration representing the *eccentricity* and the angle of *perihelion* of the orbit. The orbit is circular, elliptic, parabolic, or hyperbolic according to $e=0,\ 0<e<1,\ e=1,\ 1<e$, respectively. We shall consider

only the elliptic case $0 < e < 1$ which corresponds to a planetary motion around the Sun. Putting (3.28) into (3.27iii), we obtain the differential equation:

$$\frac{\mathrm{d}^2 Y_{(1)}(\varphi)}{\mathrm{d}\varphi^2} + Y_{(1)}(\varphi) = \frac{3m^3}{h^4}\left[1 + 2e \cdot \cos(\varphi - \tilde{\omega}) + e^2 \cdot \cos^2(\varphi - \tilde{\omega})\right]. \quad (3.29)$$

The particular integral of the above equation is

$$Y_{(1)}(\varphi) = \frac{3m^3}{h^4}\left\{1 + e\varphi \cdot \sin(\varphi - \tilde{\omega}) + e^2\left[\frac{1}{2} - \frac{1}{6}\cos 2(\varphi - \tilde{\omega})\right]\right\}, \quad (3.30\text{i})$$

$$\begin{aligned} Y(\varphi) &= Y_{(0)}(\varphi) + Y_{(1)}(\varphi) + (\text{higher order}) \\ &= \frac{m}{h^2}\left\{1 + e\cos(\varphi - \tilde{\omega}) + \frac{3m^2}{h^2} e\varphi \cdot \sin(\varphi - \tilde{\omega})\right. \\ &\qquad \left. + \frac{3e^2m^2}{h^2}\left[\frac{1}{2} - \frac{1}{6}\cos 2(\varphi - \tilde{\omega})\right]\right\} + (\text{higher order}). \end{aligned} \quad (3.30\text{ii})$$

Out of the additional terms to the Newtonian approximation, the only term that can generate an observational effect is one proportional to e. Higher order terms are deemed to be ignorable. We can combine the second and the third term on the right hand side of (3.30ii) to express

$$\begin{aligned} Y(\varphi) &= \frac{m}{h^2}\left\{1 + \widehat{e}(\varphi) \cdot \left[\cos(\varphi - \tilde{\omega} - \delta\tilde{\omega}(\varphi))\right]\right\} + \cdots, \\ \widehat{e}(\varphi) &:= e \cdot \sqrt{1 + \left(\frac{3m^2\varphi}{h^2}\right)^2} = e + (\text{higher order}), \\ \delta\tilde{\omega}(\varphi) &:= \operatorname{Arctan}\left(\frac{3m^2\varphi}{h^2}\right) = \frac{3m^2}{h^2}\varphi + (\text{higher order}). \end{aligned} \quad (3.31)$$

The change of the eccentricity is just about unobservable. However, the *perihelion shift* per single revolution is detectable and furnished by (momentarily reinstating the speed of light, c)

$$\begin{aligned} \Delta\tilde{\omega} &:= \left(\frac{3m^2}{h^2}\right) \cdot (2\pi) = \frac{24\pi^3 a^2}{(1 - e^2)c^2 T_{(0)}^2}, \\ a &:= \text{semimajor axis of the orbit}, \\ e &:= \text{eccentricity of the orbit}, \\ c &:= \text{speed of light}, \\ T_{(0)} &:= \text{period of revolution (planet's year)}. \end{aligned} \quad (3.32)$$

(Consult the reference-84, p. 89)

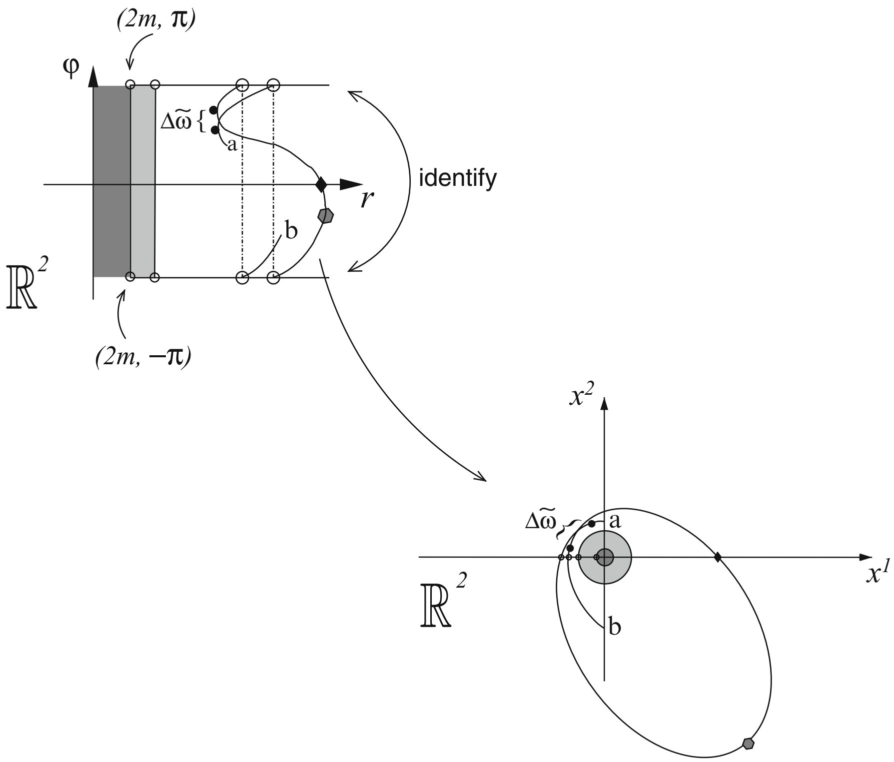


Fig. 3.2 Rosette motion of a planet and the perihelion shift

For the planet Mercury, the above equation amounts to $43 \cdot 03''$ per century, which was previously *inexplicable by Newtonian theory*. We make a coordinate transformation from polar to Cartesian (of Example 1.1.1) to visualize the planetary motion in Fig. 3.2.

Remark: Within the scope of the differential equation (3.26ii), there exist orbits which spiral around inward and eventually fall into the central body [35]. *Inspiralling orbits are simply nonexistent in the Newtonian theory of gravitation due to a central point mass*.

Now we shall explore null geodesics in the Schwarzschild space–time characterized by (3.9). We use *an affine parameter α for the null curve*. By changing from the parameter s to α, virtually all (3.16i–iv), (3.17i,ii), (3.18)–(3.21) can be *repeated*. However, (3.22) has to be *changed* to

$$\left[\left(1 - \frac{2m}{r}\right)^{-1} \cdot (u^r)^2 + \frac{h^2}{r^2} - E^2 \cdot \left(1 - \frac{2m}{r}\right)^{-1}\right]_{|..} = 0. \tag{3.33}$$

Therefore, (3.26i,ii) are modified to

$$\left[\frac{\mathrm{d}Y(\varphi)}{\mathrm{d}\varphi}\right]^2 + [Y(\varphi)]^2 = h^{-2}\cdot E^2 + 2m\cdot[Y(\varphi)]^3\,, \tag{3.34i}$$

$$\frac{\mathrm{d}^2Y(\varphi)}{\mathrm{d}\varphi^2} + Y(\varphi) = 3m\cdot[Y(\varphi)]^2\,. \tag{3.34ii}$$

We solve the equations by the same perturbative method, namely,

$$Y(\varphi) = Y_{(0)}(\varphi) + Y_{(1)}(\varphi) + \cdots\,, \tag{3.35i}$$

$$\frac{\mathrm{d}^2Y_{(0)}(\varphi)}{\mathrm{d}\varphi^2} + Y_{(0)}(\varphi) = 0\,, \tag{3.35ii}$$

$$\frac{\mathrm{d}^2Y_{(1)}(\varphi)}{\mathrm{d}\varphi^2} + Y_{(1)}(\varphi) = 3m\cdot\left[Y_{(0)}(\varphi)\right]^2. \tag{3.35iii}$$

The general solution of (3.35ii) is given by

$$Y_{(0)}(\varphi) = [R_0]^{-1}\cdot\cos(\varphi-\varphi_0)\,. \tag{3.36}$$

Here, $R_0 > 0$ and φ_0 are constants of integration. For the sake of simplicity, we set $\varphi_0 = 0$ so that

$$Y_{(0)}(\varphi) = [R_0]^{-1}\cdot\cos\varphi. \tag{3.37}$$

With the equation above, the particular integral (3.35iii) is furnished by

$$Y_{(1)}(\varphi) = m\,[R_0]^{-2}\cdot\left(\cos^2\varphi + 2\sin^2\varphi\right).$$

Therefore,

$$\begin{aligned} Y(\varphi) &= Y_{(0)}(\varphi) + Y_{(1)}(\varphi) + \cdots\,, \\ &= [R_0]^{-1}\cdot\cos\varphi + m\,[R_0]^{-2}\left(\cos^2\varphi + 2\sin^2\varphi\right) + \cdots\,, \end{aligned} \tag{3.38i}$$

or $$(rR_0)\cdot Y(\varphi) = r\cos\varphi + m[R_0]^{-1}\cdot\cos\varphi\cdot\left(r\cos^2\varphi + 2r\sin^2\varphi\right) + \cdots\,, \tag{3.38ii}$$

or $$\begin{aligned} x^1 = r\cos\varphi &= [rY(\varphi)]\cdot R_0 - m[R_0]^{-1}\cdot\left(r\cos^2\varphi + 2r\sin^2\varphi\right) + \cdots \\ &= R_0 - m\,[R_0]^{-1}\cdot\left[\frac{\left(x^1\right)^2 + 2\left(x^2\right)^2}{\sqrt{\left(x^1\right)^2 + \left(x^2\right)^2}}\right] + \cdots\,. \end{aligned} \tag{3.38iii}$$

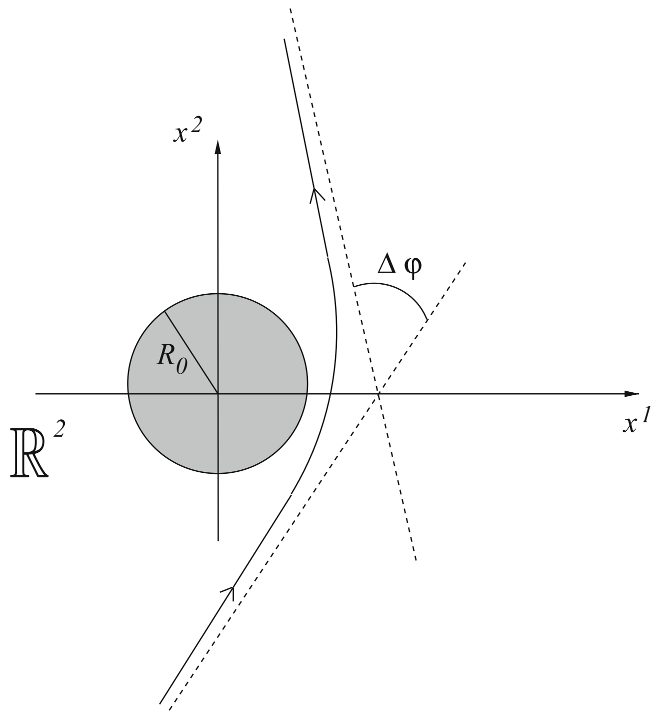


Fig. 3.3 The deflection of light around the Sun

Neglecting the higher order terms, *two asymptotes for the null curve* above are provided by the straight lines:

$$x^1 = R_0 \mp 2m\,[R_0]^{-1} \cdot x^2. \tag{3.39}$$

Figure 3.3 depicts the deflection of light from a star around the Sun from the geometry of asymptotes.

The deflection $\Delta\varphi$ can be calculated from (3.39), and it is approximately given by

$$\Delta\varphi = \frac{4m}{R_0}\,. \tag{3.40}$$

The predicted value $\Delta\varphi$ in (3.40) amounts to $1.75''$ for a light ray that barely misses the surface of the sun, and it has been confirmed experimentally.

Now we shall deal with the third topic of physical interest. Optical or other electromagnetic wave emissions from the Sun or other distant stars reveal shifts of the standard wavelengths of known atomic spectral lines. *The spectral shifts* can be usefully split into a part due to relative motions (between the star and the detector) and a part due to gravitational fields. Since measured wavelengths or frequencies are related to the proper times of the sources or the observers, we look again into the Schwarzschild metric in (3.9). It will be simpler for our investigations to

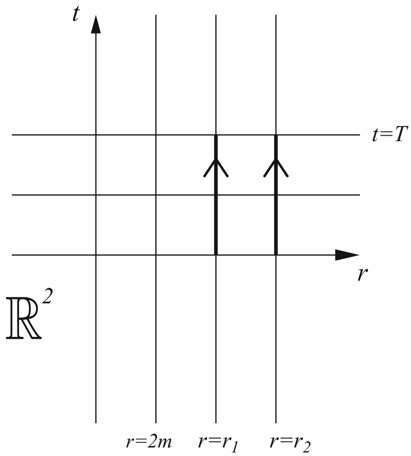


Fig. 3.4 Two t-coordinate lines endowed with ideal clocks

study the induced metric of the submanifold M_2 characterized by $\theta = \pi/2$ and $\varphi = \varphi_0 = \text{const}$. The induced metric is given by

$$\overline{\mathbf{g}}_{..}\left(x^1, x^4\right) = \left(1 - \frac{2m}{x^1}\right)^{-1} \mathrm{d}x^1 \otimes \mathrm{d}x^1 - \left(1 - \frac{2m}{x^1}\right) \mathrm{d}x^4 \otimes \mathrm{d}x^4 \,,$$

$$-\mathrm{d}\tau^2 = \left(1 - \frac{2m}{r}\right)^{-1} (\mathrm{d}r)^2 - \left(1 - \frac{2m}{r}\right) (\mathrm{d}t)^2 \,. \tag{3.41}$$

In Fig. 3.4, we explore proper times along two straight world lines specified by $r = r_1 > 2m$ and $r = r_2 > r_1$.

By (2.91) and (3.37), proper times τ_1 and τ_2 along two t-coordinate lines are furnished by

$$\tau_1 = \int_0^T \sqrt{1 - \frac{2m}{r_1}} \cdot \mathrm{d}t = \sqrt{1 - \frac{2m}{r_1}} \cdot T \,,$$

$$\tau_2 = \int_0^T \sqrt{1 - \frac{2m}{r_2}} \cdot \mathrm{d}t = \sqrt{1 - \frac{2m}{r_2}} \cdot T \,,$$

$$(\tau_2/\tau_1) = \sqrt{\frac{1 - \frac{2m}{r_2}}{1 - \frac{2m}{r_1}}} > 1 \,. \tag{3.42}$$

Therefore, there is *a change of proper time* from one world line to another. Thus, relative frequencies (which are inversely proportional to time) or relative wavelengths (which are inversely proportional to frequency) of photons emitted from those two world lines are *shifted*. This gives rise to what is known as *gravitational redshift*. That is, electromagnetic waves emitted close to a gravitational source will reach distant observers with a smaller frequency (longer wavelength) than those emitted by a similar source further away from the gravitating object. This effect has been very well confirmed in many observations. For example, in the case of the dense companion of the star Sirius, there is agreement between observation and the prediction of general relativity in the frequency shift of spectral lines [15].

Remark: Figure 3.4 depicts the gravitational redshift in *a coordinate chart* corresponding to a submanifold of *the curved space–time manifold*.

To summarize, there exist three critical phenomena associated with the Schwarzschild space–time. These are given by:

1. Advance of the perihelion shift of Mercury,
2. Deflection of a light ray grazing the Sun,
3. Spectral redshift of light emission from a distant star.

The theory of general relativity is *validated by all observations regarding the above three phenomena and other experimental tests* [264, 265].

Exercises 3.1

1. Show that the following metric components belong to different coordinate charts of the Schwarzschild space–time:

   (i) *The isotropic coordinate chart* is characterized by

$$ds^2 = \left(1 + \frac{m}{2\widehat{r}}\right)^4 \cdot \left[\left(d\widehat{r}\right)^2 + \widehat{r}^2 \left(\left(d\widehat{\theta}\right)^2 + \sin^2\widehat{\theta}\left(d\widehat{\varphi}\right)^2\right)\right] - \left[\frac{1 - \frac{m}{2\widehat{r}}}{1 + \frac{m}{2\widehat{r}}}\right]^2 \cdot \left(d\widehat{t}\,\right)^2;$$

$$\widehat{D} := \left\{\left(\widehat{r}, \widehat{\theta}, \widehat{\varphi}, \widehat{t}\,\right) : (m/2) < \widehat{r},\ 0 < \widehat{\theta} < \pi,\ -\pi < \widehat{\varphi} < \pi,\ -\infty < \widehat{t} < \infty\right\}.$$

   (ii) In another coordinate chart, the metric tensor components are furnished by

$$ds^2 = \left[\delta_{\alpha\beta} + \frac{2m}{\overline{r}^2\left(\overline{r} - 2m\right)} \cdot \delta_{\alpha\mu} \cdot \delta_{\beta\nu} \cdot x^\mu x^\nu\right] dx^\alpha dx^\beta - \left(1 - \frac{2m}{\overline{r}}\right)\left(d\overline{t}\,\right)^2;$$

$$\overline{r} := \sqrt{\delta_{\alpha\beta} x^\alpha x^\beta}\ .$$

(iii) In still another chart, the metric tensor components are given by

$$\begin{aligned}
ds^2 = e^{-x^1} \cdot \Big\{ & \left(a e^{x^1/2} - b e^{-x^1/2}\right)^{-4} \cdot \left(dx^1\right)^2 \\
& + \left(a e^{x^1/2} - b e^{-x^1/2}\right)^{-2} \cdot \left[1 + \left(\frac{ab}{4}\right) \cdot \left(x^{2^2} + x^{3^2}\right)\right]^{-2} \\
& \cdot \left(dx^{2^2} + dx^{3^2}\right)\Big\} - e^{x^1} \cdot (dt)^2 .
\end{aligned}$$

Here, the constants satisfy $m = \left(4a \cdot b^{-3}\right)^{-1/2} > 0$.

2. Let the nonlinear, first-order ordinary differential equation (3.26i) be expressed as

$$\begin{aligned}
\left(y'\right)^2 &= f(y) := h^{-2} \cdot \left(E^2 - 1\right) + 2mh^{-2}y - y^2 + 2my^3 \\
&=: 2m \cdot (y - y_1)(y - y_2)(y - y_3) .
\end{aligned}$$

Here, the roots y_1, y_2, y_3 of $f(y) = 0$ are *assumed* to be real and to satisfy inequalities $y_1 < y_2 < y_3$. Prove that the general solution of (3.26i) is furnished by

$$\sqrt{\frac{y - y_1}{y_2 - y_1}} = \operatorname{sn}\left[(1/2)\sqrt{2m\,(y_3 - y_1)} \cdot \varphi + c_0\right].$$

Here, “sn” is *the Jacobian elliptic function* and c_0 is the constant of integration.

3. The three-dimensional spatial hypersurface M_3 (for $t = t_0$) of the Schwarzschild space–time is given by the induced metric:

$$dl^2 = \left(1 - \frac{2m}{r}\right)^{-1} (dr)^2 + r^2\left[(d\theta)^2 + \sin^2\theta (d\varphi)^2\right].$$

Show that the geodesics of M_3 admitting this metric are also (spacelike) geodesics of the Schwarzschild space–time with M_4.

4. Consider the static, spherically metric in (3.1). Solve the field equations $G_{ij}(x) = \Lambda g_{ij}(x)$ (with the cosmological constant Λ). Deduce that the resulting metric can be transformed into the *Kottler solution*:

$$\begin{aligned}
ds^2 = &\left(1 - \frac{2m}{r} - \frac{1}{3} \cdot \Lambda r^2\right)^{-1} (dr)^2 + r^2\left[(d\theta)^2 + \sin^2\theta (d\varphi)^2\right] \\
&- \left(1 - \frac{2m}{r} - \frac{1}{3} \cdot \Lambda r^2\right)(dt)^2, 1 - \frac{2m}{r} - \frac{1}{3} \cdot \Lambda r^2 > 0.
\end{aligned}$$

5. Show that a domain of the Schwarzschild space–time can be embedded locally in a six-dimensional flat manifold of signature +2.

6. Consider a "spherically symmetric" N-dimensional ($N \geq 4$) pseudo-Riemannian metric:

$$ds^2 = e^{\alpha(r)}(dr)^2 + r^2 \cdot \left[d\Omega_{(N-2)}\right]^2 - e^{\gamma(r)} dt^2 \,,$$

$$\left[d\Omega_{(N-2)}\right]^2 := (d\theta_2)^2 + \sum_{n=3}^{N-1} \left[\left(\prod_{m=2}^{n-1} \sin^2 \theta_m \right) \cdot (d\theta_n)^2 \right];$$

$$D_{(N)} := \Big\{ (r, \theta_2, \theta_3, \ldots, \theta_{N-1}, t) : 0 < (2m)^{1/N-3} < r,$$
$$0 < \theta_2, \theta_3, \ldots, \theta_{N-2} < \pi, \; -\pi < \theta_{N-1} < \pi, \; -\infty < t < \infty \Big\}.$$

Prove that field equations $G_{ij}(x) = 0$ yield

$$e^{-\alpha(r)} = 1 - \frac{2m}{r^{N-3}} > 0,$$

$$e^{\gamma(r)} = \left(1 - \frac{2m}{r^{N-3}} \right) \cdot e^k > 0 \,.$$

This is the higher dimensional generalization of the Schwarzschild metric.

Answers and Hints to Selected Exercises

1. (i) Transformations to the usual Schwarzschild coordinates are given by

$$r = \left(1 + \frac{m}{2\hat{r}}\right)^2 \cdot \widehat{r}\,, \qquad \theta = \widehat{\theta}\,, \quad \varphi = \widehat{\varphi}\,, \qquad t = \widehat{t}\,.$$

(iii)

$$\frac{\widehat{r}}{2m} \left(1 + \frac{m}{2\hat{r}}\right)^2 = \left(1 - \frac{a}{b} \cdot e^{x^1}\right)^{-1},$$

$$\widehat{\theta} = 2 \cdot \text{Arctan} \sqrt{\frac{ab}{4} \cdot \left(x^{2^2} + x^{3^2}\right)},$$

$$\widehat{\varphi} = \text{arc}\left(x^2, x^3\right), \quad \widehat{t} = \sqrt{\frac{b}{a}} \cdot x^4.$$

2. Put

$$z := \frac{1}{2}\sqrt{2m\,(y_3 - y_1)} \cdot \varphi\,, \quad w := \sqrt{\frac{y - y_1}{y_2 - y_1}}\,,$$

$$k := \sqrt{\frac{y_2 - y_1}{y_3 - y_1}} < 1\,.$$

Then the nonlinear ordinary differential equation yields

$$w' = \frac{\mathrm{d}W(z)}{\mathrm{d}z} = \sqrt{(1-w^2)\cdot(1-k^2w^2)}\,.$$

3. Consider the function $F(r)$ which satisfies the differential equation:

$$\left[\frac{\mathrm{d}F(r)}{\mathrm{d}r}\right]^2 = \frac{1}{(r-2m)}\cdot\left(\frac{m^2}{r^3}+r\right).$$

Let the four-dimensional Schwarzschild space–time domain be expressed as the parametrized submanifold:

$$\begin{aligned}
y^1 &= \xi^1(r,\theta,\varphi,t) := \sqrt{1-\frac{2m}{r}}\cdot\cos t\,,\\
y^2 &= \xi^2(\cdot\cdot) := \sqrt{1-\frac{2m}{r}}\cdot\sin t\,,\\
y^3 &= \xi^3(\cdot\cdot) := F(r)\,,\\
y^4 &= \xi^4(\cdot\cdot) := r\sin\theta\,\cos\varphi\,,\\
y^5 &= \xi^5(\cdot\cdot) := r\sin\theta\,\sin\varphi\,,\\
y^6 &= \xi^6(\cdot\cdot) := r\cos\theta\,;\\
\tilde{\mathbf{g}}_{..}(y) &= -\mathrm{d}y^1\otimes\mathrm{d}y^1 - \mathrm{d}y^2\otimes\mathrm{d}y^2 + \mathrm{d}y^3\otimes\mathrm{d}y^3 + \mathrm{d}y^4\otimes\mathrm{d}y^4\\
&\qquad + \mathrm{d}y^5\otimes\mathrm{d}y^5 + \mathrm{d}y^6\otimes\mathrm{d}y^6\,.
\end{aligned}$$

4. Equations reduce to the following:

$$\begin{aligned}
&(N-3)\left[1-\mathrm{e}^{-\alpha(r)}\right] + r\mathrm{e}^{-\alpha(r)}\cdot\partial_1\alpha = 0\,,\\
&(N-3)\left[1-\mathrm{e}^{-\alpha(r)}\right] - r\mathrm{e}^{-\alpha(r)}\cdot\partial_1\gamma = 0\,,\\
&l\,\frac{\mathrm{e}^{-\alpha(r)}}{4}\cdot\Bigg[2\partial_1\partial_1\gamma + (\partial_1\gamma)^2 + \frac{2(N-3)}{r}\,(\partial_1\gamma-\partial_1\alpha)\\
&\qquad\qquad - \partial_1\alpha\cdot\partial_1\gamma + \frac{2}{r^2}\,(N-3)(N-4)\Bigg] - \frac{2(N-3)(N-4)}{r^2} = 0\,.
\end{aligned}$$

3.2 Spherically Symmetric Static Interior Solutions

Here we consider spherically symmetric space–time domains which *are not* vacuum space–times. These space–times are useful approximations to the gravitational fields produced inside of stars, for example.

As we are still dealing with spherical symmetry, the metric can still be taken to be that of (3.1). The domain of consideration is now

$$
\begin{aligned}
D := \big\{(r, \theta, \varphi, t) :\ & 0 < r_1 < r < r_2,\ 0 < \theta < \pi, \\
& -\pi < \varphi < \pi,\ -\infty < t < \infty\big\}.
\end{aligned} \tag{3.43}
$$

We often consider $r = r_2$ to be the boundary of the star so that the metric for $r > r_2$ is given by the vacuum Schwarzschild metric (3.9). Alternatively, the matter field, as manifest in the components T_{ij}, can fall off smoothly so that the solution asymptotically approaches the Schwarzschild solution.

The interior field equations and conservation equations reduce to

$$
\mathcal{E}^1{}_1(r) = r^{-2} \cdot \left[1 - \mathrm{e}^{-(\alpha+\gamma)} \cdot \partial_1 \left(r\mathrm{e}^{\gamma}\right)\right] + \kappa\, T^1_{\ 1}(r) = 0\,, \tag{3.44i}
$$

$$
\begin{aligned}
\mathcal{E}^2{}_2(r) = \mathrm{e}^{-\alpha} \cdot \Big[& -(1/2)\,\partial_1\partial_1\gamma - (1/4)(\partial_1\gamma)^2 \\
& + (1/2)\, r^{-1}\,(\partial_1\alpha - \partial_1\gamma) + (1/4)\,\partial_1\alpha \cdot \partial_1\gamma\Big] \\
& + \kappa\, T^2_{\ 2}(r) = 0\,,
\end{aligned} \tag{3.44ii}
$$

$$
\mathcal{E}^3{}_3(r) \equiv \mathcal{E}^2{}_2(r)\,, \tag{3.44iii}
$$

$$
\mathcal{E}^4{}_4(r) = r^{-2} \cdot \left[1 - \partial_1 \left(r\mathrm{e}^{-\alpha}\right)\right] + \kappa\, T^4_{\ 4}(r) = 0\,, \tag{3.44iv}
$$

$$
\begin{aligned}
& T^3_{\ 3}(r) \equiv T^2_{\ 2}(r)\,, \\
& T^1_{\ 2}(r) = T^1_{\ 3}(r) = T^1_{\ 4}(r) = T^2_{\ 3}(r) = T^2_{\ 4}(r) = T^3_{\ 4}(r) = 0\,;
\end{aligned} \tag{3.44v}
$$

$$
\begin{aligned}
(r/2) \cdot \mathcal{T}_1(r) = & -T^2_{\ 2}(r) + (r/2) \cdot \partial_1 T^1_{\ 1} + \left[1 + (r/4) \cdot \partial_1\gamma\right] T^1_{\ 1}(r) \\
& - (r/4) \cdot \partial_1\gamma \cdot T^4_{\ 4}(r) = 0\,,
\end{aligned} \tag{3.44vi}
$$

$$
\mathcal{T}_2(r) = \mathcal{T}_3(r) = \mathcal{T}_4(r) \equiv 0\,. \tag{3.44vii}
$$

We solve the above system of semilinear ordinary differential equations by *the mixed method* outlined in the following steps:

(i) *Prescribe* $T^4_{\ 4}(r)$ and solve the equation $\mathcal{E}^4{}_4(r) = 0$ to obtain $g^{11}(r) = \mathrm{e}^{-\alpha(r)}$.
(ii) *Prescribe* $T^1_{\ 1}(r)$ and solve the equation $\mathcal{E}^1{}_1(r) - \mathcal{E}^4{}_4(r) = 0$ to get $g_{44}(r) = -\mathrm{e}^{\gamma(r)}$.
(iii) *Define* $T^2_{\ 2}(r)$ by (3.44vi) namely $(r/2) \cdot \mathcal{T}_1(r) = 0$.

At this stage, *all the field equations and conservation equations are satisfied*. With the above prescriptions, we can furnish *the most general solution* of the system of differential equations in (3.44i–vii). It is stated in the following theorem. (See [63].)

Theorem 3.2.1. *Consider the spherically symmetric, static metric* (3.1) *and the domain* D *of equation* (3.43)*. The system of semilinear ordinary differential equations* (3.44i–vii) *yield the most general solution as*

$$e^{-\alpha(r)} = 1 - \frac{1}{r}\left[2m_0 - \kappa \int_{r_1}^{r} T^4{}_4(y)\, y^2 \mathrm{d}y\right] > 0\,, \tag{3.45i}$$

$$e^{\gamma(r)} = e^{-\alpha(r)} \cdot \exp\left\{k + \kappa \int_{r_1}^{r} \left[T^1{}_1(y) - T^4{}_4(y)\right] \cdot e^{\alpha(y)} \cdot y\, \mathrm{d}y\right\}, \tag{3.45ii}$$

$$\begin{aligned} T^2{}_2(r) \equiv T^3{}_3(r) &:= (r/2)\cdot \partial_1 T^1{}_1 + \left[1 + (r/4)\cdot \partial_1\gamma\right]\cdot T^1{}_1(r) \\ &\quad - (r/4)\cdot \partial_1\gamma \cdot T^4{}_4(r)\,. \end{aligned} \tag{3.45iii}$$

Here, m_0 *and* k *are two arbitrary constants of integration. Moreover, two differentiable functions* $T^1{}_1(r)$ *and* $T^4{}_4(r)$ *are prescribed.*

Proof. The field equation (3.44iv) implies that

$$\partial_1\left(r e^{-\alpha}\right) = 1 + \kappa\, r^2\, T^4{}_4(r)\,.$$

The above equation is a linear, first-order, nonhomogeneous ordinary differential equation for the function $r e^{-\alpha(r)}$. The general solution of *the corresponding homogeneous equation*, $\partial_1\left[r e^{-\alpha(r)}\right] = 0$, is given by $\left[r e^{-\alpha(r)}\right]_{\text{hom.}} = -2m_0$. (Here, m_0 is the arbitrary constant of integration.) The particular integral of the same equation is

$$\left[r e^{-\alpha(r)}\right]_{\text{part.}} = r + \kappa \int_{r_1}^{r} T^4{}_4(y)\, y^2 \mathrm{d}y\,.$$

Therefore, the most general solution is *the linear combination* $\left[r e^{-\alpha(r)}\right]_{\text{hom.}} + \left[r e^{-\alpha(r)}\right]_{\text{part.}}$ yielding (3.45i).

Now, the field equation $\mathcal{E}^1{}_1(r) - \mathcal{E}^4{}_4(r) = 0$ yields

$$\partial_1\gamma = -\partial_1\alpha + \kappa\, r e^{\alpha(r)} \cdot \left[T^1{}_1(r) - T^4{}_4(r)\right].$$

The above equation is a linear, first-order, nonhomogeneous ordinary differential equation for $\gamma(r)$. The most general solution is provided by (3.45ii) (where k is the arbitrary constant of integration). By the prescriptions suggested in p. 246, all other field equations are satisfied. ∎

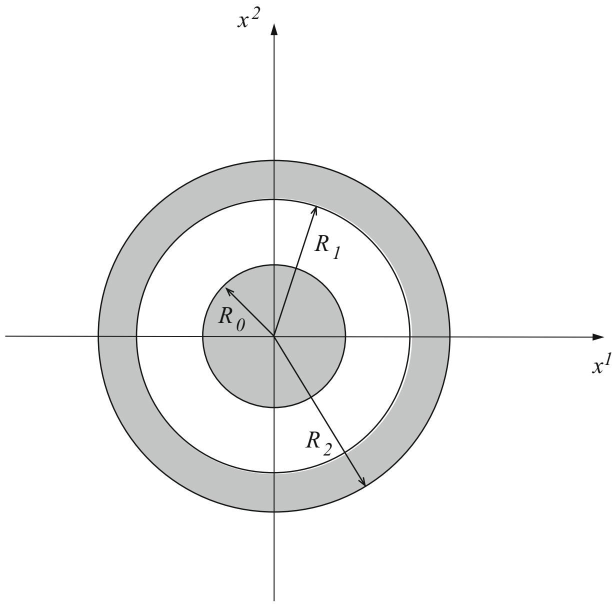


Fig. 3.5 Qualitative representation of a spherical body inside a concentric shell

Remarks: (i) It is evident from (3.44v) that $[T^i{}_j(r)]$ must be *diagonalizable*. Moreover, only Segre characteristic $[1,(1,1),1]$, $[(1,1,1),1]$, $[(1,1,1,1)]$ and *the stiff case* $T^1_1(r) = -T^4_4(r)$ are allowed.

(ii) Weak energy conditions (2.194i) reduce to

$$\begin{aligned} &- T^4_4(r) \geq 0\,, \quad -T^4_4(r) + T^1_1(r) \geq 0\,, \\ &- T^4_4(r) + T^2_2(r) \geq 0\,. \end{aligned} \tag{3.46}$$

(iii) Synge's junction conditions (2.170) on the outer boundary of a spherical body (at $r = r_2$) imply the continuity of $\alpha(r)$ and $\gamma(r)$ across $r = r_2$. Furthermore, the conditions imply that

$$T^1_1(r)_{|r_2} = 0, \tag{3.47}$$

in the case where the boundary joins to the vacuum. For boundaries which join two different types of matter, the radial pressure must be continuous.

We notice in the proof of Theorem 3.2.1 that two relevant ordinary differential equations are *linear*. Therefore, a superposition of distinct solutions to these equations is *also a solution*. This feature gives rise to interesting physical consequences. Consider two-dimensional coordinate charts of Figs. 3.2 and 3.3. In such a chart, now we represent a spherical body of mass m_0 and a concentric, thick spherical shell of mass m_1 as in Fig. 3.5.

We express explicitly the metric function $e^{\alpha(r)}$ for the above two bodies in the following:

$$r\left[e^{-\alpha(r)} - 1\right] = \begin{cases} -2m_0 \quad \text{for} \quad 0 < 2m_0 < R_0 < r < R_1\,, \\ -2m_0 + \kappa \displaystyle\int_{R_1}^{r} T^4{}_4(y)\, y^2 \mathrm{d}y \quad \text{for} \quad R_1 < r < R_2\,, \\ -2(m_0 + m_1) \quad \text{for} \quad R_2 < r < \infty\,; \end{cases}$$

$$m_1 := -\frac{\kappa}{2} \int_{R_1}^{R_2} T^4{}_4(y)\, y^2 \mathrm{d}y\,. \tag{3.48}$$

Similarly, the metric function $e^{\gamma(r)}$ can be determined. Usually in general relativity, the space–time metric generated by two massive bodies interacting only gravitationally *cannot be static*. The reason for this is that the nonlinear gravitational equations inherently imply nontrivial equations of motion for bodies. (See [17].). As well, physically, two distinct bodies interacting *solely gravitationally* cannot apply forces on each other which counteract the gravitational attraction. However, the spherically symmetric bodies in Fig. 3.5 are already in equilibrium (in some sense) due to the fact that the shell, if self-interacting, can support itself against the gravitational pull of the interior sphere. Hence, a static solution is possible. In Newtonian gravitational theory, the equilibrium among concentric spherically symmetric bodies is well known. It so happens that equilibria in Newtonian scenarios may lead to exact static solutions of Einstein's field equations. (Consult Example 2.2.9.)

Now we shall apply Theorem 3.2.1 to the case of a spherically symmetric, perfect fluid body. (We have touched upon the case of a perfect fluid in (2.182), (2.257i–iv), and Example 2.3.1.) We will reiterate the energy–momentum–stress tensor components in the static case as

$$T^i{}_j(\mathbf{x}) = \left[\rho(\mathbf{x}) + p(\mathbf{x})\right] \cdot U^i(\mathbf{x})\, U_j(\mathbf{x}) + p(\mathbf{x})\, \delta^i{}_j\,, \tag{3.49i}$$

$$U^\alpha(\mathbf{x}) \equiv 0\,, \quad U^4(\mathbf{x})\, U_4(\mathbf{x}) \equiv -1\,, \tag{3.49ii}$$

$$T^1{}_1(\mathbf{x}) \equiv T^2{}_2(\mathbf{x}) \equiv T^3{}_3(\mathbf{x}) = p(\mathbf{x})\,, \quad T^4{}_4(\mathbf{x}) = -\rho(\mathbf{x})\,. \tag{3.49iii}$$

With the spherically symmetric metric of (3.1), we consider the domain

$$D := \{(r, \theta, \varphi, t) : 0 < r < b,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ -\infty < t < \infty\}\,.$$

Furthermore, we obtain from (3.49i–iii),

$$U^\alpha(\mathbf{x}) \equiv 0\,, \quad U^4(\mathbf{x}) = U^4(r) = \exp\left[\gamma(r)/2\right] > 0\,,$$

$$T^1{}_1(r) \equiv T^2{}_2(r) \equiv T^3{}_3(r) = p(r)\,, \quad T^4{}_4(r) = -\rho(r)\,. \tag{3.50}$$

Solutions in (3.45i,ii) yield:

$$
\begin{aligned}
e^{-\alpha(r)} &= 1 - \frac{2M(r)}{r}\,, \\
M(r) &:= \frac{\kappa}{2}\int_{0+}^{r} \rho(y)\cdot y^2 \mathrm{d}y\,, \\
e^{\gamma(r)} &= \left[1 - \frac{2M(r)}{r}\right]\cdot \exp\left\{\kappa \int_{0+}^{r}\left[\frac{\rho(y)+p(y)}{y-2M(y)}\right]\cdot y^2\mathrm{d}y\right\}.
\end{aligned} \tag{3.51}
$$

(Here, we have set the constant $k = 0$.) Unlike in (3.45i,ii), the function $T^1{}_1(r) = p(r)$ is *no more freely prescribable*. The reason for this complication is that (3.50) now implies that $T^1{}_1(r) = T^2{}_2(r)$. Therefore, we *cannot define* $T^2{}_2(r)$ by (3.45iii). Instead, we have to solve the conservation equation (3.45iii) expressed as

$$
\partial_1 p = -(1/2)\cdot\left[\rho(r) + p(r)\right]\cdot \partial_1\gamma\,. \tag{3.52}
$$

(This is sometimes known as *the isotropy equation*.) By the use of the equation for $\gamma(r)$ in (3.51), (3.52) above implies that

$$
\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{[\rho(r)+p(r)]\cdot[M(r) + (\kappa/2)\,p(r)\cdot r^3]}{r[r-2M(r)]}\,. \tag{3.53}
$$

The above equation is called the *Oppenheimer–Volkoff equation* of *hydrostatic equilibrium*. (See [184].)

Example 3.2.2. Let us consider the following special case involving a constant mass density "star." We choose

$$
\begin{aligned}
\kappa\rho(r) &= 3q = \text{positive const.}\,, \\
2M(r) &= 3q\int_{0+}^{r} y^2\mathrm{d}y = qr^3 > 0\,, \\
e^{-\alpha(r)} &= 1 - qr^2 > 0\,.
\end{aligned} \tag{3.54}
$$

(Although constant density is not totally physical, such simple solutions help shed light on more realistic scenarios.)

Assuming, $\rho(r) + p(r) \neq 0$, (3.52) yields

$$
\begin{aligned}
&\partial_1 \ln\left[3\kappa^{-1}q + p(r)\right] = -(1/2)\cdot\partial_1\gamma\,, \\
\text{or,}\qquad &3\kappa^{-1}q + p(r) = K_0\,e^{-\gamma(r)/2}\,.
\end{aligned} \tag{3.55}
$$

Here, K_0 is the arbitrary constant of integration.

The equation $\mathcal{E}^1{}_1(r) - \mathcal{E}^4{}_4(r) = 0$, with (3.55), implies that

$$\mathrm{e}^{\gamma/2} \cdot \mathrm{e}^{-\alpha} \left[\partial_1 \alpha + \partial_1 \gamma\right] = \kappa K_0 r \,,$$

$$\text{or,} \qquad 2\left[1 - qr^2\right]^{3/2} \cdot \frac{\mathrm{d}}{\mathrm{d}r}\left[\mathrm{e}^{\gamma/2} \cdot (1 - qr^2)^{-1/2}\right] = \kappa K_0 r \,,$$

$$\text{or,} \qquad \mathrm{e}^{[\gamma(r)/2]} = \frac{\kappa K_0}{2q} - C_0 \sqrt{1 - qr^2} \,. \tag{3.56}$$

Here, $C_0 \neq 0$ is the constant of integration. Substituting (3.56) into (3.55), we obtain

$$p(r) = \left[\frac{3C_0 \kappa^{-1} q \sqrt{1 - qr^2} - (K_0/2)}{(\kappa K_0/2q) - C_0 \sqrt{1 - qr^2}}\right]. \tag{3.57}$$

Inserting the boundary condition $p(b) = 0$, (which is *Synge's junction condition* in (3.47) at $r = b$, the stellar boundary separating the material from the vacuum), the equation above yields

$$(\kappa K_0/2q) = 3C_0 \sqrt{1 - qb^2} \,, \tag{3.58i}$$

$$p(r) = 3q\kappa^{-1}\left[\frac{\sqrt{1 - qr^2} - \sqrt{1 - qb^2}}{3\sqrt{1 - qb^2} - \sqrt{1 - qr^2}}\right]. \tag{3.58ii}$$

Therefore, $p(r)$ is a positive-valued, monotonically decreasing function provided

$$3\sqrt{1 - qb^2} > \sqrt{1 - qr^2} \,. \tag{3.59}$$

Putting (3.58i) into (3.56), we get

$$\mathrm{e}^{\gamma(r)} = (C_0)^2 \cdot \left[3\sqrt{1 - qb^2} - \sqrt{1 - qr^2}\right]^2 . \tag{3.60}$$

Stipulating $\lim\limits_{r \to 0+} \mathrm{e}^{\gamma(r)} = 1$, we obtain

$$(C_0)^2 = \left[3\sqrt{1 - qb^2} - 1\right]^{-2} . \tag{3.61}$$

Now, the total mass of the fluid body is furnished by

$$m := \lim_{r \to b_-} M(r) = (\kappa/2) \int_{0+}^{b_-} \rho(r)\, r^2 \mathrm{d}r = (qb^3/2) > 0. \tag{3.62}$$

By the inequality (3.59), we derive that

$$\begin{aligned} & 9\left(1 - qb^2\right) > 1\,, \\ \text{or,}\qquad & \frac{8b}{9} > qb^3 = 2m\,, \\ \text{or,}\qquad & b > (9/8)(2m) > 2m\,. \end{aligned} \tag{3.63}$$

Therefore, the radius of the fluid body must *exceed the Schwarzschild radius*. Otherwise, there will be instability and the exact solution will not hold. (The above inequality is sometimes known as the *Buchdahl inequality* [31].)

The metric component $g_{44}(r) = -\mathrm{e}^{\gamma(r)}$ can be deduced, both in the interior and in the exterior to the fluid body, from (3.60), (3.61), and the vacuum solution (3.7). It is explicitly furnished as

$$\mathrm{e}^{\gamma(r)} = \begin{cases} \left[\dfrac{3\sqrt{1-qb^2} - \sqrt{1-qr^2}}{3\sqrt{1-qb^2} - 1}\right]^2 & \text{for } 0 < r < b\,, \\ \left(1 - \dfrac{2m}{r}\right) \cdot e^k & \text{for } b < r < \infty\,. \end{cases} \tag{3.64}$$

Demanding continuity of $\mathrm{e}^{\gamma(r)}$ across the boundary $r = b$, we derive that

$$\lim_{r \to b_-} \mathrm{e}^{\gamma(r)} = \left[\frac{2\sqrt{1-qb^2}}{3\sqrt{1-qb^2} - 1}\right]^2 = \lim_{r \to b_+} \mathrm{e}^{\gamma(r)} = \left(1 - \frac{2m}{b}\right) \cdot e^k,$$

or,

$$e^k = 4\left[3\sqrt{1-qb^2} - 1\right]^{-2}. \tag{3.65}$$

Summarizing, we furnish both the interior and exterior metrics as:

Interior:

$$\begin{aligned} \mathrm{d}s^2 = {} & \left[1 - qr^2\right]^{-1} (\mathrm{d}r)^2 + r^2 \left[(\mathrm{d}\theta)^2 + \sin^2\theta (\mathrm{d}\varphi)^2\right] \\ & - \left[\frac{3\sqrt{1-qb^2} - \sqrt{1-qr^2}}{3\sqrt{1-qb^2} - 1}\right]^2 \cdot (\mathrm{d}t)^2\,; \end{aligned} \tag{3.66i}$$

Exterior:

$$\begin{aligned} \mathrm{d}s^2 = {} & \left(1 - \frac{2m}{r}\right)^{-1} (\mathrm{d}r)^2 + r^2 \left[(\mathrm{d}\theta)^2 + \sin^2\theta (\mathrm{d}\varphi)^2\right] \\ & - \frac{4\left(1 - \frac{2m}{r}\right)}{\left[3\sqrt{1-qb^2} - 1\right]^2} \cdot (\mathrm{d}t)^2\,. \end{aligned} \tag{3.66ii}$$

The exterior metric in (3.66ii) *can be transformed into the Schwarzschild chart* by

$$r^{\#} = r\,, \quad \theta^{\#} = \theta\,, \quad \varphi^{\#} = \varphi\,,$$
$$t^{\#} = \left[\frac{2}{3\sqrt{1-qb^2}-1}\right] \cdot t\,.$$

The interior solution (3.66i) is known as *Schwarzschild's interior solution*.[2] □

Remark: For the case of a static, spherically symmetric perfect fluid, *many exact solutions* are cited in the books [118, 239].

Example 3.2.3. Now we shall consider a domain of a spherically symmetric, static, *electro-vac universe* (containing gravity and electrostatic fields only). The pertinent field equations are provided by *the Einstein–Maxwell equations* (2.290i–vi). In the present context, we choose

$$\begin{aligned} &F_{(1)(4)}(\mathbf{x}) \neq 0\,, \\ &F_{(2)(4)}(\mathbf{x}) = F_{(3)(4)}(\mathbf{x}) \equiv 0\,, \\ &F_{(\alpha)(\beta)}(\mathbf{x}) \equiv 0\,, \\ &\partial_4 F_{(i)(j)} \equiv 0 = \partial_3 F_{(i)(j)}\,. \end{aligned} \tag{3.67}$$

The above equations indicate that *only the radial component of the electric field is nonzero*. Other electromagnetic field components are *assumed to vanish*.

Maxwell's equations ${}^*\mathcal{M}^3(\cdot\cdot) = {}^*\mathcal{M}^2(\cdot\cdot) = 0$ imply that

$$\partial_2 F_{41} = 0 = \partial_3 F_{41}\,,$$
$$\text{or,} \qquad F_{14}(\mathbf{x}) = F_{14}(x^1) = F_{14}(r) \neq 0\,. \tag{3.68}$$

The metric in (3.1) provides

$$\sqrt{-g} = \mathrm{e}^{(\alpha+\gamma)/2} \cdot r^2 \cdot \sin\theta.$$

Therefore, Maxwell's equation $\mathcal{M}^4(\cdot\cdot) = 0$ in (2.290i) yields

$$0 = \partial_j\left[\sqrt{-g}\,F^{4j}\right] = \partial_1\left[r^2 \sin\theta \cdot \mathrm{e}^{(\alpha+\gamma)/2} \cdot F^{41}\right] = 0$$

$$\text{or,} \qquad F^{41}(r) = \frac{e_0 \cdot \mathrm{e}^{-(\alpha+\gamma)/2}}{r^2}\,,$$

[2] It should be noted that sometimes the domain $0 < r < 2m$ of the metric (3.9) also goes by this name.

$$F_{(1)(4)}(r) = \frac{e_0}{r^2} \neq 0\,,$$

$$F_{ij}(\mathbf{x}) F^{ij}(\mathbf{x}) < 0\,. \tag{3.69}$$

Here, e_0 is the constant of integration representing *the charge parameter*. All other Maxwell's equations in (2.290ii,iii) *are satisfied*. The components of the electromagnetic energy–momentum–stress tensor from (2.290i) reduce to

$$\begin{aligned}
T^1{}_1(r) &= -T^2{}_2(r) = -T^3{}_3(r) = T^4{}_4(r) = -\frac{(e_0)^2}{2r^4}\,, \\
\text{other} \qquad T^i{}_j(\cdot\cdot) &\equiv 0\,, \\
T^1{}_1(r) - T^4{}_4(r) &\equiv 0\,, \\
T^i{}_i(r) &\equiv 0\,.
\end{aligned} \tag{3.70}$$

Note that the above components imply the Segre characteristic to be $[(1,1),(1,1)]$. (Consult Theorem 2.5.6.)

The "energy density" of the electric field is furnished by

$$-T^{(4)}{}_{(4)}(r) = -T^4{}_4(r) = \frac{e_0^2}{2r^4} > 0\,. \tag{3.71}$$

The "total energy" of the spherically symmetric electric field present between radii $r_1 < r < r_2$ is

$$-(\kappa/2)\int_{r_1}^{r_2} T^{(4)}{}_{(4)}(r)\, r^2 \mathrm{d}r = (\kappa/2)\cdot(e_0^2/2)\left[\frac{1}{r_1} - \frac{1}{r_2}\right] > 0\,. \tag{3.72}$$

The exact solutions of (2.290i–v) from (3.67), (3.69), (3.70), and (3.45i,ii) are given by

$$\begin{aligned}
\mathrm{e}^{-\alpha(r)} = \mathrm{e}^{\gamma(r)} &= 1 - \frac{2m_0}{r} + \frac{\kappa}{r}\int_{r_1}^{r} T^4{}_4(y)\, y^2 \mathrm{d}y \\
&= 1 - \frac{1}{r}\left(2m_0 + \frac{\kappa e_0^2}{2r_1}\right) + \frac{\kappa e_0^2}{2r^2} \\
&=: 1 - \frac{2m}{r} + \frac{e^2}{r^2} > 0\,.
\end{aligned} \tag{3.73}$$

(Here, we have set the constant $k = 0$ in (3.45ii).)

Thus, from (3.73), we express the exact solution as

$$\begin{aligned}
\mathbf{g}_{..}(x) = {} & \left[1 - \frac{2m}{x^1} + \frac{e^2}{(x^1)^2}\right]^{-1} (\mathrm{d}x^1 \otimes \mathrm{d}x^1) \\
& + \left(x^1\right)^2 \left[(\mathrm{d}x^2 \otimes \mathrm{d}x^2) + \sin^2 x^2 \cdot (\mathrm{d}x^3 \otimes \mathrm{d}x^3)\right] \\
& - \left[1 - \frac{2m}{x^1} + \frac{e^2}{(x^1)^2}\right] (\mathrm{d}x^4 \otimes \mathrm{d}x^4)\,, \\
\mathrm{d}s^2 = {} & \left[1 - \frac{2m}{r} + \frac{e^2}{r^2}\right]^{-1} (\mathrm{d}r)^2 + r^2 \left[(\mathrm{d}\theta)^2 + \sin^2\theta (\mathrm{d}\varphi)^2\right] \\
& - \left[1 - \frac{2m}{r} + \frac{e^2}{r^2}\right] (\mathrm{d}t)^2\,.
\end{aligned} \tag{3.74}$$

The above metric for $r^2 - 2mr + e^2 > 0$ is known as the *Reissner-Nordström-Jeffery metric* [145, 196, 218] (also see [84]). □

In the preceding considerations, we have assumed that the material energy density is *strictly positive*. However, in the arena of static, spherically symmetric space–time domains, assuming that energy conditions are *not necessarily respected*, there may exist exotic materials which open up a "hollow tunnel" between two branches of the same space–time. Such a scenario is provided by the existence of the (so-called) *wormhole*. (See Fig. 2.17c.) The topic of wormholes will be discussed briefly in Appendix 6.

Exercises 3.2

1. Prove that for an incoherent dust (of (2.256)) with $\rho(\mathbf{x}) > 0$ and $U^\alpha(\mathbf{x}) \equiv 0$, the static metric (3.1) does *not admit any solution* of the gravitational field equations. Physically, why would you expect no static solutions for such a system?
2. Consider the following transformations:

$$\begin{aligned}
x &\in \mathbb{R}\,, \\
x &:= r^2 > 0\,, \\
\zeta(x) &:= \mathrm{e}^{\gamma(r)/2}\,, \\
w(x) &:= M(r) \cdot r^{-3}.
\end{aligned}$$

(i) Prove that field equations (3.51), (3.53), and (3.44ii) go over into

$$\begin{aligned} e^{-\alpha(\sqrt{x})} &= 1 - 2x \cdot w(x)\,, \\ \kappa\,\rho\left(\sqrt{x}\right) &= 6w(x) + 4x \cdot \partial_x w\,, \\ \kappa\,p\left(\sqrt{x}\right) &= -2w(x) + [4 - 8x\,w(x)] \cdot \partial_x(\ln\zeta)\,, \\ [2 - 4x\,w] \cdot \partial_x\partial_x\zeta - (2w &+ 2x \cdot \partial_x w) \cdot \partial_x\zeta - \zeta\partial_x w = 0. \end{aligned}$$

(ii) Show that the last equation above remains *unchanged* under the transformation

$$\widehat{\zeta} = \zeta\,, \quad \widehat{w} = C_0 \cdot [\zeta + 2x \cdot \partial_x\zeta]^{-2} \cdot \exp\left[4\int \partial_x\zeta \cdot (\zeta + 2x\partial_x\zeta)^{-1}\,dx\right].$$

3. Consider the metric (3.1) and static, spherically symmetric, *electromagneto-vac equations* (2.290i–vi). Assume that $F_{14}(\mathbf{x}) \neq 0$, $F_{23}(\mathbf{x}) \neq 0$, and *all other* $F_{ij}(\mathbf{x}) \equiv 0$.

(i) Show that exact solutions of the electromagneto-vac equations can be transformed into

$$\begin{aligned} &F_{14}(r) = \frac{e_0}{r^2} \cdot e^{(\alpha+\gamma)/2}\,, \quad F_{(1)(4)}(r) = \frac{e_0}{r^2} = E_{(1)}(r)\,, \\ &F_{23}(\theta) = \mu_0 \cdot \sin\theta\,, \quad F_{(2)(3)}(r) = \frac{\mu_0}{r^2} = H_{(1)}(r)\,, \\ &ds^2 = \left[1 - \frac{2m}{r} + \frac{(e^2 + \mu^2)}{r^2}\right]^{-1} (dr)^2 + r^2\left[(d\theta)^2 + \sin^2\theta(d\varphi)^2\right] \\ &\qquad - \left[1 - \frac{2m}{r} + \frac{(e^2 + \mu^2)}{r^2}\right](dt)^2. \end{aligned}$$

Here, m, e_0, μ_0, etc., are constants of integration.

(ii) Determine the external domain of validity for the solutions in the preceding part.

4. Consider general solutions (3.45i–iii) of the static, spherically symmetric interior field equations in

$$D := \{(r, \theta, \varphi, t) : 0 < r < b,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ -\infty < t < \infty\}.$$

Let the corresponding vacuum equations be satisfied in the exterior domain

$$D_0 := \{(r, \theta, \varphi, t) : b < r < \infty,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ -\infty < t < \infty\}.$$

Obtain the general solution of the problem satisfying (i) Synge's junction condition (2.170) at $r = b$ and (ii) weak energy conditions $-T^4_{\ 4}(r) \geq 0$ and $-T^4_{\ 4}(r) + T^1_{\ 1}(r) \geq 0$.

Answers and Hints to Selected Exercises

1. Use (3.53) with $p(r) \equiv 0$.

   (*Remark:* Eventual collapse is inevitable.)

3. (i) Maxwell's equations ${}^*\mathcal{M}^3 = {}^*\mathcal{M}^2 = \mathcal{M}^1 = 0$ yield $F_{14}(r) = \frac{e_0}{r^2} \cdot \mathrm{e}^{(\alpha+\gamma)/2} = E_{(1)}$. Maxwell's equations ${}^*\mathcal{M}^4 = \mathcal{M}^2 = \mathcal{M}^3 = 0$ imply

$$F_{23}(\theta) = \mu_0 \cdot \sin\theta\,,\ F_{(2)(3)}(r) = \frac{\mu_0}{r^2} = H_{(1)}(r),$$

$$T^1_{\ 1}(r) \equiv T^4_{\ 4}(r) \equiv -T^2_{\ 2}(r) \equiv -T^3_{\ 3}(r) = -\frac{(e_0^2 + \mu_0^2)}{2r^4}\,.$$

   (ii) *Assume* that $m \geq \sqrt{e^2 + \mu^2}$. Let $r_+ := m + \sqrt{m^2 - (e^2 + \mu^2)}$. A possible domain of validity is

$$D := \{(r, \theta, \varphi, t) : r_+ < r,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ -\infty < t < \infty\}$$
$$- \{(r, \theta, \varphi, t) : r_+ < r,\ \theta = \pi/2,\ \varphi = 0,\ -\infty < t < \infty\}\,.$$

   (*Remark:* This solution represents a combined electric as well as magnetic monopole source. The magnetic monopole has a massless, semi-infinite string attached [78].)

4.

$$-T^4_{\ 4}(r) := (b - r)^{n+2} \cdot \exp[F(r)] > 0\,,$$

$$T^1_{\ 1}(r) := (b - r)^{n+2} \cdot \left\{[h(r)]^2 - \exp\left[F(r)\right]\right\},$$

$$-T^4_{\ 4}(r) + T^1_{\ 1}(r) = (b - r)^{n+2} \cdot [h(r)]^2 \geq 0\,,$$

$$T^1_{\ 1}(b) = 0\,;$$

$$2M(r) := \kappa \int_{0+}^{r} (b - y)^{n+2} \cdot \exp[F(y)] \cdot y^2 \mathrm{d}y > 0\,,$$

$$m := \lim_{r \to b_-} M(r)\,,$$

$$e^{-\alpha(r)} = 1 - \frac{2M(r)}{r} \,,$$

$$e^{\gamma(r)} = \left[1 - \frac{2M(r)}{r}\right] \cdot \exp\left\{k + \kappa \int_{0+}^{r} \left[\frac{(b-y)^{n+2}[h(y)]^2}{y - 2M(y)}\right] \cdot y^2 \mathrm{d}y\right\}.$$

(Here, n is a positive integer and $F(r)$, $h(r)$ are thrice differentiable arbitrary *slack functions*.)

3.3 Nonstatic, Spherically Symmetric Solutions

The space–time metric $\mathbf{g}_{..}(x)$ admitting three independent Killing vectors

$$\sin x^3 \cdot \frac{\partial}{\partial x^2} + \cot x^2 \cdot \cos x^3 \cdot \frac{\partial}{\partial x^3} \,,$$
$$\cos x^3 \cdot \frac{\partial}{\partial x^2} - \cot x^2 \cdot \sin x^3 \cdot \frac{\partial}{\partial x^3} \,, \quad \text{and} \quad \frac{\partial}{\partial x^3} \,,$$

is reducible to

$$\mathbf{g}_{..}(x) = e^{\alpha(x^1,x^4)} \left(\mathrm{d}x^1 \otimes \mathrm{d}x^1\right) + \left(x^1\right)^2 \left[\left(\mathrm{d}x^2 \otimes \mathrm{d}x^2\right) + \sin^2 x^2 \cdot \left(\mathrm{d}x^3 \otimes \mathrm{d}x^3\right)\right]$$
$$- e^{\gamma(x^1,x^4)} \cdot \left(\mathrm{d}x^4 \otimes \mathrm{d}x^4\right) . \tag{3.75}$$

(Rigorous proof of the above conclusion can be found in [250].)

We can also express (3.75) as

$$\mathrm{d}s^2 = e^{\alpha(r,t)}(\mathrm{d}r)^2 + r^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\varphi)^2\right] - e^{\gamma(r,t)}(\mathrm{d}t)^2 \,;$$

$$e^i_{(1)}(r,t) = e^{-\alpha/2} \cdot \delta^i_{(1)} \,, \quad e^i_{(2)}(r) = r^{-1} \cdot \delta^i_{(2)} \,,$$

$$e^i_{(3)}(r,\theta) = (r \sin\theta)^{-1} \cdot \delta^i_{(3)} \,, \quad e^i_{(4)}(r,t) = e^{-\gamma/2} \cdot \delta^i_{(4)} \,. \tag{3.76}$$

In the present case, the ten field equations (2.161i) and four conservation equations (2.166i) reduce to the following[3]:

$$\mathcal{E}^1{}_1(r,t) := r^{-2} \cdot [1 - e^{-\alpha}(1 + r\partial_1\gamma)] + \kappa\, T^1{}_1(r,t) = 0 \,, \tag{3.77i}$$

[3]To derive (3.77vii,viii), use the following equations for a symmetric tensor S_{jk}: $\nabla_j S^j_{\ k} = \left(1/\sqrt{|g|}\right) \cdot \partial_j \left(\sqrt{|g|} \cdot S^j_k\right) - \frac{1}{2} \cdot \left(\partial_k g_{ij}\right) \cdot S^{ij}$.

$$\begin{aligned}\mathcal{E}^2{}_2(r,t) &:= (1/2)\mathrm{e}^{-\alpha}\cdot[-\partial_1\partial_1\gamma + (1/2r)\cdot(r\partial_1\gamma+2)\,\partial_1(\alpha-\gamma)] \\ &\quad + (1/2)\mathrm{e}^{-\gamma}\cdot[\partial_4\partial_4\alpha + (1/2)\cdot\partial_4\alpha\cdot\partial_4(\alpha-\gamma)] \\ &\quad + \kappa\,T^2{}_2(r,t) = 0\,,\end{aligned}\tag{3.77ii}$$

$$\mathcal{E}^3{}_3(r,t) \equiv \mathcal{E}^2{}_2(r,t) = 0\,,\tag{3.77iii}$$

$$\mathcal{E}^1{}_4(r,t) := r^{-1}\cdot\partial_4\left(e^{-\alpha}\right) + \kappa\,T^1{}_4(r,t) = 0\,,\tag{3.77iv}$$

$$\mathcal{E}^4{}_4(r,t) := r^{-2}\cdot[1-\partial_1\left(re^{-\alpha}\right)] + \kappa\,T^4{}_4(r,t) = 0\,,\tag{3.77v}$$

$$\begin{aligned}&T^2{}_2(r,t) \equiv T^3{}_3(r,t)\,, \\ &\quad T^1{}_2(\cdot\cdot) = T^1{}_3(\cdot\cdot) = T^2{}_3(\cdot\cdot) = T^2{}_4(\cdot\cdot) = T^3{}_4(\cdot\cdot) \equiv 0\,,\end{aligned}\tag{3.77vi}$$

$$\begin{aligned}\mathcal{T}_1(r,t) &:= \partial_1 T^1{}_1 + \partial_4 T^4{}_1 + (2/r)\cdot[1+(r/4)\,\partial_1\gamma]\cdot T^1{}_1(r,t) \\ &\quad + (1/2)\,\partial_4(\alpha+\gamma)\cdot T^4{}_1(r,t) - (1/2)\,\partial_1\gamma\cdot T^4{}_4(r,t) \\ &\quad - (2/r)\,T^2{}_2(r,t) = 0\,,\end{aligned}\tag{3.77vii}$$

$$\begin{aligned}\mathcal{T}_4(r,t) &:= \partial_1 T^1{}_4 + \partial_4 T^4{}_4 + (2/r)\cdot[1+(r/4)\,\partial_1(\alpha+\gamma)]\cdot T^1{}_4(r,t) \\ &\quad + (1/2)\,\partial_4\alpha\cdot\left[T^4{}_4(r,t) - T^1{}_1(r,t)\right] = 0\,,\end{aligned}\tag{3.77viii}$$

$$\mathcal{T}_2(r,t) = \mathcal{T}_3(r,t) \equiv 0\,.\tag{3.77ix}$$

We investigate the above system of partial differential equations in the following two-dimensional convex domain in the coordinate plane:

$$D : \{(r,t) :\ 0 < r < B(t),\ t_1 < t < t_2\}\,.\tag{3.78}$$

Here, the boundary curve $r = B(t)$, representing the boundary of a nonstatic spherical body, is assumed to be twice differentiable. (See Fig. 3.6.)

Now, in a convex domain,[4] the general solution of the system of partial differential equations (3.77i–ix) will be provided. (See [63].)

Theorem 3.3.1. *Let metric functions $\alpha(r,t)$ and $\gamma(r,t)$ of* (3.76) *be of class C^3 in the convex domain D of* (3.78)*. Then, the general solution of the system of partial differential equations* (3.77i–ix) *is furnished by:*

[4]Counterexamples for nonconvex domains are provided in [61].

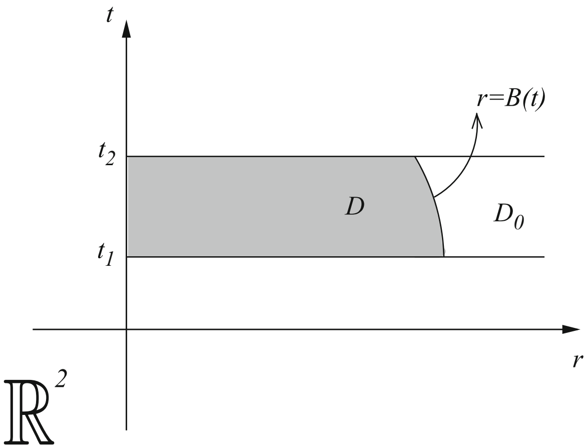


Fig. 3.6 A convex domain D in a two-dimensional coordinate plane

$$e^{-\alpha(r,t)} = 1 - \frac{\kappa}{r}\left[f(t) - \int_{0+}^{r} T^4_{\ 4}(y,t)\, y^2 \mathrm{d}y\right] > 0\,, \tag{3.79i}$$

$$e^{\gamma(r,t)} = e^{-\alpha(r,t)} \cdot \left\{\exp\left[h(t) + \kappa \int_{0+}^{r} \left[T^1_{\ 1}(y,t) - T^4_{\ 4}(y,t)\right] \cdot e^{\alpha(y,t)} \cdot y\, \mathrm{d}y\right]\right\}, \tag{3.79ii}$$

$$T^1_{\ 4}(r,t) := \frac{1}{r^2} \cdot \left[\frac{\mathrm{d}f(t)}{\mathrm{d}t} - \int_{0+}^{r} \partial_4 T^4_{\ 4} \cdot y^2 \mathrm{d}y\right], \tag{3.79iii}$$

$$\begin{aligned} T^2_{\ 2}(r,t) \equiv T^3_{\ 3}(r,t) &:= (r/2) \cdot \left[\partial_1 T^1_{\ 1} + \partial_4 T^4_{\ 1}\right] + [1 + (r/4)\partial_1 \gamma] \cdot T^1_{\ 1} \\ &\quad + (r/4) \cdot \partial_4(\alpha + \gamma) \cdot T^4_{\ 1} - (r/4) \cdot \partial_1 \gamma \cdot T^4_{\ 4}\,, \end{aligned} \tag{3.79iv}$$

where $f(t)$ and $h(t)$ are two arbitrary C^3 functions of integration.

Proof. *The mixed method* of p. 197 is used. *Prescribe* $T^4_{\ 4}(r,t)$ and solve $\mathcal{E}^4_{\ 4}(r,t) = 0$ to obtain $e^{-\alpha(r,t)}$ of (3.79i). *Prescribe* $T^1_{\ 1}(r,t)$ and solve $\mathcal{E}^1_{\ 1}(r,t) - \mathcal{E}^4_{\ 4}(r,t) = 0$ to get $e^{\gamma(r,t)}$ of (3.79ii). *Define* $T^1_{\ 4}(r,t)$ by the equation $\mathcal{E}^1_{\ 4}(r,t) = 0$ to obtain (3.79iii). Finally, *define* $T^2_{\ 2}(r,t) \equiv T^3_{\ 3}(r,t)$ by the conservation equation $\mathcal{T}_1(r,t) = 0$ to get (3.79iv). At this stage, using (3.77vi), *all other field equations are satisfied.* ■

(*Remark:* Auxiliary conditions, such as an equation of state or the condition of isotropy, may be imposed. These scenarios are also encompassed in the above solution.)

Now we investigate the nonstatic, spherically symmetric metric (3.76) and *vacuum field equations* (2.164i) to derive *Birkhoff's theorem* [20, 144, 184].

Theorem 3.3.2. *Let metric functions $\alpha(r,t)$ and $\gamma(r,t)$ of (3.76) be of class C^3 in the rectangular coordinate domain*

$$D := \{(r,t) : 0 < r_1 \leq r,\ t_1 < t < t_2\}\,.$$

Then, the general solution of the vacuum equations is transformable to the Schwarzschild metric.

Proof. For vacuum equations (2.164i), we put $T^i{}_j(r,t) \equiv 0$. Equation (3.79iii) yields

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = 0\,,$$

$$\text{or,} \qquad f(t) = 2\kappa^{-1}m = \text{const.}$$

Therefore, (3.79i,ii) imply that

$$\mathrm{e}^{-\alpha(r,t)} = 1 - \frac{2m}{r} > 0\,,$$

$$\mathrm{e}^{\gamma(r,t)} = \left(1 - \frac{2m}{r}\right) \cdot \exp[h(t)] > 0.$$

A coordinate transformation is introduced by

$$\widehat{r} = r\,, \quad \widehat{\theta} = \theta\,, \quad \widehat{\varphi} = \varphi\,,$$

$$\widehat{t} = \int_{t_1}^{t} \exp\left[\frac{h(w)}{2}\right] \mathrm{d}w\,.$$

Dropping hats subsequently, the Schwarzschild metric (3.9) is established. ■

Remark: One physical implication of Birkhoff's theorem is that a pulsating, spherically symmetric body *cannot generate external gravitational waves* (see Appendix 5).

Now we shall derive the *total mass function $M(r,t)$* and the boundary surface of the spherical body. We notice from field equations (3.77i–ix) *two additional differential identities*:

$$\partial_1\left(r^2 G^1{}_4\right) + \partial_4\left(r^2 G^4{}_4\right) \equiv 0\,, \tag{3.80i}$$

$$\partial_1(\alpha+\gamma)\cdot G^1{}_4(r,t) + \partial_4\alpha \cdot \left[G^4{}_4(\cdot\cdot) - G^1{}_1(\cdot\cdot)\right] \equiv 0\,. \tag{3.80ii}$$

(However, since $\nabla_j G^j_4 \equiv 0$, only *one of above identities* is independent.) Thus, from (3.80i) and field equations (3.77i–ix), we deduce *the new conservation equation*:

$$(\kappa/2)\cdot\left[\partial_1\left(r^2 T^1_4\right) + \partial_4\left(r^2 T^4_4\right)\right] = 0\,. \tag{3.81}$$

This conservation equation has an immediate physical interpretation. It states that the rate of change of “energy” inside a sphere of radius r is equal to the “energy” flux through the boundary of the same sphere.

Consider the 1-form $(\kappa/2)\cdot\left[r^2 T^1{}_4(r,t)\,\mathrm{d}t - r^2 T^4{}_4(r,t)\,\mathrm{d}r\right]$. It is *a closed form* due to (3.81) in *a convex coordinate domain*. Therefore, by Theorem 1.2.21, there *exists a differentiable function* $M(r,t)$ such that

$$\mathrm{d}M(r,t) = (\kappa r^2/2)\cdot\left[T^1{}_4(r,t)\,\mathrm{d}t - T^4{}_4(r,t)\,\mathrm{d}r\right], \tag{3.82i}$$

$$\partial_1 M = -(\kappa/2) r^2 \cdot T^4{}_4(r,t),$$

$$\partial_4 M = (\kappa/2) r^2 \cdot T^1{}_4(r,t). \tag{3.82ii}$$

The integration of the differential equation (3.82i) leads to the *total mass function* $M(r,t)$. Consider a level curve furnished by $M(r,t) = c > 0$. Let an interior point (r_0, t_0) of D, be such that $M(r_0,t_0) = c$ and $\partial_1 M_{|(r_0,t_0)} = -\frac{\kappa}{2}(r_0)^2\cdot T^4{}_4(r_0,t_0) > 0$. By the implicit function theorem (see [32]), there exists a differentiable curve $r = \mathsf{B}(t;c)$ in the neighborhood of (r_0,t_0) such that $M[\mathsf{B}(t;c),t] \equiv c$ and $r_0 = \mathsf{B}(t_0;c)$. Out of the one-parameter level curves $r = \mathsf{B}(t;c)$, the boundary curve of a spherically symmetric body must be defined as follows:

$$r = B(t) := \lim_{c\to m_-} \mathsf{B}(t;c),$$

$$\partial D := \{(r,t) : r = B(t),\ t_1 < t < t_2\},$$

$$M\,[B(t);t] \equiv m > c > 0,$$

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = -\left[\partial_4 M/\partial_1 M\right]_{|\partial D} = \left[T^1{}_4(\cdot\cdot)/T^4{}_4(\cdot\cdot)\right]_{|\cdot\cdot}. \tag{3.83}$$

The above equations are valid on the boundary curve ∂D of Fig. 3.6.

Let us explore Synge’s junction condition (2.170) on the boundary $r = B(t)$. By (3.79i,ii) and (3.82ii), we deduce that

$$2M(r,t) = \kappa\left[f(t) - \int_{0+}^{r} T^4{}_4(y,t)\,y^2\mathrm{d}y\right] > 0, \tag{3.84i}$$

$$\mathrm{e}^{-\alpha(r,t)} = \begin{cases} 1 - \dfrac{2M(r,t)}{r} & \text{for } 0 < r < B(t),\ t_1 < t < t_2, \\ 1 - \dfrac{2m}{r} > 0 & \text{for } B(t) < r,\ t_1 < t < t_2; \end{cases} \tag{3.84ii}$$

$$e^{\gamma(r,t)} = \begin{cases} \left[1 - \frac{2M(r,t)}{r}\right] \cdot \exp\left[h(t) + \chi(r,t)\right] & \text{for } 0 < r < B(t), \ldots, \\ \left[1 - \frac{2m}{r}\right] \cdot \exp\left[h(t) + \chi(B(t),t)\right] & \text{for } B(t) < r, \ldots, \end{cases} \tag{3.84iii}$$

$$\chi(r,t) := \kappa \int_{0+}^{r} \left[\frac{T^1_{\ 1}(y,t) - T^4_{\ 4}(y,t)}{y - 2M(y,t)}\right] \cdot y^2 \mathrm{d}y\,. \tag{3.84iv}$$

Continuity of the functions $\alpha(r,t)$ and $\gamma(r,t)$ across the boundary is obvious. Moreover, the external metric is transformable to the Schwarzschild metric. (Note that we are assuming a matter-vacuum boundary.)

Junction conditions $T^i_{\ j}(\cdot\cdot)\, n_i{}_{|\partial D} = 0$ reduce to

$$\left[T^1_{\ 1}(\cdot\cdot) \cdot \partial_1 M + T^4_{\ 1}(\cdot\cdot) \cdot \partial_4 M\right]_{|\partial D} = 0\,, \tag{3.85i}$$

$$\left[T^1_{\ 4}(\cdot\cdot) \cdot \partial_1 M + T^4_{\ 4}(\cdot\cdot) \cdot \partial_4 M\right]_{|\partial D} = 0\,. \tag{3.85ii}$$

The condition (3.85ii) is identically satisfied by (3.82ii,iii). In case the "velocity" of the boundary $\frac{\mathrm{d}B(t)}{\mathrm{d}t} \neq 0$, the other junction condition (3.85i) can be satisfied by the choice:

$$e^{h(t)} := \left[\frac{\mathrm{d}B(t)}{\mathrm{d}t}\right]^2 \cdot \left\{\left[1 - \frac{2m}{r}\right]^{-2} \cdot \exp\left[-\chi(r,t)\right] \cdot \left|\frac{T^4_{\ 4}(r,t)}{T^1_{\ 1}(r,t)}\right|\right\}_{|r=B(t)} > 0\,. \tag{3.86}$$

(See [63].) We notice that the energy–momentum–stress matrix $[T^i_{\ j}(r,t)]$ in (3.79i–iv) is *not necessarily diagonalizable*. The criterion for *diagonalization* is

$$\Delta(r,t) := \left[T^1_{\ 1}(\cdot\cdot) - T^4_{\ 4}(\cdot\cdot)\right]^2 - 4e^{\alpha-\gamma} \cdot \left[T^1_{\ 4}(\cdot\cdot)\right]^2 \geq 0\,. \tag{3.87}$$

(In case of $\Delta(r,t) < 0$, there exist *complex eigenvalues and hence exotic matter*.)

In case $\Delta(r,t) \geq 0$, (2.194i) can be employed to investigate the *weak energy conditions* (2.190).

Example 3.3.3. We shall generalize Schwarzschild's interior solution of Example 3.2.2 to the nonstatic arena. (See [67].)

We choose the total mass function and the domain of validity by the following:

$$M(r,t) := \left(\frac{q}{2}\right) \cdot \left[\frac{r}{1 - (t/c_2)}\right]^j\,,$$
$$0 < q < 1\,,\quad 2 < j \leq 3\,,\quad \sqrt{3} \leq k\,,\ k = [j/(j-2)]^{1/2}\,,$$

$$b > 0\,, \quad c_2 := kb > 0\,, \quad c_1 := c_2 - kqb^j < c_2\,;$$

$$D := \{(r,t) : 0 < r < B(t),\; t < c_1\}\,.$$

By (3.82i,ii) and *a linear equation of state* $T^4_{\ 4}(..) + (k)^2 \cdot T^1_{\ 1}(..) = 0$, we obtain

$$\kappa\, T^4_{\ 4}(r,t) = -\frac{j q\, r^{j-3}}{[1-(t/c_2)]^j}\,,$$

$$\kappa\, T^1_{\ 4}(r,t) = \frac{j\; q\, r^{j-2}}{c_2[1-(t/c_2)]^{j+1}}\,,$$

$$\kappa\, T^1_{\ 1}(r,t) = \frac{(j-2)\, q\, r^{j-3}}{[1-(t/c_2)]^j}\,,$$

$$\chi(r,t) = -2 \cdot \ln\left|1 - \frac{q\, r^{j-1}}{[1-(t/c_2)]^j}\right|\,.$$

By choosing $f(t) \equiv 0$ and choosing $h(t)$ from (3.86), we get from (3.84ii,iii):

$$\mathrm{e}^{\alpha(r,t)} = \left[1 - \frac{q\, r^{j-1}}{[1-(t/c_2)]^j}\right]^{-1} = \mathrm{e}^{\gamma(r,t)}\,.$$

The boundary of the spherical body is provided by

$$r = B(t) := \left(\frac{c_2 - t}{k}\right)\,,$$

$$\left|\frac{\mathrm{d}B(t)}{\mathrm{d}t}\right| = \left|-\frac{1}{k}\right| < 1\,,$$

$$M[B(t),t] \equiv (q/2)\cdot(c_2)^j = (q/2)\cdot(b)^j =: m > 0\,,$$

$$B(c_1) = \left(\frac{c_2 - c_1}{k}\right) = 2m\,.$$

Synge's junction condition (3.85ii) is identically satisfied. The other junction condition (3.85i) is also satisfied by

$$\kappa\left[T^1_{\ 1}(\cdot\cdot)\cdot 1 - T^1_{\ 4}(\cdot\cdot)\cdot\frac{1}{k}\right]_{|r=k^{-1}(c_2-t)}$$

$$= \left\{\frac{(j-2)\, q\, r^{j-3}}{[1-(t/c_2)]^j} - \frac{j\; q\, r^{j-2}}{k\, c_2[1-(t/c_2)]^{j+1}}\right\}_{|r=k^{-1}(c_2-t)}$$

$$\equiv 0\,.$$

Equation (3.87) yields

$$\Delta(r,t) = \left\{ \frac{2q \cdot r^{j-3}}{c_2[1-(t/c_2)]^{j+1}} \right\}^2 \cdot \{(j-1)^2 \cdot (c_2 - t)^2 - j^2 \cdot r^2\} > 0\,.$$

Therefore, the energy–momentum–stress tensor is *diagonalizable*. We can identify the material with an anisotropic fluid (of (2.264)) or with a deformable solid characterized by

$$T^i{}_j(r,t) = (\rho + p_\perp) \cdot u^i u_j + p_\perp \cdot \delta^i{}_j + \left(p_\parallel - p_\perp\right) \cdot s^i s_j\,,$$

$$u^1 = s^4\,, \quad u^4 = s^1 > 0\,,$$

$$u^2 = u^3 = s^2 = s^3 \equiv 0\,.$$

In this example, *weak energy conditions (2.190) are satisfied.* □

Now we shall explore the *Tolman–Bondi–Lemaître metric* [24, 162, 249]:

$$\begin{aligned}
\mathbf{g}_{..}(x) &:= \mathrm{e}^{2\lambda(x^1,x^4)} \left(\mathrm{d}x^1 \otimes \mathrm{d}x^1\right) + \left[Y\left(x^1, x^4\right)\right]^2 \cdot \left[\left(\mathrm{d}x^2 \otimes \mathrm{d}x^2\right)\right.\\
&\quad \left. + \sin^2 x^2 \cdot \left(\mathrm{d}x^3 \otimes \mathrm{d}x^3\right)\right] - \left(\mathrm{d}x^4 \otimes \mathrm{d}x^4\right),\\
\mathrm{d}s^2 &= \mathrm{e}^{2\lambda(r,t)}(\mathrm{d}r)^2 + [Y(r,t)]^2 \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right] - (\mathrm{d}t)^2\,;\\
e^i_{(1)}(\cdot\cdot) &= \mathrm{e}^{-\lambda} \cdot \delta^i_{(1)}\,, \quad e^i_{(2)}(\cdot\cdot) = Y^{-1} \cdot \delta^i_{(2)}\,,\\
e^i_{(3)}(\cdot\cdot) &= (Y \sin\theta)^{-1} \cdot \delta^i_{(3)}\,, \quad e^i_{(4)} = \delta^i_{(4)}\,;\\
D &:= \{(r,\theta,\varphi,t) : r_1 < r < r_2,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ t_1 < t < t_2\}\,.
\end{aligned} \tag{3.88}$$

(See [238, 250].)

The metric above implies a geodesic normal (or Gaussian normal) coordinate chart of (1.160). The t-coordinate curves represent *timelike geodesics*. Let us investigate field equations (2.161i) and (2.166i) in the presence of an incoherent dust modeled by (2.256). We *assume a comoving chart* (see p. 193) so that the 4-velocity fields are tangential to t-coordinate curves. Therefore, we can choose

$$U^\alpha(\cdot\cdot) \equiv 0\,, \quad U^4(\cdot\cdot) \equiv 1\,. \tag{3.89}$$

Moreover, we denote

$$y = Y(r,t)\,, \quad y' := \partial_1 Y\,, \quad \dot{y} := \partial_4 Y.$$

Thus, the nontrivial field equations (2.161i) and (2.166i), from (3.88) and (3.89), reduce to the following:

$$\mathcal{E}^1{}_1(r,t) := y^{-2} + 2y^{-1} \cdot \left[\ddot{y} + (\dot{y}^2/2y)\right] - y^{-2} \cdot \mathrm{e}^{-2\lambda} \cdot \left(y'\right)^2 = 0\,, \tag{3.90i}$$

$$\begin{aligned}\mathcal{E}^2{}_2(r,t) := \mathrm{e}^{-2\lambda} \cdot \left[(y'\lambda'/y) - (y''/y)\right] \\ + \left[\ddot{\lambda} + \left(\dot{\lambda}\right)^2 + (\ddot{y}/y) + \left(\dot{y}\dot{\lambda}/y\right)\right] = 0\,,\end{aligned} \tag{3.90ii}$$

$$\mathcal{E}^3{}_3(r,t) \;\equiv \mathcal{E}^2{}_2(r,t) = 0\,, \tag{3.90iii}$$

$$\begin{aligned}\mathcal{E}^4{}_4(r,t) := y^{-2} - 2\left(\mathrm{e}^{-2\lambda}/y\right) \cdot \left[y'' - y'\lambda' + \left(y'^2/2y\right)\right] \\ + 2y^{-1} \cdot \left[\dot{y}\dot{\lambda} + \left(\dot{y}^2/2y\right)\right] - \kappa\, \rho(r,t) = 0\,,\end{aligned} \tag{3.90iv}$$

$$\mathcal{E}^4{}_1(r,t) := \; - (2/y) \cdot \left[\dot{y}' - y'\dot{\lambda}\right] = 0\,, \tag{3.90v}$$

$$\mathcal{T}_4(r,t) := \; - \left[\dot{\rho} + (\lambda + 2\ln|y|)^{\dot{}} \cdot \rho(r,t)\right] = 0\,. \tag{3.90vi}$$

(See [250].)

Our strategy to solve the above system is *the following mixed method*:

(i) We prescribe $T_{\alpha\beta}(..) \equiv 0,\; T_{\alpha 4}(..) \equiv 0$.
(ii) We solve the equation $\mathcal{E}^4{}_1(\cdot\cdot) = 0$.
(iii) Next, we solve the equation $\mathcal{E}^1{}_1(\cdot\cdot) = 0$.
(iv) We *define* $\rho(r,t)$ from the equation $\mathcal{E}^4{}_4(\cdot\cdot) = 0$.
(v) At this stage $\mathcal{T}_4(\cdot\cdot) = 0$ and *all other field equations are satisfied*.

Solving (3.90v), with *an additional assumption* $y' \neq 0$, we obtain

$$\begin{aligned}\lambda(r,t) &= \ln\left|y'\right| + j(r)\,, \\ \mathrm{e}^{2\lambda(r,t)} &= \mathrm{e}^{2j(r)} \cdot (\partial_1 Y)^2 > 0\,.\end{aligned} \tag{3.91}$$

Here, $j(r)$ is the thrice differentiable, arbitrary function of integration. To cast the system in a more familiar form found in the literature [238], in terms of *another arbitrary function*, $f(r)$, we express

$$\begin{aligned}&\exp\left[-2j(r)\right] =: 1 - \varepsilon \left[f(r)\right]^2\,, \\ &\varepsilon = 0, \pm 1\,,\end{aligned}$$

$$\begin{aligned}
&\varepsilon\,[f(r)]^2 < 1\,,\\
&g_{11}(r,t) = \mathrm{e}^{2\lambda(r,t)} = \frac{[\partial_1 Y]^2}{1-\varepsilon[f(r)]^2} > 0\,.
\end{aligned} \tag{3.92}$$

Substituting (3.92) into (3.90i) and multiplying with $y^2 \cdot \dot{y}$, we derive that

$$y^2\dot{y}\cdot\mathcal{E}^1{}_1(\cdot\cdot) = \dot{y}\cdot\left[1-1+\varepsilon\, f^2(r)\right] + 2y\dot{y}\cdot\left[\ddot{y} + \left(\dot{y}^2/2y\right)\right] = 0\,,$$

or, $$\left[y\,(\dot{y})^2\right]^{\cdot} = -\varepsilon\, f^2(r)\cdot\dot{y}\,. \tag{3.93}$$

Integrating the above equation, we deduce that

$$[\partial_4 Y]^2 = \frac{2\overline{M}(r)}{Y(r,t)} - \varepsilon\,[f(r)]^2 > 0\,. \tag{3.94}$$

Here, $\overline{M}(r)$ is the thrice differentiable, positive-valued, arbitrary function of integration. At this stage, all the field equations have reduced to *one first-order, nonlinear differential equation (3.94)*. Now we shall interpret (3.94) in physical terms. In the metric (3.88), the function $Y(r,t)$ plays the role of a "radius." Consider a particular point r_0 in the interval (r_1, r_2). Equation (3.94) yields

$$\frac{1}{2}\cdot\left[\frac{\partial Y(r_0,t)}{\partial t}\right]^2 - \frac{\overline{M}(r_0)}{Y(r_0,t)} = -\left(\frac{\varepsilon}{2}\right)\cdot[f(r_0)]^2 = \text{const.} \tag{3.95}$$

It is interesting to note that the equation above *resembles the Newtonian energy conservation equation* for a test particle of unit mass following the trajectory $y = Y(r_0,t)$ in a spherically symmetric gravitational potential $-\left[\overline{M}(r_0)\big/Y(r_0,t)\right]$.

Let us *assume that*

$$\begin{aligned}
\frac{\mathrm{d}\overline{M}(r)}{\mathrm{d}r} &> 0\,,\\
\lim_{r\to r_{1+}}\left[\overline{M}(r)\right] &= 0\,,\\
y' &> 0\,.
\end{aligned} \tag{3.96}$$

Then, it follows from (3.90iv,v) and (3.94) that

$$\rho(r,t) = \left[2\frac{\mathrm{d}\,\overline{M}(r)}{\mathrm{d}r}\bigg/\kappa\, y^2 y'\right] > 0\,. \tag{3.97}$$

Thus, energy conditions (2.190)–(2.192) are all satisfied. Moreover, *the effective total mass function*

$$
\begin{aligned}
M^{\#}(r,t) &:= \int_{r_1}^{r}\int_{0+}^{\pi-}\int_{-\pi+}^{\pi-} \rho(w,t)\cdot[Y(w,t)]^2\cdot[\partial_1 Y]\cdot\sin\theta\,\mathrm{d}w\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \overline{M}(r)\,, \\
\frac{\partial M^{\#}(r,t)}{\partial t} &\equiv 0\,.
\end{aligned}
\tag{3.98}
$$

Now we shall solve *the vacuum equations* with the metric in (3.88).

Example 3.3.4. The vacuum is characterized by $\rho(r,t) \equiv 0$ in (3.90iv). Equation (3.97) implies that

$$\overline{M}(r) = m = \text{positive const.}, \tag{3.99}$$

and (3.94) reduces to

$$[\partial_4 Y]^2 = \frac{2m}{Y(r,t)} - \varepsilon\,[f(r)]^2\,. \tag{3.100}$$

Out of *three possible cases*, let us consider *the parabolic case* $\varepsilon = 0$. Equation (3.100), for the case of expansion, reduces to

$$\partial_4 Y = \sqrt{2m/Y(r,t)} > 0\,. \tag{3.101}$$

Solving the equation above, we obtain

$$
\begin{aligned}
Y(r,t) &= (3/2)^{2/3}\cdot(2m)^{1/3}\cdot[T(r)-t]^{2/3} > 0\,,\\
\partial_1 Y &= (2/3)\cdot Y(r,t)\cdot\left[T'(r)\big/(T(r)-t)\right].
\end{aligned}
\tag{3.102}
$$

Here, $T(r)$ is the thrice differentiable *arbitrary function* of integration. Thus, the metric (3.88) becomes

$$
\begin{aligned}
\mathrm{d}s^2 = {}& (2m)\cdot\left\{(3/2)\cdot\sqrt{2m}\cdot[T(r)-t]\right\}^{-2/3}\cdot\left[T'(r)\right]^2\mathrm{d}r^2 \\
&+\left\{(3/2)\cdot\sqrt{2m}\cdot[T(r)-t]\right\}^{4/3}\left[(\mathrm{d}\theta)^2+\sin^2\theta\,(\mathrm{d}\varphi)^2\right]-\mathrm{d}t^2.
\end{aligned}
\tag{3.103}
$$

Making a coordinate transformation

$$
\begin{aligned}
& r^{\#} = T(r)\,, \quad \left(\theta^{\#},\varphi^{\#},t^{\#}\right) = (\theta,\varphi,t)\,,\\
& Y^{\#}\left(r^{\#},t^{\#}\right) := \left[(3/2)\cdot\sqrt{2m}\cdot\left(r^{\#}-t^{\#}\right)\right]^{2/3} = Y(r,t)\,,
\end{aligned}
\tag{3.104}
$$

the metric (3.103) reduces to

$$\begin{aligned} ds^2 = & \left[2m/Y^{\#}\left(r^{\#},t^{\#}\right)\right]\cdot\left(dr^{\#}\right)^2 \\ & + \left[Y^{\#}\left(r^{\#},t^{\#}\right)\right]^2\cdot\left[\left(d\theta^{\#}\right)^2+\sin^2\theta^{\#}\cdot\left(d\varphi^{\#}\right)^2\right]-\left(dt^{\#}\right)^2 . \end{aligned} \tag{3.105}$$

The above metric is known as the *Lemaître metric* [58, 238]. We notice that $-(1/2)\,g^{\#}_{11}(\cdot\cdot) = -m/Y^{\#}(\cdot\cdot)$, resembling the Newtonian gravitational potential. (Glance through the comments made on p. 198.)

Now, *the nonstatic Lemaître metric* in (3.105) is an exact solution of the spherically symmetric vacuum field equations in a domain $D^{\#}_0$. By Birkhoff's Theorem 3.3.2, this metric must be transformable to the static Schwarzschild metric over *a subset of* $D^{\#}_0$. The explicit transformation to the Schwarzschild chart is provided by the following equations:

$$\begin{aligned} \widehat{r} &= \left[(3/2)\cdot\sqrt{2m}\cdot\left(r^{\#}-t^{\#}\right)\right]^{2/3}, \\ \left(\widehat{\theta},\widehat{\varphi}\right) &= \left(\theta^{\#},\varphi^{\#}\right), \\ \widehat{t} &= t^{\#}-2\sqrt{2m}\cdot\left[(3/2)\cdot\sqrt{2m}\cdot\left(r^{\#}-t^{\#}\right)\right]^{1/3} \\ &\quad + 2m\cdot\ln\left\{\frac{\left[(3/2)\sqrt{2m}\left(r^{\#}-t^{\#}\right)\right]^{1/3}+\sqrt{2m}}{\left[(3/2)\sqrt{2m}\left(r^{\#}-t^{\#}\right)\right]^{1/3}-\sqrt{2m}}\right\}, \\ & \left[(3/2)\cdot\sqrt{2m}\cdot\left(r^{\#}-t^{\#}\right)\right]^{2/3} > 2m. \end{aligned} \tag{3.106}$$

□

Now we shall touch upon the spherically symmetric, T-*domain* solutions. (See [73, 222].) The T-domain corresponds, for example, to the domain $0 < r < 2m$ of metric (3.9). Here we have changed the notation for the coordinates, as what was the radial coordinate is now *a time coordinate* and what was the time coordinate is now *a spatial coordinate*. We also denote the stress–energy–momentum tensor components by $\Theta^{i}{}_{j}(\cdot\cdot)$ to avoid confusion with the components of the stress–energy–momentum tensor in the previous domain.

The metric is *assumed to be locally reducible to*

$$\begin{aligned} ds^2 &= e^{\lambda(T,R)}(dR)^2 + T^2\left[(d\theta)^2+\sin^2\theta\,(d\varphi)^2\right]-e^{\nu(T,R)}(dT)^2, \\ D &:= \left\{(T,R):\ T_1<T<T_2,\ R_1<R<R_2,\ 0<\theta<\pi,\ -\pi<\phi\le\pi\right\}. \end{aligned} \tag{3.107}$$

The general solution of field equations $G^i{}_j(\cdot\cdot) + \kappa\,\Theta^i{}_j(\cdot\cdot) = 0$ and conservation equations $\nabla_j \Theta^j{}_i = 0$ is furnished by

$$
\begin{aligned}
\mathrm{e}^{-\nu(T,R)} &= \frac{1}{T}\left[\sigma(R) - \kappa \int_{T_1}^{T} \Theta^1{}_1(w,R) w^2 \mathrm{d}w\right] - 1 \\
&=: \frac{2\Xi(T,R)}{T} - 1 > 0\,, \qquad (3.108\mathrm{i})
\end{aligned}
$$

$$
\mathrm{e}^{\lambda(T,R)} = \left[\frac{2\Xi(T,R)}{T} - 1\right] \cdot \exp\left[\beta(R) + \kappa \int_{T_1}^{T} \mathrm{e}^{\nu(w,R)} \cdot \left[\Theta^4_4 - \Theta^1_1\right] w \mathrm{d}w\right], \qquad (3.108\mathrm{ii})
$$

$$
\Theta^4{}_1(T,R) := \frac{2}{\kappa T^2} \cdot \partial_1\left[\Xi(T,R)\right], \qquad (3.108\mathrm{iii})
$$

$$
\Theta^1{}_2(\cdot\cdot) = \Theta^1{}_3(\cdot\cdot) = \Theta^2{}_3(\cdot\cdot) = \Theta^4{}_2(\cdot\cdot) = \Theta^4{}_3(\cdot\cdot) \equiv 0\,, \qquad (3.108\mathrm{iv})
$$

$$
\begin{aligned}
\Theta^2{}_2(\cdot\cdot) \equiv \Theta^3{}_3(\cdot\cdot) &:= (T/2) \cdot \left[\partial_4 \Theta^4{}_4 + \partial_1 \Theta^1{}_4\right] + [1 + (T/4) \cdot \partial_4 \lambda] \cdot \Theta^4{}_4(\cdot\cdot) \\
&\quad + (T/4) \cdot \partial_1(\lambda + \nu) \cdot \Theta^1{}_4(\cdot\cdot) \\
&\quad - (T/4) \cdot \partial_4 \lambda \cdot \Theta^1{}_1(\cdot\cdot)\,. \qquad (3.108\mathrm{v})
\end{aligned}
$$

Here, $\sigma(R)$ and $\beta(R)$ are thrice differentiable, *arbitrary functions* of integration.

Remarks: (i) There exists a "duality" between solutions of (3.79i–iv) and the solutions of (3.108i–iv).

(ii) The function $\Xi(T,R)$, which is the "dual" of the total mass function $M(r,t)$, arises from the integral of *the local tension of the "material."*

(iii) The diagonalization of the matrix $\left[\Theta^i{}_j(\cdot\cdot)\right]$ is guaranteed if and only if

$$
\Delta^{\#}(\cdot\cdot) := \left[\Theta^4{}_4(\cdot\cdot) - \Theta^1{}_1(\cdot\cdot)\right]^2 + 4\Theta^1{}_4(\cdot\cdot) \cdot \Theta^4{}_1(\cdot\cdot) \geq 0\,.
$$

Otherwise, *complex eigenvalues* ensue.

Example 3.3.5. We study *the vacuum field equations* with the metric in (3.107). We set $\Theta^i{}_j(T,R) \equiv 0$ in the solutions in (3.108i–v). Therefore, from (3.108i), we derive

$$
2\Xi(T,R) = \sigma(R)\,. \qquad (3.109)
$$

By (3.108iii) and (3.109), we conclude that

$$\frac{\mathrm{d}\sigma(R)}{\mathrm{d}R} = 0\,,$$

or $$2\,[\Xi(T,R)] = \sigma(R) = 2m = \text{const.}\,,$$

$$0 < T < 2m\,. \tag{3.110}$$

From (3.108i,ii), we obtain the metric as

$$\begin{aligned}\mathrm{d}s^2 = {} & \left[\frac{2m}{T} - 1\right]\cdot \mathrm{e}^{\beta(R)}\cdot(\mathrm{d}R)^2 + T^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta\cdot(\mathrm{d}\varphi)^2\right] \\ & - \left[\frac{2m}{T} - 1\right]^{-1}\cdot(\mathrm{d}T)^2\,. \end{aligned}\tag{3.111}$$

By the coordinate transformation

$$\widehat{R} = \int_{R_1}^{R} \exp\left[\beta(w)/2\right]\cdot \mathrm{d}w\,,$$

$$\left(\widehat{\theta}, \widehat{\varphi}, \widehat{T}\right) = (\theta, \varphi, T)\,,$$

the metric in (3.111) reduces to

$$\begin{aligned}&\mathrm{d}s^2 = \left[\frac{2m}{\widehat{T}} - 1\right]\cdot\left(\mathrm{d}\widehat{R}\right)^2 + \widehat{T}^2\left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2\widehat{\theta}\,(\mathrm{d}\widehat{\varphi})^2\right] - \left[\frac{2m}{\widehat{T}} - 1\right]^{-1}\cdot\left(\mathrm{d}\widehat{T}\right)^2\,, \\ &\widehat{D} := \left\{\left(\widehat{R}, \widehat{\theta}, \widehat{\varphi}, \widehat{T}\right) : -\infty < \widehat{R} < \infty,\ 0 < \widehat{\theta} < \pi,\ -\pi < \widehat{\varphi} < \pi,\ 0 < \widehat{T} < 2m\right\}.\end{aligned}\tag{3.112}$$

Remarks: (i) The metric above admits an additional Killing vector $\frac{\partial}{\partial\widehat{R}}$ analogous to that in Birkhoff's Theorem 3.3.2.

(ii) The metric (3.112) in the T-domain is the "dual" of the Schwarzschild metric.

(iii) This metric represents *one possible interior for the spherically symmetric black hole*. (See the Chap. 5.) From the dual Birkhoff's theorem, this is the only *vacuum solution* for such a spherical black hole interior. (See Exercise 3.3-1ii.)

□

Section 3.3 is useful for the study of *spherically symmetric stars* according to Einstein's theory of general relativity. Moreover, this section is a prelude to the exploration of *spherically symmetric black holes* in Chap. 5.

Exercises 3.3

1. Consider the nonstatic, spherically symmetric metric in (3.76).

   (i) Investigate electromagneto-vac equations (2.290i–vi) in that metric with the additional assumptions $F_{14}(\cdot\cdot) \neq 0$, $F_{23}(\cdot\cdot) \neq 0$, and *other* $F_{ij}(\cdot\cdot) \equiv 0$. Prove the appropriate generalization of Birkhoff's Theorem 3.3.2 in this case.

   (ii) Consider spherically symmetric field equations (3.77i–ix). Assuming one additional condition $T^{1}{}_{4}(r,t) \equiv 0$, prove the interior generalization of Birkhoff's Theorem 3.3.2.

2. Consider the nonstatic, spherically symmetric metric in (3.76). Let the three-dimensional, timelike boundary hypersurface (analogous to $\partial_{(3)}D$ of Fig. 2.19) be furnished by

$$
\begin{aligned}
r &= x^1 = \xi^1(u^2, u^3, u^4) := B(u^4)\,,\\
\theta &= x^2 = \xi^2(\cdots) := u^2\,,\\
\varphi &= x^3 = \xi^3(\cdots) := u^3\,,\\
t &= x^4 = \xi^4(\cdots) := u^4\,,\\
\mathcal{D} &:= \left\{\left(u^2, u^3, u^4\right) : \, 0 < u^2 < \pi,\ -\pi < u^3 < \pi,\ t_1 < u^4 < t_2\right\}.
\end{aligned}
$$

   Obtain the nontrivial, extrinsic curvature components from (1.234).

3. Consider the nonstatic, spherically symmetric metric:

$$
\mathrm{d}s^2 = \mathrm{e}^{2\lambda(r,t)}(\mathrm{d}r)^2 + [Y(r,t)]^2 \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\varphi)^2\right] - \mathrm{e}^{2\nu(r,t)}(\mathrm{d}t)^2\,.
$$

   Assume that the material source is *a perfect fluid*. *Assume* furthermore that a comoving chart exists so that $\vec{\mathbf{U}}(\cdot\cdot) = \mathrm{e}^{-\nu(r,t)} \cdot \frac{\partial}{\partial t}$.

   (i) Obtain the 4-acceleration field, the vorticity tensor, expansion scalar, and the shear tensor from (2.199i–v).

   (ii) *Assuming the energy condition* $\rho(r,t) + p(r,t) > 0$ and *an equation of state* $\rho = \mathcal{R}(p)$, $\frac{\mathrm{d}\mathcal{R}(p)}{\mathrm{d}p} \neq 0$, or equivalently $p = \mathcal{P}(\rho)$, $\frac{\mathrm{d}\mathcal{P}(\rho)}{\mathrm{d}\rho} \neq 0$, integrate the two nontrivial conservation equations.

4. Obtain the energy–momentum–stress tensor components $\Theta^i{}_j(T,R)$ corresponding to the T-domain metric:

$$\mathrm{d}s^2 = \left[\frac{3\sqrt{|qb^2-1|}-\sqrt{|qT^2-1|}}{3\sqrt{|qb^2-1|}-1}\right]^2 \cdot (\mathrm{d}R)^2 + T^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right]$$
$$-\left[qT^2-1\right]^{-1}\cdot(\mathrm{d}T)^2\,;$$
$$q>0\,,\quad qb^2>1\,,\quad 0<T<b\,.$$

5. Consider a "nonstatic, spherically symmetric" N-dimensional ($N \geq 4$) pseudo-Riemannian metric:

$$\mathrm{d}s^2 = \mathrm{e}^{\alpha(r,t)}\cdot(\mathrm{d}r)^2 + r^2\left[\mathrm{d}\Omega_{(N-2)}\right]^2 - \mathrm{e}^{\gamma(r,t)}\cdot(\mathrm{d}t)^2\,,$$

$$\left[\mathrm{d}\Omega_{(N-2)}\right]^2 := (\mathrm{d}\theta_2)^2 + \sum_{n=3}^{N-1}\left[\left(\prod_{m=2}^{n-1}\sin^2\theta_m\right)\cdot(\mathrm{d}\theta_n)^2\right],$$
$$D_{(N)} := \left\{(r,\theta_2,\ldots,\theta_{N-1},t)\colon r_1<r<r_2,\ 0<\theta_2,\theta_3,\ldots,\theta_{N-2}<\pi,\right.$$
$$\left.-\pi<\theta_{N-1}<\pi,\ t_1<t<t_2\right\}.$$

Obtain the general solution of the system $\mathbf{G}^{\cdot}_{\cdot}(x) + \kappa\,\mathbf{T}^{\cdot}_{\cdot}(x) = \mathbf{O}^{\cdot}_{\cdot}(x)$.

6. Consider the nonstatic, spherically symmetric *Vaidya metric*:

$$\mathrm{d}s^2 = r^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta\cdot(\mathrm{d}\phi)^2\right] - 2\,\mathrm{d}u\cdot\mathrm{d}r - [1-2M(u)/r]\cdot(\mathrm{d}u)^2.$$

Prove that the corresponding 4×4 matrix $[T_{ij}(..)]$, with respect to invariant eigenvalues, is of Segre characteristic $[(1,1,1),1]$.

(*Remark:* This metric can represent the space–time outside of a spherically symmetric object which is emitting a spherically symmetric distribution of radiation consisting of massless particles.)

Answers and Hints to Selected Exercises

1. (i) Equations $\mathcal{E}^1{}_4(\cdot\cdot)=0$ and $\mathcal{E}^1{}_1(\cdot\cdot)-\mathcal{E}^4{}_4(\cdot\cdot)=0$ imply that

$$\partial_4\alpha = 0 \quad \text{and} \quad \gamma(r,t) = -\alpha(r) + h(t)\,.$$

By Maxwell's equations $\mathcal{M}^i(\cdot\cdot) = 0 = {}^*\mathcal{M}^i(\cdot\cdot)$, one obtains

$$F_{14}(r,t) = \frac{e_0}{r^2}\cdot\mathrm{e}^{h(t)/2}\,,$$

$$F_{23}(\theta) = \mu_0\,\sin\theta\,.$$

With help of a coordinate transformation

$$\left(\widehat{r}, \widehat{\theta}, \widehat{\varphi}\right) = (r, \theta, \varphi)\,,$$

$$\widehat{t} = \int_{t_1}^{t} \exp\left[h(w)/2\right] \cdot \mathrm{d}w\,,$$

the static metric of p. 256 is recovered. (See [48].)

2.

$$\begin{aligned}
K_{22}\left(u^2, u^3, u^4\right) &= \left\{ - \frac{r\, \mathrm{e}^{\alpha(r,t)}}{\sqrt{\mathrm{e}^{-\alpha} - \mathrm{e}^{-\gamma} \cdot (\partial_4 B)^2}} \right\}_{\big| r=B(u^4),\, t=u^4}\,, \\
K_{33}(\cdot\cdot) &\equiv \sin^2 u^2 \cdot K_{22}(\cdot\cdot)\,, \\
2K_{44}(\cdot\cdot) &= \left\{ \frac{1}{\sqrt{\mathrm{e}^{-\alpha} - \mathrm{e}^{-\gamma} \cdot (\partial_4 B)^2}} \cdot \big[2\partial_4\partial_4 B + \mathrm{e}^{\gamma-\alpha} \cdot \partial_1 \gamma \right. \\
&\qquad + \partial_4 B \cdot \partial_4(2\alpha - \gamma) + (\partial_4 B)^2 \cdot \partial_1(\alpha - 2\gamma) \\
&\qquad \left. - (\partial_4 B)^3 \cdot \mathrm{e}^{\alpha-\gamma} \cdot \partial_4 \alpha \big] \right\}_{\big| \cdot\cdot}\,.
\end{aligned}$$

Remark: The above equations can be utilized to satisfy I–S–L–D junction conditions in (2.171).)

3. (i)

$$\begin{aligned}
\dot{\vec{\mathbf{U}}}(\cdot\cdot) &= [\partial_1 \nu(r,t)] \cdot \frac{\partial}{\partial r}\,, \\
\boldsymbol{\omega}_{\cdot\cdot}(\cdot\cdot) &\equiv \mathbf{O}_{\cdot\cdot}(\cdot\cdot)\,, \\
\Theta(\cdot\cdot) &= e^{-\nu} \cdot \partial_4\big[\lambda + 2\ln|Y|\big], \\
\boldsymbol{\sigma}\,\dot{\cdot}\,(\cdot\cdot) &= e^{-\nu} \cdot \partial_4\big[\ln|Y| - \lambda\big] \cdot \left[\frac{1}{3} \cdot \frac{\partial}{\partial \theta} \otimes \mathrm{d}\theta + \frac{1}{3} \cdot \frac{\partial}{\partial \varphi} \otimes \mathrm{d}\varphi - \frac{2}{3} \cdot \frac{\partial}{\partial r} \otimes \mathrm{d}r \right].
\end{aligned}$$

(ii) Two nontrivial conservation equations yield

$$\partial_1 p = -(\rho + p) \cdot \partial_1 \nu\,, \quad \partial_4 \rho = -(\rho + p) \cdot \partial_4(\lambda + 2\ln|Y|)\,.$$

Integrating the above equations, one obtains

$$\int \frac{\mathrm{d}p}{\mathcal{R}(p)+p} = -\nu(r,t) + F(t)\,,$$

$$\int \frac{\mathrm{d}\rho}{\rho+\mathcal{P}(\rho)} = -\big[\lambda(r,t) + 2\ln|Y(r,t)|\big] + H(r).$$

Here, $F(t)$ and $H(r)$ are arbitrary functions of integration.

4.

$$\Theta^i{}_j(T,R) = \big[-\sigma(T,R) - 3\kappa^{-1}q\big]s^i(\cdot\cdot)\cdot s_j(\cdot\cdot) + \sigma(\cdot\cdot)\,\delta^i{}_j\,,$$

$$-\sigma(T,R) := 3\kappa^{-1}q\cdot\left[\frac{\sqrt{|qb^2-1|}-\sqrt{|qT^2-1|}}{3\sqrt{|qb^2-1|}-\sqrt{|qT^2-1|}}\right] > 0\,,$$

$$s^2(\cdot\cdot) = s^3(\cdot\cdot) = s^4(\cdot\cdot) \equiv 0\,, \quad s^1(\cdot\cdot)\,s_1(\cdot\cdot) \equiv 1.$$

(*Remark:* This metric is the "dual" solution to the Schwarzschild's interior metric (3.66i).)

5. The general solution of the system is furnished by:

$$\mathrm{e}^{-\alpha(r,t)} = 1 - \frac{\kappa}{r^{N-3}}\cdot\left[f(t) - \left(\frac{2}{N-2}\right)\cdot\int_{r_1}^{r} T^N{}_N(y,t)\cdot y^{N-2}\mathrm{d}y\right]$$

$$=:\ 1 - \frac{2M(r,t)}{r^{N-3}} > 0\,,$$

$$\mathrm{e}^{\gamma(r,t)} = \left[1-\frac{2M(r,t)}{r^{N-3}}\right]\cdot\exp\left\{h(t) + \left(\frac{2\kappa}{N-2}\right)\right.$$

$$\left.\times\int_{r_1}^{r}\left[\frac{T^1{}_1(y,t) - T^N{}_N(y,t)}{y^{N-3}-2M(y,t)}\right]\cdot y^{N-2}\mathrm{d}y\,,\right\}$$

$$T^1{}_N(r,t) := \left(\frac{N-2}{2r^{N-2}}\right)\cdot\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t} - \left(\frac{2}{N-2}\right)\cdot\int_{r_1}^{r}\partial_N T^N{}_N\cdot y^{N-2}\mathrm{d}y\right\},$$

$$T^{\theta_A}{}_{\theta_A}(r,t) := \left(\frac{r}{N-2}\right)\cdot\left\{\partial_1 T^1{}_1 + \partial_N T^N{}_1 + \frac{1}{2}\partial_N[\alpha+\gamma]\cdot T^N{}_1\right.$$

$$\left.+\left[\frac{1}{2}\partial_1\gamma + \left(\frac{N-2}{r}\right)\right]\cdot T^1{}_1 - \frac{1}{2}\partial_1\gamma\cdot T^N{}_N\right\},$$

$$A \in \{2,\ldots,N-1\}\,.$$

(Compare with Problem # 6 of Exercise 3.1 and (3.79i–iv). See [61].)

Chapter 4
Static and Stationary Space–Time Domains

4.1 Static Axially Symmetric Space–Time Domains

We assume that two commuting Killing vectors $\frac{\partial}{\partial x^3} \equiv \frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial x^4} \equiv \frac{\partial}{\partial t}$ exist. Furthermore, we assume *discrete isometries* under reflections $\widehat{x}^3 = -x^3$ and $\widehat{x}^4 = -x^4$. The metric is then locally reducible to

$$\begin{aligned}
\mathbf{g}_{..}(x) = {} & \mathrm{e}^{-2w(x^1,x^2)} \cdot \Big\{ \mathrm{e}^{2\nu(x^1,x^2)} \cdot \left[\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 \right] \\
& \qquad + \mathrm{e}^{2\beta(x^1,x^2)} \cdot \mathrm{d}x^3 \otimes \mathrm{d}x^3 \Big\} \\
& - \mathrm{e}^{2w(x^1,x^2)} \cdot \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\
\mathrm{d}s^2 = {} & \mathrm{e}^{-2w(x^1,x^2)} \cdot \Big\{ \mathrm{e}^{2\nu(x^1,x^2)} \cdot \left[\left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 \right] + \mathrm{e}^{2\beta(x^1,x^2)} \left(\mathrm{d}x^3\right)^2 \Big\} \\
& - \mathrm{e}^{2w(x^1,x^2)} \cdot \left(\mathrm{d}x^4\right)^2, \\
D := {} & \Big\{ \left(x^1, x^2, x^3, x^4\right) : \varrho_1 < x^1 < \varrho_2,\ z_1 < x^2 < z_2, \\
& -\pi < x^3 < \pi,\ -\infty < x^4 < \infty \Big\}.
\end{aligned} \tag{4.1}$$

We investigate *the vacuum equations* (2.164i) at the outset. We first form the linear combination [243]:

$$0 = R^3{}_3(\cdot\cdot) + R^4{}_4(\cdot\cdot) = \mathrm{e}^{2(w-\nu)-\beta} \cdot \left[\partial_1 \partial_1 \left(\mathrm{e}^{\beta}\right) + \partial_2 \partial_2 \left(\mathrm{e}^{\beta}\right) \right]. \tag{4.2}$$

The general solution to the above (disguised linear) equation is provided by (A2.41) of Appendix 2 as

$$\begin{aligned}
\mathrm{e}^{\beta(x^1,x^2)} &= \varrho(x^1, x^2) := \operatorname{Re}\left[f(x^1 + i x^2) \right], \\
z(x^1, x^2) &:= \operatorname{Im}\left[f(x^1 + \mathrm{i}x^2) \right], \\
\rho(\cdot\cdot) + \mathrm{i}z(\cdot\cdot) &= f(x^1 + \mathrm{i}x^2).
\end{aligned} \tag{4.3}$$

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4_4,

Here, f is *an arbitrary holomorphic function of the complex variable* $x^1 + \mathrm{i}x^2$. *Assuming* $f(\cdot\cdot)$ *to be a nonconstant function*, we derive that

$$(\mathrm{d}\varrho)^2 + (\mathrm{d}z^2) = |\mathrm{d}f(\cdot\cdot)|^2 = |f'(\cdot\cdot)|^2 \cdot \left[\left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2\right]. \tag{4.4}$$

Now, we make the coordinate transformation:

$$(\varrho, z, \varphi, t) = \left(\mathrm{Re}\, f(\cdot\cdot), \mathrm{Im}\, f(\cdot\cdot), x^3, x^4\right). \tag{4.5}$$

From (4.4) and (4.5), the metric in (4.1) yields

$$\begin{aligned} \mathrm{d}s^2 = {} & \mathrm{e}^{-2\widehat{w}(\varrho,z)} \cdot \left\{\mathrm{e}^{2\widehat{\nu}(\varrho,z)} \cdot \left[(\mathrm{d}\varrho)^2 + (\mathrm{d}z)^2\right] + \varrho^2 \cdot (\mathrm{d}\varphi)^2\right\} \\ & - \mathrm{e}^{2\widehat{w}(\varrho,z)} \cdot (\mathrm{d}t)^2. \end{aligned} \tag{4.6}$$

Dropping hats in the sequel, we obtain the static, axially symmetric metric in *Weyl's coordinate chart* [238, 261] as

$$\mathrm{d}s^2 = \mathrm{e}^{-2w(\varrho,z)} \cdot \left\{\mathrm{e}^{2\nu(\varrho,z)} \cdot \left[(\mathrm{d}\varrho)^2 + (\mathrm{d}z)^2\right] + \varrho^2 \cdot (\mathrm{d}\varphi)^2\right\} - \mathrm{e}^{2w(\varrho,z)} \cdot (\mathrm{d}t)^2. \tag{4.7}$$

The nontrivial vacuum field equations (2.164i), from the metric above, are the following:

$$\begin{aligned} (1/2)\left[R_{11}(\cdot\cdot) + R_{22}(\cdot\cdot)\right] = {} & \partial_1\partial_1\nu + \partial_2\partial_2\nu + (\partial_1 w)^2 \\ & + (\partial_2 w)^2 - \nabla^2 w = 0, \end{aligned} \tag{4.8i}$$

$$(1/2)\left[R_{11}(\cdot\cdot) - R_{22}(\cdot\cdot)\right] = (\partial_1 w)^2 - (\partial_2 w)^2 - \varrho^{-1} \cdot \partial_1\nu = 0, \tag{4.8ii}$$

$$R_{12}(\cdot\cdot) = 2\partial_1 w \cdot \partial_2 w - \varrho^{-1} \cdot \partial_2\nu = 0, \tag{4.8iii}$$

$$R^3{}_3(\cdot\cdot) - R^4{}_4(\cdot\cdot) = -2\mathrm{e}^{2(w-\nu)} \cdot \nabla^2 w = 0, \tag{4.8iv}$$

$$R^3{}_3(\cdot\cdot) + R^4{}_4(\cdot\cdot) \equiv 0, \tag{4.8v}$$

$$\nabla^2 w := \partial_1\partial_1 w + \varrho^{-1} \cdot \partial_1 w + \partial_2\partial_2 w, \tag{4.8vi}$$

$$D := \{(\varrho, z) : 0 < \varrho < \varrho_2,\ z_1 < z < z_2\}. \tag{4.8vii}$$

The two-dimensional domain D and the corresponding three-dimensional domain in "Euclidean coordinate spaces" are depicted below in Fig. 4.1.

The general solution of the system of field equations (4.8i–iv) will be provided below.

Theorem 4.1.1. *Let the metric functions* $w(\varrho, z)$ *and* $\nu(\varrho, z)$ *of* (4.7) *be of class* C^3 *in a simply connected domain* $D \subset \mathbb{R}^2$ *of Fig.* 4.1. *The general solution of*

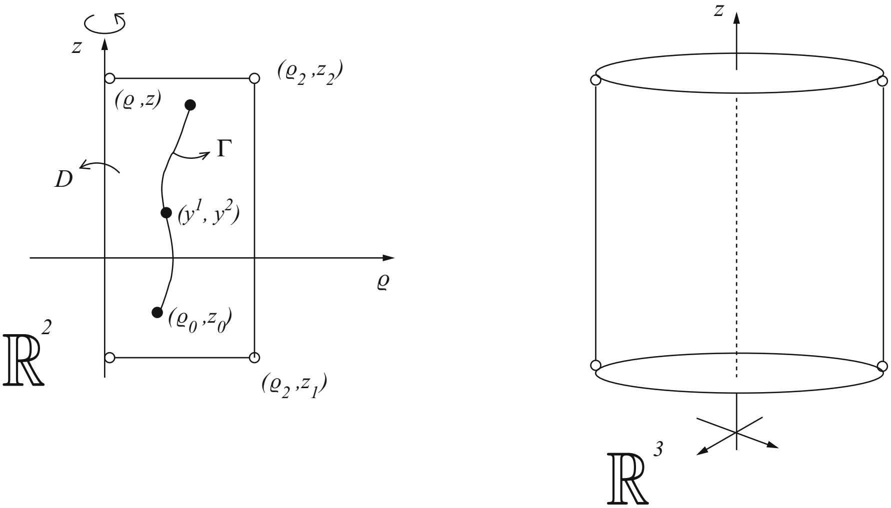


Fig. 4.1 The two-dimensional and the corresponding axially symmetric three-dimensional domain

the system of differential equations (4.8i–iv) *is furnished by an arbitrary, axially symmetric Newtonian potential* $w(\varrho, z)$ *and the following line integral:*

$$\nu(\varrho, z) := \int_{(\varrho_0, z_0)[\Gamma]}^{(\varrho, z)} y^1 \cdot \left\{\left[(\partial_1 w)^2 - (\partial_2 w)^2\right] \mathrm{d}y^1 + 2(\partial_1 w) \cdot (\partial_2 w)\, \mathrm{d}y^2\right\}. \tag{4.9}$$

Proof. Vacuum field equations (4.8i–v) reduce to

$$\nabla^2 w = \partial_1 \partial_1 w + \varrho^{-1} \cdot \partial_1 w + \partial_2 \partial_2 w = 0, \tag{4.10i}$$

$$\partial_1 \nu = \varrho \cdot \left[(\partial_1 w)^2 - (\partial_2 w)^2\right], \tag{4.10ii}$$

$$\partial_2 \nu = 2\varrho \cdot (\partial_1 w) \cdot (\partial_2 w), \tag{4.10iii}$$

$$\partial_1 \partial_1 \nu + \partial_2 \partial_2 \nu + (\partial_1 w)^2 + (\partial_2 w)^2 = 0. \tag{4.10iv}$$

Equation (4.10i) shows that $w(\varrho, z)$ is an axially symmetric Newtonian potential. The integrability condition (1.239) for the pair of first-order equations (4.10ii, 4.10iii) is $\partial_2 \partial_1 \nu - \partial_1 \partial_2 \nu = 2\varrho \cdot \nabla^2 w \cdot \partial_2 w = 0$. It is automatically satisfied by (4.10i). Thus, the solution (4.9) is obtained by a line integral consisting of an arbitrary constant. The remaining (4.10iv) is implied by the other three (due to the contracted Bianchi identities). ∎

Caution: The solution (4.9) is valid only in *the open-subset D* which excludes points on the z-axis. To analytically extend the solution to the z-axis, we need to check the existence of *tangent planes* (or "local flatness") at those points. To test the local flatness, choose a point $(0, z_c)$ as the center of a "horizontal circle" in the three-dimensional coordinate space of Fig. 4.1. The limit of the ratio of circumference divided by "radius" of the spatial geometry in (4.7) is given by $\lim_{\Delta\varrho\to 0_+}\left[\frac{2\pi\cdot\Delta\varrho}{e^{\nu(\Delta\varrho,z_c)}\cdot\Delta\varrho}\right] = 2\pi$ for the local flatness. Therefore, *we must check the condition* $\nu(0_+, z_c) = 0$ for the regularity of the solution.

Example 4.1.2. We choose the Newtonian potential function as [17]

$$
\begin{aligned}
w(\varrho, z) &:= -\frac{m}{\sqrt{\varrho^2 + z^2}} < 0,\\
D &:= \left\{(\varrho, z) : 0 < \varrho,\ \varrho^2 + z^2 > 0\right\}.
\end{aligned}
\tag{4.11}
$$

To evaluate the line integral in (4.9), we choose the curve Γ as the *circular arc*:

$$
\Gamma := \left\{(y^1, y^2) : y^1 = r_0 \sin\alpha,\ y^2 = r_0\cos\alpha,\ \theta < \alpha < \pi\right\}.
$$

The line integral (4.9) then reduces to

$$
\begin{aligned}
\nu(r_0\sin\theta, r_0\cos\theta) &= \frac{m^2}{r_0^2}\int_{\pi-}^{\theta} \sin\alpha\cdot\left[-\cos\alpha\cdot\sin^2\alpha - \cos\alpha\cdot\cos^2\alpha\right]\mathrm{d}\alpha\\
&= -\frac{m^2}{2r_0^2}\cdot\sin^2\theta,\\
\nu(\varrho, z) &= -\frac{m^2\varrho^2}{2(\varrho^2+z^2)^2},\\
\lim_{\varrho\to 0_+}\nu(\varrho, z) &\equiv 0 \quad \text{for} \quad z \neq 0.
\end{aligned}
\tag{4.12}
$$

The space–time metric of (4.7) goes over into

$$
\begin{aligned}
\mathrm{d}s^2 = \exp\left[\frac{2m}{\sqrt{\varrho^2+z^2}}\right]\cdot\Bigg\{\exp\left[-\frac{m^2\varrho^2}{(\varrho^2+z^2)^2}\right]\cdot\left[(\mathrm{d}\varrho)^2 + (\mathrm{d}z)^2\right]\\
+\varrho^2(\mathrm{d}\varphi)^2\Bigg\} - \exp\left[-\frac{2m}{\sqrt{\varrho^2+z^2}}\right]\cdot(\mathrm{d}t)^2.
\end{aligned}
\tag{4.13}
$$

The only singularity of the above metric is at $(\varrho, z) = (0, 0)$. There is a peculiarity about this metric in (4.13). Although the Newtonian potential $w(\varrho, z)$ in (4.11) is *spherically symmetric* (since it is a function of the particular combination ϱ^2+z^2),

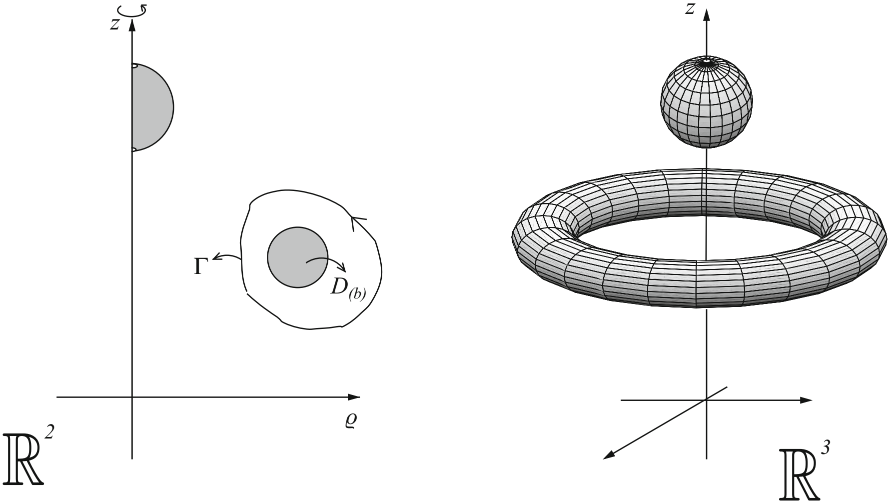


Fig. 4.2 Two axially symmetric bodies in "Euclidean coordinate spaces"

the metric (4.13) *cannot be transformed into the Schwarzschild metric (3.9)*. The reason for this disparity is that the potential $w(\varrho, z)$ in (4.11) is spherically symmetric in the "Euclidean coordinate space." However, the non-Euclidean spatial hypersurface induced by (4.13) for $t = \text{const.}$ does not admit spherical symmetry. (Compare Fig. 2.8.) □

The axially symmetric potential equation (4.10i) *is linear*. Therefore, *superposition of several potential functions* is itself a potential function. It *seems* that several axially symmetric, static bodies in Newtonian gravitation can generate a corresponding exact, static solution in Einstein's theory! But in Newtonian gravitation, several axially symmetric, massive bodies *may or may not* be in *equilibrium*. How does Einstein's theory distinguish these different cases? *One* answer is the use of the criterion $\nu(0_+, z) = 0$. Now, we shall put forward another criterion that is physically more transparent. Let us consider *two, axially symmetric, massive bodies*. One is a spherical body, another is a solid ring. (See Fig. 4.2.)

Consider the 1-form $\tilde{\boldsymbol{\omega}}(\varrho, z)$ and its exterior derivative in the following:

$$\tilde{\boldsymbol{\omega}}(\varrho, z) := \varrho \cdot \left\{\left[(\partial_1 w)^2 - (\partial_2 w)^2\right] \mathrm{d}\varrho + 2(\partial_1 w)\cdot(\partial_2 w)\,\mathrm{d}z\right\},$$

$$\mathrm{d}\tilde{\boldsymbol{\omega}}(\cdot\cdot) = 2\left[\varrho \cdot \nabla^2 w \cdot \partial_2 w\right](\mathrm{d}\varrho \wedge \mathrm{d}z). \tag{4.14}$$

Applying Stokes' Theorem 1.2.23 on the closed curve Γ of Fig. 4.2, we conclude that

$$(1/2) \cdot \oint_{[\Gamma]} \tilde{\boldsymbol{\omega}}(\cdot\cdot) = \int_{D_{(b)}} \left(\nabla^2 w\right) \cdot \partial_2 w \cdot \varrho \,\mathrm{d}\varrho\,\mathrm{d}z. \tag{4.15}$$

Since $\nabla^2 w$ is proportional to the mass density in Newtonian theory, the above equation is proportional to *the net "Newtonian gravitational force"* along the z-axis on a vertical section of the solid ring. (See the reference [173].) Since there is a massive sphere on top of the ring in Fig. 4.2, it is clear that the net force in (4.15) *cannot vanish*. Thus, from (4.14), $d\tilde{\boldsymbol{\omega}}(\cdot\cdot) \not\equiv \mathbf{0}_{..}(\cdot\cdot)$. Therefore, the integrability condition for differential equations (4.10ii,iii) is *violated*. There exists *no solution* of the field equations in this static scenario. It is in this way that the nonlinear field equations of general relativity make a physically satisfying, subtle impact on the axially symmetric case. In essence, the above static scenario, which cannot exist in equilibrium in the Newtonian sense due to the fact that the net force cannot vanish, also cannot be in equilibrium in the relativistic sense due to violation of the integrability conditions.

(*Remarks*: (i) In case the sphere and the ring are concentric, there can be equilibrium and static solutions do exist. (ii) The actual curved submanifold corresponding to Fig. 4.2 is *quite different*.)

Example 4.1.3. Next, consider the case of two "gravitational monopoles" on the vertical axis [44]. The corresponding Newtonian potential is furnished by

$$w(\varrho, z) = -\frac{m_1}{\sqrt{\varrho^2 + (z - z_1)^2}} - \frac{m_2}{\sqrt{\varrho^2 + (z - z_2)^2}},$$
$$D := \left\{(\varrho, z) : 0 < \varrho,\ 0 < \varrho^2 + (z - z_1)^2,\ 0 < \varrho^2 + (z - z_2)^2,\ z_2 > z_1\right\}. \tag{4.16}$$

Then, (4.9) yields

$$\begin{aligned} \nu(\varrho, z) = & -\frac{m_1^2 \varrho^2}{2\left[\varrho^2 + (z - z_1)^2\right]^2} - \frac{m_2^2 \varrho^2}{2\left[\varrho^2 + (z - z_2)^2\right]^2} \\ & + \left[\frac{2m_1 m_2}{(z_1 - z_2)^2}\right] \cdot \left[\frac{\varrho^2 + (z - z_1)(z - z_2)}{\sqrt{[\varrho^2 + (z - z_1)^2] \cdot [\varrho^2 + (z - z_2)^2]}} - 1\right]. \end{aligned} \tag{4.17}$$

$$\begin{aligned} \lim_{\varrho \to 0_+} \nu(\varrho, z) &= \left[\frac{2m_1 m_2}{(z_1 - z_2)^2}\right] \cdot \{[\operatorname{sgn}(z - z_1)] \cdot [\operatorname{sgn}(z - z_2)] - 1\} \\ &= \begin{cases} 0 \ \text{ for } \ z \in (-\infty, z_1) \cup (z_2, \infty), \\ -\left[\dfrac{4m_1 m_2}{(z_1 - z_2)^2}\right] < 0 \ \text{ for } \ z \in (z_1, z_2)\,. \end{cases} \end{aligned}$$

Therefore, exact static solutions *do exist on the vertical axis* on the interval $(-\infty, z_1) \cup (z_2, \infty)$. However, the solution *does not exist* on the interval (z_1, z_2). Mathematically, we can make the solution valid by *incising the interval corresponding to* (z_1, z_2) from the manifold. (Physically speaking, this process can be viewed as placing an ignored, massless, rigid, *linear strut* between two massive particles at $(0, z_1)$ and $(0, z_2)$ in order to balance the gravitational attraction.) □

Example 4.1.4. In Example 4.1.2, the spherically symmetric Newtonian potential *failed* to generate the Schwarzschild metric out of the Weyl metric in (4.7). Naturally, a question arises as to which of the Newtonian potentials will generate then the Schwarzschild metric. This query will be answered shortly. Consider the Newtonian potential of a finite, massive, linear rod of length $l = 2m > 0$ on the z-axis. This potential is provided by

$$
\begin{aligned}
w(\varrho, z) &:= \frac{1}{2} \ln \left| \frac{A(\varrho, z) + B(\varrho, z) - 2m}{A(\cdot\cdot) + B(\cdot\cdot) + 2m} \right|, \\
A(\varrho, z) &:= \sqrt{\varrho^2 + (z+m)^2} > 0, \\
B(\varrho, z) &:= \sqrt{\varrho^2 + (z-m)^2} > 0, \\
D &:= \{(\varrho, z) : 0 < \varrho,\ A(\cdot\cdot) > 0,\ B(\cdot\cdot) > 0,\ A(\cdot\cdot) + B(\cdot\cdot) > 2m\}. \qquad (4.18)
\end{aligned}
$$

The corresponding $\nu(\varrho, z)$ from (4.9) yields

$$
\begin{aligned}
\nu(\varrho, z) &= \frac{1}{2} \ln \left| \frac{[A(\cdot\cdot) + B(\cdot\cdot)]^2 - 4m^2}{4A(\cdot\cdot) \cdot B(\cdot\cdot)} \right|, \\
\lim_{\varrho \to 0_+} \nu(\varrho, z) &= \frac{1}{2} \ln \left| \frac{1}{2} + \frac{1}{2} [\operatorname{sgn}(z+m)] \cdot [\operatorname{sgn}(z-m)] \right| \\
&= \begin{cases} 0 \ \text{ for } \ z \in (-\infty, -m) \cup (m, \infty), \\ \text{undefined for } \ z \in (-m, m). \end{cases} \qquad (4.19)
\end{aligned}
$$

By the coordinate transformation

$$
\begin{aligned}
r &= \frac{1}{2} [A(\varrho, z) + B(\varrho, z)] + m, \\
\tan\theta &= \sqrt{\frac{4m^2}{[A(\cdot\cdot) - B(\cdot\cdot)]^2} - 1}, \qquad (4.20)
\end{aligned}
$$

we can recover the Schwarzschild metric in (3.9). (See [239].) The inverse coordinate transformation of (4.20) is provided by

$$
\begin{aligned}
\varrho &= \sqrt{r^2 - 2mr} \cdot \sin\theta, \\
z &= (r - m) \cdot \cos\theta, \\
\widehat{D} &:= \{(r, \theta) : 0 < 2m < r,\ 0 < \theta < \pi\}. \qquad (4.21)
\end{aligned}
$$

□

Now, we shall investigate static, axially symmetric, *interior solutions*. The metric is given by (4.1) which we express as

$$\begin{aligned} ds^2 = {} & e^{-2w(\varrho,z)} \cdot \left\{ e^{2\nu(\varrho,z)} \cdot \left[(d\varrho)^2 + (dz)^2 \right] + e^{2\beta(\varrho,z)} \cdot (d\varphi)^2 \right\} \\ & - e^{2w(\varrho,z)} (dt)^2. \end{aligned} \tag{4.22}$$

The field equations (2.161i) reduce to

$$\begin{aligned} \mathcal{E}_{11}(\varrho,z) := {} & \partial_2\nu \cdot \partial_2\beta - \partial_1\nu \cdot \partial_1\beta + (\partial_1 w)^2 - (\partial_2 w)^2 \\ & - \partial_2\partial_2\beta - (\partial_2\beta)^2 + \kappa\, T_{11}(\varrho,z) = 0, \end{aligned} \tag{4.23i}$$

$$\begin{aligned} \mathcal{E}_{22}(\cdot\cdot) := {} & \partial_1\nu \cdot \partial_1\beta - \partial_2\nu \cdot \partial_2\beta + (\partial_2 w)^2 - (\partial_1 w)^2 \\ & - \partial_1\partial_1\beta - (\partial_1\beta)^2 + \kappa\, T_{22}(\cdot\cdot) = 0, \end{aligned} \tag{4.23ii}$$

$$\begin{aligned} \mathcal{E}_{12}(\cdot\cdot) := {} & 2 \cdot \partial_1 w \cdot \partial_2 w + \partial_1\partial_2\beta + \partial_1\beta \cdot \partial_2\beta \\ & - \partial_1\nu \cdot \partial_2\beta - \partial_2\nu \cdot \partial_1\beta + \kappa\, T_{12}(\cdot\cdot) = 0, \end{aligned} \tag{4.23iii}$$

$$\begin{aligned} \mathcal{E}_{33}(\cdot\cdot) := {} & -e^{2(\beta-\nu)} \cdot \left[\partial_1\partial_1\nu + \partial_2\partial_2\nu + (\partial_1 w)^2 + (\partial_2 w)^2 \right] \\ & + \kappa\, T_{33}(\cdot\cdot) = 0, \end{aligned} \tag{4.23iv}$$

$$\begin{aligned} \mathcal{E}_{44}(\cdot\cdot) := {} & -e^{2(2w-\nu)} \cdot \Big[2\big(\partial_1\partial_1 w + \partial_2\partial_2 w + \partial_1\beta \cdot \partial_1 w + \partial_2\beta \cdot \partial_2 w \\ & - (\partial_1 w)^2 - (\partial_2 w)^2 \big) - \big(\partial_1\partial_1\beta + \partial_2\partial_2\beta + (\partial_1\beta)^2 \\ & + (\partial_2\beta)^2 \big) - (\partial_1\partial_1\nu + \partial_2\partial_2\nu) \Big] + \kappa\, T_{44}(\cdot\cdot) = 0, \end{aligned} \tag{4.23v}$$

$$T_{13}(\cdot\cdot) = T_{23}(\cdot\cdot) = T_{14}(\cdot\cdot) = T_{24}(\cdot\cdot) = T_{34}(\cdot\cdot) \equiv 0, \tag{4.23vi}$$

$$\begin{aligned} \mathcal{T}_1(\cdot\cdot) := {} & e^{-2(\nu-w)-\beta} \cdot \left\{ \partial_1\left[e^{2(\nu-w)-\beta} \cdot T^1{}_1 \right] + \partial_2\left[e^{2(\nu-w)-\beta} \cdot T^2{}_1 \right] \right\} \\ & - (1/2) \cdot \left\{ \left[\partial_1 e^{2(\nu-w)} \right] \cdot \left(T^{11} + T^{22} \right) + \left[\partial_1 e^{2(\beta-w)} \right] \cdot T^{33} \right. \\ & \left. - \left[\partial_1 e^{2w} \right] \cdot T^{44} \right\} = 0, \end{aligned} \tag{4.23vii}$$

$$\begin{aligned} \mathcal{T}_2(\cdot\cdot) := {} & e^{-2(\nu-w)-\beta} \cdot \left\{ \partial_1\left[e^{2(\nu-w)+\beta} \cdot T^1{}_2 \right] + \partial_2\left[e^{2(\nu-w)+\beta} \cdot T^2{}_2 \right] \right\} \\ & - (1/2) \cdot \left\{ \left[\partial_2 e^{2(\nu-w)} \right] \cdot \left(T^{11} + T^{22} \right) + \left[\partial_2 e^{2(\beta-w)} \right] \cdot T^{33} \right. \\ & \left. - \left[\partial_2 e^{2w} \right] \cdot T^{44} \right\} = 0, \end{aligned} \tag{4.23viii}$$

$$\mathcal{T}_3(\cdot\cdot) = \mathcal{T}_4(\cdot\cdot) \equiv 0. \tag{4.23ix}$$

The equation $\mathrm{e}^{-4w} \cdot \mathcal{E}_{44} + \mathrm{e}^{-2\nu}(\mathcal{E}_{11} + \mathcal{E}_{22}) + \mathrm{e}^{-2\beta}\mathcal{E}_{33} = 0$ corresponds to the axially symmetric Poisson's equation (2.157i). Moreover, (4.23vii,viii) resemble conditions of axially symmetric equilibrium in the theory of elasticity [244].

Now, let us count the number of unknown functions versus the number of independent differential equations in (4.23i–ix).

No. of unknown functions: $1(w) + 1(\nu) + 1(\beta) + 5(T^i{}_j) = 8.$

No. of differential equations: $5(\mathcal{E}^i{}_j = 0) + 2(\mathcal{T}^i = 0) = 7.$

No. of differential identities: $2(\nabla_i\, \mathcal{E}^{ij} \equiv \kappa\, \mathcal{T}^j) = 2.$

No. of independent equations: $7 - 2 = 5.$

The system is obviously *underdetermined*. Therefore, three of the eight unknown functions can be prescribed. There are several *mixed methods* available. (However, ideally, we have to satisfy energy conditions (2.194i–iii), too!)

Now, we explore the Einstein–Maxwell equations (2.290i–iv) in *the static, axially symmetric case*. We further *restrict the analysis to static, electro-vac equations* so that

$$\begin{aligned} \partial_4 F_{ij}(\cdot\cdot) &\equiv 0, \\ F_{\alpha\beta}(\cdot\cdot) &\equiv 0, \\ \partial_\alpha A_\beta - \partial_\beta A_\alpha &\equiv 0. \end{aligned} \tag{4.24}$$

Therefore, the 1-form $A_\alpha(\cdot\cdot)\,\mathrm{d}x^\alpha$ is *closed* in a spatial, star-shaped domain. By the Theorem 1.2.21, there exists a differentiable function $\lambda(\cdot\cdot)$ such that

$$A_\alpha(\cdot\cdot) = \partial_\alpha \lambda. \tag{4.25}$$

By the gauge transformation (1.71ii) or (2.280),

$$\begin{aligned} \widehat{A}_\alpha(\cdot\cdot) &= A_\alpha(\cdot\cdot) - \partial_\alpha\lambda \equiv 0, \\ \widehat{A}_4(\cdot\cdot) &= A_4(\cdot\cdot) - \partial_4\lambda \not\equiv 0. \end{aligned} \tag{4.26}$$

We *drop hats* in the sequel and denote that (new)

$$\begin{aligned} A_4(\cdot\cdot) &=: \mathcal{A}(\cdot\cdot), \\ F_{\alpha 4}(\cdot\cdot) &= \partial_\alpha \mathcal{A}. \end{aligned} \tag{4.27}$$

From physical considerations of axial symmetry, *we set*

$$\begin{aligned} F_{34}(\cdot\cdot) &\equiv 0, \\ \text{so that} \quad \mathcal{A}(\cdot\cdot) &= \mathcal{A}(\varrho, z). \end{aligned} \tag{4.28}$$

Now, we assume Weyl's static, axially symmetric metric (4.7). *The nonzero components of* $T^{i}{}_{j}(\cdot\cdot)$ of (2.290i), with (4.24), (4.26), and (4.28), are furnished by

$$T^{1}{}_{1}(\varrho, z) = (1/2)\cdot \mathrm{e}^{-2\nu}\cdot\left[(\partial_2\mathcal{A})^2 - (\partial_1\mathcal{A})^2\right] \equiv -T^{2}{}_{2}(\cdot\cdot), \tag{4.29i}$$

$$T^{1}{}_{2}(\cdot\cdot) = -\mathrm{e}^{-2\nu}\cdot\partial_1\mathcal{A}\cdot\partial_2\mathcal{A}, \tag{4.29ii}$$

$$T^{3}{}_{3}(\cdot\cdot) = (1/2)\cdot \mathrm{e}^{-2\nu}\cdot\left[(\partial_1\mathcal{A})^2 + (\partial_2\mathcal{A})^2\right] \equiv -T^{4}{}_{4}(\cdot\cdot) > 0, \tag{4.29iii}$$

$$T^{i}{}_{i}(\cdot\cdot) \equiv 0. \tag{4.29iv}$$

The nontrivial Einstein–Maxwell equation $\mathcal{M}^i(\cdot\cdot) = 0$ and $R^{i}{}_{j}(\cdot\cdot) + \kappa\, T^{i}{}_{j}(\cdot\cdot) = 0$ reduces to the following:

$$\mathcal{M}^4(\cdot\cdot) = \varrho^{-1}\cdot \mathrm{e}^{2(w-\nu)}\cdot\left\{\partial_1\left[\varrho\cdot\partial_1\mathcal{A}\cdot \mathrm{e}^{-2w}\right] + \partial_2\left[\varrho\cdot\partial_2\mathcal{A}\cdot \mathrm{e}^{-2w}\right]\right\} = 0, \tag{4.30i}$$

$$R^{1}{}_{1}(\cdot\cdot) + \kappa\, T^{1}{}_{1}(\cdot\cdot) = \mathrm{e}^{2(w-\nu)}\cdot\left[\partial_1\partial_1\nu + \partial_2\partial_2\nu - \nabla^2 w + 2(\partial_1 w)^2 - \varrho^{-1}\cdot\partial_1\nu\right] + (\kappa/2)\cdot \mathrm{e}^{-2\nu}\cdot\left[(\partial_2\mathcal{A})^2 - (\partial_1\mathcal{A})^2\right] = 0, \tag{4.30ii}$$

$$\mathcal{E}^{1}{}_{1}(\cdot\cdot) - \mathcal{E}^{2}{}_{2}(\cdot\cdot) = 2\mathrm{e}^{2(w-\nu)}\cdot\left[(\partial_1 w)^2 - (\partial_2 w)^2 - \varrho^{-1}\cdot\partial_1\nu\right] + \kappa\, \mathrm{e}^{-2\nu}\cdot\left[(\partial_2\mathcal{A})^2 - (\partial_1\mathcal{A})^2\right] = 0, \tag{4.30iii}$$

$$R^{1}{}_{2}(\cdot\cdot) + \kappa\, T^{1}{}_{2}(\cdot\cdot) = \mathrm{e}^{2(w-\nu)}\cdot\left[2\partial_1 w\cdot\partial_2 w - \varrho^{-1}\cdot\partial_2\nu\right] - \kappa\, \mathrm{e}^{-2\nu}\cdot\partial_1\mathcal{A}\cdot\partial_2\mathcal{A} = 0, \tag{4.30iv}$$

$$R^{3}{}_{3}(\cdot\cdot) + \kappa\, T^{3}{}_{3}(\cdot\cdot) = -\mathrm{e}^{2(w-\nu)}\cdot\nabla^2 w + (\kappa/2)\cdot \mathrm{e}^{-2\nu}\left[(\partial_1\mathcal{A})^2 + (\partial_2\mathcal{A})^2\right] \equiv -\left[R^{4}{}_{4}(\cdot\cdot) + \kappa\, T^{4}{}_{4}(\cdot\cdot)\right] = 0. \tag{4.30v}$$

We shall obtain *a class of exact solutions* of the above system.

Example 4.1.5. Equations (4.30i) and (4.30v) reduce to

$$\nabla^2\mathcal{A} = 2\left[\partial_1 w\cdot\partial_1\mathcal{A} + \partial_2 w\cdot\partial_2\mathcal{A}\right], \tag{4.31i}$$

$$\nabla^2 w = (\kappa/2)\cdot \mathrm{e}^{-2w}\cdot\left[(\partial_1\mathcal{A})^2 + (\partial_2\mathcal{A})^2\right] > 0. \tag{4.31ii}$$

Assume a functional relationship between two "potentials" $w(\rho, z)$ and $\mathcal{A}(\varrho, z)$. Therefore, there exists a thrice-differentiable function F such that

$$\begin{aligned}
e^{2w(\varrho,z)} &= [F \circ \mathcal{A}]\,(\varrho, z) \equiv F\,[\mathcal{A}(\varrho, z)] > 0,\\
\partial_1\left[e^{2w(\cdot\cdot)}\right] &= F'(\mathcal{A}) \cdot \partial_1 \mathcal{A} \quad \text{etc.},\\
F'(\mathcal{A}) &:= \frac{dF(\mathcal{A})}{d\mathcal{A}} \neq 0.
\end{aligned} \tag{4.32}$$

With the above equations, (4.31i,ii), respectively, reduce to

$$\nabla^2 \mathcal{A} = F'(\cdot\cdot) \cdot F^{-1} \cdot \left[(\partial_1\mathcal{A})^2 + (\partial_2\mathcal{A})^2\right], \tag{4.33i}$$

$$\nabla^2 \mathcal{A} = \left[\frac{F'}{F} + \frac{\kappa - F''}{F'}\right] \cdot \left[(\partial_1\mathcal{A})^2 + (\partial_2\mathcal{A})^2\right]. \tag{4.33ii}$$

Subtracting (4.33ii) from (4.33i) and recalling $(\partial_1\mathcal{A})^2 + (\partial_2\mathcal{A})^2 > 0$, we deduce that

$$\begin{aligned}
&F''(\mathcal{A}) = \kappa,\\
\text{or,}\quad &F(\mathcal{A}) = a + b\mathcal{A} + (\kappa/2)(\mathcal{A})^2 > 0,\\
\text{or,}\quad &e^{2w(\varrho,\, z)} = a + b\mathcal{A}(\varrho,\, z) + \frac{\kappa}{2} \cdot [\mathcal{A}(\varrho,\, z)]^2.
\end{aligned} \tag{4.34}$$

The above quadratic function, involving two arbitrary constants, $a > 0$ and b, was first discovered by Weyl [261].

Another auxiliary function is introduced by the indefinite integral

$$\mathcal{V}(\mathcal{A}) := -\sqrt{\frac{\kappa}{2}} \cdot \int \frac{d\mathcal{A}}{[a + b\mathcal{A} + (\kappa/2)\mathcal{A}^2]}, \tag{4.35i}$$

$$V(\varrho, z) := \mathcal{V}\left(\mathcal{A}(\varrho, z)\right), \tag{4.35ii}$$

$$\partial_1 V = -\sqrt{\kappa/2} \cdot e^{-2w} \cdot \partial_1\mathcal{A} \quad \text{etc.}, \tag{4.35iii}$$

$$\nabla^2 V = -\sqrt{\kappa/2} \cdot e^{-2w} \cdot \left[\nabla^2\mathcal{A} - 2(\partial_1 w \cdot \partial_1\mathcal{A} + \partial_2 w \cdot \partial_2\mathcal{A})\right] = 0. \tag{4.35iv}$$

Now, *let us choose the constants to satisfy*

$$b^2 = 2a\kappa > 0, \tag{4.36i}$$

$$\mathcal{V}(\mathcal{A}) = \pm\left[\sqrt{a} \pm \sqrt{\kappa/2} \cdot \mathcal{A}\right]^{-1}, \tag{4.36ii}$$

$$[V(\varrho, z)]^{-2} = e^{2w(\varrho,z)}, \tag{4.36iii}$$

$$\partial_1 \mathcal{A} = -\sqrt{2/\kappa} \cdot [V(\cdot\cdot)]^{-2} \cdot \partial_1 V \quad \text{etc.}, \tag{4.36iv}$$

$$\partial_1 w = -[V(\cdot\cdot)]^{-1} \cdot \partial_1 V \quad \text{etc.} \tag{4.36v}$$

Under assumptions (4.36i) and consequent equations (4.36ii–v), the axially symmetric electro-vac equations (4.30i–iv) *dramatically reduce to the linear potential equation (4.35iv)* and

$$\begin{aligned} \partial_1 \nu &= 0, \\ \partial_2 \nu &= 0. \end{aligned} \tag{4.37}$$

Recalling $\lim_{\varrho \to 0_+} \nu(\varrho, z) = 0$, (4.37) implies that

$$\nu(\varrho, z) \equiv 0.$$

Therefore, the Weyl metric (4.7) reduces to the simple form:

$$\begin{aligned} \mathrm{d}s^2 = {} & [V(\varrho, z)]^2 \cdot \left[(\mathrm{d}\varrho)^2 + (\mathrm{d}z)^2 + \varrho^2 (\mathrm{d}\varphi)^2\right] \\ & - [V(\varrho, z)]^{-2} \cdot (\mathrm{d}t)^2, \\ \text{and} \quad \nabla^2 V = {} & \partial_1 \partial_1 V + \varrho^{-1} \cdot \partial_1 V + \partial_z \partial_z V = 0. \end{aligned} \tag{4.38}$$

The above metric depends on *a single axially symmetric Newtonian potential!* Therefore, exact, static solutions due to *several axially symmetric, massive, charged bodies are possible*. What is happening to the equilibrium of different bodies? In the next section, in a more general scenario, we shall prove that in such cases, *electrostatic repulsion is exactly cancelling the gravitational attraction*. (The metric in (4.38) is a special case of the class first discovered by Curzon [44] and Chazy [36].) □

Exercise 4.1

1. Consider the Weyl metric in (4.7) satisfying the vacuum field equations. Obtain the corresponding metric function, $\nu(\varrho, z)$, from (4.10i,ii), as well as the curvature tensor $\mathbf{R}^{\cdot}{}_{\cdot\cdot\cdot}(\cdot\cdot)$, for the following two cases:

   (i) $w(\varrho, z) = \text{const.}$
   (ii) $w(\varrho, z) = \ln \varrho$.

2. Recall the polar coordinates given by $\varrho = r \sin\theta$ and $z = r \cos\theta$; $0 < r$ and $0 < \theta < \pi$. A Newtonian potential $\widehat{w}(r, \theta) = w(\varrho, z)$ is provided by the *superposition*

of multipoles as $\widehat{w}(r,\theta) = \sum_{n=0}^{N} c_{(n)} \cdot r^{-(n+1)} \cdot P_n(\cos\theta)$. (Here, $P_n(\cos\theta)$ denotes a Legendre polynomial.) Obtain the corresponding $\widehat{\nu}(r,\theta)$ which solves the vacuum equations.

3. Consider the axially symmetric, static, electro-vac equations (4.30i–v) with assumptions (4.34) and (4.35i). In case the constants satisfy the inequality $b^2 \neq 2a\kappa$, show that the electro-vac equations can be reduced to *purely vacuum equations*.

Answers and Hints to Selected Exercises:

1. (i) $$\mathbf{R}^{\cdot}_{\ldots}(\cdot\cdot) \equiv \mathbf{O}^{\cdot}_{\ldots}(\cdot\cdot).$$

2.

$$\varrho = r\sin\theta,\ \ z = r\cos\theta;\ 0 < r,$$
$$\widehat{\omega}(r,\,\theta) := \widehat{\omega}(\varrho,\,z),\ \ \widehat{\nu}(r,\,\theta) := \widehat{\nu}(\varrho,\,z).$$

Equations (4.10ii,iii) yield

$$\begin{aligned}
&\partial_r\widehat{\nu}(r,\theta)\\
&= r\Bigg\{\sin^2\theta\cdot\Bigg[\left(\sin\theta\cdot\partial_r\widehat{\omega} + \frac{\cos\theta}{r}\cdot\partial_\theta\widehat{\omega}\right)^2 - \left(\cos\theta\cdot\partial_r\widehat{\omega} - \frac{\sin\theta}{r}\cdot\partial_\theta\widehat{\omega}\right)^2\Bigg]\\
&\qquad + 2\,\sin\theta\cdot\cos\theta\cdot\Bigg[\left(\sin\theta\cdot\partial_r\widehat{\omega} + \frac{\cos\theta}{r}\cdot\partial_\theta\widehat{\omega}\right)\\
&\qquad\qquad \cdot\left(\cos\theta\cdot\partial_r\widehat{\omega} - \frac{\sin\theta}{r}\cdot\partial_\theta\widehat{\omega}\right)\Bigg]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\partial_\theta\widehat{\nu}(r,\theta) = r^2\Bigg\{&\sin\theta\cdot\cos\theta\cdot\Bigg[\left(\sin\theta\cdot\partial_r\widehat{\omega} + \frac{\cos\theta}{r}\cdot\partial_\theta\widehat{\omega}\right)^2\\
&\qquad - \left(\cos\theta\cdot\partial_r\widehat{\omega} - \frac{\sin\theta}{r}\cdot\partial_\theta\widehat{\omega}\right)^2\Bigg]\\
&- 2\,\sin^2\theta\cdot\Bigg[\left(\sin\theta\cdot\partial_r\widehat{\omega} + \frac{\cos\theta}{r}\cdot\partial_\theta\widehat{\omega}\right)\cdot\left(\cos\theta\cdot\partial_r\widehat{\omega} - \frac{\sin\theta}{r}\cdot\partial_\theta\widehat{\omega}\right)\Bigg]\Bigg\}.
\end{aligned}$$

$$\widehat{\nu}(r,\,\theta) = \int\left(\partial_r\widehat{\nu}\,\mathrm{d}r + \partial_\theta\widehat{\nu}\,\mathrm{d}\theta\right).$$

3. Equation (4.35iv) yields
$$\nabla^2 V = 0.$$
Equations (4.35iii), (4.34), and (4.30iv) provide
$$\partial_2 \widehat{\nu} := \partial_2 \left\{ \left[(b^2/2\kappa) - a \right]^{-1} \cdot \nu \right\} = 2\varrho \cdot \partial_1 V \cdot \partial_2 V.$$
Equations (4.35iii), (4.34), and (4.30iii) give
$$\partial_1 \widehat{\nu} := \partial_1 \left\{ \left[(b^2/2\kappa) - a \right]^{-1} \cdot \nu \right\} = \varrho \cdot \left[(\partial_1 V)^2 - (\partial_2 V)^2 \right].$$
The static vacuum metric is furnished by
$$ds^2 = e^{-2V(\cdot\cdot)} \left\{ e^{2\widehat{\nu}(\cdot\cdot)} \left[(d\varrho)^2 + (dz)^2 \right] + \varrho^2 (d\varphi)^2 \right\} - e^{2V(\cdot\cdot)} (dt)^2.$$

4.2 The General Static Field Equations

We have dealt with static, spherically symmetric equations in Sects. 3.1 and 3.2. Moreover, we have examined static, axially symmetric metrics in the preceding section. Now, we shall investigate *the general, static field equations of general relativity. We assume that a timelike Killing vector field* $\frac{\partial}{\partial x^4}$ *exists*. Furthermore, we *assume that the metric admits a discrete isometry* (of time reflection) $\widehat{x}^4 = -x^4$. Such a metric can be expressed as[1]

$$\begin{aligned} \mathbf{g}_{..}(x) &:= e^{-2w(\mathbf{x})} \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x}) \cdot (dx^\alpha \otimes dx^\beta) - e^{2w(\mathbf{x})} \cdot (dx^4 \otimes dx^4), \\ ds^2 &= e^{-2w(\mathbf{x})} \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x})(dx^\alpha dx^\beta) - e^{2w(\mathbf{x})} \cdot \left(dx^4\right)^2, \\ \mathbf{x} &:= (x^1, x^2, x^3) \in \mathbf{D} \subset \mathbb{R}^3. \end{aligned} \tag{4.39}$$

The three-dimensional metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x}) := \overset{\circ}{g}_{\alpha\beta}(\mathbf{x}) \cdot (dx^\alpha \otimes dx^\beta)$ is considered to be *positive-definite*.

The field equations (2.161i) reduce to the following *three-dimensional tensor field equations*:

[1] We shall provide *a tensorial characterization of staticity*. Suppose that a tangent vector field $\vec{\mathbf{K}}(x)$ satisfies three conditions: (1) $g_{ij}(\cdot\cdot)K^i(\cdot\cdot)K^j(\cdot\cdot) < 0$, (2) $\nabla_i K_j + \nabla_j K_i = 0$, and (3) $\nabla_i \nabla_j K_l + \nabla_j \nabla_l K_i + \nabla_l \nabla_i K_j = 0$. Then, the metric is transformable to (4.39). (See [239].)

$$\overset{\circ}{\mathcal{E}}_{\mu\nu}(\mathbf{x}) := \overset{\circ}{G}_{\mu\nu}(\mathbf{x}) + 2\left[\overset{\circ}{\nabla}_{\mu} w \cdot \overset{\circ}{\nabla}_{\nu} w - (1/2)\overset{\circ}{g}_{\mu\nu}(\mathbf{x}) \cdot \overset{\circ}{g}^{\alpha\beta}(\mathbf{x}) \cdot \overset{\circ}{\nabla}_{\alpha} w \cdot \overset{\circ}{\nabla}_{\beta} w\right] + \kappa\, T_{\mu\nu}(\mathbf{x}) = 0, \tag{4.40i}$$

$$\mu(\mathbf{x}) := \overset{\circ}{g}^{\alpha\beta}(\mathbf{x}) \cdot \overset{\circ}{\nabla}_{\alpha} \overset{\circ}{\nabla}_{\beta} w - (\kappa/2)\left[\mathrm{e}^{-4w} \cdot T_{44}(\cdot\cdot) + \overset{\circ}{g}^{\mu\nu}(\cdot\cdot)\, T_{\mu\nu}(\cdot\cdot)\right] =: \overset{\circ}{\nabla}{}^2 w - (\kappa/2) \cdot \tilde{\rho}(\mathbf{x}) = 0, \tag{4.40ii}$$

$$T_{\alpha 4}(\mathbf{x}) \equiv 0, \tag{4.40iii}$$

$$\overset{\circ}{\mathcal{T}}_{\alpha}(\mathbf{x}) := \tilde{\rho}(\mathbf{x}) \cdot \overset{\circ}{\nabla}_{\alpha} w + \overset{\circ}{\nabla}{}^{\beta} T_{\alpha\beta} = 0. \tag{4.40iv}$$

(Here, covariant derivatives are defined with respect to the metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$. Consult the answer of #1 of Exercise 4.2).

Let us count now the number of unknown functions versus the number of independent equations.

No. of unknown functions: $6(\overset{\circ}{g}_{\mu\nu}) + 1(w) + 6(T_{\mu\nu}) + 1(\tilde{\rho}) = 14.$

No. of equations: $6(\overset{\circ}{\mathcal{E}}_{\mu\nu} = 0) + 1(\mu = 0) + 3(\overset{\circ}{\mathcal{T}}_{\alpha} = 0) = 10.$

No. of identities: $3(\overset{\circ}{\nabla}_{\nu}\, \mathcal{E}^{\nu}{}_{\mu} \equiv \kappa\, \overset{\circ}{\mathcal{T}}_{\mu}) = 3.$

No. of independent equations: $10 - 3 = 7.$

Thus, the system is underdetermined, and seven of the unknown functions can be prescribed. Therefore, many mixed methods are available. It must be noted that the metric (4.39) and (4.40iii) imply that the energy–momentum–stress tensor matrix $\left[T_{ij}(\mathbf{x}, x^4)\right]$ is *diagonalizable* at every event point $x = (\mathbf{x}, x^4)$.

The vacuum field equations (2.164i), with the metric (4.39), reduce to the following three-dimensional tensor field equations:

$$\tilde{\mathcal{E}}^{(0)}_{\mu\nu}(\mathbf{x}) := \overset{\circ}{R}_{\mu\nu}(\mathbf{x}) + 2\overset{\circ}{\nabla}_{\mu} w \cdot \overset{\circ}{\nabla}_{\nu} w = 0, \tag{4.41i}$$

$$\mu^{(0)}(\mathbf{x}) := \overset{\circ}{\nabla}{}^2 w = 0. \tag{4.41ii}$$

Remarks: (i) In the static case, Einstein's vacuum field equations (2.164i) have reduced to *three-dimensional tensor and scalar field equations* (4.41i,ii).

(ii) Three coordinate conditions $\mathcal{C}(\overset{\circ}{g}_{\alpha\beta},\ \partial_{\mu}\overset{\circ}{g}_{\alpha\beta}) = 0$ can be imposed.

Table 4.1 Comparison between Newtonian gravity and Einstein static gravity outside matter

Newtonian gravitational equations	Einstein's *static* gravitational equations
Lagrangian in (2.153) for a unit mass $L_{(N)}(\mathbf{x},\mathbf{v}) = (1/2)\cdot\left[\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})v^{\alpha}v^{\beta} - 2w(\mathbf{x})\right]$	Squared Lagrangian (2.156) for a timelike geodesic $L_{(2)}(\mathbf{x},u)$ $= (1/2)\cdot\left[\mathrm{e}^{-2w}\cdot\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})v^{\alpha}v^{\beta} - \mathrm{e}^{2w(\mathbf{x})}\right]\cdot(u^4)^2$ $v^{\alpha} := (u^{\alpha}/u^4)$
Flatness conditions of the Euclidean space $\mathbb{E}_3$ $\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) \equiv 0$	Static field equations (4.41i) in vacuum $\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + 2\partial_{\alpha}w\cdot\partial_{\beta}w = 0$
Potential equation outside matter $\overset{\circ}{\nabla}{}^2 w = 0$	Static field equation (4.41ii) in vacuum $\overset{\circ}{\nabla}{}^2 w = 0$

Now, Weyl's conformal tensor in *a three-dimensional manifold is identically the zero tensor*. (See # 13 of Exercise 1.3.) Thus, the curvature tensor components are linear combinations of Ricci tensor components. (See (1.169i,ii.)) Therefore, *an equivalent system to field equations (4.41i,ii)* can be expressed as

$$\begin{aligned}\tilde{\mathcal{E}}^{(0)}_{\mu\alpha\beta\gamma}(\mathbf{x}) :=& \overset{\circ}{R}_{\mu\alpha\beta\gamma}(\mathbf{x}) + 2\left[\overset{\circ}{g}_{\mu\gamma}\cdot\partial_{\beta}w - \overset{\circ}{g}_{\mu\beta}\cdot\partial_{\gamma}w\right]\cdot\partial_{\alpha}w \\ &+ 2\left[\overset{\circ}{g}_{\alpha\beta}\cdot\partial_{\gamma}w - \overset{\circ}{g}_{\alpha\gamma}\cdot\partial_{\beta}w\right]\cdot\partial_{\mu}w \\ &+ \left[\overset{\circ}{g}{}^{\nu\lambda}\cdot\partial_{\nu}w\cdot\partial_{\lambda}w\right]\cdot\left[\overset{\circ}{g}_{\alpha\gamma}\cdot\overset{\circ}{g}_{\beta\mu} - \overset{\circ}{g}_{\alpha\beta}\cdot\overset{\circ}{g}_{\gamma\mu}\right] = 0, \end{aligned} \tag{4.42i}$$

$$\mu^{(0)}(\mathbf{x}) := \overset{\circ}{\nabla}{}^2 w = 0. \tag{4.42ii}$$

(Compare and contrast with the general vacuum field equations (2.165i).)

We are now in a position to compare Newtonian gravitational theory with Einstein's static theory of general relativity. For this, we refer to Table 4.1.

Let us now explore the static field equations (4.40i–iii). Moreover, let us consider only *the vacuum scenario* by setting $T_{ij}(\cdot\cdot) \equiv 0$. We would like to derive these equations by a variational principle, as discussed in Appendix 1. An appropriate Lagrangian (A1.25) and the action integral (A1.14) for these equations can be chosen as

$$\begin{aligned}&L\left(\mathbf{x}, y^{\alpha\beta}, \gamma^{\mu}{}_{\alpha\beta}, w;\ y^{\alpha\beta}{}_{\mu}, \gamma^{\mu}{}_{\alpha\beta\nu}, w_{\mu}\right) \\ &\quad := y^{\alpha\beta}\left[\gamma^{\mu}{}_{\mu\alpha\beta} - \gamma^{\mu}{}_{\alpha\beta\mu} - \gamma^{\nu}{}_{\nu\mu}\cdot\gamma^{\mu}{}_{\alpha\beta} + \gamma^{\nu}{}_{\alpha\mu}\cdot\gamma^{\mu}{}_{\nu\beta} + 2w_{\alpha}w_{\beta}\right],\end{aligned} \tag{4.43i}$$

$$\mathcal{L}(\cdot\cdot)_{|..} = \left[\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + 2\partial_{\alpha}w\cdot\partial_{\beta}w\right]\cdot\overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x})\cdot\sqrt{\overset{\circ}{g}}, \tag{4.43ii}$$

$$J(F) = \int_{\mathbf{D}}\left[\overset{\circ}{R}(x) + 2\overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x})\cdot\partial_{\alpha}w\cdot\partial_{\beta}w\right]\sqrt{\overset{\circ}{g}}\cdot\mathrm{d}^3\mathbf{x}. \tag{4.43iii}$$

The corresponding Euler–Lagrange equations (A1.20i) yield exactly the required static, vacuum field equations. Now, we may wonder how the reduced three-dimensional action integral (4.43iii) is deduced from the original *four-dimensional* Einstein–Hilbert action integral from the Lagrangian in (A1.25). To answer this question, we first choose a (simplified) four-dimensional domain of Cartesian product:

$$D := \mathbf{D} \times (t_1, t_2). \tag{4.44}$$

Moreover, the four-dimensional scalar invariant $R(x)$, as calculated from the static metric of (4.39), turns out to be

$$R(\cdot\cdot) = \mathrm{e}^{2w(\mathbf{x})} \cdot \left[\overset{\circ}{R}(\mathbf{x}) + 2\overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x}) \cdot \partial_\alpha w \cdot \partial_\beta w - 2\overset{\circ}{\nabla}_\alpha \left(\overset{\circ}{\nabla}{}^\alpha w \right) \right]. \tag{4.45}$$

(Consult the answer to #1 of Exercise 4.2.)

Therefore, the Einstein–Hilbert action integral reduces to

$$\begin{aligned}
&\int\limits_{\mathbf{D}\times(t_1,t_2)} R(x) \cdot \sqrt{-g} \cdot \mathrm{d}^4 x \\
&\quad = (t_2 - t_1) \cdot \int\limits_{\mathbf{D}} \left[\overset{\circ}{R}(\mathbf{x}) + 2\overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\alpha w \cdot \partial_\beta w - 2\overset{\circ}{\nabla}_\alpha \left(\overset{\circ}{\nabla}{}^\alpha w \right) \right] \sqrt{\overset{\circ}{g}} \cdot \mathrm{d}^3\mathbf{x} \\
&\quad = (\text{const.}) \cdot \int\limits_{\mathbf{D}} \left[\overset{\circ}{R}(\mathbf{x}) + 2\overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\alpha w \cdot \partial_\beta w \right] \sqrt{\overset{\circ}{g}} \cdot \mathrm{d}^3\mathbf{x} + (\text{a boundary term}).
\end{aligned} \tag{4.46}$$

Thus, action integrals (4.43ii) and (4.46) are *variationally equivalent*.

The second part in the action integral (4.43iii) gives rise to *the general relativistic Dirichlet integral* [43]:

$$J(F_0) := \int\limits_{\mathbf{D}} \overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x}) \cdot \partial_\alpha w \cdot \partial_\beta w \cdot \sqrt{\overset{\circ}{g}} \cdot \mathrm{d}^3\mathbf{x}. \tag{4.47}$$

The above action integral leads to the "potential equation" (4.41ii). The integral above naturally induces a "metric" in a one-dimensional *potential manifold* by introducing the line element:

$$\mathrm{d}\,\Sigma^2 = (\mathrm{d}w)^2. \tag{4.48}$$

Note that the above metric remains invariant under the nonhomogeneous, linear transformation:

$$\widehat{w} = \pm w + k, \tag{4.49}$$

where k is an arbitrary constant. Thus, it is clear that the action integral (4.47) remains invariant under the transformation (4.49) from $w \to \widehat{w}$. Therefore, a vacuum, exact solution with $\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})$ and $w(\mathbf{x})$ can *generate a new vacuum, exact solution* with $\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})$ and $\widehat{w}(\mathbf{x})$.

Example 4.2.1. Consider the particular transformation

$$\mathrm{e}^{2\widehat{w}(\mathbf{x})} = \left[\mathrm{e}^{2w(\mathbf{x})}\right]^{-1}$$

from (4.49). It is called *an inversion* [30]. Let us consider the static metric:

$$\mathrm{d}s^2 = \delta_{\alpha\beta} \cdot \left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - \left(x^1\right)^2 \cdot \left(\mathrm{d}x^4\right)^2, \quad x^1 > 0.$$

It turns out that the above metric is *flat*. Applying the inversion to this metric, we generate another static metric:

$$\mathrm{d}s^2 = \left(x^1\right)^4 \cdot \delta_{\alpha\beta} \cdot \left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - \left(x^1\right)^{-2} \cdot \left(\mathrm{d}x^4\right)^2, \quad x^1 > 0.$$

It is *a nonflat, static, vacuum metric*. (This is *the simplest of all known, nonflat vacuum solutions!*) *Static*, *Kasner metrics* [239] contain this particular metric. □

Now, we shall *classify the system of six partial differential equations (4.41i)*. (See Appendix 2.) Let us choose three harmonic coordinate conditions $\partial_\alpha\left(\sqrt{\overset{\circ}{g}} \cdot \overset{\circ}{g}{}^{\alpha\beta}\right) = 0$. (Consult Problem # 10 of Exercise 2.2.) We have already used four-dimensional harmonic coordinate conditions to classify the general field equations (2.223i) in Example 2.4.6. An exactly similar analysis of the static, vacuum equations (4.41i) leads to the conclusion that the static system is *elliptic*. Now, let us use an alternate set of coordinate conditions, namely, $\overset{\circ}{g}_{12}(\mathbf{x}) = \overset{\circ}{g}_{23}(\mathbf{x}) = \overset{\circ}{g}_{31}(\mathbf{x}) \equiv 0$. Therefore, we are employing an orthogonal coordinate chart to express

$$\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x}) = \mathrm{e}^{2P(\mathbf{x})} \cdot \left(\mathrm{d}x^1 \otimes \mathrm{d}x^1\right) + \mathrm{e}^{2Q(\mathbf{x})} \cdot \left(\mathrm{d}x^2 \otimes \mathrm{d}x^2\right) + \mathrm{e}^{2R(\mathbf{x})} \cdot \left(\mathrm{d}x^3 \otimes \mathrm{d}x^3\right). \tag{4.50}$$

Vacuum equations (4.41i) reduce to

$$\overset{\circ}{R}_{12}(\mathbf{x}) = \partial_1\partial_2 R + \partial_1 R\cdot\partial_2 R - \partial_1 R\cdot\partial_2 P - \partial_1 Q\cdot\partial_2 R = -2\cdot\partial_1 w\cdot\partial_2 w, \tag{4.51i}$$

$$\overset{\circ}{R}_{23}(\mathbf{x}) = \partial_2\partial_3 P + \partial_2 P\cdot\partial_3 P - \partial_2 P\cdot\partial_3 Q - \partial_2 R\cdot\partial_3 P = -2\cdot\partial_2 w\cdot\partial_3 w, \tag{4.51ii}$$

$$\overset{\circ}{R}_{31}(\mathbf{x}) = \partial_3\partial_1 Q + \partial_3 Q\cdot\partial_1 Q - \partial_3 Q\cdot\partial_1 R - \partial_3 P\cdot\partial_1 Q = -2\cdot\partial_3 w\cdot\partial_1 w, \tag{4.51iii}$$

$$\overset{\circ}{R}_{11}(\mathbf{x}) = \partial_1\partial_1(Q+R) + \partial_1 Q \cdot \partial_1(Q-P) + \partial_1 R \cdot \partial_1(R-P)$$
$$+ e^{2(P-Q)} \cdot [\partial_2\partial_2 P + \partial_2 P \cdot \partial_2(P-Q+R)]$$
$$+ e^{2(P-R)} \cdot [\partial_3\partial_3 P + \partial_3 P \cdot \partial_3(P-R+Q)] = -2(\partial_1 w)^2, \quad (4.51\text{iv})$$

$$\overset{\circ}{R}_{22}(\mathbf{x}) = \partial_2\partial_2(R+P) + \partial_2 R \cdot \partial_2(R-Q) + \partial_2 P \cdot \partial_2(P-Q)$$
$$+ e^{2(Q-R)} \cdot [\partial_3\partial_3 Q + \partial_3 Q \cdot \partial_3(Q-R+P)]$$
$$+ e^{2(Q-P)} \cdot [\partial_1\partial_1 Q + \partial_1 Q \cdot \partial_1(Q-P+R)] = -2(\partial_2 w)^2, \quad (4.51\text{v})$$

$$\overset{\circ}{R}_{33}(\mathbf{x}) = \partial_3\partial_3(P+Q) + \partial_3 P \cdot \partial_3(P-R) + \partial_3 Q \cdot \partial_3(Q-R)$$
$$+ e^{2(R-P)} \cdot [\partial_1\partial_1 R + \partial_1 R \cdot \partial_1(R-P+Q)]$$
$$+ e^{2(R-Q)} \cdot [\partial_2\partial_2 R + \partial_2 R \cdot \partial_2(R-Q+P)] = -2(\partial_3 w)^2. \quad (4.51\text{vi})$$

Let us classify the system of three semilinear, second order p.d.e.s (4.51i–iii). Each of these three p.d.e.s is *hyperbolic* according to the criterion (A2.27i), (A2.49), and (A2.51). Since (principal) terms with second derivatives occur *singly in each p.d.e.*, the system is *decoupled* for the purpose of classification. Therefore, the characteristic matrix and the determinant are furnished by $[\Gamma(..)] = \operatorname{diag}[\partial_1\phi, \partial_2\phi, \partial_2\phi, \partial_3\phi, \partial_3\phi, \partial_1\phi]$, $\det[\Gamma(..)] = (\partial_1\phi\partial_2\phi\partial_3\phi)^2$. Thus, the system is clearly *hyperbolic* with characteristic surfaces governed by

$$\partial_1\phi \cdot \partial_2\phi \cdot \partial_3\phi = 0. \quad (4.52)$$

(Across any of the characteristic surfaces, there may occur jump discontinuities of the metric functions. See (A2.5).) Previously, we concluded with a harmonic coordinate chart that the system of equations (4.41i,ii) is *elliptic*. Now with an orthogonal coordinate chart, we infer that the system of (4.51i–vi) is *mixed elliptic-hyperbolic*. On the other hand, we have concluded after (A2.23iv) that classification of a p.d.e. remains *unchanged under a coordinate transformation*. To resolve this dilemma, we note that in the case of a single p.d.e. (A2.23ii), the unknown function $w(x)$ is treated as *an invariant*, *scalar field*. But in the case of the system of field equation (4.41i), the unknown metric functions $\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})$ undergo *the transformation of a* $(0+2)$*th order tensor field*. It is this additional set of degrees of freedom that allows the system to change character.

Now, we shall deal with a star-shaped domain $D \subset \mathbb{R}^N$ and *a second-order, linear, elliptic differential operator*:

$$\mathbf{L}(w) := g^{ij}(x) \cdot \partial_i\partial_j w + h^j(x) \cdot \partial_j w, \quad x \in D \subset \mathbb{R}^N. \quad (4.53)$$

Here, the metric $\mathbf{g}_{..}(x)$ is assumed to be continuous and *positive-definite*. Moreover, the vector field $\vec{\mathbf{h}}(x)$ is assumed to be continuous. Clearly, *the Laplacian operator*

$$\nabla^2 w := \nabla^i \nabla_i w = g^{ij}(x) \cdot \partial_i \partial_j w + \left(1/\sqrt{g}\right) \cdot \partial_i \left(\sqrt{g}\, g^{ij}\right) \cdot \partial_j w \tag{4.54}$$

is a special case of (4.53). A function $w(x)$ satisfying $\nabla^2 w = 0$ is called a *harmonic function*. (In physical applications, we often call such a function *a potential function*.)

We shall now state *Hopf's theorem* and also the *maximum–minimum principle*.

Theorem 4.2.2. *Let a function* $w \in C^2(D \subset \mathbb{R}^N; \mathbb{R})$ *satisfy the weak inequality* $\mathbf{L}[w(x)] \geq 0$ *for all* $x \in D$*. If there exists an interior point* $x_0 \in D$ *such that* $w(x) \leq w(x_0)$ *for all* $x \in D$*, then* $w(x) \equiv w(x_0) =$ *constant. On the other hand, if* $L[w(x)] \leq 0$ *and* $w(x) \geq w(x_0)$ *for an interior point* x_0*, then* $w(x) \equiv w(x_0) =$ *constant.*

For proof, consult [268].

Theorem 4.2.3. *Let a function* $w(x)$ *be harmonic in* $D \subset \mathbb{R}^N$*. Moreover, let the function* $w(x)$ *be continuous up to and on the boundary* ∂D*. Then its maximum and minimum are attained on boundary points. Furthermore, the maximum or the minimum are attained at an interior point if and only if* $w(x)$ *is constant-valued.*

For proof, see [268].

Example 4.2.4. For this example, we shall employ the same Kasner metric as in Example 4.2.1. It is furnished by

$$\begin{aligned} ds^2 &= \left(x^1\right)^4 \cdot \left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right] - \left(x^1\right)^{-2} \cdot \left(dx^4\right)^2, \\ \mathbf{D} &:= \left\{\mathbf{x} : 0 < a_1 < x^1 < b_1,\ a_2 < x^2 < b_2,\ a_3 < x^3 < b_3\right\}. \end{aligned}$$

In this case, we have

$$\begin{aligned} e^{2w(\mathbf{x})} &= \left(x^1\right)^{-2}, \\ w(\mathbf{x}) &= -\ln\left(x^1\right). \end{aligned}$$

The function $w(\mathbf{x})$ is nonconstant, real analytic, and *harmonic* with respect to *the non-Euclidean metric*:

$$dl^2 = \left(x^1\right)^2 \cdot \left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right].$$

Since $\partial_1 w = -\left(x^1\right)^{-1} < 0$, according to (A1.1), the function $w(\mathbf{x})$ *does not have any critical point or extrema* inside $\mathbf{D}$. *The maximum value* $-\ln(a_1)$ and the *minimum value* $-\ln(b_1)$ are attained on some points *on the boundary* $\partial\mathbf{D}$. □

We shall now apply Theorem 4.2.3 to the static field equations (4.41i,ii).

Theorem 4.2.5. *Let static field equations* (4.41i) *and Laplace's equation* (4.41ii) *hold in a star-shaped domain* $D \subset \mathbb{R}^3$. *Let* $w(\mathbf{x})$ *be continuous up to and on* $\partial\mathbf{D}$. *Moreover, let* $w(\mathbf{x})$ *attain an extremum at an interior point* $\mathbf{x}_0 \in \mathbf{D}$. *Then, the four-dimensional domain* $\mathbf{D} \times \mathbb{R}$ *is flat.*

Proof. Consider Laplace's equation (4.41ii). By Theorem 4.2.3, $w(\mathbf{x})$ is constant-valued in $\mathbf{D} \cup \partial\mathbf{D}$. Therefore, field equation (4.41i) implies that $\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) \equiv 0$ in $\mathbf{D}$. Thus, the three-dimensional metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ yields $\overset{\circ}{\mathbf{R}}{}^{\cdot}_{\cdot\cdot\cdot}(\mathbf{x}) \equiv \mathbf{O}^{\cdot}_{\cdot\cdot\cdot}(\mathbf{x})$. Consequently, *the four-dimensional metric in (4.39) implies that the domain* $D := \mathbf{D} \times \mathbb{R}$ *is flat.* ∎

Now, we shall discuss *a special class of static metrics*. Consider the static metric given in (4.39). Let the three-dimensional, positive-definite metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ be *conformally flat*. Then, the four-dimensional metric in (4.39) is called a *conformastat metric*. By the definition in page 71, a conformastat metric can locally be reduced to

$$
\begin{aligned}
&\mathrm{d}s^2 = [U(\mathbf{x})]^4 \cdot \left[\left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2\right] - \mathrm{e}^{2w(\mathbf{x})} \cdot \left(\mathrm{d}x^4\right)^2,\\
&U(\mathbf{x}) \neq 0 \quad \text{for} \quad \mathbf{x} \in \mathbf{D} \subset \mathbb{R}^3.
\end{aligned}
\tag{4.55}
$$

The criterion for conformal flatness for dimension $N \geq 4$ was given in Theorem 1.3.33. We need to provide a criterion for conformal flatness for the dimension $N = 3$. For that reason, we introduce the *Cotton-Schouten-York tensor*[2] [16, 42, 90, 228]:

$$
\overset{\circ}{R}_{\alpha\beta\gamma}(\mathbf{x}) := \overset{\circ}{\nabla}_\gamma \overset{\circ}{R}_{\alpha\beta} - \overset{\circ}{\nabla}_\beta \overset{\circ}{R}_{\alpha\gamma} + (1/4) \cdot \left[\overset{\circ}{g}_{\alpha\gamma} \cdot \overset{\circ}{\nabla}_\beta \overset{\circ}{R} - \overset{\circ}{g}_{\alpha\beta} \cdot \overset{\circ}{\nabla}_\gamma R\right]. \tag{4.56}
$$

Lemma 4.2.6. *Consider the thrice-differentiable metric* $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ *in* $\mathbf{D} \subset \mathbb{R}^3$. *It is conformally flat if and only if*

$$
\overset{\circ}{R}_{\alpha\beta\gamma}(\mathbf{x}) \equiv 0. \tag{4.57}
$$

For proof, see [90, 171].

We shall now derive *all the conformastat, vacuum solutions*.

Theorem 4.2.7. *Consider the thrice-differentiable, static metric in equation* (4.39) *for the domain* $D := \mathbf{D} \times \mathbb{R} \subset \mathbb{R}^4$. *Let the three-dimensional metric* $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ *be conformally flat. Moreover, let the static field equations* (4.42i,ii) *prevail.* (i) *Then, the two-dimensional level surface* $w(\mathbf{x}) = const.$ *is a surface of constant curvature in the domain* $\mathbf{D} \subset \mathbb{R}^3$. (ii) *Furthermore, the four-dimensional conformastat metric reduces to one of the* (a) *Kasner,* (b) *Schwarzschild, and* (c) *pseudo-Schwarzschild metrics.*

[2] York's tensor is actually a rank-two tensor related to the one in (4.56) [269].

Proof. Here, we shall provide an abridged version of the proof. (See [52].) We have to solve the three-dimensional tensor equations (4.57), (4.41i), and (4.42i). We employ three possible coordinate conditions:

$$w(\mathbf{x}) = (x^1/2), \quad \overset{\circ}{g}_{12}(\mathbf{x}) = \overset{\circ}{g}_{13}(\mathbf{x}) \equiv 0. \tag{4.58}$$

In this coordinate chart, (4.41i) yields

$$\begin{aligned}
&\overset{\circ}{R}_{11}(\cdot\cdot) = -(1/2), \\
&\overset{\circ}{R}_{12}(\cdot\cdot) = \overset{\circ}{R}_{13}(\cdot\cdot) = \overset{\circ}{R}_{22}(\cdot\cdot) = \overset{\circ}{R}_{23}(\cdot\cdot) = \overset{\circ}{R}_{33}(\cdot\cdot) \equiv 0.
\end{aligned} \tag{4.59}$$

Equations (4.57) can be solved to obtain

$$\begin{aligned}
\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})\, \mathrm{d}x^\alpha \mathrm{d}x^\beta &= \left[V\left(x^1\right)\right]^2 \cdot \left(\mathrm{d}x^1\right)^2 + \left[V\left(x^1\right)\right] \cdot \Big[f\left(x^2, x^3\right)\left(\mathrm{d}x^2\right)^2 \\
&\qquad + g\left(x^2, x^3\right)\left(\mathrm{d}x^3\right)^2 + 2h\left(x^2, x^3\right)\left(\mathrm{d}x^2 \mathrm{d}x^3\right) \Big], \\
V^4\left(x^1\right) \cdot \Big[f\left(x^2, x^3\right) &\cdot g\left(x^2, x^3\right) - \left(h\left(x^2, x^3\right)\right)^2 \Big] > 0.
\end{aligned} \tag{4.60}$$

Here, $V(x^1)$, $f(x^2, x^3)$, $g(x^2, x^3)$, and $h(x^2, x^3)$ are thrice-differentiable arbitrary functions of integration obeying the inequality above.

Substituting (4.58)–(4.60) into field equations (4.42i), we derive that $\tilde{\mathcal{E}}^{(0)}_{1221}(\cdot\cdot) = 0 = \tilde{\mathcal{E}}^{(0)}_{1331}(\cdot\cdot)$ implies

$$V(x^1) = \left[a\, \mathrm{e}^{x^1/2} - b\, \mathrm{e}^{-x^1/2}\right]^{-2} > 0. \tag{4.61}$$

Here, a and b are two constants of integration satisfying $a^2 + b^2 > 0$. The remaining nontrivial equation $\tilde{\mathcal{E}}^{(0)}_{2332} = 0$ leads to

$$\rho_{2332}(x^2, x^3) = (ab) \cdot \Big[\left(h(x^2, x^3)\right)^2 - f(x^2, x^3) \cdot g(x^2, x^3) \Big]. \tag{4.62}$$

Here, $\rho_{2332}(x^2, x^3)$ is *a curvature tensor* component of *the surface metric*:

$$\mathrm{d}\sigma^2 = f(x^2, x^3) \cdot \left(\mathrm{d}x^2\right)^2 + g(x^2, x^3) \cdot \left(\mathrm{d}x^3\right)^2 + 2h\left(x^2, x^3\right)\left(\mathrm{d}x^2\right) \cdot \left(\mathrm{d}x^3\right). \tag{4.63}$$

By (1.164i), it follows from (4.62) that the surface metric is that of *constant curvature* equal to (ab).

There exist *three subcases*, namely, (i) $ab = 0$, (ii) $ab > 0$, and (iii) $ab < 0$. In subcase (i), the only nonflat solution is reducible to the Kasner metric:

$$
\begin{aligned}
ds^2 &= \left(1 + mx^1\right)^4 \cdot \delta_{\alpha\beta}\, dx^\alpha dx^\beta - \left(1 + mx^1\right)^{-2} \cdot \left(dx^4\right)^2, \\
&\quad 1 + mx^1 > 0.
\end{aligned} \tag{4.64i}
$$

In the subcase $ab > 0$, the solution is reducible to the isotropic form of the Schwarzschild metric:

$$
\begin{aligned}
ds^2 &= \left[1 + \frac{m}{2r}\right]^4 \cdot \left[dr^2 + r^2\left((d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right)\right] - \left[\frac{1-(m/2r)}{1+(m/2r)}\right]^2 \cdot (dt)^2, \\
&\quad 0 < m < 2r.
\end{aligned} \tag{4.64ii}
$$

(Compare with Problem # 3.ii of Exercise 3.1.)

In the subcase $ab < 0$, the solution is reducible to *the pseudo-Schwarzschild metric*:

$$
\begin{aligned}
ds^2 &= \left(\frac{2m}{R} - 1\right)^{-1} \cdot dR^2 + R^2 \cdot \left[(d\psi)^2 + \sinh^2\psi \cdot (d\varphi)^2\right] \\
&\quad - \left(\frac{2m}{R} - 1\right) \cdot (dt)^2, \\
&\quad 0 < R < 2m.
\end{aligned} \tag{4.64iii}
$$

Thus, the condensed proof is completed. ■

Remarks: (i) Synge has proved earlier [243] that $\sqrt[4]{g_{11}}(\mathbf{x})$ is a Newtonian potential for a conformastat vacuum solution. That conclusion is obvious for metrics (4.64i,ii). It is not so obvious for the metric in (4.64iii).

(ii) In a conformastat domain of the space–time, as characterized by (4.55), the "coordinate speed" of a photon is independent of the spatial direction of propagation.

Now, we shall study static, electrovac domains with the help of metric (4.39). For a static electric field, we have, from (4.24), (4.26), and (4.27),

$$
\begin{aligned}
A_\alpha(\mathbf{x}) &\equiv 0, \\
A_4(\mathbf{x}) &=: \mathcal{A}(\mathbf{x}) \neq 0, \\
F_{\alpha\beta}(\mathbf{x}) &\equiv 0, \\
F_{\alpha 4}(\mathbf{x}) &= \overset{\circ}{\nabla}_\alpha \mathcal{A} = \partial_\alpha \mathcal{A} \not\equiv 0.
\end{aligned} \tag{4.65}
$$

The nonzero components of the energy–momentum–stress tensor from (2.290i) are given by

$$T_{\alpha\beta}(\mathbf{x}) = -\mathrm{e}^{-2w} \cdot \overset{\circ}{\nabla}_{\alpha}\mathcal{A} \cdot \overset{\circ}{\nabla}_{\beta}\mathcal{A} + (1/2)\,\mathrm{e}^{-2w} \cdot \overset{\circ}{g}_{\alpha\beta} \cdot \overset{\circ}{\nabla}{}^{\mu}\mathcal{A} \cdot \overset{\circ}{\nabla}_{\mu}\mathcal{A},$$

$$T_{44}(\mathbf{x}) = (1/2)\,\mathrm{e}^{2w} \cdot \overset{\circ}{\nabla}{}^{\mu}\mathcal{A} \cdot \overset{\circ}{\nabla}_{\mu}\mathcal{A} > 0. \tag{4.66}$$

Nontrivial field equations (2.290i–vi), with the metric (4.39), electric field (4.65), and $T^{i}_{\ i}(\cdot\cdot) \equiv 0 \equiv R(\cdot\cdot)$, are reduced to the following:

$$\mathcal{M}^{4}(\mathbf{x}) = \overset{\circ}{\nabla}{}^{2}\mathcal{A} - 2\,\overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_{\alpha}w \cdot \partial_{\beta}\mathcal{A} = 0, \tag{4.67i}$$

$$\tilde{\mathcal{E}}_{\alpha\beta}(\cdot\cdot) - \overset{\circ}{g}_{\alpha\beta}\mathrm{e}^{-4w} \cdot \tilde{\mathcal{E}}_{44}(\cdot\cdot) = \overset{\circ}{R}_{\alpha\beta}(\cdot\cdot) + 2\overset{\circ}{\nabla}_{\alpha}w \cdot \overset{\circ}{\nabla}_{\beta}w$$
$$- \kappa\,\mathrm{e}^{-2w} \cdot \overset{\circ}{\nabla}_{\alpha}\mathcal{A} \cdot \overset{\circ}{\nabla}_{\beta}\mathcal{A} = 0, \tag{4.67ii}$$

$$\tilde{\mathcal{E}}_{44}(\mathbf{x}) = -\mathrm{e}^{4w} \cdot \overset{\circ}{\nabla}{}^{2}w + (\kappa/2) \cdot \mathrm{e}^{2w} \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \overset{\circ}{\nabla}_{\alpha}\mathcal{A} \cdot \overset{\circ}{\nabla}_{\beta}\mathcal{A} = 0, \tag{4.67iii}$$

$$\tilde{\mathcal{E}}_{\alpha\beta}(\cdot\cdot) = R_{\alpha\beta}(\cdot\cdot) + \kappa\left[T_{\alpha\beta}(\cdot\cdot) - 0_{\cdot\cdot}\right] = 0. \tag{4.67iv}$$

For a special class of exact solutions, *assume a functional relationship analogous to (4.32)*, namely,

$$\mathrm{e}^{2w(\mathbf{x})} = F\left[\mathcal{A}(\mathbf{x})\right] > 0,$$
$$F'(\mathcal{A}) := \frac{\mathrm{d}F[\mathcal{A}]}{\mathrm{d}\mathcal{A}} \neq 0,$$
$$\partial_{\alpha}\partial_{\beta}\left[\mathrm{e}^{2w}\right] = F'(\cdot\cdot) \cdot \partial_{\alpha}\partial_{\beta}\mathcal{A} + F''(\cdot\cdot) \cdot \partial_{\alpha}\mathcal{A} \cdot \partial_{\beta}\mathcal{A},$$
$$\overset{\circ}{\nabla}{}^{2}\omega = \frac{1}{2} \cdot \left\{[\ln|F(\cdot\cdot)|]' \cdot \overset{\circ}{\nabla}{}^{2}\mathcal{A} + [\ln|F(\cdot\cdot)|]'' \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_{\alpha}\mathcal{A} \cdot \partial_{\beta}\mathcal{A}\right\}. \tag{4.68}$$

Thus, (4.67i) and (4.67iii) reduce, respectively, to

$$\overset{\circ}{\nabla}{}^{2}\mathcal{A} = F'(\cdot\cdot) \cdot F^{-1}(\cdot\cdot) \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_{\alpha}\mathcal{A} \cdot \partial_{\beta}\mathcal{A}, \tag{4.69i}$$

$$\overset{\circ}{\nabla}{}^{2}\mathcal{A} = [F(\cdot\cdot)]^{-1} \cdot \left\{\kappa - F(\cdot\cdot) \cdot [\ln|F(\cdot\cdot)|]''\right\} \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_{\alpha}\mathcal{A} \cdot \partial_{\beta}\mathcal{A}$$
$$= \left[\frac{F'(\cdot\cdot)}{F(\cdot\cdot)} + \frac{\kappa - F''(\cdot\cdot)}{F'(\cdot\cdot)}\right] \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_{\alpha}\mathcal{A} \cdot \partial_{\beta}\mathcal{A}. \tag{4.69ii}$$

Subtracting (4.69i) from (4.69ii) and recalling $\overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\alpha \mathcal{A} \cdot \partial_\beta \mathcal{A} > 0$, we obtain

$$\begin{aligned} & F''(\mathcal{A}) = \kappa, \\ \text{or} \quad & F(\mathcal{A}) = a + b\mathcal{A} + (\kappa/2) \cdot (\mathcal{A})^2, \\ \text{or} \quad & \mathrm{e}^{2w(\mathbf{x})} = a + b\mathcal{A}(\mathbf{x}) + (\kappa/2) \cdot [\mathcal{A}(\mathbf{x})]^2 > 0. \end{aligned} \tag{4.70}$$

Here, $a > 0$ and b are arbitrary constants of integration. The equation above is the generalization of Weyl's condition (4.34) in axial symmetry.

Now, let us consider the special case:

$$\begin{aligned} b^2 &= 2a\kappa > 0, \\ \mathrm{e}^{2w(\mathbf{x})} &= \left[\sqrt{a} \pm \sqrt{\kappa/2} \cdot \mathcal{A}(\mathbf{x})\right]^2, \\ \partial_\alpha w &= \frac{\pm\sqrt{\kappa/2} \cdot \partial_\alpha \mathcal{A}}{\left[\sqrt{a} \pm \sqrt{\kappa/2} \cdot \mathcal{A}(\cdot\cdot)\right]}. \end{aligned} \tag{4.71}$$

The field equations (4.67ii) reduce to

$$\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) = 0. \tag{4.72}$$

Therefore, the three-dimensional metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ is flat! The remaining field equations, with

$$V(\mathbf{x}) := \left[\sqrt{a} \pm \sqrt{\kappa/2} \cdot \mathcal{A}(\mathbf{x})\right]^{-1}, \tag{4.73i}$$

reduce to

$$\overset{\circ}{\nabla}{}^2 V = \overset{\circ}{\nabla}{}^\alpha \overset{\circ}{\nabla}_\alpha V = 0. \tag{4.73ii}$$

Thus, $V(\mathbf{x})$ is a harmonic function in a three-dimensional Euclidean space (or it is a Newtonian potential). Using a Cartesian coordinate chart, *the resulting conformastat metric* and the covariant component of the electrostatic potential can be expressed as

$$\begin{aligned} \mathrm{d}s^2 &= [V(\mathbf{x})]^2 \cdot \left[\left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2\right] - [V(\mathbf{x})]^{-2} \cdot \left(\mathrm{d}x^4\right)^2, \\ \nabla^2 V &:= \delta^{\alpha\beta} \partial_\alpha \partial_\beta V = 0, \\ A_4(\mathbf{x}) &\equiv \mathcal{A}(\mathbf{x}) = \pm\sqrt{2/\kappa} \cdot \left\{-\sqrt{a} + [V(\mathbf{x})]^{-1}\right\}. \end{aligned} \tag{4.74}$$

This metric was discovered by Majumdar [173] and Papapetrou [204]. (Compare the above metric with the axially symmetric equation (4.38).) Equation (4.74) admits exact solutions due to many massive, charged bodies such that *gravitational attractions are exactly balanced by electrostatic repulsions*.

Example 4.2.8. We consider *the multicenter solution*:

$$V(\mathbf{x}) := 1 + \sqrt{\frac{2}{\kappa}} \cdot \sum_{A=1}^{n} \frac{e_{(A)}}{\| \mathbf{x} - \mathbf{x}_{(A)} \|},$$

$$\| \mathbf{x} - \mathbf{x}_{(A)} \| := \sqrt{\left(x^1 - x^1_{(A)}\right)^2 + \left(x^2 - x^2_{(A)}\right)^2 + \left(x^3 - x^3_{(A)}\right)^2},$$

$$\begin{aligned} \mathrm{d}s^2 = &\left[1 + \sqrt{\frac{2}{\kappa}} \cdot \sum_A \frac{e_{(A)}}{\| \mathbf{x} - \mathbf{x}_{(A)} \|}\right]^2 \cdot \delta_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta \\ &- \left[1 + \sqrt{\frac{2}{\kappa}} \cdot \sum_A \frac{e_{(A)}}{\| \mathbf{x} - \mathbf{x}_{(A)} \|}\right]^{-2} \cdot \left(\mathrm{d}x^4\right)^2, \end{aligned}$$

$$A_4(\mathbf{x}) = -\frac{2}{\kappa} \left[\sum_A \frac{e_{(A)}}{\| \mathbf{x} - \mathbf{x}_{(A)} \|}\right] \cdot \left[1 + \sqrt{\frac{2}{\kappa}} \cdot \sum_A \frac{e_{(A)}}{\| \mathbf{x} - \mathbf{x}_{(A)} \|}\right]^{-1}. \tag{4.75}$$

(Here, we have chosen $a = 1$ and the positive sign in (4.74).)

At a very large coordinate distance $\| \mathbf{x} - \mathbf{x}_{(A)} \|$, we obtain the expressions

$$\begin{aligned} \mathrm{d}s^2 = &\left[1 + 2\sqrt{\frac{2}{\kappa}} \cdot \sum_A \frac{e_{(A)}}{\| \mathbf{x} - \mathbf{x}_{(A)} \|} + \cdots\right] \cdot \delta_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta \\ &- \left[1 - 2\sqrt{\frac{2}{\kappa}} \cdot \sum_A \frac{e_{(A)}}{\| \mathbf{x} - \mathbf{x}_{(A)} \|} + \cdots\right] \cdot \left(\mathrm{d}x^4\right)^2, \end{aligned}$$

$$A_4(\mathbf{x}) = -\frac{2}{\kappa} \sum_A \frac{e_{(A)}}{\| \mathbf{x} - \mathbf{x}_{(A)} \|} - \cdots. \qquad \square$$

Now, we shall discuss static, electromagneto-vac equations employing the metric in (4.39). (See [54].) The assumption of staticity, and Maxwell's equations $\mathcal{M}^4(\cdot\cdot)=0$, ${}^*\mathcal{M}^\alpha(\cdot\cdot)=0$ in (2.290i–vi), implies that

$$F_{\alpha 4}(\mathbf{x}) = \partial_\alpha A_4 =: \partial_\alpha \mathcal{A}, \tag{4.76i}$$

$$\overset{\circ}{\nabla}_\mu \left[\mathrm{e}^{-2w} \cdot \overset{\circ}{\nabla}{}^\mu \mathcal{A}\right] = 0. \tag{4.76ii}$$

Now, we introduce the magnetic field vector by

$$\begin{aligned}
\overset{\circ}{g}{}^{\alpha\beta}(\cdot\cdot)\, H_\beta(\cdot\cdot) &= \overset{\circ}{H}{}^{\alpha}(\mathbf{x}) := (1/2)\, \mathrm{e}^{2w} \cdot \overset{\circ}{\eta}{}^{\alpha\beta\gamma}(\mathbf{x}) \cdot F_{\beta\gamma}(\mathbf{x}), \\
\mathrm{e}^{2\omega} \cdot \overset{\circ}{F}{}^{\mu\nu}(\cdot\cdot) &= \overset{\circ}{\eta}{}^{\alpha\mu\nu}(\cdot\cdot) \cdot H_\alpha(\cdot\cdot).
\end{aligned} \tag{4.77}$$

Here, $\overset{\circ}{\eta}{}^{\alpha\mu\nu}(\cdot\cdot)$ are the components of antisymmetric, oriented tensor in (1.110ii). The remaining Maxwell's equations, namely, $\mathcal{M}^\alpha(\cdot\cdot) = 0$ and $^*\mathcal{M}^4(\cdot\cdot) = 0$ yield

$$\begin{aligned}
H_\alpha(\mathbf{x}) &= \partial_\alpha \mathcal{B}, \\
\overset{\circ}{\nabla}_\mu \left[\mathrm{e}^{-2w} \cdot \overset{\circ}{\nabla}{}^{\mu} \mathcal{B} \right] &= 0.
\end{aligned} \tag{4.78}$$

Here, $\mathcal{B}(\mathbf{x})$ represents the magnetic potential which must satisfy the integral condition by Gauss' Theorem 1.3.27:

$$\int_{\partial \mathbf{D}} \mathrm{e}^{-2w} \cdot \overset{\circ}{\nabla}{}^{\mu} \mathcal{B} \cdot n_\mu \cdot \mathrm{d}^2\sigma = 0. \tag{4.79}$$

Here, $\partial\mathbf{D}$ is a continuous, piecewise-differentiable, orientable closed-surface outside matter with the areal element $\mathrm{d}^2\sigma$. Physically speaking, (4.79), by (4.77), implies that the "total magnetic flux across a regular, closed surface must be zero."

Field equations (4.40i) and (2.290i) provide

$$\begin{aligned}
& T^{\alpha}{}_{4}(\cdot\cdot) = -\mathrm{e}^{-2\omega} \cdot \overset{\circ}{\eta}{}^{\alpha\beta\gamma}(\cdot\cdot) \cdot \partial_\beta \mathcal{A} \cdot \partial_\gamma \mathcal{B}, \quad \mathcal{E}^{\alpha}{}_{4}(\mathbf{x}) = 0, \\
\text{or} \qquad & \overset{\circ}{\eta}{}^{\alpha\beta\gamma}(\mathbf{x}) \cdot \partial_\alpha \mathcal{A} \cdot \partial_\beta \mathcal{B} = 0.
\end{aligned} \tag{4.80}$$

The physical meaning of above equation is that *the Poynting vector of a static electromagnetic field in general relativity must be the zero vector.*

For the nontrivial electromagneto-vac cases, (4.80) yields three possibilities:

$$\begin{aligned}
&\text{(i)} \quad \partial_\alpha \mathcal{A} \not\equiv 0, \;\; \partial_\alpha \mathcal{B} \equiv 0; \qquad \text{(ii)} \quad \partial_\alpha \mathcal{A} \equiv 0, \;\; \partial_\alpha \mathcal{B} \not\equiv 0; \\
&\text{(iii)} \quad \partial_\alpha \mathcal{A} \not\equiv 0 \quad \text{and} \quad \partial_\alpha \mathcal{B} \not\equiv 0.
\end{aligned}$$

The first possibility is the electro-vac case. The second possibility is the magneto-vac case. (The magneto-vac case is mathematically equivalent to the electro-vac case, *except the condition in* (4.79).) The third possibility is the nontrivial electromagneto-vac case. In such a scenario, let us investigate equation (4.80) more closely. It leads to the linear dependence:

$$\begin{aligned}
& \partial_\alpha \mathcal{B} - \Lambda(\mathbf{x}) \cdot \partial_\alpha \mathcal{A} = 0, \;\; \mathrm{d}\mathcal{B} = \Lambda(\cdot\cdot) \cdot \mathrm{d}\mathcal{A}, \\
& \mathcal{B} = \mathsf{B}(\mathcal{A}), \;\; \frac{\mathrm{d}\mathsf{B}}{\mathrm{d}\mathcal{A}} = \mathsf{B}'(\mathcal{A}(\cdot\cdot)) = \Lambda(\cdot\cdot).
\end{aligned} \tag{4.81}$$

Here, $\Lambda(\mathbf{x}) \neq 0$ is some twice-differentiable function. Substituting (4.81) into (4.78) and subtracting it from (4.76ii), we derive that

$$\mathsf{B}''(\mathcal{A}) = 0, \quad k\,\mathcal{B}(\mathbf{x}) = \mathcal{A}(\mathbf{x}) + d. \tag{4.82}$$

Here, $k \neq 0$ and d are arbitrary constants of integration. The physical meaning of (4.82) is that "the static electric field vector is a constant-multiple of the static magnetic field vector."

The electromagneto-vac equations (2.290ii–iv) with (4.82), are reduced to

$$\widehat{\mathcal{B}}(\mathbf{x}) := \sqrt{1+k^2}\cdot\mathcal{B}(\mathbf{x}), \tag{4.83i}$$

$$\overset{\circ}{\nabla}{}^2\widehat{\mathcal{B}} - 2\overset{\circ}{\nabla}_\mu\widehat{\mathcal{B}}\cdot\overset{\circ}{\nabla}{}^\mu\widehat{\mathcal{B}} = 0, \tag{4.83ii}$$

$$\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + 2\overset{\circ}{\nabla}_\alpha w\cdot\overset{\circ}{\nabla}_\beta w - \kappa\,\mathrm{e}^{-2w}\cdot\overset{\circ}{\nabla}_\alpha\widehat{\mathcal{B}}\cdot\overset{\circ}{\nabla}_\beta\widehat{\mathcal{B}} = 0, \tag{4.83iii}$$

$$\overset{\circ}{\nabla}{}^2 w - (\kappa/2)\,\mathrm{e}^{-2w}\cdot\overset{\circ}{g}{}^{\alpha\beta}\cdot\overset{\circ}{\nabla}_\alpha\widehat{\mathcal{B}}\cdot\overset{\circ}{\nabla}_\beta\widehat{\mathcal{B}} = 0. \tag{4.83iv}$$

Equations (4.83ii–iv) are *formally magneto-vac equations* (*as if the electric field has disappeared*).

Feynman has already noticed in special relativistic electrodynamics [97] that an electrically charged magnet generates a stationary Poynting vector $\vec{\mathbf{E}}(\mathbf{x}) \times \vec{\mathbf{H}}(\mathbf{x})$ around it, which implies that the space–time generated by such a system must be *nonstatic* (as there is energy flow around the magnet). This can be deduced from (4.82) where we can see that Einstein's *static field equations* do not admit the existence of an electrically charged static magnet. (There is a possibility of experimental verification of this effect [212] in the context of conservation of angular momentum. Also, see [54].)

Example 4.2.9. We shall discuss here the static, magneto-vac solution of Bonnor [26] and Melvin [180]. The metric and the magnetic potential in Weyl coordinates are furnished by

$$\begin{aligned}\mathrm{d}s^2 = {} & \left[1+(b\varrho/2)^2\right]^{-2}\cdot\left\{\left[1+(b\varrho/2)^2\right]^4\cdot\left[(\mathrm{d}\varrho)^2+(\mathrm{d}z)^2\right]+\varrho^2(\mathrm{d}\varphi)^2\right\}\\ & -\left[1+(b\varrho/2)^2\right]^2\cdot(\mathrm{d}t)^2,\end{aligned} \tag{4.84i}$$

$$\widehat{\mathcal{B}}(\cdot\cdot) = \sqrt{2/\kappa}\cdot b\cdot z. \tag{4.84ii}$$

Here, b is a nonzero constant. By (4.82) and (4.83i), we can generate a new static, electromagneto-vac solution by the same metric (4.84i) and potentials:

$$\mathcal{A}(\cdot\cdot) = \sqrt{\frac{2}{\kappa(1+k^2)}} \cdot kb \cdot z + \text{const.},$$

$$\mathcal{B}(\cdot\cdot) = \sqrt{\frac{2}{\kappa(1+k^2)}} \cdot b \cdot z. \tag{4.85}$$

Both the electric and magnetic vector fields are along the z-axis.

Another possibility of yielding the same metric is choosing the potentials as

$$\mathcal{A}(\cdot\cdot) = \sqrt{\frac{2}{\kappa(1+k^2)}} \cdot b \cdot z,$$

$$\mathcal{B}(\cdot\cdot) = \sqrt{\frac{2}{\kappa(1+k^2)}} \cdot kb \cdot z. \tag{4.86}$$

In the limit $k \to 0$, we obtain from (4.86) the same metric (4.84i) as a solution of the electro-vac equations with

$$\mathcal{A}(\cdot\cdot) = \sqrt{2/\kappa} \cdot b \cdot z. \tag{4.87}$$

The metric (4.84i) with (4.87) provides an example of electro-vac solutions *outside the Weyl class* characterized by (4.70). □

Now, we shall study incoherent charged dust as discussed in (2.296i–vii). We shall use the static metric (4.39). Moreover, we assume that *only the static electric field* is present. Therefore, we have

$$F_{\alpha\beta}(\mathbf{x}) \equiv 0, \quad A_\alpha(\mathbf{x}) \equiv 0, \quad F_{\alpha 4}(\mathbf{x}) = \partial_\alpha A_4 =: \partial_\alpha \mathcal{A} \neq 0,$$
$$\sigma(\mathbf{x}) \not\equiv 0, \quad \rho(\mathbf{x}) > 0, \quad U^\alpha(\mathbf{x}) \equiv 0, \quad U^4(\mathbf{x}) = \mathrm{e}^{-w(\mathbf{x})}.$$

Using (4.66), (4.67i–iii), the non-trivial field equations reduce to

$$\overset{\circ}{\nabla}{}^2 \mathcal{A} - 2\, \overset{\circ}{g}{}^{\alpha\beta} \cdot \overset{\circ}{\nabla}_\alpha \omega \cdot \overset{\circ}{\nabla}_\beta \mathcal{A} - \sigma(\mathbf{x}) \cdot \mathrm{e}^{-w} = 0, \tag{4.88i}$$

$$\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + 2\overset{\circ}{\nabla}_\alpha w \cdot \overset{\circ}{\nabla}_\beta w - \kappa\, \mathrm{e}^{-2w} \left[\overset{\circ}{\nabla}_\alpha \mathcal{A} \cdot \overset{\circ}{\nabla}_\beta \mathcal{A} \right] = 0, \tag{4.88ii}$$

$$-\, \mathrm{e}^{4w} \cdot \overset{\circ}{\nabla}{}^2 w + (\kappa/2)\, \mathrm{e}^{2w} \left[\rho + \overset{\circ}{g}{}^{\alpha\beta} \cdot \overset{\circ}{\nabla}_\alpha \mathcal{A} \cdot \overset{\circ}{\nabla}_\beta \mathcal{A} \right] = 0, \tag{4.88iii}$$

$$\rho(\mathbf{x}) \cdot \partial_\alpha \left[\mathrm{e}^w\right] = \sigma(\mathbf{x}) \cdot \partial_\alpha \mathcal{A}, \tag{4.88iv}$$

$$\rho(\mathbf{x}) \cdot \mathrm{d}\left[\mathrm{e}^\omega\right] = \sigma(\mathbf{x}) \cdot \mathrm{d}\mathcal{A}. \tag{4.88v}$$

Now we shall try to solve the above system under some reasonable assumptions [49].

Theorem 4.2.10. *Consider field equations* (4.88i–iv) *with* (4.87) *in a star-shaped domain* $\mathbf{D} \subset \mathbb{R}^3$. *Let functions* $\rho(\mathbf{x}) > 0$ *and* $\sigma(\mathbf{x}) \neq 0$ *be of class* C^1; *the function* $\mathcal{A}(\mathbf{x})$ *be of class* C^2; *and functions* $w(\mathbf{x})$, $\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})$ *be of class* C^3 *in* $\mathbf{D}$. (i) *Then,* $w(\mathbf{x})$ *and* $\mathcal{A}(\mathbf{x})$ *are functionally related.* (ii) *In case this functional relationship is given by the quadratic equation* (4.70)*, the relationship has to be a perfect square.* (iii) *Moreover, the three-dimensional metric* $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ *is flat Euclidean, and the field equations reduce to one elliptic equation:*

$$\overset{\circ}{g}{}^{\alpha\beta}\overset{\circ}{\nabla}_\alpha\overset{\circ}{\nabla}_\beta V =: \overset{\circ}{\nabla}{}^2 V = -(\kappa/2)\,\rho(\mathbf{x})\cdot[V(\mathbf{x})]^3\,,$$
$$V(\mathbf{x}) := \mathrm{e}^{-\omega(\mathbf{x})}. \tag{4.89}$$

Proof. Equation (4.88iv) implies that $w(\mathbf{x})$ and $\mathcal{A}(\mathbf{x})$ are functionally dependent. Therefore, there exists a function F such that

$$\mathrm{e}^{2w(\mathbf{x})} = F\,[\mathcal{A}(\mathbf{x})] > 0,$$
$$\frac{\mathrm{d}\,[\mathrm{e}^{\omega}]}{\mathrm{d}\mathcal{A}}\Big|_{\mathcal{A}(\mathbf{x})} = \frac{\mathrm{d}\,[F(\mathcal{A})]^{1/2}}{\mathrm{d}\mathcal{A}}\Big|_{\mathcal{A}(\mathbf{x})} = \frac{\sigma(\mathbf{x})}{\rho(\mathbf{x})} \neq 0. \tag{4.90}$$

The field equations (4.88i) and (4.88iii) reduce to

$$\overset{\circ}{\nabla}{}^2\mathcal{A} = [F(\cdot\cdot)]^{-1}\cdot F'(\cdot\cdot)\cdot \overset{\circ}{g}{}^{\alpha\beta}\cdot\partial_\alpha\mathcal{A}\cdot\partial_\beta\mathcal{A} + \sigma(\cdot\cdot)\cdot[F(\cdot\cdot)]^{-1/2}\,, \tag{4.91i}$$

$$\overset{\circ}{\nabla}{}^2\mathcal{A} = \left[\frac{F'(\cdot\cdot)}{F(\cdot\cdot)} + \frac{\kappa - F''(\cdot\cdot)}{F'(\cdot\cdot)}\right]\cdot \overset{\circ}{g}{}^{\alpha\beta}\cdot\partial_\alpha\mathcal{A}\cdot\partial_\beta\mathcal{A} + \frac{\kappa\,\rho(\cdot\cdot)}{F'(\cdot\cdot)}. \tag{4.91ii}$$

Subtracting (4.91i) from (4.91ii), we obtain

$$\left[\frac{\kappa - F''(\cdot\cdot)}{F'}\right]\cdot \overset{\circ}{g}{}^{\alpha\beta}\cdot\partial_\alpha\mathcal{A}\cdot\partial_\beta\mathcal{A} + \left[\frac{\kappa\,\rho}{F'} - \frac{\sigma}{F^{1/2}}\right] = 0. \tag{4.92}$$

Assuming Weyl's condition (4.70), the first term of the equation above becomes zero. Therefore, from the second term, and (4.90), we deduce that

$$\frac{\sigma(\mathbf{x})}{\rho(\mathbf{x})} = \frac{\kappa\,F^{1/2}}{F'}\Big|_{..} = \frac{1}{2}\,\frac{F'}{F^{1/2}}\Big|_{..}. \tag{4.93}$$

Substituting $F[\mathcal{A}] = a + b\mathcal{A} + (\kappa/2)\mathcal{A}^2$ into (4.93), we derive both equations

$$b^2 = 2\kappa a > 0,$$
$$\text{and}\quad F[\mathcal{A}] = \left[\sqrt{a} \pm \sqrt{\kappa/2}\cdot\mathcal{A}\right]^2. \tag{4.94}$$

Putting (4.94) into (4.93), we obtain that

$$\frac{|\sigma(\mathbf{x})|}{\rho(\mathbf{x})} = \sqrt{\frac{\kappa}{2}} = \text{const.} \tag{4.95}$$

Using (4.95) and (4.94), the field equation (4.88ii) yields

$$\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) = 0. \tag{4.96}$$

Therefore, the three-dimensional metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ *is flat Euclidean*. Moreover, we define, as in (4.73i),

$$V(\mathbf{x}) := \left[\sqrt{a} \pm \sqrt{\kappa/2} \cdot \mathcal{A}(\mathbf{x})\right]^{-1}. \tag{4.97}$$

The remaining field equations reduce to *one elliptic equation* (4.89). Choosing a Cartesian chart, the metric (4.39) and (4.89) go over into

$$\begin{aligned} \mathrm{d}s^2 =& V^2(\cdot\cdot) \cdot \delta_{\alpha\beta} \left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - V^{-2}(\cdot\cdot) \left(\mathrm{d}x^4\right)^2 \\ \text{and} \quad & \nabla^2 V := \delta^{\alpha\beta} \partial_\alpha \partial_\beta V = -\frac{\kappa}{2} \rho(\mathbf{x}) \left[V(\mathbf{x}\right]^3. \end{aligned}$$

■

Remarks: (i) In the preceding theorem, we have proved that inside a charged dust (a), a functional relationship between $w(\mathbf{x})$ and $\mathcal{A}(\mathbf{x})$ *must hold*, and (b) Assuming Weyl's equation (4.70), the Majumdar–Papapetrou condition (4.94) is a consequence and (c) the conformastat metric is valid. (Similar conclusions will be proved in Chap. 8 dealing with general relativistic wave fields, as originally shown by Das [49].) Therefore, we shall call the metric in (4.97), the *Weyl–Majumdar–Papapetrou–Das* (or *W–M–P–D*) *metric*.

(ii) De and Raychaudhuri [71] have proved the condition $|\sigma(\mathbf{x})| = \sqrt{\kappa/2} \cdot \rho(\mathbf{x})$ under weaker assumptions.

(iii) We can satisfy the underdetermined equation (4.89) by defining $\rho(\mathbf{x}) := -\frac{2}{\kappa} \left[V(\cdot\cdot)\right]^{-3} \cdot \overset{\circ}{\nabla}{}^2 V$.

Example 4.2.11. Let us consider a massive, charged particle and an extended massive, charged body in *Newtonian physics*. We use a Cartesian coordinate chart. (See Fig. 4.3.)

The gravitational and electrostatic potentials $W(\mathbf{x})$ and $\Phi(\mathbf{x})$ due to the particle at $\mathbf{x}_{(1)}$ are furnished by

$$\begin{aligned} W(\mathbf{x}) &= -\frac{G\, m_{(1)}}{\|\,\mathbf{x} - \mathbf{x}_{(1)}\,\|}, \\ \Phi(\mathbf{x}) &= \frac{e_{(1)}}{4\pi} \cdot \frac{1}{\|\,\mathbf{x} - \mathbf{x}_{(1)}\,\|} = -\mathcal{A}(\mathbf{x}). \end{aligned} \tag{4.98}$$

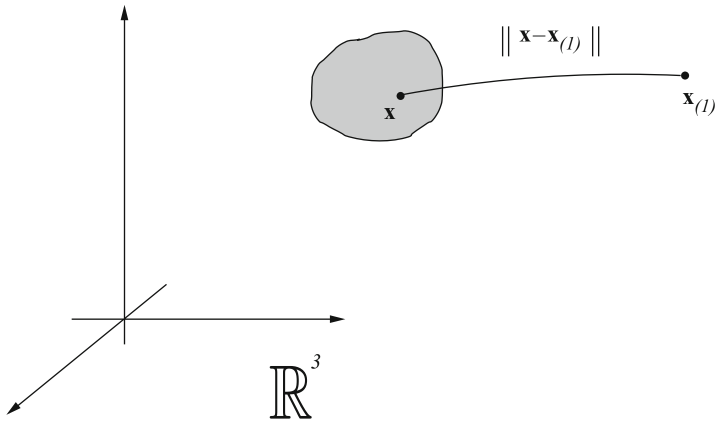


Fig. 4.3 A massive, charged particle at $\mathbf{x}_{(1)}$ and a point $\mathbf{x}$ in the extended body

(Here, we have reinstated temporarily the Newtonian gravitational constant G.) The functions W and Φ are *linearly related*. The (static) equilibrium at the point $\mathbf{x}$ of the extended body due to *external forces* is governed by equations

$$-\rho(\mathbf{x}) \cdot \nabla W = \sigma(\mathbf{x}) \cdot \nabla \Phi,$$

$$\text{or} \qquad G\, m_{(1)} \cdot \rho(\mathbf{x}) = (4\pi)^{-1} \cdot e_{(1)} \cdot \sigma(\mathbf{x}),$$

$$\text{or} \qquad \frac{\sigma(\mathbf{x})}{\rho(\mathbf{x})} = \left(\frac{\kappa}{2}\right) \cdot \frac{m_{(1)}}{e_{(1)}}.$$

$$\text{Now,} \qquad e_{(1)} = \int_{\mathbf{D}} \sigma(\mathbf{x})\, \mathrm{d}^3\mathbf{x}, \; m_{(1)} = \int_{\mathbf{D}} \rho(\mathbf{x})\, \mathrm{d}^3\mathbf{x}.$$

$$\text{Therefore,} \qquad e_{(1)} = \frac{\kappa}{2} \cdot \left(\frac{m_{(1)}}{e_{(1)}}\right) \cdot m_{(1)}, \; \text{or,} \quad \frac{\sigma(\mathbf{x})}{\rho(\mathbf{x})} = \pm\sqrt{\frac{\kappa}{2}}. \tag{4.99}$$

The last equation is the analogue of the general relativistic equation (4.95). □

Now, we shall explore the static, vacuum equations (4.42i,ii) in terms of an orthonormal triad, physical or orthonormal components, and *Ricci rotation coefficients* [53].

The static metric is furnished by (4.39). Now, we shall consider three-dimensional tensor field equations arising from the metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$. The corresponding orthonormal triad and directional derivatives from (1.105) and (1.114ii) are given by

$$\vec{\mathbf{e}}_{(\alpha)}(\mathbf{x}) = \lambda^{\mu}_{(\alpha)}(\mathbf{x}) \cdot \frac{\partial}{\partial x^{\mu}}, \tag{4.100i}$$

$$\overset{\circ}{g}_{\mu\nu}(\mathbf{x}) \cdot \lambda^{\mu}_{(\alpha)}(\mathbf{x}) \cdot \lambda^{\nu}_{(\beta)}(\mathbf{x}) = \delta_{(\alpha)(\beta)}, \tag{4.100ii}$$

$$\partial_{(\alpha)} f := \lambda^{\mu}_{(\alpha)}(\mathbf{x}) \cdot \frac{\partial f}{\partial x^{\mu}}. \tag{4.100iii}$$

Equation (1.139iv) for $f(\mathbf{x}) := x^{\mu}$ yields *metric equations*

$$\lambda^{\nu}_{(\alpha)} \cdot \partial_{\nu} \lambda^{\mu}_{(\beta)} - \lambda^{\nu}_{(\beta)} \cdot \partial_{\nu} \lambda^{\mu}_{(\alpha)} = \lambda^{\mu}_{(\delta)} \cdot \left[\gamma^{(\delta)}{}_{(\alpha)(\beta)} - \gamma^{(\delta)}{}_{(\beta)(\alpha)} \right]. \tag{4.101}$$

Equation (1.138) provides *the Ricci rotation coefficients*

$$\begin{aligned} \gamma_{(\alpha)(\beta)(\delta)}(\mathbf{x}) &= \left[\overset{\circ}{\nabla}_{\nu} \lambda_{\mu(\alpha)} \right] \cdot \lambda^{\mu}_{(\beta)}(\mathbf{x}) \cdot \lambda^{\nu}_{(\delta)}(\mathbf{x}) \\ &\equiv -\gamma_{(\beta)(\alpha)(\delta)}(\mathbf{x}). \end{aligned} \tag{4.102}$$

Equation (1.141iv) furnishes curvature tensor components as

$$\begin{aligned} \overset{\circ}{R}_{(\alpha)(\beta)(\gamma)(\delta)}(\mathbf{x}) = {} & \partial_{(\delta)} \gamma_{(\alpha)(\beta)(\gamma)} - \partial_{(\gamma)} \gamma_{(\alpha)(\beta)(\delta)} \\ & + \delta^{(\sigma)(\varepsilon)} \big[\gamma_{(\sigma)(\alpha)(\delta)} \cdot \gamma_{(\varepsilon)(\beta)(\gamma)} - \gamma_{(\sigma)(\alpha)(\gamma)} \cdot \gamma_{(\varepsilon)(\beta)(\delta)} \\ & + \gamma_{(\alpha)(\beta)(\sigma)} \cdot \big(\gamma_{(\varepsilon)(\gamma)(\delta)} - \gamma_{(\varepsilon)(\delta)(\gamma)} \big) \big], \end{aligned} \tag{4.103}$$

and the potential equation (4.42ii) yields

$$\begin{aligned} \mu^{(0)}(\mathbf{x}) &= \delta^{(\alpha)(\beta)} \cdot \overset{\circ}{\nabla}_{(\alpha)} \overset{\circ}{\nabla}_{(\beta)} w \\ &= \delta^{(\alpha)(\beta)} \cdot \left[\partial_{(\alpha)} \partial_{(\beta)} w + \gamma^{(\delta)}{}_{(\beta)(\alpha)} \partial_{(\delta)} w \right] = 0. \end{aligned} \tag{4.104}$$

The field equations in (4.42i) go over into

$$\begin{aligned} \tilde{\mathcal{E}}^{(0)}_{(\alpha)(\beta)(\gamma)(\delta)}(\mathbf{x}) = {} & \overset{\circ}{R}_{(\alpha)(\beta)(\gamma)(\delta)}(\mathbf{x}) + 2 \left[\delta_{(\alpha)(\delta)} \cdot \partial_{(\gamma)} w - \delta_{(\alpha)(\gamma)} \cdot \partial_{(\delta)} w \right] \cdot \partial_{(\beta)} w \\ & + 2 \left[\delta_{(\beta)(\gamma)} \cdot \partial_{(\delta)} w - \delta_{(\beta)(\delta)} \cdot \partial_{(\gamma)} w \right] \cdot \partial_{(\alpha)} w \\ & + \delta^{(\sigma)(\varepsilon)} \left[\partial_{(\sigma)} w \cdot \partial_{(\varepsilon)} w \right] \cdot \left[\delta_{(\beta)(\delta)} \cdot \delta_{(\gamma)(\alpha)} - \delta_{(\beta)(\gamma)} \cdot \delta_{(\delta)(\alpha)} \right] = 0. \end{aligned} \tag{4.105}$$

Let us choose a local orthogonal coordinate chart. Therefore, we have

$$\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x}) = [p(\mathbf{x})]^2 \cdot (\mathrm{d}x^1 \otimes \mathrm{d}x^1) + [q(\mathbf{x})]^2 \cdot (\mathrm{d}x^2 \otimes \mathrm{d}x^2) + [r(\mathbf{x})]^2 \cdot (\mathrm{d}x^3 \otimes \mathrm{d}x^3),$$
$$p(\mathbf{x}) > 0, \quad q(\mathbf{x}) > 0, \quad r(\mathbf{x}) > 0; \tag{4.106i}$$

$$\lambda^{\mu}_{(1)}(\mathbf{x}) = [p(\mathbf{x})]^{-1} \cdot \delta^{\mu}_{(1)}, \quad \lambda^{\mu}_{(2)}(\mathbf{x}) = [q(\mathbf{x})]^{-1} \cdot \delta^{\mu}_{(2)},$$
$$\lambda^{\mu}_{(3)}(\mathbf{x}) = [r(\mathbf{x})]^{-1} \cdot \delta^{\mu}_{(3)}; \tag{4.106ii}$$

$$\partial_{(1)} f = p^{-1} \cdot \partial_1 f, \quad \partial_{(2)} f = q^{-1} \cdot \partial_2 f, \quad \partial_{(3)} f = r^{-1} \cdot \partial_3 f. \tag{4.106iii}$$

The nonzero Ricci rotation coefficients from (4.106ii) and (4.102) are furnished by

$$\begin{aligned}
a(\mathbf{x}) &:= \gamma_{(1)(2)(1)}(\mathbf{x}) = -q^{-1} \cdot \partial_2 \ln p,\\
b(\mathbf{x}) &:= \gamma_{(1)(2)(2)}(\mathbf{x}) = p^{-1} \cdot \partial_1 \ln q,\\
l(\mathbf{x}) &:= \gamma_{(1)(3)(1)}(\mathbf{x}) = -r^{-1} \cdot \partial_3 \ln p,\\
n(\mathbf{x}) &:= \gamma_{(1)(3)(3)}(\mathbf{x}) = p^{-1} \cdot \partial_1 \ln r,\\
s(\mathbf{x}) &:= \gamma_{(2)(3)(2)}(\mathbf{x}) = -r^{-1} \cdot \partial_3 \ln q,\\
t(\mathbf{x}) &:= \gamma_{(2)(3)(3)}(\mathbf{x}) = q^{-1} \cdot \partial_2 \ln r.
\end{aligned} \tag{4.107}$$

Now, we compute the nontrivial field equation (4.105) by using (4.106ii), (4.107), and (4.103). These are the following first-order p.d.e.s:

$$\tilde{\mathcal{E}}^{(0)}_{(1)(2)(2)(3)}(\cdot\cdot) = r^{-1} \cdot \partial_3 b + s(n - b) + 2r^{-1} \cdot p^{-1} \cdot \partial_3 w \cdot \partial_1 w = 0, \tag{4.108i}$$

$$\tilde{\mathcal{E}}^{(0)}_{(2)(3)(3)(1)}(\cdot\cdot) = p^{-1} \cdot \partial_1 t + n(a + t) + 2p^{-1} \cdot q^{-1} \cdot \partial_1 w \cdot \partial_2 w = 0, \tag{4.108ii}$$

$$\tilde{\mathcal{E}}^{(0)}_{(1)(3)(1)(2)}(\cdot\cdot) = q^{-1} \cdot \partial_2 l + a(s - l) - 2q^{-1} \cdot r^{-1} \cdot \partial_2 w \cdot \partial_3 w = 0, \tag{4.108iii}$$

$$\begin{aligned}
\tilde{\mathcal{E}}^{(0)}_{(1)(2)(2)(1)}(\cdot\cdot) = {} & p^{-1} \cdot \partial_1 b - q^{-1} \cdot \partial_2 a + a^2 + b^2 + ls + p^{-2} \cdot (\partial_1 w)^2\\
& + q^{-2} \cdot (\partial_2 w)^2 - r^{-2} \cdot (\partial_3 w)^2 = 0,
\end{aligned} \tag{4.108iv}$$

$$\begin{aligned}
\tilde{\mathcal{E}}^{(0)}_{(2)(3)(3)(2)}(\cdot\cdot) = {} & q^{-1} \cdot \partial_2 t - r^{-1} \cdot \partial_3 s + s^2 + t^2 + bn - p^{-2} \cdot (\partial_1 w)^2\\
& + q^{-2} \cdot (\partial_2 w)^2 + r^{-2} \cdot (\partial_3 w)^2 = 0,
\end{aligned} \tag{4.108v}$$

$$\begin{aligned}
\tilde{\mathcal{E}}^{(0)}_{(3)(1)(1)(3)}(\cdot\cdot) = {} & p^{-1} \cdot \partial_1 n - r^{-1} \cdot \partial_3 l + l^2 + n^2 - at + p^{-2} \cdot (\partial_1 w)^2\\
& - q^{-2} \cdot (\partial_2 w)^2 + r^{-2} \cdot (\partial_3 w)^2 = 0.
\end{aligned} \tag{4.108vi}$$

Algebraic identities $\overset{\circ}{R}_{(\gamma)(\delta)(\alpha)(\beta)}(\cdot\cdot) \equiv \overset{\circ}{R}_{(\alpha)(\beta)(\gamma)(\delta)}(\cdot\cdot)$ yield, from (4.108i–iii), equations:

$$r^{-1} \cdot \partial_3 b + p^{-1} \cdot \partial_1 s - bl + ns = 0, \tag{4.109i}$$

$$r^{-1} \cdot \partial_3 a - q^{-1} \cdot \partial_2 l - lt - as = 0, \tag{4.109ii}$$

$$p^{-1} \cdot \partial_1 t - q^{-1} \cdot \partial_2 n + bt + an = 0. \tag{4.109iii}$$

The potential equation (4.104) provides

$$\begin{aligned}\mu^{(0)}(\cdot\cdot) = {} & p^{-2} \cdot \partial_1\partial_1 w + q^{-2} \cdot \partial_2\partial_2 w + r^{-2} \cdot \partial_3\partial_3 w \\ & + p^{-1} \cdot \left[\partial_1(p^{-1}) + b + n\right] \cdot \partial_1 w + q^{-1} \cdot \left[\partial_2(q^{-1}) + t - a\right] \cdot \partial_2 w \\ & + r^{-1} \cdot \left[\partial_3(r^{-1}) - l - s\right] \cdot \partial_3 w = 0.\end{aligned} \tag{4.110}$$

Equations (4.108i–vi) and (4.109i–iii) are *equivalent to the static, vacuum field equations* (4.51i–vi). A special exact solution of this system will be investigated in the following example.

Example 4.2.12. Consider *the following special functions*:

$$p(\mathbf{x}) \equiv q(\mathbf{x}) \equiv r(\mathbf{x}) := x^1 > 0,$$

$$w(\mathbf{x}) := -\ln(x^1).$$

The nonzero Ricci rotation coefficients from (4.107) are furnished by

$$b(\mathbf{x}) = (x^1)^{-2} > 0,$$

$$n(\mathbf{x}) = (x^1)^{-2} > 0.$$

Equations (4.108i–iii) and (4.109i–iii) are *identically satisfied*, and (4.108iv) reduces to

$$\frac{1}{x^1} \cdot \left(-\frac{2}{(x^1)^3}\right) + \frac{1}{(x^1)^4} + \left(\frac{1}{(x^1)^2}\right) \cdot \left(\frac{1}{(x^1)^2}\right) = 0.$$

Similarly, *all other equations are satisfied*. Therefore, we have a vacuum metric which is none other than *the Kasner metric* of Example 4.2.4. $\square$

Exercise 4.2

1. Consider the general static metric:

$$g_{ij}(\cdot\cdot)\,\mathrm{d}x^i\mathrm{d}x^j := \mathrm{e}^{-2w(\mathbf{x})}\cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x})\left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - \mathrm{e}^{2w(\mathbf{x})}\cdot\left(\mathrm{d}x^4\right)^2.$$

   Derive static field equations (4.40i–iv).
2. Prove that there exists no solution for the static metric in (4.39) in case of an incoherent dust with

$$T_{ij}(\cdot\cdot) = \rho(\cdot\cdot)\cdot U_i(\cdot\cdot)\cdot U_j(\cdot\cdot) \quad \text{and} \quad \rho(\mathbf{x}) > 0.$$

   Provide a physical reason why this is so.
3. Consider the potential equation (4.41ii). Let the two-dimensional boundary $\partial\mathbf{D}$ be a continuous, piecewise-differentiable, orientable, and closed surface. Moreover, let the boundary-value $w(\mathbf{x})_{|\partial\mathbf{D}} = F(\cdot\cdot)$ be a prescribed, continuous function. Prove that, in case this Dirichlet problem has a solution for the potential $w(\mathbf{x})$, this solution must be *unique*.
4. Consider static, electro-vac equations (4.67i–iii) and assume Weyl's condition (4.70). In case the constants satisfy the inequality $b^2 \neq 2a\,\kappa$, show that the field equations can be formally reduced to *static, vacuum equations*:

$$\mathrm{d}s^2 = \mathrm{e}^{-2\sqrt{(2\kappa)^{-1}(b^2-2a\kappa)}\cdot V}\overset{\circ}{g}_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta - \mathrm{e}^{2\sqrt{(2\kappa)^{-1}(b^2-2a\kappa)}\cdot V}(\mathrm{d}x^4)^2,$$

$$\overset{\circ}{\nabla}{}^2\left[\sqrt{(2\kappa)^{-1}(b^2-2a\kappa)}\cdot V\right] = 0.$$

5. Consider static, electro-vac equations (4.67i–iii). Let the function $w(\mathbf{x})$ attain *the maximum* at an interior point $\mathbf{x}_0 \in \mathbf{D}$. Furthermore, let $\mathcal{A}(\mathbf{x})$ attain a minimum at an interior point of $\mathbf{D} \subset \mathbb{R}^3$. Then, prove that (i) $F_{ij}(\cdot\cdot) \equiv 0$ and (ii) $R^i{}_{jkl}(\cdot\cdot) \equiv 0$ in $\mathbf{D}\times\mathbb{R}$.
6. Let a static, orthogonal metric be furnished by

$$\begin{aligned}\mathrm{d}s^2 = {} & \left(x^1\right)^{2n/p}\cdot\left(x^2\right)^{m(p-2)}\cdot\left(\mathrm{d}x^1\right)^2 + \left(x^1\right)^{2n(1-p)/p}\cdot\left(x^2\right)^{mp}\cdot\left(\mathrm{d}x^2\right)^2 \\ & + \left(x^1\right)^{2n(1-p)/p}\cdot\left(x^2\right)^{m(p-2)}\cdot\left(\mathrm{d}x^3\right)^2 - \left(x^1\right)^{-2n/p}\cdot\left(x^2\right)^{-mp}\cdot\left(\mathrm{d}x^4\right)^2,\end{aligned}$$

$$2 < p < 6, \quad 0 < \left[2p/(p^3-p^2+3p+1)\right] \leq n \leq 1/2,$$

$$-\left[4p/(p-2)^3\right] \leq m < 0;$$

$$\mathbf{D} := \left\{(x^1,x^2,x^3) : 0 < c_1 < x^1 < c_2,\ 0 < c_3 < x^2 < c_4,\ 0 < c_5 < x^3 < c_6\right\}.$$

Show that:

(i) $\left[T_{ij}(\cdot\cdot)\right]$ is of Segre characteristic $[1, 1, 1, 1]$.
(ii) Strong energy conditions (2.268iii) hold.

7. Express the general, static metric as [243]

$$\mathrm{d}s^2 = \overline{g}_{\alpha\beta}(\mathbf{x})\left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - [1 + W(\mathbf{x})]^2 \cdot \left(\mathrm{d}x^4\right)^2, \quad \mathbf{x} \in \mathbf{D} \subset \mathbb{R}^3.$$

(i) Show that the vacuum field equations reduce to

$$R_{\alpha\beta}(\cdot\cdot) = \overline{R}_{\alpha\beta}(\mathbf{x}) + [1 + W(\mathbf{x})]^{-1} \cdot \overline{\nabla}_\alpha \overline{\nabla}_\beta W = 0,$$
$$-[1 + W(\mathbf{x})]^{-1} \cdot R_{44}(\cdot\cdot) = \left(\overline{\nabla}\right)^2 W = 0.$$

(ii) Prove that vacuum equations above imply that $\overline{R}(\mathbf{x}) \equiv 0$.
(iii) Let $\left(\overline{\nabla}_\alpha W\right) \mathrm{d}x^\alpha$ be a Killing 1-form (with respect to the metric $\overline{\mathbf{g}}_{\cdot\cdot}(\mathbf{x})$). Prove that the vacuum field equations (of part (i)) imply that the space–time domain $\mathbf{D} \times \mathbb{R}$ is flat.

8. Consider the general, static metric of #7.

(i) Let twice-differentiable functions $x^\alpha = \mathcal{X}^\alpha(l)$ *and* $l_1 < l < l_2$ satisfy geodesic equations of the metric $\overline{\mathbf{g}}_{\cdot\cdot}(\mathbf{x})$. Show that the same functions, together with $\mathcal{X}^4(l) = t_0 =$ const., satisfy (spacelike) geodesic equations of the *four-dimensional static metric*.[3]
(ii) The squared Lagrangian of (2.155) for a geodesic yields

$$L_{(2)}(\cdot\cdot) = (1/2) \cdot \left\{\overline{g}_{\alpha\beta}(\mathbf{x})\, u^\alpha u^\beta - [1 + W(\mathbf{x})]^2 \cdot (u^4)^2\right\}.$$

Noting that x^4 is an *ignorable coordinate*, reduce the Lagrangian by Routh's Theorem 3.1.3 to

$$\overline{L}_{(2)}(\cdot\cdot) = (1/2) \cdot \left\{\overline{g}_{\alpha\beta}(\mathbf{x})\, u^\alpha u^\beta + E^2 \cdot [1 + W(\mathbf{x})]^{-2}\right\}.$$

(Here, E is the constant representing energy.)
(iii) The relativistic Hamiltonian (2.141i) (with $\mathcal{W}(x, u) \equiv 0$) corresponding to the Lagrangian in part (ii), for $m = 1$, is given by

$$\mathcal{H}(\cdot\cdot) = (1/2) \cdot \left\{\overline{g}^{\alpha\beta}(\mathbf{x}) \cdot p_\alpha p_\beta - [1 + W(\mathbf{x})]^{-2} \cdot (p_4)^2 + 1\right\}.$$

[3] Such a hypersurface is called a *totally geodesic hypersurface* [90].

Reduce the alternate Lagrangian (2.143) given by $\mathcal{L}_{\mathrm{I}}(\cdots) = p_i u^i - \lambda\,\mathcal{H}(\cdot\cdot)$ to obtain

$$\overline{\mathcal{L}}_{\mathrm{I}}(\mathbf{x}, \boldsymbol{u}; \mathbf{p}; \lambda) =: \mathcal{L}_{\mathrm{I}}(\cdot\cdot) - p_4 u^4 = p_\alpha u^\alpha - \lambda\,\overline{\mathcal{H}}(\mathbf{x}, \mathbf{p}),$$

$$\overline{\mathcal{H}}(\mathbf{x}, \mathbf{p}) := (1/2) \cdot \left\{ \overline{g}^{\alpha\beta}(\mathbf{x}) \cdot p_\alpha p_\beta - E^2\,[1 + W(\mathbf{x})]^{-2} + 1 \right\}.$$

9. Consider the system of the first-order partial differential equations (4.108iv–vi). Prove that the determinant of the characteristic matrix implies that *the system is hyperbolic*.
10. Consider the Euclidean potential equation and consequent electro-vac metric in (4.74).

    (i) Let f be an arbitrary holomorphic function of the complex variable $\zeta := x^1 + \mathrm{i}\left(\cos\varphi \cdot x^2 + \sin\varphi \cdot x^3\right)$. Show that

    $$V(\mathbf{x}) := \mathrm{Re}\left\{ \int_{-\pi}^{\pi} f\,(\zeta)\;\mathrm{d}\varphi \right\}$$

    solves the potential equation and thus generates a general class of static electro-vac metrics.

    (ii) Define

    $$V(\mathbf{x}) := \mathrm{Re}\left\{ \exp\left[x^1 + (i/5) \cdot (3x^2 + 4x^3)\right]^{-1} \right\}$$

    which generates a static, electro-vac metric. Prove that $V(\mathbf{x})$ takes all real values, except zero, in the neighborhood of the line given by $x^1 = 0$ and $3x^2 + 4x^3 = 0$.

Answers and Hints to Selected Exercises:

1. The static metric is given by

$$g_{ij}(\cdot\cdot)\,\mathrm{d}x^i \mathrm{d}x^j = \mathrm{e}^{-2w(\mathbf{x})} \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x})\left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - \mathrm{e}^{2w(\mathbf{x})} \cdot \left(\mathrm{d}x^4\right)^2.$$

Nonzero Christoffel symbols are provided by

$$\begin{aligned} \left\{ {\mu \atop \alpha\,\beta} \right\} &= \overset{\circ}{\left\{ {\mu \atop \alpha\,\beta} \right\}} + b^{\mu}_{\alpha\beta} \\ &=: \overset{\circ}{\left\{ {\mu \atop \alpha\,\beta} \right\}} - \overset{\circ}{g}{}^{\mu\nu} \cdot \left[\overset{\circ}{g}_{\beta\nu} \cdot \partial_\alpha w + \overset{\circ}{g}_{\nu\alpha} \cdot \partial_\beta w - \overset{\circ}{g}_{\alpha\beta} \cdot \partial_\nu w \right], \end{aligned}$$

$$\left\{ \begin{matrix} 4 \\ \alpha\, 4 \end{matrix} \right\} = \partial_\alpha w,$$

$$\left\{ \begin{matrix} \alpha \\ 4\, 4 \end{matrix} \right\} = \mathrm{e}^{4w} \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\beta w.$$

Denoting *antisymmetrization* by $A_{[\alpha\beta]} := (1/2) \cdot \left[A_{\alpha\beta} - A_{\beta\alpha} \right]$, nonzero components of the curvature tensor are given by

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} &= \overset{\circ}{R}{}^{\alpha}{}_{\beta\gamma\delta} + 2\overset{\circ}{\nabla}_{[\gamma} b^{\alpha}{}_{\delta]\beta} + 2b^{\alpha}{}_{\mu[\gamma} b^{\mu}{}_{\delta]\beta} \\ &= \overset{\circ}{R}{}^{\alpha}{}_{\beta\gamma\delta} + 2\overset{\circ}{g}_{\beta[\delta} \overset{\circ}{\nabla}_{\gamma]} \overset{\circ}{\nabla}{}^{\alpha} w + 2\delta^{\alpha}{}_{[\gamma} \overset{\circ}{\nabla}_{\delta]} \overset{\circ}{\nabla}_{\beta} w \\ &\quad + 2\overset{\circ}{g}_{\beta[\gamma} \delta^{\alpha}{}_{\delta]} \cdot \overset{\circ}{\nabla}_{\mu} w \cdot \overset{\circ}{\nabla}{}^{\mu} w + 2\overset{\circ}{g}_{\beta[\delta} \overset{\circ}{\nabla}_{\gamma]} w \cdot \overset{\circ}{\nabla}{}^{\alpha} w \\ &\quad + 2\delta^{\alpha}{}_{[\gamma} \overset{\circ}{\nabla}_{\delta]} w \cdot \overset{\circ}{\nabla}_{\beta} w, \\ R^{4}{}_{\beta\gamma 4} &= \overset{\circ}{\nabla}_{\beta} \overset{\circ}{\nabla}_{\gamma} w + 3\overset{\circ}{\nabla}_{\beta} w \cdot \overset{\circ}{\nabla}_{\gamma} w - \overset{\circ}{g}_{\beta\gamma} \cdot \overset{\circ}{g}{}^{\mu\nu} \overset{\circ}{\nabla}_{\mu} w \cdot \overset{\circ}{\nabla}_{\nu} w. \end{aligned}$$

The nonzero components of the Ricci tensor, and the curvature scalar, are furnished by

$$\begin{aligned} R_{\beta\gamma} &= \overset{\circ}{R}_{\beta\gamma} + 2\overset{\circ}{\nabla}_{\beta} w \cdot \overset{\circ}{\nabla}_{\gamma} w - \overset{\circ}{g}_{\beta\gamma} \cdot \overset{\circ}{\nabla}_{\alpha} \overset{\circ}{\nabla}{}^{\alpha} w, \\ R_{44} &= -\mathrm{e}^{4w} \cdot \overset{\circ}{\nabla}_{\alpha} \overset{\circ}{\nabla}{}^{\alpha} w = -\mathrm{e}^{4w} \cdot \overset{\circ}{\nabla}{}^{2} w, \\ R &= \mathrm{e}^{2w} \left[\overset{\circ}{R} + 2\overset{\circ}{g}{}^{\alpha\beta} \overset{\circ}{\nabla}_{\alpha} w \cdot \overset{\circ}{\nabla}_{\beta} w - 2\overset{\circ}{\nabla}{}^{2} w \right]. \end{aligned}$$

2. Use the comoving coordinate chart so that $U^{\alpha}(\mathbf{x}) \equiv 0$. Then use conservation equation (4.40iv). Gravitational collapse is inevitable.
3. Assume to the contrary that there exist *two solutions* such that

$$W_1(\mathbf{x}) \not\equiv W_2(\mathbf{x}) \quad \text{in} \quad \mathbf{D} \quad \text{with} \quad W_1(\mathbf{x})_{|\partial\mathbf{D}} \equiv W_2(\mathbf{x})_{|\partial\mathbf{D}}.$$

Define the function $W(\mathbf{x}) := W_1(\mathbf{x}) - W_2(\mathbf{x})$ so that $\overset{\circ}{\nabla}{}^{2} W = 0$ and $W(\mathbf{x})_{|\partial\mathbf{D}} \equiv 0$. By Theorem 4.2.3, the extremum of $W(\mathbf{x})$ is attained on some boundary points in $\partial\mathbf{D}$. Therefore, the function $W(\mathbf{x}) = W_1(\mathbf{x}) - W_2(\mathbf{x}) \equiv 0$ in $\mathbf{D} \cup \partial\mathbf{D}$. Thus, a contradiction is reached.

4. Introduce the function $\mathcal{V}(\mathcal{A})$ as in (4.35i):

$$\begin{aligned}
&V(\mathbf{x}) := \mathcal{V}(\mathcal{A}(\mathbf{x})),\\
&\partial_\alpha V = -\sqrt{\kappa/2}\cdot \mathrm{e}^{-2w}\cdot \partial_\alpha \mathcal{A},\\
&\overset{\circ}{\nabla}{}^2 V = 0,\\
&\overset{\circ}{\nabla}{}^2\left[\sqrt{(2\kappa)^{-1}\cdot(b^2-2a\kappa)}\cdot V\right] = 0,\\
&\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + \kappa^{-1}\cdot\left(b^2-2a\kappa\right)\cdot\partial_\alpha V\cdot\partial_\beta V = 0.
\end{aligned}$$

(Compare with the answer to #3 of Exercise 4.1.)

5. Consider the static, electro-vac equation (4.67iii). It follows that

$$\overset{\circ}{\nabla}{}^2 w = (\kappa/2)\cdot \mathrm{e}^{-2w}\cdot \overset{\circ}{g}{}^{\alpha\beta}\cdot\partial_\alpha\mathcal{A}\cdot\partial_\beta\mathcal{A}\ge 0.$$

Using Hopf's Theorem 4.2.2, it follows that $w(\mathbf{x})$ is constant-valued. Equation (4.67i) implies that $\mathcal{A}(\mathbf{x})$ is constant-valued. Therefore, (4.67ii) implies that $\overset{\circ}{R}{}^{\alpha}{}_{\beta\gamma\delta} \equiv 0$. Thus, from the answer to #1 of these exercises, it is implied that $R^{i}{}_{jkl} \equiv 0$.

6. Use the g-method of p. 197. (See [18].)

7.

$$\begin{aligned}
R_{\alpha\beta\gamma\delta}(\cdot\cdot) &= \bar{R}_{\alpha\beta\gamma\delta}(\mathbf{x}),\\
R_{\alpha\beta\gamma 4}(\mathbf{x}) &\equiv 0,\\
R_{\alpha 44\beta}(\cdot\cdot) &= -\left[1+W(\mathbf{x})\right]\cdot\bar{\nabla}_\alpha\bar{\nabla}_\beta W.
\end{aligned}$$

8. (ii)

$$\left.\frac{\partial L_{(2)}(\cdot\cdot)}{\partial u^4}\right|_{\cdot\cdot} = -\left[1+W(\mathbf{x})\right]^2\cdot u^4\big|_{\cdot\cdot} = -E = \text{const.}$$

Routh's Theorem 3.1.3 implies that

$$\begin{aligned}
\bar{L}_{(2)}(\cdot\cdot) &= L_{(2)}(\cdot\cdot) + Eu^4\\
&= (1/2)\cdot\left\{\overline{g}_{\alpha\beta}(\mathbf{x})\cdot u^\alpha u^\beta + E^2\cdot\left[1+W(\mathbf{x})\right]^{-2}\right\}.
\end{aligned}$$

10. (ii) Consider the complex variable ζ and the holomorphic function $\exp[1/\zeta]$ in the neighborhood of the isolated, essential singularity at $\zeta = 0$. It takes *every complex value except the value zero*. (See [191].)

4.3 Axially Symmetric Stationary Space–Time Domains

It is *assumed that an axially symmetric, stationary space–time domain admits one spacelike and another timelike Killing vector which commute*. The metric can be adapted to a coordinate chart so that the Killing vectors are $\frac{\partial}{\partial x^3}$ and $\frac{\partial}{\partial x^4}$. Thus, the metric can be expressed as

$$\begin{aligned}\mathbf{g}_{..}(x) = \mathrm{e}^{-2w(x^1,x^2)} \cdot \Big\{ &\mathrm{e}^{2\nu(x^1,x^2)} \cdot \left[(\mathrm{d}x^1 \otimes \mathrm{d}x^1) + (\mathrm{d}x^2 \otimes \mathrm{d}x^2) \right] \\ &+ \mathrm{e}^{2\beta(x^1,x^2)} \cdot \left(\mathrm{d}x^3 \otimes \mathrm{d}x^3 \right) \Big\} \\ &- \mathrm{e}^{2w(x^1,x^2)} \cdot \Big\{ \left[\alpha(x^1,x^2)\,\mathrm{d}x^3 + \mathrm{d}x^4 \right] \otimes \left[\alpha(x^1,x^2)\,\mathrm{d}x^3 + \mathrm{d}x^4 \right] \Big\},\end{aligned} \tag{4.111i}$$

$$\begin{aligned}\mathrm{d}s^2 = \mathrm{e}^{-2w(\varrho,z)} \cdot \left\{ \mathrm{e}^{2\nu(\varrho,z)} \cdot \left[(\mathrm{d}\varrho)^2 + (\mathrm{d}z)^2 \right] + \mathrm{e}^{2\beta(\varrho,z)} \cdot (\mathrm{d}\varphi)^2 \right\} \\ - \mathrm{e}^{2w(\varrho,z)} \cdot \left[\alpha(\varrho,z)\,\mathrm{d}\varphi + \mathrm{d}t \right]^2 .\end{aligned} \tag{4.111ii}$$

Some of the vacuum field equation (2.164i) for the metric (4.111ii) yields *Weyl–Lewis–Papapetrou (WLP) charts* [238] characterized by

$$\begin{aligned}\mathrm{d}s^2 = \mathrm{e}^{-2w(\varrho,z)} \cdot \left\{ \mathrm{e}^{2\nu(\varrho,z)} \cdot \left[(\mathrm{d}\varrho)^2 + (\mathrm{d}z)^2 \right] + \varrho^2 \cdot (\mathrm{d}\varphi)^2 \right\} \\ - \mathrm{e}^{2w(\varrho,z)} \cdot \left[\alpha(\varrho,z)\,\mathrm{d}\varphi + \mathrm{d}t \right]^2 .\end{aligned} \tag{4.112}$$

Nontrivial vacuum field equations from (4.112) emerge as

$$\partial_1 \nu = \varrho \cdot \left[(\partial_1 w)^2 - (\partial_2 w)^2 \right] - \frac{\mathrm{e}^{4w}}{4\varrho} \cdot \left[(\partial_1 \alpha)^2 - (\partial_2 \alpha)^2 \right], \tag{4.113i}$$

$$\partial_2 \nu = 2\varrho \cdot \partial_1 w \cdot \partial_2 w - \frac{\mathrm{e}^{4w}}{2\varrho} \cdot \partial_1 \alpha \cdot \partial_2 \alpha, \tag{4.113ii}$$

$$\nabla^2 w := \partial_1 \partial_1 w + \frac{1}{\varrho} \cdot \partial_1 w + \partial_2 \partial_2 w = -\frac{1}{2} \cdot \frac{\mathrm{e}^{4w}}{\varrho^2} \cdot \left[(\partial_1 \alpha)^2 + (\partial_2 \alpha)^2 \right], \tag{4.113iii}$$

$$\partial_1 \left[\varrho^{-1} \cdot \mathrm{e}^{4w} \cdot \partial_1 \alpha \right] + \partial_2 \left[\varrho^{-1} \cdot \mathrm{e}^{4w} \cdot \partial_2 \alpha \right] = 0. \tag{4.113iv}$$

Remarks: (i) Equations (4.113i–iv) reduce to static equations (4.10i–iii) in case $\alpha(\varrho, z) = \text{const.}$

(ii) The integrability of (4.113i,ii) is guaranteed by (4.113iii,iv).

Now, we pose *pseudo-Cauchy–Riemann equations* [43]:

$$\begin{aligned}\partial_1\phi &= -(\varrho)^{-1}\cdot e^{4w}\cdot\partial_2\alpha,\\ \partial_2\phi &= (\varrho)^{-1}\cdot e^{4w}\cdot\partial_1\alpha.\end{aligned} \tag{4.114}$$

The integrability of the above equations is guaranteed by (4.113iv). Substituting (4.114) into (4.113i–iv), we obtain an equivalent system:

$$\partial_1\nu = \varrho\cdot\left\{(\partial_1 w)^2-(\partial_2 w)^2+\frac{e^{-4w}}{4}\cdot\left[(\partial_1\phi)^2-(\partial_2\phi)^2\right]\right\}, \tag{4.115i}$$

$$\partial_2\nu = \varrho\cdot\left(2\partial_1 w\cdot\partial_2 w+\frac{e^{-4w}}{2}\cdot\partial_1\phi\cdot\partial_2\phi\right], \tag{4.115ii}$$

$$\nabla^2 w = -(1/2)\cdot e^{-4w}\cdot\left[(\partial_1\phi)^2+(\partial_2\phi)^2\right]. \tag{4.115iii}$$

The second integrability condition of (4.114) is

$$\partial_2\left(\partial_1\alpha\right)-\partial_1\left(\partial_2\alpha\right)=0.$$

By the equation above, (4.114) yields

$$\nabla^2\phi = 4\left(\partial_1 w\cdot\partial_1\phi+\partial_2 w\cdot\partial_2\phi\right). \tag{4.115iv}$$

In case the coupled potential equations, (4.115iii,iv), are satisfied, the functions $\nu(\varrho,z)$ and $\alpha(\varrho,z)$ can be determined by line integrals. These are furnished from (4.115i,ii) and (4.114) as

$$\begin{aligned}\nu(\varrho,z) = \int_{(\varrho_0,z_0)[\Gamma]}^{(\varrho,z)} y^1\cdot\Big\{&\left[(\partial_1 w)^2-(\partial_2 w)^2+(1/4)\cdot e^{-4w}\cdot\left((\partial_1\phi)^2-(\partial_2\phi)^2\right)\right]_{|..}\,dy^1\\ &+\left[2\partial_1 w\cdot\partial_2 w+(1/2)\cdot e^{-4w}\cdot\partial_1\phi\cdot\partial_2\phi\right]_{|..}\,dy^2\Big\},\end{aligned} \tag{4.116i}$$

$$\alpha(\varrho,z) = \int_{(\varrho_0,z_0)[\Gamma]}^{(\varrho,z)} y^1\cdot\left\{\left[e^{-4w}\cdot\partial_2\phi\right]_{|..}\,dy^1-\left[e^{-4w}\cdot\partial_1\phi\right]_{|..}\,dy^2\right\}. \tag{4.116ii}$$

(Equation (4.116i) *generalizes the static equation* (4.9). See Fig. 4.1.)

Now, we shall introduce *a complex potential* by

$$F(\varrho, z) := e^{2w(\varrho,z)} + i\,\phi(\varrho, z). \tag{4.117}$$

The vacuum equations (4.115i–iv) reduce to

$$\partial_1 \nu = (\varrho/4) \cdot (\text{Re}\, F)^{-2} \cdot \left[|\partial_1 F|^2 - |\partial_2 F|^2\right], \tag{4.118i}$$

$$\partial_2 \nu = (\varrho/4) \cdot (\text{Re}\, F)^{-2} \cdot \left[\partial_1 F \cdot \partial_2 \overline{F} + \partial_2 F \cdot \partial_1 \overline{F}\right], \tag{4.118ii}$$

$$\nabla^2 F = (\text{Re}\, F)^{-1} \cdot \left[(\partial_1 F)^2 + (\partial_2 F)^2\right]. \tag{4.118iii}$$

Here, *the bar denotes complex conjugation*.
We introduce another complex potential:

$$\xi(\varrho, z) := \left[\frac{1 + F(\varrho, z)}{1 - F(\varrho, z)}\right]. \tag{4.119}$$

(The above potential is useful in obtaining exact solutions more easily.)
The field equations (4.118i–iii) go over into

$$\partial_1 \nu = \varrho \cdot \left[|\xi|^2 - 1\right]^{-2} \cdot \left[|\partial_1 \xi|^2 - |\partial_2 \xi|^2\right], \tag{4.120i}$$

$$\partial_2 \nu = \varrho \cdot \left[|\xi|^2 - 1\right]^{-2} \cdot \left[\partial_1 \xi \cdot \partial_2 \overline{\xi} + \partial_2 \xi \cdot \partial_1 \overline{\xi}\right], \tag{4.120ii}$$

$$\nabla^2 \xi = 2\overline{\xi} \cdot \left[|\xi|^2 - 1\right]^{-1} \cdot \left[(\partial_1 \xi)^2 + (\partial_2 \xi)^2\right]. \tag{4.120iii}$$

Now, we introduce *a prolate spheroidal coordinate chart* [256, 273] by

$$\begin{aligned}
\varrho &= k\sqrt{(x^2-1)(1-y^2)},\\
z &= kxy,\\
k &= \text{positive const.}; \quad x > 1,\ 0 < y < 1;\\
d\varrho^2 + dz^2 &= k^2 \cdot (x^2 - y^2) \cdot \left[\frac{(dx)^2}{x^2-1} + \frac{(dy)^2}{1-y^2}\right].
\end{aligned} \tag{4.121}$$

The inverse transformation is furnished by

$$\begin{aligned}
2k \cdot x &= A(\varrho, z) + B(\varrho, z),\\
2k \cdot y &= A(\varrho, z) - B(\varrho, z),\\
A(\varrho, z) &:= \sqrt{\varrho^2 + (z+k)^2},\\
B(\varrho, z) &:= \sqrt{\varrho^2 + (z-k)^2}.
\end{aligned} \tag{4.122}$$

Remarks: (i) The prolate spheroidal coordinate x *must not be confused with the point* $x \in \mathbb{R}^4$.

(ii) Prolate spheroidal coordinates do facilitate extracting exact solutions of the static, axially symmetric field equations.

The complex potential equation (4.120iii) is equivalent to

$$\partial_x \left[(x^2 - 1) \cdot \partial_x \widehat{\xi} \right] + \partial_y \left[(1 - y^2) \cdot \partial_y \widehat{\xi} \right]$$
$$= \frac{2\overline{\widehat{\xi}}(x, y)}{\left[\left| \widehat{\xi} \right|^2 - 1 \right]} \cdot \left[(x^2 - 1) \cdot \left(\partial_x \widehat{\xi} \right)^2 + (1 - y^2) \cdot \left(\partial_y \widehat{\xi} \right)^2 \right],$$
$$\widehat{\xi}(x, y) := \xi(\varrho, z). \tag{4.123}$$

This second-order semilinear elliptic equation is known as the *Ernst equation* [92, 93].

Example 4.3.1. We shall try to find a linear solution of the type:

$$\widehat{\xi}(x, y) = px + \mathrm{i}qy. \tag{4.124}$$

Here, p and q are assumed to be real constants such that $p^2 + q^2 > 0$. The left-hand side of (4.123) yields

$$\text{L.H.S.} = 2\,(px - \mathrm{i}qy). \tag{4.125i}$$

The right-hand side of (4.123) provides

$$\text{R.H.S.} = \frac{2\,(px - \mathrm{i}qy)}{(p^2x^2 + q^2y^2 - 1)} \cdot \left[p^2x^2 + q^2y^2 - (p^2 + q^2) \right]. \tag{4.125ii}$$

Equating (4.125i) with (4.125ii), we derive that

$$p^2 + q^2 = 1. \tag{4.126}$$

(Therefore, we have $|p| \leq 1,\ |q| \leq 1$.) By (4.119), we deduce that

$$\widehat{F}(x, y) = \left[\frac{\widehat{\xi}(x, y) - 1}{\widehat{\xi}(x, y) + 1} \right] = \left[\frac{px + \mathrm{i}qy - 1}{px + \mathrm{i}qy + 1} \right]$$
$$= \left\{ \frac{\left[(px)^2 + (qy)^2 - 1 \right] + 2\mathrm{i}qy}{\left[(px + 1)^2 + (qy)^2 \right]} \right\}. \tag{4.127}$$

Using (4.117), we obtain

$$\begin{aligned} e^{2\widehat{w}(x,y)} &= \left[\frac{(px)^2 + (qy)^2 - 1}{(px+1)^2 + (qy)^2}\right], \\ \widehat{\phi}(x,y) &= \left[\frac{2qy}{(px+1)^2 + (qy)^2}\right]. \end{aligned} \tag{4.128}$$

For physical considerations, we identify the constants in the above as

$$\begin{aligned} p &= \frac{\sqrt{m^2-a^2}}{m} > 0, \\ q &= \frac{a}{m}, \\ k &= \sqrt{m^2-a^2} > 0. \end{aligned} \tag{4.129}$$

Thus, (4.128) yields by (4.122),

$$\begin{aligned} e^{2w(\varrho,z)} &= \left\{\frac{(m^2-a^2)\cdot[A(\cdot\cdot)+B(\cdot\cdot)]^2 + a^2\cdot[A(\cdot\cdot)-B(\cdot\cdot)]^2 - 4m^2\cdot(m^2-a^2)}{(m^2-a^2)\cdot[A(\cdot\cdot)+B(\cdot\cdot)+2m]^2 + a^2\cdot[A(\cdot\cdot)-B(\cdot\cdot)]^2}\right\}, \\ \phi(\varrho,z) &= \left\{\frac{4am\cdot\sqrt{m^2-a^2}\cdot[A(\cdot\cdot)-B(\cdot\cdot)]}{(m^2-a^2)\cdot[A(\cdot\cdot)+B(\cdot\cdot)+2m]^2 + a^2\cdot[A(\cdot\cdot)-B(\cdot\cdot)]^2}\right\}. \end{aligned} \tag{4.130}$$

From (4.116i,ii), we derive the corresponding four-dimensional metric as

$$\begin{aligned} ds^2 = &\left[\frac{(m^2-a^2)\cdot(A+B+2m)^2 + a^2\cdot(A-B)^2}{(m^2-a^2)\cdot(A+B)^2 + a^2\cdot(A-B)^2 - 4m^2\cdot(m^2-a^2)}\right] \\ &\times\left\{\left[\frac{(m^2-a^2)\cdot(A+B)^2 + a^2\cdot(A-B)^2 - 4m^2\cdot(m^2-a^2)}{4(m^2-a^2)\cdot A\cdot B}\right]\right. \\ &\qquad\left.\times\left[(d\varrho)^2 + (dz)^2\right] + \varrho^2\cdot(d\varphi)^2\right\} \\ &-\left[\frac{(m^2-a^2)\cdot(A+B)^2 + a^2\cdot(A-B)^2 - 4m^2\cdot(m^2-a^2)}{(m^2-a^2)\cdot(A+B+2m)^2 + a^2\cdot(A-B)^2}\right] \\ &\times\left\{dt + \left[\frac{am\cdot(A+B+2m)\cdot\left[4(m^2-a^2)-(A-B)^2\right]}{(m^2-a^2)\cdot(A+B)^2 + a^2\cdot(A-B)^2 - 4m^2\cdot(m^2-a^2)}\right]\cdot d\varphi\right\}^2. \end{aligned} \tag{4.131}$$

The metric above is *the famous Kerr metric* [148] expressed in the W–L–P coordinate chart. (The stationary metric in (4.131) generalizes the static metric in (4.18).) □

The *Boyer–Lindquist coordinate chart* [28] is introduced by the transformation

$$\begin{aligned} r &= (1/2)\cdot\left[A(\varrho,z)+B(\varrho,z)\right]+m, \\ \cos\theta &= \left(1/2\sqrt{m^2-a^2}\right)\cdot[A(\cdot\cdot)-B(\cdot\cdot)], \\ \varrho &= \sqrt{r^2-2mr+a^2}\cdot\sin\theta, \\ z &= (r-m)\cdot\cos\theta. \end{aligned} \tag{4.132}$$

The Kerr metric in (4.131) goes over into

$$\begin{aligned} ds^2 = {} & \left[1-\frac{2mr}{r^2+a^2\cos^2\theta}\right]^{-1}\cdot\left\{(r^2-2mr+a^2\cos^2\theta)\right. \\ & \left.\times\left[\frac{(dr)^2}{(r^2-2mr+a^2)}+(d\theta)^2\right]+(r^2-2mr+a^2)\cdot\sin^2\theta\cdot(d\varphi)^2\right\} \\ & -\left[1-\frac{2mr}{r^2+a^2\cos^2\theta}\right]\cdot\left\{dt+\left[\frac{2amr\cdot\sin^2\theta}{r^2-2mr+a^2\cos^2\theta}\right]\cdot d\varphi\right\}^2; \end{aligned}$$

$$\begin{aligned} D := {} & \left\{(r,\theta,\varphi,t) : r > m+\sqrt{m^2-a^2\cos^2\theta},\right. \\ & \left. 0<\theta<\pi,\ -\pi<\varphi<\pi,\ -\infty<t<\infty\right\}. \end{aligned} \tag{4.133}$$

The line element above is *more popularly expressed as*

$$\begin{aligned} ds^2 = {} & \Sigma(r,\theta)\cdot\left[\frac{(dr)^2}{\Delta(r)}+(d\theta)^2\right]+(r^2+a^2)\cdot\sin^2\theta\cdot(d\varphi)^2 \\ & -(dt)^2+\left[\frac{2mr}{\Sigma(r,\theta)}\right]\cdot\left(dt-a\ \sin^2\theta\cdot d\varphi\right)^2; \\ & \Sigma(r,\theta) := r^2+a^2\cos^2\theta, \quad \Delta(r) := r^2-2mr+a^2. \end{aligned} \tag{4.134}$$

In the limit $a\to 0$, the above metric reduces to *the Schwarzschild metric* of (3.9).

The physical meanings of the parameters "m" and "ma" are taken to be the total mass and the total angular momentum of the rotating object, respectively, (so that "a" represents the angular momentum per unit mass).

The linear approximation (linear in $1/r$) of the metric (4.133) is provided by

$$
\begin{aligned}
\mathrm{d}s^2 = {} & \left[1 + \frac{2m}{r} + 0\left(\frac{1}{r^2}\right)\right] \cdot \left[(\mathrm{d}r)^2 + r^2 \cdot (\mathrm{d}\theta)^2 + r^2 \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right] \\
& - \left[1 - \frac{2m}{r} + 0\left(\frac{1}{r^2}\right)\right](\mathrm{d}t)^2 - \left[\frac{4am}{r} \cdot \sin^2\theta + 0\left(\frac{1}{r^2}\right)\right] \mathrm{d}\varphi\, \mathrm{d}t.
\end{aligned}
\tag{4.135}
$$

This approximate solution was considered long before the exact metric of Kerr was discovered. (Originally, this approximate metric was studied by Lense and Thirring [165] and others [39].) There are some interesting physical consequences of the metric in (4.135). Some of these are listed in the following:

(i) The bending of light rays around the rotating sun are *slightly asymmetric about two opposite sides*. (See [46].)
(ii) The orbital plane of a planet shifts very slowly. The actual nodal shift of the planet Venus was originally thought not to be totally explained by (4.135). (See [84].) However, more refined calculations and analysis of more accurate data seem to indicate that the observed value is within experimental agreement with general relativity [81, 139, 211].
(iii) An artificial satellite orbiting around the Earth can experimentally verify the time dilatation inherent in (4.135) [47]. In fact, modern *Global Positioning Systems* (*G.P.S.*) exploit (4.135)! (See [9].)
(iv) *The dragging of inertial frames* is a consequence of (4.135). Einstein has commented: " ... Mach was on the right road in his thought that inertia depends on a mutual action of matter... A rotating hollow body must generate inside of itself a "Coriolis field," which deflects bodies in the sense of rotation, ... " [89].

Now, we shall touch upon briefly the charged version of the Kerr solution. It is known as the *Kerr-Newman solution* [193] and is furnished by

$$
\begin{aligned}
\mathrm{d}s^2 = {} & \Sigma(r,\theta) \cdot \left[\frac{(\mathrm{d}r)^2}{\tilde{\Delta}(r)} + (\mathrm{d}\theta)^2\right] + (r^2 + a^2) \cdot \sin^2\theta \cdot (\mathrm{d}\varphi)^2 \\
& - (\mathrm{d}t)^2 + \left[\frac{2mr - e^2}{\Sigma(r,\theta)}\right] \cdot \left(a \sin^2\theta \cdot \mathrm{d}\varphi - \mathrm{d}t\right)^2;
\end{aligned}
$$

$$
\begin{aligned}
\Sigma(r,\theta) &:= r^2 + a^2\cos^2\theta, \\
\tilde{\Delta}(r) &:= r^2 - 2mr + a^2 + e^2, \\
A_i(\cdot\cdot)\,\mathrm{d}x^i &= \left[\frac{er}{\Sigma(r,\theta)}\right] \cdot \left(a \sin^2\theta \cdot \mathrm{d}\varphi - \mathrm{d}t\right),
\end{aligned}
\tag{4.136}
$$

where e represents the charge parameter of the gravitating body. Note that in the limit $e \to 0$, the metric (4.136) reduces to the Kerr metric in (4.134). On the other hand, in the limit $a \to 0$, the Kerr–Newman metric of (4.136) goes over into the *Reissner–Nordström–Jeffery metric* of (3.74).

A possible interior for the Kerr metric was originally investigated by Hogan [134]. However, junction conditions were not satisfied. Others are summarized in [40]. We explore now the same problem in the cylindrical coordinates of (4.111ii). We write the interior metric in a slightly modified form as

$$\begin{aligned} \mathrm{d}s^2 = {} & \left[\mathsf{W}(\varrho, z)\right]^{-1} \cdot \left\{N(\varrho, z) \cdot \left[(\mathrm{d}\varrho)^2 + (\mathrm{d}z)^2\right] + \mathrm{e}^{2\beta(\varrho,z)} \cdot (\mathrm{d}\varphi)^2\right\} \\ & - \mathsf{W}(\cdot\cdot)\left[\alpha(\varrho, z) \cdot \mathrm{d}\varphi + \mathrm{d}t\right]^2 ; \end{aligned} \tag{4.137i}$$

$$\mathsf{W}(\varrho, z) > 0, \quad N(\varrho, z) > 0; \tag{4.137ii}$$

$$D_{(2)} := \{(\varrho, z) : \ 0 < \varrho_1 \leq \varrho < b(z), \ z_1 < z < z_2\}. \tag{4.137iii}$$

Here, $\varrho = b(z)$ denotes the exterior, differentiable, boundary curve of the axially symmetric body, projected onto the $\varrho - z$ coordinate plane.

We choose for the material body an axially symmetric, *deformable solid*. (See (2.270).) The axially symmetric stress components $\sigma_{\alpha\beta}(\cdot\cdot)$ and material current components $s_\alpha(\cdot\cdot)$ satisfy

$$\begin{aligned} \sigma_{13}(\cdot\cdot) = \sigma_{23}(\cdot\cdot) \equiv 0, \\ s_1(\cdot\cdot) = s_2(\cdot\cdot) \equiv 0. \end{aligned} \tag{4.138}$$

Denoting the mass density by $\tilde{\mu}(\varrho, z)$, we can express the nonzero components of the energy–momentum–stress tensor as

$$\begin{aligned} & T_{11}(\cdot\cdot) = -\sigma_{11}(\cdot\cdot), \quad T_{22}(\cdot\cdot) = -\sigma_{22}(\cdot\cdot), \quad T_{12}(\cdot\cdot) = -\sigma_{12}(\cdot\cdot), \\ & T_{33}(\cdot\cdot) = -\left(1 - \alpha \cdot \mathsf{W}^2 \cdot \mathrm{e}^{2\beta}\right) \cdot \sigma_{33} + \alpha \cdot s_3 + \alpha^2 \cdot \mathsf{W}^2 \cdot \left[\tilde{\mu} + N^{-1} \cdot (\sigma_{11} + \sigma_{22})\right], \\ & T_{34}(\cdot\cdot) = (1/2) \cdot s_3 + \alpha \cdot \mathsf{W}^2 \left[\tilde{\mu} + N^{-1} \cdot (\sigma_{11} + \sigma_{22}) + \mathrm{e}^{-2\beta} \cdot \sigma_{33}\right], \\ & T_{44}(\cdot\cdot) = \mathsf{W}^2 \cdot \left[\tilde{\mu} + N^{-1} \cdot (\sigma_{11} + \sigma_{22}) + \mathrm{e}^{-2\beta} \cdot \sigma_{33}\right]. \end{aligned} \tag{4.139}$$

(The above linear combination will be explained clearly in the next section.)

The nontrivial interior field equation (2.161i) for metric (4.137i) and the energy–momentum–stress tensor components in (4.139) are furnished as follows:

$$\partial_2\partial_2\beta + (\partial_2\beta)^2 + \left(N^{-1}/2\right)\cdot(\partial_1\beta\cdot\partial_1 N - \partial_2\beta\cdot\partial_2 N)$$
$$+ \left(\mathsf{W}^{-2}/4\right)\cdot\left[(\partial_1\mathsf{W})^2 - (\partial_2\mathsf{W})^2\right]$$
$$+ \left(\mathsf{W}^2/4\right)\cdot \mathrm{e}^{-2\beta}\cdot\left[(\partial_1\alpha)^2 - (\partial_2\alpha)^2\right] = -\kappa\,\sigma_{11}, \tag{4.140i}$$

$$\partial_1\partial_1\beta + \partial_2\partial_2\beta + (\partial_1\beta)^2 + (\partial_2\beta)^2 = -\kappa\,(\sigma_{11} + \sigma_{22})\,, \tag{4.140ii}$$

$$\partial_1\partial_2\beta + \partial_1\beta\cdot\partial_2\beta - \left(N^{-1}/2\right)\cdot(\partial_1\beta\cdot\partial_2 N + \partial_2\beta\cdot\partial_1 N)$$
$$+ \left(\mathsf{W}^{-2}/2\right)\cdot\partial_1\mathsf{W}\cdot\partial_2\mathsf{W} - \left(\mathsf{W}^2/2\right)\cdot \mathrm{e}^{-2\beta}\cdot\partial_1\beta\cdot\partial_2\beta = \kappa\,\sigma_{12}, \tag{4.140iii}$$

$$\left(N^{-1}/2\right)\cdot \mathrm{e}^{2\beta}\cdot\left\{\left[N^{-1}\cdot(\partial_1\partial_1 N + \partial_2\partial_2 N) - N^{-2}\cdot\left((\partial_1 N)^2 + (\partial_2 N)^2\right)\right]\right.$$
$$+ \left(\mathsf{W}^{-2}/2\right)\cdot\left[(\partial_1\mathsf{W})^2 + (\partial_2\mathsf{W})^2\right]$$
$$\left. + \left(\mathsf{W}^2/2\right)\cdot \mathrm{e}^{-2\beta}\cdot\left[(\partial_1\alpha)^2 + (\partial_2\alpha)^2\right]\right\} = -\kappa\,\sigma_{33}, \tag{4.140iv}$$

$$N^{-1}\cdot \mathrm{e}^{-\beta}\cdot\left\{\partial_1\left[\mathrm{e}^{\beta}\cdot\mathsf{W}^{-1}\cdot\partial_1\mathsf{W}\right] + \partial_2\left[\mathrm{e}^{\beta}\cdot\mathsf{W}^{-1}\cdot\partial_2\mathsf{W}\right]\right.$$
$$\left. + \mathsf{W}^2\cdot \mathrm{e}^{-\beta}\cdot\left[(\partial_1\alpha)^2 + (\partial_2\alpha)^2\right]\right\} = \kappa\,\tilde{\mu}, \tag{4.140v}$$

$$N^{-1}\cdot \mathrm{e}^{-\beta}\cdot\left\{\partial_1\left[\mathsf{W}^2\cdot \mathrm{e}^{-\beta}\cdot\partial_1\alpha\right] + \partial_2\left[\mathsf{W}^2\cdot \mathrm{e}^{-\beta}\cdot\partial_2\alpha\right]\right\} = \kappa\,\mathrm{e}^{-2\beta}\cdot s_3. \tag{4.140vi}$$

The nontrivial conservation equation (2.166i) reduces to

$$\tilde{\mu}\cdot\partial_1(\ln\mathsf{W}) = 2N^{-1}\cdot \mathrm{e}^{-\beta}\cdot\left[\partial_1\left(\mathrm{e}^{\beta}\cdot\sigma_{11}\right) + \partial_2\left(\mathrm{e}^{\beta}\cdot\sigma_{21}\right)\right]$$
$$+ \partial_1\left(N^{-1}\right)\cdot(\sigma_{11} + \sigma_{22}) + \mathrm{e}^{-2\beta}\cdot\partial_1\alpha\cdot s_3, \tag{4.141i}$$

$$\tilde{\mu}\cdot\partial_2(\ln\mathsf{W}) = 2N^{-1}\cdot \mathrm{e}^{-\beta}\cdot\left[\partial_1\left(\mathrm{e}^{\beta}\cdot\sigma_{12}\right) + \partial_2\left(\mathrm{e}^{\beta}\cdot\sigma_{22}\right)\right]$$
$$+ \partial_2\left(N^{-1}\right)\cdot(\sigma_{11} + \sigma_{22}) + \mathrm{e}^{-2\beta}\cdot\partial_2\alpha\cdot s_3, \tag{4.141ii}$$

The above equations represent *the equilibrium of an axially symmetric, rotating, solid body in general relativity*.

Let us represent the Kerr metric of (4.131) as

$$\begin{aligned}
ds^2 &= [\mathsf{W}_0]^{-1} \cdot \{N_0 \cdot [(d\varrho)^2 + (dz)^2] + \varrho^2 \cdot (d\varphi)^2\} \\
&\quad - [\mathsf{W}_0] \cdot [dt + \varepsilon \cdot \alpha_0 \cdot d\varphi]^2 ; \\
\mathsf{W}_0(\cdot\cdot) &:= \left[\frac{(1-\varepsilon^2)\cdot(A+B)^2 + \varepsilon^2\cdot(A-B)^2 - 4m^2(1-\varepsilon^2)}{(1-\varepsilon^2)\cdot(A+B+2m)^2 + \varepsilon^2\cdot(A-B)^2} \right], \\
N_0(\cdot\cdot) &:= \left[\frac{(1-\varepsilon^2)\cdot(A+B)^2 + \varepsilon^2\cdot(A-B)^2 - 4m^2(1-\varepsilon^2)}{4(1-\varepsilon^2)\cdot A \cdot B} \right], \\
\varepsilon\alpha_0(\cdot\cdot) &:= \left\{ \frac{\varepsilon\cdot(A+B+2m)\cdot[4m^2(1-\varepsilon^2) - (A-B)^2]}{(1-\varepsilon^2)\cdot(A+B)^2 + \varepsilon^2(A-B)^2 - 4m^2(1-\varepsilon^2)} \right\}, \\
\varepsilon &:= (a/m) = q.
\end{aligned} \tag{4.142}$$

We choose the parameter $|\varepsilon|$ to be a sufficiently small, positive number. Therefore, the body is *assumed to rotate very slowly*.

Consider the interior metric:

$$\begin{aligned}
ds^2 &= \{\mathsf{W}_0(\cdot\cdot) + [b(z)-\varrho]^n \cdot \mathsf{W}^{\#}(\cdot\cdot)\}^{-1} \cdot \{[N_0(\cdot\cdot) + |\varepsilon_1|\,(b(z)-\varrho)^{n_1} \cdot N^{\#}(\cdot\cdot)] \\
&\quad \times [(d\varrho)^2 + (dz)^2] + \varrho^2 \cdot \exp[2|\varepsilon_2| \cdot (b(z)-\varrho)^{n_2} \cdot \beta^{\#}(\cdot\cdot)] \cdot (d\varphi)^2\} \\
&\quad - \{\mathsf{W}_0(\cdot\cdot) + [b(z)-\varrho]^n \cdot \mathsf{W}^{\#}(\cdot\cdot)\} \\
&\quad \times \{dt + \varepsilon\,[\alpha_0(\cdot\cdot) + \varepsilon_3\,(b(z)-\varrho)^{n_3} \cdot \alpha^{\#}(\cdot\cdot)] \cdot d\varphi\}^2 , \\
D_{(2)} &:= \{(\varrho, z) : 0 \le b_0(z) < \varrho < b(z),\ z_1 < z < z_2\} .
\end{aligned} \tag{4.143}$$

We denote by $\overline{C}^{\,j}\left((\varrho, z) \in D_{(2)}; \mathbb{R}\right)$ the set of all real-valued functions possessing continuous and bounded partial derivatives up to and including the order $j \ge 1$. Now, we are in a position to state and prove a theorem about a slowly rotating, deformable, solid shell, joined very smoothly across a prescribed external boundary to the vacuum Kerr metric.

Theorem 4.3.2. *Let the functions* $b, \mathsf{W}^{\#}, N^{\#}, \beta^{\#}$, *and* $\alpha^{\#}$ *appearing in metric* (4.143) *belong to the class* $\overline{C}^{\,3}\left(D_{(2)} \subset \mathbb{R}^2; \mathbb{R}\right)$. *Moreover, let* $\mathsf{W}_0(\cdot\cdot) + [b(z) - \varrho]^n \cdot \mathsf{W}^{\#}(\cdot\cdot) > 0$, *and* $N_0(\cdot\cdot) + |\varepsilon_1| \cdot [b(z) - \varrho]^{n_1} \cdot N^{\#}(\cdot\cdot) > 0$. *Furthermore, let positive integers* $n \ge 4$, $n_1 \ge 4$, $n_2 \ge 4$, *and* $n_3 \ge 4$. *Also, let positive numbers* $|\varepsilon|$, $|\varepsilon_1|$, *and* $|\varepsilon_3|$ *be sufficiently small. Let the interior field equation* (2.161i) *be satisfied with the metric in* (4.143) *in the domain* $D := D_{(2)} \times (-\pi, \pi) \times \mathbb{R}$ *by defining* $T_{ij}(\cdot\cdot) := -\kappa^{-1} G_{ij}$. *Then, interior metric* (4.143) *joins continuously to the Kerr metric across the exterior boundary* $\varrho = b(z)$. *Moreover, Synge's junction conditions* (2.170) *and the I–S–L–D junction conditions* (2.171) *are satisfied on the exterior boundary.*

Proof. It is clear that the interior field equations (4.140i–vi) (and equilibrium conditions (4.141i,ii)) are satisfied. From the structure of the interior metric (4.143), it follows that the interior metric components and their partial derivatives (up to and including the third order) join continuously across the external boundary to those of the Kerr metric. Thus, the junction conditions are automatically satisfied. ■

Remarks: (i) By the Riemann mapping theorem of complex analysis [191], the boundary $\varrho = b(z)$ can *always be mapped into a circular arc*.

(ii) By choosing sufficiently small positive numbers $|\varepsilon|$, $|\varepsilon_1|$, $|\varepsilon_2|$, and $|\varepsilon_4|$, the energy–momentum–stress tensor matrix $\left[T_{ij}(\cdot\cdot)\right]$ becomes *diagonalizable*. (Consult Problem # 7 of Exercise 4.3.)

(iii) Since $\mathsf{W}^{\#}$, $N^{\#}$, $\beta^{\#}$, and $\alpha^{\#}$ are *arbitrary functions of class* $\overline{C}^{3}(\cdot\cdot)$, the metric (4.143) provides *a class of infinitely many interior solutions*.

(iv) The behavior of the metric in (4.143) *on and inside the inner boundary* $\varrho = b_0(z)$ *needs further investigations*.

Example 4.3.3. Consider the external, circular boundary $\varrho = \sqrt{1-z^2}$, $|z| < 1$. It generates a spherically symmetric solid body in the three-dimensional coordinate space. Let us choose positive integers as $n = n_1 = n_2 = n_3 = 4$. Moreover, choose for small positive numbers as $|\varepsilon| = |\varepsilon_1| = |\varepsilon_3| = (1/9)$. Furthermore, let the total mass be $m = 1$. Now, we choose the auxiliary functions as

$$\begin{aligned}
N^{\#}(\varrho, z) &:= \operatorname{sech}\left(\varrho + z^2\right), \\
\beta^{\#}(\cdot\cdot) &:= \tanh^2(\varrho \cdot z), \\
\alpha^{\#}(\cdot\cdot) &:= \tanh\left(\varrho^2 - z^2\right), \\
\mathsf{W}^{\#}(\cdot\cdot) &:= \mathsf{W}_0(\cdot\cdot) \cdot \frac{\left[\exp\left(-1/\sqrt{\varrho^2+z^2}\right) - 1\right]}{\left[\sqrt{1-z^2} - \varrho\right]^4}.
\end{aligned}$$

The corresponding metric in (4.143) joins very smoothly with the Kerr metric on the boundary $\varrho = \sqrt{1-z^2}$. Moreover, in the interior vicinity of the boundary, the energy–momentum–stress tensor behaves regularly. □

Exercise 4.3

1. Let the functions $\omega(\varrho, z)$, $\nu(\varrho, z)$, and $F(\varrho, z)$ solve axially symmetric vacuum field equations (4.118i–iii). Prove that functions $\omega(\varrho, z)$, $\nu(\varrho, z)$, and $F^{\#}(\varrho, z) := [F(\varrho, z)]^{-1}$ also solve *the same field equations*.

   (Compare with the Example 4.2.1.)
2. Consider an analog of the complex, second-order, semilinear p.d.e. (4.120iii). Let it be furnished by

$$\nabla^2\mathcal{W} = 2\mathcal{W}\cdot\left[\mathcal{W}^2-1\right]^{-1}\cdot\left[(\partial_1\mathcal{W})^2+(\partial_2\mathcal{W})^2\right].$$

Show that a general class of exact, analytic solutions of the equation above is furnished by

$$\mathcal{W}(\varrho, z) := \tanh\left[\int_0^\pi f\,(z+\mathrm{i}\varrho\cdot\cos\theta)\cdot \mathrm{d}\theta\right].$$

(Here, f is an arbitrary, holomorphic function of the complex variable $z+\mathrm{i}\varrho\cdot\cos\theta$.)

3. Consider the Kerr metric in (4.133). By investigating timelike geodesics (with proper time parameter), obtain the following first integrals:

$$\left[1-\frac{2mr}{r^2+a^2\cos^2\theta}\right]\cdot\left\{u^t+\left[\frac{2amr\cdot\sin^2\theta}{r^2-2mr+a^2\cos^2\theta}\right]\cdot u^\varphi\right\}_{|..} = E = \text{const.};$$

$$\left[\frac{r^2-2mr+a^2\cos^2\theta}{r^2+a^2\cos^2\theta}\right]^{-1}\cdot\left(r^2-2mr+a^2\right)\cdot\sin^2\theta\cdot u^\varphi{}_{|..}$$
$$-\frac{2amr\sin^2\theta}{r^2-2mr+a^2\cos^2\theta}{}_{|..}\cdot E = \text{const.}$$

(Compare the above with (3.17i,ii).)

4. Consider the prolate spheroidal coordinate chart in (4.121). Prove that the Kerr metric of (4.131) transforms into

$$\mathrm{d}s^2 = \left[\frac{\left(\sqrt{m^2-a^2}\cdot x+m\right)^2+a^2y^2}{(m^2-a^2)\cdot x^2+a^2y^2-m^2}\right]\cdot\left\{\left[\left(m^2-a^2\right)\cdot x^2+a^2y^2-m^2\right]\right.$$
$$\left.\times\left[\frac{(\mathrm{d}x)^2}{(x^2-1)}+\frac{(\mathrm{d}y)^2}{(1-y^2)}\right]+\left(m^2-a^2\right)\cdot\left(x^2-1\right)\cdot\left(1-y^2\right)\cdot(\mathrm{d}\varphi)^2\right\}$$
$$-\left[\frac{\left(m^2-a^2\right)\cdot x^2+a^2y^2-m^2}{\left(\sqrt{m^2-a^2}\cdot x+m\right)^2+a^2y^2}\right]$$
$$\times\left\{\mathrm{d}t+\left[\frac{2am\cdot\left(\sqrt{m^2-a^2}\cdot x+m\right)\cdot\left(1-y^2\right)}{(m^2-a^2)\cdot x^2+a^2y^2-m^2}\right]\cdot\mathrm{d}\varphi\right\}^2.$$

5. Consider the Ernst equation (4.123). Show explicitly that the *Tomimatsu–Sato solution* [239]

$$\widehat{\xi}(x, y) := \left[\frac{p^2 \cdot (x^4 - 1) - 2\mathrm{i}pqxy \cdot (x^2 - y^2) + q^2(y^4 - 1)}{2px \cdot (x^2 - 1) - 2\mathrm{i}qy \cdot (1 - y^2)}\right],$$

with $p^2 + q^2 = 1$, exactly solves the Ernst equation.
6. Consider the Kerr metric in the Boyer–Lindquist coordinate chart as given in (4.134). In the limit $m \to 0_+$, (which goes beyond the original domains of consideration), the metric reduces to the regular metric:

$$\mathrm{d}s^2 = \left(r^2 + a^2 \cos^2\theta\right) \cdot \left[\frac{(\mathrm{d}r)^2}{(r^2 + a^2)} + (\mathrm{d}\theta)^2\right] + \left(r^2 + a^2\right) \cdot \sin^2\theta \cdot (\mathrm{d}\varphi)^2 - (\mathrm{d}t)^2.$$

Verify that the above metric yields *a flat space–time domain*.
7. Consider the interior metric of (4.143) with the external boundary $\varrho = \sqrt{1 - z^2}$, $|z| < 1$. Suppose that three functions $N^\#$, $\beta^\#$, and $\alpha^\#$ belong to the class $\overline{C}^3(\cdot\cdot)$. Moreover, let positive constants $|\varepsilon|$, $|\varepsilon_1|$, $|\varepsilon_2|$, and $|\varepsilon_3|$ be sufficiently small. Furthermore, let the function $\mathsf{W}(\cdot\cdot) := \mathsf{W}_0 + \left(\sqrt{1 - z^2} - \varrho\right)^n \cdot \mathsf{W}^\#(\cdot\cdot)$ satisfy the potential equation (4.140v) for a prescribed, *very large, positive mass density* $\tilde{\mu}(\varrho, z)$. Prove that the interior energy–momentum–stress tensor, in the neighborhood of the boundary, is *diagonalizable satisfying the weak energy conditions*.
8. The "axially symmetric" N-dimensional ($N \geq 4$) pseudo-Riemannian metric is characterized by

$$\begin{aligned}\mathrm{d}s^2 = \Sigma(r, \theta) \cdot \left[\frac{(\mathrm{d}r)^2}{\Delta^\#(r)} + (\mathrm{d}\theta)^2\right] + \left(r^2 + a^2\right) \cdot \sin^2\theta \cdot (\mathrm{d}\varphi)^2 \\ + r^2 \cos^2\theta \cdot \left(\mathrm{d}\,\Omega_{(N-4)}\right)^2 - (\mathrm{d}t)^2 + \chi(r, \theta) \cdot \left[\mathrm{d}t - a \sin^2\theta \, \mathrm{d}\varphi\right]^2.\end{aligned}$$

Obtain the Ricci-flatness conditions (of Myers–Perry [188]) from the above metric as

$$\Sigma(r, \theta) := r^2 + a^2 \cos^2\theta, \qquad \Delta^\#(r) := r^2 + a^2 - \left(2m / r^{N-5}\right),$$

$$\chi(r, \theta) := \left[2m / \left(r^{N-5} \cdot \Sigma(r, \theta)\right)\right].$$

(*Remarks*:

(i) Compare the above problem with # 6 of Exercise 3.1.
(ii) Caution must be taken with the $N - 4$-sphere line element, $\left(\mathrm{d}\,\Omega_{(N-4)}\right)^2$. Specifically, the angle φ *does not appear in this line element*.
(iii) The solution above is valid for rotation in a single plane. A generalization is also available for rotation in several planes [188].
(iv) This is the higher dimensional generalization of the Kerr metric.)

Answers and Hints to Selected Exercises:

2. Define a complex potential:

$$\mathcal{V}(\varrho, z) := \arg\tanh\left[\mathcal{W}(\varrho, z)\right].$$

The partial differential equation under consideration, which is a disguised linear equation, reduces to

$$\nabla^2\mathcal{V} = \partial_1\partial_1\mathcal{V} + (\varrho)^{-1} \cdot \partial_1\mathcal{V} + \partial_2\partial_2\mathcal{V} = 0.$$

$\mathcal{V}(\varrho, z) := \int_0^{\pi} f(z + \mathrm{i}\varrho \cdot \cos\theta) \cdot \mathrm{d}\theta$ solves the axially symmetric Laplace's equation for any arbitrary holomorphic function f of the complex variable $z + \mathrm{i}\varrho \cdot \cos\theta$.

(*Remark*: Consult Example A2.8 of Appendix 2.)

7. Introduce another potential function:

$$V(\varrho, z) := \ln\left[1 + \left(\sqrt{1 - z^2} - \varrho\right)^n \cdot \left(\mathsf{W}^{\#}/\,\mathsf{W}_0\right)\right],$$
$$V(\cdot\cdot)_{|\varrho=\sqrt{1-z^2}} = 0.$$

The potential equation (4.140v) reduces to

$$\nabla^2 V + (\text{lower order}) = \kappa\,\tilde{\mu}(\varrho, z).$$

The above axially symmetric Poisson equation, with the boundary condition, can be solved by the method of Green's functions [77].

Field equations (4.140ii) and (4.140vi) and the metric in (4.143) yield

$$\kappa\,(\sigma_{11} + \sigma_{22}) = 0\,(|\varepsilon_2|)\,,$$
$$\mathrm{e}^{-\lambda} \cdot \mathsf{W}^{-1} \cdot s_3 = 0\,(\varepsilon)\,.$$

Therefore, conclude that

$$\tilde{\mu} + N^{-1} \cdot (\sigma_{11} + \sigma_{22}) \mp \mathrm{e}^{-\lambda} \cdot \mathsf{W}^{-1} \cdot s_3 > 0.$$

Invariant eigenvalues of the energy–momentum–stress tensor are furnished by the eigenvalue equation as

$$\det\left[T^i{}_j - \lambda\,\delta^i{}_j\right] = 0,$$

$$\text{or} \qquad \det\begin{bmatrix} T^1_{\ 1}-\lambda & T^1_{\ 2} \\ T^2_{\ 1} & T^2_{\ 2}-\lambda \end{bmatrix} = 0,$$

$$\text{and} \qquad \det\begin{bmatrix} T^3_{\ 3}-\lambda & T^3_{\ 4} \\ T^4_{\ 3} & T^4_{\ 4}-\lambda \end{bmatrix} = 0.$$

Therefore, real, invariant eigenvalues are given by

$$-2\,\lambda_{(1)} = \mathsf{W}\cdot N^{-1}\cdot\left[\sigma_{11}+\sigma_{22}+\sqrt{(\sigma_{11}-\sigma_{22})^2+4\,(\sigma_{12})^2}\right],$$

$$-2\,\lambda_{(2)} = \mathsf{W}\cdot N^{-1}\cdot\left[\sigma_{11}+\sigma_{22}-\sqrt{(\sigma_{11}-\sigma_{22})^2+4\,(\sigma_{12})^2}\right],$$

$$\begin{aligned} -2\,\lambda_{(3)} = \mathsf{W}\cdot\Big\{ & [\,\tilde{\mu}+N^{-1}(\sigma_{11}+\sigma_{22})] \\ & -\sqrt{[\,\tilde{\mu}+N^{-1}(\sigma_{11}+\sigma_{22})]^2-\mathrm{e}^{-2\beta}\cdot\mathsf{W}^{-2}\cdot s_3^2+2\mathrm{e}^{-2\beta}\cdot\sigma_{33}}\Big\}, \end{aligned}$$

$$\begin{aligned} -2\,\lambda_{(4)} = \mathsf{W}\cdot\Big\{ & [\,\tilde{\mu}+N^{-1}(\sigma_{11}+\sigma_{22})] \\ & +\sqrt{[\,\tilde{\mu}+N^{-1}(\sigma_{11}+\sigma_{22})]^2-\mathrm{e}^{-2\beta}\cdot\mathsf{W}^{-2}\cdot s_3^2+2\mathrm{e}^{-2\beta}\cdot\sigma_{33}}\Big\}. \end{aligned}$$

It is clear that $-\lambda_{(4)} > 0$ and, moreover, $-\lambda_{(4)}$ *dominates* $\lambda_{(1)}$, $\lambda_{(2)}$, and $\lambda_{(3)}$.

4.4 The General Stationary Field Equations

We have investigated axially symmetric, stationary field equations in the preceding section. Now, we shall deal with the general, stationary space–time domains. *The only assumption is the existence of a timelike Killing vector field.* Coordinate charts exist such that the metric is expressible as

$$\begin{aligned} \mathbf{g}_{..}(\mathbf{x}) = {} & \mathrm{e}^{-2w(\mathbf{x})}\cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x})\cdot\left(\mathrm{d}x^\alpha\otimes\mathrm{d}x^\beta\right) \\ & -\mathrm{e}^{2w(\mathbf{x})}\cdot\left[a_\alpha(\mathbf{x})\,\mathrm{d}x^\alpha+\mathrm{d}x^4\right]\otimes\left[a_\beta(\mathbf{x})\,\mathrm{d}x^\beta+\mathrm{d}x^4\right] \end{aligned}$$

$$
\begin{aligned}
\mathrm{d}s^2 = \mathrm{e}^{-2w(\mathbf{x})} \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x}) \left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - \mathrm{e}^{2w(\mathbf{x})} \cdot \left[a_\alpha(\mathbf{x})\, \mathrm{d}x^\alpha + \mathrm{d}x^4\right] \\
\times \left[a_\beta(\mathbf{x})\, \mathrm{d}x^\beta + \mathrm{d}x^4\right]; \\
\mathbf{x} \in \mathbf{D} \subset \mathbb{R}^3, \ x \in D := \mathbf{D} \times \mathbb{R}.
\end{aligned} \tag{4.144}
$$

The above metric clearly admits the timelike Killing vector field $\frac{\partial}{\partial x^4}$ in the space–time domain $D := \mathbf{D} \times \mathbb{R}$. (The three-dimensional metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ is assumed to be positive-definite.)

Consider restricted coordinate transformations:

$$
\begin{aligned}
\widehat{x}^\alpha &= \widehat{X}^\alpha(\mathbf{x}), \\
\widehat{x}^4 &= x^4, \\
\widehat{w}(\widehat{x}) &= w(\mathbf{x}), \\
\widehat{a}_\alpha(\widehat{x})\, \mathrm{d}\widehat{x}^\alpha &= a_\alpha(\mathbf{x})\, \mathrm{d}x^\alpha, \quad \text{etc.}
\end{aligned} \tag{4.145i}
$$

The line element in (4.144) goes over into

$$
\mathrm{d}s^2 = \mathrm{e}^{-2\widehat{\hat{w}(x)}} \cdot \widehat{g}_{\alpha\beta}(\widehat{x}) \cdot \mathrm{d}\widehat{x}^\alpha \cdot \mathrm{d}\widehat{x}^\beta - \mathrm{e}^{2\widehat{\hat{w}(x)}} \cdot \left[\widehat{a}_\alpha(\hat{\mathbf{x}})\, \mathrm{d}\widehat{x}^\alpha + \mathrm{d}\widehat{x}^4\right] \left[\widehat{a}_\beta(\hat{\mathbf{x}})\, \mathrm{d}\widehat{x}^\beta + \mathrm{d}\widehat{x}^4\right].
$$

However, under another restricted transformation,

$$
\begin{aligned}
x'^\alpha &= x^\alpha, \\
x'^4 &= x^4 + \lambda(\mathbf{x}), \\
a'_\alpha(\mathbf{x}') &= a_\alpha(\mathbf{x}) - \partial_\alpha \lambda.
\end{aligned} \tag{4.145ii}
$$

The line element in (4.144) goes over into

$$
\mathrm{d}s^2 = \mathrm{e}^{-2w(\mathbf{x}')} \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x}') \cdot \mathrm{d}x'^\alpha \cdot \mathrm{d}x'^\beta - \mathrm{e}^{2w(\mathbf{x}')} \cdot \left[a'_\alpha\, \mathrm{d}x'^\alpha - \mathrm{d}x'^4\right] \left[a'_\beta(\mathbf{x}')\, \mathrm{d}x'^\beta - \mathrm{d}x'^4\right].
$$

(In both of the transformations (4.145i) and (4.145ii), the stationary form of the metric in (4.144) is preserved. The transformation (4.145ii) is reminiscent of the gauge transformation (1.71ii) of the electromagnetic 4 potential.)

In the sequel, we adopt the notations

$$
\overset{\circ}{a}{}^\alpha(\mathbf{x}) := \overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x}) \cdot a_\beta(\mathbf{x}), \text{ etc.}, \tag{4.146i}
$$

$$
f_{\beta\alpha}(\mathbf{x}) := \partial_\beta a_\alpha - \partial_\alpha a_\beta \equiv -f_{\alpha\beta}(\mathbf{x}). \tag{4.146ii}
$$

The vacuum field equations in (2.169) reduce to

$$\overset{\circ}{G}_{\mu\nu}(\mathbf{x}) + 2\left(\overset{\circ}{\nabla}_\mu w \cdot \overset{\circ}{\nabla}_\nu w - (1/2)\cdot \overset{\circ}{g}_{\mu\nu} \cdot \overset{\circ}{g}^{\alpha\beta} \cdot \overset{\circ}{\nabla}_\alpha w \cdot \overset{\circ}{\nabla}_\beta w\right)$$
$$+ (1/2)\cdot \mathrm{e}^{4w} \cdot \left[-\overset{\circ}{f}_\mu{}^\alpha \cdot f_{\nu\alpha} + (1/4)\cdot \overset{\circ}{g}_{\mu\nu} \cdot \overset{\circ}{f}{}^{\alpha\beta} \cdot f_{\alpha\beta}\right] = 0, \tag{4.147i}$$

$$\overset{\circ}{\nabla}{}^2 w + (1/4)\cdot \mathrm{e}^{4w} \cdot \overset{\circ}{f}{}^{\alpha\beta} \cdot f_{\alpha\beta} = 0, \tag{4.147ii}$$

$$\overset{\circ}{\nabla}_\nu \left[\mathrm{e}^{4w} \cdot \overset{\circ}{f}_\mu{}^\nu\right] = 0, \tag{4.147iii}$$

$$\overset{\circ}{\nabla}_\lambda f_{\mu\nu} + \overset{\circ}{\nabla}_\mu f_{\nu\lambda} + \overset{\circ}{\nabla}_\nu f_{\lambda\mu} \equiv 0. \tag{4.147iv}$$

(Here, $\overset{\circ}{\nabla}{}^2 w := \overset{\circ}{\nabla}{}^\alpha \overset{\circ}{\nabla}_\alpha w$.)

The above equations are actually *three-dimensional tensor field equations* derivable from the Lagrangian density in (4.165). These equations are covariant under restricted coordinate transformations in (4.145i).

Defining $\phi_\alpha(\mathbf{x}) := (1/2)\cdot \mathrm{e}^{4w} \cdot \sqrt{\overset{\circ}{g}} \cdot \varepsilon_{\alpha\beta\gamma} \cdot \overset{\circ}{f}{}^{\beta\gamma}$, it can be shown from (4.147iii), (1.61), and Theorem 1.2.21 that $\phi_\alpha(\mathbf{x}) = \partial_\alpha \phi$ for some function $\phi(\mathbf{x})$ called the *twist potential*. With the help of $\phi(\mathbf{x})$, field equations (4.147i–iv) reduce to

$$\overset{\circ}{G}_{\mu\nu}(\mathbf{x}) + 2\left(\overset{\circ}{\nabla}_\mu w \cdot \overset{\circ}{\nabla}_\nu w - (1/2)\cdot \overset{\circ}{g}_{\mu\nu} \cdot \overset{\circ}{g}^{\alpha\beta} \cdot \overset{\circ}{\nabla}_\alpha w \cdot \overset{\circ}{\nabla}_\beta w\right)$$
$$+ (1/2)\cdot \mathrm{e}^{-4w} \cdot \left(\overset{\circ}{\nabla}_\mu \phi \cdot \overset{\circ}{\nabla}_\nu \phi - (1/2)\cdot \overset{\circ}{g}_{\mu\nu} \cdot \overset{\circ}{g}^{\alpha\beta} \cdot \overset{\circ}{\nabla}_\alpha \phi \cdot \overset{\circ}{\nabla}_\beta \phi\right) = 0, \tag{4.148i}$$

$$\overset{\circ}{\nabla}{}^2 w + (1/2)\cdot \mathrm{e}^{-4w} \cdot \overset{\circ}{g}^{\alpha\beta} \cdot \overset{\circ}{\nabla}_\alpha \phi \cdot \overset{\circ}{\nabla}_\beta \phi = 0, \tag{4.148ii}$$

$$\overset{\circ}{\nabla}{}^2 \phi - 4\, \overset{\circ}{g}^{\alpha\beta} \cdot \overset{\circ}{\nabla}_\alpha w \cdot \overset{\circ}{\nabla}_\beta \phi = 0. \tag{4.148iii}$$

(In case $\phi(\mathbf{x}) \equiv$ const., the equations above reduce to the general static, vacuum field equations (4.40i,ii) with $\rho(\mathbf{x}) \equiv 0$ and $T_{\alpha\beta}(\mathbf{x}) \equiv 0$.)

Next, we define a complex potential by

$$F(\mathbf{x}) := \mathrm{e}^{2w(\mathbf{x})} + \mathrm{i}\,\phi(\mathbf{x}). \tag{4.149}$$

(Compare with (4.117).)

Using (4.149), we can express field equations (4.148i–iii) as

$$\overset{\circ}{R}_{\mu\nu}(\mathbf{x}) + (1/4)\cdot (\mathrm{Re}\, F)^{-2} \cdot \left[\overset{\circ}{\nabla}_\mu F \cdot \overset{\circ}{\nabla}_\nu \overline{F} + \overset{\circ}{\nabla}_\nu F \cdot \overset{\circ}{\nabla}_\mu \overline{F}\right] = 0, \tag{4.150i}$$

$$\overset{\circ}{\nabla}{}^2 F - (\text{Re } F)^{-1} \cdot \overset{\circ}{g}{}^{\mu\nu} \cdot \overset{\circ}{\nabla}_\mu F \cdot \overset{\circ}{\nabla}_\nu F = 0. \tag{4.150ii}$$

(Equations (4.150i,ii) reduce to (4.118i–iii) for the axially symmetric case.)

We denote *antisymmetrization* by $A_{[\mu\nu]} := (1/2) \cdot \left[A_{\mu\nu} - A_{\nu\mu} \right]$. Field equations (4.150i) are *equivalent to*

$$\begin{aligned} &\overset{\circ}{R}_{\mu\nu\lambda\rho}(\mathbf{x}) - (1/2) \cdot (\text{Re } F)^{-2} \cdot \Big[\overset{\circ}{g}_{\mu[\lambda} \cdot \left(\partial_{\rho]} \overline{F} \cdot \partial_\nu F + \partial_\rho F \cdot \partial_\nu \overline{F} \right) \\ &\quad + \overset{\circ}{g}_{\nu[\rho} \cdot \left(\partial_{\lambda]} \overline{F} \cdot \partial_\mu F + \partial_{\lambda]} F \cdot \partial_\mu \overline{F} \right) \\ &\quad + \overset{\circ}{g}{}^{\sigma\tau} \cdot \partial_\sigma F \cdot \partial_\tau \overline{F} \cdot \overset{\circ}{g}_{\mu[\rho} \cdot \overset{\circ}{g}_{\lambda]\nu} \Big] = 0. \end{aligned} \tag{4.151}$$

(The equations above are a generalization of the static, vacuum field equations (4.42i).)

Now, we introduce *an orthonormal triad* and a corresponding directional derivatives as

$$\begin{aligned} &\vec{\mathbf{e}}_{(\alpha)}(\mathbf{x}) := \lambda^\mu{}_{(\alpha)}(\mathbf{x}) \cdot \frac{\partial}{\partial x^\mu}, \\ &\overset{\circ}{g}_{\mu\nu}(\mathbf{x}) \cdot \lambda^\mu{}_{(\alpha)}(\mathbf{x}) \cdot \lambda^\nu{}_{(\beta)}(\mathbf{x}) = \delta_{(\alpha)(\beta)}, \\ &\partial_{(\alpha)} f := \lambda^\mu{}_{(\alpha)}(\mathbf{x}) \cdot \frac{\partial f}{\partial x^\mu}. \end{aligned} \tag{4.152}$$

(These equations are *identical to* (4.100i–iii)!)

The corresponding Ricci rotation coefficients and orthonormal curvature tensor components are furnished in (4.102) and (4.103). Thus, stationary field equations (4.151) and (4.150ii) go over into

$$\begin{aligned} &\overset{\circ}{R}_{(\alpha)(\beta)(\gamma)(\delta)}(\mathbf{x}) - (1/2) \cdot (\text{Re } F)^{-2} \cdot \Big[\delta_{(\alpha)[(\gamma)} \cdot \left(\partial_{(\delta)]} \overline{F} \cdot \partial_{(\beta)} F + \partial_{(\delta)]} F \cdot \partial_{(\beta)} \overline{F} \right) \\ &\quad + \delta_{(\beta)[(\delta)} \cdot \left(\partial_{(\gamma)]} \overline{F} \cdot \partial_{(\alpha)} F + \partial_{(\gamma)]} F \cdot \partial_{(\alpha)} \overline{F} \right) \\ &\quad + \delta^{(\sigma)(\tau)} \cdot \partial_{(\sigma)} F \cdot \partial_{(\tau)} \overline{F} \cdot \delta_{(\alpha)[(\delta)} \cdot \delta_{(\gamma)](\beta)} \Big] = 0, \end{aligned} \tag{4.153i}$$

$$\begin{aligned} &\delta^{(\alpha)(\beta)} \cdot \left[\partial_{(\alpha)} \partial_{(\beta)} F + \gamma^{(\delta)}{}_{(\beta)(\alpha)} \cdot \partial_{(\delta)} F \right] \\ &\quad - (\text{Re } F)^{-1} \cdot \delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)} F \cdot \partial_{(\beta)} F = 0. \end{aligned} \tag{4.153ii}$$

The above field equations are generalizations of static field equations (4.105) and (4.104).

The stationary vacuum field equations (4.148i–iii) are *analogous to the static, electro-vac equations* (4.67i–iii) (and static magneto-vac equations (4.83ii–iv)). Therefore, for a special class of exact solutions, *we assume a functional relationship*:

$$\begin{aligned} e^{2w(\mathbf{x})} &= f\,[\phi(\mathbf{x})] > 0, \\ f' &:= \frac{d\,f[\phi]}{d\phi} \neq 0. \end{aligned} \tag{4.154}$$

Thus, (4.148iii) and (4.148ii) reduce, respectively, to

$$\overset{\circ}{\nabla}{}^2\phi - 2\,\overset{\circ}{g}{}^{\alpha\beta} \cdot f^{-1} \cdot f' \cdot \partial_\alpha\phi \cdot \partial_\beta\phi = 0, \tag{4.155i}$$

$$\overset{\circ}{\nabla}{}^2\phi + \left[\frac{f''}{f'} - \frac{f'}{f} + \frac{1}{ff'}\right] \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\alpha\phi \cdot \partial_\beta\phi = 0. \tag{4.155ii}$$

Subtracting (4.155i) from (4.155ii) and assuming $\partial_\alpha\phi \not\equiv 0$, we derive that

$$\frac{f''}{f'} + \frac{f'}{f} + \frac{1}{ff'} = 0. \tag{4.156}$$

The general solution of the second-order o.d.e. above is furnished by

$$\begin{aligned} [f(\phi)]^2 &= c_0 + 2c_1\phi - \phi^2 = \left(c_0 + c_1^2\right) - (\phi - c_1)^2 > 0, \\ e^{4w(\mathbf{x})} &= c_0 + 2c_1\phi(\mathbf{x}) - [\phi(\mathbf{x})]^2\,. \end{aligned} \tag{4.157}$$

Here, c_0 and c_1 are constants of integration satisfying $c_0 + c_1^2 > 0$.

Assuming the functional relationship (4.157), let us try to solve stationary field equations (4.148i–iii). For that purpose, we define

$$\begin{aligned} h(\phi) &:= k \int \frac{d\,\phi}{[c_0 + 2c_1\phi - \phi^2]}, \\ k &:= \sqrt{c_0 + c_1^2} > 0. \end{aligned} \tag{4.158}$$

(Compare the equation above with (4.35i).) We infer from (4.158) and (4.157) that

$$\begin{aligned} \chi(\mathbf{x}) &:= h[\phi(\mathbf{x})], \\ \phi(\mathbf{x}) &= c_1 + k\,\tanh\left[\,\chi(\mathbf{x})\right], \\ e^{2w(\mathbf{x})} &= k\,\operatorname{sech}\left[\,\chi(\mathbf{x})\right]. \end{aligned} \tag{4.159}$$

Therefore, field equations (4.148i–iii) reduce to

$$\overset{\circ}{R}_{\mu\nu}(\mathbf{x}) + (1/2) \cdot \partial_\mu \chi \cdot \partial_\nu \chi = 0,$$
$$\overset{\circ}{\nabla}{}^2 \chi = 0. \tag{4.160}$$

These equations are equivalent to the static, vacuum field equations (4.41i,ii) for the metric:

$$\mathrm{d}s^2 = \mathrm{e}^{-\chi(\mathbf{x})} \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x}) \left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - \mathrm{e}^{\chi(\mathbf{x})} \cdot \left(\mathrm{d}x^4\right)^2. \tag{4.161}$$

Thus, the static vacuum metric (4.161) can generate the stationary vacuum metric:

$$\begin{aligned} \mathrm{d}s^2 = {} & k^{-1} \cdot [\cosh \chi(\mathbf{x})] \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x}) \left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) \\ & - k \cdot [\operatorname{sech} \chi(\mathbf{x})] \cdot \left[a_\alpha(\mathbf{x})\, \mathrm{d}x^\alpha + \mathrm{d}x^4\right] \cdot \left[a_\beta(\mathbf{x})\, \mathrm{d}x^\beta + \mathrm{d}x^4\right]. \end{aligned} \tag{4.162}$$

This derivation is possible, provided we can solve[4] the linear, first-order system of p.d.e.s:

$$\partial_\alpha a_\beta - \partial_\beta a_\alpha = k^{-1} \cdot \overset{\circ}{\eta}_{\alpha\beta\gamma} \cdot \overset{\circ}{\nabla}{}^\gamma \chi. \tag{4.163}$$

The class of exact, stationary vacuum solutions in (4.162) is called the *Papapetrou-Ehlers class*.

Example 4.4.1. Consider the static metric (3.9) of Schwarzschild, namely,

$$\begin{aligned} \mathrm{d}s^2 = {} & \left(1 - \frac{2m}{r}\right)^{-1} \cdot \left\{(\mathrm{d}r)^2 + (r^2 - 2mr) \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right]\right\} \\ & - \left(1 - \frac{2m}{r}\right) \cdot (\mathrm{d}t)^2 \\ =: {} & \mathrm{e}^{-\chi(\cdot\cdot)} \cdot \overset{\circ}{g}_{\mu\nu}(\cdot\cdot) \left(\mathrm{d}x^\mu \mathrm{d}x^\nu\right) - \mathrm{e}^{\chi(\cdot\cdot)} \left(\mathrm{d}x^4\right)^2. \end{aligned}$$

Therefore, we derive that

$$\sqrt{\overset{\circ}{g}} = \left(r^2 - 2mr\right) \cdot \sin\theta,$$
$$\chi(r) = \ln\left(1 - \frac{2m}{r}\right),$$
$$\cosh\left[\chi(r)\right] = \frac{1}{2} \cdot \left[\left(1 - \frac{2m}{r}\right) + \left(1 - \frac{2m}{r}\right)^{-1}\right].$$

[4]The solution technique for (4.163) is explained in [149].

Equation (4.163) yields (with $a_1(\cdot\cdot) = a_2(\cdot\cdot) \equiv 0$)

$$\partial_2 a_3 - 0 = k^{-1} \cdot \overset{\circ}{\eta}_{231} \cdot \partial_1 \chi = k^{-1} \cdot 2m \cdot \sin\theta,$$

$$\text{or} \qquad a_3(\theta) = -(2m/k) \cdot \cos\theta + \text{const.}$$

(We ignore the above constant of integration in the sequel.)

Thus, from (4.162), we deduce a particular example of the Papapetrou-Ehlers class of stationary metrics as

$$\begin{aligned} \mathrm{d}s^2 = {} & (2k)^{-1} \cdot \left[\left(1 - \frac{2m}{r}\right) + \left(1 - \frac{2m}{r}\right)^{-1}\right] \\ & \times \left\{(\mathrm{d}r)^2 + (r^2 - 2mr) \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right]\right\} \\ & - (2k) \cdot \left[\left(1 - \frac{2m}{r}\right) + \left(1 - \frac{2m}{r}\right)^{-1}\right]^{-1} \cdot [\mathrm{d}t - (2m/k) \cdot \cos\theta \cdot \mathrm{d}\varphi]^2, \end{aligned}$$

$$D := \{(r, \theta, \varphi, t) : 0 < 2m < r,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ -\infty < t < \infty\}. \quad \square$$

We should mention that the Papapetrou-Ehlers class of solutions can involve "exotic sources." This is because, for this class, either the "total mass" is zero, or there exist wire (or string) singularities. (See Problem #4ii of Exercise 4.4.)

Now, consider the quadratic function of ϕ in (4.157). Let us try to pursue the Majumdar-Papapetrou "perfect square" condition in (4.73i) for (4.157). If we set $k^2 = c_0 + c_1^2 = 0$ in (4.157), then we obtain $\mathrm{e}^{4w} = -(\phi - c_1)^2 < 0$. But this condition is *impossible in the realm of real numbers*. However, in case the four-dimensional differential manifold is *a positive-definite Riemannian manifold*, such exact solutions *are mathematically viable*. Such solutions can give rise to *gravitational instantons*, to be discussed in Appendix 7. (These solutions are important for some approaches to quantum gravity, as well as in some aspects of cosmology [127].)

Now, we shall explore an exact solution of the stationary field equations (4.153i,ii) employing *an orthonormal triad*.

Example 4.4.2. Consider the three-dimensional metric furnished by

$$\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x}) := \left(x^1\right)^2 \cdot \delta_{\mu\nu}\left(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu\right); \quad x^1 > 1,\ \ k = 1,\ \ \mathrm{e}^{-\chi(\cdot\cdot)} = (x^1)^2.$$

The natural orthonormal triad and directional derivatives are provided by

$$\vec{\mathbf{e}}_{(\alpha)}(\mathbf{x}) = \left(x^1\right)^{-1} \cdot \frac{\partial}{\partial x^\alpha},$$
$$\partial_{(\alpha)} f = \left(x^1\right)^{-1} \cdot \frac{\partial f}{\partial x^\alpha}.$$

(See (4.106i–iii) and (4.107).) The nonvanishing Ricci rotation coefficients are given by

$$b(\mathbf{x}) := \gamma_{(1)(2)(2)}(\mathbf{x}) = \left(x^1\right)^{-2} > 0,$$
$$n(\mathbf{x}) := \gamma_{(1)(3)(3)}(\mathbf{x}) = \left(x^1\right)^{-2} > 0.$$

A particular, orthonormal component of the curvature tensor is provided by

$$\begin{aligned}\overset{\circ}{R}_{(1)(2)(2)(1)}(\cdot\cdot) &= \left(x^1\right)^{-1} \cdot \partial_1 b + \left[b(x^1)\right]^2 + 0 \\ &= -\left(x^1\right)^{-4}.\end{aligned}$$

Let us choose the complex potential as

$$F(x^1) := \left\{ \frac{2 - \mathrm{i}\left[\left(x^1\right)^2 - \left(x^1\right)^{-2}\right]}{\left[(x^1)^2 + (x^1)^{-2}\right]} \right\}.$$

Then,

$$\begin{aligned}&-(1/2)\cdot[\mathrm{Re}(F)]^{-2}\cdot\Big\{\delta_{(1)[(2)}\cdot\left[\partial_{(1)]}\overline{F}\cdot\partial_{(2)}F + \partial_{(1)]}F\cdot\partial_{(2)}\overline{F}\,\right] \\ &\qquad + \delta_{(2)[(1)}\cdot\left[\partial_{(2)]}\overline{F}\cdot\partial_{(1)}F + \partial_{(2)]}F\cdot\partial_{(1)}\overline{F}\,\right] \\ &\qquad - \partial_{(1)}F\cdot\partial_{(1)}\overline{F}\cdot\delta_{(1)[(1)}\cdot\delta_{(2)](2)}\Big\} \\ &= \left(-\frac{1}{8}\right)\cdot\left(-\frac{1}{2}\right)\cdot\left[\left(x^1\right)^2 + \left(x^1\right)^{-2}\right]^2 \\ &\quad \times \left(x^1\right)^{-2}\cdot\left[\left(\frac{16}{(x^1)^2}\right)\cdot\left(\frac{4 + \left(\left(x^1\right)^2 - \left(x^1\right)^{-2}\right)^2}{\left(\left(x^1\right)^2 + \left(x^1\right)^{-2}\right)^4}\right)\right] \\ &= \left(x^1\right)^{-4}.\end{aligned}$$

Therefore, among the field equations in (4.153i), this particular one vanishes, namely,

$$\overset{\circ}{R}_{(1)(2)(2)(1)}(\cdot\cdot) - (\cdots) = 0.$$

Similarly, *all other field equations are satisfied*.

The corresponding four-dimensional metric is furnished by

$$\mathrm{d}s^2 = \frac{1}{2}\left[1 + \left(x^1\right)^4\right] \cdot \delta_{\mu\nu} \cdot (\mathrm{d}x^\mu \mathrm{d}x^\nu) - 2\left[\left(x^1\right)^2 + \left(x^1\right)^{-2}\right]^{-1}$$
$$\times \left[\mathrm{d}x^4 + (x^3 \cdot \mathrm{d}x^2 - x^2 \cdot \mathrm{d}x^3)\right]^2.$$

The above metric is of the Papapetrou-Ehlers class, which is generated from the static, Kasner metric in Example 4.2.1. □

Now, we shall investigate variational derivations of stationary field equations (4.148i–iii) and (4.150i,ii). We need to use the variational equation (A1.20i) of Appendix 1. The appropriate Lagrangian is furnished by

$$\mathcal{L}(\cdot\cdot)_{|\cdot\cdot} = \left[\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + 2 \cdot \partial_\alpha w \cdot \partial_\beta w + (1/2)\mathrm{e}^{-4w} \cdot \partial_\alpha \phi \cdot \partial_\beta \phi\right] \cdot \overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x}) \cdot \sqrt{\overset{\circ}{g}}$$
$$= \left[\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + (1/2) \cdot (\mathrm{Re}\, F)^{-2} \cdot \partial_\alpha F \cdot \partial_\beta \overline{F}\right] \cdot \overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x}) \cdot \sqrt{\overset{\circ}{g}}. \qquad (4.164)$$

(The above Lagrangian is a generalization of the static Lagrangian in (4.43ii).[5])

Now, we inquire how the reduced action integral corresponding to (4.164) is deduced from the original Einstein–Hilbert action integral corresponding to (A1.25). Using the stationary metric in (4.144), the Einstein–Hilbert action integral reduces to

$$\int\limits_{\mathbf{D}\times(t_1,t_2)} R(x) \cdot \sqrt{-g} \cdot \mathrm{d}^4 x = (t_2 - t_1) \cdot \int\limits_{\mathbf{D}} \left[\overset{\circ}{R}(\mathbf{x}) + 2 \cdot \overset{\circ}{\nabla}_\alpha w \cdot \overset{\circ}{\nabla}{}^\alpha w - 2 \cdot \overset{\circ}{\nabla}_\alpha \overset{\circ}{\nabla}{}^\alpha w\right.$$
$$\left. - (1/4)\,\mathrm{e}^{4w} \cdot f_{\alpha\beta}(\mathbf{x}) \cdot \overset{\circ}{f}{}^{\alpha\beta}(\mathbf{x})\right] \cdot \sqrt{\overset{\circ}{g}} \cdot \mathrm{d}^3\mathbf{x}$$
$$= (\text{const.}) \cdot \int\limits_{\mathbf{D}} \left[\overset{\circ}{R}(\mathbf{x}) + 2 \cdot \overset{\circ}{\nabla}_\alpha w \cdot \overset{\circ}{\nabla}{}^\alpha w - (1/4)\,\mathrm{e}^{4w} \cdot f_{\alpha\beta}(\mathbf{x}) \cdot \overset{\circ}{f}{}^{\alpha\beta}(\mathbf{x})\right] \cdot \sqrt{\overset{\circ}{g}} \cdot \mathrm{d}^3\mathbf{x}$$
$$+ \text{boundary terms}. \qquad (4.165)$$

[5] We use the following variational formula for the N-dimensional, pseudo-Riemannian manifold: $\delta\left(\sqrt{|g|} \cdot g^{ij}\right) = \sqrt{|g|}\left[\delta g^{ij} - \frac{1}{2} g_{kl} \cdot g^{ij} \cdot \delta(g^{kl})\right]$.

Now, field equation (4.147iii) has been solved to obtain

$$f_{\alpha\beta}(\mathbf{x}) = \mathrm{e}^{-4w(\mathbf{x})} \cdot \overset{\circ}{\eta}_{\alpha\beta\gamma} \cdot \overset{\circ}{\nabla}{}^{\gamma} \phi. \tag{4.166}$$

(Here, $\phi(\mathbf{x})$ is *the twist potential*.) We refer to *Routh's Theorem* 3.1.3 on the reduction of a Lagrangian in case of a cyclic or ignorable coordinate. Similarly, a Lagrangian for partial differential equations admits reductions in case some of the field equations are *solved*. (See [65].)

The reduced Lagrangian from (4.164) and (4.166) is provided by the use of the Legendre transformation of Example A2.4 as

$$\begin{aligned}
\overline{L}(\cdot\cdot) &= \left[L(\cdot\cdot) - \frac{\partial L(\cdot\cdot)}{\partial f_{\alpha\beta}} \cdot f_{\alpha\beta}(\cdot\cdot) \right]_{\big| f_{\alpha\beta} = \mathrm{e}^{-4w} \cdot \overset{\circ}{\eta}_{\alpha\beta\gamma} \cdot \overset{\circ}{\nabla}{}^{\gamma} \phi} \\
&= \left[\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + 2 \cdot \partial_\alpha w \cdot \partial_\beta w + (1/2) \cdot \mathrm{e}^{-4w} \cdot \partial_\alpha \phi \cdot \partial_\beta \phi \right] \cdot \overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x}) \\
&= \left[\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + (1/2) \cdot (\mathrm{Re}\, F)^{-2} \cdot \partial_\alpha F \cdot \partial_\beta \overline{F} \right] \cdot \overset{\circ}{g}{}^{\alpha\beta}(\mathbf{x}).
\end{aligned} \tag{4.167}$$

Thus, we have recovered the Lagrangian's in (4.164) from the original Einstein–Hilbert Lagrangian.

The "potential manifold," inherent in the Lagrangian (4.167), induces the metric:

$$\begin{aligned}
\mathrm{d}\,\Sigma^2 &= (\mathrm{Re}\, F)^{-2} \cdot \left(\mathrm{d}F \, \mathrm{d}\overline{F} \right) \\
&= \mathsf{W}^{-2} \left[(\mathrm{d}\,\mathsf{W})^2 + (\mathrm{d}\,\phi)^2 \right], \\
\mathsf{W} &:= \mathrm{e}^{2w} > 0.
\end{aligned} \tag{4.168}$$

It represents a non-Euclidean (Riemannian) surface of constant negative curvature. The metric in the above equation is attributable to Poincaré. Obviously, $\frac{\partial}{\partial \phi}$ is a Killing vector field.

Now, we shall explore the group of transformations which leave (4.168) invariant.

Theorem 4.4.3. *Let stationary field equations* (4.150i,ii) *hold in a regular domain* $\mathbf{D} \subset \mathbb{R}^3$. *Moreover, let the metric field* $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ *and the complex potential* $F(\mathbf{x})$ *belong to the class* $C^3(\mathbf{D})$. *Let* $h(F)$ *be a differentiable function that preserves solutions in* (4.150i,ii).

(i) *If, furthermore,* $h(F)$ *is holomorphic, then*

$$h(F) = \left[\frac{aF + \mathrm{i}b}{\mathrm{i}cF + d} \right]$$

such that real constants a, b, *and* d *satisfy* $ad + bc > 0$.

(ii) *On the other hand, if h is a conjugate holomorphic function, then*

$$h(\cdot\cdot) = \left[\frac{a\overline{F} + \mathrm{i}b}{\mathrm{i}c\overline{F} + d}\right] \quad \textit{with} \quad ad + bc > 0$$

(iii) *In case f is neither holomorphic, nor non-conjugate-holomorphic, there does not exist any such non-constant-valued f.*

Proof. For part (i), we have to integrate the nonlinear, first-order differential equation:

$$\frac{h'(F)\cdot\overline{h'(F)}}{\left[h(\cdot\cdot) + \overline{h(\cdot\cdot)}\right]^2} = \frac{1}{\left(F + \overline{F}\right)^2}.$$

(Consult (A2.38), (A2.39i,ii), and (A2.41) of Appendix 2.)

Integrating with respect to the complex variable $\overline{F}$, we obtain

$$\frac{h'(F)}{\left[h(F) + \overline{h(F)}\right]} = \frac{1}{\left(F + \overline{F}\right)} + \left[\ln K(F)\right]'.$$

Here, K is an arbitrary holomorphic function. Integrating with respect to F, we conclude that

$$\left(F + \overline{F}\right)\cdot K(F)\cdot\overline{G(F)} = h(F) + \overline{h(F)}.$$

Here, $\overline{G(F)}$ is an arbitrary conjugate-holomorphic function. Since $h(F) + \overline{h(F)}$ is real, we conclude that $G(F) = \hat{c}K(F)$ where $\hat{c} \neq 0$ is a real constant. Now, differentiating with respect to F and then $\overline{F}$, we derive that

$$F + \frac{K(F)}{K'(F)} = -\overline{\left[F + \frac{K(F)}{K'(F)}\right]} = i\hat{d} = \text{ imaginary const.}$$

Thus, we deduce that

$$[K(F)]^{-1}\cdot K'(F) = (\mathrm{i}\widehat{d} - F)^{-1},$$

$$K(F) = \alpha_0\cdot[F - \mathrm{i}\widehat{d}]^{-1},$$

$$\begin{aligned} h(F) + \overline{h(F)} &= \widehat{c}\cdot|\alpha_0|^2\cdot|F + \overline{F}|\cdot|F - \mathrm{i}\widehat{d}|^{-2} \\ &= \widehat{c}\cdot|\alpha_0|^2\cdot\left\{(F - \mathrm{i}\widehat{d})^{-1} + (\overline{F} + \mathrm{i}\widehat{d})^{-1}\right\}, \end{aligned}$$

$$\text{or,}\quad h(F) = \left[\frac{aF + \mathrm{i}b}{icF + d}\right], \qquad ad + bc > 0.$$

(ii) The proof of this part is exactly similar to that of part (i).
(iii) For the proof of this part, we refer to [149] . ■

Remarks: (i) In complex analysis, the set of all Möbius conformal transformations,

$$h(F) := \left[\frac{\alpha F + \beta}{\gamma F + \delta}\right], \qquad \alpha\delta - \beta\gamma \neq 0,$$

forms a group isomorphic to $GL(2, \mathbb{C})$. The transformations derived in Theorem 4.4.3 constitute *a subgroup* of the *Möbius group* [191].

(ii) These transformations can generate *new exact solutions from the old ones*. (Compare with Example 4.2.1 in the static case.)

Now, we shall deal with the stationary case of the Einstein–Maxwell equations (or electromagneto-vac equations) of (2.290i–iii). For this, we adopt the general stationary metric (4.144). The electromagnetic field tensor can be adapted to the Killing vector $\frac{\partial}{\partial x^4}$ so that $\partial_4 F_{ij} \equiv 0$. It has been proved that in such a scenario [65], there exist two potentials $\mathcal{A}$ and $\mathcal{B}$ so that

$$F_{\alpha 4}(\mathbf{x}) = \partial_\alpha \mathcal{A}, \tag{4.169i}$$

$$F_{\alpha\beta}(\mathbf{x}) = \mathrm{e}^{-2w} \cdot \overset{\circ}{\eta}_{\alpha\beta\gamma} \cdot \overset{\circ}{\nabla}{}^{\gamma} \mathcal{B}. \tag{4.169ii}$$

We introduce *the complex electromagnetic potential* by

$$\Phi(\mathbf{x}) := \mathcal{A}(\mathbf{x}) + \mathrm{i}\, \mathcal{B}(\mathbf{x}). \tag{4.170}$$

Now, we define two three-dimensional vector fields by

$$\tau^\alpha(\mathbf{x}) := (1/2) \cdot \mathrm{e}^{4w} \cdot \overset{\circ}{\eta}{}^{\alpha\beta\gamma} \cdot f_{\beta\gamma}, \tag{4.171i}$$

$$\Theta_\alpha(\mathbf{x}) := \tau_\alpha(\mathbf{x}) + \mathrm{i}\,(\kappa/2) \cdot \left(\overline{\Phi} \cdot \partial_\alpha \Phi - \Phi \cdot \partial_\alpha \overline{\Phi}\right). \tag{4.171ii}$$

Equation $\mathcal{E}^{\alpha 4} = 0$ in (2.290iv) implies the existence of *the twist potential* $\Theta(\mathbf{x})$ such that

$$\Theta_\alpha(\mathbf{x}) = \partial_\alpha \Theta. \tag{4.172}$$

Now, we introduce *the complex gravo-electromagnetic potential* by:

$$\Gamma(\mathbf{x}) := \mathrm{e}^{2w(\mathbf{x})} - (\kappa/2) \cdot |\Phi(\mathbf{x})|^2 + \mathrm{i}\, \Theta(\mathbf{x}). \tag{4.173}$$

The nontrivial, stationary electromagneto-vac equations reduce to [65]

$$\begin{aligned} &\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + (1/2) \cdot \mathrm{e}^{-4w} \cdot \mathrm{Re}\left[\left(\partial_\alpha \overline{\Gamma} + \kappa\, \Phi \cdot \partial_\alpha \overline{\Phi}\right) \cdot \left(\partial_\beta \Gamma + \kappa\, \overline{\Phi} \cdot \partial_\alpha \Phi\right)\right] \\ &\quad - \kappa \cdot \mathrm{e}^{-2w} \cdot \mathrm{Re}\left[\partial_\alpha \overline{\Phi} \cdot \partial_\beta \Phi\right] = 0, \end{aligned} \tag{4.174i}$$

$$\overset{\circ}{\nabla}{}^2 \Gamma - \mathrm{e}^{-2w} \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\alpha \Gamma \cdot \left(\partial_\beta \Gamma + \kappa \cdot \overline{\Phi} \cdot \partial_\beta \Phi \right) = 0, \tag{4.174ii}$$

$$\overset{\circ}{\nabla}{}^2 \Phi - \mathrm{e}^{-2w} \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\alpha \Phi \cdot \left(\partial_\beta \Gamma + \kappa \cdot \overline{\Phi} \cdot \partial_\beta \Phi \right) = 0. \tag{4.174iii}$$

Example 4.4.4. We shall try to derive a special class of solutions of the system of field equations in (4.174i–iii). We *assume* a linear, functional relationship [142, 208]:

$$\Gamma(\mathbf{x}) = \gamma \cdot \Phi(\mathbf{x}) + \delta. \tag{4.175}$$

Here, $\gamma \neq 0$ and δ are some complex constants. The assumption (4.175) reduces *both* (4.174ii,iii) into

$$\overset{\circ}{\nabla}{}^2 \Phi - \mathrm{e}^{-2w} \cdot \left(\gamma + \kappa \, \overline{\Phi} \right) \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\alpha \Phi \cdot \partial_\beta \Phi = 0. \tag{4.176}$$

Moreover, field equations (4.174i) imply that

$$\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + \left[(1/2) \cdot \mathrm{e}^{-4w} \cdot |\overline{\gamma} + \kappa \, \Phi|^2 - \kappa \, \mathrm{e}^{-2w} \right] \cdot [\cdots] = 0. \tag{4.177}$$

Choosing

$$\mathrm{e}^{2w(\mathbf{x})} = \left| \frac{\overline{\gamma}}{\sqrt{2\kappa}} + \sqrt{\frac{\kappa}{2}} \cdot \Phi(\mathbf{x}) \right|^2, \tag{4.178}$$

we obtain that $\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) \equiv 0$. Thus, the metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ is *flat*. Introducing another complex potential by

$$\mathcal{W}(\mathbf{x}) := \sqrt{2\kappa} \cdot [\, \overline{\gamma} + \kappa \, \Phi(\mathbf{x})]^{-1}, \tag{4.179}$$

the equation (4.177) boils down to *the Euclidean-Laplace's equation*:

$$\overset{\circ}{\nabla}{}^2 \mathcal{W} = \nabla^2 \mathcal{W} = 0. \tag{4.180}$$

(Here, we have used a Cartesian coordinate chart so that $\overset{\circ}{g}_{\alpha\beta}(\mathbf{x}) = \delta_{\alpha\beta}$.)

To obtain the full metric, we need to satisfy the linear system:

$$\vec{\nabla} \times \vec{\mathbf{a}} = \operatorname{curl} \vec{\mathbf{a}} = i \left[\mathcal{W} \cdot \vec{\nabla} \overline{\mathcal{W}} - \overline{\mathcal{W}} \cdot \vec{\nabla} \mathcal{W} \right]. \tag{4.181}$$

The integrability of above equations requires

$$\mathcal{W}^{-1} \cdot \nabla^2 \mathcal{W} = \overline{\mathcal{W}}^{-1} \cdot \nabla^2 \overline{\mathcal{W}}. \tag{4.182}$$

This class of exact solutions represents the case of many spinning, charged bodies. However, wire or string singularities may be present. (See # 4(ii) of Exercise 4.4.)

This class of stationary solutions was discovered by Neugebaur [192], Perjes [208], and Israel–Wilson [142], (or N–P–I–W). These are generalizations of the Majumdar-Papapetrou solutions of (4.74). □

We shall now define a *conformastationary metric* by

$$\mathrm{d}s^2 = [U(\mathbf{x})]^4 \cdot \delta_{\mu\nu} \cdot (\mathrm{d}x^\mu \mathrm{d}x^\nu) - \mathrm{e}^{2w(\mathbf{x})} \cdot \left[a_\alpha(\mathbf{x})\,\mathrm{d}x^\alpha + \mathrm{d}x^4\right]^2,$$
$$U(\mathbf{x}) \neq 0 \quad \text{for} \quad \mathbf{x} \in \mathbf{D} \subset \mathbb{R}^3. \tag{4.183}$$

(It is the stationary generalization of *the conformastat metric* in (4.55).) The N–P–I–W solutions of the preceding Example 4.4.4 constitute a class of conformastationary metrics (emerging out of the Einstein–Maxwell field equations).

Now, we propose to furnish stationary field equations inside a material continuum. We use the general stationary metric in (4.144). Moreover, we generate *ten physically relevant material tensor components* of continuum mechanics out of *the ten components* of $T_{ij}(\cdot\cdot)$. These are provided by

$$\sigma_{\mu\nu}(\mathbf{x}) := -T_{\mu\nu}(\mathbf{x}) - a_\mu(\mathbf{x}) \cdot a_\nu(\mathbf{x}) \cdot T_{44}(\mathbf{x})$$
$$+ a_\mu(\mathbf{x}) \cdot T_{\nu 4}(\mathbf{x}) + a_\nu(\mathbf{x}) \cdot T_{\mu 4}(\mathbf{x}) \equiv \sigma_{\nu\mu}(\mathbf{x}), \tag{4.184i}$$

$$s_\mu(\mathbf{x}) := 2\left[T_{\mu 4}(\mathbf{x}) - a_\mu(\mathbf{x}) \cdot T_{44}(\mathbf{x})\right], \tag{4.184ii}$$

$$\tilde{\mu}(\mathbf{x}) := \left[\mathrm{e}^{-4w(\mathbf{x})} + \overset{\circ}{a}{}^\mu(\mathbf{x}) \cdot a_\mu(\mathbf{x})\right] \cdot T_{44}(\mathbf{x})$$
$$+ \overset{\circ}{g}{}^{\mu\nu}(\mathbf{x}) \cdot T_{\mu\nu}(\mathbf{x}) - 2\overset{\circ}{a}{}^\mu(\mathbf{x}) \cdot T_{\mu 4}(\mathbf{x}). \tag{4.184iii}$$

Here, $\tilde{\mu}(\mathbf{x})$, $s_\mu(\mathbf{x})$, and $\sigma_{\mu\nu}(\mathbf{x})$ represent respectively the mass density, the material current density, and stress components of a deformable solid body. (*Caution:* $\sigma_{\mu\nu}$ is different from the rate of strain tensor σ_{ij} in (2.199v).) (Compare with the corresponding axially symmetric equation (4.139).) The interior field equation (2.161i) in the stationary arena yields the following:

$$\overset{\circ}{G}_{\mu\nu}(\mathbf{x}) + 2 \cdot \left[\partial_\mu w \cdot \partial_\nu w - (1/2) \cdot \overset{\circ}{g}_{\mu\nu} \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\alpha w \cdot \partial_\beta w\right]$$
$$+ \left(\mathrm{e}^{4w}/2\right) \cdot \left[-\overset{\circ}{f}{}_\mu{}^\alpha \cdot f_{\nu\alpha} + (1/4) \cdot \overset{\circ}{g}_{\mu\nu} \cdot \overset{\circ}{f}{}^{\alpha\beta} \cdot f_{\alpha\beta}\right] = \kappa\, \sigma_{\mu\nu}(\mathbf{x}), \tag{4.185i}$$

$$\overset{\circ}{\nabla}_\nu \left[\mathrm{e}^{4w} \cdot \overset{\circ}{f}{}_\mu{}^\nu\right] = \kappa\, s_\mu(\mathbf{x}), \tag{4.185ii}$$

$$\overset{\circ}{\nabla}{}^2 w + \left(\mathrm{e}^{4w}/4\right) \cdot \overset{\circ}{f}{}^{\mu\nu} \cdot f_{\mu\nu} = (\kappa/2) \cdot \tilde{\mu}(\mathbf{x}). \tag{4.185iii}$$

(These equations are generalizations of static equations (4.40i–iii).)

The conservation equation (2.166i) leads to

$$-\tilde{\mu}(\mathbf{x}) \cdot \overset{\circ}{\nabla}_{\mu} w + \overset{\circ}{\nabla}^{\nu} \sigma_{\mu\nu} + (1/2) \cdot \overset{\circ}{s}^{\nu}(\mathbf{x}) \cdot f_{\nu\mu}(\mathbf{x}) = 0. \tag{4.186}$$

The equations above have direct physical interpretations. Consider the three terms in the left-hand side of the equations in (4.186). The first term corresponds to the "gravitational force density." The second term stands for *the elastic force density*. The third term indicates the "gravo-magnetic force density." Therefore, (4.186) represents *general relativistic equilibrium conditions* for a deformable solid body in *a state of steady motion. These equations are consequences of the Einstein's field equations*.

We have now completed investigations of the stationary field equations of general relativity. In closing, *we want to emphasize that the whole of this chapter deals only with local solutions of Einstein's field equations.*

Exercise 4.4

1. Consider the stationary metric of (4.144):
   (i) Express $\det\left[g_{ij}\right]$ in terms of $\overset{\circ}{g}_{\alpha\beta}$, a_α, w etc.
   (ii) Express components $g^{ij}(\cdot\cdot)$ in terms of $\overset{\circ}{g}_{\alpha\beta}$, a_α, w etc.
2. Examine stationary vacuum field equations (4.147i–iii). Let the function $w(\mathbf{x})$ attain the minimum at an interior point $\mathbf{x}_0 \in \mathbf{D}$. Then, prove that the curvature tensor $\mathbf{R}^{\cdot}{}_{\cdots}(x) = \mathbf{O}^{\cdot}{}_{\cdots}(x)$ for all $x \in \mathbf{D} \times \mathbb{R}$.
3. Consider the three-dimensional invariant eigenvalue problem $\det\left[\overset{\circ}{R}_{(\alpha)(\beta)}(\mathbf{x}) - \lambda(\mathbf{x})\,\delta_{(\alpha)(\beta)}\right] = 0$ in a domain $\mathbf{D}$ where vacuum field equations (4.150i,ii) hold. Show that real, invariant eigenvalues are furnished by

$$\begin{aligned}
&\lambda_{(1)}(\mathbf{x}) \equiv 0,\\
&\lambda_{(2)}(\mathbf{x}) = -(1/4) \cdot (\mathrm{Re}\ F)^{-2} \cdot \delta^{(\alpha)(\beta)} \cdot \left[\partial_{(\alpha)} F \cdot \partial_{(\beta)} \overline{F} + |\partial_{(\alpha)} F \cdot \partial_{(\beta)} F|\right] \leq 0,\\
&\lambda_{(3)}(\mathbf{x}) = -(1/4) \cdot (\mathrm{Re}\ F)^{-2} \cdot \Big[\delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)} F \cdot \partial_{(\beta)} \overline{F}\\
&\qquad\qquad - |\delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)} F \cdot \partial_{(\beta)} F|\Big] \leq 0.
\end{aligned}$$

4. Investigate the Papapetrou-Ehlers class of stationary metrics given by (4.162):

   (i) Prove that in case the invariant eigenvalue $\lambda_{(3)}(\mathbf{x})$ of the preceding example is identically zero, the metric must be of the Papapetrou-Ehlers class.

   (ii) Moreover, the total mass defined by the integral $\int_{\mathbf{D}}(\overset{\circ}{\nabla}{}^2\chi)\cdot\sqrt{\overset{\circ}{g}}\cdot d^3\mathbf{x}$ must either vanish, or else, there must exist a wire singularity.

5. Obtain *every nonflat, stationary vacuum metric which is both conformastationary and of the Papapetrou-Ehlers class*.
6. Consider the system of second-order, semilinear partial differential equation (4.150i):

   (i) In case a harmonic coordinate chart is used (so that $\partial_\mu[\sqrt{\overset{\circ}{g}}\cdot\overset{\circ}{g}{}^{\mu\nu}]=0$), show that the system is *elliptic*.
   (ii) In case an orthogonal coordinate chart is used, prove that the subsystem $\overset{\circ}{R}_{12}(\cdot\cdot)+(\cdot\cdot)=\overset{\circ}{R}_{23}(\cdot\cdot)+(\cdot\cdot)=\overset{\circ}{R}_{31}(\cdot\cdot)+(\cdot\cdot)=0$ is *hyperbolic*.

7. Consider the stationary Einstein–Maxwell equations (4.174i–iii):

   (i) Show that these equations are variationally derivable from the Lagrangian density:

$$
\begin{aligned}
\mathcal{L}(\cdots)_{|..} = \Big\{ & \overset{\circ}{R}(\mathbf{x}) + (1/2)\cdot\left[\mathrm{Re}\,\Gamma + (\kappa/2)\cdot|\Phi|^2\right]^{-2} \\
& \times\left(\partial_\alpha\overline{\Gamma}+\kappa\,\Phi\cdot\partial_\alpha\overline{\Phi}\right)\cdot\left(\overset{\circ}{\nabla}{}^\alpha\Gamma+\kappa\,\overline{\Phi}\cdot\overset{\circ}{\nabla}{}^\alpha\Phi\right) \\
& -\kappa\left(\mathrm{Re}\,\Gamma+\frac{\kappa}{2}\cdot|\Phi|^2\right)^{-1}\cdot\partial_\alpha\overline{\Phi}\cdot\overset{\circ}{\nabla}{}^\alpha\Phi\Big\}\cdot\sqrt{\overset{\circ}{g}}\,.
\end{aligned}
$$

   (ii) Prove that corresponding metric of the four-dimensional potential manifold, given by

$$
\begin{aligned}
d\Sigma^2 = {} & (1/2)\cdot\left[\mathrm{Re}\Gamma+(\kappa/2)\cdot|\Phi|^2\right]^{-2}\cdot\left[d\,\overline{\Gamma}+\kappa\,\Phi\cdot d\,\overline{\Phi}\right]\cdot\left[d\,\Gamma+\kappa\,\overline{\Phi}\cdot d\,\Phi\right] \\
& -\kappa\left(\mathrm{Re}\,\Gamma+\frac{\kappa}{2}\cdot|\Phi|^2\right)^{-1}\cdot\left(d\,\Phi\cdot d\,\overline{\Phi}\right),
\end{aligned}
$$

   yields a four-dimensional pseudo-Riemannian manifold.

(*Remark*: The isometry group of this space generates new solutions out of old ones. Consult the references of [65, 152].)

Answers and Hints to Selected Exercises:

1.
(i)
$$g = -\mathrm{e}^{-4w} \cdot \overset{\circ}{g}.$$

(ii)
$$\begin{aligned}
g^{\alpha\beta} &= \mathrm{e}^{2w} \cdot \overset{\circ}{g}{}^{\alpha\beta}, \\
g^{\alpha 4} &= -\,\mathrm{e}^{2w} \cdot \overset{\circ}{a}{}^{\alpha}, \\
g^{44} &= -\,\mathrm{e}^{-2w} + \mathrm{e}^{2w} \cdot a_{\alpha} \cdot \overset{\circ}{a}{}^{\alpha}.
\end{aligned}$$

2. The potential equation (4.147ii) implies that

$$\overset{\circ}{\nabla}{}^2 w = -(1/4) \cdot \mathrm{e}^{4w} \cdot \overset{\circ}{f}{}^{\alpha\beta} \cdot f_{\alpha\beta} \leq 0.$$

Using Hopf's Theorem 4.2.2, it follows that $w(\mathbf{x})$ is constant-valued. Therefore, by (4.147ii), $\overset{\circ}{f}{}^{\alpha\beta} \cdot f_{\alpha\beta} = 0$, $f_{\alpha\beta} = 0$, and $a_\alpha = \partial_\alpha \lambda$. By (4.147i), $\overset{\circ}{G}_{\alpha\beta} = 0$; thus, $\overset{\circ}{R}{}^{\alpha}{}_{\beta\gamma\delta} = 0$. (Consult [166].) The metric in (4.144) reduces to $\mathrm{d}s^2 = \mathrm{e}^{-2c} \cdot \delta_{\alpha\beta} \cdot \mathrm{d}\widehat{x}^{\alpha}\mathrm{d}\widehat{x}^{\beta} - \mathrm{e}^{2c} \cdot \left[\mathrm{d}(\lambda(\cdot\cdot) + x^4)\right]^2$ which is flat.

3.

$$\begin{aligned}
&\det\left[\overset{\circ}{R}_{(\alpha)(\beta)} - \lambda\, \delta_{(\alpha)(\beta)}\right] \\
&\quad = -\lambda \left\{ \lambda^2 + \left[(1/2) \cdot (\mathrm{Re}\, F)^{-2} \cdot \delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)} F \cdot \partial_{(\beta)} \overline{F}\right] \cdot \lambda \right. \\
&\qquad + \left[(1/16) \cdot (\mathrm{Re}\, F)^{-4} \cdot \left(\left(\delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)} F \cdot \partial_{(\beta)} \overline{F}\right)^2 \right.\right. \\
&\qquad \left.\left.\left. - \left|\delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)} F \cdot \partial_{(\beta)} F\right|^2 \right)\right]\right\}.
\end{aligned}$$

4. (i) Note that $\lambda_{(3)}(x) = 0$ implies

$$\left\{\delta^{(\alpha)(\beta)} \cdot \left[\partial_{(\alpha)} F \cdot \partial_{(\beta)} \overline{F}\right]\right\}^2 = \delta^{(\alpha)(\beta)} \cdot \left|\partial_{(\alpha)} F \cdot \partial_{(\beta)} F\right|^2.$$

Therefore,

$$\begin{aligned}
&\left[\delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)}(\mathrm{Re} F) \cdot \partial_{(\beta)}(\mathrm{Re} F)\right]^2 \cdot \left[\delta^{(\mu)(\nu)} \cdot \partial_{(\mu)}(\mathrm{Im} F) \cdot \partial_{(\nu)}(\mathrm{Im} F)\right]^2 \\
&\quad = \left[\delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)}(\mathrm{Re} F) \cdot \partial_{(\beta)}(\mathrm{Im} F)\right]^2,
\end{aligned}$$

or

$$\left[\delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)}(\mathrm{Re}F) \cdot \partial_{(\beta)}(\mathrm{Re}F)\right] \cdot \left[\delta^{(\mu)(\nu)} \cdot \partial_{(\mu)}(\mathrm{Im}F) \cdot \partial_{(\nu)}(\mathrm{Im}F)\right]$$
$$= \left[\delta^{(\alpha)(\beta)} \cdot \partial_{(\alpha)}(\mathrm{Re}F) \cdot \partial_{(\beta)}(\mathrm{Im}F)\right].$$

By the *Cauchy–Schwartz (weak) inequality* in (1.94) and (1.95), it follows that the $\mathrm{Re}(F)$ and $\mathrm{Im}(F)$ are *functionally related*:

$$\partial_{(\alpha)}(\mathrm{Re}F) = \lambda(\cdot\cdot) \cdot \partial_{(\alpha)}(\mathrm{Im}F),$$

for some scalar function $\lambda(\cdot\cdot)$.

Thus, by (4.154) and (4.157), the desired conclusion is reached.

(ii) Assume that the boundary $\partial\mathbf{D}$ is a continuous, piecewise-smooth, orientable, simply connected surface so that Gauss' Theorem 1.3.27 can be applied. The "total mass" reduces to

$$\int_{\mathbf{D}} \overset{\circ}{\nabla}{}^2 \chi \, \mathrm{d}^3 v = \int_{\partial\mathbf{D}} \overset{\circ}{\nabla}_\alpha \chi \cdot n^\alpha \, \mathrm{d}^2 v$$
$$= \frac{k}{2} \cdot \int_{\partial\mathbf{D}} \overset{\circ}{\eta}{}^{\alpha\beta\gamma} \cdot f_{\beta\gamma} \cdot n_\alpha \, \mathrm{d}^2 v = k \cdot \int_{\partial\mathbf{D}} \mathrm{d}\left[a_\mu \cdot \mathrm{d}x^\mu\right]$$
$$= k \cdot \int_{\partial^2\mathbf{D}} \left[a_\mu \cdot \mathrm{d}x^\mu\right] = 0.$$

The last equation follows from (1.206), Stokes' Theorem 1.2.23, and the topological fact that the boundary of a closed boundary, $\partial\mathbf{D}$, is the null subset! The only way to avoid this conclusion is to attach a massless semi-infinite wire singularity to the material body so that *there exists no closed surface enclosing the material body*.

5. Use (4.64i–iii) and Example 4.4.1. There exist exactly three such metrics shown below:

(i)

$$\mathrm{d}s^2 = (2k)^{-1} \cdot \left[\left(1 - \frac{2m}{r}\right) + \left(1 - \frac{2m}{r}\right)^{-1}\right]$$
$$\times \left\{(\mathrm{d}r)^2 + (r^2 - 2mr) \cdot \left[(\mathrm{d}\theta)^2 + (\sin\theta)^2 \cdot (\mathrm{d}\varphi)^2\right]\right\}$$
$$- (2k) \cdot \left[\left(1 - \frac{2m}{r}\right) + \left(1 - \frac{2m}{r}\right)^{-1}\right]^{-1} \cdot \left[\mathrm{d}t - (2m/k) \cdot \cos\theta \cdot \mathrm{d}\varphi\right]^2;$$

(ii)

$$\mathrm{d}s^2 = (2k)^{-1} \cdot \left[1 + \left(1 + mx^1\right)^4\right] \cdot \delta_{\mu\nu} \cdot (\mathrm{d}x^\mu \mathrm{d}x^\nu)$$

$$- (2k) \cdot \left[\left(1+mx^1\right)^2 + \left(1 + mx^1\right)^{-2}\right]^{-1} \cdot \left[\mathrm{d}t + (m/k) \cdot \left(x^3 \mathrm{d}x^2 - x^2 \mathrm{d}x^3\right)\right]^2 ;$$

(iii)

$$\mathrm{d}s^2 = (2k)^{-1} \cdot \left[\left(\frac{2m}{R} - 1\right) + \left(\frac{2m}{R} - 1\right)^{-1}\right]$$

$$\times \left\{(\mathrm{d}R)^2 + (2mR - R^2) \cdot \left[(\mathrm{d}\psi)^2 + (\sinh\psi)^2 \cdot (\mathrm{d}\varphi)^2\right]\right\}$$

$$- (2k) \cdot \left[\left(\frac{2m}{R} - 1\right) + \left(\frac{2m}{R} - 1\right)^{-1}\right]^{-1} \cdot [\mathrm{d}t - (2m/k) \cdot \cosh\psi \cdot \mathrm{d}\varphi]^2 .$$

Chapter 5
Black Holes

5.1 Spherically Symmetric Black Holes

The phenomenon of a massive body imploding into a *black hole* is a fascinating topic to investigate. Nowadays, the possible detection of supermassive black holes in the centers of galaxies [96, 187] makes these studies extremely relevant. A historical survey and review of current research in black hole physics may be found in [72].

Investigations on black holes historically started from the Schwarzschild metric of (3.9) expressed, with slightly different notation, as

$$ds^2 = \left(1 - \frac{2m}{\widehat{r}}\right)^{-1} \cdot (d\widehat{r})^2 + \left(\widehat{r}\right)^2 \cdot \left[\left(d\widehat{\theta}\right)^2 + \sin^2\widehat{\theta} \cdot (d\widehat{\varphi})^2\right] - \left(1 - \frac{2m}{\widehat{r}}\right) \cdot (d\widehat{t})^2,$$
$$\widehat{D} := \left\{\left(\widehat{r}, \widehat{\theta}, \widehat{\varphi}, \widehat{t}\right) : 0 < 2m < \widehat{r}, 0 < \widehat{\theta} < \pi, -\pi < \widehat{\varphi} < \pi, -\infty < \widehat{t} < \infty\right\}. \tag{5.1}$$

Clearly, some of the metric tensor components in (5.1) are *undefined for* $\widehat{r} = 2m$, the *Schwarzschild radius*. However, the corresponding orthonormal components of the curvature tensor in (3.12) are *real-analytic* for $0 < \widehat{r}$. Therefore, the space–time geometry is *completely smooth* at $\widehat{r} = 2m$, but the Schwarzschild coordinate chart is *unable to cover* $\widehat{r} = 2m$. (Recall that the spherical polar coordinate chart of Example 1.1.2 cannot cover the North or the South pole.)

Eddington [82], Painlevé [203], and Gullstrand [119] extended the Schwarzschild chart on and beyond the Schwarzschild radius. Eddington employed *one null coordinate* in his construction. (Finkelstein [100] also similarly extended the same with one null coordinate.) Lemaître [163] used the geodesic normal time coordinate (or comoving coordinates) for the extension whereas Synge [241] devised *a doubly null coordinate chart* to extend all the previous charts for *the maximal extension* of the original Schwarzschild chart. A modification of Synge's chart was constructed independently by Kruskal [153] and Szekeres [245]. It is the commonly used chart now for the maximal extension of the Schwarzschild chart.

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4_5,

We shall start our investigations on black holes with the Lemaître metric of (3.105), with a change of notation, given by

$$\begin{aligned}
\mathrm{d}s^2 &= \left[\frac{2m}{Y(r,t)}\right] \cdot (\mathrm{d}r)^2 + [Y(r,t)]^2 \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right] - (\mathrm{d}t)^2\,,\\
Y(r,t) &:= \left[(3/2) \cdot \sqrt{2m} \cdot (r-t)\right]^{2/3} > 0,\\
D &:= \{(r,\theta,\varphi,t) : 0 < r - t,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi\}\,.
\end{aligned} \tag{5.2}$$

The above metric and the orthonormal components of the curvature tensor diverge in the limit $(r-t) \to 0_+$. The transformation from a subset of the Lemaître chart to the Schwarzschild chart is furnished by

$$\begin{aligned}
\widehat{r} &= Y(r,t),\\
\left(\widehat{\theta}, \widehat{\varphi}\right) &= (\theta,\varphi),\\
\widehat{t} &= t - \left(2\sqrt{2m}\right) \cdot \sqrt{Y(r,t)} + (2m) \cdot \ln\left|\frac{\sqrt{Y(r,t)} + \sqrt{2m}}{\sqrt{Y(r,t)} - \sqrt{2m}}\right|,\\
D_s &:= \{(r,\theta,\varphi,t) : (4m/3) < r - t,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi\}\,,\\
\widehat{D}_s &:= \left\{\left(\widehat{r}, \widehat{\theta}, \widehat{\varphi}, \widehat{t}\right) : 2m < \widehat{r},\ 0 < \widehat{\theta} < \pi,\ -\pi < \widehat{\varphi} < \pi,\ \widehat{t} \in \mathbb{R}\right\},\\
\frac{\partial\left(\widehat{r}, \widehat{t}\right)}{\partial(r,t)} &= \sqrt{\frac{2m}{\widehat{r}}} > 0.
\end{aligned} \tag{5.3}$$

From the sub-Jacobian above (which has the same value as the Jacobian), it is clear that the mapping is *one-to-one*. (Compare the transformation in (5.3) with (3.106).)

Now, consider the subset:

$$\begin{aligned}
D'_s := D - \overline{\widehat{D}}_s = \big\{&(r,\theta,\varphi,t) :\ 0 < r - t < (4m/3),\\
&0 < \theta < \pi,\ -\pi < \varphi < \pi\big\}.
\end{aligned} \tag{5.4}$$

A coordinate mapping from D'_s onto $D^{\#}_s$ of a T-domain chart is provided by

$$\begin{aligned}
R &= t - \left(2\sqrt{2m}\right) \cdot \sqrt{Y(r,t)} + (2m) \cdot \ln\left|\frac{\sqrt{2m} + \sqrt{Y(r,t)}}{\sqrt{2m} - \sqrt{Y(r,t)}}\right|,\\
T &= Y(r,t),
\end{aligned}$$

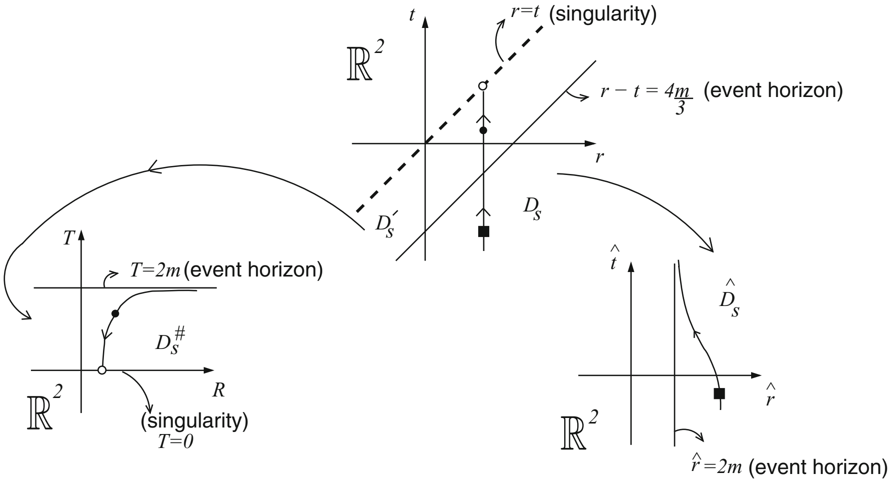


Fig. 5.1 Qualitative picture depicting two mappings from the Lemaître chart

$$\left(\theta^{\#}, \varphi^{\#}\right) = (\theta, \varphi),$$

$$D_s^{\#} := \left\{\left(R, \theta^{\#}, \varphi^{\#}, T\right) : R \in \mathbb{R},\ 0 < \theta^{\#} < \pi,\ -\pi < \varphi^{\#} < \pi,\ 0 < T < 2m\right\},$$

$$\frac{\partial(T, R)}{\partial(r, t)} = -\sqrt{\frac{2m}{T}} < 0. \tag{5.5}$$

From the sub-Jacobian above, it is evident that the mapping is *one-to-one*.

In $D_s^{\#}$, the metric (5.2) goes over into (3.112), namely (in a different notation),

$$\begin{aligned} ds^2 = {} & \left[\frac{2m}{T} - 1\right] \cdot (dR)^2 + T^2 \cdot \left[\left(d\theta^{\#}\right)^2 + \sin^2\theta^{\#} \cdot \left(d\varphi^{\#}\right)^2\right] \\ & - \left[\frac{2m}{T} - 1\right]^{-1} \cdot (dT)^2 . \end{aligned} \tag{5.6}$$

Figure 5.1 depicts mappings (5.3) and (5.5) from the Lemaître chart into the Schwarzschild chart and the T-domain chart (with $\theta = \text{const.}$, $\varphi = \text{const.}$).

Let us elaborate more on the physical aspects of Fig. 5.1. There is an oriented, semi-infinite, straight line segment in the Lemaître chart. It is *parallel to the t-axis*. Thus, it can represent a test particle following *a timelike geodesic*. The segment between two solid dots is *a finite interval*, physically representing *the finite proper time* elapsed between those two events. The part of the line segment that is mapped into the Schwarzschild chart is a semi-infinite curve asymptotic to the vertical line $\widehat{r} = 2m$. Thus, an observer in the Schwarzschild universe concludes that the test particle takes "infinite coordinate time $\widehat{t}$" to reach $\widehat{r} = 2m$. (But from the preimage in the Lemaître chart, it is evident that only a finite proper time is involved.) Therefore, an observer "at infinity" of the Schwarzschild space–time never sees a massive particle freely falling into the region characterized by $\widehat{r} \leq 2m$. Similarly, the other finite part of the line segment is mapped into a semi-infinite curve in the T-domain chart. It is asymptotic to the horizontal straight line $T = 2m$. The curve terminates on the R-axis where there is *a singularity of the curvature tensor*. The proper time along this curve is also finite.

Now, we shall explore the inclined straight line $r - t = (4m/3)$ in the Lemaître chart. It is the image of the parametrized curve:

$$\begin{aligned}
r &= \mathcal{X}^1(\alpha) := f(\alpha),\\
\theta &= \mathcal{X}^2(\alpha) := \theta_0 = \text{const.},\\
\varphi &= \mathcal{X}^3(\alpha) := \varphi_0 = \text{const.},\\
t &= \mathcal{X}^4(\alpha) := f(\alpha) - (4m/3),\\
\alpha &\in \mathbb{R}.
\end{aligned} \tag{5.7}$$

It is assumed that $\frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha}$ exists and nonzero for $\alpha \in \mathbb{R}$. Using the Lemaître metric in (5.2), we conclude that

$$g_{ij}\left(\mathcal{X}(\alpha)\right) \cdot \frac{\mathrm{d}\mathcal{X}^i(\alpha)}{\mathrm{d}\alpha} \cdot \frac{\mathrm{d}\mathcal{X}^j(\alpha)}{\mathrm{d}\alpha} = \left[\frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha}\right]^2 + 0 + 0 - \left[\frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha}\right]^2 \equiv 0.$$

Therefore, $r - t = (4m/3)$ in the Lemaître chart represents *a null* hypersurface $\mathcal{N}_3$. The metric (5.2) restricted to $\mathcal{N}_3$ yields

$$\mathbf{g}_{..}(\cdot\cdot)_{|\mathcal{N}_3} = (2m)^2 \cdot \left[\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \cdot (\mathrm{d}\varphi \otimes \mathrm{d}\varphi)\right]. \tag{5.8}$$

The above equation reveals *a loss of one dimension* as in the null hypersurface of Example 2.1.17. This null hypersurface $\mathcal{N}_3$ is called an *event horizon*.[1]

We shall show later that some massive particles propagating in the Schwarzschild space–time can cross the event horizon $\mathcal{N}_3$ only to eventually end their journey into the ultimate singularity at $r = t$. Moreover, we shall demonstrate later that from the domain D_s' in Fig. 5.1, photons pursuing null geodesics *cannot come out* of event horizon $\mathcal{N}_3$ into the external Schwarzschild universe. That is why the domain D_s' is called a (spherically symmetric) *black hole*.

[1] Note that by (3.42), the event horizon is a null hypersurface of "infinite redshift."

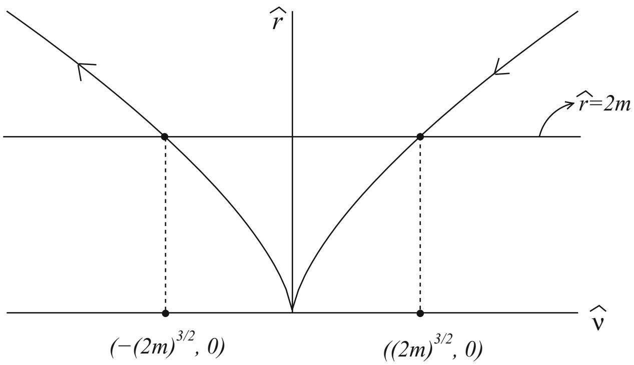


Fig. 5.2 The graph of the semicubical parabola $(\hat{r})^3 = (\hat{\nu})^2$

Now, let us go back to (5.2) and (5.3). We can derive that

$$\begin{aligned} \widehat{r} &= \left[(3/2)\cdot\sqrt{2m}\cdot(r-t)\right]^{2/3} = Y(r,t), \\ \widehat{\nu} &:= (3/2)\cdot\sqrt{2m}\cdot(r-t), \\ (\widehat{r})^3 &= (\widehat{\nu})^2. \end{aligned} \tag{5.9}$$

The function $Y(r,t)$ is one of the solutions of the differential equation:

$$\partial_4 Y = \pm\sqrt{[2m/Y(r,t)]}\,. \tag{5.10}$$

In Newtonian gravitation, the above equation represents the radial velocity of a test particle in the vicinity of a spherically symmetric gravitating body. (Consult (3.101).) This equation also indicates that the "Newtonian kinetic energy" is exactly balanced by the "gravitational potential energy." A possible graph of such a motion can be provided by (5.9). Particularly, the equation $(\widehat{r})^3 = (\widehat{\nu})^2$ yields the continuous, piecewise real-analytic, semi-cubical parabola (with a cusp). (See Fig. 5.2.)

In the graph of Fig. 5.2, the segment for $\widehat{\nu} > 0$ represents a particle coming from "infinity" that merges with a (idealized) central body. However, the segment of the curve for $\widehat{\nu} < 0$ indicates the trajectory of a particle, ejected from the center, which possesses exactly the "escape velocity" (reaches "infinity" with zero kinetic energy). The mathematics of Newtonian gravitation *coincides* with that of the general relativity in this scenario. However, in general relativity, there exist *four distinct domains* corresponding to (1) $(2m)^{3/2} < \widehat{\nu}$, (2) $0 < \widehat{\nu} < (2m)^{3/2}$, (3) $-(2m)^{3/2} < \widehat{\nu} < 0$, and $\widehat{\nu} < -(2m)^{3/2}$. The first domain with $0 < 2m < \widehat{r}$ corresponds to the Schwarzschild space–time. The second domain with $0 < \widehat{\nu} < (2m)^{3/2}$ indicates *a spherically symmetric black hole*. The other two domains will be explained later.

Let us go back to the T-domain metric (5.6). Two Killing vectors are $\frac{\partial}{\partial\varphi}$ and $\frac{\partial}{\partial R}$. The corresponding squared Lagrangian is given by

$$L_{(2)}(\cdot\cdot) = (1/2)\cdot\left\{\left[\frac{2m}{T}-1\right]\cdot\left(u^R\right)^2 + T^2\left[\left(u^{\theta^\#}\right)^2 + \sin^2\theta^\#\cdot\left(u^{\varphi^\#}\right)^2\right]\right.$$
$$\left. -\left[\frac{2m}{T}-1\right]^{-1}\cdot\left(u^T\right)^2\right\}. \tag{5.11}$$

Timelike geodesics admit a class of solutions with

$$R = R_0 = \text{const.},$$
$$\theta^\# = \pi/2,$$
$$\varphi^\# = \varphi_0^\# = \text{const.}$$

The T-geodesic equation implies that

$$\left[\frac{2m}{T}-1\right]^{-1}\cdot\left(\frac{\mathrm{d}T(s)}{\mathrm{d}s}\right)^2 = 1,$$

$$\text{or}\qquad \frac{\mathrm{d}T(s)}{\mathrm{d}s} = \pm\left[\frac{2m-T}{T}\right]^{1/2},$$

$$0 < T < 2m. \tag{5.12}$$

By Example A2.10, it is evident that a particle pursuing a geodesic along a T-coordinate line toward the singularity $T = 0_+$, exhibits *chaotic behavior*. (See [151].)

Example 5.1.1. Historically, the first chart that extended the Schwarzschild chart into $\widehat{r} \leq 2m$ was due to Painlevé [203] and Gullstrand [119]. It is obtained via the coordinate transformation:

$$\left(\tilde{r},\tilde{\theta},\tilde{\varphi}\right) = \left(\widehat{r},\widehat{\theta},\widehat{\varphi}\right),$$

$$\tilde{t} = \widehat{t} + f\left(\widehat{r}\right),$$

$$\text{where,}\quad \frac{\mathrm{d}f\left(\widehat{r}\right)}{\mathrm{d}\widehat{r}} = \left(1-\frac{2m}{\widehat{r}}\right)^{-1}\cdot\sqrt{\frac{2m}{\widehat{r}}}.$$

The Schwarzschild metric goes over into

$$\mathrm{d}s^2 = (\mathrm{d}\tilde{r})^2 + (\tilde{r})^2 \cdot \left[\left(\mathrm{d}\tilde{\theta}\right)^2 + \sin^2 \tilde{\theta} \cdot (\mathrm{d}\tilde{\varphi})^2 \right]$$
$$- \left(1 - \frac{2m}{\tilde{r}}\right) \cdot \left(\mathrm{d}\tilde{t}\,\right)^2 + 2 \cdot \sqrt{\frac{2m}{\tilde{r}}} \cdot (\mathrm{d}\tilde{r}) \cdot \left(\mathrm{d}\tilde{t}\,\right).$$

It is *a nonorthogonal coordinate chart* known as the *Painlevé–Gullstrand coordinate chart*.

For the three-dimensional hypersurface defined by $\tilde{r} = 2m$, the above equation yields the two-dimensional metric:

$$\mathrm{d}s^2{}_{|\tilde{r}=2m} = (2m)^2 \cdot \left[\left(\mathrm{d}\tilde{\theta}\right)^2 + \sin^2 \tilde{\theta} \cdot (\mathrm{d}\tilde{\varphi})^2 \right].$$

Thus, the hypersurface is null (and represents the event horizon). The nontrivial curvature components are furnished by

$$\begin{aligned}
R_{1212} &= -\frac{m}{\tilde{r}}, & R_{1224} &= \frac{\sqrt{\frac{2m}{\tilde{r}}} \cdot m}{\tilde{r}}\,, \\
R_{1313} &= -\frac{m \cdot \sin^2 \tilde{\theta}}{\tilde{r}}, & R_{1334} &= \frac{\sqrt{\frac{2m}{\tilde{r}}} \cdot m \cdot \sin^2 \tilde{\theta}}{\tilde{r}}\,, \\
R_{1414} &= -\frac{2m}{\tilde{r}^3}, & R_{2323} &= 2m \cdot \tilde{r} \cdot \sin^2 \tilde{\theta}, \\
R_{2424} &= \frac{(\tilde{r} - 2m) \cdot m}{\tilde{r}^2}, & R_{3434} &= \frac{(\tilde{r} - 2m) \cdot m \cdot \sin^2 \tilde{\theta}}{\tilde{r}^2}\,.
\end{aligned}$$

Therefore, the space–time geometry is *very smooth for* $\tilde{r} > 0$. □

Now, we shall introduce null coordinates into the picture. Consider *a radial, null geodesic* in the Schwarzschild metric (5.1). It is governed by the ordinary differential equations:

$$\left(1 - \frac{2m}{\widehat{r}}\right)^{-1} \cdot \left(\frac{\mathrm{d}\widehat{r}}{\mathrm{d}\alpha}\right) = \pm \frac{\mathrm{d}\widehat{t}}{\mathrm{d}\alpha}\,,$$

$$\text{or} \qquad \int \left(\frac{\widehat{r}}{\widehat{r} - 2m}\right) \cdot \mathrm{d}\widehat{r} = \pm \widehat{t} + \text{const.},$$

$$\text{or} \qquad \widehat{r}_* := \widehat{r} + (2m) \cdot \ln \left| \frac{\widehat{r}}{2m} - 1 \right| = \pm \widehat{t} + \text{const.} \tag{5.13}$$

(Here, $\widehat{r}_*$ is called the *Regge-Wheeler tortoise coordinate* [217].) We introduce *the retarded null coordinate* $\widehat{u}$ *and the advanced null coordinate* $\widehat{v}$ by the transformations:

$$\begin{aligned}\widehat{u} &= \widehat{t} - \widehat{r}_* \,, \\ \widehat{v} &= \widehat{t} + \widehat{r}_* \,.\end{aligned} \tag{5.14}$$

Remarks: (i) In the lim $m \to 0_+$, we obtain the null coordinates of Example 2.1.17.
(ii) Compare with the Example A2.6.

The Schwarzschild's metric, in terms of each of the null coordinates introduced above, can be transformed into

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1 - \frac{2m}{\widehat{r}}\right) \cdot (\mathrm{d}\widehat{u})^2 - 2\,(\mathrm{d}\widehat{r}) \cdot (\mathrm{d}\widehat{u}) \\ &+ (\widehat{r}\,)^2 \cdot \left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2\widehat{\theta} \cdot (\mathrm{d}\widehat{\varphi})^2\right],\end{aligned} \tag{5.15i}$$

$$\begin{aligned}\text{or}\quad \mathrm{d}s^2 = &-\left(1 - \frac{2m}{\widehat{r}}\right) \cdot (\mathrm{d}\widehat{v})^2 + 2\,(\mathrm{d}\widehat{r}) \cdot (\mathrm{d}\widehat{v}) \\ &+ (\widehat{r}\,)^2 \cdot \left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2\widehat{\theta} \cdot (\mathrm{d}\widehat{\varphi})^2\right].\end{aligned} \tag{5.15ii}$$

The metrics (5.15i) and (5.15ii) were studied by Eddington [82] and Finkelstein [100]. Both of the metrics in (5.15i) and (5.15ii) are regular on the event horizon, $\widehat{r} = 2m$, with a loss of two dimensions.

Now, we shall introduce *both of the null coordinates* $\widehat{u}$ and $\widehat{v}$. Note that by (5.13) and (5.14) we obtain

$$\begin{aligned}\widehat{t} &= (1/2) \cdot (\widehat{u} + \widehat{v})\,, \\ \widehat{r}_* &= (1/2) \cdot (\widehat{v} - \widehat{u})\,, \\ \frac{\widehat{r}}{2m} + \ln\left|\frac{\widehat{r}}{2m} - 1\right| &= \left(\frac{1}{4m}\right) \cdot (\widehat{v} - \widehat{u})\,, \\ \widehat{r} &= \widehat{\mathcal{Y}}\,(\widehat{u}, \widehat{v})\,.\end{aligned} \tag{5.16}$$

Here, $\widehat{\mathcal{Y}}\,(\widehat{u}, \widehat{v})$ is an implicitly defined function.

The Schwarzschild metric (5.1), with the use of (5.16), transforms into

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{\widehat{r}}\right)\cdot(\mathrm{d}\widehat{u})\cdot(\mathrm{d}\widehat{v}) + (\widehat{r})^2 \cdot \left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2\widehat{\theta}\cdot(\mathrm{d}\widehat{\varphi})^2\right] \tag{5.17i}$$

$$= -\left[1 - \frac{2m}{\widehat{\mathcal{Y}}(\widehat{u},\widehat{v})}\right]\cdot(\mathrm{d}\widehat{u})\cdot(\mathrm{d}\widehat{v}) + \left[\widehat{\mathcal{Y}}(\widehat{u},\widehat{v})\right]^2\left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2\widehat{\theta}\cdot[\mathrm{d}\widehat{\varphi})^2\right],$$

$$\mathrm{d}s^2_{|\widehat{r}=2m} = 4m^2\cdot\left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2\widehat{\theta}\cdot\left(\mathrm{d}\widehat{\phi}\right)^2\right]. \tag{5.17ii}$$

The above metric was first derived by Synge [241]. *It is regular at* $\widehat{r} = 2m$. Moreover, the metric represented the maximal extension of the Schwarzschild chart for the first time (as far as we know).[2] There is *a loss of two dimensions* on the hypersurface $\widehat{r} = 2m$, indicating that it is *null*. (Compare with (3.8).)

We now introduce another transformation by

$$u = -\mathrm{e}^{-\left(\widehat{u}/4m\right)},$$
$$v = \mathrm{e}^{\left(\widehat{v}/4m\right)},$$
$$\mathcal{Y}(u,v) \equiv \widehat{\mathcal{Y}}(\widehat{u},\widehat{v}). \tag{5.18}$$

With help of (5.18) and (5.16), the Schwarzschild metric (5.1) goes over into

$$\mathrm{d}s^2 = -\frac{32\cdot(m)^3}{\mathcal{Y}(u,v)}\cdot\exp\left[-\frac{\mathcal{Y}(u,v)}{2m}\right]\cdot(\mathrm{d}u)\cdot(\mathrm{d}v)$$
$$+ [\mathcal{Y}(u,v)]^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta\cdot(\mathrm{d}\varphi)^2\right],$$
$$\widehat{r} = \mathcal{Y}(u,v) > 2m > 0. \tag{5.19}$$

Suppressing the θ *and* φ *coordinates* for simplicity, the transformation of the doubly null coordinate chart from the Schwarzschild chart can be summarized below as

$$u = -\sqrt{\frac{\widehat{r}}{2m} - 1}\cdot\exp\left[\left(\widehat{r} - \widehat{t}\right)/\,4m\right] < 0,$$
$$v = \sqrt{\frac{\widehat{r}}{2m} - 1}\cdot\exp\left[\left(\widehat{r} + \widehat{t}\right)/\,4m\right] > 0,$$
$$\frac{\partial(u,v)}{\partial\left(\widehat{r},\widehat{t}\right)} = -\left[\frac{\widehat{r}}{16\cdot m^3}\right]\cdot\exp\left[\widehat{r}/2m\right] < 0,$$
$$\text{codomain } D_{\mathrm{I}} := \left\{(u,v)\in\mathbb{R}^2 : u < 0,\ v > 0\right\}. \tag{5.20}$$

[2] Consult the reference of Schmidt [227].

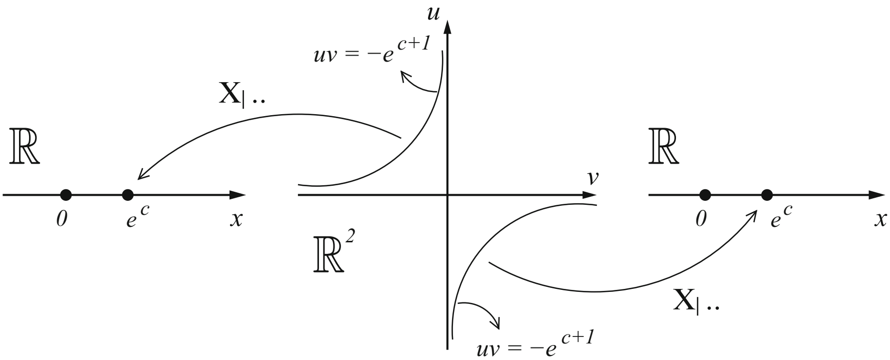


Fig. 5.3 The mapping X and its restrictions $X_{|..}$

The above transformation is *one-to-one*. The inverse transformation is furnished by

$$\left(\frac{\widehat{r}}{2m} - 1\right) \cdot \exp\left[\widehat{r}/2m\right] = -uv,$$
$$\widehat{r} = \mathcal{Y}(u, v),$$
$$\widehat{t} = (2m) \cdot \ln\left[-(v/u)\right]. \tag{5.21}$$

Note that the function $\widehat{r} = \mathcal{Y}(u, v)$ is *implicitly defined above*.[3] However, it can be *expressed explicitly* in terms of known functions. Consider firstly the polynomial function from $\mathbb{R}$ to $\mathbb{R}^2$ as

$$x = X(u, v) := -(e)^{-1} \cdot uv,$$
$$(u, v) \in \mathbb{R}^2,$$
$$e := \exp(1). \tag{5.22}$$

It is *not a one-to-one function* from $\mathbb{R}$ into $\mathbb{R}^2$. The function X maps the rectangular hyperbola $uv = k$ (of two branches) into the single point $x = -k \cdot (e)^{-1} = \text{const.}$ (See Fig. 5.3 with $k = -\mathrm{e}^{c+1}$.)

[3]Note that from (5.21), we can derive that $\mathcal{Y}(u, v) = \widehat{r} = 2m \Leftrightarrow u \equiv 0,$ and/or $v \equiv 0 \Rightarrow \mathrm{d}u \equiv 0,$ and/or $\mathrm{d}v \equiv 0$; thus, $\mathrm{d}u \otimes \mathrm{d}v_{|\widehat{r}=2m} \equiv \mathbf{O}_{..|\widehat{r}=2m}$.

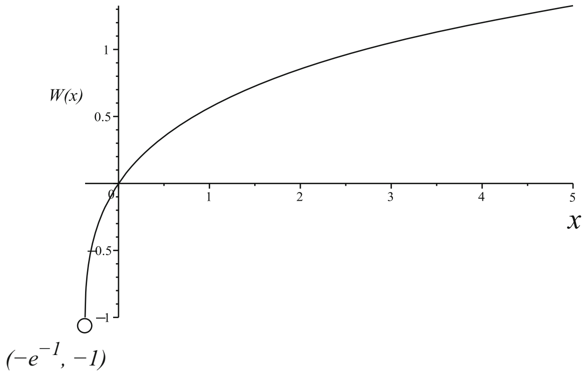


Fig. 5.4 The graph of the Lambert W-function

We need another function, namely, the *Lambert* W-*Function*. (See [41].) This function has the following properties:

$$\begin{aligned}
&\mathsf{W}(x) \cdot \exp\left[\mathsf{W}(x)\right] \equiv x, \\
&\mathsf{W}(0) = 0, \\
&\lim_{x \to (-\mathrm{e}^{-1})_+} \left[\mathsf{W}(x)\right] = -1, \\
&\lim_{x \to \infty} \left[\mathsf{W}(x)\right] \longrightarrow \infty, \\
&\frac{\mathrm{d}\mathsf{W}(x)}{\mathrm{d}x} = \mathrm{e}^{-\mathsf{W}(x)} \cdot \left[1 + \mathsf{W}(x)\right]^{-1} > 0, \\
&x \in \left(-\mathrm{e}^{-1}, \infty\right). \qquad (5.23)
\end{aligned}$$

The graph of the above function is displayed in Fig. 5.4.

Now we introduce another real-analytic function by

$$\begin{aligned}
\tilde{r} = \widetilde{\mathcal{Y}}(u,v) &:= (2m) \cdot \{1 + \mathsf{W}\left[X(u,v)\right]\} \\
&= (2m) \cdot \left[1 + \mathsf{W}\left(-\mathrm{e}^{-1} uv\right)\right] > 0, \\
\widetilde{D} &:= \left\{(u,v) \in \mathbb{R}^2 : -\infty < uv < 1\right\}, \\
\tilde{t} &= 2m \cdot \ln|v/u| \quad \text{for} \quad uv \neq 0 \quad \text{and} \quad uv < 1. \qquad (5.24)
\end{aligned}$$

Four restrictions on the above function over four domains of validity will be presented in the following. *The first domain* and a related (one-to-one) coordinate transformation are given by

$$
\begin{aligned}
& D_{\mathrm{I}} := \left\{(u,v) \in \mathbb{R}^2 : \ u < 0, \ v > 0\right\}; \\
& \widehat{r} = \mathcal{Y}(u,v) := \widetilde{\mathcal{Y}}(u,v)_{|D_{\mathrm{I}}} = (2m) \cdot \left[1 + \mathsf{W}\left(-\mathrm{e}^{-1} uv\right)\right]_{|D_{\mathrm{I}}}, \\
& \widehat{t} = (2m) \cdot \ln\left[-(v/u)\right]_{|..} \ .
\end{aligned}
\tag{5.25}
$$

According to (5.21), the above transformation leads to the Schwarzschild metric (5.1).

The second domain and a related (one-to-one) coordinate transformation are provided by

$$
\begin{aligned}
& D_{\mathrm{II}} := \left\{(u,v) \in \mathbb{R}^2 : \ u > 0, \ v > 0, \ uv < 1\right\}; \\
& T = \mathcal{Y}^{\#}(u,v) := \widetilde{\mathcal{Y}}(u,v)_{|D_{\mathrm{II}}} = (2m) \cdot \left[1 + \mathsf{W}\left(-\mathrm{e}^{-1} uv\right)\right]_{|D_{\mathrm{II}}}, \\
& R = (2m) \cdot \ln(v/u)_{|..} \ .
\end{aligned}
\tag{5.26}
$$

Equation (5.26) leads to the four-dimensional, T-domain metric in (5.6).

The third domain and a related (one-to-one) coordinate transformation are furnished by

$$
\begin{aligned}
& D_{\mathrm{III}} := \left\{(u,v) \in \mathbb{R}^2 : \ u > 0, \ v < 0\right\}; \\
& r' = \mathcal{Y}'(u,v) := \widetilde{\mathcal{Y}}(u,v)_{|D_{\mathrm{III}}} = (2m) \cdot \left[1 + \mathsf{W}\left(-\mathrm{e}^{-1} uv\right)\right]_{|D_{\mathrm{III}}} \ , \\
& t' = (2m) \cdot \ln\left[-(v/u)\right]_{|..} \ .
\end{aligned}
\tag{5.27}
$$

The above transformation leads to the (dual) Schwarzschild metric:

$$
\begin{aligned}
\mathrm{d}s^2 = {} & \left(1 - \frac{2m}{r'}\right)^{-1} \cdot (\mathrm{d}r')^2 + (r')^2 \cdot \left[(\mathrm{d}\theta')^2 + \sin^2\theta' \cdot (\mathrm{d}\varphi')^2\right] \\
& - \left(1 - \frac{2m}{r'}\right) \cdot (\mathrm{d}t')^2.
\end{aligned}
\tag{5.28}
$$

Finally, *the fourth domain* and the corresponding coordinate transformation are given by

$$D_{\mathrm{IV}} := \left\{(u,v) \in \mathbb{R}^2 : u < 0,\ v < 0,\ uv < 1\right\};$$

$$T' = \mathcal{Y}'^{\#}(u,v) := \widetilde{\mathcal{Y}}(u,v)_{|D_{\mathrm{IV}}} = (2m) \cdot \left[1 + \mathsf{W}\left(-\mathrm{e}^{-1}uv\right)\right]_{|D_{\mathrm{IV}}},$$

$$R' = (2m) \cdot \ln(v/u)_{|..}. \tag{5.29}$$

The above equations imply the four-dimensional, (dual) T-domain metric:

$$\mathrm{d}s^2 = \left(\frac{2m}{T'} - 1\right) \cdot (\mathrm{d}R')^2 + (T')^2 \cdot \left[(\mathrm{d}\theta')^2 + \sin^2\theta' \cdot (\mathrm{d}\varphi')^2\right]$$
$$- \left(\frac{2m}{T'} - 1\right)^{-1} \cdot (\mathrm{d}T')^2. \tag{5.30}$$

The four domains of (5.25)–(5.27), and (5.29) and the corresponding coordinate transformations are succinctly depicted in Fig. 5.5.

Remarks: (i) In Fig. 5.5, the const. $\varepsilon > 0$ denotes a sufficiently small, positive constant.

(ii) An equation like "$\widehat{r} = 2m$" indicates the image of some boundary points in the corresponding $\widehat{r} - \widehat{t}$ plane.

Now, we shall interpret Fig. 5.5 in physical terms. For that purpose, we make another coordinate transformation by

$$\overset{\circ}{x} = (1/2) \cdot (v - u),$$

$$\overset{\circ}{t} = (1/2) \cdot (v + u),$$

$$\overset{\circ}{D} := \left\{\left(\overset{\circ}{x}, \overset{\circ}{t}\right) \in \mathbb{R}^2 : -1 < \left(\overset{\circ}{x}\right)^2 - \left(\overset{\circ}{t}\right)^2 < \infty\right\}. \tag{5.31}$$

Evidently, $\overset{\circ}{x}$ is a spacelike coordinate and, $\overset{\circ}{t}$ is a timelike coordinate for $(\overset{\circ}{x})^2 > (\overset{\circ}{t})^2$ in $\overset{\circ}{D} \subset \mathbb{R}^2$.

The domain $\overset{\circ}{D}$ can be obtained from Fig. 5.5 by *rotating the whole figure* by 45° and relabelling four subsets. Thus, Fig. 5.6 below originates.

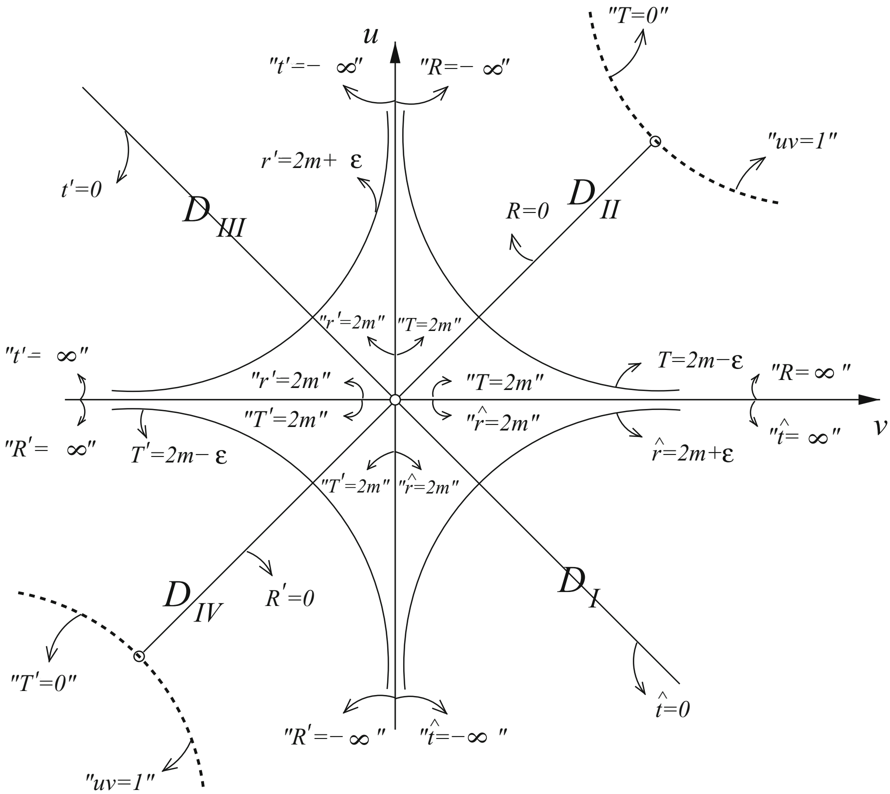


Fig. 5.5 Four domains covered by the doubly null, $u - v$ coordinate chart

We shall now physically interpret Fig. 5.6. We note that radial, null geodesics from (5.19) and (5.31) are governed by the differential equations:

$$(\cdots)_{|..} \cdot \left\{ \left[\frac{\mathrm{d}\overset{\circ}{\mathcal{X}}(\alpha)}{\mathrm{d}\alpha} \right]^2 - \left[\frac{\mathrm{d}\overset{\circ}{\mathcal{T}}(\alpha)}{\mathrm{d}\alpha} \right]^2 \right\} = 0,$$

$$\text{or} \qquad u = \overset{\circ}{t} - \overset{\circ}{x} = \text{const.},$$

$$\text{and} \qquad v = \overset{\circ}{t} + \overset{\circ}{x} = \text{const.} \tag{5.32}$$

Therefore, straight lines, parallel to u-axis or v-axis in Fig. 5.6, represent *all possible radial, null geodesics*. Among these, straight lines parallel to the u-axis (oriented upward) represent *incoming null rays* falling toward the center. On the

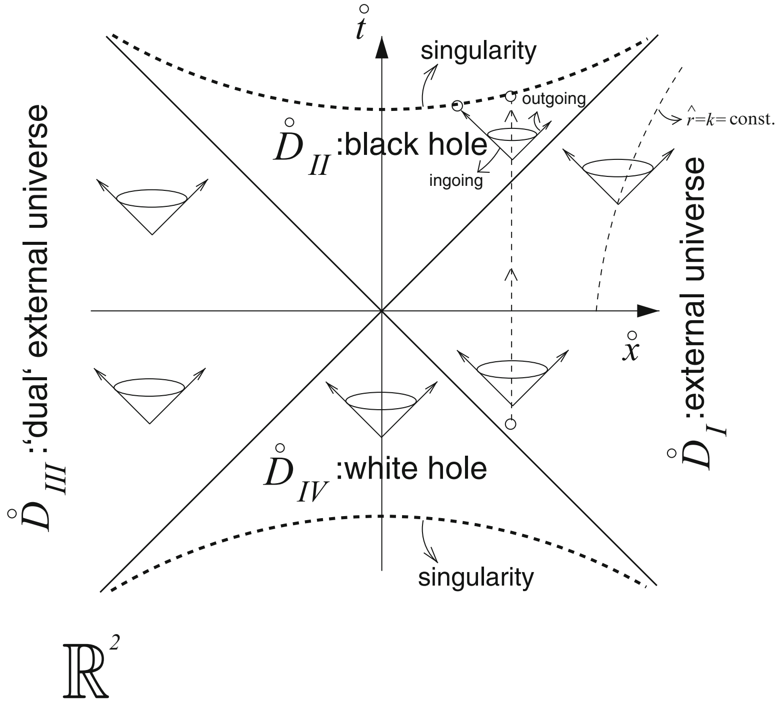


Fig. 5.6 The maximal extension of the Schwarzschild chart

other hand, straight lines parallel to the v-axis represent *outgoing null rays*. (These interpretations of "incoming" and "outgoing" are valid only on *the right-hand side* of the u-axis.)

If an (idealized) observer (not necessarily geodesic) is represented by the oriented vertical line (on the right-hand side of the u-axis in Fig. 5.6), then it is clear to visualize forward light cones, incoming light rays, and outgoing light rays emanating along this timelike world line. It is evident from Fig. 5.6 that all incoming rays originating in $\overset{\circ}{D}_{\mathrm{I}}$ and crossing the event horizon *end up at the singularity* at $(\overset{\circ}{x})^2 - (\overset{\circ}{t})^2 = -1$. On the other hand, outgoing rays emitted in $\overset{\circ}{D}_{\mathrm{I}}$, *cannot* cross the event horizon and remain forever in the external (Schwarzschild) universe. However, an incoming ray or an outgoing ray, emitted in the domain $\overset{\circ}{D}_{\mathrm{II}}$, remains *inside the domain* $\overset{\circ}{D}_{\mathrm{II}}$. Or, in other words, neither incoming nor outgoing rays can cross the event horizon to come out of the domain $\overset{\circ}{D}_{\mathrm{II}}$ into the external universe. Now, consult Fig. 5.1 in the Lemaître chart. There, timelike radial geodesics cross the event horizon to terminate on the singularity. Also, in Fig. 5.6, the world line (parallel to the $\overset{\circ}{t}$-axis) crosses the event horizon never to come out of the domain $\overset{\circ}{D}_{\mathrm{II}}$. Therefore, this domain $\overset{\circ}{D}_{\mathrm{II}}$ is appropriately called a *black hole*. However, not all timelike world lines originating in $\overset{\circ}{D}_{\text{I}}$ end up inside the black hole $\overset{\circ}{D}_{\text{II}}$. For example, consider an accelerated observer in a rocket who traverses the world line $\widehat{r} = k > 2m$, $\widehat{\theta} = \text{const.}$, $\widehat{\varphi} = \text{const.}$ His or her world line is depicted by a rectangular hyperbola with asymptote $\widehat{r} = 2m$ in $\overset{\circ}{D}_{\text{I}}$. (Compare the image of $\widehat{r} = 2m + \varepsilon$ in Fig. 5.5.) Let the world line start from the event $\widehat{r} = k$, $\widehat{t} = 0$. The proper time along the world line is given by (3.42) as

$$\tau = \mathcal{T}\big(\widehat{t}\,\big) = \left(1 - \frac{2m}{k}\right)^{1/2} \cdot \widehat{t}\,.$$

Therefore, $\lim\limits_{\widehat{t}\to\infty} \mathcal{T}\big(\widehat{t}\,\big) \longrightarrow \infty$. Thus, this observer will *never reach the event horizon* nor will he observe the black hole in his life time! (This statement is pictorially obvious in Fig. 5.5.)

Now, let us discuss the domain $\overset{\circ}{D}_{\text{IV}}$. Light rays come out into both external universes $\overset{\circ}{D}_{\text{I}}$ and $\overset{\circ}{D}_{\text{III}}$. However, no light ray can enter into $\overset{\circ}{D}_{\text{IV}}$ from other domains. Similarly, massive particles can only come out of this domain and cannot enter into it. Therefore, the domain $\overset{\circ}{D}_{\text{IV}}$ is appropriately named as a *white hole*.

Every photon originating inside the black hole $\overset{\circ}{D}_{\text{II}}$ must end up into the singularity (at $(\overset{\circ}{x})^2 - (\overset{\circ}{t})^2 = -1$). No photon or massive particle from $\overset{\circ}{D}_{\text{I}}$ or $\overset{\circ}{D}_{\text{II}}$ can travel into the dual universe $\overset{\circ}{D}_{\text{III}}$ or the white hole $\overset{\circ}{D}_{\text{IV}}$. (However, in case tachyons exist in nature, they can cross these causal barriers!)

Finally, we note that the singularities in the black hole $\overset{\circ}{D}_{\text{II}}$ and the white hole $\overset{\circ}{D}_{\text{IV}}$ are *spacelike*.

Now let us express explicitly the four-dimensional, maximal extension of the Schwarzschild chart. We use (5.19), (5.24)–(5.27), and (5.29) to write

$$\mathrm{d}s^2 = -\frac{32\cdot(m)^3}{\tilde{r}} \cdot \exp\left[-\frac{\tilde{r}}{2m}\right] \cdot (\mathrm{d}u)(\mathrm{d}v) + (\tilde{r})^2 \cdot \left[\left(\mathrm{d}\tilde{\theta}\,\right)^2 + \sin^2\tilde{\theta} \cdot (\mathrm{d}\tilde{\varphi})^2\right], \tag{5.33i}$$

$$= -\frac{32\cdot(m)^3}{\widetilde{\mathcal{Y}}(u,v)} \cdot \exp\left[-\frac{\widetilde{\mathcal{Y}}(u,v)}{2m}\right] \cdot (\mathrm{d}u)(\mathrm{d}v) + \left[\widetilde{\mathcal{Y}}(u,v)\right]^2 \cdot \left[\left(\mathrm{d}\tilde{\theta}\,\right)^2 + \sin^2\tilde{\theta} \cdot (\mathrm{d}\tilde{\varphi})^2\right], \tag{5.33ii}$$

$$\widetilde{D}_4 := \left\{\left(u, v, \tilde{\theta}, \tilde{\varphi}\right) : -\infty < uv < 1,\ 0 < \tilde{\theta} < \pi,\ -\pi < \tilde{\varphi} < \pi\right\}, \tag{5.33iii}$$

$$\mathrm{d}s^2_{|uv=0} = = 4m^2 \cdot \left[\left(\mathrm{d}\tilde{\theta}\,\right)^2 + \sin^2\tilde{\theta} \cdot (\mathrm{d}\tilde{\varphi})^2\right]. \tag{5.33iv}$$

Remarks: (i) The symbol $\tilde{r}$ in (5.33i) *can be misconstrued as a spacelike coordinate*. In fact, $\tilde{r}$ is *spacelike for* $0 < 2m < \tilde{r}$ and *timelike for* $0 < \tilde{r} < 2m$.

(ii) There is a loss of two dimensions in (5.33iv) since the event horizon characterized by $uv = 0$ is *a null hypersurface*. (Compare with (5.8) and (5.17i).)

(iii) By the footnote on page 360, (5.33iv) can be deduced.

Now, we use the coordinate transformation in (5.31) to obtain from (5.33ii):

$$\begin{aligned} ds^2 = {} & \frac{32 \cdot (m)^3}{\mathcal{Y}\left(\mathring{x}, \mathring{t}\right)} \cdot \exp\left[-\frac{\mathring{\mathcal{Y}}\left(\mathring{x}, \mathring{t}\right)}{2m}\right] \cdot \left[-\left(d\mathring{t}\right)^2 + \left(d\mathring{x}\right)^2\right] \\ & + \left[\mathring{\mathcal{Y}}\left(\mathring{x}, \mathring{t}\right)\right]^2 \cdot \left[\left(d\mathring{\theta}\right)^2 + \sin^2\mathring{\theta} \cdot \left(d\mathring{\varphi}\right)^2\right], \end{aligned} \tag{5.34i}$$

$$\mathring{D}_4 := \left\{\left(\mathring{t}, \mathring{x}, \mathring{\theta}, \mathring{\varphi}\right) : -1 < \left(\mathring{x}\right)^2 - \left(\mathring{t}\right)^2 < \infty, 0 < \mathring{\theta} < \pi, -\pi < \mathring{\varphi} < \pi\right\}. \tag{5.34ii}$$

Remarks: (i) The domain $\mathring{D}_4$ is an unbounded, simply connected, proper subset of *the coordinate space* $\mathbb{R}^4$.

(ii) The metrics in (5.33i,ii) and (5.34i) are all called the *Kruskal–Szekeres metrics*. (See [184].)

(iii) The Kruskal–Szekeres extensions of the Schwarzschild chart are all non-static. However, both the domains corresponding to $\mathring{D}_{\mathrm{I}}$ and $\mathring{D}_{\mathrm{III}}$ (of Fig. 5.6) are transformable to *static metrics* by Birkhoff's Theorem 3.3.2.

(iv) The Kruskal–Szekeres metric (5.33i) is not asymptotically flat in the sense that $g_{uv} \to 0$ as $\tilde{r} \to \infty$. However, Synge's doubly null coordinate chart in (5.17i) is asymptotically flat in the sense that $g_{\hat{u}\hat{v}} \to 1$ as $\widehat{r} \to \infty$.

Finally, we express the metric (5.34i) *explicitly in terms of the Lambert* W-*Function* of (5.24) as

$$\begin{aligned} ds^2 = {} & \frac{16 \cdot (m)^2}{\left\{1 + \mathsf{W}\left[e^{-1}\left(\mathring{x}^2 - \mathring{t}^2\right)\right]\right\}} \cdot \exp\left\{-\left[1 + \mathsf{W}\left[e^{-1}\left(\mathring{x}^2 - \mathring{t}^2\right)\right]\right]\right\} \\ & \times \left[-\left(d\mathring{t}\right)^2 + \left(d\mathring{x}\right)^2\right] + \left(4 \cdot m^2\right) \cdot \left\{1 + \mathsf{W}\left[e^{-1}\left(\mathring{x}^2 - \mathring{t}^2\right)\right]\right\}^2 \\ & \times \left[\left(d\mathring{\theta}\right)^2 + \sin^2\mathring{\theta} \cdot \left(d\mathring{\varphi}\right)^2\right]. \end{aligned} \tag{5.35}$$

(The domain of validity of the metric above is exactly the same as in (5.34ii).)

Figures 5.1, 5.5, and 5.6 are all drawn in coordinate planes $\mathbb{R}^2$. Thus, these figures "appear" as if they belong to "Euclidean planes." The actual curved geometries, associated with these domains, are worthwhile to visualize. (Compare Figs. 1.4, 2.8, and 3.1.) That is why we shall investigate various, two-dimensional, curved submanifolds of the four-dimensional pseudo-Riemannian manifold corresponding to the metric in (5.35).

Firstly, consider the two-dimensional submanifold characterized by $\overset{\circ}{\theta} = \pi/2$ and $\overset{\circ}{t} = 0$. It is constructed from (5.35), (1.205), (1.217), and (A6.3) as the (positive-definite) Riemannian surface metric:

$$\begin{aligned} \mathrm{d}\sigma^2 = {} & \frac{16 \cdot (m)^2}{\left\{1 + \mathsf{W}\left[\mathrm{e}^{-1} \cdot \left(\overset{\circ}{x}\right)^2\right]\right\}} \cdot \exp\left\{-\left[1 + \mathsf{W}\left[\mathrm{e}^{-1}\left(\overset{\circ}{x}\right)^2\right]\right]\right\} \cdot \left(\mathrm{d}\overset{\circ}{x}\right)^2 \\ & + \left(4 \cdot m^2\right) \cdot \left\{1 + \mathsf{W}\left[\mathrm{e}^{-1}\left(\overset{\circ}{x}\right)^2\right]\right\}^2 \cdot \left(\mathrm{d}\overset{\circ}{\varphi}\right)^2; \\ \overset{\circ}{D} := {} & \left\{\left(\overset{\circ}{x}, \overset{\circ}{\varphi}\right) : -\infty < \overset{\circ}{x} < \infty,\ -\pi < \overset{\circ}{\varphi} < \pi\right\}. \end{aligned} \tag{5.36}$$

The metric above yields a surface of revolution around the x^3-axis in $\mathbb{R}^3$ by the following profile curve:

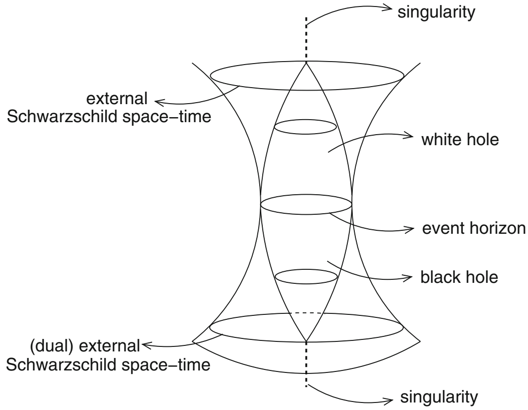
$$\begin{aligned} & \left(x^1, x^2, x^3\right) = \left(f\left(\overset{\circ}{x}\right), 0, h\left(\overset{\circ}{x}\right)\right) \in \mathbb{R}^3, \\ & f\left(\overset{\circ}{x}\right) := (2m) \cdot \left\{1 + \mathsf{W}\left[\mathrm{e}^{-1}\left(\overset{\circ}{x}\right)^2\right]\right\} > 0, \\ & h\left(\overset{\circ}{x}\right) := (4m) \cdot \int_0^{\overset{\circ}{x}} \frac{\mathrm{e}^{-(1/2)\cdot(1+\mathsf{W})}}{[1 + \mathsf{W}]} \cdot \sqrt{1 + \mathsf{W} - y^2 \cdot \mathrm{e}^{-(1+\mathsf{W})}} \cdot \mathrm{d}y. \end{aligned} \tag{5.37}$$

The qualitative picture of the surface is provided by *a wormhole* in Fig. A6.2.

Now, consider the two-dimensional, *pseudo-Riemannian surface* of revolution characterized by $\overset{\circ}{\theta} = \pi/2$, $\overset{\circ}{x} = 0$. The metric is provided by

$$\begin{aligned} \mathrm{d}\tau^2 = {} & -\frac{16 \cdot (m)^2}{\left\{1 + \mathsf{W}\left[-\mathrm{e}^{-1}\left(\overset{\circ}{t}\right)^2\right]\right\}} \cdot \exp\left\{-\left[1 + \mathsf{W}\left[-\mathrm{e}^{-1}\left(\overset{\circ}{t}\right)^2\right]\right]\right\} \cdot \left(\mathrm{d}\overset{\circ}{t}\right)^2 \\ & + \left(4 \cdot m^2\right) \cdot \left\{1 + \mathsf{W}\left[-\mathrm{e}^{-1}\left(\overset{\circ}{t}\right)^2\right]\right\}^2 \cdot \left(\mathrm{d}\overset{\circ}{\varphi}\right)^2. \end{aligned} \tag{5.38}$$

Fig. 5.7 Intersection of two surfaces of revolution in the maximally extended Schwarzschild universe



The two surfaces associated with the metrics in (5.36) and (5.38) *intersect along the curve* characterized by $\overset{\circ}{x} = \overset{\circ}{t} = 0$, providing the one-dimensional metric:

$$\begin{aligned} \mathrm{d}l^2 &= \left(4 \cdot m^2\right) \cdot \{1 + \mathsf{W}[0]\}^2 \cdot \left(\mathrm{d}\overset{\circ}{\varphi}\right)^2 \\ &= (2m)^2 \cdot \left(\mathrm{d}\overset{\circ}{\varphi}\right)^2 . \end{aligned} \tag{5.39}$$

The above yields a closed "circle," after we topologically identify the end points of the interval $(-\pi, \pi)$ for the coordinate $\overset{\circ}{\varphi}$. (In higher dimension, the intersection is on a "sphere" belonging to *the event horizon*.) Therefore, we can represent *qualitatively* the intersecting surfaces of revolution in Fig. 5.7

Now, we go back to the Eddington–Finkelstein metric of (5.15ii) for some physical clarification of the spherically symmetric black hole. The two-dimensional pseudo-Riemannian metric arising out of (5.15ii), under the restrictions $\theta = \pi/2$ and $\phi =$ const., is furnished by

$$(\mathrm{d}\tau)^2 = -\left(1 - \frac{2m}{\widehat{r}}\right) \cdot (\mathrm{d}\widehat{v})^2 + 2 \cdot (\mathrm{d}\widehat{r})(\mathrm{d}\widehat{v}). \tag{5.40}$$

Radial null geodesics parametrized by $\widehat{r}$ are provided by the ordinary differential equations:

$$-\left(1 - \frac{2m}{\widehat{r}}\right) \cdot \left(\frac{\mathrm{d}\widehat{v}}{\mathrm{d}\widehat{r}}\right)^2 + 2 \cdot \left(\frac{\mathrm{d}\widehat{v}}{\mathrm{d}\widehat{r}}\right) = 0, \tag{5.41i}$$

$$\frac{\mathrm{d}\widehat{v}}{\mathrm{d}\widehat{r}} = 0, \tag{5.41ii}$$

$$\text{or} \qquad \frac{\mathrm{d}\widehat{v}}{\mathrm{d}\widehat{r}} = 2 \cdot \left(1 - \frac{2m}{\widehat{r}}\right)^{-1} . \tag{5.41iii}$$

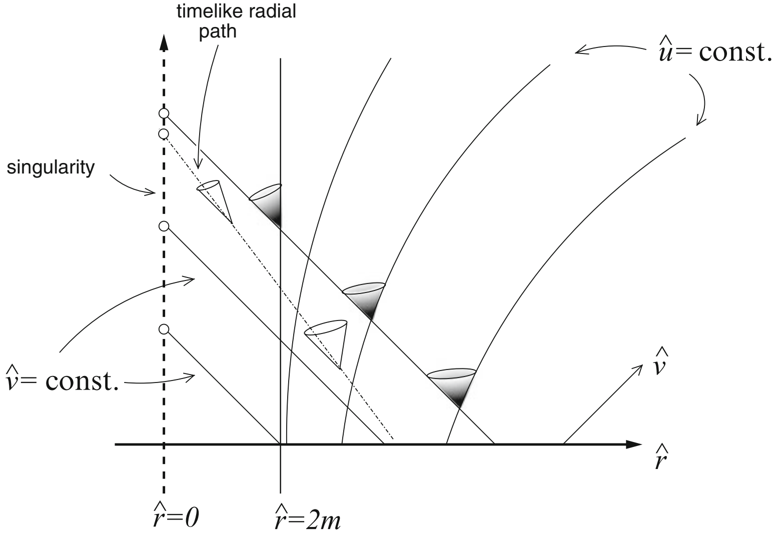


Fig. 5.8 Eddington–Finkelstein coordinates ($\hat{u}$, $\hat{v}$) describing the black hole. The *vertical lines* $\hat{r} = 2m$ and $\hat{r} = 0$ indicate the event horizon and the singularity, respectively

Equation (5.41ii) yielding $\widehat{v} =$ const. provides *ingoing null geodesics*. On the other hand, the solutions of (5.41iii) provide *outgoing null geodesics*. We shall depict the phenomenon of the spherically symmetric black hole in Eddington–Finkelstein coordinates $(\widehat{u}, \widehat{v})$ in Fig. 5.8.

Now, we shall investigate a spherically symmetric stellar body collapsing into a black hole. For the sake of simplicity, we choose the material content of the body as an incoherent dust of (2.256). A convenient coordinate chart for this investigation is the Tolman–Bondi–Lemaître chart (3.88) yielding

$$\begin{aligned} ds^2 &= e^{2\lambda(r,t)} \cdot (dr)^2 + [Y(r,t)]^2 \cdot \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right] - (dt)^2, \\ D_4 &:= \{(r,\theta,\varphi,t) : c < r < b,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ t_1 < t < t_2\}. \end{aligned} \tag{5.42}$$

We choose *comoving coordinates* (3.89) for the incoherent dust so that

$$\begin{aligned} T_{ij}(\cdot\cdot) &= \rho(\cdot\cdot) \cdot U_i(\cdot\cdot) \cdot U_j(\cdot\cdot), \\ U^\alpha(r,t) &\equiv 0, \\ U^4(r,t) &\equiv 1. \end{aligned} \tag{5.43}$$

The corresponding field equations are already written in (3.90i–vi). *A general class of solutions* are characterized in (3.92), dropping the bar on $\overline{M}(r)$, as

$$e^{2\lambda(r,t)} = \left\{ \frac{(\partial_1 Y)^2}{1 - \varepsilon[f(r)]^2} \right\} > 0, \tag{5.44i}$$

$$\frac{1}{2}(\partial_4 Y)^2 - \left[\frac{M(r)}{Y(r,t)} \right] = -\frac{\varepsilon}{2}[f(r)]^2, \tag{5.44ii}$$

$$\varepsilon = 0, \pm 1, \tag{5.44iii}$$

$$\rho(r,t) := \left\{ \frac{2 \cdot [\partial_1 M(r)]}{\kappa \cdot [Y(\cdot\cdot)]^2 \cdot [\partial_1 Y]} \right\} > 0. \tag{5.44iv}$$

Here, $f \in C^3(\cdot\cdot)$, $M \in C^4(\cdot\cdot)$ are arbitrary functions of integration. Moreover, all the field equations have reduced to *one differential equation* (5.44ii). (This equation physically implies the conservation of the total energy for a unit mass test particle.)

We consider here only the parabolic case with $\varepsilon = 0$. (Physically, it means that the potential energy *exactly cancels* the kinetic energy.) The field equation (5.44ii) reduces to *two first-order differential equations*:

$$\partial_4 Y = \pm\sqrt{2M(r)/Y(r,t)}\,. \tag{5.45}$$

For the phenomenon of gravitational *collapse*, we must choose *the negative sign* for the right-hand side of (5.45). Thus, the general solution of the differential equation is furnished by

$$Y(r,t) = (3/2)^{2/3} \cdot [2M(r)]^{1/3} \cdot [T(r) - t]^{2/3}\,. \tag{5.46}$$

Here, $T \in C^3(\cdot\cdot)$ is the arbitrary function of integration. For the sake of simplicity, we choose $T(r) = r$ so that (5.46) yields

$$Y(r,t) = (3/2)^{2/3} \cdot [2M(r)]^{1/3} \cdot (r-t)^{2/3}, \tag{5.47i}$$

$$\partial_4 Y = -(3/2)^{-1/3} \cdot [2M(r)]^{1/3} \cdot (r-t)^{-1/3}, \tag{5.47ii}$$

$$\begin{aligned}\partial_1 Y &= (3/2)^{2/3} \cdot [2M(r)]^{1/3} \cdot (r-t)^{2/3} \cdot \left\{ \frac{2}{3(r-t)} + \frac{1}{3M(r)} \cdot \frac{dM(r)}{dr} \right\} \\ &= \frac{1}{3} \cdot Y(r,t) \cdot \left\{ [\ln M(r)]' + \frac{2}{(r-t)} \right\} > 0,\end{aligned} \tag{5.47iii}$$

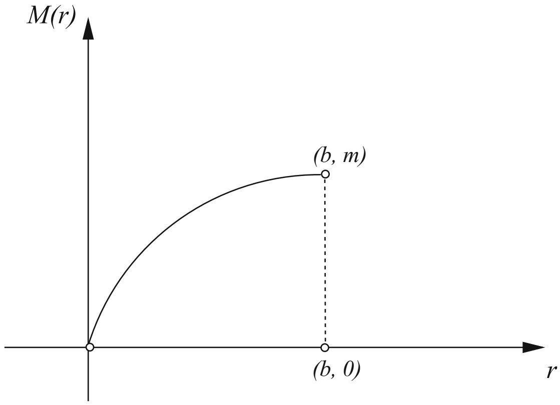


Fig. 5.9 Qualitative graph of $M(r)$

$$(\partial_1 Y)^2 = \left[\frac{2M(r)}{Y(r,t)}\right] \cdot \left\{1 + \frac{1}{2} \cdot (r-t) \cdot [\ln M(r)]'\right\}^2, \tag{5.47iv}$$

$$\lim_{r \to 0_+} Y(r,t) \equiv 0, \quad \lim_{(r-t) \to 0_+} Y(r,t) \equiv 0, \quad \lim_{t \to -\infty} [Y(r,t)] \longrightarrow \infty. \tag{5.47v}$$

Here, we have assumed that the total mass function $M(r)$ is a positive-valued, monotone-increasing function of class C^4 in the interval $(0, b) \subset \mathbb{R}$ with the following properties:

$$\begin{aligned}
&\lim_{r \to 0_+} [M(r)] = 0, \\
&\lim_{r \to 0_+} \left[M'(r)\right] > 0, \\
&\lim_{r \to b_-} \left[M'(r)\right] = 0, \\
&\lim_{r \to b_-} [M(r)] = m = \text{positive const.}
\end{aligned} \tag{5.48}$$

(See Fig. 5.9.)

Now, we shall extend the interior mass function $M(r)$ and the interior solution $Y(r,t)$ to the exterior domains by the following:

$$\widetilde{M}(r) := \begin{cases} M(r) & \text{for } 0 < r < b, \\ m & \text{for } b < r < \infty\,; \end{cases} \tag{5.49i}$$

$$\widetilde{Y}(r,t) := \begin{cases} (3/2)^{2/3} \cdot [2M(r)]^{1/3} \cdot (r-t)^{2/3} & \text{for } 0 < r < b \text{ and } r > t, \\[2ex] (3/2)^{2/3} \cdot [2m]^{1/3} \cdot (r-t)^{2/3} & \text{for } b < r < \infty \text{ and } r > t\,; \end{cases} \tag{5.49ii}$$

$$D_{\text{int}} := \left\{(r,t) \in \mathbb{R}^2 : 0 < r < b,\ r > t\right\}; \tag{5.49iii}$$

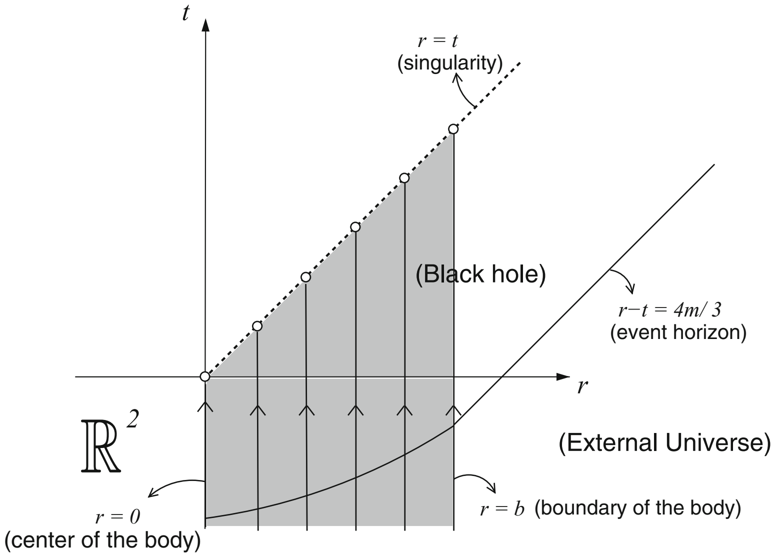


Fig. 5.10 Collapse of a dust ball into a black hole in a Tolman-Bondi-Lemaître chart

$$D_{\text{ext}} := \left\{(r,t) \in \mathbb{R}^2 : \ b < r < \infty, \ r > t\right\}; \tag{5.49iv}$$

$$\begin{aligned} D &:= \left\{(r,t) \in \mathbb{R}^2 : \ 0 < r < \infty, \ r > t\right\} \\ &= D_{\text{int}} \cup D_{\text{ext}} \cup \{\text{boundary}\}. \end{aligned} \tag{5.49v}$$

The four-dimensional metric (5.42) reduces to

$$\begin{aligned} ds^2 = {} & \left[\frac{2\widetilde{M}(r)}{\widetilde{Y}(r,t)}\right] \cdot \left\{1 + \frac{1}{2} \cdot (r-t) \cdot \left[\ln \widetilde{M}(r)\right]'\right\}^2 \cdot (dr)^2 \\ & + \left[\widetilde{Y}(r,t)\right]^2 \cdot \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right] - (dt)^2, \end{aligned} \tag{5.50}$$

$$D_4 := \{(r,\theta,\varphi,t) : \ 0 < r < \infty, \ 0 < \theta < \pi, \ -\pi < \varphi < \pi, \ -\infty < t < r\}.$$

Note that the above metric, restricted to the external domain D_{ext} of (5.49iv), *reduces exactly to the Lemaître metric* in (5.2). Moreover, the interior metric components, and their derivatives up to the third order, *join continuously* to those of the exterior metric components.

Now, we utilize (5.42), (5.43), (5.49i)–(5.49iii), and (5.50) to depict the collapse of a general class of dust balls into black holes in Fig. 5.10.

Remarks: (i) The coordinates r and t are *spacelike and timelike, respectively, everywhere* in the space–time domain D_4 of (5.50).

(ii) The congruence of oriented, vertical lines in Fig. 5.10 indicate *timelike, radial geodesics*. These represent world lines of a special class of dust particles collapsing into a black hole and eventually ending up into the singularity at $r = t$. (Note that $\lim_{t \to r_-} \left[\widetilde{Y}(r,t)\right] \equiv 0$.)

(iii) By (5.48) and (5.47i), it is clear that $\lim_{r \to 0_+} [Y(r,t)] \equiv 0$ *for* $t < 0$. Therefore, the corresponding interval on the t-axis represents physically *the center of the material body*.

(iv) The spherically symmetric body is starting to collapse from rest in the "infinite past."

Next we shall investigate coordinate transformations in *the external universe*. Using the inverse coordinate transformation of (5.3), the coordinate transformation from the Kruskal–Szekeres coordinate chart of equation (5.33ii) (suppressing angular coordinates) into the Lemaître chart (5.2) is furnished by:

$$\begin{aligned} r &= (4m) \cdot \sqrt{1 + \mathsf{W}\left(-\mathrm{e}^{-1}uv\right)} + \left(\frac{4m}{3}\right) \cdot [1 + \mathsf{W}(\cdot\cdot)]^{3/2} \\ &\quad + (2m) \cdot \ln\left\{\frac{v \cdot \left[1 - \sqrt{1+\mathsf{W}(\cdot\cdot)}\right]}{u \cdot \left[1 + \sqrt{1+\mathsf{W}(\cdot\cdot)}\right]}\right\} \quad \text{for } u \neq 0,\ v > 0, \\ &\quad \left(\frac{10m}{3}\right) + (4m) \cdot \ln(v/2) \quad \text{for} \quad u = 0,\ v > 0\,; \end{aligned} \tag{5.51}$$

$$\begin{aligned} t &= (4m) \cdot \sqrt{1 + \mathsf{W}(\cdot\cdot)} + (2m) \cdot \ln\left\{\frac{v \cdot \left[1 - \sqrt{1+\mathsf{W}(\cdot\cdot)}\right]}{u \cdot \left[1 + \sqrt{1+\mathsf{W}(\cdot\cdot)}\right]}\right\} \quad \text{for } u \neq 0,\ v > 0, \\ &\quad (2m) + (4m) \cdot \ln(v/2) \quad \text{for} \quad u = 0,\ v > 0. \end{aligned}$$

Using (5.51) and Fig. 5.10, we demonstrate the implosion of the spherically symmetric stellar body into a black hole in the arena of the Kruskal–Szekeres coordinate chart (with suppressed angular coordinates) in Fig. 5.11.

Remark: In the Kruskal–Szekeres chart of Fig. 5.11, *the collapse into a black hole is fully covered in domain* $D_{\mathrm{I}} \cup D_{\mathrm{II}} \cup \{\text{boundary}\}$. The domains D_{III} and D_{IV} of Fig. 5.5 are *not needed*.

Figures 5.10 and 5.11 are drawn in domains of the (Euclidean) coordinate plane $\mathbb{R}^2$. The actual curved geometry of the corresponding space–time is *not revealed*. For that purpose, consider the two-dimensional (positive-definite) surface of revolution characterized by $\theta = \pi/2$ and $t = t_0 < 0$. The corresponding metric

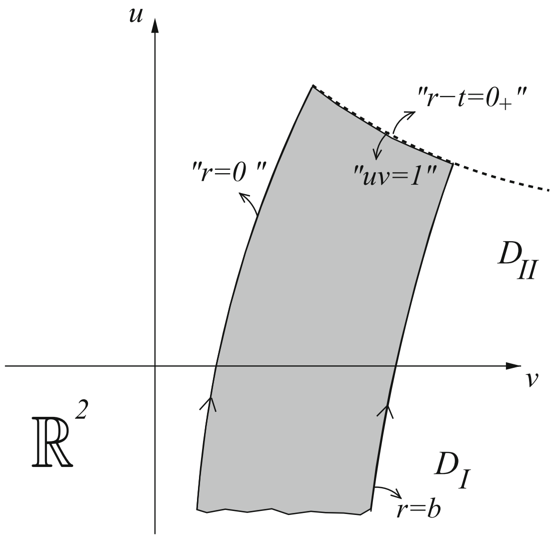


Fig. 5.11 Collapse of a dust ball into a black hole in Kruskal–Szekeres coordinates

from (5.50) is given by

$$d\sigma^2 = \left[\frac{2\widetilde{M}(r)}{\widetilde{Y}(r,t_0)}\right] \cdot \left\{1 + \frac{1}{2}\cdot(r-t_0)\cdot\left[\ln M(r)\right]'\right\}^2 \cdot (dr)^2 + \left[\widetilde{Y}(r,t_0)\right]^2 \cdot (d\varphi)^2. \tag{5.52}$$

By Example 3.1.2, we can work out, from (5.52), the following ratio:

$$\frac{\text{"Circumference of a circle"}}{\text{"Radial distance"}}$$

$$= \frac{(2\pi)\cdot\widetilde{Y}(r,t_0)}{\int\limits_{0_+}^{r}\sqrt{\frac{2\widetilde{M}(w)}{\widetilde{Y}(w,t_0)}}\cdot\left\{1+\frac{1}{2}\cdot(w-t_0)\cdot\left[\ln\widetilde{M}(w)\right]'\right\}\cdot dw}. \tag{5.53}$$

The qualitative pictures of the surface of revolution from (5.53) in three different "times" are plotted in Fig. 5.12.

The spherically symmetric boundary of the collapsing star and *the event horizon* are furnished, respectively, by metrics (recall $\theta = \pi/2$):

$$d\tau^2 = [Y(b,t)]^2 \cdot (d\varphi)^2 - dt^2, \tag{5.54i}$$

$$\text{and} \quad dl^2 = (2m)^2 \cdot (d\varphi)^2. \tag{5.54ii}$$

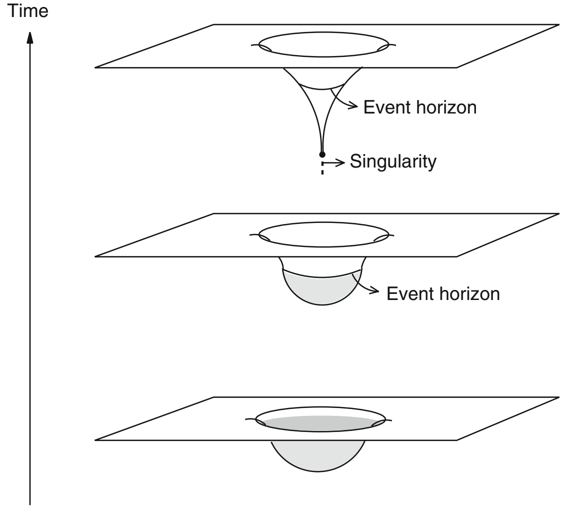


Fig. 5.12 Qualitative representation of a collapsing spherically symmetric star in three instants

Remarks: (i) In Fig. 5.13, the spherically symmetric body is collapsing from the "infinite past."

(ii) We have *tacitly assumed* that there exists no curvature singularity at the center of the body at early times.

We have a general class of space–time metrics in (5.50) representing the collapse of a dust ball into a black hole. To check the regularity of the space–time domain, we need to examine the *orthonormal components* of the curvature tensor. The nonzero components are furnished by the following:

$$R_{(2)(3)(2)(3)}(r,t) = \left[\frac{2}{3(r-t)}\right]^2, \tag{5.55i}$$

$$R_{(2)(4)(2)(4)}(r,t) = \frac{1}{2}\cdot\left[\frac{2}{3(r-t)}\right]^2 = R_{(3)(4)(3)(4)}(r,t), \tag{5.55ii}$$

$$-R_{(1)(4)(1)(4)}(r,t) = \left[\frac{2}{3(r-t)}\right]^2 - \frac{2M'(r)}{3(r-t)\cdot[2M(r)+(r-t)\cdot M'(r)]}, \tag{5.55iii}$$

$$\begin{aligned} -2R_{(1)(2)(1)(2)}(r,t) &= \left[\frac{2}{3(r-t)}\right]^2 - \frac{4M'(r)}{3(r-t)\cdot[2M+(r-t)\cdot M']} \\ &= -2R_{(1)(3)(1)(3)}(r,t). \end{aligned} \tag{5.55iv}$$

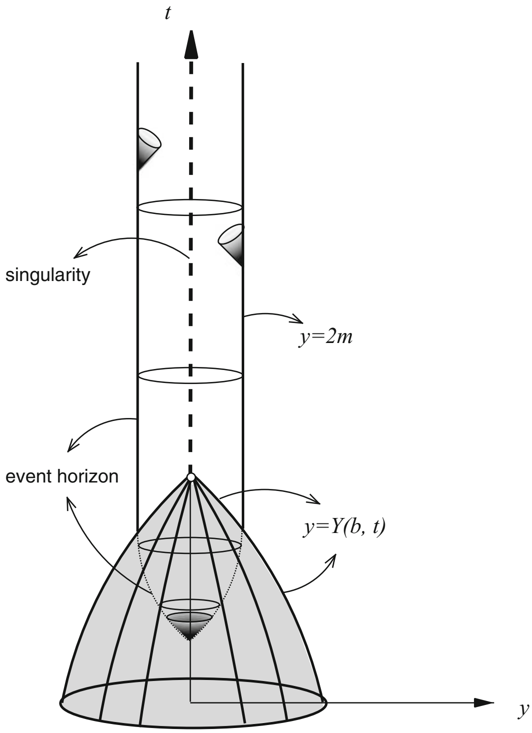


Fig. 5.13 Boundary of the collapsing surface and the (absolute) event horizon

Moreover, the mass density is provided by (3.97) as

$$\frac{\kappa}{2} \cdot \rho(r,t) = \frac{2M'(r)}{3\,[2 \cdot (r-t) \cdot M + (r-t)^2 \cdot M']} > 0. \tag{5.56}$$

Every orthonormal component of the curvature tensor and the mass density remain very smooth functions in the domain of consideration D_{int} given in (5.49iii). The conditions $\lim_{r\to 0_+} M(r) = 0$ and $\lim_{r\to 0_+} M'(r) > 0$ suffice to ensure the regularities of the curvature tensor and the mass density *at the center of the body*. It is interesting to note that *every nonzero component of the curvature tensor, and the mass density, diverges in the limit* $\lim (r-t) \to 0_+$, indicating the existence of the ultimate singularity!

Consider a bounded function:

$$F(r) := \mathrm{e}^{k_1} \cdot \tanh\,[h(r)]^2 + \mathrm{e}^{k_2} \cdot \mathrm{sech}\,[\,g(r)]\,. \tag{5.57}$$

Here, k_1 is an arbitrary constant of integration and two *slack functions* $h(r)$ and $g(r)$ are assumed to be continuous in $0 \leq r \leq b$ but otherwise arbitrary. We generate another function of class C^4 by successive integrations:

$$V(r) := \int_{0_+}^{r} \left[\int_{0_+}^{w_4} \left[\int_{0_+}^{w_3} \left[\int_{0_+}^{w_2} F\,(w_1) \cdot \mathrm{d}w_1 \right] \mathrm{d}w_2 \right] \mathrm{d}w_3 \right] \mathrm{d}w_4. \tag{5.58}$$

Finally, we define a class of total mass functions by

$$\begin{aligned} &M(r) := m \cdot \sin\left(\frac{\pi r}{2b}\right) \cdot \left[1 + \varepsilon (b-r)^{n+5} \cdot V(r)\right], \\ &n \in \{0, 1, 2, \ldots\}, \\ &\varepsilon > 0. \end{aligned} \tag{5.59}$$

We shall now state and prove a theorem about the class of mass functions in (5.59).

Theorem 5.1.2. *Let the total mass function* $M(r)$ *be defined by* (5.59) (*which involves one arbitrary constant and two arbitrary slack functions*)*. Then in the open interval* $(0, b)$*, the following equations and inequalities hold:*

$$\text{(i) } M(r) > 0, \qquad \text{(ii) } \lim_{r \to 0_+} M(r) = 0, \qquad \text{(iii) } M'(r) > 0,$$

$$\text{(iv) } \lim_{r \to b_-} M(r) = m, \qquad \text{(v) } \lim_{r \to b_-} M'(r) = 0.$$

Moreover, $\lim_{r \to 0_+} \left| R_{(a)(b)(c)(d)}(r,t) \right| < \mathrm{e}^{\beta}$*, thus, are bounded for* $t < 0$*. Furthermore, the resulting interior metric inherent in* (5.50) *joins very smoothly to the external Lemaître metric in* (5.2) *at the boundary* $r = b$*,* $t < -b$*. Synge's junction conditions* (2.170) *and I–S–L–D jump conditions* (2.171) *are all satisfied on the boundary. Finally, all three energy conditions* (2.190)– (2.192) *are valid inside the material body collapsing into a black hole.*

Proof. The parts (i)–(v) can be proved from the definition in (5.59) and (5.57) and (5.58). The part involving boundedness of the curvature tensor is provable from equations (5.55i–iv). Smooth joining of the interior to the exterior is evident from (5.59), (5.49i–v), and the resultant metric in (5.50).

Now, we shall check Synge's junction conditions at the boundary. The unit normal to this hypersurface is given by

$$n_i(\cdot\cdot)_{|..} = \sqrt{\frac{2m}{\widetilde{Y}(b,t)}} \cdot \delta^{(1)}{}_i.$$

But the four-velocity components are provided by $U^i(\cdot\cdot)_{|..} = \delta^i{}_{(4)}$. Thus, by (5.56),

$$T^{ij}(\cdot\cdot) \cdot n_j(\cdot\cdot)_{|..} = \rho(\cdot\cdot)_{|..} \cdot \sqrt{\frac{2m}{\widetilde{Y}(b,t)}} \cdot \delta^{(1)}{}_{(4)} \equiv 0.$$

To check the I–S–L–D junction conditions, we need to express boundary hypersurface parametrically as

$$\begin{aligned} r &= \xi^1\left(u^1, u^2, u^3\right) := b, \\ \theta &= \xi^2\left(\cdots\right) := u^1, \\ \varphi &= \xi^3\left(\cdots\right) := u^2, \\ t &= \xi^4\left(\cdots\right) := u^3. \end{aligned}$$

The induced hypersurface metric is furnished by

$$\overline{g}_{\alpha\beta}\left(u^1, u^2, u^3\right) \cdot (\mathrm{d}u^\alpha)\left(\mathrm{d}u^\beta\right) = \left[\widetilde{Y}\left(b, u^3\right)\right]^2 \cdot \left[\left(\mathrm{d}u^1\right)^2 + \sin^2 u^1 \cdot \left(\mathrm{d}u^2\right)^2\right] - \left(\mathrm{d}u^3\right)^2.$$

By (1.234), the nonvanishing components of the extrinsic curvature components are provided by

$$\begin{aligned} K_{11}\left(u^1, u^2, u^3\right) &= -\widetilde{Y}\left(b, u^3\right), \\ K_{22}\left(u^1, u^2, u^3\right) &= -\left(\sin u^1\right)^2 \cdot \widetilde{Y}\left(b, u^3\right). \end{aligned}$$

Obviously, the above functions are continuous across the boundary. Thus, I-S-L-D jump conditions hold.

Finally, to check the three energy conditions of (2.190)–(2.192), we note from (5.43) that *the only nonzero component* of $T_{ij}(\cdot\cdot)$ is given by

$$T_{44}(r,t) = \rho(r,t).$$

By (5.56) and the function $M(r)$ defined in (5.59), clearly $\rho(r,t) > 0$ in $D_{\text{int}} := \left\{(r,t) \in \mathbb{R}^2 : 0 < r < b,\ r > t\right\}$. Thus, all three energy conditions are validated. ■

Exercises 5.1

1. Consider the spherically symmetric doubly null coordinate chart specified by

$$\mathrm{d}s^2 = -4 \cdot [f(u,v)]^2 \cdot (\mathrm{d}u)(\mathrm{d}v) + \left[\widetilde{\mathcal{Y}}(u,v)\right]^2 \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right].$$

(i) Solve the vacuum field equations *explicitly* to obtain the Synge metric of (5.17i).

(ii) In the same coordinate chart, express Einstein's interior field equations for the case of a perfect fluid.

(iii) Show that the preceding field equations (of part (ii)) imply that the 1-form $\left[f(\cdot\cdot)\right]^{-2} \cdot \left[\partial_v \widetilde{\mathcal{Y}} \cdot \mathrm{d}u + \partial_u \widetilde{\mathcal{Y}} \cdot \mathrm{d}v\right]$ is *closed*.

2. Consider *the elliptic case* of the Tolman–Bondi–Lemaître metric in (3.92) (with $\varepsilon = +1$) yielding

$$g_{11}(r,t) = \mathrm{e}^{2\lambda(r,t)} = \frac{[\partial_1 Y]^2}{1 - [f(r)]^2}\,,$$

$$0 < f(r) < 1.$$

(i) Show that the remaining field equation (3.94) (for the collapse) is given by

$$\partial_4 Y = -\sqrt{\frac{2M(r)}{Y(r,t)} - [f(r)]^2} \,<\, 0.$$

(ii) Derive the general solution of the above differential equation in *the implicit form* as

$$t - T_0(r) = \left[\frac{Y(r,t)}{f(r)}\right] \cdot \sqrt{\frac{2M(r)}{[f(r)]^2 \cdot Y(r,t)} - 1}$$

$$+ \frac{2M(r)}{[f(r)]^3} \cdot \mathrm{Arctan}\sqrt{\frac{2M(r)}{[f(r)]^2 \cdot Y(r,t)} - 1}\,.$$

(Here, $T_0(r)$ is a smooth, arbitrary function arising out of integration.)

(iii) Smoothly join the interior black hole solution of part (ii) to the exterior Schwarzschild universe.

3. Consider the particular Tolman–Bondi–Lemaître metric

$$\mathrm{d}s^2 = \left[\partial_1 \widetilde{Y}\right]^2 \cdot (\mathrm{d}r)^2 + \left[\widetilde{Y}(r,t)\right]^2 \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right] - (\mathrm{d}t)^2$$

with

$$\widetilde{Y}(r,t) := \begin{cases} \left(\dfrac{3}{2}\right)^{2/3} \cdot (2m)^{1/3} \cdot \sin\left(\dfrac{\pi r}{2b}\right) \cdot [r - t - q\cdot(b-r)^n \cdot (t - \ln\cosh(t))]^{2/3} \\ \qquad \text{for } D_{\text{int}} := \{(r,t) : 0 < r < b, -\infty < t < \mathcal{T}(r)\}, \\ \left[(3/2)\cdot\sqrt{2m}\cdot(r-t)\right]^{2/3} \\ \qquad \text{for } D_{\text{ext}} := \{(r,t) : b < r, -\infty < t < \mathcal{T}(r)\}. \end{cases}$$

Here, $n \geq 5$, $0 < q < \frac{1}{2nb^n}$, and the curve $t = \mathcal{T}(r)$ is defined *implicitly by* $r - t + q\cdot(b-r)^n\cdot[t - \ln\cosh(t)] = 0$.

(i) Define a total mass function by

$$\widetilde{M}(r,t) := \begin{cases} m\cdot\sin^3\left(\dfrac{\pi r}{2b}\right)\cdot\{1 - q\cdot(b-r)^n\cdot[1-\tanh(t)]\}^2 & \text{for } D_{\text{int}}, \\ m \quad \text{for} & D_{\text{ext}}. \end{cases}$$

Show that the function $\widetilde{Y}(r,t)$ satisfies the collapse equation:

$$\partial_4\widetilde{Y} = -\sqrt{\frac{2\widetilde{M}(r,t)}{\widetilde{Y}(r,t)}} < 0.$$

(ii) Prove that the Tolman–Bondi–Lemaître metric of this problem represents the collapse of a spherically symmetric body of *Segre characteristic* $[1,(1,1),1]$ into a black hole.

4. Consider the Reissner–Nordström–Jeffery metric in (3.74), written with a different notation as

$$\mathrm{d}s^2 = \left[1 - \frac{2m}{\widehat{r}} + \left(\frac{e}{\widehat{r}}\right)^2\right]^{-1}\cdot(\mathrm{d}\widehat{r})^2 + (\widehat{r})^2\cdot\left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2\widehat{\theta}\cdot(\mathrm{d}\widehat{\varphi})^2\right]$$
$$-\left[1 - \frac{2m}{\widehat{r}} + \left(\frac{e}{\widehat{r}}\right)^2\right]\cdot\left(\mathrm{d}\widehat{t}\right)^2$$

for $0 < |e| < m$.

(i) Prove that orthonormal components of the corresponding curvature tensor are regular and differentiable at $\widehat{r} = \widehat{r}_+$ and $\widehat{r} = \widehat{r}_-$. (Here, $\widehat{r}_+$ and $\widehat{r}_-$ are two, distinct real roots of the quadratic equation $(\widehat{r})^2 - 2m\cdot\widehat{r} + (e)^2 = 0$.)

(ii) Construct the generalization of the Synge's doubly null coordinate chart in (5.17i) for the present metric.

Answers and Hints to Selected Exercises

1. (i) To obtain the vacuum field equations, put $\rho(\cdot\cdot) = p(\cdot\cdot) \equiv 0$ into the following answer to part (ii). The equations $G_{11}(\cdot\cdot) = G_{22}(\cdot\cdot) = 0$ can be solved to obtain $[f(u, v)]^2 = A(u) \cdot \partial_v \widetilde{\mathcal{Y}} = B(v) \cdot \partial_u \widetilde{\mathcal{Y}}$, where $A(u)$ and $B(v)$ are arbitrary functions of integration. Other field equations yield $[f(\cdot\cdot)]^2 = -[1 - 2m/\widetilde{\mathcal{Y}}(\cdot\cdot)] \cdot A(u) \cdot B(v)$.

   (ii)

$$G_{11} = \frac{4 \cdot \left(\partial_u \widetilde{\mathcal{Y}}\right) \cdot (\partial_u f)}{\widetilde{\mathcal{Y}}(\cdot\cdot) \cdot f(\cdot\cdot)} - \frac{2 \cdot \partial_u \partial_u \widetilde{\mathcal{Y}}}{\widetilde{\mathcal{Y}}(\cdot\cdot)} = \kappa \left[\rho(\cdot\cdot) + p(\cdot\cdot)\right] \cdot [f(\cdot\cdot)]^2,$$

$$G_{22} = \frac{4 \cdot \left(\partial_v \widetilde{\mathcal{Y}}\right) \cdot (\partial_v f)}{\widetilde{\mathcal{Y}}(\cdot\cdot) \cdot f(\cdot\cdot)} - \frac{2 \cdot \partial_v \partial_v \widetilde{\mathcal{Y}}}{\widetilde{\mathcal{Y}}(\cdot\cdot)} = \kappa \left[\rho(\cdot\cdot) + p(\cdot\cdot)\right] \cdot [f(\cdot\cdot)]^2,$$

$$G_{12} = \frac{2 \cdot [f(\cdot\cdot)]^2}{\left[\widetilde{\mathcal{Y}}(\cdot\cdot)\right]^2} + \frac{2 \cdot \left(\partial_u \widetilde{\mathcal{Y}}\right) \cdot \left(\partial_v \widetilde{\mathcal{Y}}\right)}{\left[\widetilde{\mathcal{Y}}(\cdot\cdot)\right]^2} + \frac{2 \cdot \partial_u \partial_v \widetilde{\mathcal{Y}}}{\widetilde{\mathcal{Y}}(\cdot\cdot)}$$

$$= \kappa \left[\rho(\cdot\cdot) - p(\cdot\cdot)\right] \cdot [f(\cdot\cdot)]^2,$$

$$G_{33} = -\frac{\widetilde{\mathcal{Y}}(\cdot\cdot) \cdot \partial_u \partial_v \widetilde{\mathcal{Y}}}{[f(\cdot\cdot)]^2} + \frac{\left[\widetilde{\mathcal{Y}}(\cdot\cdot)\right]^2 \cdot (\partial_u f) \cdot (\partial_v f)}{[f(\cdot\cdot)]^4} - \frac{\left[\widetilde{\mathcal{Y}}(\cdot\cdot)\right]^2 \cdot (\partial_u \partial_v f)}{[f(\cdot\cdot)]^3}$$

$$= \kappa p(\cdot\cdot) \left[\widetilde{\mathcal{Y}}(\cdot\cdot)\right]^2.$$

   (*Comment*: The above equations have been studied in detail in [19].)

2. (iii) Extend the function $M(r)$ into the external domain by (5.49i). Extend functions $f(r)$, $T_0(r)$ very smoothly similarly. Thus, extend the solution in part (ii) outside too. Make a coordinate transformation in the outside domain by

$$\xi = r,$$

$$\eta = 2 \cdot \text{Arctan} \sqrt{\frac{2m}{\left[\tilde{f}(r)\right]^2 \cdot \widetilde{Y}(r, t)} - 1}\,.$$

Make another successive transformation by

$$\widehat{r} = \frac{m \cdot (1 + \cos \eta)}{\left[\tilde{f}(\xi)\right]^2},$$

$$\widehat{t} = \int_b^{\xi} \frac{\widetilde{T}_0'(x)}{\sqrt{1 - \left[\tilde{f}(x)\right]^2}}\,\mathrm{d}x + m \cdot \sqrt{1 - \left[\tilde{f}(\xi)\right]^2} \cdot \left[\frac{\eta + \sin \eta}{\left[\tilde{f}(\xi)\right]^3} + \frac{2\eta}{\tilde{f}(\xi)}\right]$$

$$+ (2m) \cdot \ln \left[\frac{\tan(\eta_h(\xi)/2) + \tan(\eta/2)}{\tan(\eta_h(\xi)/2) - \tan(\eta/2)}\right],$$

$$\tan(\eta_h(\xi)/2) := \left[\tilde{f}(\xi)\right]^{-1} \cdot \sqrt{1 - \left[\tilde{f}(\xi)\right]^2},$$

$$\overset{\circ}{D}_{\text{ext}} := \{(\xi, \eta) : b < \xi,\ 0 < \eta < \eta_h < \pi\}.$$

The corresponding coordinate chart exactly yields the Schwarzschild metric in (5.1).

3. (ii) The nonzero components of the energy–momentum–stress tensor are provided by

$$\kappa\, T^1{}_1(\cdot\cdot) = -\frac{2 \cdot \partial_4 \widetilde{M}}{\left[\widetilde{Y}(\cdot\cdot)\right]^2 \cdot \partial_4 \widetilde{Y}},$$

$$\kappa\, T^2{}_2(\cdot\cdot) \equiv \kappa T^3{}_3(\cdot\cdot) = -\frac{1}{\left[\widetilde{Y}(\cdot\cdot)\right] \cdot \partial_1 \widetilde{Y}} \cdot \partial_1 \left[\frac{\partial_4 \widetilde{M}}{\partial_4 \widetilde{Y}}\right],$$

$$\kappa\, T^4{}_4(\cdot\cdot) = -\frac{2 \cdot \partial_1 \widetilde{M}}{\left[\widetilde{Y}(\cdot\cdot)\right]^2 \cdot \partial_1 \widetilde{Y}} < 0.$$

(*Comment*: To visualize many graphs on this collapse, see [66].)

4. (ii)

$$\widehat{r}_* := \int \left[1 - \frac{2m}{\widehat{r}} + \frac{\mathrm{e}^2}{\widehat{r}^2}\right]^{-1} \cdot \mathrm{d}\widehat{r}$$

$$= \widehat{r} + \frac{(\widehat{r}_+)^2}{(\widehat{r}_+ - \widehat{r}_-)} \cdot \ln|\widehat{r} - \widehat{r}_+| - \frac{(\widehat{r}_-)^2}{(\widehat{r}_+ - \widehat{r}_-)} \cdot \ln|\widehat{r} - \widehat{r}_-|,$$

$$r_\pm := m \pm \sqrt{m^2 - \mathrm{e}^2}$$

$$\widehat{u} = \widehat{t} - \widehat{r}_*, \qquad \widehat{v} = \widehat{t} + \widehat{r}_*,$$

$$\widehat{r} = \widehat{\mathcal{Y}}(\widehat{u}, \widehat{v}),$$

$$\mathrm{d}s^2 = -\left\{1 - \frac{2m}{\widehat{\mathcal{Y}}(\widehat{u}, \widehat{v})} + \left[\frac{e}{\widehat{\mathcal{Y}}(\widehat{u}, \widehat{v})}\right]^2\right\} \cdot (\mathrm{d}\widehat{u})(\mathrm{d}\widehat{v})$$
$$+ \left[\widehat{\mathcal{Y}}(\widehat{u}, \widehat{v})\right]^2 \cdot \left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2 \widehat{\theta} \cdot (\mathrm{d}\widehat{\varphi})^2\right].$$

Remark: *Two horizons* of the Reissner–Nordström–Jeffery black hole are given by $\widehat{r} = \widehat{\mathcal{Y}}(\widehat{u}, \widehat{v}) = \widehat{r}_+$ and $\widehat{r} = \widehat{\mathcal{Y}}(\widehat{u}, \widehat{v}) = \widehat{r}_-$. These are solutions of the special Hamilton–Jacobi equation of page 591 from the R–N–J metric as

$$\widehat{g}^{ij}(\hat{x}) \cdot \frac{\partial \widehat{r}}{\partial \widehat{x}^i} \cdot \frac{\partial \widehat{r}}{\partial \widehat{x}^j} = \widehat{g}^{11}(\widehat{x}) = \left[1 - \frac{2m}{\widehat{r}} + \frac{e^2}{\widehat{r}^2}\right] = 0.$$

5.2 Kerr Black Holes

One of the most important axially symmetric, stationary vacuum solutions is provided by the Kerr metric in Sect. 4.3. In the Boyer–Lindquist coordinate chart, the Kerr metric is furnished by (4.134) as

$$\mathrm{d}s^2 = \Sigma(r, \theta) \cdot \left[\frac{(\mathrm{d}r)^2}{\Delta(r)} + (\mathrm{d}\theta)^2\right] + \left(r^2 + a^2\right) \cdot \sin^2 \theta \cdot (\mathrm{d}\varphi)^2$$
$$- (\mathrm{d}t)^2 + \left[\frac{2mr}{\Sigma(r, \theta)}\right] \cdot \left(\mathrm{d}t - a \sin^2 \theta \cdot \mathrm{d}\varphi\right)^2, \tag{5.60i}$$

$$\Sigma(r, \theta) := r^2 + a^2 \cos^2 \theta > 0, \quad \Delta(r) := r^2 - 2mr + a^2, \tag{5.60ii}$$

$$0 < |a| < m, \tag{5.60iii}$$

$$D_4 := \left\{(r, \theta, \varphi, t) : 0 < m + \sqrt{m^2 - a^2 \cos^2 \theta} < r < \infty,\ 0 < \theta < \pi,\right.$$
$$\left. -\pi < \varphi < \pi,\ -\infty < t < \infty\right\}. \tag{5.60iv}$$

The above metric may represent the gravitational field outside of certain rotating, axially symmetric, massive stars. Moreover, it can stand for the external metric due to a rotating, axially symmetric, *black hole*. This metric admits two, commuting Killing vectors $\frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial t}$.

(*Caution*: The r and t coordinates of (5.60i) *differ* from Lemaître's r and t coordinates of the preceding section.)

Now, we shall explore the curvature tensor components of the Kerr metric of (5.60i) in the following example.

Example 5.2.1. Let us express (5.60i) in a different coordinate chart given by

$$y = \cos\theta, \quad |y| < 1, \quad \widehat{\Sigma}(r, y) := \Sigma(r, \theta).$$

Also, using (r, φ, t) for the new coordinates (for the sake of simplicity), we get

$$\mathrm{d}s^2 = \left[r^2 + a^2 y^2\right] \cdot \left[\frac{(\mathrm{d}r)^2}{\Delta(r)} + \frac{(\mathrm{d}y)^2}{(1-y^2)}\right] + (r^2 + a^2)(1-y^2) \cdot \mathrm{d}\varphi^2 - (\mathrm{d}t)^2 \tag{5.61i}$$

$$+ \frac{2mr}{(r^2 + a^2 y^2)} \cdot \left[\mathrm{d}t - a \cdot (1-y^2) \cdot \mathrm{d}\varphi\right]^2, \tag{5.61ii}$$

$$g := \det\left[g_{ij}(\cdot\cdot)\right] = -\left(r^2 + a^2 y^2\right)^2 \cdot \left(1 - y^2\right). \tag{5.61iii}$$

Thus, the metric in (5.61ii) is given by *polynomial functions*. The Kerr space–time in various coordinate systems is also reviewed in [255].

Remark: Computer-based, symbolic computation is easier with polynomial functions than with trigonometric functions. (See the Appendix 8.))

The nonzero, orthonormal components of the curvature tensor from (5.61ii) are furnished by

$$\begin{aligned}
R_{(1)(4)(1)(4)}(\cdot\cdot) &= -2R_{(2)(4)(2)(4)}(\cdot\cdot) = -2R_{(3)(4)(3)(4)}(\cdot\cdot) = 2R_{(1)(2)(1)(2)}(\cdot\cdot) \\
&= 2R_{(1)(3)(1)(3)}(\cdot\cdot) = -R_{(2)(3)(2)(3)}(\cdot\cdot) = -\frac{2mr \cdot (r^2 - 3a^2 y^2)}{(r^2 + a^2 y^2)^3}, \\
R_{(2)(3)(4)(1)}(\cdot\cdot) &= R_{(1)(3)(4)(2)}(\cdot\cdot) = -R_{(1)(2)(4)(3)}(\cdot\cdot) = \frac{2ma \cdot y \cdot (3r^2 - a^2 y^2)}{(r^2 + a^2 y^2)^3}.
\end{aligned} \tag{5.62}$$

Remarks: (i) In the limit $a \to 0$, the above expressions reduce to those of the Schwarzschild metric. (See (3.12).)

(ii) In the limit $m \to 0_+$, the curvature tensor vanishes. (See the Problem # 6 of the Exercise 4.3.)

The Kretschmann scalar for the Kerr metric reduces to

$$\begin{aligned}
&R^{(a)(b)(c)(d)}(\cdot\cdot) \cdot R_{(a)(b)(c)(d)}(\cdot\cdot) \\
&\quad = \frac{48 \cdot m^2 \cdot (r^2 - a^2 y^2) \cdot \left[\left(r^2 + a^2 y^2\right)^2 - 16 r^2 a^2 y^2\right]}{(r^2 + a^2 y^2)^6}.
\end{aligned} \tag{5.63}$$

Note that the above orthonormal components in (5.62) and the Kretschmann scalar in (5.63) are *undefined for points satisfying*

$$\begin{aligned}
& r^2 + a^2 y^2 = 0, \\
\text{or} \qquad & r = 0 \quad \text{and} \quad y = 0 \quad \text{for } a \neq 0, \\
\text{or} \qquad & r = 0 \quad \text{and} \quad \theta = \pi/2 \quad \text{for } \ a \neq 0.
\end{aligned} \tag{5.64}$$

These points constitute *singularities*. □

Now consider the two roots of the quadratic equation[4]

$$\begin{aligned}
& \Delta(r) = r^2 - 2mr + a^2 = 0, \\
\text{or} \quad & r = r_\pm := m \pm \sqrt{m^2 - a^2} = \text{consts.}
\end{aligned} \tag{5.65}$$

Similarly, two solutions of the second-degree equation

$$\Sigma(r, \theta) - 2mr = r^2 - 2mr + a^2 \cos^2\theta = 0 \tag{5.66}$$

are provided by functions

$$r = \mathcal{R}_\pm(\theta) := m \pm \sqrt{m^2 - a^2 \cos^2\theta} \, . \tag{5.67}$$

Assuming the physically reasonable inequality $0 < |a| < m$, we derive from (5.65) and (5.67) more inequalities:

$$0 < \mathcal{R}_-(\theta) < r_- < r_+ < \mathcal{R}_+(\theta). \tag{5.68}$$

In terms of $r_\pm$ and $\mathcal{R}_\pm(\theta)$, the nonzero metric tensor components in (5.60i) are provided by

$$\begin{aligned}
g_{11}(\cdot\cdot) &= \frac{\Sigma(\cdot\cdot)}{(r - r_+)\cdot(r - r_-)} \, , \\
g_{22}(\cdot\cdot) &= \Sigma(\cdot\cdot) > 0, \\
g_{33}(\cdot\cdot) &= \left[r^2 + a^2 + \frac{2ma^2 \cdot r \cdot \sin^2\theta}{\Sigma(\cdot\cdot)} \right] \cdot \sin^2\theta > 0, \\
g_{34}(\cdot\cdot) &= -\frac{2ma \cdot r \cdot \sin^2\theta}{\Sigma(\cdot\cdot)} \, , \\
g_{44}(\cdot\cdot) &= -\frac{[r - \mathcal{R}_+(\theta)] \cdot [r - \mathcal{R}_-(\theta)]}{\Sigma(\cdot\cdot)} \, .
\end{aligned} \tag{5.69}$$

[4] *Caution*: The two roots $r_\pm$ in (5.66) are different from $r_\pm$ in Problem #4 of Exercise 5.1 (regarding the Reissner–Nordström–Jeffery metric).

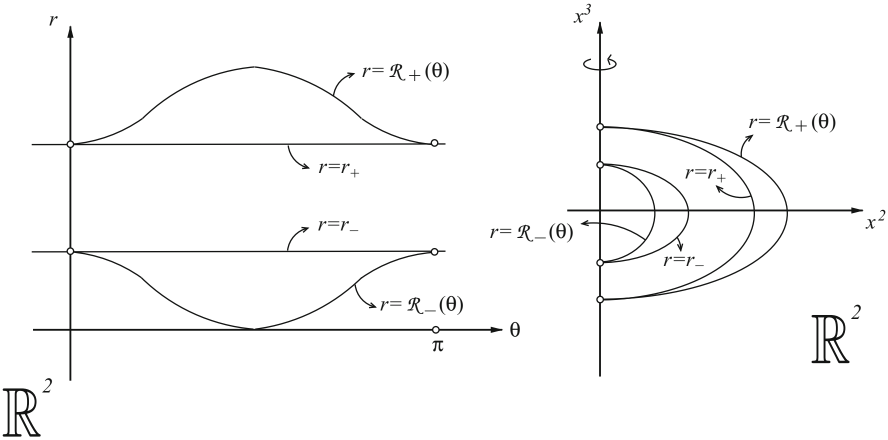


Fig. 5.14 Various profile curves representing horizons in the submanifold $\varphi = \pi/2, t = \text{const}$ in the Kerr space–time

Furthermore, real, usual eigenvalues of $\left[g_{ij}(\cdot\cdot)\right]$, characterized by $\det[g_{ij}(\cdot\cdot) - \lambda\delta_{ij}] = 0$, are furnished by

$$\begin{aligned}
\lambda_{(1)}(\cdot\cdot) &= \frac{\Sigma(\cdot\cdot)}{(r-r_+)\cdot(r-r_-)}\,,\\
\lambda_{(2)}(\cdot\cdot) &= \Sigma(\cdot\cdot) > 0,\\
2\lambda_{(3)}(\cdot\cdot) &= g_{33}(\cdot\cdot) + g_{44}(\cdot\cdot) + \sqrt{(g_{33}-g_{44})^2 + 4\,(g_{34})^2}\,,\\
2\lambda_{(4)}(\cdot\cdot) &= g_{33}(\cdot\cdot) + g_{44}(\cdot\cdot) - \sqrt{(g_{33}-g_{44})^2 + 4\,(g_{34})^2}\,. \qquad (5.70)
\end{aligned}$$

We notice in the domain (5.60iv) that the signature of the metric is $+2$. However, some of the metric tensor components and *some of the eigenvalues change sign* in the interior domains but in such a way that *the signature remains the same!*

Now let us try to visualize the various functions in (5.68) and the corresponding graphs in the $\theta - r$ coordinate plane. Moreover, we also map these graphs into the $x^2 - x^3$ plane by the transformation $x^2 = r\sin\theta,\ x^3 = r\cos\theta$. The resulting graphs are exhibited in Fig. 5.14.

Now let us explore the physical interpretations of the various horizons. Consider the Killing vector field $\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial x^4}$ for the Kerr metric in (5.60i). We derive from (5.69) that

$$\mathbf{g}_{..}(\cdot\cdot)\left[\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4}\right] = g_{44}(\cdot\cdot) = -\frac{[r-\mathcal{R}_+(\theta)]\cdot[r-\mathcal{R}_-(\theta)]}{\Sigma(\cdot\cdot)}\,. \qquad (5.71)$$

Therefore, the Killing vector $\frac{\partial}{\partial x^4}$ becomes *spacelike inside* $\mathcal{R}_-(\theta) < r < \mathcal{R}_+(\theta)$. Thus, the hypersurface $r = \mathcal{R}_+(\theta)$ is called the *stationary limit surface*. (It is also called the *outer ergosurface*.) The domain denoted as the *ergosphere* is characterized by inequalities $r_+ < r < \mathcal{R}_+(\theta)$. (See [106, 253].)

Next, let us derive *the event horizons* for the Kerr space–time. It is well known that a three-dimensional null hypersurface $S(\cdot\cdot) = \text{const.}$ (with *a two-dimensional metric*) must satisfy the Hamilton–Jacobi equation (A2.19ii):

$$\mathbf{g}^{\cdot\cdot}(\cdot\cdot)\,[\mathrm{d}S, \mathrm{d}S] = g^{ij}(\cdot\cdot) \cdot \partial_i S \cdot \partial_j S = 0$$
$$\text{with} \quad \delta^{ij} \cdot \partial_i S \cdot \partial_j S > 0. \tag{5.72}$$

From the Kerr space–time metric in (5.60i), the hypersurface $r =: S(\cdot\cdot) = \text{const.}$ satisfies the null condition in (5.72), provided

$$g^{11}(\cdot\cdot) \cdot (\partial_1 r)^2 = \frac{(r - r_+) \cdot (r - r_-)}{\Sigma(\cdot\cdot)} = 0. \tag{5.73}$$

Therefore, the three-dimensional hypersurfaces $r = r_+$ and $r = r_-$ are *null*. The null hypersurface $r = r_+$ turns out to be the event horizon.[5]

Now, let us go back to Example 5.2.1 and (5.64). The curvature singularity of the Kerr metric in (5.60i) is given by $r = 0_+,\ \theta = \pi/2$. The restriction of the metric (5.60i) for the submanifold $r = 0_+,\ \theta = \pi/2,\ t \equiv x^4 = \text{const.}$ yields *the one-dimensional* metric:

$$(\mathrm{d}l)^2 = \left(a^2\right) \cdot (\mathrm{d}\varphi)^2\,. \tag{5.74}$$

The above metric is that of a "ring" of radius "a" in the equatorial plane of the three-dimensional coordinate space $\mathbb{R}^3$.

In the three-dimensional submanifold $x^4 = t = \text{const.}$, we can visualize various horizons as surfaces of revolution and the singularity. Profile curves for four surfaces of revolution are already shown in Fig. 5.14. The equations governing the outer event horizon, inner horizon, outer ergosurface (or stationary limit surface), and inner ergosurface are furnished, respectively, by the following:

$$\left(x^1, x^2, x^3\right) = (r_+ \cdot \sin\theta \cdot \cos\varphi,\ r_+ \cdot \sin\theta \cdot \sin\varphi,\ r_+ \cdot \cos\theta), \tag{5.75i}$$

$$\left(x^1, x^2, x^3\right) = (r_- \cdot \sin\theta \cdot \cos\varphi,\ r_- \cdot \sin\theta \cdot \sin\varphi,\ r_- \cdot \cos\theta), \tag{5.75ii}$$

$$\left(x^1, x^2, x^3\right) = (\mathcal{R}_+(\theta) \cdot \sin\theta \cdot \cos\varphi, \mathcal{R}_+(\theta) \cdot \sin\theta \cdot \sin\varphi, \mathcal{R}_+(\theta) \cdot \cos\theta), \tag{5.75iii}$$

[5] For a precise definition of an event horizon, consult the book by Hawking and Ellis [126]. The other horizon, at $r = r_-$, is called a *Cauchy horizon*.

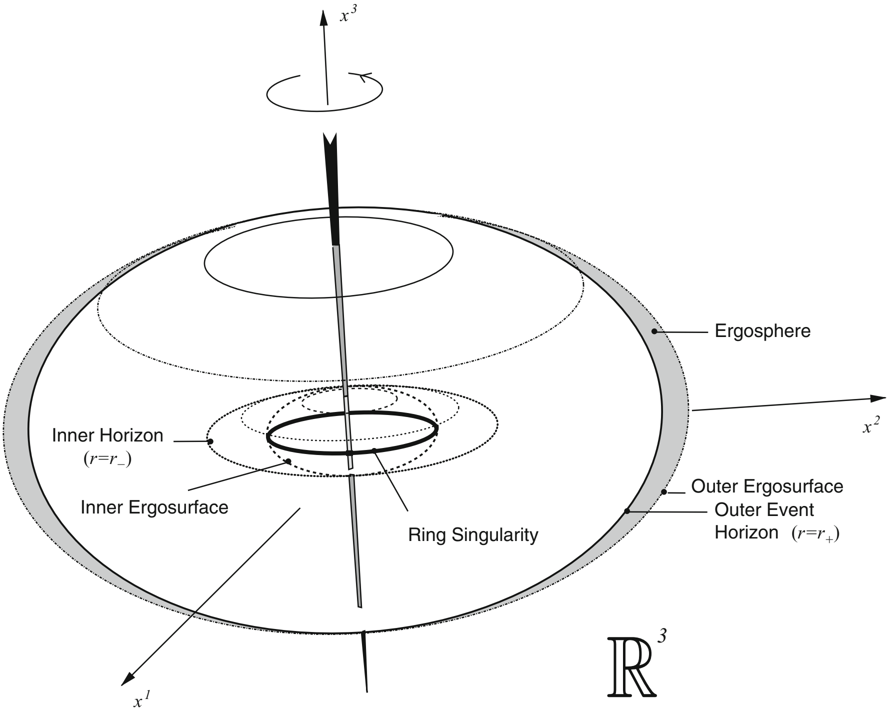


Fig. 5.15 Locations of horizons, ergosphere, ring singularity, etc., in the Kerr-submanifold $x^4 =$ const

$$\left(x^1, x^2, x^3\right) = (\mathcal{R}_-(\theta)\cdot \sin\theta \cdot \cos\varphi, \mathcal{R}_-(\theta)\cdot \sin\theta\cdot \sin\varphi, \mathcal{R}_-(\theta)\cdot \cos\theta), \quad (5.75\text{iv})$$

$$\mathcal{D}_2 := \left\{(\theta, \varphi) \in \mathbb{R}^2 : 0 < \theta < \pi,\ -\pi < \varphi < \pi\right\}. \quad (5.75\text{v})$$

Following Example 1.4.1, four surfaces of revolution are depicted in Fig. 5.15. (Recall that this figure is depicted in *a coordinate chart*.)

Particles entering into the ergosphere from the external domain in (5.60iv) *can go back* to the external universe. That is why $r = \mathcal{R}_+(\theta)$ is *not an event horizon*. (However, $r = r_+$ is an event horizon [126].)

Consider two two-dimensional domains defined by

$$\mathcal{D}_2^+ := \left\{\left(r, \theta, \varphi, x^4\right) : r = 0_+,\ 0 < \theta < \pi/2,\ -\pi < \varphi < \pi,\ x^4 = \text{const.}\right\},$$

$$\mathcal{D}_2^- := \left\{\left(r, \theta, \varphi, x^4\right) : r = 0_+,\ \pi/2 < \theta < \pi,\ -\pi < \varphi < \pi,\ x^4 = \text{const.}\right\}. \quad (5.76)$$

In both of these mutually disconnected domains, the two-dimensional metric induced by (5.60i) is given as

$$\begin{aligned} d\sigma^2 &:= ds^2{}_{|..} = \Sigma_0(\theta) \cdot (d\theta)^2 + a^2 \sin^2\theta \cdot (d\varphi)^2, \\ \Sigma_0(\theta) &:= a^2 \cos^2\theta. \end{aligned} \tag{5.77}$$

The two-dimensional metric in (5.77) is *flat*. *The* r*-coordinate curves can be analytically extended through domains* $\mathcal{D}_2^{\pm}$ *into negative values* $r < 0$.

The preceding treatments of the Kerr black hole are mathematically unsatisfactory due to the use of the Boyer–Lindquist coordinate chart beyond its valid domain D_4 in (5.60iv). In the case of the spherically symmetric black hole of the preceding section, the original Schwarzschild chart had to be extended to the Kruskal–Szekeres chart of (5.33i) and (5.35). Similarly, we need to extend the Boyer–Lindquist chart in a suitable manner into the Kerr black hole. For that purpose, we extend (5.13) into[6]

$$\begin{aligned} r_* &:= \int \frac{(r^2 + a^2)}{\Delta(r)}\, dr = r + \frac{1}{2p_+} \cdot \ln|r - r_+| + \frac{1}{2p_-} \cdot \ln|r - r_-| + \text{const.}, \\ p_+ &:= \frac{(r_+ - r_-)}{2(r_+^2 + a^2)} = \text{const.}, \quad p_- := \frac{(r_- - r_+)}{2(r_-^2 + a^2)} = \text{const.} \end{aligned} \tag{5.78}$$

One generalization of (5.14) is provided by

$$\widehat{v} = t + r_* = t + \int \frac{(r^2 + a^2) \cdot dr}{\Delta(r)}\ . \tag{5.79}$$

The remaining coordinate transformations are furnished by

$$\begin{aligned} \widehat{\varphi} &= \varphi + a \cdot \int \frac{dr}{\Delta(r)}, \\ \widehat{\theta} &= \theta. \end{aligned} \tag{5.80}$$

[6]Consider the two-dimensional submanifold of the Kerr space–time characterized by $\theta = 0$ and $\varphi = \text{const.}$ The corresponding metric from (5.61ii–v) is furnished by $ds^2_{|..} = \left[(r^2 + a^2)/\Delta(r)\right] \cdot (dr)^2 - \left[\Delta(r)/(r^2 + a^2)\right] \cdot (dt)^2$. The null geodesics from the preceeding metric yield (5.78), (5.79), and (5.82).

The metric (5.60i), under transformations (5.79) and (5.80), goes over into

$$\begin{aligned} ds^2 = & -\frac{\Delta(r)}{\Sigma\left(r,\widehat{\theta}\right)} \cdot \left[d\widehat{v} - a \cdot \sin^2\widehat{\theta} \cdot d\widehat{\varphi}\right]^2, \\ & + \frac{\sin^2\widehat{\theta}}{\Sigma\left(r,\widehat{\theta}\right)} \cdot \left[\left(r^2+a^2\right) \cdot d\widehat{\varphi} - a d\widehat{v}\right]^2 + \Sigma(\cdot\cdot) \cdot \left(d\widehat{\theta}\right)^2 \\ & + 2dr \cdot \left[d\widehat{v} - a \cdot \sin^2\widehat{\theta} \cdot d\widehat{\varphi}\right]. \end{aligned} \tag{5.81}$$

The above is the generalization of the spherically symmetric Eddington–Finkelstein metric in (5.15ii). The metric in (5.81) is *very smooth in the neighborhood of the event horizon* $r = r_+$.

We can make another transformation, namely,

$$\begin{aligned} u^{\#} &= t - r_*, \\ \varphi^{\#} &= \varphi - a \cdot \int \frac{dr}{\Delta(r)} \; , \\ \theta^{\#} &= \theta. \end{aligned} \tag{5.82}$$

The metric in (5.60i) transforms into

$$\begin{aligned} ds^2 = & -\frac{\Delta(r)}{\Sigma\left(r,\theta^{\#}\right)} \cdot \left[du^{\#} - a \cdot \sin^2\theta^{\#} \cdot d\varphi^{\#}\right]^2 \\ & + \frac{\sin^2\theta^{\#}}{\Sigma(\cdot\cdot)} \cdot \left[\left(r^2+a^2\right) \cdot d\varphi^{\#} - a \cdot du^{\#}\right]^2 \\ & + \Sigma(\cdot\cdot) \cdot \left(d\theta^{\#}\right)^2 - 2dr \cdot \left[du^{\#} - a \cdot \sin^2\theta^{\#} \cdot d\varphi^{\#}\right]. \end{aligned} \tag{5.83}$$

(Note that (5.83) can be obtained from (5.82) by the transformation $\left(\widehat{v},\widehat{\varphi}\right) \rightarrow \left(-u^{\#}, -\varphi^{\#}\right)$.) Equation (5.83) generalizes the equation of the Eddington–Finkelstein metric (5.15i).

Now, let us try to express the metric (5.60i) in terms of both $\widehat{u} \equiv u^{\#}$ and $\widehat{v}$ coordinates. For that purpose, it is convenient to write (5.60i) as

$$\begin{aligned} ds^2 = & \frac{\Sigma(r,\theta)}{\Delta(r)} \cdot (dr)^2 + \Sigma(r,\theta) \cdot (d\theta)^2 - \frac{\Delta(\cdot\cdot)}{\Sigma(\cdot\cdot)} \cdot \left(dt - a \cdot \sin^2\theta \cdot d\varphi\right)^2 \\ & + \frac{\sin^2\theta}{\Sigma(\cdot\cdot)} \cdot \left[\left(r^2+a^2\right) \cdot d\varphi - a \cdot dt\right]^2. \end{aligned} \tag{5.84}$$

We next make the following coordinate transformation:

$$\widehat{u} = t - r_*, \quad \widehat{v} = t + r_*, \quad \widetilde{\varphi} = \varphi - \frac{a \cdot t}{2mr_+}\,,$$
$$\widetilde{\theta} = \theta. \tag{5.85}$$

The inverse transformation is characterized by

$$\begin{aligned} \mathrm{d}r &= \frac{\Delta(r)}{(r^2+a^2)} \cdot \mathrm{d}r_* = \frac{1}{2} \cdot \frac{\Delta(r)}{(r^2+a^2)} \left(\mathrm{d}\widehat{v} - \mathrm{d}\widehat{u}\right), \\ \mathrm{d}t &= \frac{1}{2} \cdot \left(\mathrm{d}\widehat{v} + \mathrm{d}\widehat{u}\right), \quad \mathrm{d}\varphi = \mathrm{d}\widetilde{\varphi} + \frac{a \cdot \mathrm{d}t}{2mr_+} \\ &= \mathrm{d}\widetilde{\varphi} + \frac{a \cdot (\mathrm{d}\widehat{v} + \mathrm{d}\widehat{u})}{4mr_+}\,, \\ \mathrm{d}\theta &= \mathrm{d}\widetilde{\theta}. \end{aligned} \tag{5.86}$$

By (5.85) and (5.86), it follows that

$$\begin{aligned} \mathrm{d}t - a \cdot \sin^2\theta \cdot \mathrm{d}\varphi &= \frac{1}{2} \cdot \left[\frac{\Sigma\left(r_+, \widetilde{\theta}\right)}{r_+^2 + a^2}\right] \cdot \left(\mathrm{d}\widehat{u} + \mathrm{d}\widehat{v}\right) - a \cdot \sin^2\widetilde{\theta} \cdot \mathrm{d}\widetilde{\varphi}, \\ \left(r^2 + a^2\right) \cdot \mathrm{d}\varphi - a \cdot \mathrm{d}t &= \left(r^2 + a^2\right) \cdot \mathrm{d}\widetilde{\varphi} + \frac{a}{2} \cdot \left[\frac{r^2 - r_+^2}{r_+^2 + a^2}\right] \cdot \left(\mathrm{d}\widehat{u} + \mathrm{d}\widehat{v}\right). \end{aligned} \tag{5.87}$$

Therefore, the Kerr metric in (5.84) goes over into

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{\Sigma\left(r, \widetilde{\theta}\right)}{4\Delta(r)} \cdot \left[\frac{\Delta(r)}{(r^2+a^2)} \cdot \left(\mathrm{d}\widehat{v} - \mathrm{d}\widehat{u}\right)\right]^2 + \Sigma(\cdot\cdot) \cdot \left(\mathrm{d}\widetilde{\theta}\right)^2 \\ & + \frac{\sin^2\widetilde{\theta}}{\Sigma(\cdot\cdot)} \cdot \left[\left(r^2 + a^2\right)\mathrm{d}\widetilde{\varphi} + \frac{1}{2}a \cdot \left(\frac{r^2 - r_+^2}{r_+^2 + a^2}\right) \cdot \left(\mathrm{d}\widehat{u} + \mathrm{d}\widehat{v}\right)\right]^2 \\ & - \frac{\Delta(\cdot\cdot)}{\Sigma(\cdot\cdot)} \cdot \left\{\frac{1}{2} \cdot \left(\frac{r_+^2 + a^2 \cdot \cos^2\widetilde{\theta}}{r_+^2 + a^2}\right) \cdot \left(\mathrm{d}\widehat{u} + \mathrm{d}\widehat{v}\right) - a \cdot \sin^2\widetilde{\theta} \cdot \mathrm{d}\widetilde{\varphi}\right\}^2. \end{aligned} \tag{5.88}$$

The above metric reduces to *Synge's chart* in (5.17i,ii) for $a \to 0$.

Now, we make another coordinate transformation[7] for $uv < 0$ by

$$
\begin{aligned}
&u = -\mathrm{e}^{p_+\cdot(r_*-t)} < 0, \quad v = \mathrm{e}^{p_+\cdot(r_*+t)} > 0, \\
&\theta = \widetilde{\theta}, \quad \varphi = \widetilde{\varphi}.
\end{aligned}
\tag{5.89}
$$

The inverse transformation for $r > r_+ > r_-$ is furnished by

$$
\begin{aligned}
r_* &= r + \frac{1}{2p_+}\cdot\ln(r-r_+) + \frac{1}{2p_-}\cdot\ln(r-r_-) = \frac{1}{2p_+}\cdot\ln(-uv), \\
t &= \frac{1}{2p_+}\cdot\ln(-v/u), \quad \text{etc.}
\end{aligned}
\tag{5.90}
$$

Note that the function

$$
r = \mathcal{Y}(u, v) \tag{5.91}
$$

is *implicitly defined by*

$$
\mathrm{e}^{2p_+r}\cdot(r-r_+)\cdot(r-r_-)^{(p_+/p_-)} = -(uv). \tag{5.92}
$$

The metric in (5.88) goes over into

$$
\begin{aligned}
\mathrm{d}s^2 = &\frac{a^2\cdot[G_+(\cdot\cdot)]^2\cdot\sin^2\theta}{4p_+^2\cdot\Sigma(\cdot\cdot)}\cdot\frac{(r-r_-)\cdot(r+r_+)}{(r^2+a^2)\cdot(r_+^2+a^2)}\cdot\left[\frac{\Sigma(\cdot\cdot)}{r^2+a^2}+\frac{\Sigma(r_+,\theta)}{r_+^2+a^2}\right] \\
&\times\left(u^2\mathrm{d}v^2+v^2\mathrm{d}u^2\right) \\
&+\frac{G_+(\cdot\cdot)\cdot(r-r_-)}{2p_+^2\cdot\Sigma(\cdot\cdot)}\cdot\left[\frac{(\Sigma(\cdot\cdot))^2}{(r^2+a^2)^2}+\frac{(\Sigma(r_+,\theta))^2}{\left(r_+^2+a^2\right)^2}\right]\cdot\mathrm{d}u\cdot\mathrm{d}v \\
&+\frac{a^2\cdot[G_+(\cdot\cdot)]^2\cdot\sin^2\theta}{4p_+^2\cdot\Sigma(\cdot\cdot)}\cdot\frac{(r+r_+)^2}{\left(r_+^2+a^2\right)^2}\cdot(u\mathrm{d}v-v\mathrm{d}u)^2 \\
&+\frac{a\cdot G_+(\cdot\cdot)\cdot\sin^2\theta}{p_+\cdot\Sigma(\cdot\cdot)\cdot(r_+^2+a^2)}\cdot\left[\Sigma(r_+,\theta)\cdot(r-r_-)+(r^2+a^2)\cdot(r+r_+)\right] \\
&\times(u\mathrm{d}v-v\mathrm{d}u)\cdot\mathrm{d}\varphi
\end{aligned}
$$

[7] *Caution*: The φ-coordinate in (5.89) is *different* from the φ-coordinate in (5.60i)!

$$+\,\Sigma(r,\theta)\cdot(\mathrm{d}\theta)^2+\left[\frac{\left(r^2+a^2\right)^2-a^2\cdot\Delta(r)\cdot\sin^2\theta}{\Sigma(\cdot\cdot)}\right]\cdot\sin^2\theta\cdot(\mathrm{d}\varphi)^2,$$

$$G_+(u,v):=\frac{r-r_+}{uv}=\frac{\mathcal{Y}(u,v)-r_+}{uv}\,. \tag{5.93}$$

The metric above is the generalization of the Kruskal–Szekeres chart in (5.19) [198]. Moreover, the function $r=\mathcal{Y}(u,v)$ in (5.91) admits natural, analytic extensions beyond the original domain characterized by $u<0$ and $v>0$. Thus, in case the function $\mathcal{Y}$ of (5.91) is extended to $\widetilde{\mathcal{Y}}$ in the metric of (5.93), we obtain the generalization of the Kruskal–Szekeres chart in (5.33ii) for the Kerr metric. However, the Kerr metric can be analytically extended *also for negative values of* $r=\widetilde{\mathcal{Y}}(u,v)$. (See the discussions after (5.77).)

To depict the maximal analytic extension of the Kerr metric in (5.93), let us restrict our analysis to the two-dimensional submanifold characterized by $\theta=$ const. and $\varphi=$ const. In the degenerate case of $\theta=0_+$ (and $\varphi=$ const.), the induced metric from (5.93) drastically reduces to

$$\mathrm{d}\sigma^2:=\mathrm{d}s^2|_{\theta=0_+,\,\varphi=\text{const.}}=\frac{\Delta(r)}{p_+^2\cdot(r^2+a^2)\cdot uv}\cdot(\mathrm{d}u)\cdot(\mathrm{d}v),$$

$$r=\widetilde{\mathcal{Y}}(u,v). \tag{5.94}$$

It is more convenient to extend the simpler metric above [35]. To continue the analysis, we shall now have to digress to a relevant topic. We shall *discuss briefly* the *Penrose-Carter compactification* of a space–time manifold. Let us consider *a toy model*, namely, the real-analytic function “Arctan.” It can be summarized as

$$\begin{aligned}&\text{Arctan}:\ \mathbb{R}\longrightarrow\mathbb{R},\\&\text{Domain}\,(\text{Arctan})=\mathbb{R},\\&\text{Codomain}\,(\text{Arctan})=(-\pi/2,\pi/2)\subset\mathbb{R}.\end{aligned} \tag{5.95}$$

The codomain (Arctan) is *an open subset* and, thus, *noncompact* [32]. We want to extend both the domain and the function “Arctan” to obtain a compact domain. For that purpose, we define the set of *extended real numbers* as

$$\mathbb{R}^*:=\mathbb{R}\cup\{-\infty,\infty\}\,. \tag{5.96}$$

The extended function “Arctan*” is defined by

$$\begin{aligned}&\text{Arctan}^*:\ \mathbb{R}^*\longrightarrow\mathbb{R}^*,\\&\text{Domain}\,(\text{Arctan}^*)=\mathbb{R}^*,\\&\text{Codomain}\,(\text{Arctan}^*)=[-\pi/2,\pi/2]\subset\mathbb{R}^*.\end{aligned} \tag{5.97}$$

Thus, the codomain is a *closed subset and compact*.

The two-dimensional submanifold characterized by (5.94) is compactified by the transformations:

$$U_+ = \text{Arctan}^*(u) \in [-\pi/2, \pi/2],$$
$$V_+ = \text{Arctan}^*(v) \in [-\pi/2, \pi/2]. \tag{5.98}$$

The line element in (5.94) then goes over into

$$\left(d\sigma_+^*\right)^2 = 4\,[\text{cosec}\,(2U_+)] \cdot [\text{cosec}\,(2V_+)] \cdot \left[\frac{\Delta(r)}{p_+^2 (r^2 + a^2)}\right] \cdot (dU_+) \cdot (dV_+),$$
$$r = \mathcal{Y}^*\,(U_+, V_+) := \widetilde{\mathcal{Y}}\,(\tan U_+, \tan V_+)\,. \tag{5.99}$$

We can express *the conformally flat metric* in (5.99) as

$$\left(d\sigma_+^*\right)^2 = \Omega(\cdot\cdot) \cdot \left(d\bar{\sigma}^*\right)^2, \tag{5.100i}$$

$$\Omega(\cdot\cdot) := \frac{4\Delta(r)}{p_+^2 \cdot (r^2 + a^2)} \cdot [\text{cosec}\,(2U_+)] \cdot [\text{cosec}\,(2V_+)]\,, \tag{5.100ii}$$

$$\left(d\bar{\sigma}^*\right)^2 := (dU_+) \cdot (dV_+), \tag{5.100iii}$$

$$(U_+, V_+) \in D_{\mathrm{I}} \cup D_{\mathrm{II}} \cup D_{\mathrm{I'}} \cup D_{\mathrm{II'}} \cup \{\text{some bd. points}\}\,. \tag{5.100iv}$$

The region[8] of validity in (5.100iv) is depicted in Fig. 5.16.

Note that the metrics in (5.93), (5.99), and (5.100iii) are *valid on the outer horizon* at $r = r_+$. However, these metrics *do not cover the inner horizon* at $r = r_-$. The chart that covers the domain corresponding to $-\infty < r < r_+$, including the inner horizon, can be obtained from (5.93) by interchanging "+" by "−." The corresponding metric for the two-dimensional submanifold (characterized by $\theta = 0_+$) is furnished by

$$\left(d\sigma_-^*\right)^2 = \frac{4 \cdot \Delta(r)}{p_-^2 \cdot (r^2 + a^2)} \cdot (\text{cosec}\,2U_-) \cdot (\text{cosec}\,2V_-) \cdot (dU_-) \cdot (dV_-),$$
$$(U_-, V_-) \in D_{\mathrm{II}} \cup D_{\mathrm{III}} \cup D_{\mathrm{II'}} \cup D_{\mathrm{III'}} \cup \{\text{some boundary points}\}\,. \tag{5.101}$$

The corresponding region of validity is shown in Fig. 5.17.

[8] We define *a region* as a domain plus some (or all) of its boundary points. *Dashed line segments* in Fig. 5.16 indicate nonexistence of boundary points.

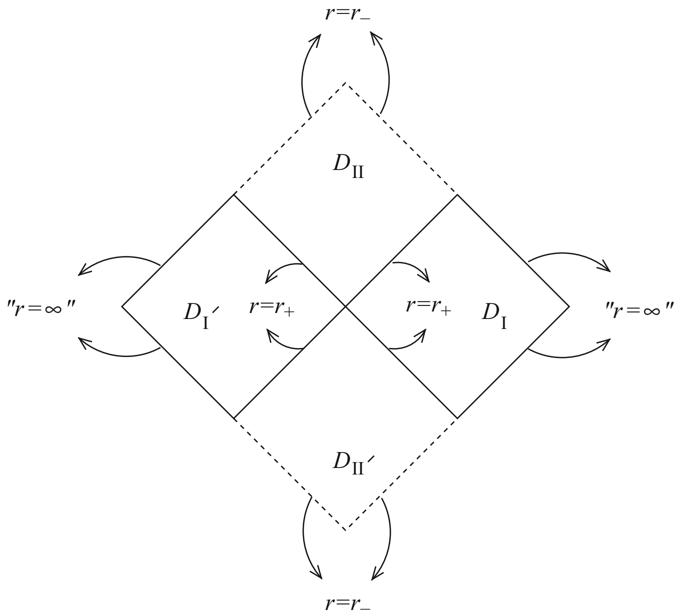


Fig. 5.16 The region of validity for the metric in (5.100iii) and (5.99)

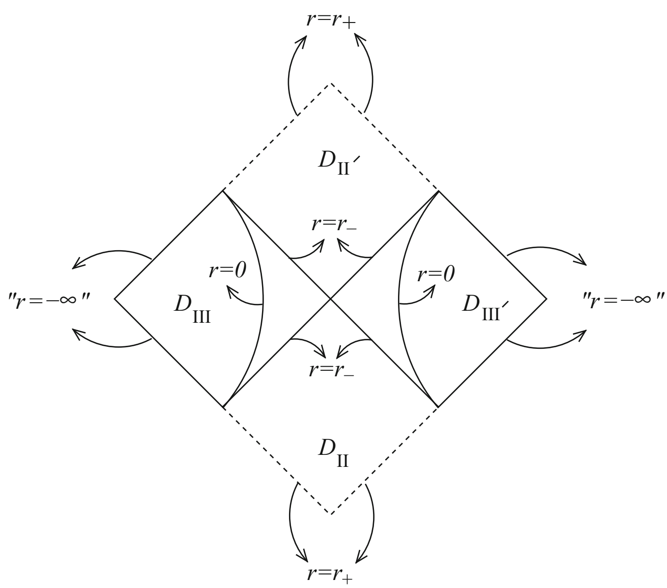


Fig. 5.17 The region of validity for the metric in (5.101)

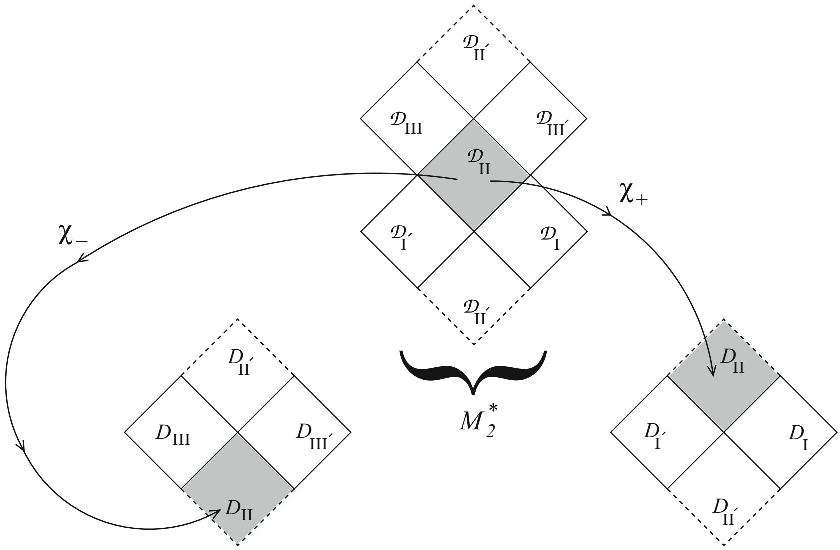


Fig. 5.18 The submanifold M_2^* and its two coordinate charts

Remarks: (i) The geometry along the curve $r = 0$ (in Fig. 5.17) is *smooth*.
(ii) However, the geometry along $r = 0$ for the case of $\theta = \pi/2$ is *singular*.
(iii) In subsets of $D_{\mathrm{III}} \cup D_{\mathrm{III}'}$, corresponding to $r < 0$, *closed timelike curves exist*! (See [197] and Appendix 6.)

Consider now the two-dimensional Kerr submanifold M_2 with $\theta = 0_+$ and $\varphi =$ const. In case it is also compactified by (5.97), we denote it by the symbol M_2^*. Two distinct coordinate charts, with some overlap, are provided by (5.99) and (5.101). Using Figs. 1.2, 5.16, and 5.17, we exhibit the submanifold M_2^* and two coordinate charts in Fig. 5.18.

Remarks: (i) Note that in the shaded domain $\mathcal{D}_{\mathrm{II}}$ of M_2^* in Fig. 5.18, two coordinate charts *overlap*.
(ii) In Fig. 5.18, the intrinsic curvature of the submanifold M_2^* is not evident. (Consult Figs. 1.4 and 2.8.)
(iii) In Fig. 5.18, we note that some of boundary points (along dashed line segments) are missing.
(iv) We can take two exact replicas of M_2^* and (topologically) glue them on the top and bottom of M_2^* in Fig. 5.18 to complete boundary points and further extend the manifold.
(v) However, to obtain *the maximally extended Kerr submanifold* (and the corresponding Kerr manifold), we must add *a denumerably infinite number of exact replicas* of M_2^* and glue them together on the top and bottom of M_2^* in Fig. 5.18.

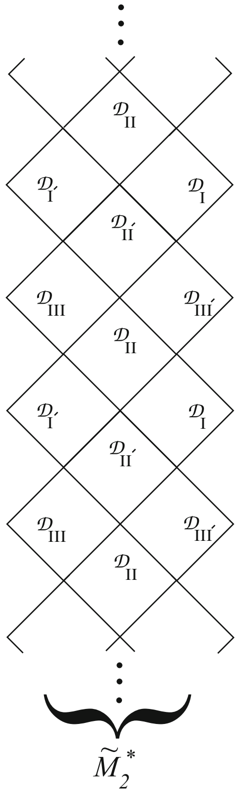


Fig. 5.19 The maximally extended Kerr submanifold $\widetilde{M}_2^*$

(vi) The resulting *maximally extended Kerr submanifold* $\widetilde{M}_2^*$ is qualitatively demonstrated in Fig. 5.19.

Before concluding this section, we would like to make some comments on the black holes discussed in Sects. 5.1 and 5.2. The extended Schwarzschild black hole in the preceding section, and the extended Kerr black hole, enjoys *privileged status*. Among all externally static, asymptotically flat vacuum solutions yielding black holes, *the only possibility* is the case of the (extended) Schwarzschild black hole involving *one parameter* $m > 0$. (See [35, 141].) Similarly, among all stationary asymptotically flat vacuum solutions yielding black holes, the only possibility is the extended Kerr black hole with *two parameters* $0 < |a| < m$. (See [35].) Similar uniqueness theorems hold for stationary, asymptotically flat electromagneto-vac black holes. The only solution admitted must be an extended *Kerr–Newman black hole* (of Problem #8 of Exercises 5.2) with *three parameters* m, "a," *and* e [35,220]. An external observer can determine these three parameters by observing trajectories of uncharged and charged test particles outside a Kerr–Newman black hole. All other physical informations of the charged, material body before the collapse are *lost after the collapse of the body inside the event horizon!* This fact is popularly known as the *no hair theorem*. (See [141, 220].)

Exercises 5.2

1. Show that for the Kerr metric in (5.60i–iv) an orthonormal 1-form basis set is provided by

$$\tilde{\mathbf{e}}^1(\cdot\cdot) := \sqrt{\frac{\Sigma(\cdot\cdot)}{\Delta(\cdot\cdot)}} \cdot \mathrm{d}r,$$

$$\tilde{\mathbf{e}}^2(\cdot\cdot) := \sqrt{\Sigma(\cdot\cdot)} \cdot \mathrm{d}\theta,$$

$$\tilde{\mathbf{e}}^3(\cdot\cdot) := \frac{\sin\theta}{\sqrt{\Sigma(\cdot\cdot)}} \cdot \left[\left(r^2 + a^2\right) \cdot \mathrm{d}\varphi - a \cdot \mathrm{d}t\right],$$

$$\tilde{\mathbf{e}}^4(\cdot\cdot) := \sqrt{\frac{\Delta(\cdot\cdot)}{\Sigma(\cdot\cdot)}} \cdot \left(\mathrm{d}t - a \cdot \sin^2\theta \cdot \mathrm{d}\varphi\right).$$

2. Prove that for eigenvalues in (5.70), the following equation

$$\det\left[g_{ij}(\cdot\cdot)\right] = \lambda_{(1)}(\cdot\cdot) \cdot \lambda_{(2)}(\cdot\cdot) \cdot \lambda_{(3)}(\cdot\cdot) \cdot \lambda_{(4)}(\cdot\cdot) = -\left[\Sigma(\cdot\cdot)\right]^2 \cdot \sin^2\theta$$

must hold.

3. A second-order *Killing tensor* field $k_{ij}(\cdot\cdot)\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ satisfies:

$$k_{ij}(\cdot\cdot) \equiv k_{ji}(\cdot\cdot), \quad \nabla_l k_{ij} + \nabla_i k_{jl} + \nabla_j k_{li} = 0.$$

Show that the Kerr metric in (5.60i,iv) admits a Killing tensor with nonzero components:

$$k_{11}(\cdot\cdot) = -\frac{a^2 \cdot \cos^2\theta \cdot \Sigma(\cdot\cdot)}{\Delta(\cdot\cdot)},$$

$$k_{22}(\cdot\cdot) = r^2 \cdot \Sigma(\cdot\cdot),$$

$$k_{33}(\cdot\cdot) = \frac{\sin^2\theta}{\Sigma(\cdot\cdot)} \cdot \left[r^2\left(r^2 + a^2\right)^2 + \frac{1}{4} \cdot a^4 \cdot \Delta(\cdot\cdot) \cdot \sin^2 2\theta\right],$$

$$k_{34}(\cdot\cdot) = -\frac{a \cdot \sin^2\theta}{\Sigma(\cdot\cdot)} \cdot \left[a^2 \cdot \Delta(\cdot\cdot) \cdot \cos^2\theta + r^2\left(r^2 + a^2\right)\right],$$

$$k_{44}(\cdot\cdot) = a^2 \cdot \left[1 - \frac{2mr \cdot \cos^2\theta}{\Sigma(\cdot\cdot)}\right].$$

4. (i) Transform the Kerr metric in (5.60i,iv) into the *Kerr–Schild coordinate chart*:

$$\begin{aligned}
ds^2 &= \left(d_{ij} + l_i l_j\right) \cdot \left(dx^i\right) \cdot \left(dx^j\right) \\
&:= d_{ij} \cdot \left(dx^i\right) \cdot \left(dx^j\right) + \frac{2mr}{r^4 - a^2\left(x^3\right)^2} \\
&\quad \times \left[\frac{r \cdot \left(x^1 dx^1 + x^2 dx^2\right) - a \cdot \left(x^1 dx^2 - x^2 dx^1\right)}{r^2 + a^2} + \frac{x^3 dx^3}{r} + dx^4\right]^2.
\end{aligned}$$

Here, $r = \mathcal{R}\left(x^1, x^2, x^3\right)$ is defined implicitly by the roots of the equation:

$$2 \cdot r^2 = \delta_{\alpha\beta} \cdot x^\alpha \cdot x^\beta + \sqrt{\left(\delta_{\alpha\beta} x^\alpha x^\beta\right)^2 + 4a^2 \cdot [(x^3)^2 - r^2]}\,.$$

(ii) Show that for l_i in part (i), the "null condition" $d_{ij} l^i l^j = 0$ holds.

5. The *Doran coordinate chart* [80] for the Kerr metric is furnished by

$$\begin{aligned}
ds^2 &= \left[\frac{\Sigma(\cdot\cdot)}{r^2 + a^2}\right] \cdot \left[dr + \frac{\sqrt{2mr \cdot (r^2 + a^2)}}{\Sigma(\cdot\cdot)} \cdot \left(dt - a \sin^2\theta \cdot d\varphi\right)\right]^2 \\
&\quad + \Sigma(\cdot\cdot) \cdot (d\theta)^2 + \left(r^2 + a^2\right) \cdot \sin^2\theta \cdot (d\varphi)^2 - (dt)^2.
\end{aligned}$$

Prove that in the limit $a \to 0$, the above metric goes over into the Painlevé–Gullstrand chart (of Example 5.1.1).

6. Show that the event horizon at $r = r_+$ (which is a three-dimensional, null hypersurface of two-dimensional metric) has the area

$$\mathcal{A}_H^+ = 4\pi \cdot \left(r_+^2 + a^2\right) = 8\pi m \cdot \left(m + \sqrt{m^2 - a^2}\right).$$

7. Consider the extended mapping of the set of extended real numbers furnished by

$$\tanh^* : \mathbb{R}^* \longrightarrow \mathbb{R}^*,$$

$$\text{codomain}\left(\tanh^*\right) = [-1, 1] \subset \mathbb{R}^*.$$

Let a two-dimensional compactification be defined by

$$U_+ = \tanh^*(u), \quad V_+ = \tanh^*(v), \quad (u, v) \in \mathbb{R}^* \times \mathbb{R}^*.$$

Show that metric (5.94) of the two-dimensional submanifold yields

$$\left(\mathrm{d}\sigma_+^*\right)^2 = \frac{\Delta(r)\cdot(\mathrm{d}U_+)\cdot(\mathrm{d}V_+)}{p_+^2\cdot(r^2+a^2)\cdot \operatorname{Arg}\tanh^*(U_+)\cdot \operatorname{Arg}\tanh^*(V_+)\cdot\left(1-U_+^2\right)\cdot\left(1-V_+^2\right)}.$$

8. Consider *the Kerr–Newman metric* of (4.136) stating

$$\mathrm{d}s^2 = \Sigma(\cdot\cdot)\cdot\left[\frac{(\mathrm{d}r)^2}{\widetilde{\Delta}(\cdot\cdot)} + (\mathrm{d}\theta)^2\right] + \left(r^2+a^2\right)\cdot\sin^2\theta\cdot(\mathrm{d}\varphi)^2$$

$$-(\mathrm{d}t)^2 + \left[\frac{2mr-e^2}{\Sigma(\cdot\cdot)}\right]\cdot\left[a\cdot\sin^2\theta\cdot\mathrm{d}\varphi - \mathrm{d}t\right]^2,$$

$$\Sigma(\cdot\cdot) := r^2 + a^2\cdot\cos^2\theta, \quad \widetilde{\Delta}(r) := r^2 - 2mr + a^2 + e^2,$$

$$0 < \sqrt{a^2+e^2} < m,$$

$$D_4 := \left\{(r,\theta,\varphi,t) :\ m + \sqrt{m^2 - e^2 - a^2\cos^2\theta} < r,\ 0<\theta<\pi,\right.$$
$$\left. -\pi < \varphi < \pi,\ -\infty < t < \infty\right\}.$$

(Recall that e is the charge parameter here.)

(i) Prove that two horizons and two ergosurfaces are furnished by

$$r = r_\pm := m \pm \sqrt{m^2 - a^2 - e^2}\,,$$

$$r = \mathcal{R}_\pm(\theta) := m \pm \sqrt{m^2 - e^2 - a^2\cos^2\theta}\,.$$

(ii) Show that Kerr–Newman metric can be analytically extended by two charts (covering $r = r_+$ and $r = r_-$, respectively) characterized in the following:

$$(\mathrm{d}s_\pm)^2 = \frac{a^2\cdot[G_\pm(\cdot\cdot)]^2\cdot\sin^2\theta\cdot(r-r_\mp)\cdot(r+r_\pm)}{4p_\pm^2\cdot\Sigma(\cdot\cdot)\cdot(r^2+a^2)\cdot(r_\pm^2+a^2)}$$

$$\times\left[\frac{\Sigma(\cdot\cdot)}{r^2+a^2} + \frac{\Sigma(r_\pm,\theta)}{r_\pm^2+a^2}\right]\cdot\left(u_\pm^2\cdot\mathrm{d}v_\pm^2 + v_\pm^2\cdot\mathrm{d}u_\pm^2\right)$$

$$+\frac{G_\pm(\cdot\cdot)\cdot(r-r_\mp)}{2p_\pm^2\cdot\Sigma(\cdot\cdot)}\cdot\left[\frac{(\Sigma(\cdot\cdot))^2}{(r^2+a^2)^2} + \frac{(\Sigma(r_\pm,\theta))^2}{\left(r_\pm^2+a^2\right)^2}\right]\cdot(\mathrm{d}u_\pm)\cdot(\mathrm{d}v_\pm)$$

$$+\frac{a^2\cdot[G_\pm(\cdot\cdot)]^2\cdot\sin^2\theta\cdot(r+r_\pm)^2}{4p_\pm^2\cdot\Sigma(\cdot\cdot)\cdot\left(r_\pm^2+a^2\right)}\cdot(u_\pm\cdot\mathrm{d}v_\pm - v_\pm\cdot\mathrm{d}u_\pm)^2$$

$$
\begin{aligned}
&+ \frac{a \cdot G_{\pm}(\cdot\cdot) \cdot \sin^2\theta}{p_{\pm} \cdot \Sigma(\cdot\cdot) \cdot \left(r_{\pm}^2 + a^2\right)} \cdot \big[\Sigma(r_{\pm}, \theta) \cdot (r - r_{\mp}) \\
&\qquad\qquad\qquad\qquad + (r^2 + a^2) \cdot (r + r_{\pm})\big] \\
&\times (u_{\pm} \cdot \mathrm{d}v_{\pm} - v_{\pm} \cdot \mathrm{d}u_{\pm}) \cdot \mathrm{d}\varphi_{\pm} \\
&+ \Sigma(\cdot\cdot) \cdot (\mathrm{d}\theta)^2 + \left[\frac{\left(r^2 + a^2\right)^2 - a^2 \cdot \widetilde{\Delta}(r) \cdot \sin^2\theta}{\Sigma(\cdot\cdot)}\right] \cdot \sin^2\theta \cdot (\mathrm{d}\varphi_{\pm})^2;
\end{aligned}
$$

$$
G_{\pm}(\cdot\cdot) := \frac{r - r_{\pm}}{u_{\pm} \cdot v_{\pm}} \,.
$$

Remarks: (i) The upper and lower signs are to be read *separately*.
(ii) Here, $r_{\pm} = m \pm \sqrt{m^2 - a^2 - \mathrm{e}^2}$ are *different from those* in (5.66).
(iii) Here, $r = r_+$ denotes an event horizon of the Kerr–Newman solution, whereas $r = r_-$ represents a Cauchy horizon [106].

Answers and Hints to Selected Exercises

2.

$$
\begin{aligned}
\lambda_{(3)}(\cdot\cdot) \cdot \lambda_{(4)}(\cdot\cdot) &= g_{33} \cdot g_{44} - (g_{34})^2 \\
&= -\Delta(\cdot\cdot) \cdot \sin^2\theta.
\end{aligned}
$$

4. (i)

$$
\begin{aligned}
x^1 + \mathrm{i}x^2 &= (r + \mathrm{i}a) \cdot \sin\theta \cdot \exp\left[\mathrm{i}\int \left(\mathrm{d}\varphi + a \cdot \Delta^{-1} \cdot \mathrm{d}r\right)\right], \\
x^3 &= r\cos\theta, \\
x^4 &= \int \left[\mathrm{d}t + \left(r^2 + a^2\right) \cdot \Delta^{-1} \cdot \mathrm{d}r\right] - r.
\end{aligned}
$$

6. Consider the metric in (5.93). Noting that $\Delta(r_+) = 0 = G(\cdot\cdot)_{|r_+}$, the induced metric is given by

$$
\mathrm{d}\sigma^2 := \mathrm{d}s^2{}_{|r=r_+} = \Sigma(r_+, \theta) \cdot (\mathrm{d}\theta)^2 + \left[\frac{\left(r_+^2 + a^2\right)^2}{\Sigma(r_+, \theta)}\right] \cdot \sin^2\theta \cdot (\mathrm{d}\varphi)^2.
$$

Therefore,

$$\mathcal{A}_H^+ = \int_{0_+}^{\pi_-} \int_{-\pi_+}^{\pi_-} \sqrt{g} \cdot d\theta \cdot d\varphi = \int_{0_+}^{\pi_-} \int_{-\pi_+}^{\pi_-} \left(r_+^2 + a^2\right) \cdot \sin\theta \cdot d\theta \cdot d\varphi.$$

7.

$$(du) \cdot (dv) = \frac{1}{\left(1 - U_+^2\right) \cdot \left(1 - V_+^2\right)} \cdot (dU_+) \cdot (dV_+).$$

8. (ii)

$$r_* := r + \frac{1}{2p_+} \cdot \ln|r - r_+| + \frac{1}{2p_-} \cdot \ln|r - r_-|,$$

$$|U_\pm| = \exp\left[p_\pm \cdot (r_* - t)\right],$$

$$|V_\pm| = \exp\left[p_\pm \cdot (r_* + t)\right],$$

$$\varphi_\pm = \varphi - \frac{a \cdot t}{\left(r_\pm^2 + a^2\right)},$$

$$\theta_\pm = \theta.$$

5.3 Exotic Black Holes

This section lies outside the main focus of this chapter and may be considered optional for an introductory curriculum. Several of the models presented in this section may apparently not seem particularly physical. However, one example we present below may represent one class of elastic body, an anisotropic fluid, or a perfect fluid with charge. This section also serves to illustrate how many of the techniques studied earlier are employed when considering the T-domain of a black hole space–time and such models are therefore pedagogically useful. Another important issue illustrated here is that physical measurements are *local*. For example, it is shown below that demanding a well-behaved space–time structure in the exterior of the event horizon does not preclude exotic material with bizarre properties as the source of the gravitational field is *inside the event horizon*.

From the preceding sections, it is evident that cases of massive objects imploding into black holes may be physically plausible, according to the theory of general relativity. (Compare Figs. 5.10–5.13.) However, there are examples of *eternal black holes* and *eternal white holes* [106], where only "singularities" reside inside horizons for all time. Thus, these "material contents" are always *completely hidden* from the view of external observers. The prime example of an eternal black hole (together with *an eternal white hole*) is provided by the maximally extended Schwarzschild metric. (See Figs. 5.5 and 5.6.) Furthermore, there exist maximally extended Kerr metrics for which horizons hide denumerably infinite number of connected universes (containing black holes) from any observer in the external space–time vacuum. (See Fig. 5.19.)

We have other examples of eternal black holes, where the "material" inside the event horizon may be *exotic* (*violating energy conditions in some way*). To explore such black holes, let us go back to T-domain solutions in (3.107) and (3.108i–v). (See, e.g., [73].) The corresponding spherically symmetric metric is furnished by

$$\mathrm{d}s^2 = \mathrm{e}^{\lambda(T,R)} \cdot (\mathrm{d}R)^2 + T^2 \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right] - \mathrm{e}^{\nu(T,R)} \cdot (\mathrm{d}T)^2, \quad (5.102\text{i})$$

$$D_4 := \big\{(R,\theta,\varphi,T) : R_1 < R < R_2,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ T_1 < T < T_2\big\}. \quad (5.102\text{ii})$$

Consider Einstein's interior field equations containing exotic material as

$$\mathcal{E}^i{}_j(\cdot\cdot) := G^i{}_j(\cdot\cdot) + \kappa\, \Theta^i{}_j(\cdot\cdot) = 0, \quad (5.103\text{i})$$

$$\mathcal{T}^i(\cdot\cdot) := \nabla_j \Theta^{ij} = 0. \quad (5.103\text{ii})$$

The spherically symmetric metric in (5.102i) yields, from (5.103i,ii), the following nontrivial differential equations:

$$\frac{1}{T^2} - \frac{\mathrm{e}^{-\nu(\cdot\cdot)}}{T} \cdot \left[\partial_4\nu - \frac{1}{T}\right] = -\kappa\, \Theta^1{}_1(T,R), \quad (5.104\text{i})$$

$$\frac{1}{T^2} - \frac{\mathrm{e}^{-\nu(\cdot\cdot)}}{T} \cdot \left[\partial_4\lambda + \frac{1}{T}\right] = -\kappa\, \Theta^4{}_4(\cdot\cdot), \quad (5.104\text{ii})$$

$$\frac{1}{T} \cdot \partial_1 \mathrm{e}^{-\nu} = -\kappa\, \Theta^4{}_1(\cdot\cdot), \quad (5.104\text{iii})$$

$$\frac{\mathrm{e}^{-\nu}}{2} \cdot \left[\partial_4\partial_4\lambda + \frac{1}{2}(\partial_4\lambda)^2 - \partial_4\lambda \cdot \partial_4\nu + \frac{1}{T}(\partial_4\lambda - \partial_4\nu)\right] - \frac{\mathrm{e}^{-\lambda}}{2} \cdot \left[\partial_1\partial_1\nu + \frac{1}{2} \cdot (\partial_1\nu)^2 - \partial_1\nu \cdot \partial_1\lambda\right] = -\kappa\, \Theta^2{}_2(\cdot\cdot) \equiv -\kappa\, \Theta^3{}_3(\cdot\cdot). \quad (5.104\text{iv})$$

The conservation equation (5.103ii) leads to

$$\partial_4\Theta^4{}_4 + \partial_1\Theta^1{}_4 + \frac{1}{2} \cdot (\partial_1\lambda + \partial_1\nu) \cdot \Theta^1{}_4(\cdot\cdot) - \frac{1}{2}\partial_4\lambda \cdot \Theta^1{}_1(\cdot\cdot) - \frac{2}{T} \cdot \Theta^2{}_2(\cdot\cdot) + \left(\frac{1}{2}\partial_4\lambda + \frac{2}{T}\right) \cdot \Theta^4{}_4(\cdot\cdot) = 0, \quad (5.105\text{i})$$

$$\partial_1 \Theta^1{}_1 + \partial_4 \Theta^4{}_1 + \left[\frac{1}{2} \cdot (\partial_4 \lambda + \partial_4 \nu) + \frac{2}{T}\right] \cdot \Theta^4{}_1(\cdot\cdot)$$
$$+ \frac{1}{2} \partial_1 \nu \cdot \left[\Theta^1{}_1(\cdot\cdot) - \Theta^4{}_4(\cdot\cdot)\right] = 0. \tag{5.105ii}$$

The general solution of the system of partial differential equations (5.104i–iv) and (5.105i,ii) is furnished by

$$\mathrm{e}^{-\nu(T,R)} = \frac{1}{T} \cdot \left[\sigma(R) - \kappa \int_{T_0}^{T} \Theta^1{}_1(\cdot\cdot) \cdot \left(T'\right)^2 \cdot \mathrm{d}T'\right] - 1$$
$$=: \frac{2\Xi(T,R)}{T} - 1, \tag{5.106i}$$

$$\mathrm{e}^{\lambda(T,R)} = \left[\frac{2\Xi(\cdot\cdot)}{T} - 1\right] \cdot \exp\left\{\beta(R) + \kappa \int_{T_0}^{T} \left[\frac{\Theta^4{}_4 - \Theta^1{}_1}{2\Xi(\cdot\cdot) - T'}\right] \cdot \left(T'\right)^2 \cdot \mathrm{d}T'\right\}, \tag{5.106ii}$$

$$\Theta^4{}_1(T,R) := \frac{2}{\kappa T^2} \cdot \partial_1 \Xi, \tag{5.106iii}$$

$$\Theta^1_2(\cdot\cdot) = \Theta^1_3(\cdot\cdot) = \Theta^2_3(\cdot\cdot) = \Theta^4_2(\cdot\cdot) = \Theta^4_3(\cdot\cdot) = 0, \tag{5.106iv}$$

$$\Theta^2{}_2(T,R) \equiv \Theta^3{}_3(T,R) := \frac{T}{2} \cdot \left(\partial_4 \Theta^4{}_4 + \partial_1 \Theta^1{}_4\right) + \left(1 + \frac{T}{4} \cdot \partial_4 \lambda\right) \cdot \Theta^4{}_4(\cdot\cdot)$$
$$+ \frac{T}{4} \cdot (\partial_1 \lambda + \partial_1 \nu) \cdot \Theta^1{}_4(\cdot\cdot) - \frac{T}{4} \cdot \partial_4 \lambda \cdot \Theta^1{}_1(\cdot\cdot)\,; \tag{5.106v}$$

$$D_4 := \{(R, \theta, \varphi, T) : \ R_1 < R < R_2, \ 0 < \theta < \pi, \ -\pi < \varphi < \pi,$$
$$0 < T_1 \leq T_0 < T < T_2 < 2\Xi(T,R)\}. \tag{5.106vi}$$

Here, two functions of integration $\sigma(R)$ and $\beta(R)$ are of class C^4 and otherwise arbitrary. We note that the arbitrary function $\beta(R)$ can be eliminated by the coordinate transformation:

$$\widehat{R} = \int \exp\left[\beta(R)/2\right] \cdot \mathrm{d}R,$$
$$\left(\widehat{\theta}, \widehat{\varphi}, \widehat{T}\right) = (\theta, \varphi, T). \tag{5.107}$$

Subsequently, we will be *dropping hats* on the coordinates. We now have for (5.106ii):

$$e^{\lambda(T,R)} = \left[\frac{2\Xi(\cdot\cdot)}{T} - 1\right] \cdot \exp\left\{\kappa \int_{T_0}^{T} \left[\frac{\Theta^4{}_4 - \Theta^1{}_1}{2\Xi(\cdot\cdot) - T'}\right] \cdot \left(T'\right)^2 \cdot dT'\right\}. \qquad (5.108)$$

It is evident that there exist dualities between the set of solutions in (5.106i–iv) and the "R-domain" solutions of the Theorem 3.3.1. However, the present T-domain solutions in (5.106i–iv) yield completely new, *exotic physics*.

We shall discuss a *special case* of such a black hole inherent in #4 of Exercises 3.3. We recapitulate the metric as

$$ds^2 = \left[\frac{3\sqrt{qb^2-1} - \sqrt{qT^2-1}}{3\sqrt{qb^2-1} - 1}\right]^2 \cdot (dR)^2 + T^2 \cdot \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right]$$

$$- \left[qT^2 - 1\right]^{-1} \cdot (dT)^2 ; \qquad (5.109i)$$

$$q > 0, \quad qb^2 > 1 ; \qquad (5.109ii)$$

$$\mathcal{D}_{4b} := \{(R,\theta,\varphi,T) : -\infty < R < \infty,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,$$
$$(1/\sqrt{q}) < T < b\}. \qquad (5.109iii)$$

The metric in (5.109i) satisfies Einstein's interior field equations:

$$\mathcal{E}_{ij}(\cdot\cdot) := G_{ij}(\cdot\cdot) + \kappa\, \Theta_{ij}(\cdot\cdot) = 0, \qquad (5.110i)$$

$$\mathcal{T}^i(\cdot\cdot) := \nabla_j \Theta^{ji} = 0, \qquad (5.110ii)$$

$$\Theta^{ij}(\cdot\cdot) := -\left[\sigma(\cdot\cdot) + 3\kappa^{-1} \cdot q\right] \cdot s^i(\cdot\cdot) \cdot s^j(\cdot\cdot) + \sigma(\cdot\cdot) g^{ij}(\cdot\cdot), \qquad (5.110iii)$$

$$s^1(\cdot\cdot) s_1(\cdot\cdot) \equiv 1, \quad s^2(\cdot\cdot) = s^3(\cdot\cdot) = s^4(\cdot\cdot) \equiv 0, \qquad (5.110iv)$$

$$\sigma(T) := -3\kappa^{-1} \cdot q \cdot \left[\frac{\sqrt{qb^2-1} - \sqrt{qT^2-1}}{3\sqrt{qb^2-1} - \sqrt{qT^2-1}}\right]. \qquad (5.110v)$$

It is clear from (5.110iv) that $s^i(\cdot\cdot)\frac{\partial}{\partial x^i}$ is *a spacelike vector field*, and thus, *no energy condition is satisfied*. We are dealing with an exotic material of Segre characteristic $[1, (1, 1), 1]$. *For the outside vacuum metric*, we choose

$$\mathrm{d}s^2 = \left[\frac{2}{3\sqrt{qb^2-1}-1}\right]^2 \cdot \left[\frac{2m}{T}-1\right] \cdot (\mathrm{d}R)^2 + T^2 \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right]$$
$$- \left[\frac{2m}{T}-1\right]^{-1} \cdot (\mathrm{d}T)^2, \tag{5.111i}$$

$$\mathcal{D}_4 := \{(R, \theta, \varphi, T) : -\infty < R < \infty,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,$$
$$b < T < 2m\}, \tag{5.111ii}$$

$$2m := qb^3 > 0. \tag{5.111iii}$$

It is evident that the metric tensor components from (5.109i) and (5.111i) are *continuous across the boundary* $T = b$. This boundary is now *spacelike* and denotes an instant in time. Although a spacelike matter-vacuum boundary such as this may not be particularly physical, this boundary is *inside the event horizon*. It may also be noted that these techniques (the implementation of these boundary conditions) could be used to join *two different matter fields* at a spacelike boundary. This could be useful, for example, if the material undergoes an abrupt phase transition at some instant in time.

Synge's junction conditions (2.170) on the boundary hypersurface $T = b$ reduce to

$$\Theta^4{}_4(b_-) = \sigma(b_-) = 0. \tag{5.112}$$

Thus, Synge's junction conditions are *satisfied* on the hypersurface of jump discontinuities by (5.110v).

Now, let us examine I–S–L–D junction conditions (2.171) on the boundary $T = b$. The nonzero components of the extrinsic curvature tensor from the interior metric (5.109i) are furnished by (1.249) as

$$K_{11}(\cdot\cdot)_{|b_-} = \frac{-2qb \cdot \sqrt{qb^2-1}}{\left[3\sqrt{qb^2-1}-1\right]^2}, \tag{5.113i}$$

$$K_{22}(\cdot\cdot)_{|b_-} = b\sqrt{qb^2-1}, \quad K_{33}(\cdot\cdot)_{|b_-} = \sin^2\theta \cdot K_{22}(\cdot\cdot)_{|b_+}. \tag{5.113ii}$$

On the other hand, from the vacuum metric in (5.111i), the nonzero components of the extrinsic curvature, in the limit $T \to b_+$, are provided by

$$K_{11}(\cdot\cdot)_{|b_+} = -\frac{4m}{b^2} \cdot \sqrt{\frac{2m}{b}-1} \cdot \left[3\sqrt{\frac{2m}{b}-1}-1\right]^{-2}, \tag{5.114i}$$

$$K_{22}(\cdot\cdot)_{|b_+} = b\sqrt{\frac{2m}{b}-1}, \quad K_{33}(\cdot\cdot)_{|b_+} = \sin^2\theta \cdot K_{22}(\cdot\cdot)_{|b_+}. \tag{5.114ii}$$

Comparing (5.113i,ii) with (5.114i,ii), noting that $2m = qb^3$, we conclude that *the I–S–L–D junction conditions are satisfied on* $T = b$.

Now, we shall transform the T-domain vacuum metric of (5.111i,ii) into the black hole metric in the Kruskal–Szekeres charts of (5.19) and (5.26). It is easier to work out the details in the case of the two-dimensional submanifold for $\theta =$ const., $\varphi =$ const. The corresponding two-dimensional metric from (5.109i,iii) is

$$(\mathrm{d}\sigma)^2 := \left[\frac{3\sqrt{qb^2-1} - \sqrt{qT^2-1}}{3\sqrt{qb^2-1}-1}\right]^2 \cdot \mathrm{d}R^2 - \left[qT^2 - 1\right]^{-1} \cdot (\mathrm{d}T)^2, \quad (5.115\text{i})$$

$$\mathcal{D}_b^{\#} := \left\{(T, R) \in \mathbb{R}^2 : -\infty < R < \infty,\ \left(1/\sqrt{q}\right) < T < b\right\}. \quad (5.115\text{ii})$$

Similarly, the vacuum metric in (5.111i,ii) reduces to

$$(\mathrm{d}\sigma)^2 = \left[\frac{2}{3\sqrt{qb^2-1}-1}\right]^2 \cdot \left[\frac{2m}{T} - 1\right] \cdot (\mathrm{d}R)^2 - \left[\frac{2m}{T} - 1\right]^{-1} \cdot (\mathrm{d}T)^2, \quad (5.116\text{i})$$

$$\mathcal{D}_{\mathrm{II}} := \left\{(T, R) \in \mathbb{R}^2 : -\infty < R < \infty,\ b < T < 2m\right\}. \quad (5.116\text{ii})$$

Now, using (5.26), we make the following coordinate transformation from the Kruskal–Szekeres chart:

$$R = m\left[3\sqrt{qb^2-1} - 1\right] \cdot \ln(v/u), \quad (5.117\text{i})$$

$$T = (2m) \cdot \left[1 + \mathsf{W}\left(-\mathrm{e}^{-1} \cdot uv\right)\right]; \quad (5.117\text{ii})$$

$$D_{\mathrm{II}} := \left\{(u, v) \in \mathbb{R}^2 : u > 0,\ v > 0,\ uv < b^{\#}\right\}, \quad (5.117\text{iii})$$

$$b^{\#} := -e \cdot \mathsf{W}^{-1}\left(\frac{b}{2m} - 1\right). \quad (5.117\text{iv})$$

Here, W is the Lambert W-Function of Fig. 5.4, and W^{-1} is its inverse.

We can analytically continue the Kruskal–Szekeres chart into the usual domain D_{I} characterized by $u < 0$ and $v > 0$. (See Fig. 5.5.)

We summarize the spherically symmetric, eternal, exotic black hole of (5.115i,ii) and (5.116i,ii) with this analytic extension in Fig. 5.20.

(In the preceding example, we have *not explored the maximal analytical extension* of the manifold or the chart.)

Now we shall investigate another case of spherically symmetric, exotic black hole, namely, a fluid or an elastic body collapsing to form a black hole [67]. We assume that the energy–momentum–stress tensor is due to an anisotropic fluid (or an elastic body with symmetries). This class of fluid has been discussed in (2.181)

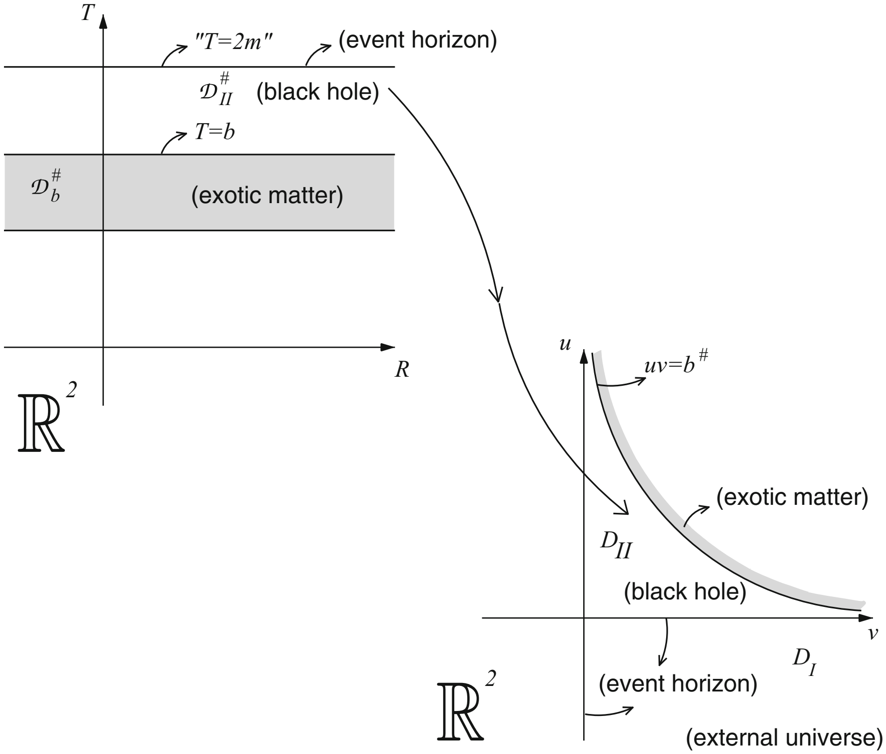


Fig. 5.20 Qualitative representation of *an exotic black hole* in the T-domain and the Kruskal–Szekeres chart

and (2.264). We choose a special exact solution mentioned in Example 3.3.3. It is furnished by the following equations:

$$M(r,t) := \left(\frac{q}{2}\right) \cdot \left\{ \frac{r}{\left[1 - \frac{t}{c_2}\right]} \right\}^j, \tag{5.118i}$$

$$\begin{aligned} ds^2 = {} & \left\{1 - \frac{q r^{j-1}}{[1-(t/c_2)]^j}\right\}^{-1} \cdot (dr)^2 + r^2 \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right] \\ & - \left\{1 - \frac{q r^{j-1}}{[1-(t/c_2)]^j}\right\}^{-1} \cdot (dt)^2, \end{aligned} \tag{5.118ii}$$

$$0 < q < 1, \quad 2 < j \leq 3, \tag{5.118iii}$$

$$k := \sqrt{j/(j-2)}, \quad \sqrt{3} \leq k < \infty, \tag{5.118iv}$$

$$b > 0, \;\; c_2 := kb > 0, \;\; 2m := qb^j > 0, \;\; c_1 := c_2 - 2mk < c_2. \tag{5.118v}$$

The two-dimensional, conformally flat metric for the submanifold characterized by $\theta = \text{const.}$, $\varphi = \text{const.}$ (and interior to the body) is given by

$$(\mathrm{d}\sigma)_b^2 = \left\{1 - \frac{qr^{j-1}}{[1-(t/c_2)]^j}\right\}^{-1} \cdot \left[(\mathrm{d}r)^2 - (\mathrm{d}t)^2\right], \tag{5.119i}$$

$$\begin{aligned} D_b := &\{(r,t) : 0 < r < B(t), \ -\infty < t < c_1\} \cup \\ &\{(r,t) : 0 < r < \Sigma(t), \ c_1 < t < c_2\} \cup \{\text{boundary pt.s}\}, \end{aligned} \tag{5.119ii}$$

$$B(t) := k^{-1} \cdot (c_2 - t), \quad \left|B'(t)\right| = k^{-1} < 1, \tag{5.119iii}$$

$$\Sigma(t) := \left\{q^{-1} \cdot [1-(t/c_2)]^j\right\}^{1/j-1}. \tag{5.119iv}$$

Here, $r = B(t)$ represents the boundary of the spherical body, and $r = \Sigma(t)$ stands for the curvature singularity (to be discussed later). The exterior metric joining continuously with (5.119i) is provided by

$$(\mathrm{d}\sigma)_{\mathrm{I}}^2 = \left[1 - \frac{2m}{r}\right]^{-1} \cdot (\mathrm{d}r)^2 - \left[1 - \frac{2m}{r}\right] \cdot \left[\frac{c_2 - t}{c_1 - t}\right]^2 \cdot (\mathrm{d}t)^2, \tag{5.120i}$$

$$D_{\mathrm{I}} := \{(r,t) : 0 < B(t) < r < \infty, \ -\infty < t < c_1\}. \tag{5.120ii}$$

Note that the vacuum metric from (5.120i), according to Birkhoff's Theorem 3.3.2, *is locally transformable to the Schwarzschild metric* by the coordinate transformation:

$$\begin{aligned} \left(\widehat{r}, \widehat{\theta}, \widehat{\varphi}\right) &= (r, \theta, \varphi), \\ \widehat{t} &= t - (c_2 - c_1) \cdot \ln|1 - (t/c_1)| + \text{const.} \end{aligned} \tag{5.121}$$

Now, the nonzero components of the energy–momentum–stress tensor from (5.118ii) are furnished by

$$\kappa\, T^1{}_1(r,t) = \frac{\left[(j-2) \cdot q \cdot r^{j-3}\right]}{[1-(t/c_2)]^j} > 0, \tag{5.122i}$$

$$\kappa\, T^1{}_4(\cdot\cdot) = \frac{jq \cdot r^{j-2}}{c_2 \cdot [1-(t/c_2)]^{j+1}} \equiv -\kappa T^4{}_1(\cdot\cdot), \tag{5.122ii}$$

$$\kappa\, T^4{}_4(\cdot\cdot) = -\frac{jq \cdot r^{j-3}}{[1-(t/c_2)]^j} < 0, \tag{5.122iii}$$

$$T^4{}_4(\cdot\cdot) + k^2 \cdot T^1{}_1(\cdot\cdot) \equiv 0, \tag{5.122iv}$$

$$-\kappa \cdot \int_{0+}^{B(t)} T^4{}_4(r,t) \cdot r^2 \cdot \partial r = qb^j = 2m. \tag{5.122v}$$

As well, from the energy–momentum–stress tensor of (2.181) and (2.264), we derive that

$$T_{ij}(\cdot\cdot) = (\rho + p_\perp)\, u_i u_j + p_\perp \cdot g_{ij} + \left(p_\parallel - p_\perp\right) s_i s_j, \tag{5.123i}$$

$$u^i u_i = -1, \quad s^i s_i = 1, \quad u^i s_i = 0, \tag{5.123ii}$$

$$T_{ij}(\cdot\cdot)\, u^j = -\rho(\cdot\cdot)\, g_{ij} u^j, \tag{5.123iii}$$

$$T_{ij}(\cdot\cdot)\, s^j = p_\parallel(\cdot\cdot)\, g_{ij} s^j. \tag{5.123iv}$$

The above equations illustrate that the invariant eigenvalues of $[T^i{}_j]$ are $-\rho,\ p_\parallel$, and $p_\perp$, respectively. Moreover, we satisfy a particular field equation by defining

$$\kappa\, p_\perp(r,t) \equiv \kappa\, T^2{}_2(r,t) := -G^2{}_2(r,t). \tag{5.124}$$

Using (5.122i–iii), (5.123i–iv), and (5.124), we deduce that

$$\rho(r,t) = \frac{q r^{j-3} \cdot \left[\sqrt{(j-1)^2 \cdot (c_2 - t)^2 - (jr)^2} + c_2 - t\right]}{\kappa\, c_2 \cdot [1 - (t/c_2)]^{j+1}}, \tag{5.125i}$$

$$p_\parallel(\cdot\cdot) = \frac{q r^{j-3} \cdot \left[\sqrt{(j-1)^2 \cdot (c_2 - t)^2 - (jr)^2} - c_2 + t\right]}{\kappa\, c_2 \cdot [1 - (t/c_2)]^{j+1}}, \tag{5.125ii}$$

$$\begin{aligned} p_\perp(\cdot\cdot) = {} & \frac{q r^{j-3} \cdot \left[(j-1) \cdot (j-2) \cdot (c_2 - t)^2 - j(j+1) \cdot r^2\right]}{2\kappa \cdot (c_2)^2 \cdot [1 - (t/c_2)]^{j+2}} \\ & + \frac{q^2 r^{2(j-2)} \cdot \left[(j-1)^2 \cdot (c_2 - t)^2 - (jr)^2\right]}{2\kappa \cdot (c_2)^2 \cdot [1 - (t/c_2)]^{2j+2} \cdot [1 - (q r^{j-1})/(1 - t/c_2)^j]}. \end{aligned} \tag{5.125iii}$$

Using (5.125i–iii), we can prove that, in this case, the weak energy conditions of (2.268i), namely,

$$\rho \geq 0, \quad \rho + p_\perp \geq 0, \quad \rho + p_\parallel \geq 0$$

are satisfied by $\rho(\cdot\cdot),\ p_\perp(\cdot\cdot)$, and $p_\parallel(\cdot\cdot)$ in (5.125i–iii).

Now, we shall investigate Synge's junction conditions (2.170) and (3.85i,ii). The four equations reduce to the following two nontrivial conditions:

$$\begin{aligned} & \kappa \left[T^1{}_1(r,t) \cdot 1 + T^4{}_1(\cdot\cdot) \cdot k^{-1}\right]_{|r=B(t)} \\ & = \left\{ \frac{(j-2) \cdot q \cdot r^{j-3}}{[1 - (t/c_2)]^j} - \frac{j \cdot q \cdot r^{j-2}}{k\, c_2\, [1 - (t/c_2)]^{j+1}} \right\}_{|r=k^{-1}\cdot(c_2 - t)} \equiv 0, \end{aligned} \tag{5.126i}$$

$$\kappa\left[T^{1}{}_{4}(\cdot\cdot)\cdot 1\ +T^{4}{}_{4}(\cdot\cdot)\cdot k^{-1}\right]_{|\cdot\cdot}$$
$$=\left\{\frac{1\cdot q\cdot r^{j-2}}{c_2\cdot[1-(t/c_2)]^{j+1}}-\frac{j\cdot q\cdot r^{j-3}}{k\cdot[1-(t/c_2)]^{j}}\right\}_{|\cdot\cdot}\equiv 0. \tag{5.126ii}$$

Thus, *Synge's junction conditions hold on the boundary* of the collapsing body.

Let us consider the two-dimensional submanifold metric outside matter, joining continuously with the interior metric of (5.119i). It is provided by

$$(\mathrm{d}\sigma)^2_{\mathrm{I}}=\left(1-\frac{2m}{r}\right)^{-1}\cdot(\mathrm{d}r)^2-\left(1-\frac{2m}{r}\right)\cdot\left(\frac{c_2-t}{c_1-t}\right)^2\cdot(\mathrm{d}t)^2, \tag{5.127i}$$

$$D_{\mathrm{I}}:=\{(r,t):\ 0<B(t)<r,\ -\infty<t<c_1\}. \tag{5.127ii}$$

It follows from (5.119ii) and (5.119iii) that

$$\lim_{t\to c_{1-}}[B(t)]=k^{-1}\cdot(c_2-c_1)=2m. \tag{5.128}$$

Thus, the collapsing spherical body reaches *the event horizon* in the limit $t\to c_{1-}$. Other horizons are furnished from the equations:

$$g^{ij}(\cdot\cdot)\cdot(\partial_i r)\cdot(\partial_j r)=1-\frac{qr^{j-1}}{[1-(t/c_2)]^j}=0, \tag{5.129i}$$

$$g^{ij}(\cdot\cdot)\cdot(\partial_i t)\cdot(\partial_j t)=-\left\{1-\frac{qr^{j-1}}{[1-(t/c_2)]^j}\right\}=0, \tag{5.129ii}$$

$$\text{or}\qquad r=\Sigma(t):=\left\{q^{-1}\cdot[1-(t/c_2)]^j\right\}^{1/j-1}. \tag{5.129iii}$$

We shall call this *the singular horizon*. (In the $\lim\limits_{r\to\Sigma(t)}\left|R^{ijkl}(\cdot\cdot)\cdot R_{ijkl}(\cdot\cdot)\right|\longrightarrow\infty$.)

We now analytically extend the two-dimensional manifold and the chart in (5.119i) and (5.127i). The metric (5.127i) is obviously *undefined* for $t=c_1$ and $t=c_2$. Extending beyond "$t=c_1$," we have a metric:

$$(\mathrm{d}\sigma)^2_{\mathrm{III}}:=\left(1-\frac{2m}{r^{\#}}\right)^{-1}\cdot\left(\mathrm{d}r^{\#}\right)^2-\left(1-\frac{2m}{r^{\#}}\right)\cdot\left(\frac{c_2-t^{\#}}{t^{\#}-c_1}\right)^2\cdot\left(\mathrm{d}t^{\#}\right)^2, \tag{5.130i}$$

$$D_{\mathrm{III}}:=\left\{\left(r^{\#},t^{\#}\right):\ 2m<r^{\#}<\infty,\ c_1<t^{\#}<c_2\right\}. \tag{5.130ii}$$

(See Fig. 5.21 for different domains.) Extending beyond "$t^{\#}={c_2}^{\#}$," we have another external vacuum domain characterized by

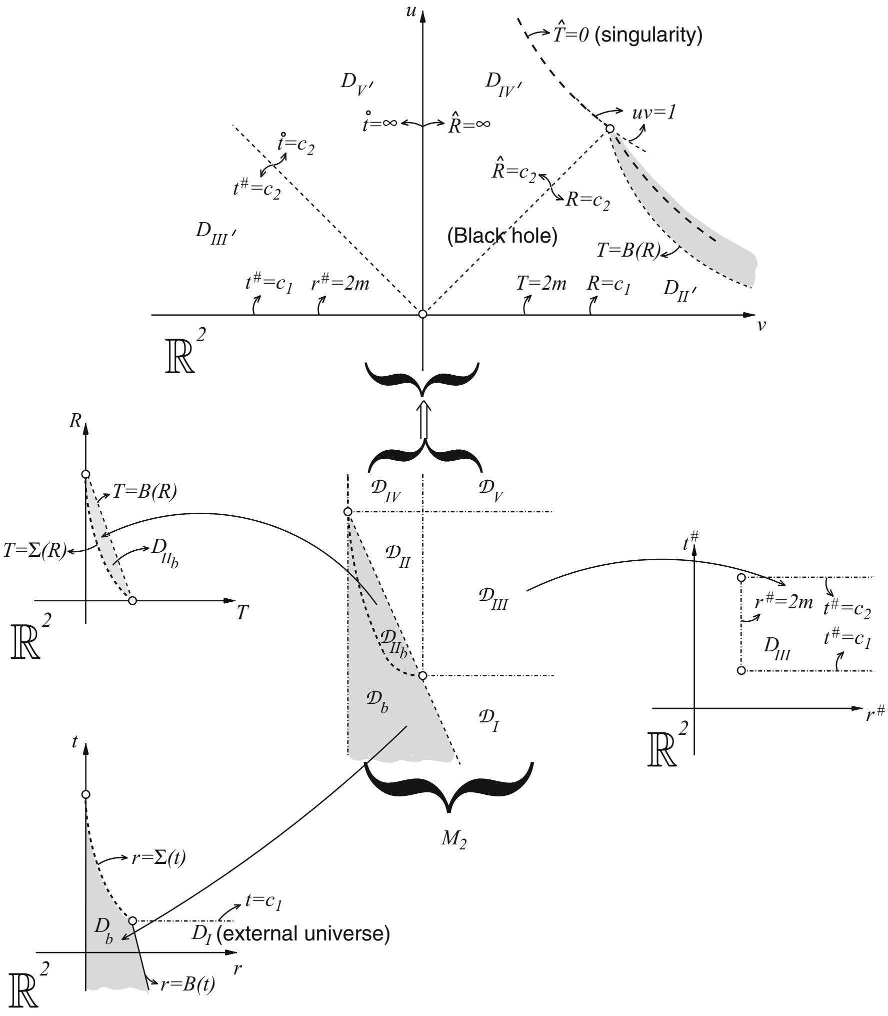


Fig. 5.21 Collapse into an exotic black hole depicted by four coordinate charts

$$(\mathrm{d}\sigma)^2_{\mathrm{V}} := \left(1 - \frac{2m}{\overset{\circ}{r}}\right)^{-1} \cdot \left(\mathrm{d}\overset{\circ}{r}\right)^2 - \left(1 - \frac{2m}{\overset{\circ}{r}}\right) \cdot \left(\frac{\overset{\circ}{t} - c_2}{\overset{\circ}{t} - c_1}\right)^2 \cdot \left(\mathrm{d}\overset{\circ}{t}\right)^2, \qquad (5.131\mathrm{i})$$

$$D_{\mathrm{V}} := \left\{\left(\overset{\circ}{r}, \overset{\circ}{t}\right) : \, 2m < \overset{\circ}{r} < \infty, \; c_2 < \overset{\circ}{t} < \infty\right\}. \qquad (5.131\mathrm{ii})$$

Now, let us investigate the domains representing *the black hole*. The vacuum black hole domain D_{II} is characterized by

$$(\mathrm{d}\sigma)^2_{\mathrm{II}} := \left(\frac{2m}{T} - 1\right)^{-1} \cdot \left(\frac{c_2 - R}{R - c_1}\right)^2 \cdot (\mathrm{d}R)^2 - \left(\frac{2m}{T} - 1\right)^{-1} \cdot (\mathrm{d}T)^2, \quad (5.132\mathrm{i})$$

$$D_{\mathrm{II}} := \{(T, R) : \, c_1 < R < c_2, \; B(R) < T < 2m\}. \quad (5.132\mathrm{ii})$$

The domain representing collapsing material *inside* the black hole is provided by

$$(\mathrm{d}\sigma)^2_{\mathrm{IIb}} := \left\{ \frac{q \cdot T^{j-1}}{[1 - (R/c_2)]^j} - 1 \right\}^{-1} \cdot \left[(\mathrm{d}R)^2 - (\mathrm{d}T)^2\right], \quad (5.133\mathrm{i})$$

$$D_{\mathrm{IIb}} := \{(T, R) : \, c_1 < R < c_2, \; \Sigma(R) < T < B(R)\}. \quad (5.133\mathrm{ii})$$

The other vacuum domain D_{IV}, inside the black hole, is described by

$$(\mathrm{d}\sigma)^2_{\mathrm{IV}} := \left(\frac{2m}{\widehat{T}} - 1\right) \cdot \left(\frac{\widehat{R} - c_2}{\widehat{R} - c_1}\right)^2 \cdot \left(\mathrm{d}\widehat{R}\right)^2$$

$$- \left(\frac{2m}{\widehat{T}} - 1\right)^{-1} \cdot \left(\mathrm{d}\widehat{T}\right)^2, \quad (5.134\mathrm{i})$$

$$D_{\mathrm{IV}} := \left\{\left(\widehat{R}, \widehat{T}\right) : \, c_2 < \widehat{R} < \infty, \; 0 < \widehat{T} < 2m\right\}. \quad (5.134\mathrm{ii})$$

Finally, we shall introduce *the Kruskal–Szekeres chart* for the vacuum domain $D_{\mathrm{II}} \cup D_{\mathrm{III}} \cup D_{\mathrm{IV}} \cup D_{\mathrm{V}} \cup \{\text{some boundary points}\}$.

The coordinate transformation from the domain D_{III} into the K–S chart is furnished by

$$u = \sqrt{\left(r^\#/2m\right) - 1} \cdot \left[\left(t^\#/c_1\right) - 1\right]^{k/2} \cdot \mathrm{e}^{\left(r^\# - t^\#\right)/4m}, \quad (5.135\mathrm{i})$$

$$v = -\sqrt{\left(r^\#/2m\right) - 1} \cdot \left[\left(t^\#/c_1\right) - 1\right]^{-k/2} \cdot \mathrm{e}^{\left(r^\# + t^\#\right)/4m}, \quad (5.135\mathrm{ii})$$

$$r^\# = \widetilde{\mathfrak{Y}}(u, v) = (2m) \cdot \left[1 + \mathsf{W}\left(-\mathrm{e}^{-1}uv\right)\right], \quad (5.135\mathrm{iii})$$

$$\frac{\mathrm{e}^{t^\#/2m}}{\left[\left(t^\#/c_1\right) - 1\right]^k} = -\,(v/u). \quad (5.135\mathrm{iv})$$

The transformation from the domain D_{II} into the K–S chart is accomplished by

$$u = \sqrt{1 - (T/2m)} \cdot [(R/c_1) - 1]^{k/2} \cdot \mathrm{e}^{(T-R)/4m}, \quad (5.136\mathrm{i})$$

$$v = \sqrt{1 - (T/2m)} \cdot [(R/c_1) - 1]^{-k/2} \cdot \mathrm{e}^{(T+R)/4m}, \quad (5.136\mathrm{ii})$$

$$T = (2m) \cdot \left[1 + \mathsf{W}\left(-\mathrm{e}^{-1} uv\right)\right], \tag{5.136iii}$$

$$\frac{\mathrm{e}^{R/2m}}{[(R/c_1) - 1]^k} = (v/u), \tag{5.136iv}$$

$$D_{\mathrm{II}'} := \left\{(u, v) : 0 < u < \beta'(v),\ 0 < v < \Gamma(u)\right\}, \tag{5.136v}$$

$$\beta(v) := \mathrm{e}^{-c_2/2m} \cdot [(c_2/c_1) - 1]^k \cdot v, \tag{5.136vi}$$

$$v = \Gamma(u) : \tag{5.136vii}$$

$$\left[\begin{array}{l} v = \mathrm{e}^{R/4m} \cdot [(R/c_1) - 1]^{-k/2} \cdot \sqrt{1 - (c_2 - R/2mk)} \cdot \mathrm{e}^{(c_2 - R)/4km}, \\ u = \mathrm{e}^{-R/4m} \cdot [(R/c_1) - 1]^{+k/2} \cdot \sqrt{1 - (c_2 - R/2mk)} \cdot \mathrm{e}^{(c_2 - R)/4km}, \\ c_1 < R < c_2. \end{array}\right. \tag{5.136viii}$$

Similarly, the transformation from D_{IV} is furnished by

$$u = \sqrt{1 - \left(\widehat{T}/2m\right)} \cdot \left[\left(\widehat{R}/c_1\right) - 1\right]^{k/2} \cdot \mathrm{e}^{\left(\widehat{T} - \widehat{R}\right)/4m}, \tag{5.137i}$$

$$v = \sqrt{1 - \left(\widehat{T}/2m\right)} \cdot \left[\left(\widehat{R}/c_1\right) - 1\right]^{-k/2} \cdot \mathrm{e}^{\left(\widehat{T} + \widehat{R}\right)/4m}, \tag{5.137ii}$$

$$D_{\mathrm{IV}'} := \left\{(u, v) : 0 < u < \infty,\ 0 < v < u^{-1}\right\}. \tag{5.137iii}$$

The two-dimensional metric in the K–S chart for *all the above domains* is given by (5.33ii) as

$$(\mathrm{d}\sigma)^2 = \frac{-16(m)^2}{[1 + \mathsf{W}(-euv)]} \cdot \exp\left\{-[1 + \mathsf{W}(-euv)]\right\} \cdot (\mathrm{d}u) \cdot (\mathrm{d}v), \tag{5.138i}$$

$$D' := D_{\mathrm{II}'} \cup D_{\mathrm{III}'} \cup D_{\mathrm{IV}'} \cup D_{\mathrm{V}'} \cup \{\text{some boundary pts.}\}. \tag{5.138ii}$$

Coordinate transformations, starting from the (5.127i) and ending with (5.137iii), are depicted in Fig. 5.21.

Let us try to understand this complicated collapse into an exotic black hole in physical terms. In this solution of Einstein's equations, a spherically symmetric anisotropic fluid (or elastic) body, satisfying energy and jump conditions, is imploding under its own gravitational attraction. As soon as the external boundary hypersurface contracts into *the Schwarzschild radius* $r = 2m$, *a singularity* appears. The singular hypersurface starts to collapse faster than the outer boundary surface (*inside the black hole*). However, both of the hypersurfaces collapse with superluminous speed! Moreover, the interior to the singular hypersurface still holds *regular, anisotropic fluid*. Thus, the singular hypersurface keeps on absorbing regular fluid from the inside and *the tachyonic fluid* from the outer shell. Eventually, the whole of the exotic material inside the event horizon contracts into the ultimate singularity at $\widehat{T} = 0$. (See Fig. 5.21.)

In the realm of exotic black holes, we briefly mention the fact that, with the addition of a negative cosmological constant, black hole solutions to the field equations also exist with cylindrical, planar, toroidal, and higher genus topology [164, 235, 252].

The black holes considered so far all possess *curvature singularities*. One might be tempted to believe that all T-domain solutions must be singular. However, this is not necessarily the case. One example of a nonsingular T-domain is that present in the Reissner–Nordström–Jeffery solution (see Problem 4 in the exercises of Sect. 5.1) where, although there is a singularity, the singularity is *not in the T-domain*. There also exist examples of exotic, spherically symmetric T-domain solutions without curvature singularities. We cite the following as a simple example [73]:

$$\begin{aligned} \mathrm{d}s^2 = \left(2k \cdot T^{2+\nu} - 1\right) \cdot (\mathrm{d}R)^2 + T^2 \cdot \left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right] \\ - \left(2k \cdot T^{2+\nu} - 1\right)^{-1} \cdot (\mathrm{d}T)^2, \end{aligned} \tag{5.139i}$$

$$k > 0, \quad \nu > 0, \tag{5.139ii}$$

$$\begin{aligned} D := \big\{(R, \theta, \varphi, T) : -\infty < R < \infty,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi, \\ (2k)^{-1/(2+\nu)} < T < \infty\big\}. \end{aligned} \tag{5.139iii}$$

The nonzero, natural orthonormal components of the Riemann tensor are given by

$$R_{(1)(4)(1)(4)}(\cdot\cdot) = -k \cdot (2+\nu) \cdot (1+\nu) \cdot T^{\nu}, \tag{5.140i}$$

$$R_{(2)(4)(2)(4)}(\cdot\cdot) = -k \cdot (2+\nu) \cdot T^{\nu}, \tag{5.140ii}$$

$$R_{(1)(2)(1)(2)}(\cdot\cdot) = k \cdot (2+\nu) \cdot T^{\nu}, \tag{5.140iii}$$

$$R_{(2)(3)(2)(3)}(\cdot\cdot) = 2k \cdot T^{\nu}, \tag{5.140iv}$$

as well as those related by symmetry.

The above components are all defined inside the T-domain of the manifold (for $\nu > 0$) and therefore provide an example of a regular T-domain solution with a horizon (not necessarily a black hole, however). It can be shown that the matter field supporting the solution obeys an extreme (stiff) anisotropic *polytropic equation of state* [9] between $\rho(T)$ and the pressures. In the limit that $\nu \to 0$, one retrieves the *de Sitter space–time* of constant curvature.

---

[9] A polytropic equation of state is of the form $p \propto \rho^{\gamma_0}$, with γ_0 constant.

In regard to singularities, we have only considered *curvature singularities*. However, in general relativity, subtler criteria for the existence of singularities, involving *incompleteness of timelike, null, or spacelike geodesics* exist. There are many *singularity theorems* in the literature (see [126]). We shall conclude this section and this chapter by stating and proving *one of many singularity theorems* involving the collapse of material bodies satisfying energy conditions.

Theorem 5.3.1. *Let field equations* $G_{ij}(x) + \kappa\, T_{ij}(x) = 0$ *be satisfied in a domain* $D \subset \mathbb{R}^4$ *such that the energy–momentum–tensor is diagonalizable. Moreover, let strong energy conditions prevail. Furthermore, let the vorticity tensor* $\omega_{ij}(x) \cdot \mathrm{d}x^i \wedge \mathrm{d}x^j \equiv \mathbf{O}_{..}(x)$. *Suppose that a timelike geodesic congruence has a negative expansion scalar* $\Theta(x) < 0$ *in a neighborhood of* $D \subset \mathbb{R}^4$. *Then, each of the timelike geodesic* $x = \mathcal{X}(s)$ *will terminate at a singularity* $x^* = \mathcal{X}(s^*) \in \partial D$ *after a finite proper time.*

Proof. We consider the Raychaudhuri–Landau equation (2.202ii). In case of *a timelike geodesic and vanishing vorticity*, (2.202ii) reduces to

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \big\{ -\sigma^{jk}(x) \cdot \sigma_{jk}(x) - (1/3)\,[\Theta(x)]^2 + R_{jk}(x) \cdot U^j(x) \cdot U^k(x) \big\}_{|x=\mathcal{X}(s)}, \tag{5.141i}$$

$$y^{-1} = \theta(s) := [\Theta \circ \mathcal{X}]\,(s), \tag{5.141ii}$$

$$y = [\theta(s)]^{-1}, \tag{5.141iii}$$

$$s \in \big(0, s^*\big)\,. \tag{5.141iv}$$

Recalling *the strong energy conditions* (2.194iii) for a diagonalizable energy-momentum-tensor, we obtain from (5.141i) and #5.ii of Exercises 2.3

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} + \frac{1}{3}\,[\theta(s)]^2 = -\left\{ \sigma^{jk}(x) \cdot \sigma_{jk}(x) + \frac{\kappa}{2} \cdot \left[\rho(x) + \sum_{\mu} p_{(\mu)}(x) \right] \right\}_{|x=\mathcal{X}(s)}$$
$$=: -[f(s)]^2 \leq 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\,[\theta(s)]^{-1} = \frac{1}{3} + [f(s)]^2 \cdot [\theta(s)]^{-2} \geq 0. \tag{5.142}$$

We impose an initial condition:

$$\theta(s_0) = (y_0)^{-1} < 0,\qquad s_0 \in (0, s^*). \tag{5.143}$$

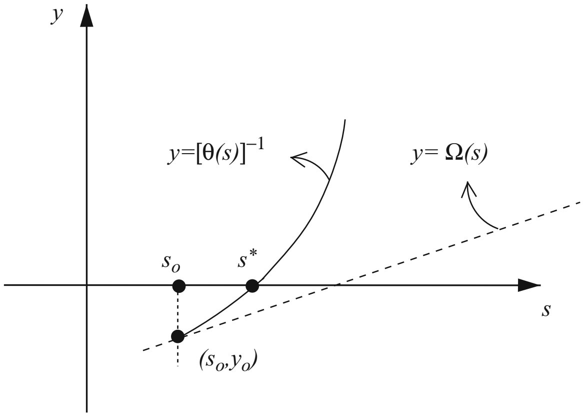


Fig. 5.22 Qualitative graphs of $y = [\theta(s)]^{-1}$ and the straight line $y = \Omega(s) := y_0 + (1/3) \cdot (s - s_0)$

The differential equation (5.142) and the initial condition (5.143) can be incorporated in the following nonlinear, integral equation:

$$y = [\theta(s)]^{-1} = (y_0) + (1/3) \cdot (s - s_0) + \int_{s_0}^{s} [f(w)]^2 \cdot [\theta(w)]^{-2} \cdot \mathrm{d}w,$$

$$(y_0) + (1/3) \cdot (s - s_0) =: \Omega(s) \leq [\theta(s)]^{-1},$$

$$y_0 = [\theta(s_0)]^{-1} = \Omega(s_0) < 0. \tag{5.144}$$

The graphs of the function $y = [\theta(s)]^{-1}$ and the straight line $y = \Omega(s)$ are depicted in Fig. 5.22.

It is evident from the continuity, differentiability, monotonicity, and the mean value theorem [32] that there exists a point $s^* > s_0 \geq 0$ such that $[\theta(s^*)]^{-1} = 0$. Therefore, we can conclude that

$$\lim_{s \to s^*_-} [\theta(s)]^{-1} = 0, \tag{5.145i}$$

$$\text{and} \quad \lim_{s \to s^*_-} [\theta(s)] \longrightarrow -\infty. \tag{5.145ii}$$

Thus, *the timelike geodesic reaches a singularity in a finite proper time.* ■

Chapter 6
Cosmology

6.1 Big Bang Models

General relativity has impacted the physical sciences in the most profound way with its cosmological models. Einstein himself proposed the first cosmological model [88]. The main motivations behind his static model were the following aspects of cosmology:

1. Isotropy of the spatial submanifold is dictated from large-scale observations. (This assumption is part of what is known as the *cosmological principle*.)
2. A finite average mass density is a reasonable expectation.
3. Moreover, *a finite spatial volume* is aesthetically appealing and a real possibility.

With these considerations, Einstein put forward the following "hypercylindrical," static metric [88]:

$$ds^2 = (R_0)^2 \cdot \left\{(d\chi)^2 + \sin^2\chi \cdot \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right]\right\} - (dt)^2, \tag{6.1i}$$

$$D := \{(\chi, \theta, \varphi, t) : 0 < \chi < \pi,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ -\infty < t < \infty\}. \tag{6.1ii}$$

The three-dimensional spatial hypersurface for $t = \text{const.}$ is *a space of (positive) constant curvature* as governed by (1.164i,ii). The finite total volume of the spatial hypersurface is given by

$$\int_{0+}^{\pi-}\int_{0_+}^{\pi-}\int_{-\pi+}^{\pi-} (R_0)^3 \cdot \sin^2\chi \cdot \sin\theta \cdot (d\chi)(d\theta)(d\varphi) = 2(\pi)^2 \cdot (R_0)^3 > 0. \tag{6.2}$$

Metric (6.1i) yields a three-dimensional spatial hypersurface admitting a six-parameter group of isometries. Moreover, *the hypercylindrical universe* is

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(topologically) homeomorphic to $S^3 \times \mathbb{R}$. The nontrivial Einstein tensor components from (6.1i) are provided by

$$G_{11}(\cdot\cdot) = (R_0)^{-2} \cdot g_{11}(\cdot\cdot), \tag{6.3i}$$

$$G_{22}(\cdot\cdot) = (R_0)^{-2} \cdot g_{22}(\cdot\cdot), \tag{6.3ii}$$

$$G_{33}(\cdot\cdot) = (R_0)^{-2} \cdot g_{33}(\cdot\cdot), \tag{6.3iii}$$

$$G_{44}(\cdot\cdot) = 3 \cdot (R_0)^{-2} \cdot g_{44}(\cdot\cdot), \tag{6.3iv}$$

$$G_{\alpha 4}(\cdot\cdot) \equiv 0. \tag{6.3v}$$

The interior field equations for a perfect fluid (2.257ii),

$$G_{ij}(\cdot\cdot) = -\kappa\left[(\rho(\cdot\cdot) + p(\cdot\cdot)) \cdot u_i(\cdot\cdot) \cdot u_j(\cdot\cdot) + p(\cdot\cdot) \cdot g_{ij}(\cdot\cdot)\right],$$

yield, from (6.3i–v) (noting that equation (6.3v) yields the comoving condition $u_\alpha \equiv 0$), that

$$\rho(x) + 3p(x) \equiv 0. \tag{6.4}$$

The above equation seemed to be *physically unacceptable* at the time, due to the fact that neither negative energy densities nor large negative pressures are observed in ordinary matter. Therefore, rather than relaxing the assumption of staticity, Einstein modified his equations to the following:

$$\mathcal{E}'_{ij}(\cdot\cdot) := G_{ij}(\cdot\cdot) - \Lambda\, g_{ij}(\cdot\cdot) + \kappa\, T_{ij}(\cdot\cdot) = 0. \tag{6.5}$$

Here, Λ is called the *cosmological constant.*[1] (Note that this modification is permissible by Theorem 2.2.8.) *One possibility* is choosing the cosmological constant $\Lambda = (R_0)^{-2}$. Then, the field equations (6.5) with metric (6.1i) yield

$$T_{\alpha\beta}(\cdot\cdot) \equiv 0, \quad T_{\alpha 4}(\cdot\cdot) \equiv 0, \quad \kappa T_{44}(\cdot\cdot) = 2\,(R_0)^{-2} > 0. \tag{6.6}$$

The above energy–momentum–stress tensor is due to an acceptable incoherent dust. This reasonable energy–momentum–stress tensor, supplemented with the cosmological constant, would still yield a static universe as desired by Einstein at the time. (However, it was later discovered that this state of static equilibrium is unstable to small perturbations [83].)

[1] Later, with the discovery that the universe was expanding (and therefore not static), Einstein *retracted* the cosmological constant from the field equation (6.5). He also supposedly stated that introducing it was the biggest blunder of his life. Interestingly, today, the cosmological constant is again in vogue as a possible mechanism to drive the observed accelerating expansion of the universe [13, 34, 209].

In 1927 and 1929, observations revealed that the universe is *not static* [136, 161]. In the meanwhile, Friedmann had already discovered in 1922 [105], from Einstein's usual field equations, *a nonstatic metric* which yields a finite (positive) mass density of the universe *without the use of a cosmological constant* Λ. He introduced the metric

$$ds^2 = [a(t)]^2 \cdot \left\{ \left(1 - k_0 \cdot r^2\right)^{-1} \cdot (dr)^2 + r^2 \cdot \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right] \right\} - (dt)^2, \quad (6.7i)$$

$$k_0 \in \{-1, 0, 1\}, \quad (6.7ii)$$

$$D := \begin{cases} \{(r, \theta, \varphi, t) :\ 0 < r < \infty,\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ t > 0\} \\ \qquad\qquad\qquad\qquad \text{for} \quad k_0 \in \{-1, 0\}, \\ \left\{(r, \theta, \varphi, t) : 0 < r < (k_0)^{-1/2},\ 0 < \theta < \pi,\ -\pi < \varphi < \pi,\ t > 0\right\} \\ \qquad\qquad\qquad\qquad \text{for} \quad k_0 = 1. \end{cases} \quad (6.7iii)$$

Remarks: (i) In the metric (6.7i), t is a geodesic normal coordinate (of (1.160)). Therefore, t-coordinate curves represent *timelike geodesics*.

(ii) The metric (6.7i) is a special case of the Tolman–Bondi–Lemaître metric of (3.88).

(iii) The function $a(t)$ governs *the proper distance* between points on a spatial (t = const.) hypersurface. Therefore, this function is sometimes known as *the scale factor*. When working with these types of metrics, one is often interested in solving for this function, as it dictates the time evolution of the universe.

(iv) The constant k_0 governs *the spatial curvature* of the spatial hypersurfaces. By a simple rescaling of the r coordinate, it can always be taken to have one of the values $+1$ (positive curvature), -1 (negative curvature), or 0 (flat). Note that in the $k_0 = 0$ case, although the spatial sections are flat, the space–time is *not flat*, as can be seen by calculating the Riemann curvature tensor components.

Example 6.1.1. Consider the "flat spatial case" of (6.7i), characterized by $k_0 = 0$. Thus, (6.7i) yields

$$ds^2 = [a(t)]^2 \cdot \left\{(dr)^2 + r^2 \cdot \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right]\right\} - (dt)^2. \quad (6.8)$$

Using a Cartesian coordinate chart for the spatial hypersurface, we obtain

$$ds^2 = [a(t)]^2 \cdot \delta_{\alpha\beta} \cdot (dx^\alpha)\left(dx^\beta\right) - (dt)^2. \quad (6.9)$$

The nonvanishing components of the Christoffel symbols of the second kind, from (6.9), are computed to be

$$\left\{ \begin{matrix} 4 \\ \alpha\ \beta \end{matrix} \right\} = a(t) \cdot \frac{\mathrm{d}a(t)}{\mathrm{d}t} \cdot \delta_{\alpha\beta}, \tag{6.10i}$$

$$\left\{ \begin{matrix} \alpha \\ \beta\ 4 \end{matrix} \right\} = \frac{1}{a(t)} \cdot \frac{\mathrm{d}a(t)}{\mathrm{d}t} \cdot \delta^{\alpha}{}_{\beta}. \tag{6.10ii}$$

We make the following abbreviations:

$$\dot{a} := \frac{\mathrm{d}a(t)}{\mathrm{d}t},$$
$$\ddot{a} := \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2}. \tag{6.11}$$

The nonzero components of the Ricci tensor, the scalar invariant, and the Einstein tensor from (6.9) and (6.10i,ii) are provided by

$$R_{\alpha\beta}(\cdot\cdot) = -\left[a \cdot \ddot{a} + 2\,(\dot{a})^2\right] \cdot \delta_{\alpha\beta}, \tag{6.12i}$$

$$R_{44}(\cdot\cdot) = 3 \cdot a^{-1} \cdot \ddot{a}, \tag{6.12ii}$$

$$R(\cdot\cdot) = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right], \tag{6.12iii}$$

$$G_{\alpha\beta}(\cdot\cdot) = \left[2a \cdot \ddot{a} + (\dot{a})^2\right] \cdot \delta_{\alpha\beta}, \tag{6.12iv}$$

$$G_{44}(\cdot\cdot) = -3 \cdot \left(\frac{\dot{a}}{a}\right)^2. \tag{6.12v}$$

Now, consider an incoherent dust as the idealized material filling the cosmos. Stream lines, which are timelike geodesics, are chosen to pursue t-coordinate lines. Thus, we choose the energy–momentum–stress tensor

$$T_{ij}(\cdot\cdot) = \rho(\cdot\cdot) \cdot U_i(\cdot\cdot) \cdot U_j(\cdot\cdot), \tag{6.13i}$$

$$\text{with} \qquad U^{\alpha}(\cdot\cdot) \equiv 0, \quad U^4(\cdot\cdot) \equiv 1. \tag{6.13ii}$$

(The comoving conditions (6.13ii) follow from the field equation $\mathcal{E}_{\alpha 4} = 0$.)

The interior field equations $\mathcal{E}_{ij}(\cdot\cdot) := G_{ij}(\cdot\cdot) + \kappa\, T_{ij}(\cdot\cdot) = 0$ reduce to

$$\mathcal{E}_{\alpha\beta}(\cdot\cdot) = \left[2a \cdot \ddot{a} + (\dot{a})^2\right] \cdot \delta_{\alpha\beta} = 0, \tag{6.14i}$$

$$\mathcal{E}_{44}(\cdot\cdot) = -3\left(\frac{\dot{a}}{a}\right)^2 + \kappa\,\rho(\cdot\cdot) = 0. \tag{6.14ii}$$

As well, the conservation equations $\mathcal{T}^i(\cdot\cdot) := \nabla_j\, T^{ij} = 0$ reduce to

$$\mathcal{T}_4(\cdot\cdot) = -\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\,\rho(\cdot\cdot) + 3\ln|a(\cdot\cdot)|\,\right]\right\}\cdot\rho(t) = 0. \tag{6.15}$$

The above equation can be integrated to obtain

$$\rho(t)\cdot[a(t)]^3 = M = \text{positive const.} \tag{6.16}$$

Here, $M > 0$ is interpreted as the "total mass" of the universe. Substituting (6.16) into (6.14ii), we deduce that

$$(\dot{a})^2 = \frac{\kappa M}{3a}. \tag{6.17}$$

Compare the above equation with (3.101). Assuming $\dot{a} > 0$, the general solution of (6.17) is furnished by

$$a(t) = (3/2)^{2/3}\cdot(\kappa M/3)^{1/3}\cdot(t - t_0)^{2/3}\,. \tag{6.18}$$

(Compare the solution above with (5.9).) The function $a(t)$ is *a semi-cubical parabola*, and its qualitative graph is depicted in Fig. 5.2. We put $t_0 = 0$ for simplicity to write

$$a(t) = \left[(3/2)\cdot\sqrt{(\kappa M)/3}\cdot t\right]^{2/3} > 0, \tag{6.19i}$$

$$\dot{a} = \frac{\mathrm{d}a(t)}{\mathrm{d}t} = \left[\frac{2\kappa M}{9t}\right]^{1/3} > 0, \tag{6.19ii}$$

$$\lim_{t\to 0_+}[a(t)] = 0. \tag{6.19iii}$$

Therefore, in this (flat) model, from (6.9), (6.16), and (6.19i), we derive that

$$\lim_{t\to 0_+}\left[g_{\alpha\beta}(\cdot\cdot)\right] = \lim_{t\to 0_+}\left\{[a(t)]^2\cdot\delta_{\alpha\beta}\right\} = 0, \tag{6.20i}$$

$$\lim_{t\to 0_+}\left[\rho(t)\right] = \lim_{t\to 0_+}\left[\frac{4}{3\kappa}\cdot\frac{1}{t^2}\right] \longrightarrow \infty. \tag{6.20ii}$$

Thus, we have a *big bang cosmological model*! That is, an expanding model that emerged from a (singular) event of infinite density at some finite time in the past. □

Now, we shall go back to the more general metric in (6.7i). We shall investigate the field equations with the metric in (6.7i) in the presence of an incoherent dust of equations (6.13i,ii). The nontrivial field equations and conservation equations turn out to be

$$\mathcal{E}^{\alpha}{}_{\beta}(\cdot\cdot) = \left[2 \cdot \frac{\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right] \cdot \delta^{\alpha}{}_{\beta} = 0, \tag{6.21i}$$

$$\mathcal{E}^{4}{}_{4}(\cdot\cdot) = 3 \cdot \left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - \kappa\,\rho(\cdot\cdot) = 0, \tag{6.21ii}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\ln[\rho(t)] + 3\ln|a(t)|\} = 0. \tag{6.21iii}$$

Equation (6.21iii) can be integrated to obtain

$$\rho(t) \cdot [a(t)]^3 = M = \text{positive const.} \tag{6.22}$$

(This equation is identical to (6.16).) Substituting (6.22) into (6.21ii), we get

$$(\dot{a})^2 - \frac{\kappa}{3} \cdot \rho \cdot (a)^2 = -k_0, \tag{6.23i}$$

$$\frac{1}{2}(\dot{a})^2 - \frac{\kappa \cdot M}{6 \cdot a(t)} = -(k_0/2)\,. \tag{6.23ii}$$

The above equation is *an exact analogue* of the nonrelativistic conservation of energy (of a unit mass particle) in the presence of a spherically symmetric gravitating mass M. (Compare (6.23ii) with (3.95).) Now, differentiating (6.23ii) with respect to t, and cancelling $\dot{a} \neq 0$, we derive that

$$\ddot{a} = \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} = -\frac{\kappa M}{6 \cdot [a(t)]^2}. \tag{6.24}$$

The above equation is analogous to Newton's inverse square law of gravitation!

Let us solve (6.23i) for the case of positive spatial curvature characterized by $k_0 = 1$ with $\dot{a} > 0$. The general solution is *implicitly provided by*[2]

$$t_0 - t = a \cdot \sqrt{\frac{\kappa M}{3a} - 1} + \left(\frac{\kappa M}{3}\right) \cdot \operatorname{Arctan}\sqrt{\frac{\kappa M}{3a} - 1}. \tag{6.25}$$

Here, t_0 is an arbitrary constant of integration. To cast the solution in a more recognizable form, we introduce the parameter

[2]Compare (6.25) with Problem #2 ii of Exercises 5.1.

$$\eta := 2 \cdot \operatorname{Arc\,cot} \sqrt{\frac{\kappa M}{3a} - 1}. \tag{6.26}$$

Choosing the constant $t_0 = -\frac{\kappa M \pi}{6}$, (6.25) and (6.26) yield *the parametric representation*

$$a = A(\eta) := \frac{\kappa M}{6}(1 - \cos\eta), \tag{6.27i}$$

$$t = T(\eta) := \frac{\kappa M}{6}(\eta - \sin\eta), \tag{6.27ii}$$

$$\eta \in (0, 2\pi). \tag{6.27iii}$$

The graph of (6.27i,ii) is *a cycloid*.

Similarly, for the case of negative constant spatial curvature, that is, $k_0 = -1$, the ordinary differential equation to be solved is

$$\dot{a} = \sqrt{\frac{\kappa M}{3a} + 1}. \tag{6.28}$$

The implicitly defined general solution is furnished by

$$t - t_0 = a \cdot \sqrt{\frac{\kappa M}{3a} + 1} - \left(\frac{\kappa M}{3}\right) \cdot \operatorname{arg\,coth} \sqrt{\frac{\kappa M}{3a} + 1}. \tag{6.29}$$

By suitable choices of the parameter η and the constant t_0, the graph of the function $a(t)$ can be *parametrically expressed* as

$$a = A(\eta) := \frac{\kappa M}{6} \cdot (\cosh\eta - 1), \tag{6.30i}$$

$$t = T(\eta) := \frac{\kappa M}{6} \cdot (\sinh\eta - \eta), \tag{6.30ii}$$

$$\eta \in (0, \infty). \tag{6.30iii}$$

Graphs of the three functions in (6.19i), (6.27i,ii), and (6.30i,ii) are shown in Fig. 6.1.

The two-dimensional submanifold M_2, characterized by $r = r_0,\ \theta = \pi/2$, and $k_0 = 1$, is given by the metric from (6.7i) as

$$d\sigma^2 = (r_0)^2 \cdot [a(t)]^2 \cdot (d\varphi)^2 - (dt)^2, \tag{6.31i}$$

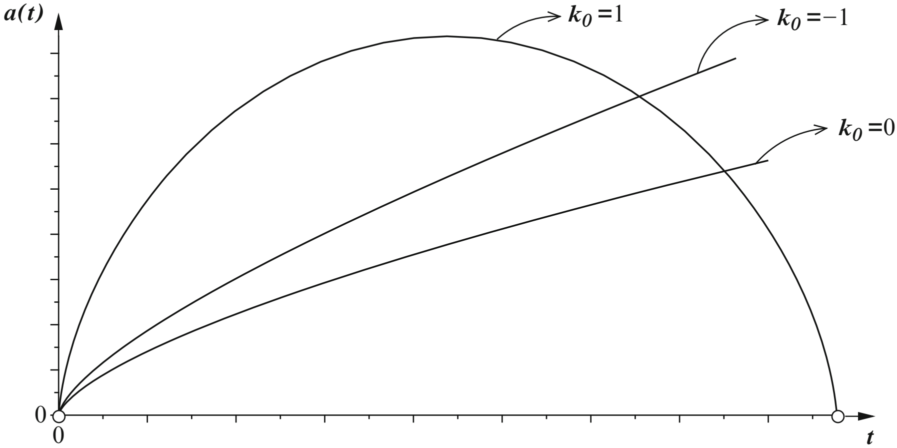
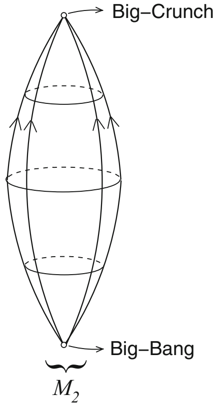


Fig. 6.1 Qualitative graphs for the "radius of the universe" as a function of time in three Friedmann (or standard) models

Fig. 6.2 Qualitative representation of a submanifold M_2 of *the spatially closed space–time* M_4



$$D_2 := \{(\varphi, t) : -\pi < \varphi < \pi,\ 0 < t < (\kappa\pi M/3\}. \tag{6.31ii}$$

Using Fig. 6.1, a qualitative representation of this manifold M_2 is provided in Fig. 6.2.

In Fig. 6.2 above, oriented longitudinal curves represent timelike geodesics which are trajectories of the dust particles. Moreover, closed horizontal circles represent three-dimensional hyperspheres S^3 with "evolving radii." Note that this universe emerges from a big bang, expands, and eventually recollapses to an event of infinite density known as the *big crunch*.

After their studies of the above models, Friedmann and Lemaître studied field equations (6.5) with the cosmological constant Λ reinserted, using the same metric as (6.7i) for the field equations. Next, Robertson [219] and Walker [259] investigated cosmological models from the point of view of *isotropy and homogeneity*. Let us try to explain the concepts of isotropy and homogeneity, both intuitively and mathematically. In case we look at a star-studded, dark night sky, all directions seem to look alike. This observation is the basis for isotropy. Again, if we make similar observations from different locations of the entire globe, the same notion of isotropy persists. In other words, in the spatial fabric of the cosmos, no single point appears to be privileged. That is the intuitive notion of homogeneity.

In an N-dimensional Riemannian or pseudo-Riemannian manifold, the mathematical condition of a point $x_0 \in D \subset \mathbb{R}^N$ to be of *isotropic sectional curvature* [56, 90, 244] is that

$$R_{lijk}(x_0) = \chi(x_0) \cdot \left[g_{lj}(x_0) \cdot g_{ik}(x_0) - g_{lk}(x_0) \cdot g_{ij}(x_0)\right]. \tag{6.32}$$

Now, we shall state and prove a theorem due to Schur [56, 90, 244] on this topic.

Theorem 6.1.2. *Let M_N be a Riemannian or a pseudo-Riemannian manifold with $N > 2$. Moreover, let the metric field $\mathbf{g}_{..}(x)$ in a chart be of class C^3. Furthermore, let every point $x \in D \subset \mathbb{R}^N$ be endowed with the isotropic sectional curvature $\chi(x)$. Then, $\chi(x)$ must be constant-valued.*

Proof. From (6.32), it is deduced that

$$\begin{aligned} R_{lijk}(x) &= \chi(x) \cdot \left[g_{lj}(x) \cdot g_{ik}(x) - g_{lk}(x) \cdot g_{ij}(x)\right], \\ & x \in D \subset \mathbb{R}^N. \end{aligned} \tag{6.33}$$

By Bianchi's identities (1.143i), it can be derived that

$$\begin{aligned} &(\partial_m \chi) \cdot \left(g_{lj} \cdot g_{ik} - g_{lk} \cdot g_{ij}\right) + \left(\partial_j \chi\right) \cdot \left(g_{lk} \cdot g_{im} - g_{lm} \cdot g_{ik}\right) \\ &\quad + (\partial_k \chi) \cdot \left(g_{lm} \cdot g_{ij} - g_{lj} \cdot g_{im}\right) \equiv 0. \end{aligned}$$

By multiplying $g^{lj}(x) \cdot g^{ik}(x)$ and contracting, the above differential identities yield

$$(N-1) \cdot (N-2) \cdot (\partial_m \chi) \equiv 0.$$

Thus, for $N > 2$, $\partial_m \chi \equiv 0$ and $\chi(x)$ must be *constant-valued.* ■

Remarks: (i) The constancy of sectional curvatures is *a sufficient mathematical criterion* of *homogeneity*.[3]

(ii) With $\chi(x) = K_0 =$ const., (6.33) yields *a space of constant curvature* (discussed in (1.164i,ii)).

[3] A domain $D \subset \mathbb{R}^N$ is *homogeneous* in case there exists a chart there with N Killing vectors: $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^N}$.

Thus, by the assumption of spatial isotropy everywhere, we conclude that the three-dimensional hypersurface of the cosmological universe admits *a six-parameter group of isometries* (or group of motions) by a theorem discussed in [90, 171]. The three-dimensional spatial metric $\mathbf{g}^{\#}_{..}(\mathbf{x})$ of constant curvature is characterized by (6.33) as

$$R^{\#}_{\alpha\beta\gamma\delta}(\mathbf{x}) = K_0 \cdot \left(g^{\#}_{\alpha\gamma}(\mathbf{x}) \cdot g^{\#}_{\beta\delta}(\mathbf{x}) - g^{\#}_{\alpha\delta}(\mathbf{x}) \cdot g^{\#}_{\beta\gamma}(\mathbf{x}) \right). \tag{6.34}$$

By a suitable scale transformation, we can simplify (6.34) to

$$\begin{aligned} R^{\#}_{\alpha\beta\gamma\delta}(\mathbf{x}) &= k_0 \cdot \left(g^{\#}_{\alpha\gamma}(\mathbf{x}) \cdot g^{\#}_{\beta\delta}(\mathbf{x}) - g^{\#}_{\alpha\delta}(\mathbf{x}) \cdot g^{\#}_{\beta\gamma}(\mathbf{x}) \right), \\ k_0 &\in \{-1, 0, 1\}. \end{aligned} \tag{6.35}$$

Another assumption made about the cosmological metric is that the time coordinate t is *a geodesic normal coordinate* of (1.160). Therefore, the metric for the cosmological universe is chosen to be

$$\mathbf{g}_{..}(x) = \left[a\left(x^4\right) \right]^2 \cdot \mathbf{g}^{\#}_{..}(\mathbf{x}) - \left(\mathrm{d}x^4\right) \otimes \left(\mathrm{d}x^4\right), \tag{6.36i}$$

$$\mathrm{d}s^2 = [a(t)]^2 \cdot g^{\#}_{\alpha\beta}(\mathbf{x}) \left(\mathrm{d}x^{\alpha}\right) \left(\mathrm{d}x^{\beta}\right) - (\mathrm{d}t)^2. \tag{6.36ii}$$

(Here, $\mathbf{g}^{\#}_{..}(\mathbf{x})$ is assumed to satisfy (6.35).)

Now, we shall compute the Einstein tensor from the metric in (6.36i,ii). We shall state and prove the following theorem on this topic.

Theorem 6.1.3. *Let the metric* $\mathbf{g}_{..}(x)$ *in* (6.36i) *be a Lorentz metric of class* C^3 *in a domain* $D \subset \mathbb{R}^4$. *Then, the corresponding* 4×4 *Einstein matrix* $\left[G_{ij}(x)\right]$ *must be of Segre characteristic* $[(1, 1, 1), 1]$.

Proof. Define $\overline{\mathbf{g}}_{..}(x) := [a(t)]^2 \cdot \mathbf{g}^{\#}_{..}(\mathbf{x})$. Using (2.226), (2.228), (2.230), (6.34), and (6.36i), the following equations are derived:

$$\begin{aligned} G_{\mu\nu}(x) &= G^{\#}_{\mu\nu}(\mathbf{x}) + \left[2a(t) \cdot \ddot{a} + (\dot{a})^2 \right] \cdot g^{\#}_{\mu\nu}(\mathbf{x}) \\ &= \left[k_0 + 2a(t) \cdot \ddot{a} + (\dot{a})^2 \right] \cdot g^{\#}_{\mu\nu}(\mathbf{x}), \end{aligned} \tag{6.37i}$$

$$G_{\mu 4}(x) \equiv 0, \tag{6.37ii}$$

$$G_{44}(x) = -3 \cdot \left[\frac{k_0}{(a(t))^2} + \left(\frac{\dot{a}}{a} \right)^2 \right]. \tag{6.37iii}$$

Consider *the invariant eigenvalue problem*

$$\det\left[G_{ij}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot g_{ij}(\cdot\cdot)\right] = 0. \tag{6.38}$$

Using (6.36ii), (6.38) reduces to

$$\det\left[G_{\mu\nu}(x) - \lambda(\cdot\cdot) \cdot (a(t))^2 \cdot g^{\#}_{\mu\nu}(\mathbf{x})\right] = 0, \tag{6.39i}$$

$$\textit{and} \qquad G_{44}(x) + \lambda(\cdot\cdot) = 0. \tag{6.39ii}$$

Therefore, using (6.39i,ii) and (6.37i–iii), four, real, invariant eigenvalues are furnished by

$$\lambda_{(1)}(\cdot\cdot) \equiv \lambda_{(2)}(\cdot\cdot) \equiv \lambda_{(3)}(\cdot\cdot) = \left[k_0 + 2 \cdot a \cdot \ddot{a} + (\dot{a})^2\right], \tag{6.40i}$$

$$\lambda_{(4)}(\cdot\cdot) = 3\left[\frac{k_0 + (\dot{a})^2}{a^2}\right]. \tag{6.40ii}$$

Thus, the theorem is proven. ■

The metric (6.36ii), with help of (6.35) and Theorem 1.3.30, can be expressed as

$$\begin{aligned} \mathrm{d}s^2 &= [a(t)]^2 \cdot \left[1 + (k_0/4) \cdot \delta_{\mu\nu} x^\mu x^\nu\right]^{-2} \cdot \left[\delta_{\alpha\beta} \cdot \mathrm{d}x^\alpha \mathrm{d}x^\beta\right] - (\mathrm{d}t)^2, \\ D &:= \left\{x : \mathbf{x} \in \mathbf{D} \subset \mathbb{R}^3,\ t > 0\right\}. \end{aligned} \tag{6.41}$$

Example 6.1.4. Consider metric (6.41) for the case $k_0 = 0$. It reduces to the metric in (6.9), which can be transformed to the flat Friedmann metric in (6.8).

Now, consider the metric (6.41) for the case $k_0 = 1$ of positive spatial curvature. In spherical polar coordinates, it can be transformed into

$$\begin{aligned} \mathrm{d}s^2 = &\left[\widehat{a}\left(\widehat{t}\right)\right]^2 \cdot \left[1 + \left(\widehat{r}/2\right)^2\right]^{-2} \\ &\times \left\{\left(\mathrm{d}\widehat{r}\right)^2 + \left(\widehat{r}\right)^2 \cdot \left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2\widehat{\theta} \cdot \left(\mathrm{d}\widehat{\varphi}\right)^2\right]\right\} - \left(\mathrm{d}\widehat{t}\right)^2. \end{aligned} \tag{6.42}$$

Making another coordinate transformation

$$\begin{aligned} r &= \widehat{r} \cdot \left[1 + \left[\widehat{r}/2\right)^2\right]^{-1}, \\ (\theta, \varphi, t) &= \left(\widehat{\theta}, \widehat{\varphi}, \widehat{t}\right), \end{aligned} \tag{6.43}$$

the metric (6.42) goes over into

$$ds^2 = [a(t)]^2 \cdot \left\{ \left(1 - r^2\right)^{-1} \cdot (dr)^2 + r^2 \cdot \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right] \right\} - (dt)^2. \quad (6.44)$$

Similarly, in the case of $k_0 = -1$, the metric in (6.41) is transformable to

$$ds^2 = [a(t)]^2 \cdot \left\{ \left(1 + r^2\right)^{-1} \cdot (dr)^2 + r^2 \cdot \left[(d\theta)^2 + \sin^2\theta \cdot (d\varphi)^2\right] \right\} - (dt)^2. \quad (6.45)$$

Thus, metrics in (6.41) are transformable to metrics in (6.7i–iii) (for appropriate domains). □

Metrics in (6.7i), as well as in (6.41) are called *Friedmann–Lemaître–Robertson–Walker* (or in short, *F–L–R–W*) *metrics*.

By Theorem 6.1.3 and the usual field equations, it follows that the corresponding 4×4 energy–momentum–stress matrix $\left[T_{ij}(x)\right]$ must be of Segre characteristic $[(1, 1, 1), 1]$. The most popular example of such an energy–momentum–stress tensor is that of *a perfect fluid* (in (2.257ii) and (3.49i)). Therefore, in the cosmological universe, the usual field equations considered are

$$\mathcal{E}_{ij}(\cdot\cdot) := G_{ij}(\cdot\cdot) + \kappa\left\{[\rho(\cdot\cdot) + p(\cdot\cdot)] \cdot U_i(\cdot\cdot) \cdot U_j(\cdot\cdot) + p(\cdot\cdot) \cdot g_{ij}(\cdot\cdot)\right\} = 0. \quad (6.46)$$

With help of the metric (6.7i) (or, (6.41)), and the comoving conditions $U^\alpha(\cdot\cdot) \equiv 0$, $U^4(\cdot\cdot) \equiv 1$, implied by $\mathcal{E}_{\alpha 4}(\cdot\cdot) = 0$, the field equations (6.46) and conservation equations $\mathcal{T}^i(\cdot\cdot) = 0$ yield the following nontrivial equations:

$$\mathcal{E}^1{}_1(\cdot\cdot) \equiv \mathcal{E}^2{}_2(\cdot\cdot) \equiv \mathcal{E}^3{}_3(\cdot\cdot) = \left[\frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right] + \kappa\, p(t) = 0, \quad (6.47\text{i})$$

$$\mathcal{E}^4{}_4(\cdot\cdot) = 3 \cdot \left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - \kappa\, \rho(t) = 0, \quad (6.47\text{ii})$$

$$\mathcal{T}^4(\cdot\cdot) = \dot{\rho} + 3 \cdot \left(\frac{\dot{a}}{a}\right) \cdot (\rho + p) = 0. \quad (6.47\text{iii})$$

There exists *one differential identity*

$$\left(\mathcal{E}^4{}_4\right)^{\cdot} - 3 \cdot \left(\frac{\dot{a}}{a}\right) \cdot \left(\mathcal{E}^1{}_1 - \mathcal{E}^4{}_4\right) + \kappa\, \mathcal{T}^4 \equiv 0 \quad (6.48)$$

among *three coupled ordinary differential equations* (6.47i–iii). Since there exist three unknown functions $a(t)$, $\rho(t)$, and $p(t)$, *the system is underdetermined*. One of these three functions can be prescribed to make the system determinate. In the present *matter-dominated era* it is common to set $p(t) \equiv 0$, as the average pressure of galactic clusters is negligibly small compared to the energy density. Otherwise, *one equation of state* can be imposed to render the system (6.47i–iii) determinate. Note that in every F–L–R–W universe, all the events have some material present.

An interesting linear combination of (6.47i–iii) can be expressed as

$$\frac{a}{6} \cdot \left[3\mathcal{E}^1{}_1 - \mathcal{E}^4{}_4\right] = \ddot{a} + (\kappa/6) \cdot (\rho + 3p) \cdot a = 0, \tag{6.49i}$$

$$\text{or,} \qquad \ddot{a}(t) = -(\kappa/6) \cdot [\,\rho(t) + 3p(t)] \cdot a(t). \tag{6.49ii}$$

Equation (6.49ii) is called *the Raychaudhuri equation* (as it is a consequence of (2.202ii)). Assuming the strong energy condition $\rho(t) + 3p(t) > 0$, (6.49ii) reveals that *the acceleration is negative* and the "gravitational force" is *attractive*.

Let us consider now what is commonly known as the *radiation era* [27,201,202] which has preceded the present matter-dominated era. It is characterized by *the equation of state*

$$p(t) = (1/3) \cdot \rho(t). \tag{6.50}$$

Substituting (6.50) into the continuity equation (6.47iii), we obtain

$$\dot{\rho} + \left(\frac{\dot{a}}{a}\right) \cdot (4\rho) = 0, \tag{6.51i}$$

$$\text{or,} \qquad \rho(t) \cdot [a(t)]^4 = C = \text{positive const.} \tag{6.51ii}$$

Putting (6.51ii) back into (6.47ii), we obtain the differential equation

$$(\dot{a})^2 = \frac{\kappa\, C}{3a^2} - k_0. \tag{6.52}$$

Integrating the above, we have *three explicit solutions*:

$$[a(t)]^2 = \begin{cases} 2\sqrt{\dfrac{\kappa\, C}{3}} \cdot (t - t_0) \quad \text{for} \quad k_0 = 0, & (6.53\text{i}) \\[2ex] 2\sqrt{\dfrac{\kappa\, C}{3}} \cdot (t - t_0) + (t - t_0)^2 \quad \text{for} \quad k_0 = -1, & (6.53\text{ii}) \\[2ex] 2\sqrt{\dfrac{\kappa\, C}{3}} \cdot (t - t_0) - (t - t_0)^2 \quad \text{for} \quad k_0 = 1. & (6.53\text{iii}) \end{cases}$$

Needless to say, the above solutions have physical consequences [190,200,238,260].

After the big bang and before the radiation era, it is commonly believed that there may have existed an extremely short *inflationary era* [27,206]. *One of various speculations* about this era, and which is also mathematically the simplest, is to set *the exotic condition* $\rho(x) + p(x) \equiv 0$. Field equations (6.47i–iii) and (6.49ii) reduce to

$$\left[\frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right] - \kappa\,\rho(t) = 0, \tag{6.54i}$$

$$3\left[\frac{\dot{a}^2 + k_0}{a^2}\right] - \kappa\,\rho(t) = 0, \tag{6.54ii}$$

$$\dot{\rho} = \frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = 0, \tag{6.54iii}$$

$$\ddot{a} = \left(\frac{\kappa\,\rho}{3}\right) \cdot a(t). \tag{6.54iv}$$

Integrating (6.54iii), we conclude that

$$\rho(t) = \rho_0 = \text{positive const.} \tag{6.55}$$

Substituting (6.55) into (6.54iv), we can integrate to obtain

$$a(t) = c_1 \cdot \mathrm{e}^{kt} + c_2 \cdot \mathrm{e}^{-kt}, \tag{6.56i}$$

$$\text{or,} \qquad a(t) = \widehat{c}_1 \cdot \cosh(kt) + \widehat{c}_2 \cdot \sinh(kt), \tag{6.56ii}$$

$$k := \sqrt{\frac{\kappa\,\rho_0}{3}} \; > \; 0. \tag{6.56iii}$$

Here, c_1, c_2 (or $\widehat{c}_1, \widehat{c}_2$) are arbitrary constants of integration.

Example 6.1.5. Choosing the constants $c_1 = 1$ and $c_2 = 0$, we get, from (6.56i),

$$a(t) = \mathrm{e}^{kt} > 0. \tag{6.57}$$

The above function causes *exponential inflation*, as the proper spatial distance between two points increases exponentially with time. Putting (6.57) into (6.54ii), we deduce that

$$3\left[k^2 + k_0\,\mathrm{e}^{-2kt}\right] = \kappa\,\rho_0. \tag{6.58}$$

Differentiating the equation above, we obtain that

$$-\,6k_0 \cdot k \cdot \mathrm{e}^{-2kt} \equiv 0, \tag{6.59i}$$

$$\text{or,} \qquad k_0 = 0. \tag{6.59ii}$$

Thus, for the case of exponential inflation (6.57), we must have *the flat F–L–R–W model*. □

Studies of the cosmological universe are greatly facilitated by the following two functions derivable from $a(t)$.

$$\mathsf{H}(t) := \frac{\dot{a}(t)}{a(t)}, \tag{6.60i}$$

$$q(t) := -\frac{a(t)\cdot \ddot{a}(t)}{[\dot{a}(t)]^2}, \tag{6.60ii}$$

$$a(t) > 0, \quad \dot{a}(t) > 0. \tag{6.60iii}$$

The function in (6.60i) is called the *Hubble function* (or the *Hubble parameter*). The other function $q(t)$ in (6.60ii) is called the *deceleration function* (or the *deceleration parameter*). Note that *in an (positive) accelerating phase of the universe,* $q(t) < 0$.

Field equations (6.47i,ii) and (6.49ii) imply that

$$3\cdot[\mathsf{H}(t)]^2 = 3\cdot\left(\frac{\dot{a}}{a}\right)^2 = \kappa\,\rho(t) - \frac{3k_0}{a^2}, \tag{6.61i}$$

$$6\cdot q(t)\cdot[\mathsf{H}(t)]^2 = -\frac{6\ddot{a}}{a} = \kappa\big[\,\rho(t) + 3p(t)\big]. \tag{6.61ii}$$

In the case of an incoherent dust, (6.61ii) reduces to

$$6\cdot q(t)\cdot[\mathsf{H}(t)]^2 = \kappa\,\rho(t). \tag{6.62}$$

Subtracting (6.61i) from (6.62), we deduce that

$$[\mathsf{H}(t)]^2\cdot[2q(t) - 1] = \frac{k_0}{[a(t)]^2}. \tag{6.63}$$

Therefore, in the dust model, the constant value $q(t) \equiv 1/2$, *the critical value*, yields an open spatial universe with $k_0 = 0$. Moreover, for $q(t) < 1/2$, we must have $k_0 = -1$ and another open spatial universe. However, for $q(t) > 1/2$, the spatial universe is *finite and closed* with $k_0 = 1$ corresponding to Fig. 6.2. Present observations and recent theories cannot definitely predict the ultimate future of the universe. The *critical energy density* for which $k_0 = 0$ is given by $\rho_c := 3\,[\mathsf{H}(t)]^2/\kappa$.

There exist other cosmological models which are *not isotropic*. Consider, for example, *Bianchi type-I models* [126, 238], admitting three commuting (independent) Killing vector fields indicating *spatial homogeneity*. The metric can be expressed in an orthogonal coordinate chart, with a hybrid notation, as

$$ds^2 = g_{11}(t) \cdot \left(dx^1\right)^2 + g_{22}(t) \cdot \left(dx^2\right)^2 + g_{33}(t) \cdot \left(dx^3\right)^2 - (dt)^2, \quad (6.64\text{i})$$

$$g_{11}(t) > 0, \quad g_{22}(t) > 0, \quad g_{33}(t) > 0, \quad \sqrt{-g(t)} > 0. \quad (6.64\text{ii})$$

(Note that $t = \text{const.}$ hypersurfaces are flat.)

In the case of an incoherent dust, the usual field equations in a comoving frame can be integrated to obtain

$$g_{11}(t) = (-g)^{1/3} \cdot \left[\frac{t}{mt + c_0}\right]^{(4/3)\cdot\sin\alpha}, \quad (6.65\text{i})$$

$$g_{22}(t) = (-g)^{1/3} \cdot \left[\frac{t}{mt + c_0}\right]^{(4/3)\cdot\sin[\alpha+(2\pi/3)]}, \quad (6.65\text{ii})$$

$$g_{33}(t) = (-g)^{1/3} \cdot \left[\frac{t}{mt + c_0}\right]^{(4/3)\cdot\sin[\alpha+(4\pi/3)]}, \quad (6.65\text{iii})$$

$$\kappa \cdot \rho(t) \cdot \sqrt{-g(t)} \equiv m = \text{positive const.}, \quad (6.65\text{iv})$$

$$\sqrt{-g(t)} = (3/4) \cdot t \cdot (mt + c_0), \quad (6.65\text{v})$$

$$\alpha \in \big(-(\pi/6), (\pi/2)\big]. \quad (6.65\text{vi})$$

Here, $c_0 > 0$ is the constant determining the magnitude of the anisotropy.

The expansion scalar $\Theta(x)$ of (2.199iv) in such a universe (with properties $\dot{U}^i(\cdot\cdot) = \omega_{ij}(\cdot\cdot) \equiv 0$) is furnished by

$$\Theta(t) = \frac{(2mt + c_0)}{t(mt + c_0)}. \quad (6.66)$$

Obviously, $\lim_{t\to 0_+} [\Theta(t)] \longrightarrow \infty$, or there exists *an initial singularity*. In fact, with a positive expansion rate, in any cosmological model obeying strong energy conditions, *the initial singularity (or big bang) cannot be avoided*.

We shall now state a theorem which is *dual (or a mirror-image) to the Singularity Theorem 5.3.1*.

Theorem 6.1.6. *Let field equations $G_{ij}(x) + \kappa\, T_{ij}(x) = 0$ prevail in a domain $D \subset \mathbb{R}^4$ such that the energy–momentum–stress tensor is diagonalizable. Moreover, let the strong energy conditions hold. Furthermore, let the vorticity tensor $\omega_{ij}(x) \cdot dx^i \wedge dx^j \equiv \mathbf{O}_{..}(x)$. Suppose that a timelike geodesic congruence has a positive*

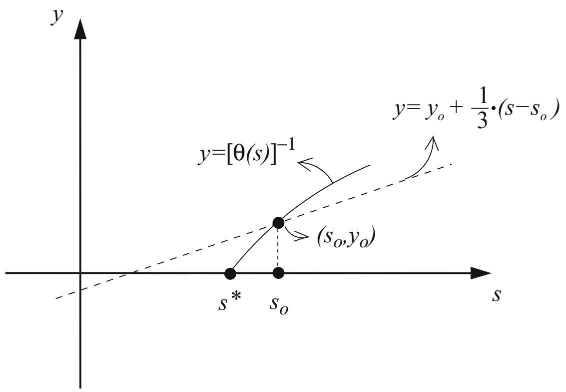


Fig. 6.3 Qualitative graphs of $y = [\theta(s)]^{-1}$ and the straight line $y = y_0 + \frac{1}{3}(s - s_0)$

expansion scalar $\Theta(x) > 0$ *in a neighborhood of* $D \subset \mathbb{R}^4$. *Then, each of the timelike geodesics* $x = \mathcal{X}(s)$ *had an initial singularity at* $x^* = \mathcal{X}(s^*) \in \partial D$ *before a finite proper time.*

Proof of this is exactly analogous to that of Theorem 5.3.1 and will be skipped. However, we shall provide Fig. 6.3, which is similar to Fig. 5.22 and depicts the initial singularity at $s = s^*$.

It is interesting to note that (6.23ii) yields

$$\dot{a} = \sqrt{\frac{\kappa M}{3a} - k_0} =: F(t, a), \tag{6.67i}$$

$$\frac{\partial F(\cdot\cdot)}{\partial a} = -\frac{\kappa M}{6a^{3/2}\left(\frac{\kappa M}{3} - k_0 a\right)^{1/2}}, \tag{6.67ii}$$

$$D_2 := \left\{(t, a) \in \mathbb{R}^2 : 0 < t,\ 0 < a\right\}. \tag{6.67iii}$$

Therefore, the $\lim\limits_{a \to 0_+} \left[\frac{\partial F(\cdot\cdot)}{\partial a}\right] \longrightarrow -\infty$ is thus *undefined*. So, we conclude that solutions of (6.67i) are *unstable and chaotic near the initial singularity*. (Compare with the Example A2.10 and (5.12) governing a black hole.)

Finally, we note in the F–L–R–W metrics of (6.7i) and (6.41) that the space and time coordinates are not on equal footings. The ten-parameter Poincaré group $\mathcal{IO}(3, 1; \mathbb{R})$ of special relativity (and its minimal extension to the de Sitter or anti-de Sitter universes) is *not a valid group of isometries* for F–L–R–W universes. (For the definition of a de Sitter metric, consult # 1 of Exercises 6.1.) However, the six-parameter isometry group $\mathcal{IO}(3, \mathbb{R})$ is still *a valid local isometry* in a F–L–R–W universe.

Exercise 6.1

1. Consider the *de Sitter universe* of constant curvature $K_0 > 0$ characterized by the metric

$$\mathrm{d}s^2 = \left[1 + (K_0/4) \cdot \left(d_{kl} x^k x^l\right)\right]^{-2} \cdot \left[d_{ij} \cdot \left(\mathrm{d}x^i\right)\left(\mathrm{d}x^j\right)\right].$$

   (i) Prove that the above metric satisfies the field equations $G_{ij}(x) = \Lambda \cdot g_{ij}(x)$ for some cosmological constant Λ.

   (ii) Show that the de Sitter metric can be transformed into another chart as

$$\mathrm{d}s^2 = (K_0)^{-1} \cdot \left\{\cosh^2 t \cdot \left[(\mathrm{d}\chi)^2 + \sin^2 \chi \cdot \left((\mathrm{d}\theta)^2 + \sin^2 \theta \cdot (\mathrm{d}\varphi)^2\right)\right] - (\mathrm{d}t)^2\right\}.$$

   (iii) Consider a five-dimensional, flat manifold M_5 with the metric

$$(\mathrm{d}\tilde{s})^2 = \left(\mathrm{d}y^1\right)^2 + \left(\mathrm{d}y^2\right)^2 + \left(\mathrm{d}y^3\right)^2 + \left(\mathrm{d}y^4\right)^2 - \left(\mathrm{d}y^5\right)^2.$$

   Let a four-dimensional *hyper-hyperboloid*, characterized by the constraint

$$\left(y^1\right)^2 + \left(y^2\right)^2 + \left(y^3\right)^2 + \left(y^4\right)^2 - \left(y^5\right)^2 = (K_0)^{-1},$$

   be embedded in M_5.

   Prove that the de Sitter metric in part (ii) can locally represent this hyper-hyperboloid.

2. (i) Show that the F–L–R–W metric (6.7i) for $k_0 = 1$, can be transformed into

$$\mathrm{d}s^2 = \left[\widehat{a}\left(\widehat{t}\,\right)\right]^2 \cdot \left\{(\mathrm{d}\widehat{\chi})^2 + \sin^2 \widehat{\chi} \cdot \left[\left(\mathrm{d}\widehat{\theta}\right)^2 + \sin^2 \widehat{\theta} \cdot (\mathrm{d}\widehat{\varphi}\,)^2\right] - \left(\mathrm{d}\widehat{t}\,\right)^2\right\}$$

   which is conformal to the Einstein metric in (6.1i).

   (ii) Prove that the F–L–R–W metric of part (i) is transformable to *the conformally flat form*:

$$\mathrm{d}s^2 = \frac{1}{4} \cdot \left[a^{\#}\left(t^{\#}\right)\right]^2 \cdot \left\{\left[\cos \mathring{\chi}\left(r^{\#}, t^{\#}\right)\right] + \left[\cos \mathring{t}\left(r^{\#}, t^{\#}\right)\right]\right\}^2 \cdot$$
$$\cdot \left\{\left(\mathrm{d}r^{\#}\right)^2 + \left(r^{\#}\right)^2 \cdot \left[\left(\mathrm{d}\theta^{\#}\right)^2 + \sin^2 \theta^{\#} \cdot \left(\mathrm{d}\varphi^{\#}\right)^2\right] - \left(\mathrm{d}t^{\#}\right)^2\right\};$$

$$\widehat{\chi} = \mathring{\chi}\left(r^{\#}, t^{\#}\right), \quad \widehat{t} = \mathring{t}\left(r^{\#}, t^{\#}\right).$$

3. Consider a mixture of n distinct perfect fluids obeying simultaneous comoving conditions:

$$U_{\alpha}^{(A)}(x) \equiv 0, \quad U_{4}^{(A)}(x) \equiv -1 \quad \text{for all} \quad A \in \{1, 2, \ldots, n\}.$$

Denote

$$\rho(x) := \sum_{A=1}^{n} \rho^{(A)}(x), \quad p(x) := \sum_{A=1}^{n} p^{(A)}(x).$$

Consider the energy–momentum–stress tensor of the mixture

$$T_{ij}(x) := \sum_{A=1}^{n} \left\{ \left[\rho^{(A)}(\cdot\cdot) + p^{(A)}(\cdot\cdot) \right] \cdot U_{i}^{(A)}(\cdot\cdot) \cdot U_{j}^{(A)}(\cdot\cdot) + p^{(A)}(\cdot\cdot) \cdot g_{ij}(\cdot\cdot) \right\}.$$

Assuming the F–L–R–W metric (6.7i), deduce that the interior field equations $G_{ij}(\cdot\cdot) + \kappa\, T_{ij}(x) = 0$ *exactly reduce to* (6.47i–iii).

4. Suppose that a perfect fluid is endowed with an equation of state

$$p = \mathcal{P}(\rho) > 0,$$

where $\mathcal{P}(\rho)$ is a differentiable function for $\rho > 0$.

(i) Imposing the above equation of state, derive from field equations (6.47i–iii),

$$\mathcal{M}\left[\rho(t)\right] \cdot [a(t)]^{3} = m_0 = \text{positive const.},$$

where $\mathcal{M}(\rho) := \exp\left[\int \frac{\mathrm{d}\rho}{[\rho + \mathcal{P}(\rho)]}\right]$.

(ii) Solve the system of field equations (6.47i–iii) with the help of part (i) to obtain the *implicit form of the general solution* as

$$\int_{a_1}^{a} \frac{\mathrm{d}y}{\left[(\kappa/3) \cdot y^{2} \cdot \mathcal{M}^{-1}\left(m_0 \cdot y^{-3}\right) - k_0\right]^{1/2}} = t - t_1, \qquad a_1 = a(t_1).$$

5. Consider the exponential inflationary model of (6.57) in a spatially flat F–L–R–W universe. By considering the trajectories of nearby galaxies and the trajectory of light emitted by the galaxies, argue that, although this scenario can represent an enormous rate of expansion, the expansion is *not superluminal* in the following sense: Any two sufficiently nearby fluid elements in this space–time are traveling away from each other at less than the speed of massless particles (speed of light).

6. Consider an inhomogeneous generalization of the F–L–R–W metric as

$$\mathrm{d}s^{2} = [a(t)]^{2} \cdot \left\{ \left[1 - k_0 r^{2} + k_1 f(r)\right]^{-1} \cdot (\mathrm{d}r)^{2} + r^{2} \cdot \left[(\mathrm{d}\theta)^{2} + \sin^{2}\theta \cdot (\mathrm{d}\varphi)^{2}\right] \right\} - (\mathrm{d}t)^{2}.$$

Let the material source be a mixture of an incoherent dust and a tachyonic dust characterized by

$$T_{ij}(r,t) := \rho(\cdot\cdot) \cdot U_i(\cdot\cdot) \cdot U_j(\cdot\cdot) + \alpha(\cdot\cdot) \cdot W_i(\cdot\cdot) \cdot W_j(\cdot\cdot),$$

$$U_\mu(\cdot\cdot) \equiv 0, \;\; U_4(\cdot\cdot) \equiv -1\,; \;\; W_1(\cdot\cdot) = \sqrt{g_{11}(\cdot\cdot)}, \;\; W_2(\cdot\cdot) = W_3(\cdot\cdot) = W_4(\cdot\cdot) \equiv 0.$$

Show that integrations of (the usual) field equations lead to

$$k_1 \cdot f(r) = (k_0 - C_0) \cdot r^2 + k_1 \cdot C_1,$$

$$\text{and} \qquad (\dot{a})^2 = \frac{2M_0}{a} - C_0.$$

(Here, C_0, C_1, and $M_0 > 0$ are constants of integration.)

Answers and Hints to Selected Exercises

1. (i) Consult Theorem 1.3.30. $\Lambda = 3 \cdot K_0$.
   (iii) A parametric representation of the hyper-hyperboloid is furnished by

$$y^1 = (K_0)^{-1/2} \cdot \cosh t \cdot \sin\chi \cdot \sin\theta \cdot \cos\varphi,$$

$$y^2 = (K_0)^{-1/2} \cdot \cosh t \cdot \sin\chi \cdot \sin\theta \cdot \sin\varphi,$$

$$y^3 = (K_0)^{-1/2} \cdot \cosh t \cdot \sin\chi \cdot \cos\theta,$$

$$y^4 = (K_0)^{-1/2} \cdot \cosh t \cdot \cos\chi,$$

$$y^5 = (K_0)^{-1/2} \cdot \sinh t\,;$$

2. (i)

$$D_4 := \{(\chi,\theta,\varphi,t) : 0 < \chi < \pi, \; 0 < \theta < \pi, \; -\pi < \varphi < \pi, \; -\infty < t < \infty\}.$$

$$\widehat{\chi} = \operatorname{Arc}\,\sin r,$$

$$\left(\widehat{\theta}, \widehat{\varphi}\right) = (\theta, \varphi),$$

$$\widehat{t} = \int \frac{\mathrm{d}t}{a(t)}\,;$$

$$\widehat{a}\left(\widehat{t}\right) := a(t).$$

(ii) The coordinate transformation and its inverse (for $\cos\widehat{\chi} + \cos\widehat{t} \neq 0$) are furnished by

$$r^{\#} = \frac{2\sin\widehat{\chi}}{\left(\cos\widehat{\chi} + \cos\widehat{t}\,\right)},$$

$$\left(\theta^{\#}, \varphi^{\#}\right) = \left(\widehat{\theta}, \widehat{\varphi}\right),$$

$$t^{\#} = \frac{2\sin\widehat{t}}{\left(\cos\widehat{\chi} + \cos\widehat{t}\,\right)}$$

and,
$$\widehat{\chi} = \mathring{\chi}\left(r^{\#}, t^{\#}\right),$$

$$\left(\widehat{\theta}, \widehat{\varphi}\right) = (\theta^{\#}, \varphi^{\#}),$$

$$\widehat{t} = \mathring{t}\left(r^{\#}, t^{\#}\right),$$

$$\widehat{a}(\widehat{t}) =: a^{\#}(t^{\#}).$$

(See the reference of Stephani [238].)

4. (i) Field equations (6.47iii) can be integrated to obtain

$$\int \frac{\mathrm{d}\rho}{[\rho + \mathcal{P}(\rho)]} + \ln\left(a^3\right) = \text{const.}$$

(ii) From part (i), it can be deduced that

$$\frac{\mathrm{d}\mathcal{M}(\rho)}{\mathrm{d}\rho} = \frac{\mathcal{M}(\rho)}{[\rho + \mathcal{P}(\rho)]} > 0.$$

Since $[\rho + \mathcal{P}(\rho)]^{-1}$ is continuous for $\rho > 0$, there exists a unique, inverse function $\mathcal{M}^{-1}$ which is differentiable. (See [32].) Thus, it can be concluded that $\rho = \mathcal{M}^{-1}\left(m_0 \cdot a^{-3}\right)$. Substituting the last equation into the field equation (6.47ii), it follows that (for $\dot{a} > 0$)

$$\dot{a} = \sqrt{(\kappa/3) \cdot a^2 \cdot \mathcal{M}^{-1}(m_0 \cdot a^{-3}) - k_0} > 0.$$

5. Use a space–time diagram and the chart of metric (6.7i) with $k_0 = 0$. Note that nearby t-coordinate lines represent stream lines for the two adjacent fluid elements (or galaxies). Also, consider the equation for a null (massless) particle, $\mathrm{d}s^2 = 0$, on a null trajectory on the same space–time diagram and, for simplicity, consider motion in the r direction only. This equation can be integrated to obtain the trajectory, $r(t)$, for a massless particle. By plotting two nearby fluid (galaxy)

stream lines, and a null trajectory originating from a point on one of these stream lines, note that the null trajectory can cross the other galaxy's stream line at a later time and hence will overtake the other galaxy.

6. Denoting by $f'(r) := \frac{df(r)}{dr}$, etc., the nontrivial field equations are [62]

$$\left[\frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right] - \frac{k_1 \cdot f'(r)}{2r \cdot a^2} = 0,$$

$$3\left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - \frac{k_1 \cdot [r \cdot f(r)]'}{r^2 \cdot a^2} = \kappa\, \rho(r,t),$$

$$-\frac{k_1 \cdot r}{2a^2} \cdot \left[\frac{f(r)}{r^2}\right]' = \kappa\, \alpha(r,t),$$

$$\dot{\rho} + \left(\frac{\dot{a}}{a}\right)(3\rho + \alpha) = 0,$$

$$\alpha' + \frac{2\alpha}{r} = 0.$$

6.2 Scalar Fields in Cosmology

This section may be considered optional for an introductory curriculum.

In the preceding section, we encountered the cosmological constant Λ that Einstein introduced and later withdrew from the field equations. The cosmological constant Λ can be interpreted as a constant-valued scalar field. Hoyle and Narlikar [135] introduced a scalar C-field in cosmology for steady-state models. Das and Agrawal [57] investigated (1) a massless, real scalar field and (2) a massive, real scalar field in F–L–R–W universes. Today, various types of scalar fields are commonplace in many studies of cosmology.

Now, let us discuss the case of *a massless, real scalar field*, $\phi(x)$, in general relativity. *The coupled scalar and gravitational field equations* are the following:

$$\mathcal{E}_{ij}(x) := G_{ij}(x) + \kappa\, T_{ij}(x) = 0, \tag{6.68i}$$

$$T_{ij}(\cdot\cdot) := \nabla_i\phi \cdot \nabla_j\phi - (1/2) \cdot g_{ij}(\cdot\cdot) \cdot \nabla^k\phi \cdot \nabla_k\phi, \tag{6.68ii}$$

$$\sigma(\cdot\cdot) := \nabla^i\nabla_i\phi = 0, \tag{6.68iii}$$

$$\mathcal{T}_i(\cdot\cdot) := \nabla_j T^{\,j}_{\;i} = 0, \tag{6.68iv}$$

$$\mathcal{C}^i\left(g_{ij}, \partial_k g_{ij}\right) = 0. \tag{6.68v}$$

Eight differential identities in the system above are

$$\nabla_j \mathcal{E}^{j}_{\ i} \equiv \kappa\, \mathcal{T}_i(\cdot\cdot), \tag{6.69i}$$

$$\mathcal{T}_i(\cdot\cdot) - \sigma(\cdot\cdot) \cdot \nabla_i \phi \equiv 0. \tag{6.69ii}$$

Let us count the number of unknown functions versus the number of independent equations.

No. of unknown functions: $10\left(g_{ij}\right) + 1(\phi) = 11.$

No. of equations: $10\left(\mathcal{E}_{ij} = 0\right) + 1(\sigma = 0) + 4\left(\mathcal{T}_i = 0\right) + 4\left(\mathcal{C}^i = 0\right) = 19.$

No. of differential identities: $4\left(\nabla_j \mathcal{E}^{j}_{\ i} \equiv \kappa\, \mathcal{T}_i\right) + 4\left(\mathcal{T}_i - \sigma \cdot \nabla_i \phi \equiv 0\right) = 8.$

No. of independent equations: 11.

Thus, the system of equations (6.68i–v) is *determinate*.

Now, consider the F–L–R–W metric in (6.7i–iii). For convenience, we rewrite it here:

$$\mathrm{d}s^2 = [a(t)]^2 \cdot \left\{\left(1 - k_0 r^2\right)^{-1} \cdot (\mathrm{d}r)^2 + r^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right]\right\} - (\mathrm{d}t)^2, \tag{6.70i}$$

$$\sqrt{-g} = r^2 \cdot \left(1 - k_0 r^2\right)^{-1/2} \cdot \sin\theta \cdot a^3. \tag{6.70ii}$$

Field equations (6.68i,ii), for the metric (6.70i), yield

$$\kappa^{-1}\, \mathcal{E}_{\alpha 4}(\cdot\cdot) = \partial_\alpha \phi \cdot \partial_4 \phi = 0, \tag{6.71i}$$

$$\kappa^{-1}\, \mathcal{E}_{12}(\cdot\cdot) = \partial_1 \phi \cdot \partial_2 \phi = 0, \tag{6.71ii}$$

$$\kappa^{-1}\, \mathcal{E}_{23}(\cdot\cdot) = \partial_2 \phi \cdot \partial_3 \phi = 0, \tag{6.71iii}$$

$$\kappa^{-1}\, \mathcal{E}_{31}(\cdot\cdot) = \partial_3 \phi \cdot \partial_1 \phi = 0, \tag{6.71iv}$$

$$\partial_\alpha G^{i}_{\ j} \equiv 0. \tag{6.71v}$$

Therefore, a time-dependent, nonconstant scalar field $\phi(\cdot\cdot)$ must satisfy

$$\partial_\alpha \phi \equiv 0, \tag{6.72i}$$

$$\dot{\phi} := \partial_4 \phi \neq 0. \tag{6.72ii}$$

The scalar wave equation $\sigma(\cdot\cdot) = 0$ in (6.68iii) with the F–L–R–W metric of (6.70i) reduces to

$$\left(a^3 \cdot \dot{\phi}\right)^{\cdot} = 0, \tag{6.73i}$$

$$\text{or,} \qquad [a(t)]^3 \cdot \dot{\phi}(t) = c_1 = \text{nonzero const.} \tag{6.73ii}$$

$$\text{or,} \qquad \phi(t) = c_1 \cdot \int \cdot [a(t)]^{-3} \mathrm{d}t. \tag{6.73iii}$$

Here, $c_1 \neq 0$ is the arbitrary constant of integration. The field equation $\mathcal{E}^4{}_4(\cdot\cdot) = 0$ provides

$$3\left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - \frac{\kappa}{2} \cdot \left(\dot{\phi}\right)^2 = 0. \tag{6.74}$$

Using (6.73ii) and (6.74), we obtain that

$$a^4 \cdot (\dot{a})^2 = (\kappa/6) \cdot c_1^2 - k_0 a^4 > 0, \tag{6.75i}$$

$$\text{or,} \qquad \int_{a_1}^{a} \frac{y^2 \cdot \mathrm{d}y}{\sqrt{(\kappa/6) \cdot c_1^2 - k_0 y^4}} = t - t_1, \tag{6.75ii}$$

$$\text{where,} \qquad a_1 = a\left(t_1\right). \tag{6.75iii}$$

Equations (6.75ii,iii) and (6.73ii) yield *general solutions of all field equations and the wave equation implicitly*.

Equations (6.75ii) and (6.73iii) provide, in the flat F–L–R–W case with $k_0 = 0$, if we choose the constants $t_1 = a_1 = 0$, the following *explicit solutions*:

$$a(t) = \left[3 \cdot \sqrt{\kappa/6} \cdot |c_1|\right]^{1/3} \cdot (t)^{1/3}, \tag{6.76i}$$

$$\phi(t) = (1/3) \cdot \sqrt{6/\kappa} \cdot \operatorname{sgn}\left(c_1\right) \cdot \ln|t|. \tag{6.76ii}$$

The equations above yield *a big bang model of cosmology*.

The integral in the left-hand side of (6.75ii) can be furnished explicitly for the cases $k_0 = +1$ and $k_0 = -1$. The following equations provide those scenarios. In case $k_0 = 1$, (6.75ii) implies that

$$E\left[\operatorname{Arc}\sin\left(\left(6/\kappa\, c_1^2\right)^{1/4} \cdot a\right), 1\right] - F\left[\operatorname{Arc}\sin\left(\left(6/\kappa\, c_1^2\right)^{1/4} \cdot a\right), -1\right]$$
$$= \left(\kappa\, c_1^2/6\right)^{1/4} \cdot t + \text{const.} \tag{6.77i}$$

Here, $E[\cdot\cdot,\cdot\cdot]$ and $F[\cdot\cdot,\cdot\cdot]$ are *elliptic functions* [91]. In case $k_0 = -1$, we can deduce that

$$- E\Big[i \cdot \mathrm{Arg}\sinh\Big(\big(-6/\kappa\, c_1^2\big)^{1/4} \cdot a\Big), -1\Big] + F\Big[i \cdot \mathrm{Arg}\sinh\Big(\big(-6/\kappa\, c_1^2\big)^{1/4} \cdot a\Big), -1\Big]$$
$$= \big(-\kappa\, c_1^2/6\big)^{-1/4} \cdot t + \text{const.} \tag{6.77ii}$$

Now, we shall investigate *the real, massive scalar field* (of mass m) in general relativity. The field equations and conservation equations are furnished by

$$\mathcal{E}_{ij}(x) := G_{ij}(\cdot\cdot) + \kappa\, T_{ij}(\cdot\cdot) = 0, \tag{6.78i}$$

$$T_{ij}(\cdot\cdot) := \nabla_i\phi \cdot \nabla_j\phi - (1/2) \cdot g_{ij}(\cdot\cdot) \cdot \left[\nabla^k\phi \cdot \nabla_k\phi + m^2 \cdot \phi^2\right] = 0, \tag{6.78ii}$$

$$\sigma(\cdot\cdot) := \nabla^i\nabla_i\phi - m^2 \cdot \phi(\cdot\cdot) = 0, \tag{6.78iii}$$

$$\mathcal{T}_i(\cdot\cdot) := \nabla_j T^j_{\ i} = 0, \tag{6.78iv}$$

$$\mathcal{C}^i\left(g_{ij}, \partial_k g_{ij}\right) = 0. \tag{6.78v}$$

The eight differential identities existing in the system are the following:

$$\nabla_j \mathcal{E}^j_{\ i} \equiv \kappa\, \mathcal{T}_i(\cdot\cdot), \tag{6.79i}$$

$$\mathcal{T}_i(\cdot\cdot) \equiv \sigma(\cdot\cdot) \cdot \nabla_i\phi. \tag{6.79ii}$$

Thus, the system of partial differential equations (6.78i–v) is *determinate*.

Let us now investigate the invariant eigenvalue problem for the 4×4 energy–momentum–stress matrix $[T_{ij}(\cdot\cdot)]$ from (6.78ii).

Theorem 6.2.1. *Let the real, differentiable, massive scalar field* $\phi(x)$ *in* (6.77i,ii) *satisfy* $g^{ij}(\cdot\cdot) \cdot \nabla_i\phi \cdot \nabla_j\phi < 0$ *in* $D \subset \mathbb{R}^4$. *Then, the* 4×4 *energy–momentum–stress matrix* $[T_{ij}(x)]$ *is of the Segre characteristic* $[(1, 1, 1), 1]$ *in* $D \subset \mathbb{R}^4$.

Proof. The assumption $g^{ij}(\cdot\cdot) \cdot \nabla_i\phi \cdot \nabla_j\phi < 0$ implies that $\nabla^i\phi \cdot \frac{\partial}{\partial x^i} \neq \vec{\mathbf{0}}(x)$. By (6.78ii), it can be deduced that

$$T_{ij}(\cdot\cdot) \cdot \nabla^j\phi = (1/2) \cdot \left[\nabla^k\phi \cdot \nabla_k\phi - m^2 \cdot \phi^2(\cdot\cdot)\right] \cdot g_{ij}(\cdot\cdot) \cdot \nabla^j\phi. \tag{6.80}$$

At each $x \in D$, there exist *three other spacelike vectors* such that $\{\vec{\mathbf{s}}_{(1)}(x), \vec{\mathbf{s}}_{(2)}(x), \vec{\mathbf{s}}_{(3)}(x), \nabla^i\phi \cdot \partial_i\}$ is *an orthogonal basis set* for the tangent vector space at x. Therefore, it follows that

$$s^j_{(\alpha)}(\cdot\cdot) \cdot \nabla_j \phi \equiv 0, \tag{6.81i}$$

$$T_{ij}(\cdot\cdot) \cdot s^j_{(\alpha)}(\cdot\cdot) = -(1/2)\left[\nabla^k\phi \cdot \nabla_k\phi + m^2 \cdot \phi^2\right] \cdot g_{ij}(\cdot\cdot) \cdot s^j_{(\alpha)}(\cdot\cdot). \tag{6.81ii}$$

Thus, four invariant eigenvalues are given by

$$\begin{aligned} &\lambda_{(1)}(\cdot\cdot) \equiv \lambda_{(2)}(\cdot\cdot) \equiv \lambda_{(3)}(\cdot\cdot) = -(1/2) \cdot \left[\nabla^k\phi \cdot \nabla_k\phi + m^2\phi^2\right], \\ &\lambda_{(4)}(\cdot\cdot) = (1/2) \cdot \left[\nabla^k\phi \cdot \nabla_k\phi - m^2\phi^2\right]. \end{aligned} \tag{6.82}$$

Therefore, the theorem is proven. ■

Remark: By Theorems 6.1.2 and 6.2.1, the energy–momentum–stress tensor of a massive, real scalar field is algebraically compatible to F–L–R–W metrics of (6.70i).

Let us investigate field equations (6.78i–iii) and the conservation equation (6.78iv) using F–L–R–W metrics in (6.70i). Equations (6.71i–v) are still valid. Therefore, for *a nonconstant scalar field*, we must choose

$$\partial_\alpha \phi \equiv 0,$$

$$\dot{\phi} := \partial_4 \phi \neq 0,$$

which are the same as in (6.72i,ii). The massive scalar field equation (6.78iii) yields

$$\left(a^3 \cdot \dot{\phi}\right)^{\cdot} + m^2 \cdot a^3 \cdot \phi(t) = 0, \tag{6.83i}$$

$$\text{or,} \qquad \left(a^3\right)^{\cdot} \cdot \dot{\phi} + a^3 \cdot \left[\ddot{\phi} + m^2\phi\right] = 0. \tag{6.83ii}$$

Equation (6.83ii) is a linear, ordinary differential equation for the function $\phi(t)$ and also for the function $[a(t)]^3$. Integrating (6.83ii), we obtain

$$\begin{aligned} &a(t) = a_1 \cdot \exp\left\{-(1/3) \cdot \int_{t_1}^{t} \left[\frac{\ddot{\phi}(y) + m^2\phi(y)}{\dot{\phi}(y)}\right] \mathrm{d}y\right\}, \\ &a_1 = a(t_1). \end{aligned} \tag{6.84}$$

Field equations $\mathcal{E}^1{}_1(\cdot\cdot) = 0$ and $\mathcal{E}^4{}_4(\cdot\cdot) = 0$, from (6.78i,ii) with F–L–R–W metrics in (6.70i), yield

$$\mathcal{E}^1{}_1(\cdot\cdot) = \left[\frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right] + (\kappa/2) \cdot \left[\left(\dot{\phi}\right)^2 - m^2(\phi)^2\right] = 0, \tag{6.85i}$$

$$\mathcal{E}^4{}_4(\cdot\cdot) = 3 \cdot \left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - (\kappa/2) \cdot \left[\left(\dot{\phi}\right)^2 + m^2(\phi)^2\right] = 0. \tag{6.85ii}$$

The Raychaudhuri equation (6.49i,ii) implies that

$$\ddot{a}(t) = \left(\frac{\kappa}{6}\right) \cdot \left[m^2\left(\phi(t)\right)^2 - 2 \cdot \left(\dot{\phi}(t)\right)^2\right] \cdot a(t). \tag{6.86}$$

The above equation shows that in the right hand side of (6.86), the term $m^2\left(\phi(t)\right)^2$ *causes repulsion, but the term* $-2 \cdot \left(\dot{\phi}(t)\right)^2$ *produces attraction!*

Rainich showed [213] how to eliminate electromagnetic field components from the Einstein–Maxwell equations (2.290i–iv) and write an equivalent system involving *only the metric tensor and its derivatives*. In the much easier setting of the coupled F–L–R–W and scalar field equations (6.82) and (6.85i,ii), we shall accomplish a similar goal by eliminating the $\phi(\cdot\cdot)$ field. First, we take linear combinations of (6.85i) and (6.85ii) to derive

$$\frac{\ddot{a}}{a} + 2\left[\frac{(\dot{a})^2 + k_0}{a^2}\right] = (\kappa/2) \cdot m^2 \cdot \phi^2 > 0, \tag{6.87i}$$

$$\frac{(\dot{a})^2 + k_0}{a^2} - \frac{\ddot{a}}{a} = (\kappa/2) \cdot \left(\dot{\phi}\right)^2 > 0. \tag{6.87ii}$$

Extracting square roots of *both* of the above equations, differentiating the *first one* and eliminating $\dot{\phi}$ from both, we deduce that

$$\left\{\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}^2 + k_0}{a^2}\right)\right]^{1/2}\right\}^{\cdot} \mp m \cdot \left[\left(\frac{\dot{a}^2 + k_0}{a^2}\right) - \frac{\ddot{a}}{a}\right]^{1/2} = 0. \tag{6.88}$$

The above nonlinear equation *is equivalent to the system (6.83i) and (6.85i,ii)*. Clearly, (6.88) is formidable. However, in the case of a flat F–L–R–W metric, with $k_0 = 0$, it is possible to simplify (6.88). Recall the Hubble function $\mathsf{H}(t)$ of (6.60i). Its derivative is provided by

$$\dot{\mathsf{H}} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2. \tag{6.89}$$

Therefore, (6.88), in the $k_0 = 0$ case, reduces to

$$\left[\sqrt{\dot{\mathsf{H}} + 3(\mathsf{H})^2}\right]^{\cdot} \mp m \cdot \sqrt{-\dot{\mathsf{H}}} = 0. \tag{6.90}$$

The above equation is *a nonlinear, autonomous, second-order equation*. We make the usual substitution:

$$G[\mathsf{H}] := \dot{\mathsf{H}} < 0, \tag{6.91i}$$

$$\ddot{\mathsf{H}} = G[\mathsf{H}] \cdot G'[\mathsf{H}]. \tag{6.91ii}$$

Then, (6.90) *reduces to the first-order o.d.e.*:

$$\frac{1}{2} \frac{\{G(\mathsf{H}) \cdot G'[\mathsf{H}] + 6\mathsf{H} \cdot G[\mathsf{H}]\}}{\sqrt{G[\mathsf{H}] + 3(\mathsf{H})^2}} \mp m \cdot \sqrt{-G(\mathsf{H})} = 0. \tag{6.92}$$

We have to solve the above equation for the function $G[\mathsf{H}]$. At this stage, only numerical solutions seem to be possible. (Note that in the $\lim m \to 0_+$, (6.92) reduces drastically to $G'[H] + 6[H] = 0$, and the solutions (6.76i,ii) *can be recovered*.)

Now, we shall introduce the *quintessence scalar field* to explain the phenomenon of *dark energy* in the cosmological universe. (For a thorough mathematical analysis, see [76].) Dark energy is the generic name given to the "material" which is thought to cause *a repulsive acceleration* of the expansion of the universe [225]. (An in-depth study of quintessence related to cosmic acceleration may be found in [174].)

The coupled perfect fluid (of (2.257i,ii)) and quintessence scalar field equations in general relativity are furnished by [76]:

$$\mathcal{E}_{ij}(x) := G_{ij}(\cdot\cdot) + \kappa\, T_{ij}(\cdot\cdot) = 0, \tag{6.93i}$$

$$\begin{aligned} T_{ij}(\cdot\cdot) &:= \left[\rho(\cdot\cdot) + p(\cdot\cdot)\right] \cdot U_i(\cdot\cdot) \cdot U_j(\cdot\cdot) + p(\cdot\cdot) \cdot g_{ij}(\cdot\cdot) \\ &\quad + \nabla_i \phi \cdot \nabla_j \phi - (1/2) \cdot g_{ij}(\cdot\cdot) \cdot \left[\nabla^k \phi \cdot \nabla_k \phi + 2V(\phi)\right], \end{aligned} \tag{6.93ii}$$

$$\mathcal{T}_i(\cdot\cdot) := \nabla_j T^{\,j}_{\ i} = 0, \tag{6.93iii}$$

$$\mathcal{U}(\cdot\cdot) := U^i(\cdot\cdot) \cdot U_i(\cdot\cdot) + 1 = 0, \tag{6.93iv}$$

$$\sigma(\cdot\cdot) := \nabla^i \nabla_i \phi - V'(\phi) = 0, \tag{6.93v}$$

$$\mathcal{K}(\cdot\cdot) := \nabla^i \left[(\rho + p) \cdot U_i\right] - U^i(\cdot\cdot) \cdot \nabla_i p = 0, \tag{6.93vi}$$

$$\mathcal{F}_i(\cdot\cdot) := (\rho + p) \cdot U^j \cdot \nabla_j U_i + \left(\delta^j_{\ i} + U^j \cdot U_i\right) \cdot \nabla_j p = 0, \tag{6.93vii}$$

$$\mathcal{C}^i\left(g_{ij}, \partial_k g_{ij}\right) = 0. \tag{6.93viii}$$

Here, $V(\phi)$ is a scalar function of ϕ known as *the scalar field potential*.

The corresponding differential and algebraic identities are the following:

$$\nabla_j \mathcal{E}^j_{\ i} \equiv \kappa\, \mathcal{T}_i(\cdot\cdot), \tag{6.94i}$$

$$U^i \mathcal{T}_i + \mathcal{K}(\cdot\cdot) - \left(U^i \nabla_i \phi\right) \cdot \sigma(\cdot\cdot) \equiv 0, \tag{6.94ii}$$

$$\mathcal{F}_i(\cdot\cdot) \equiv \left(\delta^j_{\ i} + U^j U_i\right) \cdot \left[\mathcal{T}_j - \sigma \cdot \nabla_j \phi\right]. \tag{6.94iii}$$

Remark: Physically speaking, it is obvious from (6.93i–viii) that the dark energy (represented by the scalar field $\phi(x)$) *does not interact directly with the perfect fluid*. The only interaction is via the metric, or equivalently, the interaction is strictly gravitational.

Now, we shall count the number of unknown functions versus the number of independent equations.

No. of unknown functions: $10(g_{ij})+1(\rho)+1(p)+4(U^i)+1(\phi)+1(V(\phi))=18.$

No. of equations: $10(\mathcal{E}_{ij}=0)+4(\mathcal{T}_i=0)+1(\mathcal{U}=0)+1(\sigma=0)$
$+1(\mathcal{K}=0)+4(\mathcal{F}_i=0)+4(\mathcal{C}^i=0)=25.$

No. of identities: $4\left(\nabla_j \mathcal{E}^j_{\ i} \equiv \kappa \mathcal{T}_i\right)+1\left(U^i \mathcal{T}_i+\mathcal{K}-(U^i \nabla_i \phi)\cdot\sigma \equiv 0\right)$
$+4\left(\mathcal{F}_i - \left(\delta^j_{\ i}+U^j U_i\right)\cdot\left[\mathcal{T}_j-\sigma\nabla_j\phi\right]\equiv 0\right)=9.$

No. of independent equations: $25-9=16$.

The system of partial differential equations is clearly *underdetermined.* We can impose one equation of state (as in (2.258)). The system still remains underdetermined with one degree of freedom. Therefore, we can prescribe one of the unknown functions; for example, $V(\phi)$, or else, the expansion function $a(t)$, in order to render the system *determinate*.

We shall now explore *the invariant eigenvalues* of the 4×4 matrix $\left[T_{ij}(x)\right]$ from the definition (6.93ii). We *assume that the scalar field $\phi(x)$ is nonconstant* or, in other words, the vector field $\nabla^i\phi \cdot \frac{\partial}{\partial x^i} \neq \vec{\mathbf{0}}(x)$. We choose an orthonormal basis set $\left\{\vec{\mathbf{s}}_{(1)}(x), \vec{\mathbf{s}}_{(2)}(x), \vec{\mathbf{s}}_{(3)}(x), \vec{\mathbf{U}}(x)\right\}$. We obtain from (6.93ii) and the orthonormal basis vectors that

$$T^i_{\ j} \cdot U^j = -\left\{\rho + (1/2)\cdot\left[\nabla^k\phi \cdot \nabla_k\phi\right] + V(\phi)\right\}\cdot U^i + \left(U^j \cdot \nabla_j\phi\right)\cdot \nabla^i\phi, \tag{6.95i}$$

$$T^i_{\ j} \cdot \nabla^j\phi = \left[(\rho+p)\cdot U_j \nabla^j\phi\right]\cdot U^i + \left\{p + (1/2)\cdot\left[\nabla^k\phi \cdot \nabla_k\phi\right] - V(\phi)\right\}\cdot\nabla^i\phi, \tag{6.95ii}$$

$$T^i_{\ j} \cdot s^j_{(\alpha)} = \left[p - (1/2)\cdot\nabla^k\phi \cdot \nabla_k\phi - V(\phi)\right]\cdot s^i_{(\alpha)} + \left[\nabla_j\phi \cdot s^j_{(\alpha)}\right]\cdot\nabla^i\phi. \tag{6.95iii}$$

It is obvious that *none of the vectors* $\vec{\mathbf{U}}(\cdot\cdot)$, $\vec{\mathbf{s}}_{(\alpha)}(\cdot\cdot)$, $\nabla^i \phi \cdot \frac{\partial}{\partial x^i}$ are eigenvectors in the general setting. However, *in special cases, some of these vectors are eigenvectors*, as will be shown in the following theorem. (See [76].)

Theorem 6.2.2. *Let field equations* (6.93i,ii) *hold in a domain* $D \subset \mathbb{R}^4$. *Let an orthonormal basis set* $\{\vec{\mathbf{s}}_{(1)}(x), \vec{\mathbf{s}}_{(2)}(x), \vec{\mathbf{s}}_{(3)}(x), \vec{\mathbf{U}}(x)\}$ *exist in the same domain. Moreover, let* $\nabla^i \phi \cdot \frac{\partial}{\partial x^i} = l\,(x) \cdot \vec{\mathbf{U}}(x)$ *for some nonzero, differentiable scalar field* $l\,(x)$. *Then, the* 4×4 *energy–momentum–stress matrix* $\left[T_{ij}\,(x)\right]$ *of* (6.93ii) *must be of Segre characteristic* $[(1,1,1),1]$.

Proof. Using (6.93ii) and (6.95i–iii) it follows that

$$\begin{aligned} T^i{}_j \cdot s^j_{(\alpha)} &= \left[p - (1/2)\cdot \nabla^k\phi \cdot \nabla_k\phi - V(\phi)\right] \cdot s^i_{(\alpha)} \\ &=: \lambda_{(\alpha)} \cdot s^i_{(\alpha)}, \end{aligned} \tag{6.96i}$$

$$\begin{aligned} T^i{}_j \cdot U^j &= -\left[\rho - (1/2)\cdot \nabla^k\phi \cdot \nabla_k\phi + V(\phi)\right] \cdot U^i \\ &=: \lambda_{(4)} \cdot U^i. \end{aligned} \tag{6.96ii}$$

Thus, the theorem is proven. ■

Remarks: (i) Three repeated eigenvalues $\lambda_{(\alpha)} = \left[p - (1/2)\cdot \nabla^k\phi \cdot \nabla_k\phi - V(\phi)\right]$ consist of the fluid pressure plus the isotropic pressure due to the scalar field.

(ii) The negative of the fourth eigenvalue $-\lambda_{(4)} = \left[\rho - (1/2)\cdot \nabla^k\phi \cdot \nabla_k\phi + V(\phi)\right]$ denotes the energy density of the fluid plus the energy density of the scalar field.

Now, we shall study the F–L–R–W universes containing a perfect fluid and the quintessence scalar field. We employ the usual comoving coordinate frame. The relevant assumptions are

$$\mathrm{d}s^2 = [a(t)]^2 \cdot \left\{\left(1 - k_0 r^2\right)^{-1} \cdot (\mathrm{d}r)^2 + r^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta \cdot (\mathrm{d}\varphi)^2\right]\right\} - (\mathrm{d}t)^2, \tag{6.97i}$$

$$U^\alpha(\cdot\cdot) \equiv 0, \quad U^4(\cdot\cdot) \equiv 1, \tag{6.97ii}$$

$$\partial_\alpha \phi \equiv 0, \quad \dot{\phi} := \frac{\partial \phi(\cdot\cdot)}{\partial t} \neq 0. \tag{6.97iii}$$

The nontrivial field equations from (6.93i–vi) reduce to

$$-\sigma(\cdot\cdot) = a^{-3} \cdot \left(a^{-3} \cdot \dot{\phi}\right)^{\dot{}} + V'(\phi) = 0, \tag{6.98i}$$

$$\mathcal{E}^1{}_1(\cdot\cdot) = \left[\frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right] + \kappa \cdot \left[p + (1/2)\cdot\left(\dot{\phi}\right)^2 - V(\phi)\right] = 0, \tag{6.98ii}$$

$$\mathcal{E}^4{}_4(\cdot\cdot) = 3 \cdot \left[\frac{(\dot{a})^2 + k_0}{a^2} \right] - \kappa \cdot \left[\rho + (1/2) \cdot \left(\dot{\phi}\right)^2 + V(\phi) \right] = 0, \tag{6.98iii}$$

$$\mathcal{K}(\cdot\cdot) = \dot{\rho} + 3 \cdot (\rho + p) \cdot \left[\ln|a| \right]^{\cdot} = 0. \tag{6.98iv}$$

The Raychaudhuri equation in (6.86) generalizes into

$$\ddot{a} = -(\kappa/6) \cdot \left\{ (\rho + 3p) + 2\left(\dot{\phi}\right)^2 - 2V(\phi) \right\} \cdot a. \tag{6.99}$$

In the equation above, $\rho + 3p + 2\left(\dot{\phi}\right)^2 > 0$ contributes to *gravitational attraction*. However, the term $2V(\phi) > 0$ enhances *repulsive gravitation*! Thus, the dark energy (in the form of the potential term $V(\phi)$) accelerates the expansion.

Linear combinations of field equations (6.98ii) and (6.98iii) are easier to investigate. These are furnished by

$$\left[\frac{(\dot{a})^2 + k_0}{a^2} \right] - \frac{\ddot{a}}{a} = \left(\frac{\kappa}{2} \right) \cdot \left[\rho + p + \left(\dot{\phi}\right)^2 \right], \tag{6.100i}$$

$$\frac{\ddot{a}}{a} + 2\left[\frac{(\dot{a})^2 + k_0}{a^2} \right] = \left(\frac{\kappa}{2} \right) \cdot \left[\rho - p + 2V(\phi) \right]. \tag{6.100ii}$$

Firstly, we explore the inflationary scenario with $\rho(\cdot\cdot) = p(\cdot\cdot) \equiv 0$. We shall cite some exact solutions of (6.100i,ii) with these special assumptions. The first example is that of *an exponential expansion* (similar to that of (6.57)). It is provided for the case $k_0 = 1$ as

$$a(t) = \mathrm{e}^t, \tag{6.101i}$$

$$\phi(t) = -\sqrt{2/\kappa} \cdot \mathrm{e}^{-t}, \tag{6.101ii}$$

$$V(\phi) = \kappa^{-1} \cdot \left[3 + \kappa \cdot (\phi)^2 \right]. \tag{6.101iii}$$

Another exact solution is furnished for *cos-hyperbolic expansion* as

$$a(t) = (2)^{-1/2} \cdot \cosh t, \tag{6.102i}$$

$$\phi(t) = (2/\kappa)^{1/2} \cdot \mathrm{Arc}\sin(\tanh t), \tag{6.102ii}$$

$$V(\phi) = \kappa^{-1} \cdot \left\{ 3 + 2 \cdot \cos^2\left[\sqrt{\kappa/2} \cdot \phi \right] \right\}. \tag{6.102iii}$$

Now, we shall investigate *the matter phase scenario* for the coupled field equations (6.98i–iv). The system simplifies here by *neglecting the fluid pressure* therefore setting $p(\cdot\cdot) \equiv 0$. The continuity equation (6.98iv) can be readily integrated to obtain

$$(\kappa/6) \cdot \rho(t) \cdot [a(t)]^3 = M_0 = \text{positive const.} \tag{6.103}$$

Physically speaking, M_0 is *proportional to the component of the total mass of the universe which is not attributable to the scalar field* $\phi(t)$. In this phase, using (6.103), other field equations (6.100i,ii) reduce to

$$\kappa\, V(\phi) =: \kappa\, V^{\#}(t) = \frac{\ddot{a}}{a} + 2 \cdot \left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - \frac{3M_0}{a^3}, \tag{6.104i}$$

$$(\kappa/2) \cdot \left[\dot{\phi}(t)\right]^2 = \left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - \frac{\ddot{a}}{a} - \frac{3M_0}{a^3} > 0. \tag{6.104ii}$$

(In case $a(t)$ is prescribed, $\phi(t)$ can be obtained by integration of (6.104ii). Moreover, the function $V^{\#}(t)$ can be determined from (6.104i). Thus, the function $V(\phi)$ can be expressed *parametrically in terms of* t.)

Now, we shall cite an exact solution of the system (6.104i,ii). It is characterized by the following power law of expansion:

$$a(t) = 1 + (t)^{n+2}, \quad n > 0, \tag{6.105i}$$

$$\kappa\, V^{\#}(t) = \frac{(n+1)\cdot(n+2)\, t^n}{(1+t^{n+2})} + 2 \cdot \left[\frac{(n+2)^2 \cdot t^{2(n+1)} + k_0}{(1+t^{n+2})^2}\right] - \frac{3M_0}{(1+t^{n+2})^3}, \tag{6.105ii}$$

$$\sqrt{\kappa/2} \cdot [\phi(t) - \phi_1] = \int_{t_1}^{t} \left\{ \left[\frac{(n+2)^2 \cdot y^{2(n+1)} + k_0}{(1+y^{n+2})^2}\right] - \frac{(n+1)\cdot(n+2)\cdot y^n}{(1+y^{n+2})} - \frac{3M_0}{(1+y^{n+2})^3} \right\}^{1/2} \cdot \mathrm{d}y. \tag{6.105iii}$$

Caution: In the integral above, the interval (t_1, t) is to be carefully chosen so that the "curly bracket" inside the integrand is *nonnegative*.

Before proceeding, we shall furnish an example illustrating the utility of the initial value formalism introduced in Sect. 2.4.

Example 6.2.3. We consider the evolution of *a constant quintessence scalar field in an otherwise empty, spatially flat ($k_0 = 0$) FLRW space–time*. Recall that the line element of such a space–time is given by (6.9) as

$$\mathrm{d}s^2 = [a(t)]^2 \cdot \delta_{\alpha\beta} \cdot (\mathrm{d}x^\alpha)(\mathrm{d}x^\beta) - (\mathrm{d}t)^2 ,$$

$$x \in D := \mathbf{D} \times (t_0, t_1) \subset \mathbb{R}^4.$$

The scenario will be one of exponential expansion, $a(t) = \mathrm{e}^t$. Since the scalar field is constant, the only contribution to the energy–momentum–stress tensor is from the (constant) potential function $V(\phi_0)$. (Refer to (6.93ii). Note that this is equivalent to a vacuum solution with a cosmological constant.) For this, we choose a value $V(\phi_0) = 3/\kappa$. The initial data on the surface $t = t_0$ is as follows:

$$g^{\#}_{\alpha\beta}(\mathbf{x}) = \mathrm{e}^{2t_0} \cdot \delta_{\alpha\beta},\ \psi_{\alpha\beta} = 2\mathrm{e}^{2t_0} \cdot \delta_{\alpha\beta},\ \theta_i(\mathbf{x}) = \frac{3}{\kappa} \cdot \delta_{4i},\ T^{\#}_{\alpha\beta}(\mathbf{x}) = -\frac{3}{\kappa}\mathrm{e}^{2t_0} \cdot \delta_{\alpha\beta}.$$

It can be shown that the above initial data satisfies the constraints of (2.251) on the initial hypersurface and that the scalar field equation of motion (6.93v) is trivially satisfied. (Compare with Problem #5 of Exercises 2.4)

The Einstein equations can be solved analytically for the scale factor $a(t)$. (In fact, it is very simple to do so in this case.) However, here, we wish to utilize the initial value formalism of Sect. 2.4 and compare it to the analytic solution. This example is another illustration of how to utilize the initial value scheme, and its simplicity allows us to test the scheme's convergence properties to an exact analytic result. To solve for the scale factor, it is sufficient to just evolve one nonzero component of $g^{\#}_{\alpha\beta}(\mathbf{x})$, as all nonzero components of $g_{\alpha\beta}$ in this case are equal to the scale factor squared. With the above information, we can form the Taylor series for $\overline{g}_{\alpha\beta}(x)$ as in (2.252) (utilizing the previous formalism in that section for the higher order time derivatives of the metric on the initial surface). The results are displayed in Fig. 6.4 where the dotted lines indicate the approximation at quadratic, cubic, and quartic order in $(t - t_0)$ of the Taylor series. The solid line in the figure represents the analytic result $a^2(t) = \mathrm{e}^{2t}$. One can see that retaining higher-order terms in the expansion yields better agreement with the analytic result.

Now, we shall discuss what is often called the *tachyonic scalar field* (or *Born-Infeld scalar field*), denoted by Φ, gravitationally coupled with a perfect fluid. (See [76, 232, 233].) The general relativistic field equations are governed by the following:

$$\mathcal{E}^i{}_j(x) := G^i{}_j(\cdot\cdot) + \kappa\, T^i{}_j(\cdot\cdot) = 0, \tag{6.106i}$$

$$\begin{aligned} T^i_j(\cdot\cdot) := {} & [\rho(\cdot\cdot) + p(\cdot\cdot)] \cdot U^i(\cdot\cdot) \cdot U_j(\cdot\cdot) + p(\cdot\cdot) \cdot \delta^i{}_j + V(\Phi) \\ & \times \left[\left(1 + \nabla^k \Phi \cdot \nabla_k \Phi\right)^{-1/2} \cdot \nabla^i \Phi \cdot \nabla_j \Phi - \left(1 + \nabla^k \Phi \cdot \nabla_k \Phi\right)^{1/2} \cdot \delta^i{}_j \right], \end{aligned} \tag{6.106ii}$$

$$\mathcal{T}_i(\cdot\cdot) := \nabla_j T^j_i = 0, \tag{6.106iii}$$

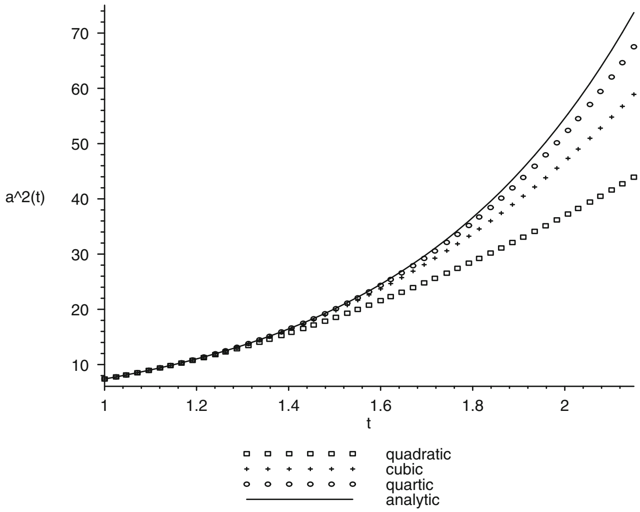


Fig. 6.4 Comparison of the square of the cosmological scale factor, $a^2(t)$. The *dotted lines* represent the numerically evolved Cauchy data utilizing the scheme outlined in Sect. 2.4 to various orders in $t - t_0$ (quadratic, cubic, quartic). The *solid line* represents the analytic result (e^{2t}) □

$$\mathcal{U}(\cdot\cdot) := U^i(\cdot\cdot) \cdot U_i(\cdot\cdot) + 1 = 0, \tag{6.106iv}$$

$$\sigma(\cdot\cdot) := \nabla^i \left[V(\Phi) \cdot \left(1 + \nabla^k \phi \cdot \nabla_k \Phi\right)^{-1/2} \cdot \nabla_i \Phi \right] - V'(\Phi) \cdot \left(1 + \nabla^k \Phi \cdot \nabla_k \Phi\right)^{1/2} = 0, \tag{6.106v}$$

$$\mathcal{K}(\cdot\cdot) := \nabla^i \left[(\rho + p) \cdot U_i\right] - U^i(\cdot\cdot) \cdot \nabla_i p = 0, \tag{6.106vi}$$

$$\mathcal{F}_i(\cdot\cdot) := (\rho + p) \cdot U^j \cdot \nabla_j U_i + \left(\delta^j_i + U^j \cdot U_i\right) \cdot \nabla_j p = 0, \tag{6.106vii}$$

$$\mathcal{C}^i \left(g_{ij}, \partial_k g_{ij}\right) = 0. \tag{6.106viii}$$

Algebraic and differential identities are provided by

$$\nabla_j \mathcal{E}^j_i \equiv \kappa\, \mathcal{T}_i(\cdot\cdot), \tag{6.107i}$$

$$U^i \mathcal{T}_i + \mathcal{K}(\cdot\cdot) - \left(U^i \nabla_i \Phi\right) \cdot \sigma(\cdot\cdot) \equiv 0, \tag{6.107ii}$$

$$\mathcal{F}_i(\cdot\cdot) \equiv \left(\delta^j_i + U^j U_i\right) \cdot \left[\mathcal{T}_j - \nabla_j \Phi \cdot \sigma(\cdot\cdot)\right]. \tag{6.107iii}$$

Just like with the quintessence scalar field, the system of the coupled perfect fluid and tachyonic fluid field equations is *underdetermined* (by two degrees of freedom).[4]

Let us now work out the invariant eigenvalue problem for the 4×4 energy–momentum–stress matrix $\left[T^i{}_j(x)\right]$. Let $\left\{\vec{\mathbf{s}}_{(1)}(x), \vec{\mathbf{s}}_{(2)}(x), \vec{\mathbf{s}}_{(3)}(x), \vec{\mathbf{U}}(x)\right\}$ be an *orthonormal basis field*. For the purpose of cosmological applications, *we restrict to the case* for which $\nabla^i \Phi \cdot \frac{\partial}{\partial x^i} = l(x) \cdot \vec{\mathbf{U}}(x) \neq \vec{\mathbf{0}}(x)$. Thus, we can deduce from (6.106ii) that

$$T^i{}_j(\cdot\cdot) \cdot s^j_{(\alpha)}(\cdot\cdot) = \left[p(\cdot\cdot) - V(\Phi) \cdot \left(1 + \nabla^k \Phi \cdot \nabla_k \Phi\right)^{1/2}\right] \cdot s^i_{(\alpha)}(\cdot\cdot), \tag{6.108i}$$

$$T^i{}_j(\cdot\cdot) \cdot U^j(\cdot\cdot) = -\left[\rho(\cdot\cdot) + V(\Phi) \cdot \left(1 + \nabla^k \Phi \cdot \nabla_k \Phi\right)^{-1/2}\right] \cdot U^i(\cdot\cdot). \tag{6.108ii}$$

Therefore, in the restricted case, the invariant eigenvalues are

$$\lambda_{(1)}(\cdot\cdot) \equiv \lambda_{(2)}(\cdot\cdot) \equiv \lambda_{(3)}(\cdot\cdot) = p(\cdot\cdot) - V(\Phi) \cdot \left(1 + \nabla^k \Phi \cdot \nabla_k \Phi\right)^{1/2}, \tag{6.109i}$$

$$\lambda_{(4)}(\cdot\cdot) = -\left[\rho(\cdot\cdot) + V(\Phi) \cdot \left(1 + \nabla^k \Phi \cdot \nabla_k \Phi\right)^{-1/2}\right]. \tag{6.109ii}$$

Clearly, the Segre characteristic is given by $[(1, 1, 1), 1]$.

We now impose the F–L–R–W metric and the comoving frame of (6.97i–iii). The coupled field equations (6.106i,iii,v,vi) reduce to

$$\mathcal{E}^1{}_1(t) = \left[\frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right] + \kappa \cdot \left[p - V(\Phi) \cdot \left(1 - (\dot{\Phi})^2\right)^{1/2}\right] = 0, \tag{6.110i}$$

$$\mathcal{E}^4{}_4(t) = 3\left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - \kappa \cdot \left[\rho + V(\Phi) \cdot \left(1 - (\dot{\Phi})^2\right)^{-1/2}\right] = 0, \tag{6.110ii}$$

[4] The scalar field component of the energy–momentum–stress tensor (6.106ii) and the equation of motion in (6.106v) may be derived from a Lagrangian density of the form $\mathcal{L}_\Phi = -\sqrt{-g} \cdot V(\Phi) \cdot \sqrt{1 + \nabla^i \Phi \cdot \nabla_i \Phi}$, which is inspired by the "square-root" particle Lagrangian of classical special relativistic mechanics of the form $L(v) = -m\sqrt{1 - \delta_{\alpha\beta} v^\alpha v^\beta}$. However, the tachyonic scalar field Φ is also utilized in *string theory*, and some string theoretic arguments place extra restrictions on $V(\Phi)$ which need to be enforced if the solutions are to decribe the phenomenon of *tachyon condensation*. (See [154, 232, 233].)

$$-\sigma(t) = a^{-3} \cdot \left[a^3 \cdot V(\Phi) \cdot \left(1 - (\dot{\Phi})^2\right)^{-1/2} \cdot \dot{\Phi}\right]^{\cdot}$$
$$+ \left(1 - (\dot{\Phi})^2\right)^{1/2} \cdot V'(\Phi) = 0, \tag{6.110iii}$$

$$\mathcal{K}(t) = \dot{\rho} + 3 \cdot (\rho + p) \cdot \left[\ln|a|\right]^{\cdot} = 0. \tag{6.110iv}$$

(It is assumed that $1 - (\dot{\Phi})^2 \geq 0$.)

The Raychaudhuri equation is furnished by

$$\ddot{a} = -\left(\frac{\kappa}{6}\right) \cdot a \cdot \left\{\rho + 3p + V(\Phi) \cdot \left[\left(1 - (\dot{\Phi})^2\right)^{-1/2} \cdot (\dot{\Phi})^2 - 2\left(1 - (\dot{\Phi})^2\right)^{1/2}\right]\right\}. \tag{6.111}$$

Now, we explore a special class of *tachyonic inflationary phases* by setting $\rho(t) = p(t) \equiv 0$. Linear combinations of field equations (6.110i,ii) lead to

$$\kappa\, V(\Phi) \cdot \left(1 - (\dot{\Phi})^2\right)^{1/2} = \frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}, \tag{6.112i}$$

$$\kappa\, V(\Phi) \cdot \left(1 - (\dot{\Phi})^2\right)^{-1/2} = 3\left[\frac{(\dot{a})^2 + k_0}{a^2}\right]. \tag{6.112ii}$$

We now provide some special solutions of (6.112i,ii). (Recall the footnote on page 453.) *Exponential inflation* for the case of $k_0 = 1$ is furnished by

$$a(t) = e^t, \tag{6.113i}$$

$$\Phi(t) = \sqrt{\frac{1}{6}} \cdot \ln\left|\frac{\sqrt{1 + e^{2t}} - 1}{\sqrt{1 + e^{2t}} + 1}\right|, \tag{6.113ii}$$

$$\kappa\, V(\Phi) = \sqrt{3} \cdot \cosh\left(\sqrt{3/2} \cdot \Phi\right) \cdot \left[2 + \cosh^2\left(\sqrt{3/2} \cdot \Phi\right)\right]^{1/2}. \tag{6.113iii}$$

The cos-hyperbolic inflation for $k_0 = 1$ is provided by the (simpler) equations

$$a(t) = \cosh t, \tag{6.114i}$$

$$\Phi(t) = \Phi_0 = \text{const.}, \tag{6.114ii}$$

$$\kappa\, V(\Phi) = 3. \tag{6.114iii}$$

Now, we shall touch upon *the tachyonic-scalar matter phase*. In this regime, we adopt the assumption of *an incoherent dust plus the scalar field*. Therefore, the continuity equation (6.110iv) yields the same integration constant $M_0 > 0$ as in (6.103). Thus, the coupled field equations (6.110i,ii) reduce to

$$\left(\dot{\Phi}\right)^2 = (2/3) \cdot \left[\frac{(\dot{a})^2 + k_0}{a^2} - \frac{\ddot{a}}{a} - \frac{3M_0}{a^3}\right] \cdot \left[\frac{(\dot{a})^2 + k_0}{a^2} - \frac{2M_0}{a^3}\right]^{-1}, \tag{6.115i}$$

$$\left[\kappa\, V^{\#}(t)\right]^2 = 3 \cdot \left[\frac{(\dot{a})^2 + k_0}{a^2} - \frac{2M_0}{a^3}\right] \cdot \left[\frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right]. \tag{6.115ii}$$

Here, we have tacitly assumed that

$$\frac{(\dot{a})^2 + k_0}{a^2} > \frac{2M_0}{a^3}, \tag{6.116i}$$

$$\frac{(\dot{a})^2 + k_0}{a^2} - \frac{\ddot{a}}{a} > \frac{3M_0}{a^3}. \tag{6.116ii}$$

We shall now provide an exact solution of the system in (6.115i,ii) involving the following *power law of expansion*:

$$a(t) = t^{n+2}, \quad n > 0, \tag{6.117i}$$

$$\begin{aligned} \left(\kappa/\sqrt{3}\right) \cdot V^{\#}(t) =: \left(\kappa/\sqrt{3}\right) \cdot V\left[\Phi(t)\right] \\ = \Bigg\{ \left[\frac{(n+2)^2 \cdot t^{2(n+1)} + k_0}{t^{2(n+2)}} - \frac{2M_0}{t^{3(n+2)}}\right] \\ \times \left[\frac{2(n+1)(n+2)}{t^2} + \frac{(n+2)^2 \cdot t^{2(n+1)} + k_0}{t^{2(n+2)}}\right] \Bigg\}^{1/2}, \end{aligned} \tag{6.117ii}$$

$$\Phi(t) - \Phi_1 = \sqrt{\frac{2}{3}} \cdot \int_{t_1}^{t} \left\{\frac{k_0 \cdot y^{n+2} + (n+2)^2 \cdot y^{3n+4} - 3M_0}{k_0 \cdot y^{n+2} + (n+2)^2 \cdot y^{3n+4} - 2M_0}\right\}^{1/2} \cdot \mathrm{d}y. \tag{6.117iii}$$

Caution: In the integral above, the interval (t_1, t) is to be so chosen that the "curly bracket" inside the integral remains *nonnegative*.

In Sects. 6.1 and 6.2, we have dealt many times with the conformally flat F–L–R–W metrics. A more in-depth analysis of conformally flat space–times is provided in Appendix 4.

6.3 Five-Dimensional Cosmological Models

This section lies outside the main focus of this chapter and may be considered optional for an introductory curriculum.

Theoretical and observational difficulties with standard F–L–R–W big bang models can be partially resolved in cosmological models with a period of inflation (either exponential or cos-hyperbolic). Inflationary models contain self-interacting scalar fields coupled with gravity. Of particular interest in elementary particle physics are scalars that are known as *Higgs fields* [120, 178]. Section 6.2 dealt with models involving various different scalar fields. Here, we present another popular extension to standard cosmology, namely, the addition of extra spatial dimensions.

Coincident with the renewal of interest sparked by inflationary models, there has been a revival of *Kaluza-Klein theory* which was an attempt at unifying gravity with electromagnetic fields in *a five-dimensional* "space–time." (For a review of the topic, consult [6, 12].) Also, modern *string theories* [117, 274] employ ten-dimensional or eleven-dimensional differential manifolds. There are some cosmological implications of these higher dimensional theories. (See, e.g., [108, 167], and references therein.) However, we shall restrict ourselves in this section only to the five-dimensional manifold for the sake of simplicity and as an introduction to higher dimensional models.

We consider *a five-dimensional pseudo-Riemannian manifold* M_5 with signature $+3$. In a general coordinate chart, we denote $y = \left(y^1, y^2, y^3, y^4, y^5\right) \in \widetilde{D} \subset \mathbb{R}^5$. The capital Roman indices take values from $\{1, 2, 3, 4, 5\}$ *with the usual summation convention* over repeated indices.

We consider the coupled Einstein equations (five-dimensional now) *with a massless scalar field and no fluid in the five-dimensional arena* [111] in what follows:

$$\widetilde{\mathbf{g}}_{..}(y) := \widetilde{g}_{AB}(y) \cdot \mathrm{d}y^A \otimes \mathrm{d}y^B, \tag{6.118i}$$

$$\widetilde{\mathcal{E}}_{AB}(y) := \widetilde{G}_{AB}(y) + \widetilde{\kappa} \cdot \left[\widetilde{\nabla}_A\phi \cdot \widetilde{\nabla}_B\phi - (1/2) \cdot \widetilde{g}_{AB}(\cdot\cdot) \cdot \widetilde{\nabla}^C\phi \cdot \widetilde{\nabla}_C\phi\right] = 0, \tag{6.118ii}$$

$$\widetilde{\sigma}(y) := \widetilde{g}^{AB}(y) \cdot \widetilde{\nabla}_A\widetilde{\nabla}_B\phi = 0. \tag{6.118iii}$$

Here, $\widetilde{\kappa}$ is the "five-dimensional" gravitational constant.

The simplest, nontrivial five-dimensional models (compatible with the previous equations (6.7i) and (6.70i)) are furnished by the metric:

$$\begin{aligned}\widetilde{\mathbf{g}}_{..}(\cdot\cdot) = {} & \left[a\left(x^4\right)\right]^2 \cdot \left[1 + (k_0/4)\cdot\delta_{\mu\nu}x^\mu x^\nu\right]^{-2} \cdot \left[\delta_{\alpha\beta}\cdot \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta\right] \\ & - \left(\mathrm{d}x^4 \otimes \mathrm{d}x^4\right) + \left[\beta\left(x^4\right)\right]^2 \cdot \mathrm{d}x^5 \otimes \mathrm{d}x^5, \end{aligned} \tag{6.119i}$$

$$\begin{aligned}(d\widetilde{s})^2 = {} & [a(t)]^2 \cdot \left\{\left(1 - k_0 r^2\right)^{-1} \cdot (\mathrm{d}r)^2 + r^2\left[(\mathrm{d}\theta)^2 + \sin^2\theta\cdot(\mathrm{d}\varphi)^2\right]\right\} \\ & - (\mathrm{d}t)^2 + [\beta(t)]^2 \cdot (\mathrm{d}\widetilde{\omega})^2 . \end{aligned} \tag{6.119ii}$$

Here, the fifth dimension, characterized by the coordinate x^5 (or $\widetilde{\omega}$) in this chart, is *assumed to be homeomorphic to a (small) circle* S^1. Therefore, *the five-dimensional pseudo-Riemannian manifold* M_5 *is assumed to be locally homeomorphic to* $\mathbb{K}^3 \times \mathbb{R} \times S^1$. (Here, $\mathbb{K}^3$ denotes *a three-dimensional space of constant curvature*.) The exterior radial function $a(t) > 0$ and the interior radial function $\beta(t) > 0$ are scale factors which specify the "sizes" of the external three-dimensional space $\mathbb{K}^3$ and the internal *compactified space* S^1 (the fifth dimension), respectively, at the time t. To be consistent with observations at the present time t_0, it must be that $\dot{a}(t_0) = \frac{\mathrm{d}a(t)}{\mathrm{d}t}_{|t_0} > 0$ and $\beta(t_0)$ must yield a diameter for the fifth dimension that is smaller than what can be probed with current high-energy accelerator experiments. (The size of the extra dimension may even be as small as of the order of the *Planck length*, the smallest scale at which classical gravity is commonly believed to apply [175].)

The metric in (6.119i,ii) admits locally *seven independent Killing vectors* including $\frac{\partial}{\partial x^5} \equiv \frac{\partial}{\partial\widetilde{\omega}}$. To be consistent with these isometries, we choose the scalar field to satisfy

$$\begin{aligned} \partial_\alpha\phi &\equiv 0, \\ \partial_5\phi &\equiv 0, \\ \phi(\cdot\cdot) &= \phi(t), \\ \dot{\phi} &\neq 0. \end{aligned} \tag{6.120}$$

The coupled field equations (6.118ii,iii), with the cosmological assumptions made in (6.119ii) and (6.120), boil down to the following nontrivial equations:

$$\frac{1}{3}\cdot\left[\widetilde{\mathcal{E}}^4{}_4 + \widetilde{\mathcal{E}}^5{}_5\right] = \frac{\ddot{a}}{a} + 2\left[\frac{(\dot{a})^2 + k_0}{a^2}\right] + \frac{\dot{a}}{a}\cdot\frac{\dot{\beta}}{\beta} = 0, \tag{6.121i}$$

$$\frac{1}{3}\cdot\left[3\,\widetilde{\mathcal{E}}^1{}_1 - 2\widetilde{\mathcal{E}}^4{}_4 + \widetilde{\mathcal{E}}^5{}_5\right] = \frac{3\ddot{a}}{a} + \frac{\ddot{\beta}}{\beta} + \widetilde{\kappa}\left(\dot{\phi}\right)^2 = 0, \tag{6.121ii}$$

$$\widetilde{\mathcal{E}}^1{}_1 + \frac{1}{3}\cdot\widetilde{\mathcal{E}}^4{}_4 - \frac{2}{3}\cdot\widetilde{\mathcal{E}}^5{}_5 = \frac{1}{\beta}\cdot\frac{1}{a^3}\left[a^3\cdot\dot{\beta}\right]^{\cdot} = 0, \tag{6.121iii}$$

$$-\widetilde{\sigma}(\cdot\cdot) = \frac{1}{a^3}\left[a^3\cdot\dot{\phi}\right]^{\cdot} + \left(\frac{\dot{\beta}}{\beta}\right)\cdot\dot{\phi} = 0. \tag{6.121iv}$$

Field equations (6.121iii) and (6.121iv) yield the following solutions as *first integrals*:

$$[a(t)]^3 \cdot [\beta(t)]^{\cdot} = c_1 = \text{const.}, \tag{6.122i}$$

$$[a(t)]^3 \cdot [\beta(t)] \cdot [\phi(t)]^{\cdot} = c_2 = \text{const.} \tag{6.122ii}$$

Here, $c_1 \neq 0$ and $c_2 \neq 0$ are otherwise arbitrary constants of integration. Hence, $\beta(t)$ and $\phi(t)$ are *functionally dependent* via the equations:

$$\beta(t) = c_3 \cdot \exp\left[(c_1/c_2) \cdot \phi(t)\right], \tag{6.123i}$$

$$\Phi(\beta(t)) := \phi(t), \tag{6.123ii}$$

$$\Phi(\beta) = (c_2/c_1) \cdot \ln|\beta/c_3|. \tag{6.123iii}$$

Here, $c_3 \neq 0$ is another arbitrary constant.

Further simplifications result when the remaining field equations (6.122i) and (6.122ii) are *rewritten with* β *as the new independent variable instead of* t. Using (6.122i), we convert derivatives as follows:

$$\begin{aligned} &\mathcal{A}(\beta(t)) := a(t), \\ &\dot{a}(t) = c_1 \cdot [\mathcal{A}(\beta)]^{-3} \cdot \mathcal{A}'(\beta) \neq 0. \end{aligned} \tag{6.124}$$

Here, the prime denotes the derivative with respect to β. Moreover, from (6.122i) and (6.123i–iii) we deduce that

$$[\phi(t)]^{\cdot} = c_2 \cdot [\mathcal{A}(\beta)]^{-3} \cdot (\beta)^{-1}. \tag{6.125}$$

Let us denote another function (related to *the Hubble function*) by the following:

$$h(\beta) := \left[\ln|\mathcal{A}(\beta)|\right]'. \tag{6.126}$$

Hence, field equations (6.121i) and (6.121ii) reduce respectively to

$$h'(\beta) + 2\left(k_0/c_1^2\right) \cdot [\mathcal{A}(\beta)]^4 + \beta^{-1} \cdot h(\beta) = 0, \tag{6.127i}$$

$$h'(\beta) - 2\,[h(\beta)]^2 - \beta^{-1} \cdot h(\beta) + (\widetilde{\kappa}/3) \cdot (c_2/c_1)^2 \cdot \beta^{-2} = 0. \tag{6.127ii}$$

Equation (6.127ii) can be integrated to obtain

$$h(\beta) = \frac{1}{\beta} \cdot \left\{ \mu \cdot \left[\frac{1 + (\beta/\beta_0)^{4\mu}}{1 - (\beta/\beta_0)^{4\mu}} \right] - \frac{1}{2} \right\}, \tag{6.128i}$$

$$\text{or,} \quad \ln(\beta/\beta_0)^2 = \frac{1}{2\mu} \cdot \ln \left| \frac{\beta \cdot h(\beta) + (1/2) - \mu}{\beta \cdot h(\beta) + (1/2) + \mu} \right|, \tag{6.128ii}$$

$$\mu := \sqrt{\frac{\widetilde{\kappa}}{6} \cdot \left(\frac{c_2}{c_1} \right)^2 + \frac{1}{4}}. \tag{6.128iii}$$

$$K := \frac{\tilde{\kappa}}{3} \cdot \left(\frac{c_1}{c_2} \right)^2 > 0, \tag{6.128iv}$$

$$\mu = \sqrt{\frac{K}{2} + \frac{1}{4}} > \frac{1}{2}. \tag{6.128v}$$

(Here, β_0 is the nonzero, otherwise arbitrary constant of integration.)

Next, we analyze the exact solutions in (6.128i–v) *quite exhaustively*. For that purpose, we introduce a couple of new functions to use in the sequel:

$$\zeta(\beta) = \beta \cdot h(\beta), \tag{6.129}$$

and

$$z(\beta) := \zeta(\beta) + \frac{1}{2} = \beta \cdot h(\beta) + \frac{1}{2}. \tag{6.130}$$

We have mentioned after (6.119ii) that we choose $\beta > 0$. Moreover, from physical considerations of the Hubble function in (6.126), we restrict $h(\beta) > 0$. Therefore, (6.129) and (6.130) yield

$$\zeta(\beta) = \beta \cdot h(\beta) > 0, \tag{6.131i}$$

$$z(\beta) = \zeta(\beta) + \frac{1}{2} > \frac{1}{2}. \tag{6.131ii}$$

Therefore, by choosing a sufficiently small positive constant $\varepsilon > 0$, we can obtain the following estimates:

$$\zeta(\beta) = \varepsilon > 0, \tag{6.132i}$$

$$z(\beta) = \frac{1}{2} + \varepsilon > \frac{1}{2}. \tag{6.132ii}$$

We shall next depict the qualitative graphs of the related two-dimensional mappings in Fig. 6.5 for *a special function* $h(\beta) := \varepsilon \cdot \beta^{-1}$.

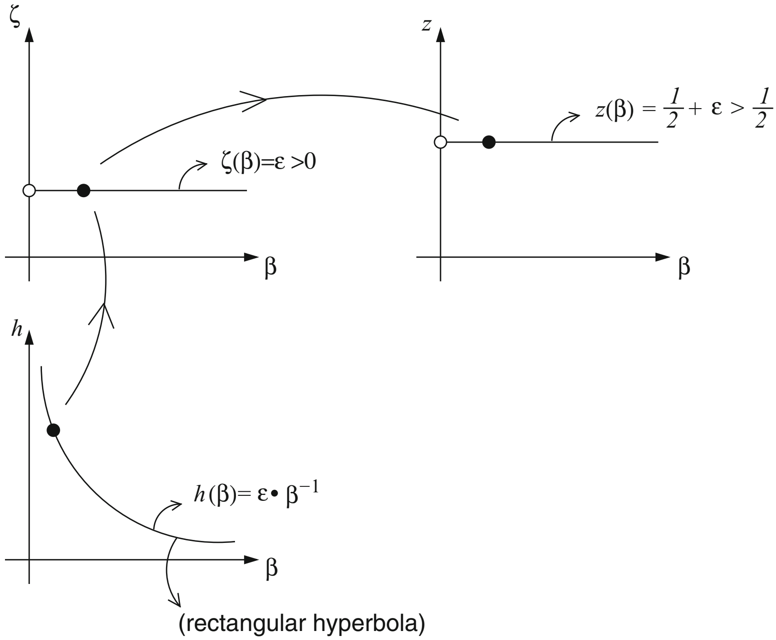


Fig. 6.5 Qualitative graphs of two functions $\zeta(\beta)$ and $z(\beta)$ corresponding to *the particular function* $h(\beta) := \varepsilon \cdot \beta^{-1}$

In the general solutions in (6.128i–v), we choose the arbitrary constant $\beta_0 = 1$ (without jeopardizing the physical contents of the solutions.) Therefore, general solutions of (6.128i–v), with the new functions in (6.131i,ii), can be cast into the forms:

$$h(\beta) = \frac{1}{\beta}\left\{\mu \cdot \left[\frac{1+\beta^{4\mu}}{1-\beta^{4\mu}}\right] - \frac{1}{2}\right\}, \tag{6.133i}$$

$$\zeta(\beta) = \beta \cdot h(\beta) = \mu \cdot \left[\frac{1+\beta^{4\mu}}{1-\beta^{4\mu}}\right] - \frac{1}{2}, \tag{6.133ii}$$

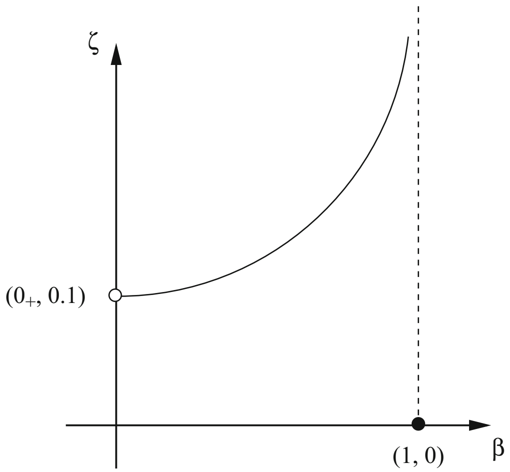
$$z(\beta) = \zeta(\beta) + \frac{1}{2} = \mu \cdot \left[\frac{1+\beta^{4\mu}}{1-\beta^{4\mu}}\right]. \tag{6.133iii}$$

For the sake of graphing exact equations, we set the numerical values of two of the parameters as

$$\varepsilon = 0.1, \tag{6.134i}$$

$$\mu = 0.6. \tag{6.134ii}$$

Fig. 6.6 The qualitative graph of the function $\zeta(\beta)$ for $0 < \beta < 1$



Then, the function in (6.133ii) is explicitly furnished as

$$\zeta(\beta) = 0.6 \cdot \left[\frac{1+\beta^{2.4}}{1-\beta^{2.4}}\right] - 0.5, \tag{6.135i}$$

$$\lim_{\beta \to 0_+} \zeta(\beta) = 0.1, \tag{6.135ii}$$

$$\lim_{\beta \to 1_-} \zeta(\beta) \to \infty. \tag{6.135iii}$$

Moreover, the derivative of the function $\zeta(\beta)$ is provided by

$$\zeta'(\beta) = \frac{\mathrm{d}\zeta(\beta)}{\mathrm{d}\beta} = \frac{8 \cdot \mu^2 \cdot \beta^{4\mu-1}}{(\beta^{4\mu}-1)^2} = \frac{8 \cdot (0.6)^2 \cdot \beta^{1.4}}{(\beta^{2.4}-1)^2} > 0, \tag{6.136i}$$

$$\lim_{\beta \to 1_-} \frac{\mathrm{d}\zeta(\beta)}{\mathrm{d}\beta} \to \infty. \tag{6.136ii}$$

With information for (6.135i–iii) and (6.136i,ii), we now plot the graph for the function $\zeta(\beta)$ in Fig. 6.6.

We note that the graph of $\zeta(\beta)$ in the figure is positive-valued. Since $\zeta(\beta) = \beta \cdot h(\beta)$ is the product of the expansion parameter of the compact dimension and the Hubble parameter, such a behavior is expected. Furthermore, the function $\zeta(\beta)$ in Fig. 6.6 *does not reveal any critical point in the interval* (0, 1). We shall see later that this indicates that the function $\zeta(\beta)$ implies monotonic expansion of the universe. However, let us investigate the general solutions in (6.128i) involving one parameter μ (with $\beta_0 = 1$). These are furnished by

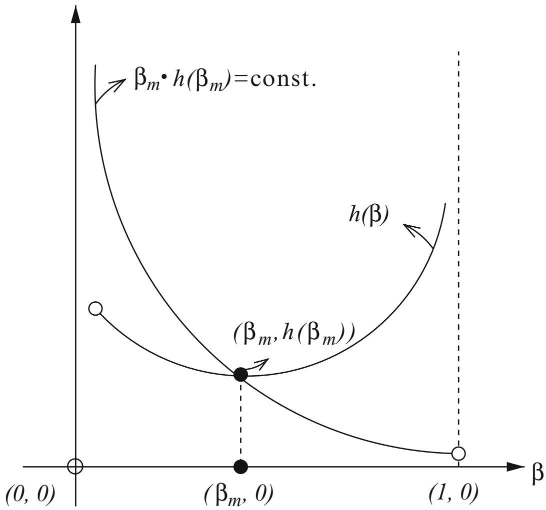


Fig. 6.7 Qualitative graphs of a typical function $h(\beta)$ and the curve comprising of minima for the one-parameter family of such functions

$$h(\beta) = \frac{1}{\beta}\left\{\mu \cdot \left[\frac{1+\beta^{4\mu}}{1-\beta^{4\mu}}\right] - \frac{1}{2}\right\}. \tag{6.137}$$

Let us examine possible critical points of this class of Hubble functions $h(\beta)$, expressed explicitly as functions of β in (6.137). We obtain from (6.127ii) that

$$h'(\beta) = \frac{d}{d\beta}\left[\frac{\zeta(\beta)}{\beta}\right] = 2[h(\beta)]^2 + \frac{h(\beta)}{\beta} - \frac{\tilde{\kappa}}{3} \cdot \left(\frac{c_1}{c_2}\right)^2 \cdot \beta^{-2} = 0, \tag{6.138i}$$

$$\text{or,} \qquad \beta^2 \cdot h'(\beta) = 2\left[\beta \cdot h(\beta)\right]^2 + \left[\beta \cdot h(\beta)\right] - K. \tag{6.138ii}$$

For the critical points β_m of the one-parameter family of functions in (6.137), we set $h'(\beta_m) = 0$ and thus deduce that

$$2\left[\beta_m \cdot h(\beta_m)\right]^2 + \left[\beta_m \cdot h(\beta_m)\right] = K. \tag{6.139}$$

Solving the above quadratic equation, we obtain the family of allowable minima as:

$$2 \cdot \beta_m \cdot h(\beta_m) = \sqrt{\frac{1}{4} + 2 \cdot K} - \frac{1}{2} = \sqrt{4\mu^2 - \frac{3}{4}} - \frac{1}{2},$$

$$\text{or,} \quad \beta_m \cdot h(\beta_m) = \sqrt{\mu^2 - \frac{3}{16}} - \frac{1}{4} = \text{const.} \tag{6.140}$$

We depict the behaviors of the function $h(\beta)$ and the one-parameter family of minima in Fig. 6.7.

Now, we shall discuss general solutions of coupled field equations (6.121i–iv). In case $h'(\beta)$ is eliminated between (6.127i) and (6.127ii) (with $k_0 = 0$), we can deduce the algebraic equation

$$[h(\beta)]^2 + \beta^{-1} \cdot h(\beta) - \frac{K}{2} \cdot \beta^{-2} = 0, \tag{6.141i}$$

$$\text{or,} \quad [\beta \cdot h(\beta)]^2 + \beta \cdot h(\beta) - \frac{K}{2} = 0. \tag{6.141ii}$$

Solving the quadratic equation (6.141ii), and discarding the negative root, we derive, from (6.126), that

$$\beta \cdot h(\beta) = \beta \cdot \{\ln [\mathcal{A}(\beta)]\}' = \mu - \frac{1}{2} > 0, \tag{6.142i}$$

$$\text{or,} \quad \mathcal{A}(\beta) = C_1 \cdot \beta^{(\mu - 1/2)}. \tag{6.142ii}$$

Here, $C_1 > 0$ is a positive, but otherwise arbitrary constant of integration.

We now define a new parameter w via

$$\mathsf{w} := \left\{3 \cdot \left[\mu - \frac{1}{2}\right] + 1\right\}^{-1},$$

$$\mathsf{w} \in (0,\, 1) \subset \mathbb{R}. \tag{6.143}$$

The general solutions of field equations (6.12i–iv) (for the subcase $k_0 = 0$) are furnished by

$$\beta = \widehat{\beta}(t,\, \mathsf{w}) := \left[-\frac{\gamma}{\mathsf{w}} \cdot (t - t_0) + (\beta_0)^{1/\mathsf{w}}\right]^{\mathsf{w}}, \tag{6.144i}$$

$$\mathcal{A}(\beta) = \widehat{\mathcal{A}}(t,\, \mathsf{w}) := C_1 \cdot \left[-\frac{\gamma}{\mathsf{w}} \cdot (t - t_0) + (\beta_0)^{1/\mathsf{w}}\right]^{(1-\mathsf{w})/3}, \tag{6.144ii}$$

$$\widehat{\phi}(t,\, \mathsf{w}) = \frac{c_1}{c_2} \cdot \mathsf{w} \cdot \ln\left|-\frac{\gamma}{\mathsf{w}} \cdot (t - t_0) + (\beta_0)^{1/\mathsf{w}}\right| + \phi_0 - \frac{c_1}{c_2} \cdot \ln|\beta_0|. \tag{6.144iii}$$

Here, $\gamma < 0$ and $C_1 > 0$ are arbitrary constants of integration. We can note the initial values from (6.144i–iii) as

$$\widehat{\beta}(t_0,\, \mathsf{w}) = \beta_0, \quad \widehat{\mathcal{A}}(t_0,\, \mathsf{w}) = C_1 \cdot (\beta_0)^{\mu - 1/2}, \quad \widehat{\phi}(t_0,\, \mathsf{w}) = \phi_0. \tag{6.145}$$

We compute the derivatives of the functions in (6.144i–iii) here as

$$\dot{\beta} := \frac{\partial}{\partial t}\widehat{\beta}(t,\, \mathsf{w}) = |\gamma| \cdot \left[\frac{|\gamma|}{\mathsf{w}} \cdot (t - t_0) + (\beta_0)^{1/\mathsf{w}}\right] > 0, \tag{6.146i}$$

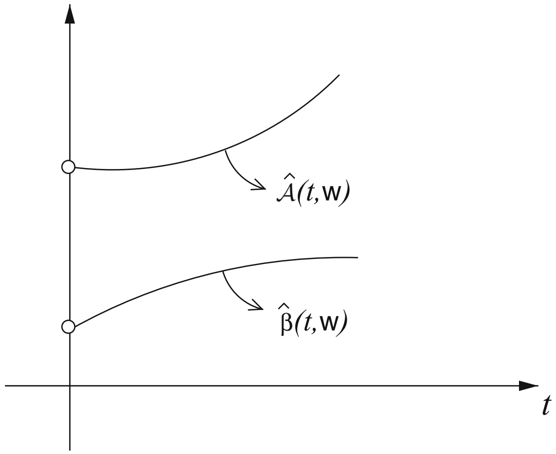


Fig. 6.8 Graphs of evolutions of functions depicting the scale factors $\hat{\mathcal{A}}(t,\,\mathsf{w})$ and $\hat{\beta}(t,\,\mathsf{w})$. (Note that at late times the compact dimension expands at a slower rate than the noncompact dimensions)

$$\dot{a} := \frac{\partial}{\partial t}\widehat{\mathcal{A}}(t,\,\mathsf{w}) = C_1 \cdot |\gamma| \cdot \frac{1-\mathsf{w}}{3\mathsf{w}} \cdot \left[\frac{|\gamma|}{\mathsf{w}} \cdot (t-t_0) + (\beta_0)^{1/\mathsf{w}}\right]^{-\frac{\mathsf{w}+2}{3}} > 0, \tag{6.146ii}$$

$$\dot{\phi} = \frac{\partial}{\partial t}\widehat{\phi}(t,\,\mathsf{w}) = |\gamma| \cdot \frac{c_1}{c_2} \cdot \left[\frac{|\gamma|}{\mathsf{w}} \cdot (t-t_0) + (\beta_0)^{1/\mathsf{w}}\right]^{-1}. \tag{6.146iii}$$

The time evolutions of $\hat{\mathcal{A}}(t,\,\mathsf{w})$ and $\hat{\beta}(t,\,\mathsf{w})$ of the above equations are depicted in Fig. 6.8.

Remarks: (i) General solutions in (6.141i–iii) satisfy field equations (6.121i,iii,iv). The initial values satisfy $\widehat{\beta}(t_0,\,\mathsf{w}) = \beta_0$, $\widehat{\mathcal{A}}(t_0,\mathsf{w}) = C_1 \cdot (\beta_0)^{\mu-1/2}$, and $\widehat{\phi}(t_0,\mathsf{w}) = \phi_0$.

(ii) In the solution in (6.141i), it is evident that the function $a(t) = \widehat{\mathcal{A}}(t,\mathsf{w})$ increases monotonically and at late times increases at a greater rate than the function $\beta(t) = \widehat{\beta}(t,\mathsf{w})$. (Compare to Fig. 6.8.) Therefore, in this five-dimensional cosmological model of the universe, the noncompact three-dimensional submanifold is expanding in size at a greater rate than the compact one-dimensional submanifold! The fifth dimension may therefore be arbitrarily "small" compared to the usual three spatial dimensions. The different expansion rate of higher dimensions is often employed in theories such as Kaluza-Klein theory and string theory.

Chapter 7
Algebraic Classification of Field Equations

7.1 The Petrov Classification of the Curvature Tensor

Some background material for this chapter is covered in Appendix 3. Glancing through that appendix may be worthwhile before reading this chapter for those who would like a review.

Let us recall the holonomic coordinate basis field $\{\partial_i\}_1^4$ or the nonholonomic orthonormal basis field $\left\{\vec{\mathbf{e}}_{(a)}(x)\right\}_1^4$ for a domain $D \subset \mathbb{R}^4$ corresponding to a coordinate chart of the space–time manifold. (The basis field $\left\{\vec{\mathbf{e}}_{(a)}(x)\right\}_1^4$ is also called *a tetrad field*.) The relevant equations are obtained from (1.36), (1.97), (1.100), (1.101), and (1.103)–(1.105). We summarize these equations here for the suitable exposition of this chapter:

$$g_{ij}(x) := \mathbf{g}_{..}(\cdot\cdot)\left(\partial_i, \partial_j\right) \equiv g_{ji}(\cdot\cdot), \tag{7.1i}$$

$$\mathbf{g}_{..}(\cdot\cdot) = g_{ij}(\cdot\cdot)\cdot \mathrm{d}x^i \otimes \mathrm{d}x^j = d_{(a)(b)}\cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot)\otimes \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot), \tag{7.1ii}$$

$$\mathbf{g}^{..}(\cdot\cdot) = g^{ij}(\cdot\cdot)\cdot \partial_i \otimes \partial_j = d^{(a)(b)}\cdot \vec{\mathbf{e}}_{(a)}(\cdot\cdot)\otimes \vec{\mathbf{e}}_{(b)}(\cdot\cdot), \tag{7.1iii}$$

$$g^{ik}(\cdot\cdot)\cdot g_{kj}(\cdot\cdot) = \delta^i{}_j, \qquad d^{(a)(c)}\cdot d_{(c)(b)} = \delta^{(a)}{}_{(b)}. \tag{7.1iv}$$

The transformation rules from one basis set to another are summarized below:

$$\widehat{\vec{\mathbf{X}}}'\left(\partial/\partial x^i\right) = \frac{\partial \widehat{X}^k(\cdot\cdot)}{\partial x^i}\cdot\frac{\partial}{\partial \widehat{x}^k}\,, \tag{7.2i}$$

$$\widehat{\widetilde{\mathbf{X}}}'(\mathrm{d}x^i) = \frac{\partial X^i(\cdot\cdot)}{\partial \widehat{x}^j}\cdot \mathrm{d}\widehat{x}^j, \tag{7.2ii}$$

$$\widehat{\widetilde{\mathbf{e}}}^{(a)}(\cdot\cdot) = A^{(a)}{}_{(b)}(\cdot\cdot)\cdot \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot), \tag{7.2iii}$$

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4_7,

$$\widehat{\vec{\mathbf{e}}}_{(a)}(\cdot\cdot) = L^{(b)}{}_{(a)}(\cdot\cdot) \cdot \vec{\mathbf{e}}_{(b)}(\cdot\cdot), \tag{7.2iv}$$

$$\left[A^{(b)}{}_{(a)}(\cdot\cdot)\right] := \left[L^{(a)}{}_{(b)}(\cdot\cdot)\right]^{-1}, \tag{7.2v}$$

$$L^{(a)}{}_{(b)}(\cdot\cdot) \cdot d_{(a)(c)} \cdot L^{(c)}{}_{(e)}(\cdot\cdot) = d_{(b)(e)}, \tag{7.2vi}$$

$$\vec{\mathbf{e}}_{(a)}(\cdot\cdot) = \lambda^{i}{}_{(a)}(\cdot\cdot) \cdot \partial_i, \qquad \partial_i = \mu^{(a)}{}_{i}(\cdot\cdot) \cdot \vec{\mathbf{e}}_{(a)}(\cdot\cdot), \tag{7.2vii}$$

$$\widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) = \mu^{(a)}{}_{i}(\cdot\cdot) \cdot \mathrm{d}x^i, \qquad \mathrm{d}x^i = \lambda^{i}{}_{(a)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot), \tag{7.2viii}$$

$$\left[\mu^{(a)}{}_{i}(\cdot\cdot)\right] := \left[\lambda^{i}{}_{(a)}(\cdot\cdot)\right]^{-1}. \tag{7.2ix}$$

It follows from (7.1i–iv) and (7.2i–ix) that

$$d_{(a)(b)} = \mathbf{g}_{..}(\cdot\cdot)\left(\vec{\mathbf{e}}_{(a)}(\cdot\cdot), \vec{\mathbf{e}}_{(b)}(\cdot\cdot)\right) = g_{ij}(\cdot\cdot) \cdot \lambda^{i}{}_{(a)}(\cdot\cdot) \cdot \lambda^{j}{}_{(b)}(\cdot\cdot), \tag{7.3i}$$

$$d^{(a)(b)} = \mathbf{g}^{..}(\cdot\cdot)\left(\widetilde{\mathbf{e}}^{(a)}(\cdot\cdot), \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot)\right) = g^{ij}(\cdot\cdot) \cdot \mu^{(a)}{}_{i}(\cdot\cdot) \cdot \mu^{(b)}{}_{j}(\cdot\cdot), \tag{7.3ii}$$

$$g_{ij}(\cdot\cdot) = \mathbf{g}_{..}(\cdot\cdot)\left(\partial_i, \partial_j\right) = d_{(a)(b)} \cdot \mu^{(a)}{}_{i}(\cdot\cdot) \cdot \mu^{(b)}{}_{j}(\cdot\cdot), \tag{7.3iii}$$

$$g^{ij}(\cdot\cdot) = \mathbf{g}^{..}(\cdot\cdot)\left(\mathrm{d}x^i, \mathrm{d}x^j\right) = d^{(a)(b)} \cdot \lambda^{i}{}_{(a)}(\cdot\cdot) \cdot \lambda^{j}{}_{(b)}(\cdot\cdot), \tag{7.3iv}$$

$$\delta^{i}{}_{j} = \mathbf{g}^{\cdot}{}_{\cdot}(\cdot\cdot)\left(\mathrm{d}x^i, \partial_j\right) = \lambda^{i}{}_{(a)}(\cdot\cdot) \cdot \mu^{(a)}{}_{j}(\cdot\cdot) =: \mathbf{I}^{\cdot}{}_{\cdot}\left(\mathrm{d}x^i, \partial_j\right), \tag{7.3v}$$

$$\delta^{(a)}{}_{(b)} = \mathbf{g}^{\cdot}{}_{\cdot}\left(\widetilde{\mathbf{e}}^{(a)}(\cdot\cdot), \vec{\mathbf{e}}_{(b)}(\cdot\cdot)\right) = \mu^{(a)}{}_{i}(\cdot\cdot) \cdot \lambda^{i}{}_{(b)}(\cdot\cdot) =: \mathbf{I}^{\cdot}{}_{\cdot}\left(\widetilde{\mathbf{e}}^{(a)}(\cdot\cdot), \vec{\mathbf{e}}_{(b)}(\cdot\cdot)\right). \tag{7.3vi}$$

Here, $\mathbf{I}^{\cdot}{}_{\cdot}(\cdot\cdot)$ is *the identity tensor field*.

Example 7.1.1. Consider the flat space–time manifold and the spherical polar spatial coordinate chart. The space–time metric is furnished by

$$\begin{aligned}
\mathbf{g}_{..}(\cdot\cdot) &= \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left(x^1\right)^2 \cdot \left[\mathrm{d}x^2 \otimes \mathrm{d}x^2 + \left(\sin x^2\right)^2 \cdot (\mathrm{d}x^3 \otimes \mathrm{d}x^3)\right] - \mathrm{d}x^4 \otimes \mathrm{d}x^4 \\
&= d_{(a)(b)} \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \otimes \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot), \\
\mathbf{g}^{..}(\cdot\cdot) &= (\partial_1 \otimes \partial_1) + \left(x^1\right)^{-2} \cdot \left[(\partial_2 \otimes \partial_2) + \left(\sin x^2\right)^{-2} \cdot (\partial_3 \otimes \partial_3)\right] - (\partial_4 \otimes \partial_4) \\
&= d^{(a)(b)} \cdot \vec{\mathbf{e}}_{(a)}(\cdot\cdot) \otimes \vec{\mathbf{e}}_{(b)}(\cdot\cdot), \\
g(\cdot\cdot) &= -\left(x^1\right)^4 \cdot \left(\sin x^2\right)^2 < 0, \\
D &:= \left\{x \in \mathbb{R}^4 : 0 < x^1,\ 0 < x^2 < \pi,\ -\pi < x^3 < \pi,\ -\infty < x^4 < \infty\right\}.
\end{aligned}$$

Transformations between the various basis sets are obtained as

$$\vec{\mathbf{e}}_{(1)}(\cdot\cdot) = \partial_1,\ \ \vec{\mathbf{e}}_{(2)}(\cdot\cdot) = \left(x^1\right)^{-1} \cdot \partial_2,\ \ \vec{\mathbf{e}}_{(3)}(\cdot\cdot) = \left(x^1 \cdot \sin x^2\right)^{-1} \cdot \partial_3,\ \ \vec{\mathbf{e}}_{(4)} = \partial_4;$$
$$\widetilde{\mathbf{e}}^{(1)}(\cdot\cdot) = \mathrm{d}x^1,\ \ \widetilde{\mathbf{e}}^{(2)}(\cdot\cdot) = \left(x^1\right) \cdot \mathrm{d}x^2,\ \ \widetilde{\mathbf{e}}^{(3)}(\cdot\cdot) = \left(x^1 \cdot \sin x^2\right) \cdot \mathrm{d}x^3,\ \ \widetilde{\mathbf{e}}^{4}(\cdot\cdot) = \mathrm{d}x^4.$$

Therefore, from (7.2vii) and (7.2viii), 4×4 diagonal transformation matrices are provided by

$$\left[\lambda^i{}_{(a)}(\cdot\cdot)\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(x^1\right)^{-1} & 0 & 0 \\ 0 & 0 & \left(x^1 \cdot \sin x^2\right)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\left[\mu^{(a)}{}_i(\cdot\cdot)\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x^1 & 0 & 0 \\ 0 & 0 & \left(x^1 \cdot \sin x^2\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

Now, we shall introduce *two null vector fields* by

$$\sqrt{2}\,\vec{\mathbf{k}}(\cdot\cdot) := \vec{\mathbf{e}}_{(4)}(\cdot\cdot) + \vec{\mathbf{e}}_{(3)}(\cdot\cdot), \tag{7.4i}$$

$$\sqrt{2}\,\vec{\mathbf{l}}(\cdot\cdot) := \vec{\mathbf{e}}_{(4)}(\cdot\cdot) - \vec{\mathbf{e}}_{(3)}(\cdot\cdot), \tag{7.4ii}$$

$$\mathbf{g}_{..}(\cdot\cdot)\left(\vec{\mathbf{l}}(\cdot\cdot), \vec{\mathbf{l}}(\cdot\cdot)\right) = \mathbf{g}_{..}\left(\vec{\mathbf{k}}(\cdot\cdot), \vec{\mathbf{k}}(\cdot\cdot)\right) = 0, \tag{7.4iii}$$

$$\mathbf{g}_{..}(\cdot\cdot)\left(\vec{\mathbf{l}}(\cdot\cdot), \vec{\mathbf{k}}(\cdot\cdot)\right) = -1. \tag{7.4iv}$$

We depict the basis field $\left\{\vec{\mathbf{e}}_{(1)}(\cdot\cdot), \vec{\mathbf{e}}_{(2)}(\cdot\cdot), \vec{\mathbf{l}}(\cdot\cdot), \vec{\mathbf{k}}(\cdot\cdot)\right\}$, involving two spacelike vector fields and two null vector fields in Fig. 7.1.

To solve a wave equation in a space–time domain, it is convenient to use two null coordinates (as discussed in Example 2.1.17 and in (A2.26)). Moreover, in Appendix 2, we have used complex conjugate coordinates to solve elliptic partial differential equations. Furthermore, for the Petrov classification of the conformal tensor, the use of a tetrad consisting of two null vectors and two complex conjugate vectors is technically convenient. Therefore, we are motivated to introduce two complex conjugate null vector fields by the following:

$$\sqrt{2}\,\vec{\mathbf{m}}(\cdot\cdot) := \vec{\mathbf{e}}_{(1)}(\cdot\cdot) - \mathrm{i}\,\vec{\mathbf{e}}_{(2)}(\cdot\cdot), \tag{7.5i}$$

$$\sqrt{2}\,\vec{\overline{\mathbf{m}}}(\cdot\cdot) = \vec{\mathbf{e}}_{(1)}(\cdot\cdot) + \mathrm{i}\,\vec{\mathbf{e}}_{(2)}(\cdot\cdot), \tag{7.5ii}$$

$$\mathbf{g}_{..}(\cdot\cdot)\left(\vec{\mathbf{m}}(\cdot\cdot), \vec{\mathbf{m}}(\cdot\cdot)\right) = \mathbf{g}_{..}(\cdot\cdot)\left(\vec{\overline{\mathbf{m}}}(\cdot\cdot), \vec{\overline{\mathbf{m}}}(\cdot\cdot)\right) = 0, \tag{7.5iii}$$

$$\mathbf{g}_{..}(\cdot\cdot)\left(\vec{\mathbf{m}}(\cdot\cdot), \vec{\overline{\mathbf{m}}}(\cdot\cdot)\right) = 1. \tag{7.5iv}$$

$$\mathbf{g}_{..}(\vec{\mathbf{l}}(\cdot\cdot),\ \vec{\mathbf{m}}(\cdot\cdot)) = \mathbf{g}_{..}(\vec{\mathbf{k}}(\cdot\cdot),\ \vec{\mathbf{m}}(\cdot\cdot)) = 0. \tag{7.5v}$$

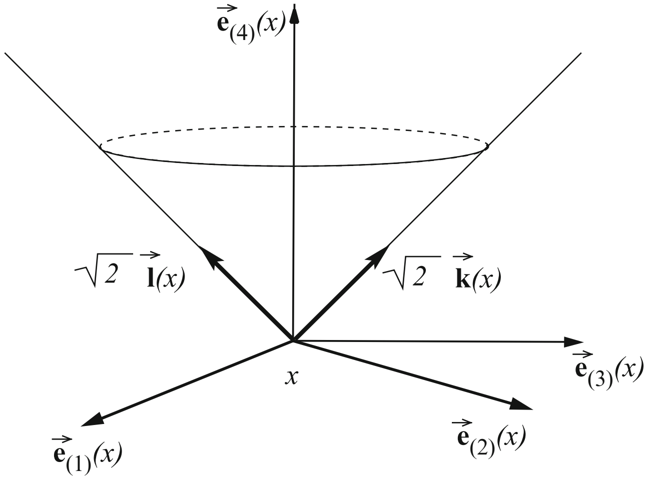


Fig. 7.1 A tetrad field containing two spacelike and two null vector fields

Thus, we can construct a *complex null tetrad field* together with $\vec{\mathbf{l}}(\cdot\cdot)$, $\vec{\mathbf{k}}(\cdot\cdot)$ defined in (7.4i,ii), by

$$\left\{\vec{\mathbf{E}}_{(a)}(x)\right\}_1^4 := \left\{\vec{\mathbf{m}}(\cdot\cdot), \vec{\overline{\mathbf{m}}}(\cdot\cdot), \vec{\mathbf{l}}(\cdot\cdot), \vec{\mathbf{k}}(\cdot\cdot)\right\}, \tag{7.6i}$$

$$\vec{\mathbf{E}}_{(a)}(x) = \Lambda^i{}_{(a)} \cdot \partial_i, \tag{7.6ii}$$

$$\begin{aligned} g_{(a)(b)}(\cdot\cdot) = g_{ij}(..) \cdot \Lambda^i{}_{(a)}(\cdot\cdot) \cdot \Lambda^j{}_{(b)}(\cdot\cdot) &= m_{(a)}(\cdot\cdot) \cdot \overline{m}_{(b)}(\cdot\cdot) + \overline{m}_{(b)}(\cdot\cdot) \cdot m_{(a)}(\cdot\cdot) \\ &- l_{(a)}(\cdot\cdot) \cdot k_{(b)}(\cdot\cdot) - l_{(b)}(\cdot\cdot) \cdot k_{(a)}(\cdot\cdot) =: \eta_{(a)(b)}, \end{aligned} \tag{7.6iii}$$

$$g^{(a)(b)}(\cdot\cdot) = \eta^{(a)(b)}, \tag{7.6iv}$$

$$\underset{4\times4}{\left[\eta_{(a)(b)}\right]} = \underset{4\times4}{\left[\eta^{(a)(b)}\right]} = \left[\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} \end{array}\right]. \tag{7.6v}$$

Here the transformation between the holonomic and the complex null basis sets is provided by $\Lambda^i{}_{(a)}$, as shown in (7.6ii).

(*Caution*: The metric tensor components above *must not be identified* with components of the totally antisymmetric oriented tensor $\eta_{(a)(b)(c)(d)}$ of (1.110i–iv)!)

Now, let us consider mappings of the complex null tetrad $\left\{\vec{\mathbf{m}}(\cdot\cdot), \vec{\overline{\mathbf{m}}}(\cdot\cdot), \vec{\mathbf{l}}(\cdot\cdot), \vec{\mathbf{k}}(\cdot\cdot)\right\}$ into another $\left\{\widehat{\vec{\mathbf{m}}}(\cdot\cdot), \widehat{\vec{\overline{\mathbf{m}}}}(\cdot\cdot), \widehat{\vec{\mathbf{l}}}(\cdot\cdot), \widehat{\vec{\mathbf{k}}}(\cdot\cdot)\right\}$. These mappings, by (7.2iv) and (7.2vi), must be mediated by linear combinations of elements of *the Lorentz group* $0(3,1;\mathbb{R})$. (See [55, 239].)

Example 7.1.2. We shall consider special Lorentz mappings in $0(3,1;\mathbb{R})$ and consequent implications on the mappings of the complex null tetrads.

Case (i): Rotations restricted to the two-dimensional vector subspace spanned by $\vec{\mathbf{e}}_{(1)}(x)$ and $\vec{\mathbf{e}}_{(2)}(x)$ in Fig. 7.1 are characterized by

$$\begin{aligned}
\widehat{\vec{\mathbf{e}}}_{(1)}(x) &= [\cos\Theta(x)]\cdot\vec{\mathbf{e}}_{(1)}(\cdot\cdot) - \sin[\Theta(x)]\cdot\vec{\mathbf{e}}_{(2)}(\cdot\cdot),\\
\widehat{\vec{\mathbf{e}}}_{(2)}(x) &= [\sin\Theta(x)]\cdot\vec{\mathbf{e}}_{(1)}(\cdot\cdot) + \cos[\Theta(x)]\cdot\vec{\mathbf{e}}_{(2)}(\cdot\cdot),\\
\widehat{\vec{\mathbf{m}}}(x) &= [\exp(-\mathrm{i}\,\Theta(x))]\cdot\vec{\mathbf{m}}(x), \widehat{\overline{\vec{\mathbf{m}}}}(x) = \exp[\mathrm{i}\Theta(x)]\cdot\widehat{\vec{\mathbf{m}}}(x).
\end{aligned} \tag{7.7i}$$

(Here, $\Theta(x)$ *is a real-valued function*.)

Case (ii): Proper, orthochronous Lorentz mappings (boosts) in the two-dimensional vector subspace spanned by $\vec{\mathbf{e}}_{(3)}(x)$ and $\vec{\mathbf{e}}_{(4)}(x)$. The consequent induced mappings on null vectors $\vec{\mathbf{l}}(x)$ and $\vec{\mathbf{k}}(x)$ are characterized by

$$\begin{aligned}
\widehat{\vec{\mathbf{l}}}(x) &= \mathrm{e}^{\alpha(..)}\cdot\left(\vec{\mathbf{l}}(x)\right),\\
\widehat{\vec{\mathbf{k}}}(x) &= \mathrm{e}^{-\alpha(..)}\cdot\left(\vec{\mathbf{k}}(x)\right),\\
\mathrm{e}^{\alpha(..)} &:= \sqrt{\frac{1+v(..)}{1-v(..)}},\\
|v(..)| &< 1.
\end{aligned} \tag{7.7ii}$$

The above transformation can be recognized as *a variable boost*.

Case (iii): Null rotations, leaving the vector $\vec{\mathbf{l}}(x)$ invariant, induce the following mappings:

$$\begin{aligned}
\widehat{\vec{\mathbf{l}}}(x) &= \vec{\mathbf{l}}(x),\\
\widehat{\vec{\mathbf{m}}}(x) &= \vec{\mathbf{m}}(x) + E(x)\cdot\vec{\mathbf{l}}(x),\\
\widehat{\vec{\mathbf{k}}}(x) &= \vec{\mathbf{k}}(\cdot\cdot) + E(\cdot\cdot)\cdot\overline{\vec{\mathbf{m}}}(\cdot\cdot) + \overline{E}(\cdot\cdot)\cdot\vec{\mathbf{m}}(\cdot\cdot) + E(\cdot\cdot)\cdot\overline{E}(\cdot\cdot)\cdot\vec{\mathbf{l}}(\cdot\cdot).
\end{aligned} \tag{7.7iii}$$

(Here, $E(x)$ is *a complex-valued function*.)

Case (iv): Null rotations, leaving the vector $\vec{\mathbf{k}}(x)$ invariant, induce the following mappings:

$$\widehat{\vec{\mathbf{k}}}(x) = \vec{\mathbf{k}}(x),$$

$$\widehat{\vec{\mathbf{m}}}(x) = \vec{\mathbf{m}}(x) + B(x) \cdot \vec{\mathbf{k}}(x),$$

$$\widehat{\vec{\mathbf{l}}}(x) = \vec{\mathbf{l}}(\cdot\cdot) + B(\cdot\cdot) \cdot \overline{\vec{\mathbf{m}}}(\cdot\cdot) + \overline{B}(\cdot\cdot) \cdot \vec{\mathbf{m}}(\cdot\cdot) + B(\cdot\cdot) \cdot \overline{B}(\cdot\cdot) \cdot \vec{\mathbf{k}}(x). \tag{7.7iv}$$

(Here, $B(x)$ is a complex-valued function.) □

Now, we shall introduce *the six-dimensional bivector space* spanned by the set of all (*real*) 2-forms defined in the space–time manifold. (Recall that a general p-form and wedge products were dealt with in (1.45), (1.46), and (1.59), Example 1.2.18, etc.) For this purpose, we can choose either the 1-form basis set $\{\mathrm{d}x^i\}_1^4$ or a dual orthonormal basis set $\{\widetilde{\mathbf{e}}^{(a)}(x)\}_1^4$. Such basis sets induce the following basis sets for *the (real) bivector space*:

$$\mathcal{B} := \{\mathrm{d}x^1 \wedge \mathrm{d}x^2,\ \mathrm{d}x^1 \wedge \mathrm{d}x^3,\ \mathrm{d}x^1 \wedge \mathrm{d}x^4,\ \mathrm{d}x^2 \wedge \mathrm{d}x^3,\ \mathrm{d}x^2 \wedge \mathrm{d}x^4,\ \mathrm{d}x^3 \wedge \mathrm{d}x^4\}, \tag{7.8i}$$

$$\begin{aligned}\mathcal{B}_{(0)} := \{&\widetilde{\mathbf{e}}^{(1)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(2)}(\cdot\cdot),\ \widetilde{\mathbf{e}}^{(1)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(3)}(\cdot\cdot),\ \widetilde{\mathbf{e}}^{(1)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(4)}(\cdot\cdot)\\ &\widetilde{\mathbf{e}}^{(2)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(3)}(\cdot\cdot),\ \widetilde{\mathbf{e}}^{(2)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(4)}(\cdot\cdot),\ \widetilde{\mathbf{e}}^{(3)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(4)}(\cdot\cdot)\}.\end{aligned} \tag{7.8ii}$$

Remark: Note that *there exist infinitely many more basis sets for the bivector fields!*

Let $\mathbf{W}_{..}(x)$ be an arbitrary 2-form belonging to the bivector fields. It is expressible as a linear combination of basis bivectors in (7.8i) or (7.8ii). Therefore, it is permissible to write

$$\begin{aligned}&\mathbf{W}_{..}(x) = (1/2)\, W_{ij}(\cdot\cdot) \cdot \mathrm{d}x^i \wedge \mathrm{d}x^j = \sum_{1 \le i < j}^{4} \sum^{4} W_{ij}(\cdot\cdot) \cdot \mathrm{d}x^i \wedge \mathrm{d}x^j,\\ &W_{ji}(x) \equiv -W_{ij}(x);\end{aligned} \tag{7.9i}$$

$$\begin{aligned}\mathbf{W}_{..}(x) &= (1/2)\, W_{(a)(b)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot)\\ &= \sum_{1 \le a < b}^{4} \sum^{4} W_{(a)(b)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot),\\ W_{(b)(a)}(x) &\equiv -W_{(a)(b)}(x).\end{aligned} \tag{7.9ii}$$

The metric tensor $\mathbf{g}_{..}(x)$ in (7.1i) and (7.1ii) induces a metric tensor $\mathbf{g}_{....}(x)$ of order $(0+4)$ for the bivector fields. (See [56, 210].) It is explicitly furnished by

$$\mathbf{g}_{....}(x) := \sum_{1\le i<j}\sum \cdot \sum_{1\le k<l}\sum g_{ijkl}(\cdot\cdot)\cdot \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l,$$

$$g_{ijkl}(x) := g_{ik}(\cdot\cdot)\cdot g_{jl}(\cdot\cdot) - g_{il}(\cdot\cdot)\cdot g_{jk}(\cdot\cdot); \tag{7.10i}$$

$$\mathbf{g}_{....}(x) := \sum_{1\le a<b}\sum \cdot \sum_{1\le c<d}\sum d_{(a)(b)(c)(d)}\cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot)\wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot)\otimes \widetilde{\mathbf{e}}^{(c)}(\cdot\cdot)\wedge \widetilde{\mathbf{e}}^{(d)}(\cdot\cdot),$$

$$d_{(a)(b)(c)(d)} := d_{(a)(c)}\cdot d_{(b)(d)} - d_{(a)(d)}\cdot d_{(b)(c)}. \tag{7.10ii}$$

Now we shall discuss *the invariant eigenvalue problem* for various 6×6 matrices arising out of the six-dimensional bivector space. (We have touched upon invariant eigenvalue problems in (1.213) and Examples A3.6 and A3.8.) Suppose that a tensor field belonging to the bivector space is denoted by

$$\mathbf{M}_{....}(x) := \sum_{i<j}\sum \cdot \sum_{k<l}\sum m_{ijkl}(\cdot\cdot)\,\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l,$$

$$m_{jikl}(\cdot\cdot) \equiv -m_{ijkl}(\cdot\cdot), \qquad m_{ijlk}(\cdot\cdot) \equiv -m_{ijkl}(\cdot\cdot). \tag{7.11}$$

A 6×6 matrix $[M(x)]$, out of entries $m_{ijkl}(x)$,

$$\underset{6\times 6}{[M(x)]} := \left[m_{ijkl}(\cdot\cdot)\right],$$

$$i < j, \qquad k < l \tag{7.12}$$

can be constructed.

The *eigenbivector* is expressed as

$$\mathbf{E}_{..}(x) := (1/2)E_{kl}(\cdot\cdot)\cdot \mathrm{d}x^k \wedge \mathrm{d}x^l = \sum_{k<l}\sum E_{kl}(\cdot\cdot)\cdot \mathrm{d}x^k \wedge \mathrm{d}x^l, \tag{7.13i}$$

$$E_{lk}(x) \equiv -E_{kl}(x); \tag{7.13ii}$$

$$\begin{aligned}\mathbf{E}_{..}(x) &:= (1/2)E_{(a)(b)}(\cdot\cdot)\cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot)\wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot)\\ &= \sum_{a<b}\sum E_{(a)(b)}(\cdot\cdot)\cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot)\wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot),\end{aligned} \tag{7.13iii}$$

$$E_{(b)(a)}(x) \equiv -E_{(a)(b)}(x). \tag{7.13iv}$$

The invariant eigenvalue problem for the 6×6 matrix in (7.12) is posed as follows:

$$m^{ij}{}_{kl}(x) := g^{ip}(\cdot\cdot)\cdot g^{jq}(\cdot\cdot)\cdot m_{pqkl}(\cdot\cdot), \tag{7.14i}$$

$$m^{ji}{}_{kl}(x) \equiv -m^{ij}{}_{kl}(x) \equiv m^{ij}{}_{lk}(x), \tag{7.14ii}$$

$$(1/2)m^{ij}{}_{kl}(\cdot\cdot) \cdot E^{kl}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot E^{ij}(\cdot\cdot), \tag{7.14iii}$$

$$\sum_{k<l}\sum m^{ij}{}_{kl}(\cdot\cdot) \cdot E^{kl}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot E^{ij}(\cdot\cdot), \tag{7.14iv}$$

$$\det\left[m^{ij}{}_{kl}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot \delta^{ij}{}_{kl}\right] = 0, \tag{7.14v}$$

$$\delta^{ij}{}_{kl} := \begin{vmatrix} \delta^i{}_k & \delta^i{}_l \\ \delta^j{}_k & \delta^j{}_l \end{vmatrix}, \tag{7.14vi}$$

$$i < j, \quad k < l, \tag{7.14vii}$$

$$\det\left[m_{ijkl}(\cdot\cdot) - \lambda(\cdot\cdot)\, g_{ijkl}(\cdot\cdot)\right] = 0\,; \tag{7.14viii}$$

$$(1/2)\, m^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) \cdot E^{(c)(d)}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot E^{(a)(b)}(\cdot\cdot), \tag{7.14ix}$$

$$\sum_{c<d}\sum m^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) \cdot E^{(c)(d)}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot E^{(a)(b)}(\cdot\cdot), \tag{7.14x}$$

$$\det\left[m^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot \delta^{(a)(b)}{}_{(c)(d)}\right] = 0, \tag{7.14xi}$$

$$a < b, \quad c < d,$$

$$\det\left[m_{(a)(b)(c)(d)}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot d_{(a)(b)(c)(d)}\right] = 0. \tag{7.14xii}$$

(Here, $\delta^{ij}{}_{kl}$ are components of *the generalized Kronecker delta* ${}^2_2\boldsymbol{\delta} \equiv \boldsymbol{\delta}^{\cdot\cdot}{}_{\cdot\cdot}$ discussed in Example 1.2.13.)

It is rather cumbersome to deal with multi-indices in (7.14i–xii), so we devise the following notations for brevity:

$$\begin{aligned} &12 \longrightarrow 1, \quad 13 \longrightarrow 2, \quad 14 \longrightarrow 3, \\ &23 \longrightarrow 4, \quad 24 \longrightarrow 5, \quad 34 \longrightarrow 6, \\ &I, J, K, L; A, B, C, D \in \{1, 2, 3, 4, 5, 6\}. \end{aligned} \tag{7.15}$$

(*We also follow summation conventions in regards to capital Roman indices.*) We also express metric tensors pertaining to the bivector space as

$$\begin{aligned} g_{ijkl}(\cdot\cdot) &\longmapsto \gamma_{IJ}(\cdot\cdot), \\ d_{(a)(b)(c)(d)} &\longmapsto D_{(A)(B)}. \end{aligned} \tag{7.16}$$

With this new notation, the invariant eigenvalue problems in (7.14viii) and (7.14xii) can be cast into the following neater equations:

$$\det\left[m_{IJ}(\cdot\cdot) - \lambda(\cdot\cdot)\cdot\gamma_{IJ}(\cdot\cdot)\right] = 0, \tag{7.17i}$$

$$\det\left[m_{(A)(B)}(\cdot\cdot) - \lambda(\cdot\cdot)\cdot D_{(A)(B)}\right] = 0, \tag{7.17ii}$$

$$\det\left[m^{(A)}{}_{(B)}(\cdot\cdot) - \lambda(\cdot\cdot)\cdot\delta^{(A)}{}_{(B)}\right] = 0. \tag{7.17iii}$$

(Here, $m^{(A)}{}_{(B)}(\cdot\cdot) := D^{(A)(C)}\cdot m_{(C)(B)}(\cdot\cdot)$.) The above equations facilitate solving the determinantal equations for 6×6 matrices!

Example 7.1.3. Let us work out explicitly the numerical metric tensor components D_{AB} given by (7.16) and (7.10ii). They are given by *the symmetric* 6×6 *matrix*

$$\left[D_{(A)(B)}\right] = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{bmatrix}. \tag{7.18}$$

The numerical matrix above is already *diagonal*. The induced metric structure of the bivector fields is governed by (7.18). For this bivector space, the corresponding metric is of *zero signature*. Explicitly ($p :=$ positive, $n :=$ negative), $p = 3,\ n = 3,\ p + n = 6$, and $p - n = 0$. Moreover,

$$\begin{aligned} \left[D^{(A)(B)}\right] &\equiv \left[D_{(A)(B)}\right], \\ D^{(A)(C)}\cdot D_{(C)(B)} &= \delta^{(A)}{}_{(B)}. \end{aligned} \tag{7.19}$$

□

Now we shall discuss canonical forms of the 6×6 matrices with *real entries*. (Consult Appendix 3 on this topic.) *The Jordan canonical form* of a 6×6 matrix field $[M(x)] \equiv [M^{(A)}{}_{(B)}(x)]$, according to Theorems A3.7 and A3.9 and (A3.10), is furnished by

$$\underset{6\times 6}{[M(x)]_{(J)}} = \left[\begin{array}{c:c} \boxed{A_{(1)}(\cdot\cdot)}\ \ddots & \\ \hdashline & \boxed{B_{(1)}(\cdot\cdot)}\ \ddots \end{array}\right], \tag{7.20i}$$

$$\left[A_{(1)}(x)\right] := \begin{bmatrix} \boxed{J^{(1)}_{(1)}(\cdot\cdot)} & & \\ & \ddots & \\ & & \ddots \end{bmatrix}, \tag{7.20ii}$$

$$\left[J^{(1)}_{(1)}(x)\right] := \underset{1\times 1}{\left[\lambda_{(1)}(x)\right]}, \quad \text{or}$$

$$\left[J^{(1)}_{(1)}(x)\right] = \begin{bmatrix} \lambda_{(1)}(\cdot\cdot) & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_{(1)}(\cdot\cdot) & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{(1)}(\cdot\cdot) & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{(1)}(\cdot\cdot) & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{(1)}(\cdot\cdot) & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_{(1)}(\cdot\cdot) \end{bmatrix} \text{etc.,} \tag{7.20iii}$$

$$\left[B_{(1)}(x)\right] := \begin{bmatrix} \boxed{C^{(1)}_{(1)}(\cdot\cdot)} & & \\ & \ddots & \\ & & \ddots \end{bmatrix}, \tag{7.20iv}$$

$$\left[C^{(1)}_{(1)}(x)\right] := \left[\begin{array}{cccc|cc} a_{(1)} & b_{(1)} & 1 & 0 & 0 & 0 \\ -b_{(1)} & a_{(1)} & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & a_{(1)} & b_{(1)} \\ 0 & 0 & 0 & 0 & -b_{(1)} & a_{(1)} \end{array}\right] \text{etc.} \tag{7.20v}$$

In (7.20v), *the complex eigenvalue* is given by $\lambda(x) = a_{(1)}(x) + i\, b_{(1)}(x)$ with $b_{(1)}(x) \neq 0$.

Example 7.1.4. Let the Jordan canonical form (7.20i) of a 6×6 matrix field be provided by

$$[M(x)]_{(J)} = \left[\begin{array}{cc|cc|cc} 2 & 1 & & & & \\ 0 & 2 & & & & \\ \hline & & x^1x^2 & 0 & & \\ & & 0 & x^3x^4 & & \\ \hline & & & & a_{(1)} & -1 \\ & & & & 1 & a_{(1)} \end{array}\right].$$

In this example, eigenvalues are given by $\lambda_{(1)}(x) = 2,\ \lambda_{(2)}(x) = x^1 \cdot x^2,\ \lambda_{(3)}(x) = x^3 \cdot x^4,\ \lambda(x) = a_{(1)} - i,\ \overline{\lambda(x)} = a_{(1)} + i$. *The corresponding Segre characteristic* of the matrix is furnished from Appendix 3 as $[2, 1, 1;\ 1, \overline{1}\,]$. $\square$

Now, we shall investigate canonical forms of the Riemann–Christoffel curvature tensor field $\mathbf{R}_{....}(x)$. Following the notational abbreviations introduced in (7.15), we denote

$$\mathbf{R}_{....}(x) = \sum_{c<d}^{4} \sum_{a<b}^{4} R_{(a)(b)(c)(d)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot) \otimes \widetilde{\mathbf{e}}^{(c)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(d)}(\cdot\cdot),$$

$$R_{(a)(b)(c)(d)}(x) \longrightarrow \mathbb{R}_{(A)(B)}(x),$$

$$1 \leq a < b, \quad 1 \leq c < d. \tag{7.21}$$

The invariant eigenvalues from (7.16) and (7.21) are furnished by

$$\det\left[\mathbb{R}_{(A)(B)}(x) - \lambda(x) \cdot D_{(A)(B)}\right] = 0. \tag{7.22}$$

We note from the algebraic identities in (1.142vii) that

$$R_{(c)(d)(a)(b)}(x) \equiv R_{(a)(b)(c)(d)}(x), \tag{7.23i}$$

$$\mathbb{R}_{(B)(A)}(x) \equiv \mathbb{R}_{(A)(B)}(x). \tag{7.23ii}$$

However, the 6×6 matrix

$$\left[\mathbb{R}^{(A)}{}_{(B)}(x)\right] := \left[D^{(A)(C)} \cdot \mathbb{R}_{(C)(B)}(x)\right] \tag{7.24}$$

is *not necessarily symmetric*. (Consult the Example A3.8.) The Jordan canonical forms of the matrix $\left[\mathbb{R}^{(A)}{}_{(B)}(x)\right]$ are analogous to those in (7.20i). But, there exist many other algebraic identities (1.142v,vi,viii) for the components $R_{(a)(b)(c)(d)}(x)$. Moreover, Einstein's field equations (2.163ii) provide additional restrictions to the Jordan forms of $\left[\mathbb{R}^{(A)}{}_{(B)}(x)\right]$. General reductions of $\left[\mathbb{R}^{(A)}{}_{(B)}(x)\right]$ to canonical forms are too involved for the scope of this book. (The reader is referred to Petrov's book [210] for detailed discussions.)

Weyl's conformal tensor $\mathbf{C}^{\cdot}{}_{...}(x)$ of (1.169i,ii) turns out to be very useful in regards to applications in the field of general relativity. (In fact, the next section, involving the Newman–Penrose equations, extensively deals with the conformal tensor.) Therefore, let us delve into the task of the algebraic classifications of the conformal tensor more exhaustively.

The components of the conformal tensor, relative to *a coordinate basis and a (real) orthonormal dual basis*, satisfy the following algebraic identities:

$$C_{jikl}(x) \equiv C_{ijlk}(x) \equiv -C_{ijkl}(x), \tag{7.25i}$$

$$C_{klij}(x) \equiv C_{ijkl}(x), \tag{7.25ii}$$

$$C_{ijkl}(\cdot\cdot) + C_{iklj}(\cdot\cdot) + C_{iljk}(\cdot\cdot) \equiv 0, \tag{7.25iii}$$

$$C^{k}{}_{ijk}(x) \equiv 0\,; \tag{7.25iv}$$

$$C_{(b)(a)(c)(d)}(x) \equiv C_{(a)(b)(d)(c)}(x) \equiv -C_{(a)(b)(c)(d)}(x), \tag{7.25v}$$

$$C_{(c)(d)(a)(b)}(x) \equiv C_{(a)(b)(c)(d)}(x), \tag{7.25vi}$$

$$C_{(a)(b)(c)(d)}(\cdot\cdot) + C_{(a)(c)(d)(b)}(\cdot\cdot) + C_{(a)(d)(b)(c)}(\cdot\cdot) \equiv 0, \tag{7.25vii}$$

$$C^{(d)}{}_{(a)(b)(d)}(x) \equiv 0. \tag{7.25viii}$$

The number of linearly independent (real) components of the conformal tensor is *exactly ten*. (Consult Problem # 14 of Exercises 1.3, as well as Problem # 3 of Exercises 2.4.)

We can express the conformal tensor, analogous to (7.11) and (7.14i,ii), as

$$\begin{aligned}\mathbf{C}_{....}(x) &= (1/4)\, C_{ijkl}(\cdot\cdot) \cdot \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l \\ &= (1/4)\, C_{(a)(b)(c)(d)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot) \otimes \widetilde{\mathbf{e}}^{(c)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(d)}(\cdot\cdot).\end{aligned} \tag{7.26}$$

We express the equivalent equations, in *the six-dimensional bivector space*, as

$$\begin{aligned}\mathbf{C}_{....}(x) &= \sum_{i<j}^{4}\sum^{4} \cdot \sum_{k<l}^{4}\sum^{4} C_{ijkl}(\cdot\cdot) \cdot \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l \\ &= \sum_{a<b}^{4}\sum^{4} \cdot \sum_{c<d}^{4}\sum^{4} C_{(a)(b)(c)(d)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot) \otimes \widetilde{\mathbf{e}}^{(c)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(d)}(\cdot\cdot).\end{aligned} \tag{7.27}$$

We define the corresponding mixed components from (7.14i) as

$$C^{ij}{}_{kl}(x) := g^{ip}(\cdot\cdot) \cdot g^{jq}(\cdot\cdot) \cdot C_{pqkl}(\cdot\cdot), \tag{7.28i}$$

$$C^{(a)(b)}{}_{(c)(d)}(x) := d^{(a)(p)} \cdot d^{(b)(q)} \cdot C_{(p)(q)(c)(d)}(\cdot\cdot). \tag{7.28ii}$$

We next define *eigenbivectors* $\mathbf{E}_{\cdot\cdot}(x)$, exactly like in (7.13i–iv), to pose the invariant eigenvalue problem for the conformal tensor. The following equations summarize this eigenvalue problem:

$$(1/2)\, C^{ij}{}_{kl}(\cdot\cdot) \cdot E^{kl}(\cdot\cdot) = \sum_{k<l}\sum C^{ij}{}_{kl}(\cdot\cdot) \cdot E^{kl}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot E^{ij}(\cdot\cdot), \tag{7.29i}$$

$$\det\left[C^{ij}{}_{kl}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot \delta^{ij}{}_{kl}\right] = 0, \tag{7.29ii}$$

$$1 \le i < j, \quad 1 \le k < l\,;$$

$$(1/2)\, C^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) \cdot E^{(c)(d)}(\cdot\cdot) = \sum_{c<d}\sum C^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) \cdot E^{(c)(d)}(\cdot\cdot)$$

$$= \lambda(\cdot\cdot) \cdot E^{(a)(b)}(\cdot\cdot), \tag{7.29iii}$$

$$\det\left[C^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot \delta^{(a)(b)}{}_{(c)(d)}\right] = 0,$$

$$1 \le a < b, \quad 1 \le c < d. \tag{7.29iv}$$

Now, we shall adopt the notations of (7.15) and (7.16) for the sake of simplicity. With these abbreviations, the invariant eigenvalue problems of (7.29ii) and (7.29iv) reduce to the following:

$$C_{ijkl}(x) \longrightarrow \mathbb{C}_{IJ}(x), \tag{7.30i}$$

$$C_{(a)(b)(c)(d)}(x) \longrightarrow \mathbb{C}_{(A)(B)}(x), \tag{7.30ii}$$

$$\det\left[\mathbb{C}_{IJ}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot \gamma_{IJ}(\cdot\cdot)\right] = 0, \tag{7.30iii}$$

$$\det\left[\mathbb{C}_{(A)(B)}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot D_{(A)(B)}\right] = 0, \tag{7.30iv}$$

$$\det\left[\mathbb{C}^{(A)}{}_{(B)}(\cdot\cdot) - \lambda(\cdot\cdot) \cdot \delta^{(A)}{}_{(B)}\right] = 0. \tag{7.30v}$$

The algebraic identities in (7.25ii), (7.25iv), (7.25vi), and (7.25viii) imply the identities:

$$\mathbb{C}_{JI}(x) \equiv \mathbb{C}_{IJ}(x), \tag{7.31i}$$

$$\mathbb{C}^{I}{}_{I}(x) \equiv 0, \quad \underset{6\times 6}{\text{Trace}}\left[\mathbb{C}^{I}{}_{J}(\cdot\cdot)\right] \equiv 0, \tag{7.31ii}$$

$$\mathbb{C}_{(B)(A)}(x) \equiv \mathbb{C}_{(A)(B)}(x), \tag{7.31iii}$$

$$\mathbb{C}^{(A)}{}_{(A)}(x) \equiv 0, \quad \underset{6\times 6}{\text{Trace}}\left[\mathbb{C}^{(A)}{}_{(B)}(\cdot\cdot)\right] \equiv 0. \tag{7.31iv}$$

However, the remaining identities of (7.25iii,vii) and (7.25iv,viii) are *not fully incorporated in identities* (7.31i–iv). (All these identities do not imply Einstein's field equations (2.163i,ii).)

Example 7.1.5. Consider a specific example of the four-dimensional space–time where the components of the curvature tensor are given by:

$$R^{(a)(b)}{}_{(c)(d)}(x) := K_0 \cdot \delta^{(a)(b)}{}_{(c)(d)}. \tag{7.32}$$

(Here, K_0 is a constant.) By (1.164ii), the above equation indicates a domain of space–time that is of *constant curvature*.

The invariant eigenvalue problem of the curvature tensor in (7.32) boils down, by (7.14x,xi), to

$$\sum_{e<d}^{4}\sum^{4}\left[R^{(a)(b)}{}_{(c)(d)}(\cdot\cdot)\right] \cdot E^{(c)(d)}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot E^{(a)(b)}(\cdot\cdot), \tag{7.33i}$$

$$\det\left\{[K_0 - \lambda(\cdot\cdot)] \cdot \delta^{(a)(b)}{}_{(c)(d)}\right\} = 0,$$

$$1 \leq a < b, \quad 1 \leq c < d. \tag{7.33ii}$$

With the notational abbreviations in (7.15) and (7.16), (7.32) and (7.33i) reduce to

$$\mathbb{R}^{(A)}{}_{(B)}(x) = K_0 \cdot \delta^{(A)}{}_{(B)}. \tag{7.34}$$

Thus, the 6×6 matrix $\left[\mathbb{R}^{(A)}{}_{(B)}(x)\right]$ is *already diagonalized* with repeated eigenvalues. We can see that the Segre characteristic of this curvature tensor is provided by $[(1, 1, 1, 1, 1, 1)]$.

Now, by (1.169ii), Problem #3 of Exercises 2.4, and Theorem 1.3.30, the conformal tensor components corresponding to (7.32) reduce to

$$C_{(a)(b)(c)(d)}(x) \equiv 0, \tag{7.35i}$$

$$\mathbb{C}^{(A)}{}_{(B)}(x) \equiv 0 \cdot \delta^{(A)}{}_{(B)}. \tag{7.35ii}$$

Therefore, every invariant eigenvalue vanishes, and the corresponding Segre characteristic of the 6×6 matrix $\left[\mathbb{C}^{(A)}{}_{(B)}(x)\right]$ is furnished by $[(1, 1, 1, 1, 1, 1)]$ as well. □

It is worthwhile to look back into the definition of 2-forms in (7.9i,ii) for characterizing canonical forms of 6×6 matrices $\left[\mathbb{C}^{(A)}{}_{(B)}(x)\right]$.

Now, the invariant eigenvalue problem for a 2-form $\mathbf{W}_{..}(x)$ is stated below as

$$W_{ij}(\cdot\cdot) \cdot \mathsf{E}^j(\cdot\cdot) = \lambda(\cdot\cdot) \cdot g_{ij}(\cdot\cdot) \cdot \mathsf{E}^j(\cdot\cdot), \tag{7.36i}$$

$$W_{(a)(b)}(\cdot\cdot) \cdot \mathsf{E}^{(b)}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot d_{(a)(b)} \cdot \mathsf{E}^{(b)}(\cdot\cdot). \tag{7.36ii}$$

We conclude from (7.36i,ii) and (7.9i,ii) that eigenvectors must satisfy

$$\lambda(\cdot\cdot) \cdot g_{ij}(\cdot\cdot) \cdot \mathsf{E}^i(\cdot\cdot) \cdot \mathsf{E}^j(\cdot\cdot) = \lambda(\cdot\cdot) \cdot d_{(a)(b)} \cdot \mathsf{E}^{(a)}(\cdot\cdot) \cdot \mathsf{E}^{(b)}(\cdot\cdot) = 0. \tag{7.37}$$

Therefore, *either the invariant eigenvalue* $\lambda(x) = 0$ *or the (nonzero) eigenvector* $\vec{\mathsf{E}}(x)$ *is a null vector, or both*. This conclusion immediately shows the importance of null eigenvectors associated with 2-forms. (Compare (2.300) and (2.301) for eigenvectors of the electromagnetic 2-form $\mathbf{F}_{..}(x)$.)

Now let us go back to the definition of the Hodge star operation or *Hodge-dual operation* on p-forms in (1.113). Particularly, for the 2-form $\mathbf{W}_{..}(x)$, it is furnished by

$$\begin{aligned} {}^*W_{(c)(d)}(x) &:= (1/2)\, \eta^{(a)(b)}{}_{(c)(d)} \cdot W_{(a)(b)}(x), \\ &= \frac{1}{2} \cdot \eta_{(c)(d)}{}^{(a)(b)} \cdot W_{(a)(b)}(x) = \frac{1}{2} \cdot \varepsilon_{(c)(d)(p)(q)} \cdot W^{(p)(q)}(x). \end{aligned} \tag{7.38}$$

Example 7.1.6. Consider the contravariant orthonormal components $W^{(c)(d)}(x) \equiv -W^{(d)(c)}(x)$. *The Hodge-dual components* from (7.38) emerge explicitly as

$$\begin{aligned} {}^*W^{(1)(2)}(\cdot\cdot) &= -W_{(3)(4)}(\cdot\cdot), & {}^*W^{(2)(3)}(\cdot\cdot) &= -W_{(1)(4)}(\cdot\cdot), \\ {}^*W^{(3)(1)}(\cdot\cdot) &= -W_{(2)(4)}(\cdot\cdot), & {}^*W^{(1)(4)}(\cdot\cdot) &= -W_{(2)(3)}(\cdot\cdot), \\ {}^*W^{(2)(4)}(\cdot\cdot) &= -W_{(3)(1)}(\cdot\cdot), & {}^*W^{(3)(4)}(\cdot\cdot) &= -W_{(1)(2)}(\cdot\cdot). \end{aligned} \tag{7.39}$$

Furthermore, it follows from (7.38) and (7.39) that *the double-dual components* are provided by

$${}^{**}W_{(a)(b)}(x) = -W_{(a)(b)}(x). \tag{7.40}$$

(Compare with the Example 1.3.6 and Problem # 5 of Exercises 1.3.) □

Now we shall investigate *complex-valued 2-forms*. We study *the special class* of complex-valued 2-forms given by the following:

$$\begin{aligned} \mathcal{S}_{(a)(b)}(x) &:= W_{(a)(b)}(x) - \mathrm{i} \cdot {}^*W_{(a)(b)}(x) \\ &\equiv -\mathcal{S}_{(b)(a)}(x). \end{aligned} \tag{7.41}$$

(Here, we assume that $\mathbf{W}_{..}(x)$ is a real 2-form.)

We define the Hodge duality for a complex 2*-form exactly as that in (7.37)*. Therefore, we express

$$^{*}\mathcal{S}_{(c)(d)}(x) := (1/2) \cdot \eta^{(a)(b)}{}_{(c)(d)} \cdot \mathcal{S}_{(a)(b)}(x). \tag{7.42}$$

It follows from (7.41) and (7.42) that *the Hodge dual* of components $\mathcal{S}_{(a)(b)}(x)$ must satisfy, by use of (7.40) and (7.41),

$$\begin{aligned} ^{*}\mathcal{S}_{(a)(b)}(x) &= {}^{*}\left[W_{(a)(b)}(\cdot\cdot) - \mathrm{i}\, {}^{*}W_{(a)(b)}(\cdot\cdot)\right] \\ &= {}^{*}W_{(a)(b)}(\cdot\cdot) + \mathrm{i}\, W_{(a)(b)}(\cdot\cdot) \\ &= \mathrm{i} \cdot \mathcal{S}_{(a)(b)}(x). \end{aligned} \tag{7.43}$$

Equation above is called[1] the *self-duality* of the corresponding complex bivector $\mathcal{S}_{(a)(b)}(x)$.

A real or complex bivector $\mathbf{X}_{..}(x)$ is called *null* provided

$$X_{(a)(b)}(\cdot\cdot) \cdot X^{(a)(b)}(\cdot\cdot) = X_{(a)(b)}(\cdot\cdot) \cdot {}^{*}X_{(a)(b)}(\cdot\cdot) = 0. \tag{7.44}$$

Therefore, a bivector $\mathbf{X}_{..}(x)$ is *nonnull* provided $X_{(a)(b)}(\cdot\cdot) \cdot X^{(a)(b)}(\cdot\cdot) \neq 0$ or $X_{(a)(b)} \cdot^{*} X^{(a)(b)} \neq 0$.

A complex bivector field $1/2 \cdot \Sigma_{(a)(b)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot)$ is defined by

$$\Sigma_{(a)(b)}(x) := X_{(a)(b)}(\cdot\cdot) - \mathrm{i}\, {}^{*}X_{(a)(b)}(\cdot\cdot) \tag{7.45}$$

and is *self-dual* on account of (7.43).

Now, suppose that $\vec{\mathbf{U}}(x)$ is a future-pointing, timelike vector field. It follows that

$$U_{(a)}(\cdot\cdot) \cdot U^{(a)}(\cdot\cdot) \equiv -1, \tag{7.46i}$$

$$U^{(4)}(\cdot\cdot) \geq 1, \tag{7.46ii}$$

$$\Sigma_{(a)(b)}(\cdot\cdot) \cdot U^{(a)}(\cdot\cdot) \cdot U^{(b)}(\cdot\cdot) \equiv 0. \tag{7.46iii}$$

A pertinent covariant vector field $\widetilde{\mathbf{X}}(x)$, defined below, satisfies

$$X_{(a)}(x) := \Sigma_{(a)(b)}(\cdot\cdot) \cdot U^{(b)}(\cdot\cdot), \tag{7.47i}$$

$$X_{(a)}(\cdot\cdot) \cdot U^{(a)}(\cdot\cdot) \equiv 0. \tag{7.47ii}$$

Here, it follows from (7.47ii) that either $X_{(a)} \cdot X^{(a)} > 0$ or $\widetilde{\mathbf{X}}(x) = \widetilde{\mathbf{0}}(x)$.

[1]Equation (7.43) *differs from those in (7.39) for a real bivector*. To call the condition (7.43) as *self-duality* is a misnomer. However, *due to the popularity of this nomenclature*, we will keep on using the same name in this chapter.

Now we shall provide an example.

Example 7.1.7. In a domain $D \subset \mathbb{R}^4$ of a coordinate chart, let a continuous vector field $\vec{\mathbf{U}}(x)$ and a nonzero covariant vector field $\widetilde{\mathbf{Y}}(x)$ exist. We define a self-dual complex bivector field by

$$\begin{aligned}\Sigma_{(a)(b)}(x) =& \left[U_{(a)}(\cdot\cdot) \cdot Y_{(b)}(\cdot\cdot) - U_{(b)}(\cdot\cdot) \cdot Y_{(a)}(\cdot\cdot)\right] \\ &- i \cdot^{*} \left[U_{(a)}(\cdot\cdot) \cdot Y_{(b)}(\cdot\cdot) - U_{(b)}(\cdot\cdot) \cdot Y_{(a)}(\cdot\cdot)\right] \\ =& \left[U_{(a)}(\cdot\cdot) \cdot Y_{(b)}(\cdot\cdot) - U_{(b)}(\cdot\cdot) \cdot Y_{(a)}(\cdot\cdot)\right] \\ &- \frac{i}{2}\eta_{(a)(b)(c)(d)} \cdot \left[U^{(c)}(\cdot\cdot) \cdot Y^{(d)}(\cdot\cdot) - U^{(d)}(\cdot\cdot) \cdot Y^{(c)}(\cdot\cdot)\right],\end{aligned}$$

$$X_{(a)} := \Sigma_{(a)(b)}(\cdot\cdot) \cdot U^{(b)}(\cdot\cdot) = U_{(a)}(\cdot\cdot) \cdot \left[Y_{(b)}(\cdot\cdot) \cdot U^{(b)}\right] + Y_{(a)}(\cdot\cdot) - 0.$$

Moreover, using the equation $-\eta^{(a)(b)(p)(q)} \cdot \eta_{(a)(b)(c)(d)} = 2 \cdot \delta^{(p)(q)}{}_{(c)(d)}$, we can derive that

$$\begin{aligned}\Sigma_{(a)(b)}(\cdot\cdot) \cdot \ \Sigma^{(a)(b)}(\cdot\cdot) =& \left[U_{(a)}(\cdot\cdot) \cdot Y_{(b)}(\cdot\cdot) - U_{(b)}(\cdot\cdot) \cdot Y_{(a)}(\cdot\cdot)\right] \\ & \cdot \left[U^{(a)}(\cdot\cdot) \cdot Y^{(b)}(\cdot\cdot) - U^{(b)}(\cdot\cdot) \cdot Y^{(a)}(\cdot\cdot)\right] - \eta^{(a)(b)(p)(q)} \\ & \cdot \eta_{(a)(b)(c)(d)} \left[U_{(p)}(\cdot\cdot) \cdot Y_{(q)}(\cdot\cdot) \cdot U^{(c)}(\cdot\cdot) \cdot Y_{(d)}(\cdot\cdot)\right] \\ =& - 2Y_{(b)} \cdot Y^{(b)} - 2\left[Y_{(a)} \cdot U^{(a)}\right]^2 \\ & + 2\left[U_{(c)}Y_{(d)} \cdot U^{(c)}Y^{(d)} - U_{(d)}Y_{(c)} \cdot U^{(c)}Y^{(d)}\right] \\ =& - 4\left[Y_{(a)} \cdot Y^{(a)}\right]^2 - 4\left[Y_{(a)} \cdot U^{(a)}\right].\end{aligned}$$

□

We shall now use the set of null tetrad fields $\left\{\vec{\mathbf{m}}(\cdot\cdot), \vec{\overline{\mathbf{m}}}(\cdot\cdot), \vec{\mathbf{l}}(\cdot\cdot), \vec{\mathbf{k}}(\cdot\cdot)\right\}$ of (7.6i) for further investigations. We define three complex bivector fields in the following:

$$\mathsf{U}_{..}(x) := 2 \cdot \widetilde{\overline{\mathbf{m}}}(\cdot\cdot) \wedge \widetilde{\mathbf{l}}(\cdot\cdot), \tag{7.48i}$$

$$\mathsf{U}_{(a)(b)}(\cdot\cdot) = -l_{(a)}(\cdot\cdot) \cdot \overline{m}_{(b)}(\cdot\cdot) + l_{(b)}(\cdot\cdot) \cdot \overline{m}_{(a)}(\cdot\cdot), \tag{7.48ii}$$

$$\mathsf{V}_{..}(x) := 2 \cdot \widetilde{\mathbf{k}}(\cdot\cdot) \wedge \widetilde{\mathbf{m}}(\cdot\cdot), \tag{7.48iii}$$

$$\mathsf{V}_{(a)(b)}(\cdot\cdot) = k_{(a)}(\cdot\cdot) \cdot m_{(b)}(\cdot\cdot) - k_{(b)}(\cdot\cdot) \cdot m_{(a)}(\cdot\cdot), \tag{7.48iv}$$

$$\mathsf{W}_{..}(x) := 2 \cdot \left[\widetilde{\mathbf{m}}(\cdot\cdot) \wedge \widetilde{\overline{\mathbf{m}}}(\cdot\cdot) - \widetilde{\mathbf{k}}(\cdot\cdot) \wedge \widetilde{\mathbf{l}}(\cdot\cdot)\right], \tag{7.48v}$$

$$\mathsf{W}_{(a)(b)}(\cdot\cdot) = m_{(a)}(\cdot\cdot)\cdot\overline{m}_{(b)}(\cdot\cdot) - m_{(b)}(\cdot\cdot)\cdot\overline{m}_{(a)}(\cdot\cdot)$$
$$- k_{(a)}(\cdot\cdot)\cdot l_{(b)}(\cdot\cdot) + k_{(b)}(\cdot\cdot)\cdot l_{(a)}(\cdot\cdot). \tag{7.48vi}$$

It can be proved that

$$^{*}\mathsf{U}_{(a)(b)}(\cdot\cdot) = \mathrm{i}\cdot\mathsf{U}_{(a)(b)}(\cdot\cdot), \tag{7.49i}$$

$$^{*}\mathsf{V}_{(a)(b)}(\cdot\cdot) = \mathrm{i}\cdot\mathsf{V}_{(a)(b)}(\cdot\cdot), \tag{7.49ii}$$

$$^{*}\mathsf{W}_{(a)(b)}(\cdot\cdot) = \mathrm{i}\cdot\mathsf{W}_{(a)(b)}(\cdot\cdot). \tag{7.49iii}$$

(See Problem #6 of Exercises 7.1.) Therefore, *each of these complex 2-forms is self-dual*, according to the definition in (7.43).

Here we work out the double contractions as the following:

$$\begin{aligned}\mathsf{U}_{(a)(b)}(\cdot\cdot)\cdot\mathsf{U}^{(a)(b)}(\cdot\cdot) = {} & \left[-l_{(a)}(\cdot\cdot)\cdot\overline{m}_{(b)}(\cdot\cdot) + l_{(b)}(\cdot\cdot)\cdot\overline{m}_{(a)}(\cdot\cdot)\right] \\ & \times\left[-l^{(a)}(\cdot\cdot)\cdot\overline{m}^{(b)}(\cdot\cdot) + l^{(b)}(\cdot\cdot)\cdot\overline{m}^{(a)}(\cdot\cdot)\right] \\ = {} & \Big[2\cdot\left(l_{(a)}(\cdot\cdot)\cdot l^{(a)}(\cdot\cdot)\cdot\overline{m}_{(b)}(\cdot\cdot)\cdot\overline{m}^{(b)}(\cdot\cdot)\right) \\ & -2\cdot\left(l_{(a)}(\cdot\cdot)\cdot\overline{m}^{(a)}(\cdot\cdot)\right)^2\Big] \\ \equiv {} & 0, \end{aligned} \tag{7.50i}$$

$$\mathsf{V}_{(a)(b)}(\cdot\cdot)\cdot\mathsf{V}^{(a)(b)}(\cdot\cdot) \equiv 0, \tag{7.50ii}$$

$$\mathsf{W}_{(a)(b)}(\cdot\cdot)\cdot\mathsf{W}^{(a)(b)}(\cdot\cdot) \equiv -4, \tag{7.50iii}$$

$$\mathsf{U}_{(a)(b)}(\cdot\cdot)\cdot\mathsf{V}^{(a)(b)}(\cdot\cdot) \equiv 2, \tag{7.50iv}$$

$$\mathsf{U}_{(a)(b)}(\cdot\cdot)\cdot\mathsf{W}^{(a)(b)}(\cdot\cdot) \equiv 0, \tag{7.50v}$$

$$\mathsf{V}_{(a)(b)}(\cdot\cdot)\cdot\mathsf{W}^{(a)(b)}(\cdot\cdot) \equiv 0. \tag{7.50vi}$$

Consider the set of all complex bivector fields which are self-dual in the sense of equations (7.49i–iii). We shall prove a useful theorem about the set of *three bivector fields* $\{\mathbf{U}_{..}(\cdot\cdot), \mathbf{V}_{..}(\cdot\cdot), \mathbf{W}_{..}(\cdot\cdot)\}$ here.

Theorem 7.1.8. *Let the set of three bivector fields* $\{\mathbf{U}_{..}(\cdot\cdot), \mathbf{V}_{..}(\cdot\cdot), \mathbf{W}_{..}(\cdot\cdot)\}$ *be defined in a coordinate domain* $D \subset \mathbb{R}^4$.

(i) *This set comprises of three linearly independent bivector fields.*
(ii) *Every linear combination of .* $\mathbf{U}_{..}(\cdot\cdot), \mathbf{V}_{..}(\cdot\cdot), \mathbf{W}_{..}(\cdot\cdot)$ *yields a self-dual bivector.*

(iii) *These three bivectors generate, by all possible linear combinations, a proper subset of all self-dual bivectors which is isomorphic to a three-dimensional, complex vector space.*

Proof. (i) Consider linear equations

$$\alpha(x)\cdot \mathsf{U}_{(a)(b)}(\cdot\cdot)+\beta(\cdot\cdot)\cdot \mathsf{V}_{(a)(b)}(\cdot\cdot)+\gamma(\cdot\cdot)\cdot \mathsf{W}_{(a)(b)}(\cdot\cdot)=0,$$

where, $\alpha(x)$, $\beta(x)$, $\gamma(x)$ are complex scalar fields. Multiplying the above equations by $\mathsf{U}^{(a)(b)}(\cdot\cdot)$, $\mathsf{V}^{(a)(b)}(\cdot\cdot)$, $\mathsf{W}^{(a)(b)}(\cdot\cdot)$ and double contracting (with use of (7.50i–vi)), it follows that $\alpha(x)=\beta(x)=\gamma(x)\equiv 0$. Thus, the 2-forms corresponding to $\mathsf{U}_{(a)(b)}(\cdot\cdot)$, $\mathsf{V}_{(a)(b)}(\cdot\cdot)$, and $\mathsf{W}_{(a)(b)}(\cdot\cdot)$ are *linearly independent*.

(ii) Consider a bivector field expressed as

$$\mathbf{Y}_{..}(x):=\xi(x)\cdot \mathbf{U}_{..}(\cdot\cdot)+\eta(\cdot\cdot)\cdot \mathbf{V}_{..}(\cdot\cdot)+\zeta(\cdot\cdot)\cdot \mathbf{W}_{..}(\cdot\cdot).$$

Here, $\xi(x)$, $\eta(x)$, and $\zeta(x)$ are *arbitrary complex scalar fields*. By use of (7.49i–iii), we derive that

$${}^{*}\mathsf{Y}_{(a)(b)}(\cdot\cdot)=\mathrm{i}\cdot \mathsf{Y}_{(a)(b)}(\cdot\cdot).$$

Therefore, the bivector field $\mathbf{Y}_{..}(x)$ is *self-dual*.

(iii) Consider the three-dimensional complex vector space $\mathbb{C}^3(x)$ of ordered, triple, complex scalar fields $\xi(x)$, $\eta(x)$, $\zeta(x)$. *An isomorphism (linear and one-to-one mapping)* can be established by

$$\mathcal{I}(x)\left[\mathbf{Y}_{..}(x)\right]:=\left(\xi(x),\eta(x),\zeta(x)\right).$$

Equation above essentially proves part (iii) of the theorem. ■

Now, we shall apply the preceding theoretical considerations first to electromagnetic fields and then to general relativity. Recall Maxwell's equations (1.69), (1.70), (2.290i,ii), (2.300), (2.309), (2.5), and (2.314). In Problem # 8 of Exercises 2.5, we introduced *the complex-valued electromagnetic field* as

$$\begin{aligned}\varphi^{kj}(x) &:= F^{kj}(x)-\mathrm{i}\,{}^{*}F^{kj}(x)\\ &\equiv -\varphi^{jk}(x).\end{aligned}\tag{7.51}$$

Remark: Since ${}^{*}F^{kj}(x)$ is *an oriented or pseudotensor* of (1.112ii), *the antisymmetric field* $\varphi^{kj}(x)$ *is not an* (*absolute*) *tensor field*!

Let us summarize Maxwell's equations here:

$$F^{(b)(a)}(x) \equiv -F^{(a)(b)}(x), \tag{7.52i}$$

$$\nabla_{(b)} F^{(a)(b)} = 0, \tag{7.52ii}$$

$$\nabla_{(a)} F_{(b)(c)} + \nabla_{(b)} F_{(c)(a)} + \nabla_{(c)} F_{(a)(b)} = 0, \tag{7.52iii}$$

$$\nabla_{(b)}\, {}^{*}F^{(a)(b)} = 0. \tag{7.52iv}$$

Equation (7.52iv) *is exactly equivalent to* (7.52iii). However, (7.52iv) is oriented tensor field equations, whereas (7.52iii) is (absolute) tensor field equations!

Now we shall introduce complex-valued, electromagnetic bivector field equations:

$$\nabla_{(b)}\varphi^{(a)(b)} = 0. \tag{7.53}$$

Remarks: (i) The set of equations above are *exactly equivalent to Maxwell's equations* (7.52i–iv). However, *these equations are not* (*absolute*) *tensor field equations*.

(ii) Equation (7.53) can represent Maxwell's equations in a curvilinear coordinate chart of *flat space–time*. Also, these equations are valid in a curved space–time with *a background metric*. Moreover, (7.53) can be *part of the Einstein–Maxwell* (*or electromagneto-vac*) *equations* in (2.277i,ii) and (2.290i–vi).

By the definition in (7.51) and (7.40) and (7.41), we obtain

$${}^{*}\varphi_{(a)(b)}(x) = \mathrm{i}\,\varphi_{(a)(b)}(x). \tag{7.54}$$

Thus, *the complex electromagnetic bivector field is self-dual*.

According to Theorem 7.1.8, the bivector components of (7.51) admit the following linear combination:

$$\varphi_{(a)(b)}(x) = \phi_{(0)}(x)\cdot \mathsf{U}_{(a)(b)}(\cdot\cdot) + \phi_{(1)}(\cdot\cdot)\cdot \mathsf{W}_{(a)(b)}(\cdot\cdot) + \phi_{(2)}(\cdot\cdot)\cdot \mathsf{V}_{(a)(b)}(\cdot\cdot). \tag{7.55}$$

Here, $\phi_{(0)}(x)$, $\phi_{(1)}(x)$, and $\phi_{(2)}(x)$ are *suitable complex scalar fields*. It can be noted that the three complex-valued fields $\phi_{(0)}(\cdot\cdot)$, $\phi_{(1)}(\cdot\cdot)$, and $\phi_{(2)}(\cdot\cdot)$ provide the same number of degrees of freedom as the six linearly independent, real-valued fields $F_{(a)(b)}(x)$. By (7.50i–vi) and (7.48ii,iv,vi) we can explicitly deduce that

$$\begin{aligned}\phi_{(0)}(x) &= (1/2)\cdot \varphi^{(a)(b)}(\cdot\cdot)\cdot \mathsf{V}_{(a)(b)}(\cdot\cdot)\\ &= (1/2)\cdot \varphi^{(a)(b)}(\cdot\cdot)\cdot \left[k_{(a)}(\cdot\cdot)\cdot m_{(b)}(\cdot\cdot) - k_{(b)}(\cdot\cdot)\cdot m_{(a)}(\cdot\cdot)\right],\end{aligned} \tag{7.56i}$$

$$\begin{aligned}\phi_{(1)}(x) = & -(1/4)\cdot\varphi^{(a)(b)}(\cdot\cdot)\cdot \mathsf{W}_{(a)(b)}(\cdot\cdot)\\ = & -(1/4)\cdot\varphi^{(a)(b)}\cdot\big[m_{(a)}(\cdot\cdot)\cdot\overline{m}_{(b)}(\cdot\cdot)-m_{(b)}(\cdot\cdot)\cdot\overline{m}_{(a)}(\cdot\cdot)\\ & -k_{(a)}(\cdot\cdot)\cdot l_{(b)}(\cdot\cdot)+k_{(b)}(\cdot\cdot)\cdot l_{(a)}(\cdot\cdot)\big],\end{aligned} \tag{7.56ii}$$

$$\begin{aligned}\phi_{(2)}(x) &= (1/2)\cdot\varphi^{(a)(b)}(\cdot\cdot)\cdot \mathsf{U}_{(a)(b)}(\cdot\cdot)\\ &= (1/2)\cdot\varphi^{(a)(b)}(\cdot\cdot)\cdot\big[-l_{(a)}(\cdot\cdot)\cdot\overline{m}_{(b)}(\cdot\cdot)+l_{(b)}(\cdot\cdot)\cdot\overline{m}_{(a)}(\cdot\cdot)\big].\end{aligned} \tag{7.56iii}$$

Direct computations from (7.50i–vi) and (7.55) lead to

$$\varphi_{(a)(b)}\varphi^{(a)(b)} = 4\cdot\left(\phi_{(0)}\phi_{(2)}-\phi^2_{(1)}\right). \tag{7.57}$$

We characterize *a nonnull or nondegenerate electromagnetic field* by the criterion

$$\varphi_{(a)(b)}(\cdot\cdot)\cdot\varphi^{(a)(b)}(\cdot\cdot)\neq 0, \tag{7.58i}$$

$$\text{or}\quad \left[\phi_{(1)}(\cdot\cdot)\right]^2\neq\phi_{(0)}(\cdot\cdot)\cdot\phi_{(2)}(\cdot\cdot). \tag{7.58ii}$$

Therefore, we define *a degenerate or null electromagnetic field* by the condition

$$\frac{1}{4}\varphi_{(a)(b)}(\cdot\cdot)\varphi^{(a)(b)}(\cdot\cdot) = \phi_{(0)}(\cdot\cdot)\cdot\phi_{(2)}(\cdot\cdot)-\left[\phi_{(1)}(\cdot\cdot)\right]^2 = 0, \tag{7.59i}$$

which is implied by

$$F_{(a)(b)}(\cdot\cdot)\cdot F^{(a)(b)}(\cdot\cdot) = F_{(a)(b)}(\cdot\cdot)\cdot {}^*F^{(a)(b)}(\cdot\cdot) = -{}^*F_{(a)(b)}(\cdot\cdot)\cdot {}^*F^{(a)(b)}(\cdot\cdot) = 0. \tag{7.59ii}$$

(Compare (7.59i,ii) with (2.5) and (7.44).)

By a transformation of the complex tetrad, as in (7.7iii) or (7.7iv), we can obtain the new coefficient $\widehat{\phi}_{(0)}(x)\equiv 0$. Dropping the hats in the sequel, the degeneracy condition (7.59ii) reduces to

$$\phi_{(0)}(x) = \phi_{(1)}(x)\equiv 0. \tag{7.60}$$

By the use of (7.55), we conclude in this case

$$\begin{aligned}\varphi_{(a)(b)}(x) &= \phi_{(2)}(\cdot\cdot)\cdot \mathsf{V}_{(a)(b)}(\cdot\cdot)\\ &= \phi_{(2)}(\cdot\cdot)\cdot\big[k_{(a)}(\cdot\cdot)\cdot m_{(b)}(\cdot\cdot)-k_{(b)}(\cdot\cdot)\cdot m_{(a)}(\cdot\cdot)\big].\end{aligned} \tag{7.61}$$

Example 7.1.9. We choose *the space–time to be flat*. Moreover, choosing the global Minkowski coordinate chart of (2.8), we write

$$\mathbf{g}_{..}(x) = d_{ij} \cdot \mathrm{d}x^i \otimes \mathrm{d}x^j\,, \tag{7.62i}$$

$$\mathbf{g}^{..}(x) = d^{ij} \cdot \partial_i \otimes \partial_j\,. \tag{7.62ii}$$

The (natural) orthonormal basis set (or tetrad) is provided by

$$\vec{\mathbf{e}}_{(a)}(x) = \delta^i{}_{(a)} \cdot \partial_i\,, \tag{7.63i}$$

$$\widetilde{\mathbf{e}}^{(a)}(x) = \delta^{(a)}_{\ i} \cdot \mathrm{d}x^i\,. \tag{7.63ii}$$

The Ricci rotation coefficients reduce to

$$\gamma_{(a)(b)(c)}(x) \equiv 0,$$

and the covariant derivatives of (1.124ii) are furnished by

$$\nabla_{(c)} T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_r)} = \partial_{(c)} T^{(a_1),\ldots,(a_r)}{}_{(b_1),\ldots,(b_r)}. \tag{7.64}$$

We choose the electromagnetic 4-potential *as a plane wave*:

$$A_{(b)}(x) = -a_{(b)} \cdot \cos\left(\nu_{(c)}\delta^{(c)}_{\ i}\, x^i\right), \tag{7.65i}$$

$$\nu_{(c)} \cdot \nu^{(c)} = 0, \quad \left[\nu_{(4)}\right]^2 = \delta^{(\alpha)(\beta)} \cdot \nu_{(\alpha)}\nu_{(\beta)}, \tag{7.65ii}$$

$$\nu_{(b)} \cdot a^{(b)} = 0. \tag{7.65iii}$$

The constants $a_{(b)}$ represent the amplitudes of the waves, whereas $\nu_{(\alpha)}$ and $\nu_{(4)}$ represent the wave numbers and frequency of the plane wave inherent in the choice (7.65i–iii). Equation (7.65iii) follows from the Lorentz gauge condition (2.282).

The electromagnetic field, from (7.65i), emerges as

$$F_{(a)(b)}(x) = \left(\nu_{(b)} \cdot a_{(a)} - \nu_{(a)} \cdot a_{(b)}\right) \cdot \sin\left(\nu_{(c)}\delta^{(c)}_{\ i}\, x^i\right). \tag{7.66}$$

We can verify explicitly that

$$\begin{aligned} F_{(a)(b)}(\cdot\cdot) \cdot F^{(a)(b)}(\cdot\cdot) = {} & \left(\nu_{(b)} \cdot a_{(a)} - \nu_{(a)} \cdot a_{(b)}\right) \cdot \left(\nu^{(b)} \cdot a^{(a)} - \nu^{(a)} \cdot a^{(b)}\right) \\ & \times \left[\sin\left(\nu_{(c)}\delta^{(c)}_{\ i}\, x^i\right)\right]^2 \equiv 0. \end{aligned} \tag{7.67i}$$

Moreover,

$$\begin{aligned} 2 \cdot F_{(a)(b)}(\cdot\cdot) \cdot {}^*F^{(a)(b)}(\cdot\cdot) = {} & \left(\nu_{(b)} \cdot a_{(a)} - \nu_{(a)} \cdot a_{(b)}\right) \cdot \eta^{(a)(b)(c)(d)} \\ & \times \left(\nu_{(d)} \cdot a_{(c)} - \nu_{(c)} \cdot a_{(d)}\right) \cdot \left[\sin\left(\nu_{(e)}\delta^{(e)}_{\ i}\, x^i\right)\right]^2 \\ = {} & \left[\eta^{(c)(a)(b)(d)} \cdot \left(\nu_{(b)} \cdot \nu_{(d)} \cdot a_{(a)} \cdot a_{(c)} + a_{(b)} \cdot a_{(d)} \cdot \nu_{(a)} \cdot \nu_{(c)}\right)\right. \end{aligned}$$

$$-\eta^{(b)(c)(a)(d)} \cdot \left(a_{(a)} \cdot a_{(d)} \cdot \nu_{(b)} \cdot \nu_{(c)} + \nu_{(a)} \cdot \nu_{(d)} \cdot a_{(b)} \cdot a_{(c)}\right)\Big]$$
$$\times \left[\sin\left(\nu_{(e)} \delta^{(e)}_{\ \ i}\, x^i\right)\right]^2 \equiv 0. \tag{7.67ii}$$

□

Now we shall investigate "gravitational waves" in general relativity. For that purpose, the classification of Weyl's conformal tensor is essential. We have already discussed such topics in (7.25i–viii), (7.26), (7.27), (7.28i,ii), (7.29i–iv), (7.30i–iv), and (7.31i–iv). However, for a more concise analysis, *the complex version* of the conformal tensor is required. For that purpose, let us define *the Hodge dual of the conformal tensor* as follows:

$$\begin{aligned} {}^*C^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) &:= \frac{1}{2}\eta_{(p)(q)}{}^{(a)(b)} \cdot C^{(p)(q)}{}_{(c)(d)}(\cdot\cdot) \\ &= (1/2) \cdot \eta^{(a)(b)}{}_{(p)(q)} \cdot C^{(p)(q)}{}_{(c)(d)}(\cdot\cdot). \end{aligned} \tag{7.68}$$

We define[2] *a complex-valued conformal tensor* as

$$\zeta^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) := C^{(a)(b)}{}_{(c)(d)}(\cdot\cdot) - \mathrm{i}\, {}^*C^{(a)(b)}{}_{(c)(d)}(\cdot\cdot). \tag{7.69}$$

Remark: The gravitational equation (7.69) resembles the electromagnetic version in (7.51).

It follows from (7.43) and (7.69) that the self-duality conditions

$$\begin{aligned} {}^*\zeta_{(a)(b)(c)(d)}(\cdot\cdot) &= {}^*C_{(a)(b)(c)(d)}(\cdot\cdot) + \mathrm{i}\, C_{(a)(b)(c)(d)}(\cdot\cdot) \\ &= \mathrm{i} \cdot \zeta_{(a)(b)(c)(d)}(\cdot\cdot) \end{aligned} \tag{7.70}$$

hold.

Consider a complex-valued "tensor field" $\boldsymbol{\xi}_{....}(x)$ that admits the following algebraic identities:

$$\xi_{(b)(a)(c)(d)}(\cdot\cdot) \equiv \xi_{(a)(b)(d)(c)}(\cdot\cdot) \equiv -\xi_{(a)(b)(c)(d)}(\cdot\cdot), \tag{7.71i}$$

$$\xi_{(c)(d)(a)(b)}(\cdot\cdot) \equiv \xi_{(a)(b)(c)(d)}(\cdot\cdot), \tag{7.71ii}$$

$$\xi_{(a)(b)(c)(d)}(\cdot\cdot) + \xi_{(a)(c)(d)(b)}(\cdot\cdot) + \xi_{(a)(d)(b)(c)}(\cdot\cdot) \equiv 0, \tag{7.71iii}$$

$$\xi^{(d)}{}_{(a)(b)(d)}(\cdot\cdot) \equiv 0. \tag{7.71iv}$$

These identities hold for $\zeta_{(a)(b)(c)(d)}(\cdot\cdot)$ in (7.69), due to the identities (7.25v–viii) of the real-valued conformal tensor components.

[2] This definition *differs* from that in [239].

Now, we introduce the following complex-valued, self-dual "tensor field" components:

$$\mathcal{E}^{(0)}{}_{(a)(b)(c)(d)}(\cdot\cdot) := \mathsf{U}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{U}_{(c)(d)}(\cdot\cdot), \tag{7.72i}$$

$$\mathcal{E}^{(1)}{}_{(a)(b)(c)(d)}(\cdot\cdot) := \mathsf{U}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{W}_{(c)(d)}(\cdot\cdot) + \mathsf{W}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{U}_{(c)(d)}(\cdot\cdot), \tag{7.72ii}$$

$$\begin{aligned}\mathcal{E}^{(2)}{}_{(a)(b)(c)(d)}(\cdot\cdot) := \ & \mathsf{V}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{U}_{(c)(d)}(\cdot\cdot) + \mathsf{U}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{V}_{(c)(d)}(\cdot\cdot) \\ & + \mathsf{W}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{W}_{(c)(d)}(\cdot\cdot),\end{aligned} \tag{7.72iii}$$

$$\mathcal{E}^{(3)}{}_{(a)(b)(c)(d)}(\cdot\cdot) := \mathsf{V}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{W}_{(c)(d)}(\cdot\cdot) + \mathsf{W}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{V}_{(c)(d)}(\cdot\cdot), \tag{7.72iv}$$

$$\mathcal{E}^{(4)}{}_{(a)(b)(c)(d)}(\cdot\cdot) := \mathsf{V}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{V}_{(c)(d)}(\cdot\cdot). \tag{7.72v}$$

It is essential to note that each of above five complex, self-dual, "tensor components" satisfies identities (7.71i–iv). (See Problem #4 of Exercises 7.1.) Therefore, analogous to Theorem 7.1.8, we can prove that the set $\left\{\boldsymbol{\mathcal{E}}^{(0)}{}_{....}(\cdot\cdot), \ldots, \boldsymbol{\mathcal{E}}^{(4)}{}_{....}(\cdot\cdot)\right\}_{|x=x_0}$ constitutes a basis set which is isomorphic to a basis set of *the five-dimensional complex vector space* $\mathbb{C}^5_{(x_0)}$. Thus, the "tensor field" in (7.69) admits the following linear combination:

$$(1/2) \cdot \zeta_{(a)(b)(c)(d)}(\cdot\cdot) = \sum_{J=0}^{4} \Psi_{(J)}(\cdot\cdot) \cdot \mathcal{E}^{(J)}{}_{(a)(b)(c)(d)}(\cdot\cdot). \tag{7.73}$$

Here, $\Psi_{(0)}(x), \ldots, \Psi_{(4)}(x)$ are *five suitable, complex-valued scalar fields*. These coefficient functions can be explicitly determined by use of (7.72i–v), (7.48ii,iv,vi), (7.50i–vi), and (7.73). Thus, these five functions are furnished by:

$$\begin{aligned}\Psi_{(0)}(x) &= (1/8) \cdot \zeta_{(a)(b)(c)(d)}(\cdot\cdot) \cdot \mathsf{V}^{(a)(b)}(\cdot\cdot) \cdot \mathsf{V}^{(c)(d)}(\cdot\cdot) \\ &= C_{(a)(b)(c)(d)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot m^{(b)}(\cdot\cdot) \cdot k^{(c)}(\cdot\cdot) \cdot m^{(d)}(\cdot\cdot),\end{aligned} \tag{7.74i}$$

$$\begin{aligned}\Psi_{(1)}(x) &= -(1/16) \cdot \zeta_{(a)(b)(c)(d)}(\cdot\cdot) \cdot \mathsf{V}^{(a)(b)}(\cdot\cdot) \cdot \mathsf{W}^{(c)(d)}(\cdot\cdot) \\ &= \frac{1}{2} \cdot C_{(a)(b)(c)(d)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot m^{(b)} \cdot \left[k^{(c)}(\cdot\cdot) \cdot l^{(d)}(\cdot\cdot) - m^{(c)}(\cdot\cdot) \cdot \overline{m}^{(d)}(\cdot\cdot)\right] \\ &= C_{(a)(b)(c)(d)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot l^{(b)}(\cdot\cdot) \cdot k^{(c)}(\cdot\cdot) \cdot m^{(d)}(\cdot\cdot),\end{aligned} \tag{7.74ii}$$

$$\begin{aligned}\Psi_{(2)}(x) &= (1/8) \cdot \zeta_{(a)(b)(c)(d)}(\cdot\cdot) \cdot \mathsf{U}^{(a)(b)}(\cdot\cdot) \cdot \mathsf{V}^{(c)(d)}(\cdot\cdot) \\ &= C_{(a)(b)(c)(d)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot m^{(b)}(\cdot\cdot) \cdot \overline{m}^{(c)}(\cdot\cdot) \cdot l^{(d)}(\cdot\cdot),\end{aligned} \tag{7.74iii}$$

$$\begin{aligned}
\Psi_{(3)}(x) &= -(1/16)\cdot\zeta_{(a)(b)(c)(d)}(\cdot\cdot)\cdot\mathsf{U}^{(a)(b)}(\cdot\cdot)\cdot\mathsf{W}^{(c)(d)}(\cdot\cdot)\\
&= \frac{1}{2}\cdot C_{(a)(b)(c)(d)}(\cdot\cdot)\cdot l^{(a)}(\cdot\cdot)\cdot\overline{m}^{(b)}\cdot\left[m^{(c)}(\cdot\cdot)\cdot\overline{m}^{(d)}(\cdot\cdot)-k^{(c)}(\cdot\cdot)\cdot\overline{l}^{(d)}(\cdot\cdot)\right]\\
&= C_{(a)(b)(c)(d)}(\cdot\cdot)\cdot k^{(a)}(\cdot\cdot)\cdot l^{(b)}(\cdot\cdot)\cdot\overline{m}^{(c)}(\cdot\cdot)\cdot l^{(d)}(\cdot\cdot), \qquad (7.74\text{iv})
\end{aligned}$$

$$\begin{aligned}
\Psi_{(4)}(x) &= (1/8)\cdot\zeta_{(a)(b)(c)(d)}(\cdot\cdot)\cdot\mathsf{U}^{(a)(b)}(\cdot\cdot)\cdot\mathsf{U}^{(c)(d)}(\cdot\cdot)\\
&= C_{(a)(b)(c)(d)}(\cdot\cdot)\cdot\overline{m}^{(a)}(\cdot\cdot)\cdot l^{(b)}(\cdot\cdot)\cdot\overline{m}^{(c)}(\cdot\cdot)\cdot l^{(d)}(\cdot\cdot). \qquad (7.74\text{v})
\end{aligned}$$

Remarks: (i) The number of *five, independent complex-valued functions* $\Psi_{(0)}(x),\ldots,\Psi_{(4)}(x)$ *matches exactly with the number of independent real-valued components of the conformal tensor* $\mathbf{C}_{....}(x)$ (*i.e., ten*).

(ii) These five, complex-valued functions for gravitational fields are *exact analogues* of the three, complex-valued components $\phi_{(0)}(x)$, $\phi_{(1)}(x)$, and $\phi_{(2)}(x)$ of the complex electromagnetic field $\boldsymbol{\varphi}_{..}(x)$ in (7.55).

(iii) In (7.74i–v), the factor $(1/2)\zeta_{(a)(b)(c)(d)}(\cdot\cdot)$ can be replaced with $C_{(a)(b)(c)(d)}(\cdot\cdot)$. (Consult the hint to #7 of Exercises 7.1.)

Now we shall proceed to classify algebraically the complex-valued conformal tensor. The relevant invariant eigenvalue problem to be investigated is given by

$$(1/4)\cdot\zeta_{(a)(b)(c)(d)}(\cdot\cdot)\cdot\mathcal{E}^{(c)(d)}(\cdot\cdot) = \lambda(\cdot\cdot)\cdot\mathcal{E}_{(a)(b)}(\cdot\cdot), \qquad (7.75\text{i})$$

$${}^{*}\mathcal{E}_{(a)(b)}(\cdot\cdot) = \mathrm{i}\cdot\mathcal{E}_{(a)(b)}(\cdot\cdot), \qquad (7.75\text{ii})$$

$$-\mathrm{i}\cdot{}^{*}C_{(a)(b)(c)(d)}(\cdot\cdot)\cdot\mathcal{E}^{(c)(d)}(\cdot\cdot) = C_{(a)(b)(c)(d)}(\cdot\cdot)\cdot\mathcal{E}^{(c)(d)}(\cdot\cdot), \qquad (7.75\text{iii})$$

$$\frac{1}{2}\cdot C_{(a)(b)(c)(d)}(\cdot\cdot)\cdot\mathcal{E}^{(c)(d)}(\cdot\cdot) = \lambda(\cdot\cdot)\cdot\mathcal{E}_{(a)(b)}(\cdot\cdot). \qquad (7.75\text{iv})$$

(Compare (7.75iv) with (7.29iii).)

We choose *a comoving frame* to render the analysis of (7.75i,ii) more tractable. Recall the future-pointing, timelike vector field $\vec{\mathbf{U}}(x)$ discussed in (7.46i,ii). We define the complex second-order tensor field $\mathbf{Q}_{..}(x)$ and some related identities in the following[3]:

$$-\,Q_{(a)(d)}(x) := \zeta_{(a)(b)(c)(d)}(\cdot\cdot)\cdot U^{(b)}(\cdot\cdot)\cdot U^{(c)}(\cdot\cdot), \qquad (7.76\text{i})$$

$$\begin{aligned}
-\,Q_{(d)(a)}(\cdot\cdot) &= \zeta_{(d)(b)(c)(a)}(\cdot\cdot)\cdot U^{(b)}(\cdot\cdot)\cdot U^{(c)}(\cdot\cdot)\\
&\equiv \zeta_{(c)(a)(d)(b)}(\cdot\cdot)\cdot U^{(b)}(\cdot\cdot)\cdot U^{(c)}(\cdot\cdot)\\
&\equiv -Q_{(a)(d)}(\cdot\cdot), \qquad (7.76\text{ii})
\end{aligned}$$

[3]We have inserted the negative sign in the left-hand sides of (7.76i–iv) so that the subsequent equations coincide with popular versions.

$$-Q^{(a)}{}_{(a)}(\cdot\cdot) = \zeta^{(a)}{}_{(b)(c)(a)}(\cdot\cdot)\cdot U^{(b)}(\cdot\cdot)\cdot U^{(c)}(\cdot\cdot) \equiv 0, \tag{7.76iii}$$

$$\begin{aligned} -Q_{(a)(d)}(\cdot\cdot)\cdot U^{(d)}(\cdot\cdot) &= \zeta_{(a)(b)(c)(d)}(\cdot\cdot)\cdot U^{(b)}(\cdot\cdot)\cdot U^{(c)}(\cdot\cdot)\cdot U^{(d)}(\cdot\cdot)\\ &= \left[\zeta_{(a)(b)(c)(d)}(\cdot\cdot)\cdot U^{(c)}(\cdot\cdot)\cdot U^{(d)}(\cdot\cdot)\right]\cdot U^{(b)}(\cdot\cdot)\\ &\equiv 0. \end{aligned} \tag{7.76iv}$$

Using (7.73), the definition (7.76i) yields

$$\begin{aligned} -\frac{1}{2}\cdot Q_{(a)(d)}(x) &= (1/2)\cdot\zeta_{(a)(b)(c)(d)}(\cdot\cdot)\cdot U^{(b)}(\cdot\cdot)\cdot U^{(c)}(\cdot\cdot)\\ &= \Big\{\Psi_{(0)}(\cdot\cdot)\cdot \mathsf{U}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{U}_{(c)(d)}(\cdot\cdot) + \Psi_{(1)}(\cdot\cdot)\\ &\quad\times\left[\mathsf{U}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{W}_{(c)(d)}(\cdot\cdot) + \mathsf{W}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{U}_{(c)(d)}(\cdot\cdot)\right]\\ &\quad + \Psi_{(2)}(\cdot\cdot)\cdot\left[\mathsf{V}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{U}_{(c)(d)}(\cdot\cdot) + \mathsf{U}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{V}_{(c)(d)}(\cdot\cdot)\right.\\ &\quad\left. + \mathsf{W}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{W}_{(c)(d)}(\cdot\cdot)\right]\\ &\quad + \Psi_{(3)}(\cdot\cdot)\cdot\left[\mathsf{V}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{W}_{(c)(d)}(\cdot\cdot) + \mathsf{W}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{V}_{(c)(d)}(\cdot\cdot)\right]\\ &\quad + \Psi_{(4)}(\cdot\cdot)\cdot\left[\mathsf{V}_{(a)(b)}\cdot V_{(c)(d)}\right]\Big\}\cdot U^{(b)}(\cdot\cdot)\cdot U^{(c)}(\cdot\cdot). \end{aligned} \tag{7.77}$$

Using the comoving frame characterized by

$$U^{(a)}(x) = \delta^{(a)}{}_{(4)}, \tag{7.78}$$

we can compute the components $Q_{(a)(b)}(\cdot\cdot)$ in (7.77) in terms of the five functions $\Psi_{(0)}(\cdot\cdot), \ldots, \Psi_{(4)}(\cdot\cdot)$. We summarize these (long) calculations in the following:

$$Q^{(1)(1)}(\cdot\cdot) = \Psi_{(2)}(\cdot\cdot) - (1/2)\cdot\left[\Psi_{(0)}(\cdot\cdot) + \Psi_{(4)}(\cdot\cdot)\right], \tag{7.79i}$$

$$Q^{(1)(2)}(\cdot\cdot) = (i/2)\cdot\left[\Psi_{(4)}(\cdot\cdot) - \Psi_{(0)}(\cdot\cdot)\right] \equiv Q^{(2)(1)}(\cdot\cdot), \tag{7.79ii}$$

$$Q^{(1)(3)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) - \Psi_{(3)}(\cdot\cdot) \equiv Q^{(3)(1)}(\cdot\cdot), \tag{7.79iii}$$

$$Q^{(2)(2)}(\cdot\cdot) = \Psi_{(2)}(\cdot\cdot) + (1/2)\cdot\left[\Psi_{(0)}(\cdot\cdot) + \Psi_{(4)}(\cdot\cdot)\right], \tag{7.79iv}$$

$$Q^{(2)(3)}(\cdot\cdot) = (i)\cdot\left[\Psi_{(1)}(\cdot\cdot) + \Psi_{(3)}(\cdot\cdot)\right] \equiv Q^{(3)(2)}(\cdot\cdot), \tag{7.79v}$$

$$Q^{(3)(3)}(\cdot\cdot) = (-2)\cdot\Psi_{(2)}(\cdot\cdot), \tag{7.79vi}$$

$$Q^{(\alpha)(4)}(\cdot\cdot) \equiv Q^{(4)(\alpha)}(\cdot\cdot) \equiv 0, \tag{7.79vii}$$

$$Q^{(4)(4)}(\cdot\cdot) \equiv 0. \tag{7.79viii}$$

(See the answer to Problem #5 of Exercise 7.1 for explicit computations.) From (7.79i–viii), we observe the following *block diagonalization*:

$$\left[\mathbb{Q}^{(\alpha)(\beta)}(\cdot\cdot)\right] := \left[Q^{(\alpha)(\beta)}(\cdot\cdot)\right], \tag{7.80i}$$

$$\underset{4\times 4}{[Q(\cdot\cdot)]} = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathbb{Q} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]. \tag{7.80ii}$$

Now, from (7.79i–vi), we can explicitly express the symmetric 3×3 complex matrix as

$$\underset{3\times 3}{[\mathbb{Q}(\cdot\cdot)]} = \left[\mathbb{Q}^{(\alpha)(\beta)}(\cdot\cdot)\right] \tag{7.81i}$$

$$= \begin{bmatrix} \Psi_{(2)} - (1/2)\cdot\left(\Psi_{(0)} + \Psi_{(4)}\right), & (i/2)\cdot\left(\Psi_{(4)} - \Psi_{(0)}\right), & \Psi_{(1)} - \Psi_{(3)} \\ & \Psi_{(2)} + (1/2)\cdot\left(\Psi_{(0)} + \Psi_{(4)}\right), & i\cdot\left(\Psi_{(1)} + \Psi_{(3)}\right) \\ & & -2\cdot\Psi_{(2)} \end{bmatrix},$$

$$\left[\mathbb{Q}^{(\beta)(\alpha)}(\cdot\cdot)\right] \equiv \left[\mathbb{Q}^{(\alpha)(\beta)}(\cdot\cdot)\right], \tag{7.81ii}$$

$$\left[\mathbb{Q}^{(\alpha)}{}_{(\alpha)}(\cdot\cdot)\right] = \text{Trace}\,[\mathbb{Q}(\cdot\cdot)] \equiv 0. \tag{7.81iii}$$

Remark: The 3×3 complex matrix $[\mathbb{Q}(\cdot\cdot)]$ involving *five complex functions* $\Psi_{(0)}(\cdot\cdot), \ldots, \Psi_{(4)}(\cdot\cdot)$ contains exactly the information in *the ten, real, independent components of the conformal tensor field* $\mathbf{C}_{....}(\cdot\cdot)$.

The algebraic classification of the conformal tensor field hinges on the classification of the complex 3×3 matrix $\left[\mathbb{Q}_{(\alpha)(\beta)}(\cdot\cdot)\right]$ in (7.81i). In constructing this matrix, we have made use of a comoving frame characterized by (7.46i,ii). What we need to note now is that the Segre characteristic of a real or complex matrix is *invariant* with respect to the choice of the underlying basis set. Therefore, the algebraic analysis of the matrix $\left[\mathbb{Q}_{(\alpha)(\beta)}(\cdot\cdot)\right]$ in (7.81i) yields *an invariant classification of the conformal tensor.*

Firstly, we notice that there exist *two classes* of the 4×4 complex matrices $\left[Q_{(a)(b)}(\cdot\cdot)\right]$ given by

$$\underset{4\times 4}{[Q(\cdot\cdot)]} \not\equiv \underset{4\times 4}{[\,0\,]}, \tag{7.82i}$$

$$\text{or else,} \qquad \underset{4\times 4}{[Q(\cdot\cdot)]} \equiv \underset{4\times 4}{[\,0\,]}. \tag{7.82ii}$$

For the class referred to in (7.82i), we must have

$$\underset{3\times 3}{[\,\mathbb{Q}\,]} = \underset{3\times 3}{\left[\mathbb{Q}_{(\alpha)(\beta)}(\cdot\cdot)\right]} \not\equiv \underset{3\times 3}{[\,0\,]}. \tag{7.83}$$

So, for this class, we need to obtain *canonical or normal forms* of 3×3 complex matrices. In Appendix 3, we have discussed Jordan canonical forms of $N \times N$ real matrices. Presently, we need to investigate canonical or normal forms of $N \times N$ *complex matrices* (specially for $N = 3$). For the sake of generality, we shall present the normal forms of an $N \times N$ *complex matrix* [133, 177] in the following[4]:

$$\underset{N\times N}{[\,M\,]_{(J)}} = \begin{bmatrix} \boxed{A_{(1)}} & & & 0 \\ & \boxed{A_{(2)}} & & \\ & & \ddots & \\ 0 & & & \boxed{A_{(k)}} \end{bmatrix}, \tag{7.84i}$$

$$[A]_{(l)} := \begin{bmatrix} \boxed{J^{(1)}_{(l)}} & & & 0 \\ & \boxed{J^{(2)}_{(l)}} & & \\ & & \ddots & \\ 0 & & & \boxed{J^{(q_l)}_{(l)}} \end{bmatrix}, \tag{7.84ii}$$

$$\left[J^{(i)}_{(l)}\right] := \begin{bmatrix} \lambda_{(l)} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{(l)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & & & \lambda_{(l)} \end{bmatrix}, \tag{7.84iii}$$

$$i \in \{1, \ldots, (q_l)\}, \qquad \sum_{i=1}^{(q_l)} n^{(i)}_{(l)} = n_{(l)},$$

$$\sum_{l=1}^{k} n_{(l)} = N.$$

Here, eigenvalues $\lambda_{(l)}$ are either real or complex.

---

[4]In Appendix 3, *only real matrices* of size $N \times N$ are discussed.

In our present case with 3×3 complex matrix $\left[\mathbb{Q}_{(\alpha)(\beta)}(\cdot\cdot)\right]$ in (7.83), the class characterized by the condition (7.81iii) implies that *three eigenvalues* must satisfy:

$$\lambda_{(1)}(\cdot\cdot) + \lambda_{(2)}(\cdot\cdot) + \lambda_{(3)}(\cdot\cdot) \equiv 0. \tag{7.85}$$

The class of complex matrices, representing the conformal tensor and satisfying (7.83) belongs to *Petrov types-I*, *D*, *II*, *N, and III*. We shall elaborate these types in greater detail below. (See [210, 239].)

The *Petrov type I* is characterized by the following equations:

$$[\mathbb{Q}(\cdot\cdot)]_{(J)} = \begin{bmatrix} \lambda_{(1)}(\cdot\cdot) & & \\ & \lambda_{(2)}(\cdot\cdot) & \\ & & \lambda_{(3)}(\cdot\cdot) \end{bmatrix}. \tag{7.86}$$

(Here, eigenvalues $\lambda_{(1)}(\cdot\cdot)$, $\lambda_{(2)}(\cdot\cdot)$, and $\lambda_{(3)}(\cdot\cdot)$ are *distinct*.)

Complex eigenbivectors from (7.75i) yield

$$\mathcal{E}^{(1)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{V}_{(a)(b)}(\cdot\cdot) - \mathsf{U}_{(a)(b)}(\cdot\cdot), \tag{7.87i}$$

$$\mathcal{E}^{(2)}{}_{(a)(b)}(\cdot\cdot) = \mathrm{i} \cdot \left[\mathsf{V}_{(a)(b)}(\cdot\cdot) + \mathsf{U}_{(a)(b)}(\cdot\cdot)\right], \tag{7.87ii}$$

$$\mathcal{E}^{(3)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{W}_{(a)(b)}(\cdot\cdot). \tag{7.87iii}$$

Complex coefficient functions:

$$\begin{aligned} &\Psi_{(0)}(\cdot\cdot) = \Psi_{(4)}(\cdot\cdot) = (1/2) \cdot \left[\lambda_{(2)}(\cdot\cdot) - \lambda_{(1)}(\cdot\cdot)\right] \neq 0, \\ &\Psi_{(1)}(\cdot\cdot) = \Psi_{(3)}(\cdot\cdot) \equiv 0, \quad \Psi_{(2)}(\cdot\cdot) = -(1/2) \cdot \lambda_{(3)}(\cdot\cdot). \end{aligned} \tag{7.88}$$

$$\textit{Complex Segre characteristic} \; = [1, 1, 1]. \tag{7.89}$$

The *Petrov type D* is specified by the following conditions:

$$[\mathbb{Q}(\cdot\cdot)]_{(J)} = \begin{bmatrix} \lambda_{(1)}(\cdot\cdot) & & \\ & \lambda_{(1)}(\cdot\cdot) & \\ & & \lambda_{(3)}(\cdot\cdot) \end{bmatrix}. \tag{7.90}$$

(Here, the relation $2\lambda_{(1)}(\cdot\cdot) = -\lambda_{(3)}(\cdot\cdot)$ holds.)

Eigenbivectors:

$$\mathcal{E}^{(1)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{V}_{(a)(b)}(\cdot\cdot) - \mathsf{U}_{(a)(b)}(\cdot\cdot), \tag{7.91i}$$

$$\mathcal{E}(\cdot\cdot) := c_{(1)} \cdot \mathcal{E}^{(1)}{}_{(a)(b)}(\cdot\cdot) + c_{(2)} \cdot \mathcal{E}^{(2)}{}_{(a)(b)}(\cdot\cdot), \tag{7.91ii}$$

$$\mathcal{E}^{(3)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{W}_{(a)(b)}(\cdot\cdot). \tag{7.91iii}$$

Complex coefficient functions:

$$\Psi_{(0)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) = \Psi_{(3)}(\cdot\cdot) = \Psi_{(4)}(\cdot\cdot) \equiv 0,$$
$$\Psi_{(2)}(\cdot\cdot) = -(1/2) \cdot \lambda_{(3)}(\cdot\cdot) \neq 0. \tag{7.92}$$
$$\textit{Complex Segre characteristic } = [(1, 1), 1]\,. \tag{7.93}$$

For the *Petrov type II*, we have the criteria

$$[\mathbb{Q}(\cdot\cdot)]_{(J)} = \begin{bmatrix} -(\lambda/2) & 1 & 0 \\ 0 & -(\lambda/2) & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{7.94i}$$

However, we shall investigate *the equivalent symmetric matrix*

$$[\mathbb{Q}(\cdot\cdot)] = \begin{bmatrix} 1-(\lambda/2) & -\mathrm{i} & 0 \\ -\mathrm{i} & -1-(\lambda/2) & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{7.94ii}$$

Eigenbivectors:

$$\mathcal{E}^{(1)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{V}_{(a)(b)}(\cdot\cdot), \tag{7.95i}$$
$$\mathcal{E}^{(3)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{W}_{(a)(b)}(\cdot\cdot). \tag{7.95ii}$$

Complex coefficient functions:

$$\Psi_{(0)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) = \Psi_{(3)}(\cdot\cdot) \equiv 0,$$
$$\Psi_{(2)}(\cdot\cdot) = -(\lambda/2) \neq 0, \quad \Psi_{(4)}(\cdot\cdot) \equiv -2. \tag{7.96}$$
$$\textit{Complex Segre characteristic } = [2, 1]\,. \tag{7.97}$$

Now we shall deal with *Petrov type N*. The required equations are furnished here.

$$[\mathbb{Q}(\cdot\cdot)]_{(J)} = \begin{array}{|cc|c|} \hline 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}\,. \tag{7.98i}$$

The equivalent symmetric matrix is given by

$$[\mathbb{Q}(\cdot\cdot)] = \begin{bmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.98ii}$$

Eigenbivectors:

$$\mathcal{E}^{(1)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{V}_{(a)(b)}(\cdot\cdot), \tag{7.99i}$$
$$\mathcal{E}^{(3)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{W}_{(a)(b)}(\cdot\cdot). \tag{7.99ii}$$

Complex coefficient functions:

$$\begin{aligned}
&\Psi_{(0)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) = \Psi_{(2)}(\cdot\cdot) = \Psi_{(3)}(\cdot\cdot) \equiv 0, \\
&\Psi_{(4)}(\cdot\cdot) \equiv -2. \qquad (7.100)
\end{aligned}$$

$$\textit{Complex Segre characteristic} \ = [(2,1)] . \qquad (7.101)$$

The *Petrov type III* requires the following conditions:

$$[\mathbb{Q}(\cdot\cdot)]_{(J)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (7.102\text{i})$$

The equivalent symmetric matrix can be represented by

$$[\mathbb{Q}(\cdot\cdot)] = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{bmatrix}. \qquad (7.102\text{ii})$$

Eigenbivectors:

$$\mathcal{E}^{(1)}{}_{(a)(b)}(\cdot\cdot) = \mathsf{V}_{(a(b)}(\cdot\cdot). \qquad (7.103)$$

Complex coefficient functions:

$$\begin{aligned}
&\Psi_{(0)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) = \Psi_{(2)}(\cdot\cdot) = \Psi_{(4)}(\cdot\cdot) \equiv 0, \\
&\Psi_{(3)}(\cdot\cdot) \equiv -\mathrm{i}. \qquad (7.104)
\end{aligned}$$

$$\textit{Complex Segre characteristic} \ = [3]. \qquad (7.105)$$

The remaining *Petrov type 0* is governed by condition (7.82ii). In this case, the conformal tensor has the properties

$$\mathbf{C}_{....}(x) \equiv \mathbf{0}_{....}(x), \quad \underset{4\times4}{[Q(\cdot\cdot)]} \equiv \underset{4\times4}{[\,0\,]}, \quad \underset{3\times3}{[\mathbb{Q}(\cdot\cdot)]} \equiv \underset{3\times3}{[\,0\,]}. \qquad (7.106\text{i})$$

$$\Psi_{(0)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) = \Psi_{(2)}(\cdot\cdot) = \Psi_{(3)}(\cdot\cdot) = \Psi_{(4)}(\cdot\cdot) \equiv 0. \qquad (7.106\text{ii})$$

Every nonzero complex three-dimensional vector is an eigenbivector for $[\mathbb{Q}] \equiv [\,0\,]$. Moreover, the complex Segre characteristic is provided by $[(1,1,1)]$.

Example 7.1.10. Let us consider the *Brinkman–Robinson–Trautman metric* provided by

$$\begin{aligned}
\mathrm{d}s^2 &= 2\,\mathrm{d}\zeta \cdot \mathrm{d}\overline{\zeta} - 2\,\mathrm{d}u \cdot \mathrm{d}v - 2\,h\left(\zeta, \overline{\zeta}, u\right) \cdot (\mathrm{d}u)^2 \\
&=: 2\,\mathrm{d}x^1 \cdot \mathrm{d}x^2 - 2\,\mathrm{d}x^3 \cdot \mathrm{d}x^4 - 2\,h\left(x^1, x^2, x^3\right) \cdot \left(\mathrm{d}x^3\right)^2 . \qquad (7.107)
\end{aligned}$$

(Here, x^1 and x^2 are *complex conjugate coordinates* of (A2.38).)

The corresponding (natural) null tetrads are furnished by

$$\vec{\mathbf{m}}(\cdot\cdot) = \partial_{(1)} = \partial_1 = \partial_\zeta, \tag{7.108i}$$

$$\vec{\overline{\mathbf{m}}}(\cdot\cdot) = \partial_{(2)} = \partial_2 = \partial_{\overline{\zeta}}, \tag{7.108ii}$$

$$\vec{\mathbf{l}}(\cdot\cdot) = \partial_3 - h(\cdot\cdot)\cdot\partial_4 = \partial_u - h(\cdot\cdot)\cdot\partial_v, \tag{7.108iii}$$

$$\vec{\mathbf{k}}(\cdot\cdot) = \partial_{(4)} = \partial_4 = \partial_v. \tag{7.108iv}$$

The nonzero Christoffel symbols, Riemann and Ricci tensor components, and curvature scalar for this metric are provided by

$$\begin{Bmatrix} 1 \\ 3\,3 \end{Bmatrix} = \frac{\partial h(\cdot\cdot)}{\partial x^2} = \begin{Bmatrix} 4 \\ 2\,3 \end{Bmatrix}, \quad \begin{Bmatrix} 2 \\ 3\,3 \end{Bmatrix} = \frac{\partial h(\cdot\cdot)}{\partial x^1}, \quad \frac{\partial h(\cdot\cdot)}{\partial x^2} = \begin{Bmatrix} 4 \\ 1\,3 \end{Bmatrix},$$
$$\begin{Bmatrix} 4 \\ 3\,3 \end{Bmatrix} = \frac{\partial h(\cdot\cdot)}{\partial x^3}. \tag{7.109i}$$

$$R_{1313} = \frac{\partial^2 h(\cdot\cdot)}{(\partial x^1)^2}, \quad R_{1323} = \frac{\partial^2 h(\cdot\cdot)}{\partial x^2 \partial x^1}, \quad R_{2323} = \frac{\partial^2 h(\cdot\cdot)}{(\partial x^2)^2}, \tag{7.109ii}$$

$$R_{33} = -2\cdot\frac{\partial^2 h(\cdot\cdot)}{\partial x^2 \partial x^1}, \qquad R = 0. \tag{7.109iii}$$

The complex conformal tensor components, from (7.107), (7.69), (7.73), (7.74i–iv), and (7.77), are given by

$$\Psi_{(0)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) = \Psi_{(2)}(\cdot\cdot) = \Psi_{(3)}(\cdot\cdot) \equiv 0, \tag{7.110i}$$

$$\frac{1}{2}\zeta_{(a)(b)(c)(d)}(\cdot\cdot) = \Psi_{(4)}(\cdot\cdot)\cdot \mathsf{V}_{(a)(b)}(\cdot\cdot)\cdot \mathsf{V}_{(c)(d)}(\cdot\cdot), \tag{7.110ii}$$

$$\Psi_{(4)}(\cdot\cdot) = \partial_2\partial_2 h(\cdot\cdot) = \partial_{\overline{\zeta}}\,\partial_{\overline{\zeta}}\, h(\cdot\cdot), \tag{7.110iii}$$

$$\mathsf{V}_{(a)(b)}(\cdot\cdot) = k_{(a)}\cdot m_{(b)} - k_{(b)}\cdot m_{(a)}. \tag{7.110iv}$$

In case

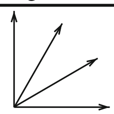
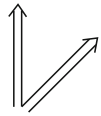
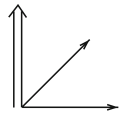
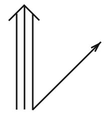
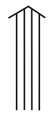
$$\left|\partial_{\overline{\zeta}}\,\partial_{\overline{\zeta}}\, h(\cdot\cdot)\right| > 0, \tag{7.111}$$

we have the Petrov type N with the principal null direction along $\vec{\mathbf{k}}(\cdot\cdot)$ (from (7.100) and (7.99i,ii)).

The differential inequality (7.111) can be solved with help of a real-valued, arbitrary *slack function* $\sigma\left(\zeta, \overline{\zeta}, u, v\right)$ as

$$h(\cdot\cdot) = \pm\int \exp\left[\sigma(\cdot\cdot)\right]\cdot \mathrm{d}\overline{\zeta}\,\mathrm{d}\overline{\zeta} + c_1(\zeta\cdot\overline{\zeta}) + c_2, \tag{7.112}$$

Table 7.1 Complex Segre characteristics and principal null directions for various Petrov types

Petrov types	Complex Segre characteristics	Multiplicity of null eigenvector $\vec{\mathbf{k}}(x)$
I	[1, 1, 1]	
D	[(1, 1), 1]	
II	[2, 1]	
III	[3]	
N	[(2, 1)]	

or

$$\left|\partial_{\bar{\zeta}} \cdot \partial_{\bar{\zeta}} h\right| = \exp[\sigma(\cdot\cdot)] > 0.$$

Here, c_1 and c_2 are two arbitrary real constants of integration. The nonvanishing $\Psi_{(4)}(\cdot\cdot)$ implies that the null eigenvector $\vec{\mathbf{k}}(x)$ is of *multiplicity four*. Thus, the metric under consideration physically describes (non-vacuum) *transverse gravitational waves*. (More detailed discussions on the metric will be provided in Example 7.2.5.)
□

The invariant eigenvalue problem for the complex-valued conformal tensor components $\zeta_{(a)(b)(c)(d)}(\cdot\cdot)$ is posed in (7.75i,ii). The corresponding eigenbivectors $\mathcal{E}_{(a)(b)}(\cdot\cdot)$ are explicitly provided in (7.87i–iii), (7.91i–iii), (7.95i,ii), (7.99i,ii), and (7.103). (We are not considering the Petrov type 0.) These eigenbivectors are furnished by the wedge products of *n*ull vectors of $\left\{\vec{\mathbf{m}}, \vec{\bar{\mathbf{m}}}, \vec{\mathbf{l}}, \vec{\mathbf{k}}\right\}$. Therefore, each of the Petrov type is intricately associated with four-dimensional (real or complex) null vectors. The Jordan canonical forms of the complex 3×3 matrix $\left[\mathbb{Q}_{(\alpha)(\beta)}(\cdot\cdot)\right]$ in (7.86), (7.90), (7.94i), (7.98i), and (7.102i) are adapted to the preferred, real null vector $\vec{\mathbf{k}}(x)$. This null vector also physically represents the direction of propagation of the corresponding "gravitational wave" disturbance. Table 7.1, we depict the Petrov type, together with the corresponding multiplicity of the null vector $\vec{\mathbf{k}}(x)$.

Now, we shall explore the canonical forms of the 4×4 real, symmetric matrix $\left[R_{(a)(b)}(\cdot\cdot)\right]$ arising out of *the Ricci tensor*. There exist *nineteen Segre characteristics* related to this matrix, which are symbolically exhibited below (from Examples A3.8 and A3.12):

$$[1,1,1,1]\,,\quad [(1,1),1,1]\,,\quad [(1,1),(1,1)]\,,\quad [(1,1,1),1]\,,\quad [(1,1,1,1)]\,;$$

$$[1,1,2]\,,\quad [(1,1),2]\,,\quad [1,(1,2)]\,,\quad [(1,1,2)]\,;$$

$$[2,2]\,,\quad [(2,2)]\,;$$

$$[1,3]\,,\quad [(1,3)]\,;$$

$$[4]\,;$$

$$\left[1,1;1,\overline{1}\right],\quad \left[(1,1);1,\overline{1}\right],\quad [2;\,1,\,\overline{1}],\quad [1,\,\overline{1};\,1,\,\overline{1}],\quad [(1,\,\overline{1};\,1,\,\overline{1})]. \tag{7.113}$$

Example 7.1.11. We will explicitly show the Jordan canonical form of a matrix endowed with Segre characteristic $\left[1,1;1,\overline{1}\right]$. It is furnished by

$$\left[d^{(a)(c)}R_{(c)(b)}\right]_{(J)} = \left[R^{(a)}{}_{(b)}\right]_{(J)} = \left[\begin{array}{cc|cc} \lambda_{(1)}(\cdot\cdot) & 0 & & \\ 0 & \lambda_{(2)}(\cdot\cdot) & \multicolumn{2}{c}{0} \\ \hline & & a(\cdot\cdot) & b(\cdot\cdot) \\ \multicolumn{2}{c|}{0} & -b(\cdot\cdot) & a(\cdot\cdot) \end{array}\right]. \tag{7.114}$$

Here, we have assumed that $\lambda_{(1)}(x) \neq \lambda_{(2)}(x)$, and the complex conjugate eigenvalues are $\lambda(x) = a(\cdot\cdot) + i\,b(\cdot\cdot)$, $\overline{\lambda}(x) = a(\cdot\cdot) - i\,b(\cdot\cdot)$, $b(\cdot\cdot) \neq 0$. □

Now, Einstein's field equations (2.162ii) state that

$$R_{(a)(b)}(\cdot\cdot) = -\kappa \cdot \left[T_{(a)(b)}(\cdot\cdot) - (1/2)\cdot d_{(a)(b)} \cdot T^{(c)}{}_{(c)}(\cdot\cdot)\right]. \tag{7.115}$$

Let the invariant eigenvalue problem for $\kappa T^{(a)}{}_{(b)}(\cdot\cdot)$ be furnished by

$$\left[\kappa \cdot T^{(a)}{}_{(b)}(\cdot\cdot) - \lambda(\cdot\cdot)\cdot \delta^{(a)}{}_{(b)}\right] \cdot E^{(b)}(\cdot\cdot) = 0. \tag{7.116}$$

From (7.115) and (7.116), we can derive that

$$\left\{R^{(a)}{}_{(b)}(\cdot\cdot) + \left[-\frac{\kappa}{2}\cdot T^{(c)}{}_{(c)}(\cdot\cdot) + \lambda(\cdot\cdot)\right]\cdot \delta^{(a)}{}_{(b)}\right\} \cdot E^{(b)} = 0. \tag{7.117}$$

Equation above demonstrates that $E^{(a)}(\cdot\cdot)$ is also an eigenvector for the matrix $\left[R^{(a)}{}_{(b)}(\cdot\cdot)\right]$ with invariant eigenvalue $-\lambda(\cdot\cdot)+\frac{\kappa}{2}\cdot T^{(c)}{}_{(c)}(\cdot\cdot)$. Of course, the corresponding invariant eigenvalue of the matrix $\left[R_{(a)(b)}(\cdot\cdot)\right]$ is exactly $-\lambda(\cdot\cdot)+\frac{\kappa}{2}\cdot T^{(c)}{}_{(c)}(\cdot\cdot)$. (We have discussed some of the canonical forms of the energy–momentum stress tensor in Example A3.11.) Therefore, the Segre characteristic of $[R_{ab}]$ is exactly the same as that of the corresponding $[T_{ab}]$ provided the field equations hold.

Next, we shall present a useful decomposition of the curvature tensor with the help of the conformal tensor of (1.169ii). It is given by

$$R_{(a)(b)(c)(d)}(x) = C_{(a)(b)(c)(d)}(x) + E_{(a)(b)(c)(d)}(x) + G_{(a)(b)(c)(d)}(x), \tag{7.118i}$$

$$S_{(a)(b)}(x) := R_{(a)(b)}(\cdot\cdot) - (1/4)\cdot R(\cdot\cdot)\cdot d_{(a)(b)} \equiv S_{(b)(a)}(\cdot\cdot),$$

$$S^{(a)}{}_{(a)}(\cdot\cdot) \equiv 0, \tag{7.118ii}$$

$$\begin{aligned} E_{(a)(b)(c)(d)}(x) &:= (1/2)\cdot\big[d_{(a)(d)}\cdot S_{(b)(c)}(\cdot\cdot) + d_{(b)(c)}\cdot S_{(a)(d)}(\cdot\cdot) \\ &\quad - d_{(a)(c)}\cdot S_{(b)(d)}(\cdot\cdot) - d_{(b)(d)}\cdot S_{(a)(c)}(\cdot\cdot)\big], \end{aligned} \tag{7.118iii}$$

$$G_{(a)(b)(c)(d)}(x) := -(1/12)\cdot R(\cdot\cdot)\cdot\big[d_{(a)(c)}\cdot d_{(b)(d)} - d_{(a)(d)}\cdot d_{(b)(c)}\big]. \tag{7.118iv}$$

Now, analogous to (7.74i–v), we shall introduce *complex components* $\Phi_{(.)(.)}(\cdot\cdot)$ which are related to the components $S_{(a)(b)}(x)$. These are furnished by

$$-\Phi_{(0)(0)}(\cdot\cdot) := (1/2)\cdot S_{(a)(b)}(\cdot\cdot)\cdot k^{(a)}(\cdot\cdot)\cdot k^{(b)}(\cdot\cdot) = -\overline{\Phi}_{(0)(0)}(\cdot\cdot), \tag{7.119i}$$

$$-\Phi_{(0)(1)}(\cdot\cdot) := (1/2)\cdot S_{(a)(b)}(\cdot\cdot)\cdot k^{(a)}(\cdot\cdot)\cdot m^{(b)}(\cdot\cdot) = -\overline{\Phi}_{(1)(0)}(\cdot\cdot), \tag{7.119ii}$$

$$-\Phi_{(0)(2)}(\cdot\cdot) := (1/2)\cdot S_{(a)(b)}(\cdot\cdot)\cdot m^{(a)}(\cdot\cdot)\cdot m^{(b)}(\cdot\cdot) = -\overline{\Phi}_{(2)(0)}(\cdot\cdot), \tag{7.119iii}$$

$$\begin{aligned} -\Phi_{(1)(1)}(\cdot\cdot) &:= (1/4)\cdot S_{(a)(b)}(\cdot\cdot)\cdot\left[k^{(a)}(\cdot\cdot)\cdot l^{(b)}(\cdot\cdot) + m^{(a)}(\cdot\cdot)\cdot\overline{m}^{(b)}(\cdot\cdot)\right] \\ &= -\overline{\Phi}_{(1)(1)}(\cdot\cdot), \end{aligned} \tag{7.119iv}$$

$$-\Phi_{(1)(2)}(\cdot\cdot) := (1/2)\cdot S_{(a)(b)}(\cdot\cdot)\cdot l^{(a)}(\cdot\cdot)\cdot m^{(b)}(\cdot\cdot) = -\overline{\Phi}_{(2)(1)}(\cdot\cdot), \tag{7.119v}$$

$$-\Phi_{(2)(2)}(\cdot\cdot) := (1/2)\cdot S_{(a)(b)}(\cdot\cdot)\cdot l^{(a)}(\cdot\cdot)\cdot l^{(b)}(\cdot\cdot) = -\overline{\Phi}_{(2)(2)}(\cdot\cdot). \tag{7.119vi}$$

Caution: Complex field components $\Phi_{(0)(0)}(\cdot\cdot), \Phi_{(0)(1)}(\cdot\cdot), \ldots, \Phi_{(2)(2)}(\cdot\cdot)$ *must not be confused with electromagnetic complex field components* $\phi_{(0)}(\cdot\cdot)$, $\phi_{(1)}(\cdot\cdot)$, *and* $\phi_{(2)}(\cdot\cdot)$ *of (7.56i–iii), or complex field* $\varphi_{(a)(b)}(\cdot\cdot)$.

We shall use the decomposition and definitions above to derive the Newman–Penrose equations in the next section.

Exercises 7.1

1. Consider the metric tensor components $d_{(a)(b)(c)(d)} = d_{(a)(c)} \cdot d_{(b)(d)} - d_{(a)(d)} \cdot d_{(b)(c)}$ for the bivector space given in (7.10ii). These components can be mapped by (7.15) into $D_{(A)(B)}$ with $1 \leq A < B \leq 6$. *Obtain explicitly the* 6×6 *matrix* $\left[D_{(A)(B)}\right]$ in terms of $d_{(a)(b)}$.
2. Consider a real 2-form $(1/2) \cdot W_{(a)(b)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot)$ and the corresponding invariant eigenvalue equation (7.36ii). Now, pose the usual, noninvariant eigenvalue problem: $W_{(a)(b)}(\cdot\cdot) \cdot \varepsilon^{(b)}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot \delta_{(a)(b)} \cdot \varepsilon^{(b)}(\cdot\cdot)$. Show that every, real eigenvalue $\lambda(x) \equiv 0$.
3. Using definition (7.41) of the complex 2-form $(1/2) \cdot \mathcal{S}_{(a)(b)}(\cdot\cdot) \cdot \widetilde{\mathbf{e}}^{(a)}(\cdot\cdot) \wedge \widetilde{\mathbf{e}}^{(b)}(\cdot\cdot)$, prove that
$$^{***}\mathcal{S}_{(a)(b)}(\cdot\cdot) = -\mathrm{i}\mathcal{S}_{(a)(b)}(\cdot\cdot).$$
4. Consider the basic self-dual tensor field components provided in (7.72i–v). Prove that each of the tensor field components $\mathcal{E}^{(J)}{}_{(a)(b)(c)(d)}(\cdot\cdot)$ satisfies the algebraic identities in (7.71i–iv).
5. Compute explicitly the complex-valued components $Q^{(a)(b)}(\cdot\cdot)$ of (7.79i–viii) from (7.82i), (7.83), (7.77), and (7.78).
6. Prove (7.49i–iii) for self-duality.
7. Verify (7.74i–v) for the five complex scalar fields $\Psi_{(0)}(\cdot\cdot)$, $\Psi_{(1)}(\cdot\cdot)$, $\Psi_{(2)}(\cdot\cdot)$, $\Psi_{(3)}(\cdot\cdot)$, and $\Psi_{(4)}(\cdot\cdot)$.
8. Verify the eigenbivectors in (7.87i–iii) for the Petrov type I.

Answers and Hints to Selected Exercises

1. *The diagonal matrix* $\left[D_{(A)(B)}\right]$ is furnished by

$$\left[D_{(A)(B)}\right] = \begin{bmatrix} d_{(1)(1)} \cdot d_{(2)(2)} & & & & & \\ & d_{(1)(1)} \cdot d_{(3)(3)} & & & & \\ & & d_{(1)(1)} \cdot d_{(4)(4)} & & & \\ & & & d_{(2)(2)} \cdot d_{(3)(3)} & & \\ & & & & d_{(2)(2)} \cdot d_{(4)(4)} & \\ & & & & & d_{(3)(3)} \cdot d_{(4)(4)} \end{bmatrix}.$$

(Note that the above matrix is *identical to that in* (7.18).)

2. $0 \equiv W_{(a)(b)}(\cdot\cdot) \cdot \varepsilon^{(b)}(\cdot\cdot) \cdot \varepsilon^{(a)}(\cdot\cdot) = \lambda(\cdot\cdot) \cdot \delta_{(a)(b)} \cdot \varepsilon^{(a)}(\cdot\cdot) \cdot \varepsilon^{(b)}(\cdot\cdot)$.

   Dividing equation above by the positive-valued function $\delta_{(a)(b)} \cdot \varepsilon^{(a)}(\cdot\cdot) \cdot \varepsilon^{(b)}(\cdot\cdot)$, the conclusion can be reached.

4. Try out $\mathcal{E}^{(4)}{}_{(a)(b)(c)(d)}(\cdot\cdot) = \mathsf{V}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{V}_{(c)(d)}(\cdot\cdot)$ for the sake of simplicity. From $\mathsf{V}_{(b)(a)}(\cdot\cdot) \equiv -\mathsf{V}_{(a)(b)}(\cdot\cdot)$, the identity (7.71i) is satisfied. By $\mathsf{V}_{(c)(d)}(\cdot\cdot) \cdot \mathsf{V}_{(a)(b)}(\cdot\cdot) \equiv \mathsf{V}_{(a)(b)}(\cdot\cdot) \cdot \mathsf{V}_{(c)(d)}(\cdot\cdot)$, the identity (7.71ii) can be established.
   Now, $\mathsf{V}_{(a)(b)}(\cdot\cdot) = k_{(a)}(\cdot\cdot) \cdot m_{(b)}(\cdot\cdot) - k_{(b)}(\cdot\cdot) \cdot m_{(a)}(\cdot\cdot)$. Therefore,

$$\begin{aligned}
&\left(k_{(a)} \cdot m_{(b)} - k_{(b)} \cdot m_{(a)}\right) \cdot \left(k_{(c)} \cdot m_{(d)} - k_{(d)} \cdot m_{(c)}\right) \\
&\quad + \left(k_{(a)} \cdot m_{(c)} - k_{(c)} \cdot m_{(a)}\right) \cdot \left(k_{(d)} \cdot m_{(b)} - k_{(b)} \cdot m_{(d)}\right) \\
&\quad + \left(k_{(a)} \cdot m_{(d)} - k_{(d)} \cdot m_{(a)}\right) \cdot \left(k_{(b)} \cdot m_{(c)} - k_{(c)} \cdot m_{(b)}\right) \equiv 0
\end{aligned}$$

   proves identity (7.71iii). Now, $\mathsf{V}^{(d)}{}_{(a)}(\cdot\cdot) = k^{(d)}(\cdot\cdot) \cdot m_{(a)}(\cdot\cdot) - k_{(a)}(\cdot\cdot) \cdot m^{(d)}(\cdot\cdot)$. Therefore,

$$\begin{aligned}
\mathsf{V}^{(d)}{}_{(a)}(\cdot\cdot) \cdot \mathsf{V}_{(b)(d)}(\cdot\cdot) &= \left(k^{(d)} \cdot m_{(a)} - k_{(a)} \cdot m^{(d)}\right) \cdot \left(k_{(b)} \cdot m_{(d)} - k_{(d)} \cdot m_{(b)}\right) \\
&= \left(m_{(a)} \cdot k_{(b)}\right) \cdot \left(k^{(d)} \cdot m_{(d)}\right) - \left(k^{(d)} \cdot k_{(d)}\right) \cdot \left(m_{(a)} \cdot m_{(b)}\right) \\
&\quad - \left(k_{(a)} \cdot k_{(b)}\right) \cdot \left(m^{(d)} \cdot m_{(d)}\right) + \left(k_{(a)} \cdot m_{(b)}\right) \cdot \left(m^{(d)} \cdot k_{(d)}\right) \equiv 0.
\end{aligned}$$

   Thus, we have proved the identity (7.71iv). Similarly, the identities (7.71i–iv) can be proved for the tensor field components $\mathcal{E}^{(0)}_{(a)(b)(c)(d)}(\cdot\cdot)$, $\mathcal{E}^{(1)}_{(a)(b)(c)(d)}(\cdot\cdot)$, $\mathcal{E}^{(2)}_{(a)(b)(c)(d)}(\cdot\cdot)$, and $\mathcal{E}^{(3)}_{(a)(b)(c)(d)}(\cdot\cdot)$.

5. Recall (consulting the answer to the next Problem #6) that

$$\begin{aligned}
\mathsf{U}^{(a)(4)}(\cdot\cdot) &= (1/2) \cdot \left(\delta^{(a)}{}_{(1)} + \mathrm{i}\,\delta^{(a)}{}_{(2)}\right), \\
\mathsf{V}^{(a)(4)}(\cdot\cdot) &= -(1/2) \cdot \left(\delta^{(a)}{}_{(1)} - \mathrm{i}\,\delta^{(a)}{}_{(2)}\right), \\
\mathsf{W}^{(a)(4)}(\cdot\cdot) &= -\delta^{(a)}{}_{(3)}.
\end{aligned}$$

   Suspending ( ) around indices for simplicity, the following equation is provided:

$$\begin{aligned}
&(1/2) \cdot Q^{ad}(\cdot\cdot) \\
&\quad = \Psi_0(\cdot\cdot) \cdot \left\{-(1/4) \cdot \left[\left(\delta^a{}_1 \cdot \delta^d{}_1 - \delta^a{}_2 \cdot \delta^d{}_2\right) + \mathrm{i} \cdot \left(\delta^a{}_1 \cdot \delta^d{}_2 + \delta^d{}_1 \cdot \delta^a{}_2\right)\right]\right\} \\
&\qquad + \Psi_1(\cdot\cdot) \cdot \left\{(1/2) \cdot \left[\left(\delta^a{}_1 \cdot \delta^d{}_3 + \delta^d{}_1 \cdot \delta^a{}_3\right) + \mathrm{i} \cdot \left(\delta^a{}_2 \cdot \delta^d{}_3 + \delta^d{}_2 \cdot \delta^a{}_3\right)\right]\right\} \\
&\qquad + \Psi_2(\cdot\cdot) \cdot \left\{(1/2) \cdot \left[\delta^a{}_1 \cdot \delta^d{}_1 + \delta^a{}_2 \cdot \delta^d{}_2 - 2\,\delta^a{}_3 \cdot \delta^d{}_3\right]\right\}
\end{aligned}$$

$$+\Psi_3(\cdot\cdot)\cdot\left\{-(1/2)\cdot\left[\left(\delta^a{}_1\cdot\delta^d{}_3+\delta^d{}_1\cdot\delta^a{}_3\right)-\mathrm{i}\cdot\left(\delta^a{}_2\cdot\delta^d{}_3+\delta^d{}_2\cdot\delta^a{}_3\right)\right]\right\}$$

$$+\Psi_4(\cdot\cdot)\cdot\left\{-(1/4)\cdot\left[\left(\delta^a{}_1\cdot\delta^d{}_1-\delta^a{}_2\cdot\delta^d{}_2\right)-\mathrm{i}\cdot\left(\delta^a{}_1\cdot\delta^d{}_2+\delta^d{}_1\cdot\delta^a{}_2\right)\right]\right\}.$$

6. Note that (7.4i,ii) and (7.5i,ii) imply that

$$\sqrt{2}\,l^{(a)}=\delta^{(a)}{}_{(4)}-\delta^{(a)}{}_{(3)},\ \sqrt{2}\,k^{(a)}=\delta^{(a)}{}_{(4)}+\delta^{(a)}{}_{(3)},$$

$$\sqrt{2}\,m^{(a)}=\delta^{(a)}{}_{(1)}-\mathrm{i}\delta^{(a)}{}_{(2)},\ \sqrt{2}\,\overline{m}^{(a)}=\delta^{(a)}{}_{(1)}+\mathrm{i}\delta^{(a)}{}_{(2)}.$$

Consider the complex bivector

$$\begin{aligned}2\mathsf{V}^{(a)(b)}(\cdot\cdot)=&\left[\delta^{(a)}{}_{(4)}\cdot\delta^{(b)}{}_{(1)}+\delta^{(a)}{}_{(3)}\cdot\delta^{(b)}{}_{(1)}-\delta^{(b)}{}_{(4)}\cdot\delta^{(a)}{}_{(1)}-\delta^{(b)}{}_{(3)}\cdot\delta^{(a)}{}_{(4)}\right]\\&+\mathrm{i}\left[\delta^{(a)}{}_{(2)}\cdot\delta^{(b)}{}_{(3)}+\delta^{(a)}{}_{(2)}\cdot\delta^{(b)}{}_{(4)}-\delta^{(b)}{}_{(2)}\cdot\delta^{(a)}{}_{(3)}-\delta^{(b)}{}_{(2)}\cdot\delta^{(a)}{}_{(3)}\right],\\=:&\ W^{(a)(b)}(\cdot\cdot)-\mathrm{i}{}^*W^{(a)(b)}(\cdot\cdot).\end{aligned}$$

Therefore, by the self-duality condition in (7.43), (7.49ii) is derived. Similarly, the other equations (7.49i, iii) are proven.

7. By (7.73), (7.77), and (7.50i–iv),

$$\begin{aligned}&\zeta_{(a)(b)(c)(d)}(\cdot\cdot)\cdot\mathsf{V}^{(a)(b)}(\cdot\cdot)\cdot\mathsf{V}^{(c)(d)}(\cdot\cdot)\\&\quad=2\cdot\left\{\Psi_{(0)}\cdot\left[\mathsf{U}_{(a)(b)}\cdot\mathsf{U}_{(c)(d)}\right]\cdot\mathsf{V}^{(a)(b)}\cdot\mathsf{V}^{(c)(d)}+0+0+0+0\right\}=8\cdot\Psi_{(0)}\end{aligned}$$

Now,

$$C_{(a)(b)(c)(d)}\cdot\mathsf{V}^{(a)(b)}\cdot\mathsf{V}^{(c)(d)}=4\cdot C_{(a)(b)(c)(d)}\cdot k^{(a)}\cdot m^{(b)}\cdot k^{(c)}\cdot m^{(d)}.$$

Moreover,

$$\begin{aligned}&-\mathrm{i}\,{}^*C_{(a)(b)(c)(d)}\cdot\mathsf{V}^{(a)(b)}\cdot\mathsf{V}^{(c)(d)}\\&\quad=-\frac{\mathrm{i}}{2}\cdot\eta_{(a)(b)(p)(q)}\cdot C^{(p)(q)}{}_{(c)(d)}\cdot\mathsf{V}^{(a)(b)}\cdot\mathsf{V}^{(c)(d)}\\&\quad=-\mathrm{i}C^{(p)(q)}{}_{(c)(d)}\cdot\mathsf{V}_{(p)(q)}\cdot V^{(c)(d)}=C_{(p)(q)(c)(d)}\mathsf{V}^{(p)(q)}\cdot\mathsf{V}^{(c)(d)}.\end{aligned}$$

Thus,

$$\begin{aligned}\Psi_{(0)}(\cdot\cdot)&=\zeta_{(a)(b)(c)(d)}\cdot\mathsf{V}^{(a)(b)}\cdot\mathsf{V}^{(c)(d)}\\&=\frac{1}{8}\cdot\left[C_{(a)(b)(c)(d)}-\mathrm{i}\,{}^*C_{(a)(b)(c)(d)}\right]\cdot\mathsf{V}^{(a)(b)}\cdot\mathsf{V}^{(c)(d)}\end{aligned}$$

$$= \frac{1}{8} [4 + 4] \cdot C_{(a)(b)(c)(d)} k^{(a)} \cdot m^{(b)} \cdot k^{(c)} \cdot m^{(d)}$$

$$= C_{(a)(b)(c)(d)} k^{(a)} \cdot m^{(b)} \cdot k^{(c)} \cdot m^{(d)}.$$

Thus, (7.74i) is proven. Similarly, other equations (7.74ii–v) can be established.

8. Equations (7.75i) and (7.77) imply that

$$\big\{\Psi_{(0)} \cdot \mathsf{U}_{ab} \cdot \mathsf{U}_{cd} + \Psi_{(1)} \cdot [\mathsf{U}_{ab} \cdot \mathsf{W}_{cd} + \mathsf{W}_{ab} \cdot \mathsf{U}_{cd}]$$

$$+ \Psi_{(2)} \cdot [\mathsf{V}_{ab} \cdot \mathsf{U}_{cd} + \mathsf{U}_{ab} \cdot \mathsf{V}_{cd} + \mathsf{W}_{ab} \cdot \mathsf{W}_{cd}] + \Psi_{(3)} \cdot [\mathsf{V}_{ab} \cdot \mathsf{W}_{cd} + \mathsf{W}_{ab} \cdot \mathsf{V}_{cd}]$$

$$+ \Psi_{(4)} \cdot \mathsf{V}_{ab} \cdot \mathsf{V}_{cd}\big\} \cdot \mathcal{E}^{cd} = 2\,\lambda \cdot \mathcal{E}_{ab}.$$

Consider the eigenbivector $\mathcal{E}^{(1)}{}_{ab} := \mathsf{V}_{ab} - \mathsf{U}_{ab}$. The eigenvalue equation above yields the following (by use of (7.50i–vi)):

$$-\Psi_{(4)} \cdot \mathsf{V}_{ab} + \Psi_{(0)} \cdot \mathsf{U}_{ab} + \Psi_{(2)} \cdot (\mathsf{V}_{ab} - \mathsf{U}_{ab}) + \big(\Psi_{(1)} - \Psi_{(3)}\big) \cdot \mathsf{W}_{ab} = \lambda_{(1)} \cdot (\mathsf{V}_{ab} - \mathsf{U}_{ab})\,.$$

By the linear independence of bivectors in (7.48i, iii, v), conclude that $\lambda_{(1)} = -\Psi_{(4)} + \Psi_{(2)} = -\Psi_{(0)} + \Psi_{(2)}$ and $\Psi_{(1)} - \Psi_{(3)} \equiv 0$. Using the eigenvector $\mathcal{E}^{(3)}{}_{ab} := \mathsf{W}_{ab}$, one can deduce from equations above that $\lambda_{(3)} = -2 \cdot \Psi_{(2)}$ and $\Psi_{(1)} = \Psi_{(3)} \equiv 0$. Similarly, employing the eigenbivector $\mathcal{E}^{(2)}{}_{ab} := \mathrm{i}\,[\mathsf{V}_{ab} + \mathsf{U}_{ab}]$, one can prove that $\lambda_{(2)} = \Psi_{(0)} + \Psi_{(2)} = \Psi_{(2)} + \Psi_{(4)}$ and $\Psi_{(1)} + \Psi_{(3)} \equiv 0$. Thus, (7.86), (7.87i–iii), and (7.88) are directly verified.

7.2 Newman–Penrose Equations

The Newman–Penrose equations [194] are *exactly equivalent to Einstein's field equations* of (2.163ii). However, (1) these are *a set of first-order partial differential equations*, and (2) *instead of a real orthonormal tetrad, the complex null tetrad of the equations* (7.6i) *is used*. (3) *These equations explicitly involve the Petrov classification of the conformal tensor*. Equations are particularly suited for extracting exact solutions involving "gravitational waves." Moreover, the Newman–Penrose equations have been exploited to discover a plethora of exact solutions in general relativity. (See [35, 118, 239].)

We have already investigated a (real) first-order version of the field equations in a harmonic coordinate chart for the classification of the partial differential equations representing the field equations in Example 2.4.6. Moreover, we have also discussed a first-order system, equivalent to the field equations, without employing special coordinates, in the answer for question #4 of Exercises 2.4. We shall recapitulate this formulation subsequently to investigate in greater depth the first-order version of the real field equations. This system of partial differential equations for *the unknown functions* $g_{ij}(x)$ *and* $\left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\}$ is furnished by

First-order p.d.e.s for unknown functions:
$$\partial_k g_{ij} = g_{jh}(\cdot\cdot) \cdot \left\{ \begin{matrix} h \\ k\ i \end{matrix} \right\} + g_{ih}(\cdot\cdot) \cdot \left\{ \begin{matrix} h \\ k\ j \end{matrix} \right\}, \tag{7.120i}$$

Integrability conditions:
$$\partial_l \left[g_{jh} \cdot \left\{ \begin{matrix} h \\ k\ i \end{matrix} \right\} + g_{ih} \cdot \left\{ \begin{matrix} h \\ k\ j \end{matrix} \right\} \right] - [l \leftrightarrow k] = 0, \tag{7.120ii}$$

Field equations: (or, first-order p.d.e.'s for unknown: $\left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\}$)
$$\partial_k \cdot \left\{ \begin{matrix} i \\ i\ j \end{matrix} \right\} - \partial_i \cdot \left\{ \begin{matrix} i \\ k\ j \end{matrix} \right\} + \left\{ \begin{matrix} i \\ k\ h \end{matrix} \right\} \cdot \left\{ \begin{matrix} h \\ i\ j \end{matrix} \right\} - \left\{ \begin{matrix} i \\ i\ h \end{matrix} \right\} \cdot \left\{ \begin{matrix} h \\ k\ j \end{matrix} \right\}$$
$$= \begin{cases} -\kappa \cdot \left[T_{kj}(\cdot\cdot) - (1/2) \cdot g_{kj}(\cdot\cdot) \cdot g^{hl}(\cdot\cdot) \cdot T_{hl}(\cdot\cdot) \right] & \text{inside sources,} \\ 0 & \text{outside sources;} \end{cases} \tag{7.120iii}$$

Contracted Bianchi identities
$$\Rightarrow \nabla_j T^{ij} = -\left(\kappa^{-1}\right) \cdot \nabla_j G^{ij} \equiv 0, \tag{7.120iv}$$

Coordinate conditions:
$$\mathcal{C}^i \left(g_{jk}, \partial_l g_{jk} \right) = 0. \tag{7.120v}$$

The above system of first-order equations is exactly equivalent to field equations (2.162i). Now, we shall provide an example of exact solutions for the system (7.120i–v).

Example 7.2.1. Consider the static, spherically symmetric metric (3.1) expressed as

$$ds^2 = e^{\alpha(x^1)} \cdot \left(dx^1\right)^2 + \left(x^1\right)^2 \cdot \left[\left(dx^2\right)^2 + \left(\sin x^2\right)^2 \cdot \left(dx^3\right)^2 \right] - e^{\gamma(x^1)} \cdot \left(dx^4\right)^2, \tag{7.121i}$$

$$g_{11}(\cdot\cdot) = e^{\alpha(x^1)}, \quad g_{22}(\cdot\cdot) = \left(x^1\right)^2, \quad g_{33}(\cdot\cdot) = \left(x^1 \cdot \sin x^2\right)^2, \quad g_{44}(\cdot\cdot) = -e^{\gamma(x^1)},$$
$$\text{other} \qquad g_{ij}(\cdot\cdot) \equiv 0. \tag{7.121ii}$$

The nonzero components of the Christoffel symbols from (3.2) are given by

$$\begin{aligned}
&\left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} = (1/2)\cdot \partial_1 \alpha, \quad \left\{ \begin{matrix} 2 \\ 1\,2 \end{matrix} \right\} = \left(x^1\right)^{-1}, \quad \left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} = -x^1 \cdot \mathrm{e}^{-\alpha}, \\
&\left\{ \begin{matrix} 3 \\ 1\,3 \end{matrix} \right\} = \left(x^1\right)^{-1}, \quad \left\{ \begin{matrix} 1 \\ 3\,3 \end{matrix} \right\} = -x^1 \cdot \left(\sin x^2\right)^2 \cdot \mathrm{e}^{-\alpha}, \\
&\left\{ \begin{matrix} 3 \\ 2\,3 \end{matrix} \right\} = \cot x^2, \quad \left\{ \begin{matrix} 2 \\ 3\,3 \end{matrix} \right\} = -\sin x^2 \cdot \cos x^2, \\
&\left\{ \begin{matrix} 4 \\ 1\,4 \end{matrix} \right\} = (1/2)\cdot \partial_1 \gamma, \quad \left\{ \begin{matrix} 1 \\ 4\,4 \end{matrix} \right\} = (1/2)\cdot \mathrm{e}^{\gamma-\alpha} \cdot \partial_1 \gamma.
\end{aligned} \tag{7.122}$$

Now, consider one of the equations in (7.120i), namely,

$$\partial_1 g_{11} = 2 g_{11}(\cdot\cdot) \cdot \left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\}.$$

By (7.121ii) and (7.122), we deduce that

$$\partial_1 g_{11} = \mathrm{e}^{\alpha(\cdot\cdot)} \cdot \partial_1 \alpha,$$

$$2 g_{11} \cdot \left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} = 2 \cdot \mathrm{e}^{\alpha(\cdot\cdot)} \cdot \left[(1/2)\cdot \partial_1 \alpha\right].$$

Thus, in this case, (7.120i) is satisfied. Now consider (7.120ii) for the case $i = j = 1$. These reduce to the following equation:

$$\partial_2 \left[2 g_{11} \cdot \left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} \right] - \partial_1 \left[g_{12} \cdot \left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} + g_{11} \cdot \left\{ \begin{matrix} 1 \\ 2\,1 \end{matrix} \right\} \right] = 0.$$

Thus, (7.120ii) is verified for $i = j = 1$. Similarly, the other equations in (7.120ii) are satisfied.

Now consider *the vacuum field equations* by choosing $T_{ij}(x) \equiv 0$ in (7.120iii), so that $R_{ij}(x) = 0$. Let us try to compute the particular component $R_{22}(\cdot\cdot)$ from (7.120iii) and (7.122). It is provided by explicit computation as

$$\begin{aligned}
R_{22}(\cdot\cdot) &= \partial_2 \left\{ \begin{matrix} 3 \\ 2\,3 \end{matrix} \right\} - \partial_1 \left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} + \left\{ \begin{matrix} 3 \\ 2\,3 \end{matrix} \right\}^2 \\
&\quad - \left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} \cdot \left[\left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} + \left\{ \begin{matrix} 4 \\ 1\,4 \end{matrix} \right\} + \left\{ \begin{matrix} 2 \\ 1\,2 \end{matrix} \right\} + \left\{ \begin{matrix} 3 \\ 1\,3 \end{matrix} \right\} \right] \\
&= \partial_2 \left[\cot x^2\right] + \partial_1 \left[x^1 \mathrm{e}^{-\alpha}\right] + \left(\cot x^2\right)^2 + x^1 \cdot \mathrm{e}^{-\alpha}
\end{aligned}$$

$$\times \left[(1/2)\partial_1\alpha + (1/2)\partial_4\gamma + \frac{2}{x^1}\right]$$
$$= e^{-\alpha(\cdot\cdot)} \cdot \left[1 + \left(x^1/2\right) \cdot (\partial_1\gamma - \partial_1\alpha)\right] - 1,$$
$$R_{33}(\cdot\cdot) \equiv \left(\sin x^2\right)^2 \cdot R_{22}(\cdot\cdot).$$

Similarly, the other nonzero components, $R_{11}(\cdot\cdot)$ and $R_{44}(\cdot\cdot)$m can be computed. Solving the partial differential equations inherent in *the Ricci-flat conditions*, $R_{ij}(\cdot\cdot) = 0$, the well-known Schwarzschild metric of (3.9) emerges. □

Now we shall consider another first-order system of equations equivalent to field equations (2.163ii). These equations involve a (real) orthonormal basis and orthonormal components of tensor fields. Equations (1.104) provide *a real tetrad* $\left\{\vec{\mathbf{e}}_{(a)}(\cdot\cdot)\right\}_1^4$ characterized by

$$\vec{\mathbf{e}}_{(a)}(x) := \lambda^i{}_{(a)}(\cdot\cdot) \cdot \partial_i, \tag{7.123i}$$

$$\widetilde{\mathbf{e}}^{(a)}(x) := \mu^{(a)}{}_i(\cdot\cdot) \cdot \mathrm{d}x^i, \tag{7.123ii}$$

$$\left[\mu^{(a)}{}_i(\cdot\cdot)\right] := \left[\lambda^i{}_{(a)}(\cdot\cdot)\right]^{-1}. \tag{7.123iii}$$

Here, we treat *the 16 components of* $\lambda^i{}_{(a)}(x)$ *as unknown functions*. The metric tensor components satisfy

$$d_{(a)(b)} = g_{ij}(\cdot\cdot) \cdot \lambda^i{}_{(a)}(\cdot\cdot) \cdot \lambda^j{}_{(b)}(\cdot\cdot), \tag{7.124i}$$

$$g_{ij}(x) = d_{(a)(b)} \cdot \mu^{(a)}{}_i(\cdot\cdot) \cdot \mu^{(b)}{}_j(\cdot\cdot), \tag{7.124ii}$$

and are thus determined by the functions $\lambda^i{}_{(a)}(x)$.

The directional derivatives are defined from (1.114ii) as

$$\partial_{(a)} f := \lambda^i{}_{(a)}(\cdot\cdot) \cdot \partial_i f = \vec{\mathbf{e}}_{(a)}(\cdot\cdot)[f]. \tag{7.125}$$

The covariant derivatives of the energy–momentum–stress tensor field components from (1.124ii) are provided by

$$\begin{aligned}\nabla_{(c)} T^{(a_1)\cdots(a_r)}{}_{(b_1)\cdots(b_s)} := \partial_{(c)} T^{(a_1)\cdots(a_r)}{}_{(b_1)\cdots(b_s)} \\ - \sum_{\alpha=1}^{r} \gamma^{(a_\alpha)}{}_{(d)(c)} \cdot T^{(a_1)\cdots(a_{\alpha-1})(d)(a_{\alpha+1})\cdots(a_r)}{}_{(b_1)\cdots(b_s)} \\ + \sum_{\beta=1}^{s} \gamma^{(d)}{}_{(b_\beta)(c)} \cdot T^{(a_1)\cdots(a_r)}{}_{(b_1)\cdots(b_{\beta-1})(d)(b_{\beta+1})\cdots(b_s)}.\end{aligned} \tag{7.126}$$

The (real) Ricci rotation coefficients, which are also unknown functions, are furnished from (1.138) and (1.139i,ii,iii) as

$$\gamma_{(a)(b)(c)}(x) := d_{(a)(e)} \cdot \gamma^{(e)}{}_{(b)(c)}(\cdot\cdot), \tag{7.127i}$$

$$\gamma_{(b)(a)(c)}(x) \equiv -\gamma_{(a)(b)(c)}(x), \tag{7.127ii}$$

$$\gamma_{(a)(b)(c)}(x) = g_{jl}(\cdot\cdot) \cdot \left(\nabla_k \lambda^l{}_{(a)}\right) \cdot \lambda^j{}_{(b)}(\cdot\cdot) \cdot \lambda^k{}_{(c)}(\cdot\cdot). \tag{7.127iii}$$

The 24 components $\gamma_{(a)(b)(c)}$ are also treated as unknown functions.

Next, we have *integrability conditions* (1.139iv)

$$\partial_{(a)}\partial_{(b)} f - \partial_{(b)}\partial_{(a)} f = d^{(c)(e)} \cdot \left[\gamma_{(e)(a)(b)} - \gamma_{(e)(b)(a)}\right] \cdot \partial_{(c)} f, \tag{7.128i}$$

$$\partial_{(a)}\partial_{(b)}\left[\lambda^k{}_{(c)}\right] - \partial_{(b)}\partial_{(a)}\left[\lambda^k{}_{(c)}\right] = d^{(e)(h)} \cdot \left[\gamma_{(e)(a)(b)} - \gamma_{(e)(b)(a)}\right] \times \partial_{(h)}\left[\lambda^k{}_{(c)}\right]. \tag{7.128ii}$$

The field equations (2.163ii) yield

$$\begin{aligned} R^{(d)}{}_{(a)(b)(c)}(x) = C^{(d)}{}_{(a)(b)(c)}(x) + (\kappa/2) \cdot \Big\{&\delta^{(d)}{}_{(b)} \cdot T_{(a)(c)} - \delta^{(d)}{}_{(c)} \cdot T_{(a)(b)} \\ &+ d_{(a)(c)} \cdot T^{(d)}{}_{(b)} - d_{(a)(b)} \cdot T^{(d)}{}_{(c)} \\ &+ (2/3) \cdot \left[\delta^{(d)}{}_{(c)} \cdot d_{(a)(b)} - \delta^{(d)}{}_{(b)} \cdot d_{(a)(c)}\right] \cdot T^{(e)}{}_{(e)}\Big\} \\ &\text{inside material sources,} \end{aligned} \tag{7.129i}$$

$$R^{(d)}{}_{(a)(b)(c)}(x) = C^{(d)}{}_{(a)(b)(c)}(x) \qquad \text{outside material sources.} \tag{7.129ii}$$

Bianchi's differential identities of (1.143ii) imply that the following equations hold:

$$\begin{aligned} &\nabla_{(e)}\left[R^{(d)}{}_{(a)(b)(c)} - C^{(d)}{}_{(a)(b)(c)}\right] + \nabla_{(b)}\left[R^{(d)}{}_{(a)(c)(e)} - C^{(d)}{}_{(a)(c)(e)}\right] \\ &\quad + \nabla_{(c)}\left[R^{(d)}{}_{(a)(e)(b)} - C^{(d)}{}_{(a)(e)(b)}\right] \\ &\equiv -\left[\nabla_{(e)} C^{(d)}{}_{(a)(b)(c)} + \nabla_{(b)} C^{(d)}{}_{(a)(c)(e)} + \nabla_{(c)} C^{(d)}{}_{(a)(e)(b)}\right] \\ &= \begin{cases} (\kappa/2) \cdot \left\{\nabla_{(e)} T^{(d)}{}_{(a)(b)(c)} + \nabla_{(b)} T^{(d)}{}_{(a)(c)(e)} + \nabla_{(c)} T^{(d)}{}_{(a)(e)(b)}\right\} \\ \qquad\qquad \text{inside material sources,} \\ 0 \qquad\qquad \text{outside material sources.} \end{cases} \end{aligned} \tag{7.130}$$

Here,

$$T^{(d)}{}_{(a)(b)(c)} := \delta^{(d)}{}_{(b)} T_{(a)(c)} - \delta^{(d)}{}_{(c)} T_{(a)(b)} + d_{(a)(c)} \cdot T^{(d)}{}_{(b)} - d_{(a)(b)} \cdot T^{(d)}{}_{(c)}$$
$$+ \frac{2}{3} \cdot \left(\delta^{(d)}{}_{(c)} d_{(a)(b)} - \delta^{(d)}{}_{(b)} d_{(a)(c)} \right) \cdot T^{(f)}{}_{(f)}.$$

Four possible coordinate conditions can be stated as

$$\mathcal{C}^{(a)} \left(\lambda^{k}{}_{(b)}, \ \partial_{(c)} \lambda^{k}{}_{(b)} \right) = 0. \tag{7.131}$$

Thus, the field equations relative to a (real) orthonormal tetrad are completely stated. Now we verify the first-order system above by substituting a known metric.

Example 7.2.2. The spatially flat F–L–R–W cosmological metric of (6.9) is examined. Recall that it is provided by

$$\begin{aligned} \mathrm{d}s^2 &= \left[a \left(x^4 \right) \right]^2 \cdot \delta_{\alpha\beta} \cdot \mathrm{d}x^\alpha \mathrm{d}x^\beta - \left(\mathrm{d}x^4 \right)^2, \\ x^4 &> 0, \quad a \left(x^4 \right) > 0. \end{aligned} \tag{7.132}$$

The metric above implies that

$$\mathbf{g}_{..}(\cdot\cdot) = \delta_{(\alpha)(\beta)} \cdot \widetilde{\mathbf{e}}^{(\alpha)}(\cdot\cdot) \otimes \widetilde{\mathbf{e}}^{(\beta)}(\cdot\cdot) - \widetilde{\mathbf{e}}^{(4)}(\cdot\cdot) \otimes \widetilde{\mathbf{e}}^{(4)}(\cdot\cdot), \tag{7.133i}$$

$$\widetilde{\mathbf{e}}^{(\alpha)}(\cdot\cdot) = \mu^{(\alpha)}{}_{i}(\cdot\cdot) \cdot \mathrm{d}x^i = a \left(x^4 \right) \cdot \delta^{(\alpha)}{}_{i} \cdot \mathrm{d}x^i, \tag{7.133ii}$$

$$\widetilde{\mathbf{e}}^{(4)}(\cdot\cdot) = \mu^{(4)}{}_{i}(\cdot\cdot) \cdot \mathrm{d}x^i = \delta^{(4)}{}_{i} \cdot \mathrm{d}x^i, \tag{7.133iii}$$

$$\vec{\mathbf{e}}_{(\alpha)}(\cdot\cdot) = \lambda^{i}{}_{(\alpha)}(\cdot\cdot) \cdot \partial_i = \left[a \left(x^4 \right) \right]^{-1} \cdot \delta^{i}{}_{(\alpha)} \cdot \partial_i, \tag{7.133iv}$$

$$\vec{\mathbf{e}}_{(4)}(\cdot\cdot) = \lambda^{i}{}_{(4)}(\cdot\cdot) \cdot \partial_i = \delta^{i}{}_{(4)} \cdot \partial_i. \tag{7.133v}$$

The directional derivatives of (7.125) imply that

$$\partial_{(\alpha)} f = \left[a \left(x^4 \right) \right]^{-1} \cdot \partial_\alpha f, \tag{7.134i}$$

$$\partial_{(4)} f = \partial_4 f, \tag{7.134ii}$$

$$\partial_{(\alpha)} \left[\lambda^{i}{}_{(a)}(\cdot\cdot) \right] \equiv 0. \tag{7.134iii}$$

The nonzero components of the Ricci rotation coefficients from (7.127i–iii), (7.132), and (7.133i–v) are calculated to be

$$\begin{aligned} \gamma_{(4)(\alpha)(\beta)}(\cdot\cdot) &\equiv -\gamma_{(\alpha)(4)(\beta)}(\cdot\cdot) = \left\{ 0 + [kh, j] \cdot \delta^{h}{}_{(4)} \right\} \cdot \left(a^{-2} \right) \cdot \delta^{j}{}_{(\alpha)} \cdot \delta^{k}{}_{(\beta)} \\ &= [\dot{a}/a] \cdot \delta_{(\alpha)(\beta)}, \\ \dot{a} &:= \frac{\mathrm{d}}{\mathrm{d}x^4} a \left(x^4 \right). \end{aligned} \tag{7.135}$$

The nonzero components of the directional derivatives of the Ricci rotation coefficients are given by

$$\partial_{(4)}\gamma_{(4)(\alpha)(\beta)} \equiv -\partial_{(4)}\gamma_{(\alpha)(4)(\beta)} = \left[\frac{\mathrm{d}}{\mathrm{d}x^4}\left(\frac{\dot{a}}{a}\right)\right] \cdot \delta_{(\alpha)(\beta)}. \tag{7.136}$$

The integrability conditions (7.128ii) *are identically satisfied* due to (7.135) and (7.136). The nonzero, orthonormal components of the curvature tensor, from (1.141i,ii), (7.135), and (7.136), are provided by

$$R^{(\rho)}{}_{(4)(4)(\sigma)}(\cdot\cdot) = 0 - \partial_{(4)}\gamma^{(\rho)}{}_{(4)(\sigma)} + \gamma^{(\rho)}{}_{(4)(\alpha)}\gamma^{(\alpha)}{}_{(4)(\sigma)} + 0 = [\ddot{a}/a] \cdot \delta^{(\rho)}{}_{(\sigma)},$$

$$R^{(\rho)}{}_{(\alpha)(\sigma)(\beta)}(\cdot\cdot) = \left(\frac{\dot{a}}{a}\right)^2 \cdot \left[\delta^{(\rho)}{}_{(\sigma)} \cdot \delta_{(\alpha)(\beta)} - \delta^{(\rho)}{}_{(\beta)} \cdot \delta_{(\alpha)(\sigma)}\right],$$

$$\ddot{a} := \frac{\mathrm{d}^2}{\left(\mathrm{d}x^4\right)^2}\, a\left(x^4\right). \tag{7.137}$$

By the prior knowledge of (6.13i,ii) and (6.14i,ii) and the assumption of an incoherent dust model, we obtain

$$C^{(a)}{}_{(b)(c)(d)}(\cdot\cdot) \equiv 0, \tag{7.138i}$$

$$T_{(\alpha)(b)}(\cdot\cdot) \equiv 0, \tag{7.138ii}$$

$$T_{(4)(4)}(\cdot\cdot) > 0. \tag{7.138iii}$$

Field equation (7.129i), with (7.137) and (7.138i–iii), boils down to

$$R^{(\rho)}{}_{(4)(4)(\sigma)}(\cdot\cdot) = \left[\frac{\ddot{a}}{a}\right] \cdot \delta^{(\rho)}{}_{(\sigma)} = -\left(\frac{\kappa}{6}\right) \cdot T_{(4)(4)} \cdot \delta^{(\rho)}{}_{(\sigma)}, \tag{7.139i}$$

$$\begin{aligned} R^{(\rho)}{}_{(\alpha)(\sigma)(\beta)}(\cdot\cdot) &= \left[\frac{\dot{a}}{a}\right]^2 \cdot \left[\delta^{(\rho)}{}_{(\sigma)} \cdot \delta_{(\alpha)(\beta)} - \delta^{(\rho)}{}_{(\beta)} \cdot \delta_{(\alpha)(\sigma)}\right] \\ &= \left(\frac{\kappa}{3}\right) \cdot \left[\delta^{(\rho)}{}_{(\sigma)} \cdot \delta_{(\alpha)(\beta)} - \delta^{(\rho)}{}_{(\beta)} \cdot \delta_{(\alpha)(\sigma)}\right] \cdot T_{(4)(4)}(\cdot\cdot), \end{aligned} \tag{7.139ii}$$

$$\left[2a \cdot \ddot{a} + (\dot{a})^2\right] \cdot \delta_{(\alpha)(\beta)} = 0, \tag{7.139iii}$$

$$3 \cdot (\dot{a}/a)^2 = \kappa \cdot T_{(4)(4)}(\cdot\cdot). \tag{7.139iv}$$

Bianchi's differential identities (7.130) reduce to the conservation equation

$$\frac{\mathrm{d}}{\mathrm{d}x^4}\left\{\ln\left[T_{(4)(4)}(\cdot\cdot)\cdot(a(\cdot\cdot))^3\right]\right\} = 0. \tag{7.140}$$

By (6.21i–iii), it is clear that (7.138ii,iii), (7.139i,ii), and (7.140) are exactly equivalent to (6.12i,ii,iv) and (6.15) governing the F–L–R–W cosmological models containing incoherent dust. □

Now, we shall introduce *the complex, null tetrad of (7.5i–iv) and (7.6i–v)*. We summarize the relevant equations in the following:

$$\left\{\vec{\mathbf{E}}_{(a)}(\cdot\cdot)\right\}_1^4 := \left\{\vec{\mathbf{m}}(\cdot\cdot), \vec{\overline{\mathbf{m}}}(\cdot\cdot), \vec{\mathbf{l}}(\cdot\cdot), \vec{\mathbf{k}}(\cdot\cdot)\right\}, \tag{7.141i}$$

$$\left\{\widetilde{\mathbf{E}}^{(a)}(\cdot\cdot)\right\}_1^4 = \left\{\widetilde{\overline{\mathbf{m}}}(\cdot\cdot), \widetilde{\mathbf{m}}(\cdot\cdot), -\widetilde{\mathbf{k}}(\cdot\cdot), -\widetilde{\mathbf{l}}(\cdot\cdot)\right\}, \tag{7.141ii}$$

$$\underset{4\times4}{\left[\eta_{(a)(b)}\right]} \equiv \underset{4\times4}{\left[\eta^{(a)(b)}\right]} = \left[\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array}\right], \tag{7.141iii}$$

$$\vec{\mathbf{E}}_{(a)}(\cdot\cdot) = \Lambda^i{}_{(a)}(\cdot\cdot)\cdot\partial_i\,, \quad \widetilde{\mathbf{E}}^{(a)}(\cdot\cdot) = \mathsf{M}^{(a)}{}_i(\cdot\cdot)\,\mathrm{d}x^i, \tag{7.141iv}$$

$$\left[\mathsf{M}^{(a)}{}_i(\cdot\cdot)\right] := \left[\Lambda^i{}_{(a)}(\cdot\cdot)\right]^{-1}, \tag{7.141v}$$

$$\eta_{(a)(b)} = g_{ij}(\cdot\cdot)\cdot\Lambda^i{}_{(a)}(\cdot\cdot)\cdot\Lambda^j{}_{(b)}(\cdot\cdot), \tag{7.141vi}$$

$$\eta^{(a)(b)} = g^{ij}(\cdot\cdot)\cdot\mathsf{M}^{(a)}{}_i(\cdot\cdot)\cdot\mathsf{M}^{(b)}{}_j(\cdot\cdot), \tag{7.141vii}$$

$$g^{ij}(\cdot\cdot) = \eta^{(a)(b)}\cdot\Lambda^i{}_{(a)}(\cdot\cdot)\cdot\Lambda^j{}_{(b)}(\cdot\cdot), \tag{7.141viii}$$

$$g_{ij}(\cdot\cdot) = \eta_{(a)(b)}\cdot\mathsf{M}^{(a)}{}_i(\cdot\cdot)\cdot\mathsf{M}^{(b)}{}_j(\cdot\cdot), \tag{7.141ix}$$

$$\delta^i{}_j = \Lambda^i{}_{(a)}(\cdot\cdot)\cdot\mathsf{M}^{(a)}{}_j(\cdot\cdot), \tag{7.141x}$$

$$\delta^{(a)}{}_{(b)} = \mathsf{M}^{(a)}{}_i(\cdot\cdot)\cdot\Lambda^i{}_{(b)}(\cdot\cdot). \tag{7.141xi}$$

The directional derivatives involving complex-valued functions $\Lambda^i{}_{(a)}(x)$ are defined similar to (7.125) as

$$\partial_{(a)}f := \Lambda^i{}_{(a)}(\cdot\cdot)\cdot\partial_i f. \tag{7.142}$$

Here, f may be *a real or a complex-valued differentiable function*.

The covariant derivatives of complex-valued components related to a complex tensor field are defined analogous to (7.126) as:

$$\begin{aligned}\nabla_{(c)}\mathbb{T}^{(a_1)\cdots(a_r)}{}_{(b_1)\cdots(b_s)} := \partial_{(c)}\mathbb{T}^{(a_1)\cdots(a_r)}{}_{(b_1)\cdots(b_s)} \\ -\sum_{\alpha=1}^{r}\mathbb{\Gamma}^{(a_\alpha)}{}_{(d)(c)}\cdot\mathbb{T}^{(a_1)\cdots(a_{\alpha-1})(d)(a_{\alpha+1})\cdots(a_r)}{}_{(b_1)\cdots(b_s)} \\ +\sum_{\beta=1}^{s}\mathbb{\Gamma}^{(d)}{}_{(b_\beta)(c)}\cdot\mathbb{T}^{(a_1)\cdots(a_r)}{}_{(b_1)\cdots(b_{\beta-1})(d)(b_{\beta+1})\cdots(b_s)}.\end{aligned} \tag{7.143}$$

The complex-valued Ricci rotation coefficients are furnished by:

$$\mathbb{\Gamma}^{(e)}{}_{(b)(c)}(\cdot\cdot) = \eta^{(e)(a)}\cdot\mathbb{\Gamma}_{(a)(b)(c)}(\cdot\cdot), \tag{7.144i}$$

$$\mathbb{\Gamma}_{(a)(b)(c)}(\cdot\cdot) = \eta_{(a)(d)}\cdot\mathbb{\Gamma}^{(d)}{}_{(b)(c)}(\cdot\cdot), \tag{7.144ii}$$

$$\mathbb{\Gamma}_{(b)(a)(c)}(\cdot\cdot) \equiv -\mathbb{\Gamma}_{(a)(b)(c)}(\cdot\cdot), \tag{7.144iii}$$

$$\mathbb{\Gamma}_{(a)(b)(c)}(\cdot\cdot) = g_{jl}(\cdot\cdot)\cdot\left(\nabla_k\Lambda^l{}_{(a)}\right)\cdot\Lambda^j{}_{(b)}(\cdot\cdot)\cdot\Lambda^k{}_{(c)}(\cdot\cdot). \tag{7.144iv}$$

We treat as unknown functions the 16 complex components $\Lambda^i{}_{(a)}(x)$ and the 24 complex components $\mathbb{\Gamma}_{(a)(b)(c)}(x)$.

Next, we present *complex integrability conditions* for the first-order partial differential equations (7.144iv) as

$$\partial_{(a)}\partial_{(b)}f - \partial_{(b)}\partial_{(a)}f = \eta^{(c)(e)}\cdot\left[\mathbb{\Gamma}_{(e)(a)(b)} - \mathbb{\Gamma}_{(e)(b)(a)}\right]\cdot\partial_{(c)}f, \tag{7.145i}$$

$$\begin{aligned}\partial_{(a)}\partial_{(b)}\left[\Lambda^k{}_{(c)}\right] - \partial_{(b)}\partial_{(a)}\left[\Lambda^k{}_{(c)}\right] = \eta^{(h)(e)}\cdot\left[\mathbb{\Gamma}_{(e)(a)(b)} - \mathbb{\Gamma}_{(e)(b)(a)}\right] \\ \times\ \partial_{(h)}\left[\Lambda^k{}_{(c)}\right].\end{aligned} \tag{7.145ii}$$

(See #3 of Exercises 7.2 for the derivation of the integrability conditions (7.145i).)

Field equations (2.163ii) and (7.129i,ii) yield *complex-valued field equations*:

$$\begin{aligned}\mathbb{R}_{(d)(a)(b)(c)}(x) = \mathbb{C}_{(d)(a)(b)(c)}(\cdot\cdot) + (\kappa/2)\cdot\Big\{\big[\eta_{(d)(b)}\cdot\mathbb{T}_{(a)(c)} \\ - \eta_{(d)(c)}\cdot\mathbb{T}_{(a)(b)} + \eta_{(a)(c)}\cdot\mathbb{T}_{(d)(b)} - \eta_{(a)(b)}\cdot\mathbb{T}_{(d)(c)}\big] \\ + (2/3)\cdot\left[\eta_{(d)(c)}\cdot\eta_{(a)(b)} - \eta_{(d)(b)}\cdot\eta_{(a)(c)}\right]\cdot\mathbb{T}^{(e)}{}_{(e)}\Big\}\end{aligned}$$

inside material sources, (7.146i)

$$\mathbb{R}_{(d)(a)(b)(c)}(x) = \mathbb{C}_{(d)(a)(b)(c)}(x) \qquad \text{outside material sources.} \tag{7.146ii}$$

The complex-valued Bianchi's differential identities from (1.143ii), (7.130), and consequent equations are furnished by:

$$\begin{aligned}
&\nabla_{(e)}\left[\mathbb{R}_{(d)(a)(b)(c)} - \mathbb{C}_{(d)(a)(b)(c)}\right] + \nabla_{(b)}[\cdot\cdot] + \nabla_{(c)}[\cdot\cdot] \\
&\quad\equiv -\left[\nabla_{(e)}\mathbb{C}_{(d)(a)(b)(c)} + \nabla_{(b)}\mathbb{C}_{(d)(a)(c)(e)} + \nabla_{(c)}\mathbb{C}_{(d)(a)(e)(b)}\right] \\
&\quad= \begin{bmatrix} (\kappa/2)\cdot\Big\{\nabla_{(e)}\left[\mathbb{T}_{(d)(b)(a)(c)}\right] & \\ +\nabla_{(b)}[\mathbb{T}_{(d)(a)(c)(e)}] + \nabla_{(c)}[\mathbb{T}_{(d)(a)(e)(b)}]\Big\}, & \text{inside material sources;} \\ 0, & \text{outside material sources.} \end{bmatrix}
\end{aligned} \tag{7.147}$$

Here,

$$\begin{aligned}
\mathbb{T}_{(d)(a)(b)(c)} := \big[&\eta_{(d)(b)}\cdot\mathbb{T}_{(a)(c)} - \eta_{(d)(c)}\cdot\mathbb{T}_{(a)(b)} \quad + \eta_{(a)(c)}\cdot\mathbb{T}_{(d)(b)} \\
&- \eta_{(a)(b)}\cdot\mathbb{T}_{(d)(c)} \quad + (2/3)\cdot\left(\eta_{(d)(c)}\cdot\eta_{(a)(b)} - \eta_{(d)(b)}\cdot\eta_{(a)(c)}\right]\cdot\mathbb{T}^{(f)}{}_{(f)}.
\end{aligned}$$

(According to the T-method of Sect. 2.4, the functions $\mathbb{T}_{(a)(b)}(x)$ can be prescribed.)

At most four possible coordinate conditions, which can be imposed, are stated as

$$\mathcal{C}^{(a)}\left(\Lambda^{k}{}_{(b)},\ \partial_{(c)}\Lambda^{k}{}_{(b)}\right) = 0. \tag{7.148}$$

Now, we should mention that there exist *other alternative ways to express the complex-valued field equations* by using

$$\mathbb{R}_{(a)(b)}(\cdot\cdot) - (1/2)\cdot\eta_{(a)(b)}\cdot\mathbb{R}(\cdot\cdot) = -\kappa\cdot\mathbb{T}_{(a)(b)}(\cdot\cdot), \tag{7.149i}$$

$$\mathbb{R}(\cdot\cdot) = \kappa\cdot\mathbb{T}^{(a)}{}_{(a)}(\cdot\cdot), \tag{7.149ii}$$

$$\mathbb{R}_{(a)(b)}(\cdot\cdot) = -\kappa\cdot\left[\mathbb{T}_{(a)(b)}(\cdot\cdot) - (1/2)\cdot\eta_{(a)(b)}\cdot\mathbb{T}^{(c)}{}_{(c)}(\cdot\cdot)\right]. \tag{7.149iii}$$

(Consult [53, 271].)

We now go back to (7.146i,ii). The field equations (7.146i,ii) with (7.149i–iii) go over into

$$\begin{aligned}
\mathbb{R}_{(d)(a)(b)(c)}(x) &= \mathbb{C}_{(d)(a)(b)(c)}(\cdot\cdot) \\
&+ (1/2)\cdot\Big\{\left[\eta_{(d)(c)}\cdot\mathbb{R}_{(a)(b)} - \eta_{(d)(b)}\cdot\mathbb{R}_{(a)(c)} + \eta_{(a)(b)}\cdot\mathbb{R}_{(d)(c)} - \eta_{(a)(c)}\cdot\mathbb{R}_{(d)(b)}\right] \\
&+ (1/3)\cdot\left[\eta_{(a)(c)}\cdot\eta_{(d)(b)} - \eta_{(a)(b)}\cdot\eta_{(d)(c)}\right]\cdot\mathbb{R}\Big\}.
\end{aligned} \tag{7.150}$$

Bianchi's differential identities from (7.147) yield

$$
\begin{aligned}
0 = \nabla_{(e)} \Big\{ & \mathbb{C}_{(d)(a)(b)(c)}(\cdot\cdot) \\
& + (1/2) \cdot \big[\eta_{(d)(c)} \cdot \mathbb{R}_{(a)(b)} - \eta_{(d)(b)} \cdot \mathbb{R}_{(a)(c)} + \eta_{(a)(b)} \cdot \mathbb{R}_{(d)(c)} - \eta_{(a)(c)} \cdot \mathbb{R}_{(d)(b)}\big] \\
& + (1/6) \cdot \big[\eta_{(a)(c)} \cdot \eta_{(d)(b)} - \eta_{(a)(b)} \cdot \eta_{(d)(c)}\big] \cdot \mathbb{R}\Big\} \\
& + \nabla_{(b)}\{\cdots\} + \nabla_{(c)}\{\cdots\}.
\end{aligned} \tag{7.151}
$$

There exists *still another way to express (7.150) and equations originating from Bianchi's differential identities*. Recall (7.118ii). We define analogous complex tensor components via

$$\Sigma_{(a)(b)}(x) := \mathbb{R}_{(a)(b)}(\cdot\cdot) - (1/4) \cdot \eta_{(a)(b)} \cdot \mathbb{R}(\cdot\cdot), \tag{7.152i}$$

$$\Sigma^{(a)}{}_{(a)}(x) \equiv 0, \tag{7.152ii}$$

$$\mathbb{R}_{(a)(b)}(x) = \Sigma_{(a)(b)}(\cdot\cdot) + (1/4) \cdot \eta_{(a)(b)} \cdot \mathbb{R}(\cdot\cdot). \tag{7.152iii}$$

Equations (7.150) with (7.152i–iii) yield a newer form of the field equations; namely,

$$
\begin{aligned}
\mathbb{R}_{(d)(a)(b)(c)}(x) = {} & \mathbb{C}_{(d)(a)(b)(c)}(\cdot\cdot) \\
& + (1/2) \cdot \Big\{\big[\eta_{(d)(c)} \cdot \Sigma_{(a)(b)} - \eta_{(d)(b)} \cdot \Sigma_{(a)(c)} + \eta_{(a)(b)} \cdot \Sigma_{(d)(c)} - \eta_{(a)(c)} \cdot \Sigma_{(d)(b)}\big] \\
& + (1/6) \cdot \big[\eta_{(a)(b)} \cdot \eta_{(d)(c)} - \eta_{(a)(c)} \cdot \eta_{(d)(b)}\big] \cdot \mathbb{R}\Big\},
\end{aligned} \tag{7.153}
$$

$$\Sigma_{(a)(b)}(\cdot\cdot) = -\kappa \cdot \left[\mathbb{T}_{(a)(b)}(\cdot\cdot) - \frac{1}{4} \cdot \eta_{(a)(b)} \cdot \mathbb{T}^{(c)}{}_{(c)}(\cdot\cdot)\right].$$

Bianchi's differential identities (7.151) yield equations[5]

$$
\begin{aligned}
0 = \nabla_{(e)} \Big\{ & \mathbb{C}_{(d)(a)(b)(c)}(\cdot\cdot) \\
& + (1/2) \cdot \big[\eta_{(d)(c)} \cdot \Sigma_{(a)(b)} - \eta_{(d)(b)} \cdot \Sigma_{(a)(c)} + \eta_{(a)(b)} \cdot \Sigma_{(d)(c)} - \eta_{(a)(c)} \cdot \Sigma_{(d)(b)}\big] \\
& + (1/12) \cdot \big[\eta_{(a)(b)} \cdot \eta_{(d)(c)} - \eta_{(d)(b)} \cdot \eta_{(a)(c)}\big] \cdot \mathbb{R}\Big\} \\
& + \nabla_{(b)}\{\cdots\} + \nabla_{(c)}\{\cdots\}.
\end{aligned} \tag{7.154}
$$

---

[5]Note that

$$
\begin{aligned}
\nabla_{(e)}\mathbb{R}_{(d)(a)(b)(c)} = {} & \nabla_{(e)}\mathbb{C}_{(d)(a)(b)(c)} + \frac{1}{2} \cdot \big[\eta_{(d)(c)} \cdot \nabla_{(e)}\Sigma_{(a)(b)} - \eta_{(d)(b)} \cdot \nabla_{(e)}\Sigma_{(a)(c)} \\
& + \eta_{(a)(b)} \cdot \nabla_{(e)}\Sigma_{(d)(c)} - \eta_{(a)(c)} \cdot \nabla_{(e)}\Sigma_{(d)(b)}\big] + \frac{1}{12} \cdot \big[\eta_{(a)(b)} \cdot \eta_{(d)(c)} - \eta_{(a)(c)} \cdot \eta_{(d)(b)}\big] \cdot \nabla_{(e)}\mathbb{R}.
\end{aligned}
$$

This equation is useful in the derivation of eight of the Bianchi identities (7.182i–viii) later on.

Now, consider the set of transformations of the complex null tetrad such that integrability conditions (7.145i,ii), field equations (7.146i,ii), and Bianchi's differential identities (7.147) *all transform covariantly*. We will provide this class of transformations in the following equations:

$$\left\{\widehat{\vec{\mathbf{E}}}_{(a)}(\cdot\cdot)\right\}_1^4 := \left\{\widehat{\vec{\mathbf{m}}}(\cdot\cdot), \widehat{\vec{\overline{\mathbf{m}}}}(\cdot\cdot), \widehat{\vec{\mathbf{l}}}(\cdot\cdot), \widehat{\vec{\mathbf{k}}}(\cdot\cdot)\right\}, \tag{7.155i}$$

$$\widehat{\vec{\mathbf{E}}}_{(a)}(\cdot\cdot) = \mathbb{L}^{(b)}{}_{(a)}(\cdot\cdot) \cdot \vec{\mathbf{E}}_{(b)}(\cdot\cdot), \tag{7.155ii}$$

$$\eta_{(a)(b)} \cdot \mathbb{L}^{(a)}{}_{(c)}(\cdot\cdot) \cdot \mathbb{L}^{(b)}{}_{(d)}(\cdot\cdot) = \eta_{(c)(d)}, \tag{7.155iii}$$

$$[\mathbb{L}(\cdot\cdot)]^T \cdot [\eta] \cdot [\mathbb{L}(\cdot\cdot)] = [\eta]. \tag{7.155iv}$$

We shall give an example of such a set of transformations in the following example.

Example 7.2.3. We consider a transformation already discussed in (7.7i). It is specified by

$$\begin{aligned}
\widehat{\vec{\mathbf{m}}}(x) &= \exp\left[-\mathrm{i}\,\Theta(x)\right] \cdot \vec{\mathbf{m}}(x),\\
\widehat{\vec{\overline{\mathbf{m}}}}(x) &= \exp\left[\mathrm{i}\,\Theta(x)\right] \cdot \vec{\overline{\mathbf{m}}}(x),\\
\widehat{\vec{\mathbf{l}}}(x) &= \vec{\mathbf{l}}(x),\\
\widehat{\vec{\mathbf{k}}}(x) &= \vec{\mathbf{k}}(x).
\end{aligned}$$

The corresponding complex, Lorentz matrix from (7.155i,ii), is given by

$$\underset{4\times 4}{[\mathbb{L}(\cdot\cdot)]} = \left[\begin{array}{cc|cc}
\mathrm{e}^{-\mathrm{i}\,\Theta} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\
0 & \mathrm{e}^{\mathrm{i}\,\Theta} & & \\
\hline
\multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\
 & & 0 & 1
\end{array}\right]. \qquad \square$$

Now we shall discuss another example, which involves the null complex tetrad, complex integrability conditions, complex field equations, and complex Bianchi's differential identities.

Example 7.2.4. The following metric is investigated:

$$\mathrm{d}s^2 = 2 \cdot \left[1 + (1/2) \cdot x^1 \cdot x^2\right]^{-2} \cdot \left(\mathrm{d}x^1\right) \cdot \left(\mathrm{d}x^2\right) - 2 \cdot \left(\mathrm{d}x^3\right) \cdot \left(\mathrm{d}x^4\right), \tag{7.156i}$$

$$= 2 \cdot \left[1 + (1/2) \cdot |\zeta|^2\right]^{-2} \cdot (\mathrm{d}\zeta) \cdot \left(\mathrm{d}\overline{\zeta}\right) - 2 \cdot (\mathrm{d}u) \cdot (\mathrm{d}v), \tag{7.156ii}$$

$$\left[g^{ij}(\cdot\cdot)\right] = \left[\begin{array}{cc|cc} 0 & [\cdot\cdot]^2 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ [\cdot\cdot]^2 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -1 \\ & & -1 & 0 \end{array}\right]. \tag{7.156iii}$$

(Here, x^1 and x^2 are *complex conjugate coordinates* of (A2.38) and (A2.39i,ii).)

The corresponding natural, complex null tetrad is provided by

$$\vec{\mathbf{E}}_{(1)}(\cdot\cdot) \equiv \vec{\mathbf{m}}(\cdot\cdot) = [\cdot\cdot] \cdot \partial_1, \quad \vec{\mathbf{E}}_{(2)}(\cdot\cdot) \equiv \overline{\vec{\mathbf{m}}}(\cdot\cdot) = [\cdot\cdot] \cdot \partial_2,$$

$$\vec{\mathbf{E}}_{(3)}(\cdot\cdot) \equiv \vec{\mathbf{l}}(\cdot\cdot) = \partial_3, \quad \vec{\mathbf{E}}_{(4)}(\cdot\cdot) \equiv \vec{\mathbf{k}}(\cdot\cdot) = \partial_4.$$

Therefore, from (7.141i–xi) and (7.156iii), we derive the complex valued components as

$$\Lambda^k{}_{(1)}(\cdot\cdot) = \left[1 + (1/2) \cdot x^1 \cdot x^2\right] \cdot \delta^k{}_{(1)}, \tag{7.157i}$$

$$\Lambda^k{}_{(2)}(\cdot\cdot) = \left[1 + (1/2) \cdot x^1 \cdot x^2\right] \cdot \delta^k{}_{(2)}, \tag{7.157ii}$$

$$\Lambda^k{}_{(3)}(\cdot\cdot) = \delta^k{}_{(3)}, \tag{7.157iii}$$

$$\Lambda^k{}_{(4)}(\cdot\cdot) = \delta^k{}_{(4)}. \tag{7.157iv}$$

With help of (7.142) and (7.157i–iv), directional derivations are deduced as

$$\partial_{(1)} f = \left[1 + (1/2) \cdot x^1 \cdot x^2\right] \cdot \partial_1 f, \tag{7.158i}$$

$$\partial_{(2)} f = \left[1 + (1/2) \cdot x^1 \cdot x^2\right] \cdot \partial_2 f, \tag{7.158ii}$$

$$\partial_{(3)} f = \partial_3 f, \tag{7.158iii}$$

$$\partial_{(4)} f = \partial_4 f. \tag{7.158iv}$$

From (7.144iv), the four, nonzero, complex-valued Ricci rotation coefficients are furnished by

$$\Gamma_{(1)(2)(1)}(x) = \partial_1 \left(g_{21} \cdot \Lambda^1{}_{(1)}\right) \cdot [\cdot\cdot] = -(1/2) \cdot x^2 \equiv -\Gamma_{(2)(1)(1)}(x), \tag{7.159i}$$

$$\Gamma_{(1)(2)(2)}(x) = (1/2) \cdot x^1 \equiv -\Gamma_{(2)(1)(2)}(x). \tag{7.159ii}$$

(The other twenty Ricci rotation coefficients are identically zero!) *The complex-valued integrability conditions* (7.145i,ii) *are exactly satisfied* from (7.156i), (7.157i–iv), (7.158i–iv), and (7.159i,ii).

Now, we shall compute *the complex-valued curvature tensor components* from generalizations of (1.141iv) as:

$$\begin{aligned}\mathbb{R}_{(a)(b)(c)(d)}(x) = {} & \partial_{(d)}\Gamma_{(a)(b)(c)} - \partial_{(c)}\Gamma_{(a)(b)(d)} + \eta^{(h)(e)} \\ & \times \left[\Gamma_{(h)(a)(d)} \cdot \Gamma_{(e)(b)(c)} - \Gamma_{(h)(a)(c)} \cdot \Gamma_{(e)(b)(d)} \right. \\ & \left. + \Gamma_{(a)(b)(h)} \cdot \left(\Gamma_{(e)(c)(d)} - \Gamma_{(e)(d)(c)}\right)\right],\end{aligned} \tag{7.160i}$$

$$\mathbb{R}_{(a)(b)(c)(d)}(x) = \mathrm{I}_{(a)(b)(c)(d)}(x) + \mathrm{II}_{(a)(b)(c)(d)}(x), \tag{7.160ii}$$

$$\mathrm{I}_{(a)(b)(c)(d)}(x) := \partial_{(d)}\Gamma_{(a)(b)(c)} - \partial_{(c)}\Gamma_{(a)(b)(d)}, \tag{7.160iii}$$

$$\begin{aligned}\mathrm{II}_{(a)(b)(c)(d)}(x) := {} & \eta^{(h)(e)} \cdot \left[\Gamma_{(h)(a)(d)} \cdot \Gamma_{(e)(b)(c)} - \Gamma_{(h)(a)(c)} \cdot \Gamma_{(e)(b)(d)} \right. \\ & \left. + \Gamma_{(a)(b)(h)} \cdot \left(\Gamma_{(e)(c)(d)} - \Gamma_{(e)(d)(c)}\right)\right].\end{aligned} \tag{7.160iv}$$

Computation of the quantities $\mathrm{I}_{(a)(b)(c)(d)}(x)$ and $\mathrm{II}_{(a)(b)(c)(d)}(x)$ in (7.160iii,iv), from (7.158i–iv) and (7.159i,ii), leads to the nonzero components:

$$\mathrm{I}_{(1)(2)(1)(2)}(x) = -\left[1 + (1/2) \cdot x^1 \cdot x^2\right], \tag{7.161i}$$

$$\mathrm{II}_{(1)(2)(1)(2)}(x) = -2 \cdot \Gamma_{(1)(2)(2)}(\cdot\cdot) \cdot \Gamma_{(1)(2)(1)}(\cdot\cdot) = (1/2) \cdot \left(x^1 \cdot x^2\right), \tag{7.161ii}$$

$$\mathbb{R}_{(1)(2)(1)(2)}(x) = \mathrm{I}_{(1)(2)(1)(2)}(x) + \mathrm{II}_{(1)(2)(1)(2)}(x) = -1. \tag{7.161iii}$$

Other, linearly dependent, nonzero components of $\mathbb{R}_{(a)(b)(c)(d)}(x)$ can be determined from (7.161iii) from the algebraic identities (1.142iv–viii).

The complex-valued Riemann tensor components satisfy the following equations:

$$\mathbb{R}_{(1)(2)(1)(2)}(x) = -1 = \eta_{(1)(1)} \cdot \eta_{(2)(2)} - \eta_{(1)(2)} \cdot \eta_{(2)(1)}. \tag{7.162i}$$

Moreover,

$$\mathbb{R}_{(a)(b)(c)(d)}(x) = \left[\begin{array}{ll} \eta_{(a)(c)} \cdot \eta_{(b)(d)} - \eta_{(a)(d)} \cdot \eta_{(b)(c)} & \text{for } a, b, c, d \in \{1, 2\}; \\ 0 & \text{otherwise.} \end{array}\right. \tag{7.162ii}$$

Therefore, (7.162ii), with reference to (1.164ii), reveal that the universe represented by the metric (7.156i) is topologically homeomorphic to $S^2 \times \mathbb{R}^2$. □

Remarks: (i) We have used complex conjugate coordinates ζ, $\overline{\zeta}$ in Examples 7.1.10 and 7.2.4. (We have mentioned these coordinates also in (A2.38).) These coordinates are very effective in solving some partial differential equations. However, *the complex conjugate coordinate $\overline{\zeta}$ is not independent of the complex variable ζ*. Therefore, there exists a logical gap in using such coordinates.

(ii) Similarly, the complex vectors $\vec{\mathbf{m}}(\cdot\cdot) := \left(1/\sqrt{2}\right) \cdot \left[\vec{\mathbf{e}}_{(1)}(\cdot\cdot) - \mathrm{i}\,\vec{\mathbf{e}}_{(2)}(\cdot\cdot)\right]$ and $\vec{\overline{\mathbf{m}}}(\cdot\cdot) = \left(1/\sqrt{2}\right) \cdot \left[\vec{\mathbf{e}}_{(1)}(\cdot\cdot) + \mathrm{i}\,\vec{\mathbf{e}}_{(2)}(\cdot\cdot)\right]$ are *not independent*. However, these vectors are used quite effectively as two basis vectors of a null, complex tetrad!

(iii) The metric of (7.156i) represents a special metric in a regular *pseudo-Riemannian manifold*. However, it is not explicitly connected with the gravitational field equations as such. To render the metric as a solution of Einstein's field equations, we have to compute the tensor field components $\mathbb{R}_{(a)(b)}(\cdot\cdot) - 1/2 \cdot \eta_{(a)(b)} \cdot \mathbb{R}(\cdot\cdot)$. Next, we need to *define* $\mathbb{T}_{(a)(b)}(\cdot\cdot)$ from (7.149i).

Now, we shall proceed to derive the *Newman–Penrose equations*. (See [35, 194, 239].) For that purpose, *we introduce new notations which are used only in this section of the book*. (This notation is often used in the literature when discussing the Newman–Penrose formalism.)

The directional derivatives are denoted by the following:

$$\boldsymbol{\delta} := \vec{\mathbf{E}}_{(1)}(\cdot\cdot) = \vec{\mathbf{m}}(\cdot\cdot) = m^i(\cdot\cdot) \cdot \partial_i = m^{(a)}(\cdot\cdot) \cdot \partial_{(a)}, \tag{7.163i}$$

$$\overline{\boldsymbol{\delta}} := \vec{\mathbf{E}}_{(2)}(\cdot\cdot) = \vec{\overline{\mathbf{m}}}(\cdot\cdot) = \overline{m}^i(\cdot\cdot) \cdot \partial_i = \overline{m}^{(a)}(\cdot\cdot) \cdot \partial_{(a)}, \tag{7.163ii}$$

$$\boldsymbol{\Delta} := \vec{\mathbf{E}}_{(3)}(\cdot\cdot) = \vec{\mathbf{l}}(\cdot\cdot) = l^i(\cdot\cdot) \cdot \partial_i = l^{(a)}(\cdot\cdot) \cdot \partial_{(a)}, \tag{7.163iii}$$

$$\mathbf{D} := \vec{\mathbf{E}}_{(4)}(\cdot\cdot) = \vec{\mathbf{k}}(\cdot\cdot) = k^i(\cdot\cdot) \cdot \partial_i = k^{(a)}(\cdot\cdot) \cdot \partial_{(a)}. \tag{7.163iv}$$

We denote *complex-valued Ricci rotation coefficients* (also called *spin coefficients*) as

$$\kappa_{(0)}(\cdot\cdot) := \Gamma_{(1)(4)(4)}(\cdot\cdot) = \left(\nabla_{(b)}k_{(a)}\right)\cdot m^{(a)}(\cdot\cdot)\cdot k^{(b)}(\cdot\cdot) = m^{(a)}(\cdot\cdot)\cdot\mathbf{D}k_{(a)}, \tag{7.164i}$$

$$\overline{\kappa}_{(0)}(\cdot\cdot) = \Gamma_{(2)(4)(4)}(\cdot\cdot), \tag{7.164ii}$$

$$\rho(\cdot\cdot) := \Gamma_{(1)(4)(2)}(\cdot\cdot) = \left(\nabla_{(b)}k_{(a)}\right)\cdot m^{(a)}(\cdot\cdot)\cdot\overline{m}^{(b)}(\cdot\cdot) = m^{(a)}(\cdot\cdot)\cdot\overline{\boldsymbol{\delta}}k_{(a)}, \tag{7.164iii}$$

$$\overline{\rho}(\cdot\cdot) = \Gamma_{(2)(4)(1)}(\cdot\cdot), \tag{7.164iv}$$

$$\sigma(\cdot\cdot) := \Gamma_{(1)(4)(1)}(\cdot\cdot) = \left(\nabla_{(b)}k_{(a)}\right)\cdot m^{(a)}(\cdot\cdot)\cdot m^{(b)}(\cdot\cdot) = m^{(a)}(\cdot\cdot)\cdot\boldsymbol{\delta}k_{(a)}, \tag{7.164v}$$

$$\overline{\sigma}(\cdot\cdot) = \Gamma_{(2)(4)(2)}(\cdot\cdot), \tag{7.164vi}$$

$$\tau(\cdot\cdot) := \Gamma_{(1)(4)(3)}(\cdot\cdot) = \left(\nabla_{(b)}k_{(a)}\right)\cdot m^{(a)}(\cdot\cdot)\cdot l^{(b)}(\cdot\cdot) = m^{(a)}(\cdot\cdot)\cdot\boldsymbol{\Delta}k_{(a)}, \tag{7.164vii}$$

$$\overline{\tau}(\cdot\cdot) = \Gamma_{(2)(4)(3)}(\cdot\cdot), \tag{7.164viii}$$

$$-\nu(\cdot\cdot) := \Gamma_{(2)(3)(3)}(\cdot\cdot) = \left(\nabla_{(b)}l_{(a)}\right)\cdot\overline{m}^{(a)}(\cdot\cdot)\cdot l^{(b)}(\cdot\cdot) = \overline{m}^{(a)}(\cdot\cdot)\cdot\boldsymbol{\Delta}l_{(a)}, \tag{7.164ix}$$

$$-\overline{\nu}(\cdot\cdot) = \Gamma_{(1)(3)(3)}(\cdot\cdot), \tag{7.164x}$$

$$-\mu(\cdot\cdot) := \Gamma_{(2)(3)(1)}(\cdot\cdot) = \left(\nabla_{(b)}l_{(a)}\right)\cdot\overline{m}^{(a)}(\cdot\cdot)\cdot m^{(b)}(\cdot\cdot) = \overline{m}^{(a)}(\cdot\cdot)\cdot\boldsymbol{\delta}l_{(a)}, \tag{7.164xi}$$

$$-\overline{\mu}(\cdot\cdot) = \Gamma_{(1)(3)(2)}(\cdot\cdot), \tag{7.164xii}$$

$$-\lambda(\cdot\cdot) := \mathbb{\Gamma}_{(2)(3)(2)}(\cdot\cdot) = \left(\nabla_{(b)}l_{(a)}\right)\cdot\overline{m}^{(a)}(\cdot\cdot)\cdot\overline{m}^{(b)}(\cdot\cdot) = \overline{m}^{(a)}(\cdot\cdot)\cdot\overline{\boldsymbol{\delta}}l_{(a)}, \quad (7.164\text{xiii})$$

$$-\overline{\lambda}(\cdot\cdot) = \mathbb{\Gamma}_{(1)(3)(1)}(\cdot\cdot), \quad (7.164\text{xiv})$$

$$-\pi_{(0)}(\cdot\cdot) := \mathbb{\Gamma}_{(2)(3)(4)}(\cdot\cdot) = \left(\nabla_{(b)}l_{(a)}\right)\cdot\overline{m}^{(a)}(\cdot\cdot)\cdot k^{(b)}(\cdot\cdot) = \overline{m}^{(a)}(\cdot\cdot)\cdot\mathbf{D}l_{(a)}, \quad (7.164\text{xv})$$

$$-\overline{\pi}_{(0)}(\cdot\cdot) = \mathbb{\Gamma}_{(1)(3)(4)}(\cdot\cdot), \quad (7.164\text{xvi})$$

$$\begin{aligned}\varepsilon(\cdot\cdot) &:= (1/2)\cdot\left[\mathbb{\Gamma}_{(3)(4)(4)} - \mathbb{\Gamma}_{(2)(1)(4)}\right]\\ &= (1/2)\cdot\left[\left(\nabla_{(b)}k_{(a)}\right)\cdot l^{(a)}\cdot k^{(b)} - \left(\nabla_{(b)}m_{(a)}\right)\cdot\overline{m}^{(a)}\cdot k^{(b)}\right]\\ &= (1/2)\cdot\left[l^{(a)}\cdot\mathbf{D}k_{(a)} - \overline{m}^{(a)}\cdot\mathbf{D}m_{(a)}\right], \quad (7.164\text{xvii})\end{aligned}$$

$$\overline{\varepsilon}(\cdot\cdot) = (1/2)\left[\mathbb{\Gamma}_{(3)(4)(4)} - \mathbb{\Gamma}_{(1)(2)(4)}\right], \quad (7.164\text{xviii})$$

$$\begin{aligned}\beta(\cdot\cdot) &:= (1/2)\cdot\left[\mathbb{\Gamma}_{(3)(4)(1)} - \mathbb{\Gamma}_{(2)(1)(1)}\right]\\ &= (1/2)\cdot\left[\left(\nabla_{(b)}k_{(a)}\right)\cdot l^{(a)}\cdot m^{(b)} - \left(\nabla_{(b)}m_{(a)}\right)\cdot\overline{m}^{(a)}\cdot m^{(b)}\right]\\ &= (1/2)\cdot\left[l^{(a)}\cdot\boldsymbol{\delta}k_{(a)} - \overline{m}^{(a)}\cdot\boldsymbol{\delta}m_{(a)}\right], \quad (7.164\text{xix})\end{aligned}$$

$$\overline{\beta}(\cdot\cdot) = (1/2)\cdot\left[\mathbb{\Gamma}_{(3)(4)(2)} - \mathbb{\Gamma}_{(1)(2)(2)}\right], \quad (7.164\text{xx})$$

$$\begin{aligned}-\gamma(\cdot\cdot) &:= (1/2)\cdot\left[\mathbb{\Gamma}_{(4)(3)(3)} - \mathbb{\Gamma}_{(1)(2)(3)}\right]\\ &= (1/2)\cdot\left[\left(\nabla_{(b)}l_{(a)}\right)\cdot k^{(a)}\cdot l^{(b)} - \left(\nabla_{(b)}\overline{m}_{(a)}\right)\cdot m^{(a)}\cdot l^{(b)}\right]\\ &= (1/2)\cdot\left[k^{(a)}\cdot\boldsymbol{\Delta}l_{(a)} - m^{(a)}\cdot\boldsymbol{\Delta}\overline{m}_{(a)}\right], \quad (7.164\text{xxi})\end{aligned}$$

$$-\overline{\gamma}(\cdot\cdot) = (1/2)\cdot\left[\mathbb{\Gamma}_{(4)(3)(3)} - \mathbb{\Gamma}_{(2)(1)(3)}\right], \quad (7.164\text{xxii})$$

$$\begin{aligned}-\alpha(\cdot\cdot) &:= (1/2)\cdot\left[\mathbb{\Gamma}_{(4)(3)(2)} - \mathbb{\Gamma}_{(1)(2)(2)}\right]\\ &= (1/2)\cdot\left[\left(\nabla_{(b)}l_{(a)}\right)\cdot k^{(a)}\cdot\overline{m}^{(b)} - \left(\nabla_{(b)}\overline{m}_{(a)}\right)\cdot m^{(a)}\cdot\overline{m}^{(b)}\right]\\ &= (1/2)\cdot\left[k^{(a)}\cdot\overline{\boldsymbol{\delta}}l_{(a)} - m^{(a)}\cdot\overline{\boldsymbol{\delta}}\overline{m}_{(a)}\right], \quad (7.164\text{xxiii})\end{aligned}$$

$$-\overline{\alpha}(\cdot\cdot) = (1/2)\cdot\left[-\mathbb{\Gamma}_{(3)(4)(1)} + \mathbb{\Gamma}_{(1)(2)(1)}\right]. \quad (7.164\text{xxiv})$$

Now, we shall compute integrability conditions (7.145i) with help of the new symbols (7.163i–iv) for directional derivatives and *the spin coefficients* defined in (7.164i–xxiv). Let us work out from (7.145i) the integrability condition

$$\begin{aligned}&\boldsymbol{\delta}\mathbf{D}f - \mathbf{D}\boldsymbol{\delta}f\\ &=\partial_{(1)}\partial_{(4)}f - \partial_{(4)}\partial_{(1)}f = \mathbb{\Gamma}_{(e)(1)(4)}\cdot\partial^{(e)}f - \mathbb{\Gamma}_{(e)(4)(1)}\cdot\partial^{(e)}f\\ &=\mathbb{\Gamma}_{(2)(1)(4)}\cdot\partial^{(2)}f + \mathbb{\Gamma}_{(3)(1)(4)}\cdot\partial^{(3)}f + \mathbb{\Gamma}_{(4)(1)(4)}\cdot\partial^{(4)}f\end{aligned}$$

$$
\begin{aligned}
&- \mathbb{\Gamma}_{(1)(4)(1)} \cdot \partial^{(1)} f - \mathbb{\Gamma}_{(2)(4)(1)} \cdot \partial^{(2)} f - \mathbb{\Gamma}_{(3)(4)(1)} \cdot \partial^{(3)} f \\
= &- \mathbb{\Gamma}_{(1)(2)(4)} \cdot \boldsymbol{\delta} f + \mathbb{\Gamma}_{(1)(3)(4)} \cdot \mathbf{D} f + \mathbb{\Gamma}_{(1)(4)(4)} \cdot \boldsymbol{\Delta} f \\
&- \mathbb{\Gamma}_{(1)(4)(1)} \cdot \overline{\boldsymbol{\delta}} f - \mathbb{\Gamma}_{(2)(4)(1)} \cdot \boldsymbol{\delta} f + \mathbb{\Gamma}_{(3)(4)(1)} \cdot \mathbf{D} f \\
= &- \left(\mathbb{\Gamma}_{(1)(2)(4)} + \mathbb{\Gamma}_{(2)(4)(1)}\right) \cdot \boldsymbol{\delta} f + \left(\mathbb{\Gamma}_{(1)(3)(4)} + \mathbb{\Gamma}_{(3)(4)(1)}\right) \cdot \mathbf{D} f \\
&+ \mathbb{\Gamma}_{(1)(4)(4)} \cdot \boldsymbol{\Delta} f - \mathbb{\Gamma}_{(1)(4)(1)} \cdot \overline{\boldsymbol{\delta}} f \\
= &- (\overline{\rho} + \varepsilon - \overline{\varepsilon}) \cdot \boldsymbol{\delta} f + \left(\overline{\alpha} + \beta - \overline{\pi}_{(0)}\right) \cdot \mathbf{D} f + \kappa_{(0)} \cdot \boldsymbol{\Delta} f - \sigma \cdot \overline{\boldsymbol{\delta}} f.
\end{aligned}
\tag{7.165}
$$

Similarly, the other three nontrivial integrability conditions can be worked out. We list *all four conditions* in the sequel.

$$
\begin{aligned}
&[\boldsymbol{\Delta}\mathbf{D} - \mathbf{D}\boldsymbol{\Delta}]\,[f] \\
&\quad = \left[(\gamma + \overline{\gamma}) \cdot \mathbf{D} + (\varepsilon + \overline{\varepsilon}) \cdot \boldsymbol{\Delta} - \left(\tau + \overline{\pi}_{(0)}\right) \cdot \overline{\boldsymbol{\delta}} - \left(\overline{\tau} + \pi_{(0)}\right) \cdot \boldsymbol{\delta}\right][f], \qquad (7.166\text{i}) \\
&[\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta}]\,[f] \\
&\quad = \left[\left(\overline{\alpha} + \beta - \overline{\pi}_{(0)}\right) \cdot \mathbf{D} + \kappa_{(0)} \cdot \boldsymbol{\Delta} - \sigma \cdot \overline{\boldsymbol{\delta}} - (\overline{\rho} + \varepsilon - \overline{\varepsilon}) \cdot \boldsymbol{\delta}\right][f], \qquad (7.166\text{ii}) \\
&[\boldsymbol{\delta}\boldsymbol{\Delta} - \boldsymbol{\Delta}\boldsymbol{\delta}]\,[f] \\
&\quad = \left[-\overline{\nu} \cdot \mathbf{D} + (\tau - \overline{\alpha} - \beta) \cdot \boldsymbol{\Delta} + \overline{\lambda} \cdot \overline{\boldsymbol{\delta}} + (\mu - \gamma + \overline{\gamma}) \cdot \boldsymbol{\delta}\right][f], \qquad (7.166\text{iii}) \\
&\left[\overline{\boldsymbol{\delta}}\boldsymbol{\delta} - \boldsymbol{\delta}\overline{\boldsymbol{\delta}}\right][f] \\
&\quad = \left[(\overline{\mu} - \mu) \cdot \mathbf{D} + (\overline{\rho} - \rho) \cdot \boldsymbol{\Delta} - (\overline{\alpha} - \beta) \cdot \overline{\boldsymbol{\delta}} - \left(\overline{\beta} - \alpha\right) \cdot \boldsymbol{\delta}\right][f]. \qquad (7.166\text{iv})
\end{aligned}
$$

Now, the complex-valued components of the conformal tensor $\mathbb{C}_{(a)(b)(c)(d)}(x)$ will be computed from (7.25viii), (7.74i–v), and (7.141i–iv). These are furnished by

$$
\begin{aligned}
\Psi_{(0)}(x) &= \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot m^{(b)}(\cdot\cdot) \cdot k^{(c)}(\cdot\cdot) \cdot m^{(d)}(\cdot\cdot) \\
&= \mathbf{C}_{....}\left(\vec{\mathbf{E}}_{(4)}, \vec{\mathbf{E}}_{(1)}, \vec{\mathbf{E}}_{(4)}, \vec{\mathbf{E}}_{(1)}\right) = \mathbb{C}_{(4)(1)(4)(1)}(x), && (7.167\text{i}) \\
\Psi_{(1)}(x) &= \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot l^{(b)}(\cdot\cdot) \cdot k^{(c)}(\cdot\cdot) \cdot m^{(d)}(\cdot\cdot) \\
&= \mathbb{C}_{(4)(3)(4)(1)}(x), && (7.167\text{ii}) \\
\Psi_{(2)}(x) &= \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot m^{(b)}(\cdot\cdot) \cdot \overline{m}^{(c)}(\cdot\cdot) \cdot l^{(d)}(\cdot\cdot) \\
&= \mathbb{C}_{(4)(1)(2)(3)}(x), && (7.167\text{iii}) \\
\Psi_{(3)}(x) &= \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot l^{(b)}(\cdot\cdot) \cdot \overline{m}^{(c)}(\cdot\cdot) \cdot l^{(d)}(\cdot\cdot) \\
&= \mathbb{C}_{(4)(3)(2)(3)}(x), && (7.167\text{iv})
\end{aligned}
$$

$$\begin{aligned}\Psi_{(4)}(x) &= \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot)\cdot \overline{m}^{(a)}(\cdot\cdot)\cdot l^{(b)}(\cdot\cdot)\cdot \overline{m}^{(c)}(\cdot\cdot)\cdot l^{(d)}(\cdot\cdot)\\ &= \mathbb{C}_{(2)(3)(2)(3)}(x),\end{aligned} \tag{7.167v}$$

$$\begin{aligned}0 \equiv \mathbb{C}^{(a)}{}_{(b)(c)(a)}(\cdot\cdot) &= \mathbb{C}_{(2)(b)(c)(1)}(x) + \mathbb{C}_{(1)(b)(c)(2)}(x)\\ &\quad - \mathbb{C}_{(4)(b)(c)(3)}(x) - \mathbb{C}_{(3)(b)(c)(4)}(x)\end{aligned} \tag{7.167vi}$$

$$\begin{aligned}\mathbb{C}_{(3)(1)(1)(4)}(x) &= \mathbb{C}_{(3)(2)(2)(4)}(x) = \mathbb{C}_{(1)(3)(3)(2)}(x) = \mathbb{C}_{(1)(4)(4)(2)}(x)\\ &= 0,\end{aligned} \tag{7.167vii}$$

$$\mathbb{C}_{(1)(2)(3)(1)}(x) = \mathbb{C}_{(1)(3)(3)(4)}(x),\ \mathbb{C}_{(1)(2)(2)(3)}(x) = \mathbb{C}_{(2)(3)(3)(4)}(x), \tag{7.167viii}$$

$$\mathbb{C}_{(1)(2)(2)(4)}(x) = \mathbb{C}_{(2)(4)(4)(3)}(x),\ \mathbb{C}_{(4)(1)(1)(2)}(x) = \mathbb{C}_{(1)(4)(4)(3)}(x), \tag{7.167ix}$$

$$\begin{aligned}\mathbb{C}_{(1)(4)(2)(3)}(x) &= \frac{1}{2}\cdot\left(\mathbb{C}_{(1)(2)(2)(1)}(x) - \mathbb{C}_{(1)(2)(3)(4)}(x)\right)\\ &= -\frac{1}{2}\left(\mathbb{C}_{(3)(4)(3)(4)}(x) + \mathbb{C}_{(1)(2)(3)(4)}(x)\right),\end{aligned} \tag{7.167x}$$

$$\begin{aligned}&2\cdot\Psi_{(2)}(x) + \mathbb{C}_{(3)(4)(2)(1)}(x) + \mathbb{C}_{(1)(2)(2)(1)}(x)\\ &\quad = -2\cdot\Psi_{(2)} + \mathbb{C}_{(1)(2)(3)(4)}(x) + \mathbb{C}_{(3)(4)(3)(4)}(x) = 0.\end{aligned} \tag{7.167xi}$$

Now we shall compute the components $\mathbb{R}_{(a)(b)(c)(d)}(x)$ and $\mathbb{C}_{(a)(b)(c)(d)}(x)$ from (7.160i), (1.169ii), (7.25i–viii), (7.150), and (7.167i–v). Let us work out the particular component $\mathbb{C}_{(4)(1)(4)(1)}(\cdot\cdot)$. We obtain

$$\begin{aligned}\mathbb{C}_{(4)(1)(4)(1)}(x) &= \mathbb{R}_{(4)(1)(4)(1)}(\cdot\cdot)\\ &\quad + (1/2)\cdot\left[\eta_{(4)(4)}\cdot\mathbb{R}_{(1)(1)} - \eta_{(4)(1)}\cdot\mathbb{R}_{(1)(4)} + \eta_{(1)(1)}\cdot\mathbb{R}_{(4)(4)} - \eta_{(1)(4)}\cdot\mathbb{R}_{(4)(1)}\right]\\ &\quad + (1/6)\cdot\left[\eta_{(4)(1)}\cdot\eta_{(1)(4)} - \eta_{(4)(4)}\cdot\eta_{(1)(1)}\right]\cdot\mathbb{R}(\cdot\cdot)\\ &= \mathbb{R}_{(4)(1)(4)(1)}(\cdot\cdot) + 0 + 0 = \mathbb{R}_{(4)(1)(4)(1)}(\cdot\cdot).\end{aligned} \tag{7.168i}$$

From (7.167i) and (7.160i), we derive that

$$\begin{aligned}\Psi_{(0)}(\cdot\cdot) &= \mathbb{C}_{(4)(1)(4)(1)}(\cdot\cdot) = \mathbb{R}_{(4)(1)(4)(1)}(\cdot\cdot)\\ &= \partial_{(1)}\Gamma_{(4)(1)(4)} - \partial_{(4)}\Gamma_{(4)(1)(1)} + \eta^{(h)(e)}\\ &\quad\times\left[\Gamma_{(h)(4)(1)}\cdot\Gamma_{(e)(1)(4)} - \Gamma_{(h)(4)(4)}\cdot\Gamma_{(e)(1)(1)}\right.\\ &\quad\left. + \Gamma_{(4)(1)(h)}\cdot\left(\Gamma_{(e)(4)(1)} - \Gamma_{(e)(1)(4)}\right)\right].\end{aligned} \tag{7.168ii}$$

Thus, using (7.163i,iv) and (7.164i–xxiv), we obtain

$$\begin{aligned}
\mathbf{D}\sigma - \boldsymbol{\delta}\kappa_{(0)} &= \Psi_{(0)}(\cdot\cdot) - \eta^{(h)(e)} \cdot \Gamma_{(h)(4)(1)} \cdot \Gamma_{(e)(1)(4)} \\
&\quad + \eta^{(h)(e)} \cdot \Gamma_{(h)(4)(4)} \cdot \Gamma_{(e)(1)(1)} - \eta^{(h)(e)} \cdot \Gamma_{(4)(1)(h)} \cdot \Gamma_{(e)(4)(1)} \\
&\quad + \eta^{(h)(e)} \cdot \Gamma_{(4)(1)(h)} \cdot \Gamma_{(e)(1)(4)} \\
&= \Psi_{(0)}(\cdot\cdot) - \left[\eta^{(1)(2)} \cdot \Gamma_{(1)(4)(1)} \cdot \Gamma_{(2)(1)(4)} + \eta^{(3)(4)} \cdot \Gamma_{(3)(4)(1)} \cdot \Gamma_{(4)(1)(4)}\right] \\
&\quad + \left[\eta^{(1)(2)} \cdot \Gamma_{(1)(4)(4)} \cdot \Gamma_{(2)(1)(1)} + \eta^{(3)(4)} \cdot \Gamma_{(3)(4)(4)} \cdot \Gamma_{(4)(1)(1)}\right] \\
&\quad - \left[\eta^{(1)(2)} \cdot \Gamma_{(4)(1)(1)} \cdot \Gamma_{(2)(4)(1)} + \eta^{(2)(1)} \cdot \Gamma_{(4)(1)(2)} \cdot \Gamma_{(1)(4)(1)}\right. \\
&\quad \left. + \eta^{(4)(3)} \cdot \Gamma_{(4)(1)(4)} \cdot \Gamma_{(3)(4)(1)}\right] + \left[\eta^{(1)(2)} \cdot \Gamma_{(4)(1)(1)} \cdot \Gamma_{(2)(1)(4)}\right. \\
&\quad \left. + \eta^{(3)(4)} \cdot \Gamma_{(4)(1)(3)} \cdot \Gamma_{(4)(1)(4)} + \eta^{(4)(3)} \cdot \Gamma_{(4)(1)(4)} \cdot \Gamma_{(3)(1)(4)}\right] \\
&= \Psi_{(0)}(\cdot\cdot) + \left[\Gamma_{(1)(4)(2)} + \Gamma_{(2)(4)(1)} + \Gamma_{(3)(4)(4)} + 2 \cdot \Gamma_{(1)(2)(4)}\right] \cdot \Gamma_{(1)(4)(1)} \\
&\quad - \left[\Gamma_{(1)(4)(3)} + \Gamma_{(1)(3)(4)} + 2 \cdot \Gamma_{(3)(4)(1)} + \Gamma_{(1)(2)(1)}\right] \cdot \Gamma_{(1)(4)(4)} \\
&= (\rho + \overline{\rho}) \cdot \sigma + (3\varepsilon - \overline{\varepsilon}) \cdot \sigma \\
&\quad - \left(\tau - \overline{\pi}_{(0)} + \overline{\alpha} + 3\beta\right) \cdot \kappa_{(0)} + \Psi_{(0)}(\cdot\cdot).
\end{aligned} \tag{7.169}$$

We have just deduced one of the Newman–Penrose equations! To derive all of these equations, we need to obtain components of the complex-valued Ricci tensor $\mathbb{R}_{(a)(b)}(\cdot\cdot)$ or $\Sigma_{(a)(b)}(\cdot\cdot)$ from (7.119i–vi) and (7.152i). Let us calculate the component

$$\begin{aligned}
-\Phi_{(1)(1)}(\cdot\cdot) &= (1/4) \cdot \left[\mathbb{R}_{(a)(b)} - (1/4) \cdot \eta_{(a)(b)} \cdot \mathbb{R}\right] \cdot \left[k^{(a)} \cdot l^{(b)} + m^{(a)} \cdot \overline{m}^{(b)}\right] \\
&= (1/4) \cdot \left[\left(\mathbb{R}_{(4)(3)} - (1/4) \cdot \eta_{(4)(3)} \cdot \mathbb{R}\right) + \left(\mathbb{R}_{(1)(2)} - (1/4) \cdot \eta_{(1)(2)} \cdot \mathbb{R}\right)\right] \\
&= (1/4) \cdot \left[\mathbb{R}_{(4)(3)} + (1/4) \cdot \mathbb{R} + \mathbb{R}_{(1)(2)} - (1/4) \cdot (\mathbb{R})\right] \\
&= (1/4) \cdot \left[\mathbb{R}_{(4)(3)} + \mathbb{R}_{(1)(2)}\right] = -\overline{\Phi_{(1)(1)}}(\cdot\cdot).
\end{aligned}$$

Similarly, we compute the other components and display all of them in the following[6]:

$$\begin{aligned}
-\Phi_{(0)(0)}(\cdot\cdot) &= (1/2) \cdot \Sigma_{(a)(b)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot k^{(b)}(\cdot\cdot) = -\overline{\Phi_{(0)(0)}}(\cdot\cdot) \\
&= (1/2) \cdot \mathbb{R}_{(4)(4)}(\cdot\cdot),
\end{aligned} \tag{7.170i}$$

[6]The notation $\mathbb{R}_{(a)(b)}(\cdot\cdot)$ in this section *differs from* $\mathbb{R}_{(A)(B)}(\cdot\cdot)$ of the preceding section.

$$-\Phi_{(0)(1)}(\cdot\cdot) = (1/2)\cdot \Sigma_{(a)(b)}(\cdot\cdot)\cdot k^{(c)}(\cdot\cdot)\cdot m^{(b)}(\cdot\cdot) = -\overline{\Phi_{(1)(0)}}(\cdot\cdot)$$
$$= (1/2)\cdot \mathbb{R}_{(4)(1)}(\cdot\cdot), \tag{7.170ii}$$
$$-\Phi_{(1)(0)}(\cdot\cdot) = (1/2)\cdot \Sigma_{(a)(b)}(\cdot\cdot)\cdot \overline{m}^{(a)}(\cdot\cdot)\cdot k^{(b)}(\cdot\cdot)$$
$$= (1/2)\cdot \mathbb{R}_{(2)(4)}(\cdot\cdot), \tag{7.170iii}$$
$$-\Phi_{(0)(2)}(\cdot\cdot) = (1/2)\cdot \Sigma_{(a)(b)}(\cdot\cdot)\cdot m^{(a)}(\cdot\cdot)\cdot m^{(b)}(\cdot\cdot) = -\overline{\Phi_{(2)(0)}}(\cdot\cdot)$$
$$= (1/2)\cdot \mathbb{R}_{(1)(1)}(\cdot\cdot), \tag{7.170iv}$$
$$-\Phi_{(2)(0)}(\cdot\cdot) = (1/2)\cdot \Sigma_{(a)(b)}(\cdot\cdot)\cdot \overline{m}^{(a)}(\cdot\cdot)\cdot \overline{m}^{(b)}(\cdot\cdot)$$
$$= (1/2)\cdot \mathbb{R}_{(2)(2)}(\cdot\cdot), \tag{7.170v}$$
$$-\Phi_{(1)(1)}(\cdot\cdot) = (1/4)\cdot \Sigma_{(a)(b)}(\cdot\cdot)\cdot \left[k^{(a)}\cdot l^{(b)} + m^{(a)}\cdot \overline{m}^{(b)}\right] = -\overline{\Phi_{(1)(1)}}(\cdot\cdot)$$
$$= (1/4)\cdot \left[\mathbb{R}_{(4)(3)} + \mathbb{R}_{(1)(2)}\right], \tag{7.170vi}$$
$$-\Phi_{(1)(2)}(\cdot\cdot) = (1/2)\cdot \Sigma_{(a)(b)}(\cdot\cdot)\cdot l^{(a)}(\cdot\cdot)\cdot m^{(b)}(\cdot\cdot) = -\overline{\Phi_{(2)(1)}}(\cdot\cdot)$$
$$= (1/2)\cdot \mathbb{R}_{(3)(1)}(\cdot\cdot), \tag{7.170vii}$$
$$-\Phi_{(2)(1)}(\cdot\cdot) = (1/2)\cdot \Sigma_{(a)(b)}(\cdot\cdot)\cdot l^{(a)}(\cdot\cdot)\cdot \overline{m}^{(b)}(\cdot\cdot)$$
$$= (1/2)\cdot \mathbb{R}_{(3)(2)}(\cdot\cdot), \tag{7.170viii}$$
$$-\Phi_{(2)(2)}(\cdot\cdot) = (1/2)\cdot \Sigma_{(a)(b)}(\cdot\cdot)\cdot l^{(a)}(\cdot\cdot)\cdot l^{(b)}(\cdot\cdot) = -\overline{\Phi_{(2)(2)}}(\cdot\cdot)$$
$$= (1/2)\cdot \mathbb{R}_{(3)(3)}(\cdot\cdot), \tag{7.170ix}$$
$$0 \equiv \eta^{(a)(b)}\cdot \Sigma_{(a)(b)} = 2\left(\Sigma_{(1)(2)} - \Sigma_{(3)(4)}\right) = 0, \tag{7.170x}$$
$$\mathbb{R} = \eta^{(a)(b)}\cdot \mathbb{R}_{(a)(b)} = 2\left(\mathbb{R}_{(1)(2)} - \mathbb{R}_{(3)(4)}\right). \tag{7.170xi}$$

Here, we note that $\Phi_{(0)(0)}(\cdot\cdot)$, $\Phi_{(1)(1)}(\cdot\cdot)$, and $\Phi_{(2)(2)}(\cdot\cdot)$ are real-valued, whereas $\Phi_{(0)(1)}(\cdot\cdot)$, $\Phi_{(1)(0)}(\cdot\cdot)$, $\Phi_{(0)(2)}(\cdot\cdot)$, $\Phi_{(2)(0)}(\cdot\cdot)$, $\Phi_{(1)(2)}(\cdot\cdot)$, and $\Phi_{(2)(1)}(\cdot\cdot)$ are complex-valued. Now, recall the components $\mathbb{R}_{(a)(b)(c)(d)}(x)$ from (7.160i). These are provided by

$$\mathbb{R}_{(a)(b)(c)(d)}(x) = \partial_{(d)}\Gamma_{(a)(b)(c)} - \partial_{(c)}\Gamma_{(a)(b)(d)}$$
$$+ \eta^{(h)(e)}\cdot\left[\Gamma_{(h)(a)(d)}\cdot\Gamma_{(e)(b)(c)} - \Gamma_{(h)(a)(c)}\cdot\Gamma_{(e)(b)(d)}\right.$$
$$\left.+ \Gamma_{(a)(b)(h)}\cdot\left(\Gamma_{(e)(c)(d)} - \Gamma_{(e)(d)(c)}\right)\right]. \tag{7.171}$$

From (7.150) and (7.153), we have

$$\mathbb{R}_{(a)(b)(c)(d)}(x) = \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot) + (1/2)\cdot\big[\eta_{(a)(d)}\cdot\mathbb{R}_{(b)(c)} - \eta_{(a)(c)}\cdot\mathbb{R}_{(b)(d)} + \eta_{(b)(c)}\cdot\mathbb{R}_{(a)(d)} - \eta_{(b)(d)}\cdot\mathbb{R}_{(a)(c)} + (1/3)\cdot\big(\eta_{(a)(c)}\cdot\eta_{(b)(d)} - \eta_{(a)(d)}\cdot\eta_{(b)(c)}\big)\cdot\mathbb{R}(\cdot\cdot)\big], \tag{7.172i}$$

$$\mathbb{R}_{(a)(b)(c)(d)}(x) = \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot) + (1/2)\cdot\big[\eta_{(a)(d)}\cdot\Sigma_{(b)(c)} - \eta_{(a)(c)}\cdot\Sigma_{(b)(d)} + \eta_{(b)(c)}\cdot\Sigma_{(a)(d)} - \eta_{(b)(d)}\cdot\Sigma_{(a)(c)} + (1/6)\cdot\big(\eta_{(a)(d)}\cdot\eta_{(b)(c)} - \eta_{(a)(c)}\cdot\eta_{(b)(d)}\big)\cdot\mathbb{R}(\cdot\cdot)\big]. \tag{7.172ii}$$

Also, from (7.171) and (7.172i,ii) , we have that

$$\partial_{(d)}\Gamma_{(a)(b)(c)} - \partial_{(c)}\Gamma_{(a)(b)(d)} = \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot) + \eta^{(h)(e)}\cdot\big[\Gamma_{(h)(a)(c)}\cdot\Gamma_{(e)(b)(d)} - \Gamma_{(h)(a)(d)}\Gamma_{(e)(b)(c)} + \Gamma_{(a)(b)(h)}\cdot\big(\Gamma_{(e)(d)(c)} - \Gamma_{(e)(c)(d)}\big)\big] + (1/2)\cdot\big[\eta_{(a)(d)}\cdot\Sigma_{(b)(c)} - \eta_{(a)(c)}\cdot\Sigma_{(b)(d)} + \eta_{(b)(c)}\cdot\Sigma_{(a)(d)} - \eta_{(b)(d)}\cdot\Sigma_{(a)(c)} + (1/6)\cdot\big(\eta_{(a)(d)}\cdot\eta_{(b)(c)} - \eta_{(a)(c)}\cdot\eta_{(b)(d)}\big)\cdot\mathbb{R}(\cdot\cdot)\big]. \tag{7.173}$$

Multiplying the above equations with the "orthonormal components" of $\vec{\mathbf{m}}(\cdot\cdot), \overline{\vec{\mathbf{m}}}(\cdot\cdot), \vec{\mathbf{l}}(\cdot\cdot)$, and $\vec{\mathbf{k}}(\cdot\cdot)$, and using (7.163i–iv), (7.164i–xxiv), (7.167i–vi), (7.170i–ix), and (7.173) (and with occasional linear combinations), we can derive *the Newman–Penrose equations*. All of these equations are furnished here[7]:

$$\mathbf{D}\rho - \overline{\boldsymbol{\delta}}\kappa_{(0)} = \rho^2 + \sigma\cdot\overline{\sigma} + (\varepsilon + \overline{\varepsilon})\cdot\rho - \overline{\kappa}_{(0)}\cdot\tau - \kappa_{(0)}\cdot\Big(3\alpha + \overline{\beta} - \pi_{(0)}\Big) + \Phi_{(0)(0)}, \tag{7.174i}$$

$$\mathbf{D}\sigma - \boldsymbol{\delta}\kappa_{(0)} = (\rho + \overline{\rho})\cdot\sigma + (3\varepsilon - \overline{\varepsilon})\cdot\sigma - \big(\tau - \overline{\pi}_{(0)} + \overline{\alpha} + 3\beta\big)\cdot\kappa_{(0)} + \Psi_{(0)}, \tag{7.174ii}$$

$$\mathbf{D}\tau - \boldsymbol{\Delta}\kappa_{(0)} = \big(\tau + \overline{\pi}_{(0)}\big)\cdot\rho + \big(\overline{\tau} + \pi_{(0)}\big)\cdot\sigma + (\varepsilon - \overline{\varepsilon})\cdot\tau - (3\gamma + \overline{\gamma})\cdot\kappa_{(0)} + \Psi_{(1)} + \Phi_{(0)(1)}, \tag{7.174iii}$$

$$\mathbf{D}\alpha - \overline{\boldsymbol{\delta}}\varepsilon = (\rho + \overline{\varepsilon} - 2\varepsilon)\cdot\alpha + \beta\cdot\overline{\sigma} - \overline{\beta}\cdot\varepsilon - \kappa_{(0)}\cdot\lambda - \overline{\kappa}_{(0)}\cdot\gamma + (\varepsilon + \rho)\cdot\pi_{(0)} + \Phi_{(1)(0)}, \tag{7.174iv}$$

[7]The following equations (7.174i–xviii) coincide with those cited in most of the literature. Compared to some sources, *the sign of certain components (especially* $\mathbb{R}(x)$*) is different*. This discrepancy occurs due to the usage of different conventions.

$$\begin{aligned}
\mathbf{D}\beta - \boldsymbol{\delta}\varepsilon = {} & \left(\alpha + \pi_{(0)}\right)\cdot\sigma + (\overline{\rho} - \overline{\varepsilon})\cdot\beta - (\mu + \gamma)\cdot\kappa_{(0)} \\
& - \left(\overline{\alpha} - \overline{\pi}_{(0)}\right)\cdot\varepsilon + \Psi_{(1)}, \qquad (7.174\text{v})
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}\gamma - \boldsymbol{\Delta}\varepsilon = {} & \left(\tau + \overline{\pi}_{(0)}\right)\cdot\alpha + \left(\overline{\tau} + \pi_{(0)}\right)\cdot\beta - (\varepsilon + \overline{\varepsilon})\cdot\gamma \\
& - (\gamma + \overline{\gamma})\cdot\varepsilon + \tau\cdot\pi_{(0)} - \nu\cdot\kappa_{(0)} + \Psi_{(2)} + \Phi_{(1)(1)} + (\mathbb{R}/24)\,, \qquad (7.174\text{vi})
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}\lambda - \overline{\boldsymbol{\delta}}\pi_{(0)} = {} & \rho\cdot\lambda + \overline{\sigma}\cdot\mu + \left(\pi_{(0)}\right)^2 + \left(\alpha - \overline{\beta}\right)\cdot\pi_{(0)} \\
& - \nu\cdot\overline{\kappa}_{(0)} - (3\varepsilon - \overline{\varepsilon})\cdot\lambda + \Phi_{(2)(0)}, \qquad (7.174\text{vii})
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}\mu - \boldsymbol{\delta}\pi_{(0)} = {} & \overline{\rho}\cdot\mu + \sigma\cdot\lambda + \pi_{(0)}\cdot\overline{\pi}_{(0)} - (\varepsilon + \overline{\varepsilon})\cdot\mu \\
& - \pi_{(0)}\cdot(\overline{\alpha} - \beta) - \nu\cdot\kappa_{(0)} + \Psi_{(2)} - (\mathbb{R}/12)\,, \qquad (7.174\text{viii})
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}\nu - \boldsymbol{\Delta}\pi_{(0)} = {} & \left(\pi_{(0)} + \overline{\tau}\right)\cdot\mu + \left(\overline{\pi}_{(0)} + \tau\right)\cdot\lambda + (\gamma - \overline{\gamma})\cdot\pi_{(0)} \\
& - (3\varepsilon + \overline{\varepsilon})\cdot\nu + \Psi_{(3)} + \Phi_{(2)(1)}, \qquad (7.174\text{ix})
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Delta}\lambda - \overline{\boldsymbol{\delta}}\nu = {} & -(\mu + \overline{\mu})\cdot\lambda - (3\gamma - \overline{\gamma})\cdot\lambda \\
& + \left(3\alpha + \overline{\beta} + \pi_{(0)} - \overline{\tau}\right)\cdot\nu - \Psi_{(4)}, \qquad (7.174\text{x})
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\delta}\rho - \overline{\boldsymbol{\delta}}\sigma = {} & \rho\cdot(\overline{\alpha} + \beta) - \sigma\cdot\left(3\alpha - \overline{\beta}\right) + (\rho - \overline{\rho})\cdot\tau \\
& + (\mu - \overline{\mu})\cdot\kappa_{(0)} - \Psi_{(1)} + \Phi_{(0)(1)}, \qquad (7.174\text{xi})
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\delta}\alpha - \overline{\boldsymbol{\delta}}\beta = {} & \mu\cdot\rho - \lambda\cdot\sigma + \alpha\cdot\overline{\alpha} + \beta\cdot\overline{\beta} - 2\alpha\cdot\beta \\
& + \gamma\cdot(\rho - \overline{\rho}) + \varepsilon\cdot(\mu - \overline{\mu}) - \Psi_{(2)} + \Phi_{(1)(1)} - (\mathbb{R}/24)\,, \qquad (7.174\text{xii})
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\delta}\lambda - \overline{\boldsymbol{\delta}}\mu = {} & (\rho - \overline{\rho})\cdot\nu + (\mu - \overline{\mu})\cdot\pi_{(0)} + \mu\cdot\left(\alpha + \overline{\beta}\right) \\
& + \lambda\cdot(\overline{\alpha} - 3\beta) - \Psi_{(3)} + \Phi_{(2)(1)}, \qquad (7.174\text{xiii})
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\delta}\nu - \boldsymbol{\Delta}\mu = {} & \mu^2 + \lambda\cdot\overline{\lambda} + (\gamma + \overline{\gamma})\cdot\mu \\
& - \overline{\nu}\cdot\pi_{(0)} + (\tau - 3\beta - \overline{\alpha})\cdot\nu + \Phi_{(2)(2)}, \qquad (7.174\text{xiv})
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\delta}\gamma - \boldsymbol{\Delta}\beta = {} & (\tau - \overline{\alpha} - \beta)\cdot\gamma + \mu\cdot\tau - \sigma\cdot\nu - \varepsilon\cdot\overline{\nu} \\
& - \beta\cdot(\gamma - \overline{\gamma} - \mu) + \alpha\cdot\overline{\lambda} + \Phi_{(1)(2)}, \qquad (7.174\text{xv})
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\delta}\tau - \boldsymbol{\Delta}\sigma = {} & \mu\cdot\sigma + \overline{\lambda}\cdot\rho + (\tau + \beta - \overline{\alpha})\cdot\tau \\
& - (3\gamma - \overline{\gamma})\cdot\sigma - \kappa_{(0)}\cdot\overline{\nu} + \Phi_{(0)(2)}, \qquad (7.174\text{xvi})
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Delta}\rho - \overline{\boldsymbol{\delta}}\tau = {} & -(\rho\cdot\overline{\mu} + \sigma\cdot\lambda) + \left(\overline{\beta} - \alpha - \overline{\tau}\right)\cdot\tau \\
& + (\gamma + \overline{\gamma})\cdot\rho + \nu\cdot\kappa_{(0)} - \Psi_{(2)} + (\mathbb{R}/12)\,, \qquad (7.174\text{xvii})
\end{aligned}$$

$$\boldsymbol{\Delta}\alpha - \overline{\boldsymbol{\delta}}\gamma = (\rho + \varepsilon)\cdot\nu - (\tau + \beta)\cdot\lambda + (\overline{\gamma} - \overline{\mu})\cdot\alpha + \left(\overline{\beta} - \overline{\tau}\right)\cdot\gamma - \Psi_{(3)}. \tag{7.174xviii}$$

Remarks: (i) The 12 equations (7.174i–iii,vii,viii,ix–xi,xiii,xiv,xvi,xvii) follow directly from (7.173).

(ii) Each of the 6 equations (7.174iv–vi,xii,xv,xviii) follows from a linear combination of *two equations* in (7.173).

(iii) Each of the 18 Newman–Penrose equations follows naturally from *spinor algebra* and *spinor analysis* [122, 207].

(iv) The spinor algebra and spinor analysis justify the eighteen complex-valued partial differential equations (7.174i–xviii) in *a mathematically rigorous way*. (Compare remarks (i) and (ii) after (7.162ii).)

Now, we shall work out an example involving the Newman–Penrose equations.

Example 7.2.5. Here we explore Example 7.1.10 in greater detail with spin coefficients. The metric representing plane-fronted gravitational waves [126, 239] is provided by

$$\mathrm{d}s^2 = 2\cdot(\mathrm{d}\zeta)\cdot\left(\mathrm{d}\overline{\zeta}\right) - 2\cdot(\mathrm{d}u)\cdot(\mathrm{d}v) - 2\cdot h\left(\zeta,\overline{\zeta},u\right)\cdot(\mathrm{d}u)^2, \tag{7.175i}$$

$$=:\ 2\cdot\left(\mathrm{d}x^1\right)\cdot\left(\mathrm{d}x^2\right) - 2\cdot\left(\mathrm{d}x^3\right)\cdot\left[\mathrm{d}x^4 + h(\cdot\cdot)\cdot\mathrm{d}x^3\right]. \tag{7.175ii}$$

$$\vec{\mathbf{m}}(\cdot\cdot) = \Lambda^i{}_{(1)}(\cdot\cdot)\cdot\partial_i = \delta^i{}_{(1)}\cdot\partial_i = \partial_1, \tag{7.176i}$$

$$\vec{\overline{\mathbf{m}}}(\cdot\cdot) = \Lambda^i{}_{(2)}(\cdot\cdot)\cdot\partial_i = \delta^i{}_{(2)}\cdot\partial_i = \partial_2, \tag{7.176ii}$$

$$\vec{\mathbf{l}}(\cdot\cdot) = \Lambda^i{}_{(3)}(\cdot\cdot)\cdot\partial_i = \left[\delta^i{}_{(3)} - h(\cdot\cdot)\cdot\delta^i{}_{(4)}\right]\cdot\partial_i = \left[\partial_3 - h(\cdot\cdot)\cdot\partial_4\right], \tag{7.176iii}$$

$$\vec{\mathbf{k}}(\cdot\cdot) = \Lambda^i{}_{(4)}(\cdot\cdot)\cdot\partial_i = \delta^i{}_{(4)}\cdot\partial_i = \partial_4, \tag{7.176iv}$$

$$\widetilde{\mathbf{m}}(\cdot\cdot) = \mathsf{M}^{(1)}{}_i(\cdot\cdot)\cdot\mathrm{d}x^i = \delta^{(1)}{}_i\cdot\mathrm{d}x^i = \mathrm{d}x^1, \tag{7.176v}$$

$$\widetilde{\overline{\mathbf{m}}}(\cdot\cdot) = \mathsf{M}^{(2)}{}_i(\cdot\cdot)\cdot\mathrm{d}x^i = \delta^{(2)}{}_i\cdot\mathrm{d}x^i = \mathrm{d}x^2, \tag{7.176vi}$$

$$\widetilde{\mathbf{l}}(\cdot\cdot) = \mathsf{M}^{(3)}{}_i(\cdot\cdot)\cdot\mathrm{d}x^i = \delta^{(3)}{}_i\cdot\mathrm{d}x^i = \mathrm{d}x^3, \tag{7.176vii}$$

$$\widetilde{\mathbf{k}}(\cdot\cdot) = \mathsf{M}^{(4)}{}_i(\cdot\cdot)\cdot\mathrm{d}x^i = \left[h(\cdot\cdot)\cdot\delta^{(3)}{}_i + \delta^{(4)}{}_i\right]\cdot\mathrm{d}x^i = h(\cdot\cdot)\cdot\mathrm{d}x^3 + \mathrm{d}x^4. \tag{7.176viii}$$

The contravariant metric tensor components $g^{ij}(x)$ can be generated from (7.176i–iv) as

$$\underset{4\times 4}{\left[g^{ij}(\cdot\cdot)\right]} = \left[\eta^{(a)(b)} \cdot \Lambda^{i}{}_{(a)}(\cdot\cdot) \cdot \Lambda^{i}{}_{(b)}(\cdot\cdot)\right] = \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -1 \\ & & -1 & 2h \end{array}\right]. \tag{7.177}$$

Directional derivatives from (7.163i–iv) are provided by

$$\boldsymbol{\delta} = \Lambda^{i}{}_{(1)}(\cdot\cdot) \cdot \partial_i = \partial_1 = \partial_\zeta, \tag{7.178i}$$

$$\overline{\boldsymbol{\delta}} = \Lambda^{i}{}_{(2)}(\cdot\cdot) \cdot \partial_i = \partial_2 = \partial_{\bar{\zeta}}, \tag{7.178ii}$$

$$\boldsymbol{\Delta} = \Lambda^{i}{}_{(3)}(\cdot\cdot) \cdot \partial_i = [\partial_3 - h(\cdot\cdot) \cdot \partial_4] = [\partial_u - h(\cdot\cdot) \cdot \partial_v], \tag{7.178iii}$$

$$\mathbf{D} = \Lambda^{i}{}_{(4)}(\cdot\cdot) \cdot \partial_i = \delta^{i}{}_{(4)} \cdot \partial_i = \partial_4 = \partial_v. \tag{7.178iv}$$

Now we shall work out integrability conditions (7.166i–iv). Let us examine (7.166iii) carefully. Using (7.178i–iv), we deduce that

$$\begin{aligned}
[\boldsymbol{\delta}\boldsymbol{\Delta} - \boldsymbol{\Delta}\boldsymbol{\delta}]\,[f] &= \{\partial_1\,[\partial_3 - h(\cdot\cdot)\cdot\partial_4] - [\partial_3 - h(\cdot\cdot)\cdot\partial_4]\cdot\partial_1\}\,[f] \\
&= -(\partial_1 h)\cdot(\partial_4 f) = -(\partial_1 h)\cdot(\mathbf{D} f) \\
&= -\overline{\nu}(\cdot\cdot)\cdot(\mathbf{D} f),
\end{aligned}$$

$$\text{or} \qquad \overline{\nu}(\cdot\cdot) = \partial_1 h \quad \text{and} \quad \nu(\cdot\cdot) = \partial_2 h,$$

$$\text{also,} \qquad \tau(\cdot\cdot) - \overline{\alpha}(\cdot\cdot) - \beta(\cdot\cdot) = \overline{\lambda}(\cdot\cdot) = \mu(\cdot\cdot) - \gamma(\cdot\cdot) + \overline{\gamma}(\cdot\cdot) \equiv 0.$$

We work out the other three integrability conditions in a similar fashion. We summarize the final conclusions as

$$\begin{aligned}
\kappa_{(0)}(\cdot\cdot) &= \overline{\kappa}_{(0)}(\cdot\cdot) = \rho(\cdot\cdot) = \overline{\rho}(\cdot\cdot) = \sigma(\cdot\cdot) = \overline{\sigma}(\cdot\cdot) = \tau(\cdot\cdot) = \overline{\tau}(\cdot\cdot) \\
&= \mu(\cdot\cdot) = \overline{\mu}(\cdot\cdot) = \lambda(\cdot\cdot) = \overline{\lambda}(\cdot\cdot) = \pi_{(0)}(\cdot\cdot) = \overline{\pi}_{(0)}(\cdot\cdot) = \varepsilon(\cdot\cdot) = \overline{\varepsilon}(\cdot\cdot) \\
&= \beta(\cdot\cdot) = \overline{\beta}(\cdot\cdot) = \alpha(\cdot\cdot) = \overline{\alpha}(\cdot\cdot) = \gamma(\cdot\cdot) = \overline{\gamma}(\cdot\cdot) \equiv 0, \\
\nu(\cdot\cdot) &= \partial_2 h = \partial_{\bar{\zeta}} h, \quad \overline{\nu}(\cdot\cdot) = \partial_1 h = \partial_\zeta h.
\end{aligned} \tag{7.179}$$

Now, Newman–Penrose equation (7.174x), with help of (7.179), implies that

$$0 - \overline{\boldsymbol{\delta}}\nu = -\Psi_{(4)}(\cdot\cdot) \not\equiv 0,$$

$$\text{or} \qquad \partial_2\nu = \partial_{\bar{\zeta}}\nu = \partial_2\partial_2 h = \partial_{\bar{\zeta}}\partial_{\bar{\zeta}} h = \Psi_{(4)}(\cdot\cdot). \tag{7.180}$$

By (7.109ii) and (7.170i–xi), the only nonzero component for this metric is $\Phi_{(2)(2)}(\cdot\cdot)$. Moreover, the Newman–Penrose equation (7.174xiv) implies that

$$\boldsymbol{\delta}\nu - 0 = 0 + \Phi_{(2)(2)}(\cdot\cdot),$$

$$\text{or} \qquad \Phi_{(2)(2)}(\cdot\cdot) = \partial_1\partial_2 h = \partial_\zeta\partial_{\bar{\zeta}} h. \tag{7.181}$$

In this example, (with $\mathbb{R}(\cdot\cdot) \equiv 0$ from (7.109iii)), the complex scalar functions $\Psi_{(0)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) = \Psi_{(2)}(\cdot\cdot) = \Psi_{(3)}(\cdot\cdot) \equiv 0$. In case

$$\partial_2\partial_2 h = \partial_{\bar{\zeta}}\partial_{\bar{\zeta}} h \neq 0,$$

the metrics (7.175i,ii) belong to the complex Segre characteristic $[(2,1)]$ by (7.101) and (7.110i). Thus, *the Brinkman–Robinson–Trautman metric* is of Petrov type N (for $\partial_2\partial_2 h \neq 0$). □

We have provided (7.147), *which are consequences* of the complex-valued *Bianchi's differential identities*. We shall now express consequences of Bianchi's identities in terms of *directional derivatives* (7.163i–iv) *and spin coefficients* (7.164i–xxiv). (See [35, 239].) These equations, which take very long calculations to derive, are furnished in the following:

$$\begin{aligned}
&\overline{\boldsymbol{\delta}}\Psi_{(0)} - \mathbf{D}\Psi_{(1)} + \mathbf{D}\Phi_{(0)(1)} - \boldsymbol{\delta}\Phi_{(0)(0)} = \left(4\alpha - \pi_{(0)}\right)\cdot\Psi_{(0)} \\
&\quad - 2\,(2\rho + \varepsilon)\cdot\Psi_{(1)} + 3\kappa_{(0)}\cdot\Psi_{(2)} + \left(\overline{\pi}_{(0)} - 2\overline{\alpha} - 2\beta\right)\cdot\Phi_{(0)(0)} \\
&\quad + 2\,(\varepsilon + \overline{\rho})\cdot\Phi_{(0)(1)} + 2\sigma\cdot\Phi_{(1)(0)} - 2\kappa_{(0)}\cdot\Phi_{(1)(1)} - \overline{\kappa}_{(0)}\cdot\Phi_{(0)(2)},
\end{aligned} \tag{7.182i}$$

$$\begin{aligned}
&\boldsymbol{\Delta}\Psi_{(0)} - \boldsymbol{\delta}\Psi_{(1)} + \mathbf{D}\Phi_{(0)(2)} - \boldsymbol{\delta}\Phi_{(0)(1)} = (4\gamma - \mu)\cdot\Psi_{(0)} \\
&\quad - 2\,(2\tau + \beta)\cdot\Psi_{(1)} + 3\sigma\cdot\Psi_{(2)} + (2\varepsilon - 2\overline{\varepsilon} + \overline{\rho})\cdot\Phi_{(0)(2)} \\
&\quad + 2\left(\overline{\pi}_{(0)} - \beta\right)\cdot\Phi_{(0)(1)} + 2\sigma\cdot\Phi_{(1)(1)} - 2\kappa_{(0)}\cdot\Phi_{(1)(2)} - \overline{\lambda}\cdot\Phi_{(0)(0)},
\end{aligned} \tag{7.182ii}$$

$$\begin{aligned}
&\overline{\boldsymbol{\delta}}\Psi_{(3)} - \mathbf{D}\Psi_{(4)} + \overline{\boldsymbol{\delta}}\Phi_{(2)(1)} - \boldsymbol{\Delta}\Phi_{(2)(0)} = (4\varepsilon - \rho)\cdot\Psi_{(4)} \\
&\quad - 2\left(2\pi_{(0)} + \alpha\right)\cdot\Psi_{(3)} + 3\lambda\cdot\Psi_{(2)} + (2\gamma - 2\overline{\gamma} + \overline{\mu})\cdot\Phi_{(2)(0)} \\
&\quad + 2\,(\overline{\tau} - \alpha)\cdot\Phi_{(2)(1)} + 2\lambda\cdot\Phi_{(1)(1)} - 2\nu\cdot\Phi_{(1)(0)} - \overline{\sigma}\cdot\Phi_{(2)(2)},
\end{aligned} \tag{7.182iii}$$

$$\boldsymbol{\Delta}\Psi_{(3)} - \boldsymbol{\delta}\Psi_{(4)} + \overline{\boldsymbol{\delta}}\Phi_{(2)(2)} - \boldsymbol{\Delta}\Phi_{(2)(1)} = (4\beta - \tau)\cdot\Psi_{(4)}$$

$$- 2\,(2\mu + \gamma)\cdot\Psi_{(3)} + 3\nu\cdot\Psi_{(2)} + \left(\overline{\tau} - 2\overline{\beta} - 2\alpha\right)\cdot\Phi_{(2)(2)}$$

$$+ 2\,(\gamma + \overline{\mu})\cdot\Phi_{(2)(1)} + 2\lambda\cdot\Phi_{(1)(2)} - 2\nu\cdot\Phi_{(1)(1)} - \overline{\nu}\cdot\Phi_{(2)(0)}, \tag{7.182iv}$$

$$\mathbf{D}\Psi_{(2)} - \overline{\boldsymbol{\delta}}\Psi_{(1)} + \boldsymbol{\Delta}\Phi_{(0)(0)} - \overline{\boldsymbol{\delta}}\Phi_{(0)(1)} - (1/12)\cdot\mathbf{D}\mathbb{R} = -\lambda\cdot\Psi_{(0)}$$

$$+ 2\left(\pi_{(0)} - \alpha\right)\cdot\Psi_{(1)} + 3\rho\cdot\Psi_{(2)} - 2\kappa_{(0)}\cdot\Psi_{(3)} + (2\gamma + 2\overline{\gamma} - \overline{\mu})\cdot\Phi_{(0)(0)}$$

$$- 2\,(\overline{\tau} + \alpha)\cdot\Phi_{(0)(1)} - 2\tau\cdot\Phi_{(1)(0)} + 2\rho\cdot\Phi_{(1)(1)} + \overline{\sigma}\cdot\Phi_{(0)(2)}, \tag{7.182v}$$

$$\boldsymbol{\Delta}\Psi_{(2)} - \boldsymbol{\delta}\Psi_{(3)} + \mathbf{D}\Phi_{(2)(2)} - \boldsymbol{\delta}\Phi_{(2)(1)} - (1/12)\cdot\boldsymbol{\Delta}\mathbb{R} = \sigma\cdot\Psi_{(4)}$$

$$+ 2\,(\beta - \tau)\cdot\Psi_{(3)} - 3\mu\cdot\Psi_{(2)} + 2\nu\cdot\Psi_{(1)} + (\overline{\rho} - 2\varepsilon - 2\overline{\varepsilon})\cdot\Phi_{(2)(2)}$$

$$+ 2\left(\overline{\pi}_{(0)} + \beta\right)\cdot\Phi_{(2)(1)} + 2\pi_{(0)}\cdot\Phi_{(1)(2)} - 2\mu\cdot\Phi_{(1)(1)} - \overline{\lambda}\cdot\Phi_{(2)(0)}, \tag{7.182vi}$$

$$\mathbf{D}\Psi_{(3)} - \overline{\boldsymbol{\delta}}\Psi_{(2)} - \mathbf{D}\Phi_{(2)(1)} + \boldsymbol{\delta}\Phi_{(2)(0)} + (1/12)\cdot\overline{\boldsymbol{\delta}}\mathbb{R} = -\kappa_{(0)}\cdot\Psi_{(4)}$$

$$+ 2\,(\rho - \varepsilon)\cdot\Psi_{(3)} + 3\pi_{(0)}\cdot\Psi_{(2)} - 2\lambda\cdot\Psi_{(1)} + \left(2\overline{\alpha} - 2\beta - \overline{\pi}_{(0)}\right)\cdot\Phi_{(2)(0)}$$

$$- 2\,(\overline{\rho} - \varepsilon)\cdot\Phi_{(2)(1)} - 2\pi_{(0)}\cdot\Phi_{(1)(1)} + 2\mu\cdot\Phi_{(1)(0)} + \overline{\kappa}_{(0)}\cdot\Phi_{(2)(2)}, \tag{7.182vii}$$

$$\boldsymbol{\Delta}\Psi_{(1)} - \boldsymbol{\delta}\Psi_{(2)} - \boldsymbol{\Delta}\Phi_{(0)(1)} + \overline{\boldsymbol{\delta}}\Phi_{(0)(2)} + (1/12)\cdot\boldsymbol{\delta}\mathbb{R} = \nu\cdot\Psi_{(0)}$$

$$+ 2\,(\gamma - \mu)\cdot\Psi_{(1)} - 3\tau\cdot\Psi_{(2)} + 2\sigma\cdot\Psi_{(3)} + \left(\overline{\tau} - 2\overline{\beta} + 2\alpha\right)\cdot\Phi_{(0)(2)}$$

$$+ 2\,(\overline{\mu} - \gamma)\cdot\Phi_{(0)(1)} + 2\tau\cdot\Phi_{(1)(1)} - 2\rho\cdot\Phi_{(1)(2)} - \overline{\nu}\cdot\Phi_{(0)(0)}, \tag{7.182viii}$$

$$\mathbf{D}\Phi_{(1)(1)} - \boldsymbol{\delta}\Phi_{(1)(0)} - \overline{\boldsymbol{\delta}}\Phi_{(0)(1)} + \boldsymbol{\Delta}\Phi_{(0)(0)} - (1/8)\cdot\mathbf{D}\mathbb{R}$$

$$= (2\gamma - \mu + 2\overline{\gamma} - \overline{\mu})\cdot\Phi_{(0)(0)} + \left(\pi_{(0)} - 2\alpha - 2\overline{\tau}\right)\cdot\Phi_{(0)(1)}$$

$$+ \left(\overline{\pi}_{(0)} - 2\overline{\alpha} - 2\tau\right)\cdot\Phi_{(1)(0)} + 2\,(\rho + \overline{\rho})\cdot\Phi_{(1)(1)} + \overline{\sigma}\cdot\Phi_{(0)(2)}$$

$$+ \sigma\cdot\Phi_{(2)(0)} - \overline{\kappa}_{(0)}\cdot\Phi_{(1)(2)} - \kappa_{(0)}\cdot\Phi_{(2)(1)}, \tag{7.182ix}$$

$$\mathbf{D}\Phi_{(1)(2)} - \boldsymbol{\delta}\Phi_{(1)(1)} - \overline{\boldsymbol{\delta}}\Phi_{(0)(2)} + \boldsymbol{\Delta}\Phi_{(0)(1)} - (1/8)\cdot\boldsymbol{\delta}\mathbb{R}$$

$$= \left(-2\alpha + 2\overline{\beta} + \pi_{(0)} - \overline{\tau}\right)\cdot\Phi_{(0)(2)} + (\overline{\rho} + 2\rho - 2\overline{\varepsilon})\cdot\Phi_{(1)(2)}$$

$$+ 2\left(\overline{\pi}_{(0)} - \tau\right)\cdot\Phi_{(1)(1)} + (2\gamma - 2\overline{\mu} - \mu)\cdot\Phi_{(0)(1)} + \overline{\nu}\cdot\Phi_{(0)(0)}$$

$$- \overline{\lambda}\cdot\Phi_{(1)(0)} + \sigma\cdot\Phi_{(2)(1)} - \kappa_{(0)}\cdot\Phi_{(2)(2)}, \tag{7.182x}$$

$$\begin{aligned}
&\mathbf{D}\Phi_{(2)(2)} - \boldsymbol{\delta}\Phi_{(2)(1)} - \overline{\boldsymbol{\delta}}\Phi_{(1)(2)} + \boldsymbol{\Delta}\Phi_{(1)(1)} - (1/8)\cdot\boldsymbol{\Delta}\mathbb{R} \\
&\quad = (\rho + \overline{\rho} - 2\varepsilon - 2\overline{\varepsilon})\cdot\Phi_{(2)(2)} + \left(2\overline{\beta} + 2\pi_{(0)} - \overline{\tau}\right)\cdot\Phi_{(1)(2)} \\
&\quad + \left(2\beta + 2\overline{\pi}_{(0)} - \tau\right)\cdot\Phi_{(2)(1)} - 2\left(\mu + \overline{\mu}\right)\cdot\Phi_{(1)(1)} \\
&\quad + \nu\cdot\Phi_{(0)(1)} + \overline{\nu}\cdot\Phi_{(1)(0)} - \overline{\lambda}\cdot\Phi_{(2)(0)} - \lambda\cdot\Phi_{(0)(2)}.
\end{aligned} \tag{7.182xi}$$

Now we shall *carefully interpret the N–P equations (7.174i–xviii) and also (7.182i–xi)* (which are consequences of Bianchi's differential identities).

In N–P equations (7.174i–xviii), if we represent $\Phi_{(0)(0)}(\cdot\cdot)$, $\Phi_{(0)(1)}(\cdot\cdot)$, ... etc., and $\mathbb{R}(\cdot\cdot)$ in terms of $\mathbb{R}_{(a)(b)}(\cdot\cdot)$ according to (7.170i–xi), then the N–P equations are *mathematically equivalent to*

$$\begin{aligned}
\mathbb{C}_{(d)(a)(b)(c)}(\cdot\cdot) := \mathbb{R}_{(d)(a)(b)(c)}(\cdot\cdot) + \frac{1}{2}\Big\{ &[\,\eta_{(d)(b)}\cdot\mathbb{R}_{(a)(c)} - \eta_{(d)(c)}\cdot\mathbb{R}_{(a)(b)} \\
&+ \eta_{(a)(c)}\cdot\mathbb{R}_{(d)(b)} - \eta_{(a)(b)}\cdot\mathbb{R}_{(d)(c)}\,] \\
&+ \frac{1}{3}\cdot\left[\eta_{(a)(b)}\cdot\eta_{(d)(c)} - \eta_{(a)(c)}\cdot\eta_{(d)(b)}\right]\cdot\mathbb{R}\Big\}.
\end{aligned} \tag{7.183}$$

(Compare equation above with (1.169ii) and (7.150).)

However, the above equation (7.183) is really *an algebraic identity* involving the tetrad components $\mathbb{C}_{(d)(a)(b)(c)}(\cdot\cdot)$, $\mathbb{R}_{(d)(a)(b)(c)}(\cdot\cdot)$, $\mathbb{R}_{(a)(b)}(\cdot\cdot)$, and $\mathbb{R}(\cdot\cdot)$ (relative to the null tetrad $\left\{\vec{\mathbf{m}}, \overline{\vec{\mathbf{m}}}, \vec{\mathbf{l}}, \vec{\mathbf{k}}\right\}$) in a general, four-dimensional pseudo-Riemannian differential manifold of signature $+2$. Thus, (7.183) does not necessarily imply Einstein's gravitational field equations (2.163ii). Similarly, with *geometrical interpretations* of $\Phi_{(0)(0)}(\cdot\cdot)$, $\Phi_{(0)(1)}(\cdot\cdot)$, ... etc., and $\mathbb{R}(\cdot\cdot)$ according to (7.170i–ix), (7.182i–xi) are mathematically equivalent to

$$\begin{aligned}
&\nabla_{(e)}\Big\{\mathbb{C}_{(d)(a)(b)(c)}\frac{1}{2}\cdot\left[\eta_{(d)(c)}\cdot\mathbb{R}_{(a)(b)} - \eta_{(d)(b)}\cdot\mathbb{R}_{(a)(c)}\right. \\
&\qquad \left. + \eta_{(a)(b)}\cdot\mathbb{R}_{(d)(c)} - \eta_{(a)(c)}\cdot\mathbb{R}_{(d)(b)}\right] + \frac{1}{3}\cdot\left[\eta_{(a)(c)}\eta_{(d)(b)} - \eta_{(a)(b)}\eta_{(d)(c)}\right]\cdot\mathbb{R}\Big\} \\
&\qquad + \nabla_{(b)}\left\{\ldots\right\} + \nabla_{(c)}\left\{\ldots\right\} \\
&= \nabla_{(e)}\left\{\mathbb{R}_{(d)(a)(b)(c)}(\cdot\cdot)\right\} + \nabla_{(b)}\left\{\mathbb{R}_{(d)(a)(c)(e)}(\cdot\cdot)\right\} + \nabla_{(c)}\left\{\mathbb{R}_{(d)(a)(e)(b)}(\cdot\cdot)\right\} \equiv 0.
\end{aligned} \tag{7.184}$$

The equations above are obviously equivalent to Bianchi's differential identities in (1.143ii) and (7.151) for a general pseudo-Riemannian manifold. Again, there are no compelling gravitational implications as such in (7.184)! To apply the N–P equation

(7.174i–xviii) and (7.182i–xi) for *gravitational field equations and conservations laws, the following (physical) equations must be imposed:*

$$\Phi_{(0)(0)}(\cdot\cdot) = \frac{\kappa}{2} \cdot \mathbb{T}_{(4)(4)}(\cdot\cdot), \tag{7.185i}$$

$$\Phi_{(0)(1)}(\cdot\cdot) = \overline{\Phi}_{(1)(0)} = \frac{\kappa}{2} \cdot \mathbb{T}_{(1)(4)}(\cdot\cdot), \tag{7.185ii}$$

$$\Phi_{(1)(0)}(\cdot\cdot) = \frac{\kappa}{2} \cdot \mathbb{T}_{(2)(4)}(\cdot\cdot), \tag{7.185iii}$$

$$\Phi_{(0)(2)}(\cdot\cdot) = \overline{\Phi}_{(2)(0)} = \frac{\kappa}{2} \cdot \mathbb{T}_{(1)(1)}(\cdot\cdot), \tag{7.185iv}$$

$$\Phi_{(2)(0)}(\cdot\cdot) = \frac{\kappa}{2} \cdot \mathbb{T}_{(2)(2)}(\cdot\cdot), \tag{7.185v}$$

$$\Phi_{(1)(1)}(\cdot\cdot) = \overline{\Phi}_{(1)(1)} = \frac{\kappa}{4} \cdot \left[\mathbb{T}_{(1)(2)}(\cdot\cdot) + \mathbb{T}_{(3)(4)}(\cdot\cdot)\right], \tag{7.185vi}$$

$$\Phi_{(1)(2)}(\cdot\cdot) = \frac{\kappa}{2} \cdot \mathbb{T}_{(1)(3)}(\cdot\cdot), \tag{7.185vii}$$

$$\Phi_{(2)(1)}(\cdot\cdot) = \frac{\kappa}{2} \cdot \mathbb{T}_{(2)(3)}(\cdot\cdot), \tag{7.185viii}$$

$$\Phi_{(2)(2)}(\cdot\cdot) = \overline{\Phi}_{(2)(2)} = \frac{\kappa}{2} \cdot \mathbb{T}_{(3)(3)}(\cdot\cdot), \tag{7.185ix}$$

$$\mathbb{R}(\cdot\cdot) = \kappa \cdot \mathbb{T}^{(a)}{}_{(a)}(\cdot\cdot). \tag{7.185x}$$

Exercises 7.2

1. Consider the first-order, real field equations (7.120i–v) with $T_{ij}(\cdot\cdot) \neq 0$. Count the number of unknown functions versus the number of equations.
2. Consider the complex Lorentz transformation of (7.7iv) given by:

$$\widehat{\vec{\mathbf{k}}}(\cdot\cdot) = \vec{\mathbf{k}}(\cdot\cdot),$$

$$\widehat{\vec{\mathbf{m}}}(\cdot\cdot) = \vec{\mathbf{m}}(\cdot\cdot) + B(\cdot\cdot) \cdot \vec{\mathbf{k}}(\cdot\cdot),$$

$$\widehat{\overline{\vec{\mathbf{m}}}}(\cdot\cdot) = \overline{\vec{\mathbf{m}}}(\cdot\cdot) + \overline{B(\cdot\cdot)} \cdot \vec{\mathbf{k}}(\cdot\cdot),$$

$$\widehat{\vec{\mathbf{l}}}(\cdot\cdot) = \vec{\mathbf{l}}(\cdot\cdot) + \overline{B(\cdot\cdot)} \cdot \vec{\mathbf{m}}(\cdot\cdot) + B(\cdot\cdot) \cdot \overline{\vec{\mathbf{m}}}(\cdot\cdot) + |B(\cdot\cdot)|^2 \cdot \vec{\mathbf{k}}(\cdot\cdot).$$

   Construct the corresponding 4×4 complex Lorentz matrix $[\mathbb{L}(\cdot\cdot)]$ of (7.155iii,iv) such that the matrix equation $[\mathbb{L}(\cdot\cdot)]^T \cdot [\eta] \cdot [\mathbb{L}(\cdot\cdot)] = [\eta]$ is satisfied.
3. Let f be a real or complex-valued function of class C^2 in the coordinate domain $D \subset \mathbb{R}^4$ of consideration. Prove the integrability conditions (7.145i) stating

$$\partial_{(a)}\partial_{(b)} f - \partial_{(b)}\partial_{(a)} f = \eta^{(c)(e)} \cdot \left[\Gamma_{(e)(a)(b)} - \Gamma_{(e)(b)(a)}\right] \cdot \partial_{(c)} f$$
$$=: \left[\Gamma_{(c)(a)(b)}(\cdot\cdot) - \Gamma_{(c)(b)(a)}(\cdot\cdot)\right] \cdot \partial^{(c)} f.$$

4. Prove that another alternative formulation of a system of the first-order, complex partial differential equations equivalent to the field equations (2.161ii) is the following:

$$\nabla_k \Lambda_{j(a)} = \Gamma_{(a)(b)(c)}(\cdot\cdot) \cdot \mathsf{M}^{(b)}{}_j(\cdot\cdot) \cdot \mathsf{M}^{(c)}{}_k(\cdot\cdot)\,;$$
$$\left[\partial_{(a)}\partial_{(b)} - \partial_{(b)}\partial_{(a)}\right]\left[\Lambda^k{}_{(c)}\right] = \left[\Gamma_{(e)(a)(b)}(\cdot\cdot) - \Gamma_{(e)(b)(a)}(\cdot\cdot)\right] \cdot \partial^{(e)} \Lambda^k{}_{(c)}\,;$$
$$\mathbb{R}_{(a)(b)}(\cdot\cdot) - (1/2) \cdot \eta_{(a)(b)} \cdot \mathbb{R}(\cdot\cdot) = -\kappa \cdot \mathbb{T}_{(a)(b)}(\cdot\cdot)\,;$$
$$\nabla_{(b)} \mathbb{T}^{(a)(b)} = -(\kappa)^{-1} \cdot \nabla_{(b)} \mathbb{G}^{(a)(b)} \equiv 0\,;$$
$$\mathcal{C}^{(a)}\left(\Lambda^k{}_{(b)}, \partial_{(c)} \Lambda^k{}_{(b)}\right) = 0.$$

5. (i) Let $\vec{\mathbf{k}}(x)$ be a null geodesic congruence. Moreover, let the null geodesic congruence $\vec{\mathbf{k}}(x)$ be affinely parameterized. Show that the spin coefficients $\kappa_{(0)}(\cdot\cdot) = \varepsilon(\cdot\cdot) + \bar{\varepsilon}(\cdot\cdot) \equiv 0$.
   (ii) In case the null tetrad $\left\{\vec{\mathbf{m}}, \bar{\vec{\mathbf{m}}}, , \vec{\mathbf{l}}, \vec{\mathbf{k}}\right\}$ is parallely transported along the null geodesic congruence $\vec{\mathbf{k}}(x)$, show that the spin-coefficients $\kappa_{(0)}(\cdot\cdot) = \varepsilon(\cdot\cdot) = \pi_{(0)}(\cdot\cdot) \equiv 0$.
6. Consider the vacuum field equations $\mathbb{R}_{(a)(b)(c)(d)}(\cdot\cdot) = \mathbb{C}_{(a)(b)(c)(d)}(\cdot\cdot)$ inherent in (7.150). For such equations, simplify Newman–Penrose equations (7.174i–xviii).
7. Using (7.56i–iii), prove that Maxwell's equations (7.53) are equivalent to the following:

$$\mathbf{D}\phi_{(1)} - \bar{\boldsymbol{\delta}}\phi_{(0)} = \left(\pi_{(0)} - 2\alpha\right) \cdot \phi_{(0)} + 2\rho \cdot \phi_{(1)} - \kappa_{(0)} \cdot \phi_{(2)},$$
$$\mathbf{D}\phi_{(2)} - \bar{\boldsymbol{\delta}}\phi_{(1)} = -\lambda \cdot \phi_{(0)} + 2\pi_{(0)} \cdot \phi_{(1)} + (\rho - 2\varepsilon) \cdot \phi_{(2)},$$
$$\boldsymbol{\delta}\phi_{(1)} - \boldsymbol{\Delta}\phi_{(0)} = (\mu - 2\gamma) \cdot \phi_{(0)} + 2\tau \cdot \phi_{(1)} - \sigma \cdot \phi_{(2)},$$
$$\boldsymbol{\delta}\phi_{(2)} - \boldsymbol{\Delta}\phi_{(1)} = -\nu \cdot \phi_{(0)} + 2\mu \cdot \phi_{(1)} + (\tau - 2\beta) \cdot \phi_{(2)}.$$

8. Obtain Newman–Penrose equations (7.174i–xviii) corresponding to Einstein–Maxwell (or electromagneto-vac) equations (2.290i–vi).
9. (i) Simplify the Newman–Penrose equations (7.174i–xviii) for the algebraically special case of Petrov type 0.
   (ii) Solve the Newman–Penrose equations for the Einstein–Maxwell fields (or, electromagento-vac) for the case of a null electromagnetic field.
10. Prove (7.182i) which is a consequence of Bianchi's identities (7.115) or (7.184).

Answers and Hints to Selected Exercises

1. No. of unknown functions:

$$10\left(g_{ij}\right) + 40\left(\left\{ {k \atop i\ j} \right\}\right) + 10\left(T_{ij}\right) = 60.$$

No. of equations:

$$\begin{aligned} &40\,(7.120\text{i}) + 60\,(7.120\text{ii}) \\ &+ 10\,(7.120\text{iii}) + 4\,(7.120\text{iv}) + 4\,(7.120\text{v}) = 118. \end{aligned}$$

2.

$$\underset{4\times 4}{[\mathbb{L}(\cdot\cdot)]} = \left[\begin{array}{cc|cc} 1 & 0 & \overline{B} & 0 \\ 0 & 1 & B & 0 \\ \hline 0 & 0 & 1 & 0 \\ B & \overline{B} & |B|^2 & 1 \end{array}\right].$$

3. It is known that coordinate components satisfy identities

$$\nabla_i \nabla_j f - \nabla_j \nabla_i f \equiv 0.$$

Therefore, it follows that

$$\nabla_{(a)} \nabla_{(b)} f - \nabla_{(b)} \nabla_{(a)} f \equiv 0.$$

Using the rules (7.143) for covariant derivatives, the integrability conditions (7.145i) ensue. (Compare (7.145i) with (1.139iv).)

4. This system is mathematically equivalent to the following system of *real, first-order p.d.e.s*:

$$\nabla_k \lambda_{j(a)} = \gamma_{(a)(b)(c)}(\cdot\cdot) \cdot \mu^{(b)}{}_j(\cdot\cdot) \cdot \mu^{(c)}{}_k(\cdot\cdot)\,;$$

$$\left[\partial_{(a)}\partial_{(b)} - \partial_{(b)}\partial_{(a)}\right]\left[\lambda^k{}_{(c)}\right] = \left[\gamma_{(e)(a)(b)}(\cdot\cdot) - \gamma_{(e)(b)(a)}(\cdot\cdot)\right] \cdot d^{(e)(f)} \cdot \partial_{(f)}\lambda^k{}_{(c)}\,;$$

$$G_{(a)(b)}(\cdot\cdot) = -\kappa \cdot T_{(a)(b)}(\cdot\cdot)\,;$$

$$\nabla_{(b)} T^{(a)(b)} = -(\kappa)^{-1} \cdot \nabla_{(b)} G^{(a)(b)} \equiv 0\,;$$

$$\mathcal{C}^{(a)}\left(\lambda^k{}_{(b)}, \partial_{(c)}\lambda^k{}_{(b)}\right) = 0.$$

(See [271, 272].)

5. (i) $k^{(b)}(\cdot\cdot) \cdot \nabla_{(b)} k^{(a)} = 0 \Rightarrow \kappa_{(0)}(\cdot\cdot) = \left[k^{(b)} \cdot \nabla_{(b)} k^{(a)}\right] \cdot m_{(a)} \equiv 0$, etc.
(ii) $k^{(b)} \cdot \nabla_{(b)} k_{(a)} = k^{(b)} \cdot \nabla_{(b)} l_{(a)} = k^{(b)} \cdot \nabla_{(b)} m_{(a)} \equiv 0.$

6. Set

$$\Phi_{(0)(0)}(\cdot\cdot) = \Phi_{(0)(1)}(\cdot\cdot) = \Phi_{(1)(0)}(\cdot\cdot) = \Phi_{(2)(0)}(\cdot\cdot) = \Phi_{(0)(2)}(\cdot\cdot) = \Phi_{(1)(1)}(\cdot\cdot)$$
$$= \Phi_{(1)(2)}(\cdot\cdot) \;= \Phi_{(2)(1)}(\cdot\cdot) = \Phi_{(2)(2)}(\cdot\cdot) = \mathbb{R}(\cdot\cdot) \equiv 0.$$

Thus, $\mathbb{T}_{(a)(b)}(\cdot\cdot) \equiv 0$.

(*Remark*: Vacuum field equations can be rendered still simpler by employing the four coordinate conditions of # 5.)

7. Recall from (7.56i–iii) that

$$\phi_{(0)}(\cdot\cdot) = \varphi_{(a)(b)}(\cdot\cdot) \cdot k^{(a)}(\cdot\cdot) \cdot m^{(b)}(\cdot\cdot) = \varphi_{(4)(1)},$$
$$\phi_{(1)}(\cdot\cdot) = \frac{1}{2} \cdot \varphi_{(a)(b)}(\cdot\cdot) \cdot \left[k^{(a)} \cdot l^{(b)} + \overline{m}^{(a)} \cdot m^{(b)}\right] = \frac{1}{2}\left[\varphi_{(4)(3)}(\cdot\cdot) + \varphi_{(2)(1)}(\cdot\cdot)\right],$$
$$\phi_{(2)}(\cdot\cdot) = \varphi_{(a)(b)}(\cdot\cdot) \cdot \overline{m}^{(a)}(\cdot\cdot) \cdot l^{(b)}(\cdot\cdot) = \varphi_{(2)(3)}(\cdot\cdot).$$

Using Maxwell's equations $\nabla_{(a)}\varphi_{(b)(c)} + \nabla_{(b)}\varphi_{(c)(a)} + \nabla_{(c)}\varphi_{(a)(b)} = 0$, the following four equations can be derived:

$$\nabla_{(1)}\varphi_{(2)(3)} + \nabla_{(2)}\varphi_{(3)(1)} + \nabla_{(3)}\varphi_{(1)(2)} = 0,$$
$$\nabla_{(1)}\varphi_{(2)(4)} + \nabla_{(2)}\varphi_{(4)(1)} + \nabla_{(4)}\varphi_{(1)(2)} = 0,$$
$$\nabla_{(1)}\varphi_{(3)(4)} + \nabla_{(3)}\varphi_{(4)(1)} + \nabla_{(4)}\varphi_{(1)(3)} = 0,$$
$$\nabla_{(2)}\varphi_{(3)(4)} + \nabla_{(3)}\varphi_{(4)(2)} + \nabla_{(4)}\varphi_{(2)(3)} = 0.$$

On the other hand, using $\varphi^{(a)(b)}(\cdot\cdot) = \eta^{(a)(c)} \cdot \eta^{(b)(d)} \cdot \varphi_{(c)(d)}(\cdot\cdot)$ and the Maxwell equations $\nabla_{(b)}\varphi^{(a)(b)} = 0$, the following four equations can be obtained:

$$-\nabla_{(2)}\varphi_{(1)(2)} - \nabla_{(3)}\varphi_{(2)(4)} - \nabla_{(4)}\varphi_{(2)(3)} = 0,$$
$$\nabla_{(1)}\varphi_{(1)(2)} - \nabla_{(3)}\varphi_{(1)(4)} - \nabla_{(4)}\varphi_{(1)(3)} = 0,$$
$$\nabla_{(1)}\varphi_{(2)(4)} + \nabla_{(2)}\varphi_{(1)(4)} - \nabla_{(4)}\varphi_{(3)(4)} = 0,$$
$$\nabla_{(1)}\varphi_{(2)(3)} + \nabla_{(2)}\varphi_{(1)(3)} + \nabla_{(3)}\varphi_{(3)(4)} = 0.$$

By linear combinations of the preceding eight equations, the following four equations can be deduced:

$$2 \cdot \nabla_{(1)}\varphi_{(2)(3)} + \nabla_{(3)}\left(\varphi_{(1)(2)} + \varphi_{(3)(4)}\right) = 0,$$
$$-\nabla_{(4)}\left(\varphi_{(1)(2)} + \varphi_{(3)(4)}\right) - 2 \cdot \nabla_{(2)}\varphi_{(4)(1)} = 0,$$

$$
\begin{aligned}
&-\nabla_{(1)}\left(\varphi_{(3)(4)}+\varphi_{(1)(2)}\right)-2\cdot\nabla_{(3)}\varphi_{(4)(1)}=0,\\
&2\cdot\nabla_{(4)}\varphi_{(2)(3)}+\nabla_{(2)}\left(\varphi_{(3)(4)}+\varphi_{(1)(2)}\right)=0.
\end{aligned}
$$

The four equations above imply the original Maxwell's equations (7.53).

Now, the rules for covariant differentiations in (7.143) imply that

$$
\nabla_{(c)}\varphi_{(a)(b)}=\partial_{(c)}\varphi_{(a)(b)}+\eta^{(d)(e)}\cdot\left[\Gamma_{(e)(a)(c)}\cdot\varphi_{(d)(b)}+\Gamma_{(e)(b)(c)}\cdot\varphi_{(a)(d)}\right].
$$

The above equations *exactly yield*, from the preceding four Maxwell's equations, equations mentioned in #7. (Of course, Newman–Penrose symbols have to be used!)

8. The Einstein–Maxwell (or electromagneto-vac) equations imply that

$$
\begin{aligned}
&\mathbb{R}_{(a)(b)}(\cdot\cdot)=-\kappa\cdot\mathbb{T}_{(a)(b)}(\cdot\cdot),\qquad \mathbb{R}(\cdot\cdot)=\mathbb{T}(\cdot\cdot)\equiv 0,\\
&\Sigma_{(a)(b)}(\cdot\cdot)=\mathbb{R}_{(a)(b)}(\cdot\cdot).
\end{aligned}
$$

From (7.119i–vi) and (7.185i–x), it can be concluded that

$$
\begin{aligned}
\Phi_{(0)(0)}(\cdot\cdot)=&-(1/2)\cdot\Sigma_{(a)(b)}\cdot k^{(a)}\cdot k^{(b)}=(\kappa/2)\cdot\mathbb{T}_{(a)(b)}\cdot k^{(a)}\cdot k^{(b)},\\
\Phi_{(0)(1)}(\cdot\cdot)=&-(1/2)\cdot\Sigma_{(a)(b)}\cdot k^{(a)}\cdot m^{(b)}=(\kappa/2)\cdot\mathbb{T}_{(a)(b)}\cdot k^{(a)}\cdot m^{(b)},\\
\Phi_{(0)(2)}(\cdot\cdot)=&-(1/2)\cdot\Sigma_{(a)(b)}m^{(a)}\cdot m^{(b)}=(\kappa/2)\cdot\mathbb{T}_{(a)(b)}m^{(a)}\cdot m^{(b)},\\
\Phi_{(1)(1)}(\cdot\cdot)=&-(1/4)\cdot\Sigma_{(a)(b)}\cdot\left[k^{(a)}\cdot l^{(b)}+m^{(a)}\cdot\overline{m}^{(b)}\right]\\
=&\,(\kappa/4)\cdot\mathbb{T}_{(a)(b)}\cdot\left[k^{(a)}\cdot l^{(b)}+m^{(a)}\cdot\overline{m}^{(b)}\right],\\
\Phi_{(1)(2)}(\cdot\cdot)=&-(1/2)\cdot\Sigma_{(a)(b)}\cdot l^{(a)}\cdot m^{(b)}=(\kappa/2)\cdot\mathbb{T}_{(a)(b)}\cdot l^{(a)}\cdot m^{(b)},\\
\Phi_{(2)(2)}(\cdot\cdot)=&-(1/2)\cdot\Sigma_{(a)(b)}\cdot l^{(a)}\cdot l^{(b)}=(\kappa/2)\cdot\mathbb{T}_{(a)(b)}\cdot l^{(a)}\cdot l^{(b)}.
\end{aligned}
$$

From (2.290i) and the answer to Problem 8-ii of Exercises 2.5, it follows that

$$
\mathbb{T}_{(a)(b)}(\cdot\cdot)=\frac{1}{2}\mathrm{Re}\left[\overline{\varphi}_{(a)(c)}\cdot\varphi_{(b)}{}^{(c)}\right].
$$

Thus, it can be deduced that

$$
\Phi_{(A)(B)}(\cdot\cdot)=(\kappa/4)\cdot\phi_{(A)}(\cdot\cdot)\cdot\overline{\phi}_{(B)}(\cdot\cdot),\qquad A,B\in\{0,1,2\}.
$$

Substituting the above and $\mathbb{R}(\cdot\cdot)\equiv 0$ into the Newman–Penrose equations, the required answers are obtained.

9. (i) Set

$$\Psi_{(0)}(\cdot\cdot) = \Psi_{(1)}(\cdot\cdot) = \Psi_{(2)}(\cdot\cdot) = \Psi_{(3)}(\cdot\cdot) = \Psi_{(4)} \equiv 0.$$

(ii)

$$\begin{aligned} ds^2 &= 2d\varrho \cdot d\overline{\varrho} - 2\,du \cdot dv - 2\,H(\varrho, \overline{\varrho}, u) \cdot (du)^2 \\ &=: 2\,dx^1 \cdot dx^2 - 2\,dx^3 \cdot dx^4 - 2\,H(x^1, x^2, x^3) \cdot (dx^3)^2, \end{aligned}$$

$$H(x^1, x^2, x^3) \equiv H(\varrho, \overline{\varrho}, u) = f(\varrho, u) + \overline{f(\varrho, u)} + \kappa \cdot |F(\varrho, u)|^2 .$$

Here, $f(\varrho, u)$ and $F(\varrho, u)$ are holomorphic in the variable ϱ and of class C^3 in the variable u, but otherwise arbitrary.

The electromagnetic field is furnished by

$$\phi_{(0)}(\cdot\cdot) = \phi_{(1)}(\cdot\cdot) \equiv 0, \quad \phi_{(2)}(\cdot\cdot) = 2 \cdot \partial_\varrho F.$$

(See [239].)

10. Consider the Bianchi identity

$$\nabla_{(2)} R_{(4)(1)(4)(1)} + \nabla_{(4)} R_{(4)(1)(1)(2)} + \nabla_{(1)} R_{(4)(1)(2)(4)} \equiv 0.$$

Equations (7.153), (7.168i), and (7.141ii) imply that

$$\nabla_{(2)}\mathbb{C}_{(4)(1)(4)(1)} + \nabla_{(4)}\left[\mathbb{C}_{(4)(1)(1)(2)} - \frac{1}{2} \cdot \mathbb{R}_{(4)(1)}\right] + \frac{1}{2} \cdot \nabla_{(1)}\mathbb{R}_{(4)(4)} = 0.$$

Now, by (7.143), it can be deduced that

$$\begin{aligned} \nabla_{(2)}\mathbb{C}_{(4)(1)(4)(1)} &= \partial_{(2)}\mathbb{C}_{(4)(1)(4)(1)} + \eta^{(d)(e)} \cdot \big[\Gamma_{(e)(4)(2)} \cdot \big(\mathbb{C}_{(d)(1)(4)(1)} \\ &\quad + \mathbb{C}_{(4)(1)(d)(1)}\big) + \Gamma_{(e)(1)(2)} \cdot \big(\mathbb{C}_{(4)(d)(4)(1)} + \mathbb{C}_{(4)(1)(4)(d)}\big)\big]. \end{aligned}$$

Using (7.163i–iv), (7.164i–xxiv), and (7.168i–xi), it can be derived that

$$\nabla_{(2)}\mathbb{C}_{(4)(1)(4)(1)} = \overline{\boldsymbol{\delta}} \cdot \Psi_{(0)} - 4\alpha \cdot \Psi_{(0)} + 4\rho \cdot \Psi_{(1)}.$$

Similarly, one can show that

$$\begin{aligned} \nabla_{(4)}\mathbb{C}_{(4)(1)(1)(2)} &= \partial_{(4)}\mathbb{C}_{(4)(1)(1)(2)} + \eta^{(d)(e)} \cdot \big[\Gamma_{(e)(4)(4)} \cdot \mathbb{C}_{(d)(1)(1)(2)} \\ &\quad + \Gamma_{(e)(1)(4)} \cdot \big(\mathbb{C}_{(4)(d)(1)(2)} + \mathbb{C}_{(4)(1)(d)(2)}\big) + \Gamma_{(e)(2)(4)} \cdot \mathbb{C}_{(4)(1)(1)(d)}\big] \\ &= -\mathbf{D}\Psi_{(1)} + 2\varepsilon \cdot \Psi_{(1)} - 3\kappa_{(0)} \cdot \Psi_{(2)} + \pi_{(0)} \cdot \Psi_{(0)}. \end{aligned}$$

Now, consider the equation

$$\frac{1}{2} \cdot \left[\nabla_{(4)} \mathbb{R}_{(4)(1)} - \nabla_{(1)} \mathbb{R}_{(4)(4)} \right]$$

$$= \frac{1}{2} \cdot \Big\{ \left[\partial_{(4)} \mathbb{R}_{(4)(1)} - \partial_{(1)} \mathbb{R}_{(4)(4)} \right] + \eta^{(d)(e)} \cdot \big[\mathbb{\Gamma}_{(e)(4)(4)} \cdot \mathbb{R}_{(d)(1)}$$

$$+ \mathbb{\Gamma}_{(e)(1)(4)} \cdot \mathbb{R}_{(4)(d)} - 2\mathbb{\Gamma}_{(e)(4)(1)} \cdot \mathbb{R}_{(d)(4)} \big] \Big\}$$

$$= -\mathbf{D}\, \Phi_{(0)(1)} + \boldsymbol{\delta}\, \Phi_{(0)(0)} - 2\kappa_{(0)} \cdot \Phi_{(1)(1)} + 2\varepsilon \cdot \Phi_{(0)(1)} + 2\overline{\rho} \cdot \Phi_{(0)(1)} + 2\sigma \cdot \Phi_{(1)(0)}$$

$$+ \left(\overline{\pi}_{(0)} - 2\overline{\alpha} - 2\beta \right) \cdot \Phi_{(0)(0)} - \overline{\kappa}_{(0)} \cdot \Phi_{(0)(2)} \,.$$

Combining all these equations, (7.182i) is proven. (Consult [35].)

Chapter 8
The Coupled Einstein–Maxwell–Klein–Gordon Equations

8.1 The General E–M–K–G Field Equations

The contents of this chapter generally lie outside the focus of an introductory curriculum, and therefore, this chapter may be considered optional for an introductory course.

Various attempts have been made in the past to arrive at a completely field-theoretic description of microscopic material objects. For example, Wheeler and his collaborators [183, 263] offered a completely geometrical description of classical matter. Heisenberg and his associates [128] presented a nonlinear Dirac equation for unification of various matter fields. Finkelstein [99] has presented extended models of particles with internal rotational motions. Lanczos [158] has constructed nonsingular field-theoretical models from a quadratic action principle. Jackiw et al. [143] have shown that certain exact solutions of nonlinear field equations provide approximations of vacuum expectation values of quantized fields. In recent years, string theory has gained a lot of advances toward the ultimate unification of fields. (See [117, 274].) Das [50] introduced the massive, complex scalar field in general relativity to replace phenomenological descriptions of matter in the right-hand sides of Einstein's field equations. Although the coupled *Einstein–Maxwell–Klein–Gordon equations* (*E–M–K–G* equations) do not qualify as a unified field theory of matter, the exact solutions of the system do provide some insights on the possible nonsingular models of the complex scalar wave fields under *gravitational and electromagnetic self-interactions*. Moreover, relativistic wave fields are utilized extensively in the gravitational research literature. Therefore, a complete introduction to this field is appropriate.

The Maxwell and Klein–Gordon fields here are treated in their "first-quantized forms" and therefore are treated as "classical fields." Second-quantized fields, where the fields themselves (and their canonical momenta) are subject to the Lie algebra of quantum mechanics, and in the standard representations become operators, are not considered here. The coupling of second-quantized fields to gravity gives rise to the

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interesting subject of *quantum field theory in curved space–times* or *semiclassical gravity*, which is beyond the scope of this text. There are several well-written texts on this subject [22, 115, 186, 258].

We shall now investigate the coupled E–M–K–G field equations in what follows. We choose the physical units such that $c = G = \hbar = 1$. (Here, $2\pi\hbar$ is *Planck's constant*.)[1] This system of units is sometimes referred to as natural units, Planck units, or fundamental units. We denote the mass and charge parameters, associated with the complex scalar field, by m and e, respectively. Now, we introduce a few notations in the following:

$$D_j\Psi := \left[\nabla_j + \mathrm{i}e \cdot A_j(\cdot\cdot)\right]\Psi, \tag{8.1i}$$

$$\overline{D}_j\overline{\Psi} := \left[\nabla_j - \mathrm{i}e \cdot A_j(\cdot\cdot)\right]\overline{\Psi}. \tag{8.1ii}$$

(These are known as *gauge covariant derivatives*.) Here, $\Psi(x)$ is the complex scalar field and $A_j(x)$ are components of the electromagnetic 4-potential. Moreover, we denote the electromagnetic field tensor via $F_{ij}(x) = \nabla_i A_j - \nabla_j A_i$.

The system of coupled Einstein–Maxwell–Klein–Gordon equations is furnished as

$$\mathsf{K}(\cdot\cdot) := D^j D_j\Psi - m^2 \cdot \Psi(\cdot\cdot) = 0, \tag{8.2i}$$

$$\overline{\mathsf{K}}(\cdot\cdot) := \overline{D}^j\overline{D}_j\overline{\Psi} - m^2 \cdot \overline{\Psi}(\cdot\cdot) = 0, \tag{8.2ii}$$

$$M^j(\cdot\cdot) := \nabla_k F^{jk} - ie \cdot \left[\overline{\Psi}(\cdot\cdot) \cdot D^j\Psi - \Psi(\cdot\cdot) \cdot \overline{D}^j\overline{\Psi}\right] = 0, \tag{8.2iii}$$

$$\begin{aligned}\mathcal{E}_{ij}(\cdot\cdot) := G_{ij}(\cdot\cdot) + \kappa \cdot \Big\{\Big[&\overline{D}_i\overline{\Psi} \cdot D_j\Psi + \overline{D}_j\overline{\Psi} \cdot D_i\Psi \\ &- g_{ij}(\cdot\cdot)\left(\overline{D}^k\overline{\Psi} \cdot D_k\Psi + m^2 \cdot \overline{\Psi} \cdot \Psi\right)\Big] \\ &+ \left[F_{ik}F_j{}^k - (1/4) \cdot g_{ij} \cdot F_{kl} \cdot F^{kl}\right]\Big\} = 0,\end{aligned} \tag{8.2iv}$$

$$\mathcal{C}^j\left(g_{ik}, \partial_l g_{ik}\right) = 0, \tag{8.2v}$$

$$\mathsf{S}\left(A_i, \partial_j A_i\right) = 0. \tag{8.2vi}$$

The last equation (8.2vi) indicates *one possible subsidiary condition* for the electromagnetic 4-potential $A_i(x)$. This reflects the gauge freedom discussed in (1.71i,ii). Among the coupled equations (8.2i–vi), assuming usual differentiability conditions, the following five differential identities exist:

[1] In this chapter, all physical quantities are expressed as dimensionless pure numbers. (Therefore, as usual, $\kappa = 8\pi$ in this chapter.)

$$\nabla_j M^j \equiv 0, \tag{8.3i}$$

$$\nabla_j \mathcal{E}^{ij} \equiv 0. \tag{8.3ii}$$

Now, we are in a position to count the number of unknown functions versus the number of independent equations in the system (8.2i–vi).

No. of unknown functions:

$$1\,[\mathrm{Re}\,(\Psi)] + 1\,[\mathrm{Im}\,(\Psi)] + 4(A_j) + 10(g_{ij}) = 16.$$

No. of equations:

$$\begin{aligned} &1(\mathsf{K} = 0) + 1\big(\overline{\mathsf{K}} = 0\big) + 4\left(M^j = 0\right) + 10\left(\mathcal{E}_{ij} = 0\right) \\ &\quad + 4\left(\mathcal{C}^j = 0\right) + 1(\mathsf{S} = 0) = 21. \end{aligned}$$

No. of differential identities:

$$1\left(\nabla_j M^j \equiv 0\right) + 4\left(\nabla_j \mathcal{E}^{ij} \equiv 0\right) = 5.$$

No. of independent equations:

$$21 - 5 = 16.$$

Thus, the system of (8.2i–vi) is *determinate*.

Now, we shall derive field equations (8.2i–iv) from a variational principle. (Consult Example 2.5.5.) The appropriate Lagrangian density for this problem is the following (notations are made clear in (8.5)):

$$\begin{aligned} &\mathcal{L}\left(\gamma^{ij}, y_i, \sigma, \overline{\sigma}, \Gamma^k_{\ ij}; y_{ij}, \sigma_i, \overline{\sigma_i};\ \Gamma^k_{\ ijl}\right) \\ &\quad := \sqrt{-\det[\gamma_{ij}]} \cdot \Big\{\gamma^{ij} \cdot \Big[\Gamma^k_{\ kij} - \Gamma^k_{\ ijk} - \Gamma^l_{\ lk} \cdot \Gamma^k_{\ ij} + \Gamma^l_{\ ik} \cdot \Gamma^k_{\ lj}\Big] \\ &\qquad + 2\kappa \cdot \Big[(1/4) \cdot \gamma^{ik} \cdot \gamma^{jl} \cdot \left(y_{ij} - y_{ji}\right) \cdot (y_{kl} - y_{lk}) \\ &\qquad + (1/2) \cdot \gamma^{ij} \cdot \Big((\overline{\sigma_i} - \mathrm{i}e\, y_i \overline{\sigma}) \cdot \left(\sigma_j + \mathrm{i}e\, y_j \sigma\right) \\ &\qquad + \left(\overline{\sigma}_j - \mathrm{i}e\, y_j \overline{\sigma}\right) \cdot \left(\sigma_i + \mathrm{i}e\, y_i \sigma\right)\Big) + m^2 \overline{\sigma} \cdot \sigma\Big]\Big\}, \end{aligned} \tag{8.4i}$$

$$\begin{aligned} \mathcal{L}(\cdot\cdot)_{|..} = \sqrt{-g} \cdot \Big\{&R(\cdot\cdot) + 2\kappa \cdot \big[(1/4) \cdot F_{ij}(\cdot\cdot) \cdot F^{ij}(\cdot\cdot) \\ &+ \overline{D}^j\, \overline{\Psi} \cdot D_j\, \Psi + m^2 \cdot \overline{\Psi}(\cdot\cdot) \cdot \Psi(\cdot\cdot)\big]\Big\}. \end{aligned} \tag{8.4ii}$$

In equation (8.4ii), we denote

$$(\cdot\cdot)_{|..} := (\cdot\cdot)\Big|_{\substack{\gamma^{ij}=g^{ij},\, \Gamma^k_{ij}=\left\{{k \atop ij}\right\},\, \Gamma^k_{ijl}=\partial_l\left\{{k \atop ij}\right\}, \\ y_j=A_j,\, y_{ij}=\partial_j A_i,\, \sigma=\Psi,\, \overline{\sigma}=\overline{\Psi},\, \sigma_j=\partial_j\Psi,\, \overline{\sigma}_j=\partial_j\overline{\Psi}}}. \tag{8.5}$$

(*Here, notations differ slightly with (A1.25).*) Now, let us derive Euler–Lagrange equations (A1.20i) from the Lagrangian density (8.4i,ii). The Euler–Lagrange equation for the complex scalar field is furnished by

$$\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial\overline{\sigma}}\Big|_{..} - \frac{\mathrm{d}}{\mathrm{d}x^j}\left[\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial\overline{\sigma}_j}\Big|_{..}\right] = -2\kappa\cdot\sqrt{-g}\cdot \mathsf{K}(\cdot\cdot) = 0. \tag{8.6}$$

Now we shall obtain the Euler–Lagrange equations for the electromagnetic field from the Lagrangian density (8.4i,ii). These are provided by:

$$\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y_i}\Big|_{..} - \frac{\mathrm{d}}{\mathrm{d}x^j}\left[\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y_{ij}}\Big|_{..}\right] = -2\kappa\cdot\sqrt{-g}\cdot M^i(\cdot\cdot) = 0. \tag{8.7}$$

(Compare with Example 2.5.5.)

Next, we shall derive Einstein's field equations (8.2iv) from the Lagrangian density (8.4i,ii). We shall follow the derivation mentioned in (A1.25) with slight changes of notation. *The action integral* arising from (8.4i,ii) is denoted by

$$J(\cdot\cdot) := \int\limits_{D_4} \mathcal{L}(\cdot\cdot)_{|..}\mathrm{d}^4x. \tag{8.8}$$

We also consider variations of the relevant functions in the following:

$$\widehat{\gamma}^{ij} = g^{ij}(\cdot\cdot) + \varepsilon\cdot h^{ij}(\cdot\cdot), \tag{8.9i}$$

$$\widehat{\Gamma}^k{}_{ij} = \left\{{k \atop i\ j}\right\} + \varepsilon\cdot h^k{}_{ij}(\cdot\cdot), \tag{8.9ii}$$

$$\widehat{\Gamma}^k{}_{ijl} = \partial_l\left\{{k \atop i\ j}\right\} + \varepsilon\cdot \partial_l h^k{}_{ij}, \tag{8.9iii}$$

Using (A1.26), (A1.27), (8.4i,ii), (8.8), and (8.9i–iii), we obtain, after a long calculation

$$\begin{aligned}\frac{\Delta J(\cdot\cdot)}{\varepsilon} = \int\limits_{D_4}\Big\{&G_{ij}(\cdot\cdot) + \kappa\cdot\Big[F_{ik}F_j{}^k - (1/4)\cdot g_{ij}\cdot F_{kl}\cdot F^{kl} \\ &+ \overline{D}_i\overline{\Psi}\cdot D_j\Psi + \overline{D}_j\overline{\Psi}\cdot D_i\Psi - g_{ij}\left(\overline{D}^k\,\overline{\Psi}\cdot D_k\Psi + m^2\overline{\Psi}\cdot\Psi\right)\Big]\Big\} \\ &\times h^{ij}(x)\cdot\sqrt{-g}\cdot\mathrm{d}^4x + (\text{boundary terms}).\end{aligned} \tag{8.10}$$

Choosing variationally admissible boundary conditions of (A1.20ii), the functional derivative $\lim_{\varepsilon \to 0} \left[\frac{\Delta J(\cdot\cdot)}{\varepsilon} \right] = 0$, from (8.10), yields Einstein's field equations (8.2iv).

Remarks: (i) Consider the Lagrangian from (8.4ii) given by

$$L(\cdot\cdot)_{|..} := \frac{1}{\sqrt{-g}} \mathcal{L}(\cdot\cdot)_{|..}$$
$$= R(\cdot\cdot) + 2\kappa \cdot \Big[(1/4) \cdot F_{ij}(\cdot\cdot) \cdot F^{ij}(\cdot\cdot) + \overline{D}^j\, \overline{\Psi} \cdot D_j \Psi + m^2 \cdot \overline{\Psi}(\cdot\cdot) \cdot \Psi(\cdot\cdot) \Big].$$

It is an invariant (or a scalar field) under general coordinate transformations. Moreover, it is invariant under *the following combined gauge transformations*:

$$\widehat{A}_j(x) = A_j(x) - \nabla_j \Lambda,$$
$$\widehat{\Psi}(x) = \exp[\mathrm{i}e\, \Lambda(x)] \cdot \Psi(x),$$
$$\widehat{\overline{\Psi}}(x) = \exp[-\mathrm{i}e\, \Lambda(x)] \cdot \overline{\Psi}(x),$$
$$\widehat{D}_j \widehat{\Psi} = \exp[\mathrm{i}e\, \Lambda(\cdot\cdot)] \cdot D_j \Psi,$$
$$\widehat{\overline{D}}_j \widehat{\overline{\Psi}} = \exp[-\mathrm{i}e\, \Lambda(\cdot\cdot)] \cdot \overline{D}_j \overline{\Psi}.$$

(ii) The invariances of part (i) give rise to identities (8.3i,ii) via Noether's theorem [56]. Physically speaking, these identities are related to differential conservations of the charge–current and the energy–momentum density of the complex scalar field.

Now we shall express gravitational field equations (8.2iv) in alternate forms. Using (2.162i), (8.2iv) yield

$$\widetilde{\mathcal{E}}_{ij}(\cdot\cdot) := R_{ij}(\cdot\cdot) + \kappa \cdot \Big\{ \overline{D}_i \overline{\Psi} \cdot D_j \Psi + \overline{D}_j \overline{\Psi} \cdot D_i \Psi$$
$$+ m^2 \cdot g_{ij} \cdot \overline{\Psi} \cdot \Psi + F_{ik} \cdot F_j{}^k - (1/4) \cdot g_{ij} \cdot F_{kl} \cdot F^{kl} \Big\} = 0. \quad (8.11)$$

Next, we provide the coupled Einstein–Maxwell–Klein–Gordon field equations with orthonormal components. These are more relevant from a physical perspective. Equations (8.2i–vi) can be expressed equivalently as

$$D_{(a)}\Psi := \nabla_{(a)}\Psi + \mathrm{i}e \cdot A_{(a)}(\cdot\cdot) \cdot \Psi(\cdot\cdot), \quad (8.12\text{i})$$

$$\overline{D}_{(a)}\overline{\Psi} := \nabla_{(a)}\overline{\Psi} - \mathrm{i}e \cdot A_{(a)}(\cdot\cdot) \cdot \overline{\Psi}(\cdot\cdot), \quad (8.12\text{ii})$$

$$\mathsf{K}(\cdot\cdot) := D^{(a)} D_{(a)}\Psi - m^2 \Psi(\cdot\cdot) = 0, \quad (8.12\text{iii})$$

$$\overline{\mathsf{K}}(\cdot\cdot) := \overline{D}^{(a)} \overline{D}_{(a)}\overline{\Psi} - m^2 \overline{\Psi}(\cdot\cdot) = 0, \quad (8.12\text{iv})$$

$$M^{(a)}(\cdot\cdot) := \nabla_{(b)} F^{(a)(b)} - \mathrm{i}e \cdot \left[\overline{\Psi}(\cdot\cdot) D^{(a)} \Psi - \Psi(\cdot\cdot) \overline{D}^{(a)} \overline{\Psi} \right] = 0, \tag{8.12v}$$

$$\begin{aligned} \mathcal{E}_{(a)(b)}(\cdot\cdot) := G_{(a)(b)}(\cdot\cdot) + \kappa \cdot \Big\{ \Big[& \overline{D}_{(a)} \overline{\Psi} \cdot D_{(b)} \Psi + \overline{D}_{(b)} \overline{\Psi} \cdot D_{(a)} \Psi \\ & - d_{(a)(b)} \cdot \left(\overline{D}^{(c)} \overline{\Psi} \cdot D_{(c)} \Psi + m^2 \cdot \overline{\Psi} \cdot \Psi \right) \Big] \\ & + \left[F_{(a)(c)} F_{(b)}{}^{(c)} - (1/4) \cdot d_{(a)(b)} \cdot F_{(c)(d)} \cdot F^{(c)(d)} \right] \Big\} = 0, \end{aligned} \tag{8.12vi}$$

$$\mathcal{C}^{(a)} \left(\lambda^i{}_{(b)}, \, \partial_{(c)} \lambda^i{}_{(b)} \right) = 0, \tag{8.12vii}$$

$$\mathsf{S} \left(A_{(a)}, \, \partial_{(b)} A_{(a)} \right) = 0. \tag{8.12viii}$$

Gravitational field equations (8.12vi) can be furnished equivalently by (2.163ii) as

$$\begin{aligned} \mathcal{E}^{(d)}{}_{(a)(b)(c)}(\cdot\cdot) := {} & R^{(d)}{}_{(a)(b)(c)}(\cdot\cdot) - C^{(d)}{}_{(a)(b)(c)}(\cdot\cdot) \\ & + (\kappa/2) \cdot \Big\{ \delta^{(d)}{}_{(c)} \cdot T_{(a)(b)}(\cdot\cdot) - \delta^{(d)}{}_{(b)} \cdot T_{(a)(c)}(\cdot\cdot) \\ & + d_{(a)(b)} \cdot T^{(d)}{}_{(c)}(\cdot\cdot) - d_{(a)(c)} \cdot T^{(d)}{}_{(b)}(\cdot\cdot) \\ & + (2/3) \cdot \left[\delta^{(d)}{}_{(b)} \cdot d_{(a)(c)} - \delta^{(d)}{}_{(c)} \cdot d_{(a)(b)} \right] \cdot T^{(e)}{}_{(e)}(\cdot\cdot) \Big\} = 0, \end{aligned} \tag{8.13i}$$

$$\begin{aligned} T_{(a)(b)}(\cdot\cdot) := {} & \overline{D}_{(a)} \overline{\Psi} \cdot D_{(b)} \Psi + \overline{D}_{(b)} \overline{\Psi} \cdot D_{(a)} \Psi \\ & - d_{(a)(b)} \cdot \left(\overline{D}^{(c)} \overline{\Psi} \cdot D_{(c)} \Psi + m^2 \cdot \overline{\Psi} \cdot \Psi \right) \\ & + \left[F_{(a)(c)} \cdot F_{(b)}{}^{(c)} - (1/4) \cdot d_{(a)(b)} \cdot F_{(c)(d)} \cdot F^{(c)(d)} \right]. \end{aligned} \tag{8.13ii}$$

The Cauchy problem or the initial value problem for the system of partial differential equations (8.2i–iv) has been investigated in [50]. (Consult Theorems 2.4.7 and 2.4.9.)

The *Rainich problem* for this system of field equations has been examined in [70].

8.2 Static Space–Time Domains and the E–M–K–G Equations

We investigate in this section the Einstein–Maxwell–Klein–Gordon (E–M–K–G) equations with the general static metric. Moreover, we assume the existence of a static electric field and *absence of any magnetic field*. Furthermore, the complex scalar wave field is assumed to be in an eigenstate of energy according to the usual wave mechanics (or quantum mechanics), as this "time-harmonic" form is quite useful in physics. We express these physical assumptions mathematically as the following:

$$\Psi(x) = \chi(\mathbf{x}) \cdot e^{-iEx^4}, \quad \overline{\Psi}(x) = \overline{\chi}(\mathbf{x}) \cdot e^{iEx^4}, \tag{8.14i}$$

$$\overline{\chi}(x) = \chi(x)\,; \tag{8.14ii}$$

$$A_\alpha(x) \equiv 0, \quad A_4(x) =: \mathcal{A}(\mathbf{x}) \neq 0, \tag{8.14iii}$$

$$F_{\alpha\beta}(x) \equiv 0, \quad F_{\alpha 4}(\mathbf{x}) = \partial_\alpha \mathcal{A} \not\equiv 0\,; \tag{8.14iv}$$

$$\begin{aligned} ds^2 = {} & g_{\alpha\beta}(\mathbf{x}) \cdot (dx^\alpha) \cdot \left(dx^\beta\right) + g_{44}(\mathbf{x}) \cdot \left(dx^4\right)^2 \\ =: {} & g^{\#}_{\alpha\beta}(\mathbf{x}) \cdot (dx^\alpha) \cdot \left(dx^\beta\right) - f(\mathbf{x}) \cdot \left(dx^4\right)^2, \end{aligned} \tag{8.14v}$$

$$g_{\alpha 4}(x) \equiv 0, \quad f(\mathbf{x}) > 0, \quad \sqrt{-g} = f^{1/2} \cdot \sqrt{g^{\#}} > 0. \tag{8.14vi}$$

Here, the constant E represents the eigenvalue of energy associated with the wave field $\Psi(x)$. Moreover, (8.14ii) and (8.14iii) guarantee that there is no magnetic field. Furthermore, (8.14iv) implies that the static electric field under consideration is *not trivial*. (See [60].)

Now let us work out some of the field equations (8.2i–iv) with the assumptions made in (8.14i–v).

$$\begin{aligned} M^4(\cdot\cdot) = {} & f^{-1} \cdot \frac{1}{\sqrt{-g}} \cdot \partial_\alpha \left(\sqrt{-g} \cdot g^{\alpha\beta} \cdot \partial_\beta \mathcal{A}\right) - f^{-2} \cdot g^{\alpha\beta} \cdot \partial_\alpha f \cdot \partial_\beta \mathcal{A} \\ & + 2e \cdot f^{-1} \cdot (E - e\mathcal{A}) \cdot \chi^2 = 0. \end{aligned} \tag{8.15}$$

By the field equations (8.2iv) and (8.11), with help of the metric in (8.14v,vi), we derive that

$$\begin{aligned} -\widetilde{\mathcal{E}}_{44}(\cdot\cdot) = {} & (1/2) \cdot \left[\frac{1}{\sqrt{-g}} \cdot \partial_\alpha \left(\sqrt{-g} \cdot g^{\alpha\beta} \cdot \partial_\beta f\right) - f^{-1} \cdot g^{\alpha\beta} \cdot \partial_\alpha f \cdot \partial_\beta f\right] \\ & - \kappa \cdot \left\{\left[2\,(E - e\mathcal{A})^2 - m^2 \cdot f\right] \cdot \chi^2 + (1/2) \cdot g^{\alpha\beta} \cdot \partial_\alpha \mathcal{A} \cdot \partial_\beta \mathcal{A}\right\} = 0. \end{aligned} \tag{8.16}$$

We shall next simplify the field equations by imposing the W–M–P–D condition (4.94) (with a different notation) as

$$f = F(\mathcal{A}) := m^{-2} \cdot [E - e \cdot \mathcal{A}]^2, \tag{8.17i}$$

$$f(\mathbf{x}) = m^{-2} \cdot [E - e \cdot \mathcal{A}(\mathbf{x})]^2, \tag{8.17ii}$$

$$f^{1/2}(\mathbf{x}) = m^{-1} \cdot [E - e \cdot \mathcal{A}(\mathbf{x})] > 0, \tag{8.17iii}$$

$$\partial_\alpha f = -2 \cdot (e/m) \cdot f^{1/2} \cdot \partial_\alpha \mathcal{A}. \tag{8.17iv}$$

The following linear combination of the field equations (8.15) and (8.16) yields:

$$\begin{aligned}
&(m/e) \cdot f^{-1/2} \cdot \widetilde{\mathcal{E}}_{44} - f \cdot M^4 \\
&\quad = \left\{(m/e) \cdot \left[(e/m)^2 + (\kappa/2)\right] \cdot f^{-1/2} - 2(e/m) \cdot f^{-1/2}\right\} \cdot g^{\alpha\beta} \partial_\alpha \mathcal{A} \cdot \partial_\beta \mathcal{A} \\
&\qquad + \left\{\left(\kappa \cdot m^3 \cdot e^{-1}\right) \cdot f^{1/2} - 2e \cdot (E - e\mathcal{A})\right\} \cdot \chi^2 = 0.
\end{aligned} \tag{8.18}$$

We satisfy the equation above by setting the quantities in the above two curly brackets separately to zero. Thus, we deduce that

$$\left(\kappa \cdot m^3 \cdot e^{-1}\right) \cdot f^{1/2} - 2e \cdot (E - e\mathcal{A}) = 0, \tag{8.19i}$$

and $$(m/e) \cdot \left[(e/m)^2 + (\kappa/2)\right] - 2(e/m) = 0. \tag{8.19ii}$$

Substituting (8.17iii) into (8.19i), we deduce that

$$(e/m)^2 = (\kappa/2). \tag{8.20}$$

(Compare the equation above with *the static, equilibrium condition* (4.95).)

Putting (8.20) into (8.19i), we obtain the identity

$$(m/e) \cdot \left[(e/m)^2 + (\kappa/2)\right] - 2 \cdot (e/m) = (m/e) \cdot [\kappa - \kappa] \equiv 0. \tag{8.21}$$

Employing conditions (8.17ii,iii) and (8.20), let us compute other field equations. From (8.2i), (8.11), (8.14i–vi), (8.17i–iv), and (8.20), we derive that

$$\begin{aligned}
\mathsf{K} \cdot \mathrm{e}^{\mathrm{i}Ex^4} &= \frac{1}{\sqrt{-g}} \cdot \partial_\alpha \left[\sqrt{-g} \cdot g^{\alpha\beta} \cdot \partial_\beta \chi\right] + \left[f^{-1} \cdot (E - e\mathcal{A})^2 - m^2\right] \cdot \chi \\
&= \frac{1}{\sqrt{-g}} \cdot \partial_\alpha \left[\sqrt{-g} \cdot g^{\alpha\beta} \cdot \partial_\beta \chi\right] + 0 = 0.
\end{aligned} \tag{8.22}$$

Equation (8.16), with the help of (8.17i–iv), yields

$$\begin{aligned}
\widetilde{\mathcal{E}}_{44}(\cdot\cdot) = (e/m) \cdot f^{1/2} \cdot \frac{1}{\sqrt{-g}} \cdot \partial_\alpha \left(\sqrt{-g} \cdot g^{\alpha\beta} \cdot \partial_\beta \mathcal{A}\right) \\
+ \kappa \cdot g^{\alpha\beta} \cdot \partial_\alpha \mathcal{A} \cdot \partial_\beta \mathcal{A} + \kappa \cdot m^2 \cdot f \cdot \chi^2 = 0.
\end{aligned} \tag{8.23}$$

Moreover, the remaining field equations, from (8.11), reduce to

$$\begin{aligned}\widetilde{\mathcal{E}}_{\alpha\beta}(\cdot\cdot) = R_{\alpha\beta}(\cdot\cdot) + \kappa \cdot \Big[2 \cdot \partial_\alpha \chi \cdot \partial_\beta \chi + m^2 \cdot g_{\alpha\beta} \cdot \chi^2 \\ + f^{-1} \cdot \Big((1/2) \cdot g_{\alpha\beta} \cdot g^{\gamma\delta} \cdot \partial_\gamma \mathcal{A} \cdot \partial_\delta \mathcal{A} - \partial_\alpha \mathcal{A} \cdot \partial_\beta \mathcal{A} \Big) \Big] = 0, \quad (8.24\text{i})\end{aligned}$$

$$\widetilde{\mathcal{E}}_{\alpha 4}(\cdot\cdot) \equiv 0. \quad (8.24\text{ii})$$

Next, we use the metric tensor of (8.14v,vi). The Klein–Gordon equation of (8.22) reduces to

$$f^{1/2} \cdot \mathsf{K} \cdot \mathrm{e}^{\mathrm{i}Ex^4} = \frac{1}{\sqrt{g^\#}} \cdot \partial_\alpha \left[f^{1/2} \cdot \sqrt{g^\#} \cdot g^{\#\alpha\beta} \cdot \partial_\beta \chi \right] = 0. \quad (8.25\text{i})$$

The field equation (8.23) goes over into

$$\begin{aligned}\widetilde{\mathcal{E}}_{44}(\cdot\cdot) = (e/m) \cdot \left(1/\sqrt{g^\#} \right) \cdot \partial_\alpha \left[f^{1/2} \cdot \sqrt{g^\#} \cdot g^{\#\alpha\beta} \cdot \partial_\beta \mathcal{A} \right] \\ + \kappa \cdot g^{\#\alpha\beta} \cdot \partial_\alpha \mathcal{A} \cdot \partial_\beta \mathcal{A} \\ + \kappa \cdot m^2 \cdot f \cdot \chi^2 = 0, \quad (8.25\text{ii})\end{aligned}$$

Furthermore, field equations (8.24i) transform into

$$\begin{aligned}\widetilde{\mathcal{E}}_{\alpha\beta}(\cdot\cdot) = R_{\alpha\beta} + \kappa \Big[2\partial_\alpha \chi \cdot \partial_\beta \chi + m^2 \cdot g^\#_{\alpha\beta} \cdot \chi^2 \\ + f^{-1} \Big((1/2) \cdot g^\#_{\alpha\beta} \cdot g^{\#\gamma\delta} \cdot \partial_\gamma \mathcal{A} \cdot \partial_\delta \mathcal{A} - \partial_\alpha \mathcal{A} \cdot \partial_\delta \mathcal{A} \Big) \Big] = 0,\end{aligned} \quad (8.25\text{iii})$$

$$\text{where, } R_{\alpha\beta} = R^\#_{\alpha\beta} + (1/2) \cdot \nabla^\#_\alpha \left(f^{-1} \cdot \nabla^\#_\beta f \right) + (1/4) \cdot f^{-2} \cdot \nabla^\#_\alpha f \cdot \nabla^\#_\beta f. \quad (8.25\text{iv})$$

The linear combination of the preceding field equations yields

$$\begin{aligned}&\widetilde{\mathcal{E}}_{\alpha\beta}(\cdot\cdot) - f^{-1} \cdot g^\#_{\alpha\beta} \cdot \widetilde{\mathcal{E}}_{44}(\cdot\cdot) \\ &= \left[R^\#_{\alpha\beta} + (1/2) \cdot \nabla^\#_\alpha \left(f^{-1} \cdot \nabla^\#_\beta f \right) + (1/4) \cdot f^{-2} \cdot \nabla^\#_\alpha f \cdot \nabla^\#_\beta f \right] \\ &\quad + 2\kappa \cdot \partial_\alpha \chi \cdot \partial_\beta \chi - f^{-1/2} \cdot (e/m) \cdot \Big[g^\#_{\alpha\beta} \cdot \left(1/\sqrt{g^\#} \right) \\ &\quad \times \partial_\gamma \left(\sqrt{g^\#} \cdot g^{\#\gamma\delta} \cdot \partial_\delta \mathcal{A} \right) + 2\,(e/m) \cdot f^{-1/2} \cdot \partial_\alpha \mathcal{A} \cdot \partial_\beta \mathcal{A} \Big] = 0. \quad (8.26)\end{aligned}$$

Now we shall use a new coordinate chart for which

$$ds^2 = [V(\mathbf{x})]^2 \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x}) \cdot dx^\alpha dx^\beta - [V(\mathbf{x})]^{-2} \cdot \left(dx^4\right)^2, \tag{8.27i}$$

$$V(\mathbf{x}) := m \cdot [E - e\,\mathcal{A}(\mathbf{x})]^{-1} > 0, \tag{8.27ii}$$

$$g^{\#}_{\alpha\beta}(\mathbf{x}) = [V(\mathbf{x})]^2 \cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x}), \quad \sqrt{g^{\#}} = [V(\mathbf{x})]^3 \cdot \sqrt{\overset{\circ}{g}}.$$

(Compare the equation above with (4.97).) It is clear from (8.27ii) that

$$\partial_\alpha V = e \cdot m \cdot (E - e\,\mathcal{A})^{-2} \cdot \partial_\alpha \mathcal{A}, \tag{8.28i}$$

$$\begin{aligned} \partial_\alpha \mathcal{A} &= (e \cdot m)^{-1} \cdot (E - e\,\mathcal{A})^2 \cdot \partial_\alpha V. \\ &= (m/e) \cdot V^{-2} \cdot \partial_\alpha V. \end{aligned} \tag{8.28ii}$$

Substituting (8.27i,ii) and (8.28i,ii) into (8.25i–iii), we obtain the following simpler field equations:

$$V^2 \cdot \mathsf{K} \cdot e^{iEx^4} = \frac{1}{\sqrt{\overset{\circ}{g}}} \cdot \partial_\alpha \left[\sqrt{\overset{\circ}{g}} \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\beta\, \chi\right] = 0, \tag{8.29i}$$

$$V^5 \cdot \widetilde{\mathcal{E}}_{44}(\cdot\cdot) = \frac{1}{\sqrt{\overset{\circ}{g}}} \cdot \partial_\alpha \left[\sqrt{\overset{\circ}{g}} \cdot \overset{\circ}{g}{}^{\alpha\beta} \cdot \partial_\beta V\right] + 2e^2 \cdot \chi^2 \cdot V^3 = 0, \tag{8.29ii}$$

$$\widetilde{\mathcal{E}}_{\alpha\beta}(\cdot\cdot) - V^4 \cdot \overset{\circ}{g}_{\alpha\beta}(\cdot\cdot) \cdot \widetilde{\mathcal{E}}_{44}(\cdot\cdot) = \overset{\circ}{R}_{\alpha\beta}(\cdot\cdot) + 2\kappa \cdot \partial_\alpha\, \chi \cdot \partial_\beta\, \chi = 0. \tag{8.29iii}$$

We notice from (8.29i) that the function $\chi(\mathbf{x})$ *is harmonic in the positive-definite, three-dimensional metric* $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$. Therefore, we can apply *Hopf's Theorem* 4.2.2 for the function $\chi(\mathbf{x})$. Thus, we state and prove the following relevant theorem for field equations (8.29i–iii).

Theorem 8.2.1. *Let the function* $\chi \in C^2\left(\mathbf{D} \subset \mathbb{R}^3; \mathbb{R}\right)$ *satisfy the harmonic condition* (8.29i) *for all* $\mathbf{x} \in \mathbf{D} \subset \mathbb{R}^3$. *If there exists an interior point* $\mathbf{x}_0 \in \mathbf{D}$ *such that* $\chi(\mathbf{x}) \leq \chi(\mathbf{x}_0) =: \chi_0$, *or,* $\chi_0 = \chi(\mathbf{x}_0) \leq \chi(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbf{D}$, *then* $\chi(\mathbf{x}) \equiv \chi(\mathbf{x}_0) = \chi_0 =$ *constant. Moreover, the metric* $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ *becomes (flat) Euclidean. Furthermore, the field equation* (8.29ii) *reduces to*

$$\left(1/\sqrt{\overset{\circ}{g}}\right) \cdot \partial_\alpha \left[\sqrt{\overset{\circ}{g}} \cdot \overset{\circ}{g}{}^{\alpha\beta} \partial_\beta V\right] + 2e^2 \cdot (\chi_0)^2 \cdot V^3 = 0. \tag{8.30}$$

The proof of the above theorem is left to the reader.

Before continuing, we present here an interesting digression. Motivated by the above theorem, we define the *homogeneous state of the complex wave function* $\Psi(x)$ by the following:

$$\begin{aligned}&\Psi(x) = \chi_0 \cdot \mathrm{e}^{-\mathrm{i}Ex^4}, \quad \overline{\Psi}(x) = \chi_0 \cdot \mathrm{e}^{\mathrm{i}Ex^4},\\&\mathrm{e}^{\mathrm{i}Ex^4} \cdot \Psi(x) = \mathrm{e}^{-\mathrm{i}Ex^4} \cdot \overline{\Psi}(x) = \chi_0 = \text{ real constant.}\end{aligned} \tag{8.31}$$

Thus, attaining an interior maximum or an interior minimum, the real wave function $\chi(\mathbf{x})$ of (8.31) *must belong to a homogeneous state* by Theorem 8.2.1. Moreover, the corresponding static metric in (8.27i) is of *the conformastat class of (4.55)*. Furthermore, *the system of eight, complicated partial differential equations (8.22), (8.23), and (8.24i) reduce to one second order, semilinear partial differential equation (8.30)!*

Now let us make a slight digression on the consequences of *a more general homogeneous wave function* provided by

$$\begin{aligned}&\Psi(x) = \chi_0 \cdot \mathrm{e}^{-\mathrm{i}Ex^4}, \quad \overline{\Psi}(x) = \overline{\chi}_0 \cdot \mathrm{e}^{\mathrm{i}Ex^4},\\&\chi_0 = \text{ a complex constant,}\end{aligned} \tag{8.32}$$

with a general electromagnetic field.

We shall investigate the implications of (8.32) on the system of general E–M–K–G field equations (8.2i–iv). The Klein–Gordon equation (8.2i) goes over into

$$e^2 \cdot A^{\alpha} \cdot A_{\alpha} + m^2 + g^{44} \cdot (E - e\,A_4)^2 = 0, \tag{8.33i}$$

$$\textit{and} \qquad -\left(E/\sqrt{-g}\right) \cdot \partial_j \cdot \left(\sqrt{-g} \cdot g^{j4}\right) + e \cdot \left(\nabla_j\, A^j\right) = 0. \tag{8.33ii}$$

Two equations (8.33i) and (8.33ii) emerge from the real and the imaginary parts of the Klein–Gordon equation. The condition (8.33i) can be aptly called *the generalized W–M–P–D condition*. The second equation is *a linear combination of the Lorentz gauge and harmonic gauge conditions*.

Now, the Maxwell equation (8.2iii), with assumptions in (8.32), yields

$$\left(1/\sqrt{-g}\right) \cdot \partial_k \left[\sqrt{-g} \cdot F^{jk}\right] = 2eE \cdot |\chi_0|^2 \cdot g^{j4}(\cdot\cdot) - 2e^2 \cdot |\chi_0|^2 \cdot A^j(\cdot\cdot). \tag{8.34}$$

The equation above is *the general relativistic, nonhomogeneous, Proca equation for a massive vector boson field*. The mass parameter for the vector boson field in (8.34) is provided by $M_0 = \sqrt{2}e \cdot |\chi_0| > 0$. The vector boson field $\vec{\mathbf{A}}(x)$ in (8.34) can induce *short-ranged self-interactions* for the complex wave field $\Psi(x)$. This interaction, implicit in (8.34), is a generalization of the *London equation* and the *Meissner effect* in superconductivity [169, 179].

Remarks: (i) Reference [59] deals with special relativistic, coupled Maxwell–Klein–Gordon equations. For a homogeneous wave field $\Psi(\cdot\cdot) = \chi_0 \cdot \mathrm{e}^{-\mathrm{i}Ex^4}$, *the special relativistic versions* of (8.33i), (8.33ii), and (8.34) are derived in that paper.

(ii) Compare and contrast the vector boson equation (8.34) with the short-ranged gravitation (which is gravitation with vanishing Weyl tensor) in Appendix 4.

(iii) The thesis by Gegenberg [109] also deals with vector boson field equations associated with a homogeneous complex wave function $\Psi(x) = \chi_0 \cdot \mathrm{e}^{-\mathrm{i}Ex^4}$.

Now we shall *go back to static field equations (8.29i–iii)* and explore some effective solution strategies. Without loss of generality, we consider the static metric:

$$\begin{aligned} \mathrm{d}s_0^2 = {} & \exp\left[-2\sqrt{\kappa}\cdot\chi(\mathbf{x})\right]\cdot \overset{\circ}{g}_{\alpha\beta}(\mathbf{x})\cdot(\mathrm{d}x^\alpha)\cdot\left(\mathrm{d}x^\beta\right) \\ & - \exp\left[2\sqrt{\kappa}\cdot\chi(\mathbf{x})\right]\cdot\left(\mathrm{d}x^4\right)^2 . \end{aligned} \tag{8.35}$$

The static, vacuum field equations from (4.41i), (4.41ii), and (8.35) are furnished by

$$\overset{\circ}{R}_{\alpha\beta}(\mathbf{x}) + 2\kappa\cdot\partial_\alpha\chi\cdot\partial_\beta\chi = 0, \tag{8.36i}$$

$$\left(1\Big/\sqrt{\overset{\circ}{g}}\right)\cdot\partial_\alpha\left[\sqrt{\overset{\circ}{g}}\cdot\overset{\circ}{g}{}^{\alpha\beta}\cdot\partial_\beta\chi\right] = 0. \tag{8.36ii}$$

Equations (8.36i) and (8.36ii) above are exactly the same equations as (8.29iii) and (8.29i)! Thus, we are able to devise a solution strategy for (8.29i–iii) in the following theorem.

Theorem 8.2.2. *Let the function* $\chi \in C^2\left(\mathbf{D}\subset\mathbb{R}^3;\mathbb{R}\right)$ *and the metric tensor components* $\overset{\circ}{g}_{\alpha\beta} \in C^3\left(\mathbf{D}\subset\mathbb{R}^3;\mathbb{R}\right)$ *satisfy the static, vacuum field equations* (8.36i) *and* (8.36ii) *for all* $\mathbf{x} \in \mathbf{D} \subset \mathbb{R}^3$. *Then, the metric tensor components in* (8.35) *satisfy field equations* (8.29i) *and* (8.29ii). *Moreover,* (8.29ii) *remains unchanged as*

$$\left(1\Big/\sqrt{\overset{\circ}{g}}\right)\cdot\partial_\alpha\left[\sqrt{\overset{\circ}{g}}\cdot\overset{\circ}{g}{}^{\alpha\beta}\cdot\partial_\beta V\right] + 2e^2\cdot[\chi(\cdot\cdot)]^2\cdot[V(\cdot\cdot)]^3 = 0. \tag{8.37}$$

The proof is again left to the reader. Now we shall investigate a consequence of this theorem.

Corollary 8.2.3. *Let the complex wave function* $\Psi(x)$ *be in a homogeneous state, that is* $\chi(\mathbf{x}) = \chi_0 =$ *a real constant. Then, the metric* $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ *must be (flat) Euclidean. Furthermore, the single nontrivial field equation* (8.37)*, with help of a Cartesian coordinate chart, reduces to*

$$\nabla^2 V := \delta^{\alpha\beta} \cdot \partial_\alpha \partial_\beta V = -2e^2 \cdot (\chi_0)^2 \cdot [V(\cdot\cdot)]^3 . \tag{8.38}$$

The proof is straightforward.

Remark: If we express $[V(\cdot\cdot)]^{-2} = [1 + 2W(\cdot\cdot)]$ from Example 2.2.9, then for a weak gravitational field we can represent $V(\mathbf{x}) = 1 - W(\mathbf{x}) + 0\left(W^2\right)$. In that case, the field equation (8.38) goes over into

$$\nabla^2 W + 6e^2 \cdot (\chi_0)^2 \cdot W(\cdot\cdot) = 2e^2 \cdot [\chi_0]^2 + 0\left(W^2\right).$$

The equation above is somewhat similar to *a nonhomogeneous Yukawa equation* for the *effective* short-ranged nuclear force that binds nucleons.

Now, we shall provide two examples of exact solutions of the partial differential equation (8.38) and the consequent exact solutions of the static E–M–K–G equations (8.29i–iii).

Example 8.2.4. Consider the plane symmetric solution of (8.38) with the assumptions $\partial_1 V \neq 0,\ \partial_2 V = \partial_3 V \equiv 0$. The exact solution is furnished by

$$V\left(x^1\right) = \sqrt{\left|\frac{c_0}{e\chi_0}\right|} \cdot \mathrm{d}n\left[\sqrt{|e\chi_0 c_0|} \cdot \left(x^1 - x_0\right),\ \sqrt{2}\right]. \tag{8.39}$$

Here, $\mathrm{d}n(..,..)$ is one of *the Jacobian elliptic functions* [91]. The function (8.39) is plotted[2] in Fig. 8.1.

The corresponding metric, complex wave function, and electrostatic potential are given by (recall Corollary 8.2.3)

$$(\mathrm{d}s)^2 = \left[V\left(x^1\right)\right]^2 \cdot \delta_{\alpha\beta} \cdot (\mathrm{d}x^\alpha) \cdot \left(\mathrm{d}x^\beta\right) - \left[V\left(x^1\right)\right]^{-2} \cdot \left(\mathrm{d}x^4\right)^2, \tag{8.40i}$$

$$\Psi(\cdot\cdot) = \chi_0 \cdot \mathrm{e}^{-\mathrm{i}Ex^4}, \qquad \overline{\Psi}(\cdot\cdot) = \chi_0 \cdot \mathrm{e}^{\mathrm{i}Ex^4}, \tag{8.40ii}$$

$$\mathcal{A}\left(x^1\right) = A_4\left(x^1\right) = (e)^{-1} \cdot \left\{E - m \cdot \left[V\left(x^1\right)\right]^{-1}\right\}. \tag{8.40iii}$$

(It is of course assumed that the equilibrium condition $e^2 = (\kappa/2) \cdot m^2$ holds.) □

Example 8.2.5. This example deals with the static, axially symmetric metric and electrostatic field. (Consult Example 4.1.5. Moreover, De [69] has investigated the E–M–K–G equations with the static, axially symmetric metric and an electrostatic field.) *The vacuum metric* under consideration is furnished by (4.7), (4.9), and (4.10i–iv) as

[2] The space–time geometry at the events corresponding to the roots of $V(x^1) = 0$ needs to be investigated.

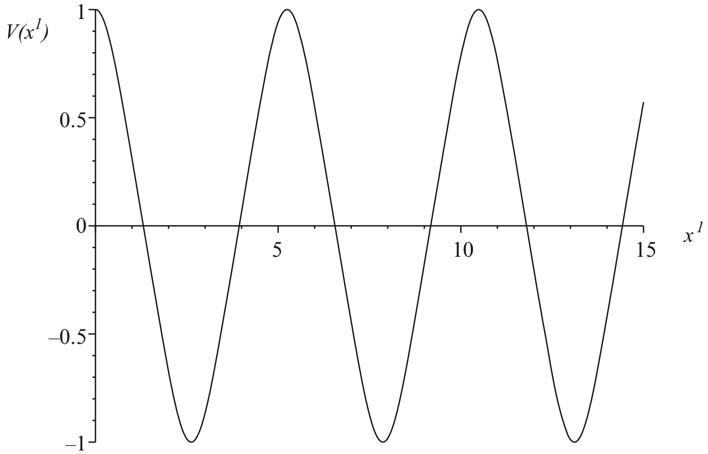

Fig. 8.1 A plot of the function in (8.39) with the following parameters: $c_0 = 1$, $e = 1$, $\chi_0 = 1$ and $x_0 = 0$

$$(\mathrm{d}s_0)^2 = \mathrm{e}^{-2\omega(x^1,x^2)} \cdot \left\{ \mathrm{e}^{2\nu(x^1,x^2)} \cdot \left[\left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 \right] + \left(x^1\right)^2 \cdot \left(\mathrm{d}x^3\right)^2 \right\} - \mathrm{e}^{2\omega(x^1,x^2)} \cdot \left(\mathrm{d}x^4\right)^2, \tag{8.41i}$$

$$\nu\left(x^1,x^2\right) = \int_{(\rho_0,z_0)[\Gamma]}^{(x^1,x^2)} y^1 \cdot \left\{ \left[(\partial_1\omega)^2 - (\partial_2\omega)^2 \right] \cdot \mathrm{d}y^1 + 2 \cdot (\partial_1\omega) \cdot (\partial_2\omega) \cdot \mathrm{d}y^2 \right\}. \tag{8.41ii}$$

$$\partial_1\partial_1\omega + \left(1/x^1\right) \cdot \partial_1\omega + \partial_2\partial_2\omega = 0. \tag{8.41iii}$$

By (8.35), we set

$$\chi\left(x^1,x^2\right) := \left(1/\sqrt{\kappa}\right) \cdot \omega\left(x^1,x^2\right). \tag{8.42}$$

Also, by Theorem 8.2.2 and (8.37), the exact solutions of the E–M–K–G equations are provided by

$$\Psi(\cdot\cdot) = \chi\left(x^1,x^2\right) \cdot \mathrm{e}^{-\mathrm{i}Ex^4}, \quad \overline{\Psi}(\cdot\cdot) = \chi\left(x^1,x^2\right) \cdot \mathrm{e}^{\mathrm{i}Ex^4}, \tag{8.43i}$$

$$(\mathrm{d}s)^2 = \left[V\left(x^1,x^2\right)\right]^2 \cdot \left\{ \mathrm{e}^{2\nu(\cdot\cdot)} \cdot \left[\left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 \right] + \left(x^1\right)^2 \cdot \left(\mathrm{d}x^3\right)^2 \right\} - \left[V\left(x^1,x^2\right)\right]^{-2} \cdot \left(\mathrm{d}x^4\right)^2, \tag{8.43ii}$$

$$\partial_1\partial_1 V + \left(1/x^1\right) \cdot \partial_1 V + \partial_2\partial_2 V + 2(e)^2 \cdot [\chi(\cdot\cdot)]^2 \cdot [V(\cdot\cdot)]^3 = 0, \tag{8.43iii}$$

$$\mathcal{A}\left(x^1,x^2\right) \equiv A_4\left(x^1,x^2\right) = (e)^{-1} \cdot \left\{ E - m \cdot \left[V\left(x^1,x^2\right)\right]^{-1} \right\}, \tag{8.43iv}$$

$$\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})\,\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta} = \mathrm{e}^{2\nu(x^1,x^2)} \cdot \left[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right] + (x^1)^2 \cdot (\mathrm{d}x^3)^2. \tag{8.43v}$$

The metric above is not conformally flat.

We have assumed the equilibrium condition $e^2 = (\kappa/2) \cdot m^2$. Furthermore, the axially symmetric, exact solutions of the E–M–K–G equations depend solely on the exact solution of the nonlinear potential equation (8.43iii). □

8.3 Spherical Symmetry and a Nonlinear Eigenvalue Problem for a Theoretical Fine-Structure Constant

In this section, we investigate the E–M–K–G equations, *assuming spherical symmetry* for the static metric and the electrostatic potential.

Firstly, let us consider the case of *the homogeneous state* for the complex wave field and the corresponding partial differential equations for the metric and electromagnetic potential in the following:

$$\Psi(x) \cdot \mathrm{e}^{\mathrm{i}Ex^4} = \overline{\Psi}(x) \cdot \mathrm{e}^{-\mathrm{i}Ex^4} = \chi_0 = \text{ a real constant}, \tag{8.44i}$$

$$\mathcal{A}\left(x^1\right) = A_4\left(x^1\right) = (e)^{-1} \cdot \left\{E - m \cdot \left[V\left(x^1\right)\right]^{-1}\right\}, \tag{8.44ii}$$

$$\begin{aligned}\mathrm{d}s^2 = &\left[V\left(x^1\right)\right]^2 \cdot \left\{\left(\mathrm{d}x^1\right)^2 + \left(x^1\right)^2 \cdot \left[\left(\mathrm{d}x^2\right)^2 + \left(\sin x^2\right)^2 \cdot \left(\mathrm{d}x^3\right)^2\right]\right\} \\ &- \left[V\left(x^1\right)\right]^{-2} \cdot \left(\mathrm{d}x^4\right)^2,\end{aligned} \tag{8.44iii}$$

$$\mathbf{D} := \left\{\left(x^1, x^2, x^3\right) \in \mathbb{R}^3 : 0 < r_0 < x^1,\ 0 < x^2 < \pi,\ -\pi < x^3 < \pi\right\}, \tag{8.44iv}$$

$$\partial_1\partial_1 V + \left(2/x^1\right) \cdot \partial_1 V + 2(e)^2 \cdot (\chi_0)^2 \cdot \left[V\left(x^1\right)\right]^3 = 0, \tag{8.44v}$$

$$(e/m) = \pm\sqrt{\kappa/2}, \tag{8.44vi}$$

$$\overset{\circ}{g}_{\alpha\beta}(\mathbf{x})\,\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta} = (\mathrm{d}x^1)^2 + (x^1)^2 \cdot \left[(\mathrm{d}x^2)^2 + \sin^2(x^2) \cdot (\mathrm{d}x^3)^2\right]. \tag{8.44vii}$$

The second order, semilinear, ordinary differential equation (8.44v) is known as one of the *Lane-Emden equations*. (See [275].) By making the transformation $W(x^1) := x^1 \cdot V(x^1)$, we can analytically reduce (8.44v) to a simpler equation:

$$\partial_1\partial_1 W + \lambda_0 \cdot \frac{W}{(x^1)^2} = 0,$$

known as a modified *Emden equation* with $\lambda_0 := 2(e)^2 \cdot (\chi_0)^2$. We analyze the function W instead of V as it more closely resembles functions later in this chapter. The function $W(x^1)$, for various boundary conditions, is plotted in Fig. 8.2. The

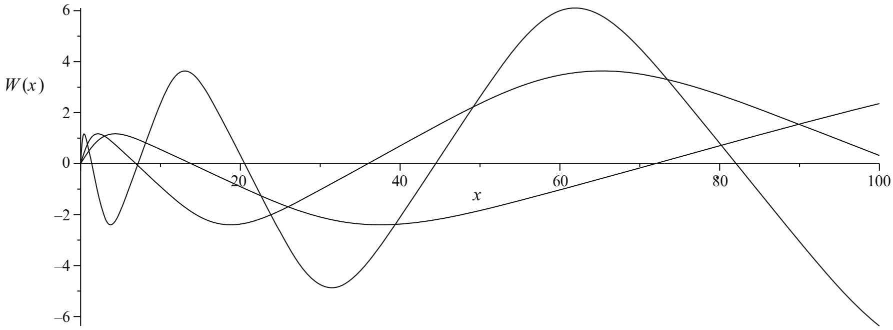

Fig. 8.2 A plot of the function $W(x^1) = x^1 \cdot V(x^1)$ subject to the boundary conditions $W(0) = 0$, $\partial_1 W(x^1)_{|x^1=0} = 0.5,\ 1,$ and 5 representing increasing frequency respectively. The constant $\lambda_{(0)}$ is set to unity

corresponding spatial universe, governed by the metric in (8.44iii,iv), is *finite and closed* [49].

Now, we shall examine more general spherical symmetric cases. First, consider the Schwarzschild universe in the isotropic coordinate chart. By Problem # 1(i) of Exercises 3.1 and the corresponding answer, it is furnished by

$$(\mathrm{d}s_0)^2 = \left[1 + \frac{k}{x^1}\right]^4 \cdot \left\{\left(\mathrm{d}x^1\right)^2 + \left(x^1\right)^2 \cdot \left[\left(\mathrm{d}x^2\right)^2 + \left(\sin x^2\right)^2 \cdot \left(\mathrm{d}x^3\right)^2\right]\right\} - \left[\frac{1 - \left(k/x^1\right)}{1 + \left(k/x^1\right)}\right]^2 \cdot \left(\mathrm{d}x^4\right)^2, \tag{8.45i}$$

$$\mathbf{D} := \left\{\left(x^1, x^2, x^3\right) \in \mathbb{R}^3 : 0 < k < x^1,\ 0 < x^2 < \pi,\ -\pi < x^3 < \pi\right\}. \tag{8.45ii}$$

We assume the complex wave field is spherically symmetric and time-harmonic:

$$\Psi(\cdot\cdot) = \chi\left(x^1\right) \cdot \mathrm{e}^{-\mathrm{i}Ex^4}, \qquad \overline{\Psi}(\cdot\cdot) = \chi\left(x^1\right) \cdot \mathrm{e}^{\mathrm{i}Ex^4}. \tag{8.46}$$

Next, (8.35) as well as Theorem 8.2.2 are applied to determine the function $\chi\left(x^1\right)$ in (8.46) as

$$\mathrm{e}^{2\sqrt{\kappa}\cdot\chi} = \left[\frac{1 - (k/x^1)}{1 + (k/x^1)}\right]^2,$$
$$\chi\left(x^1\right) = \left(1/\sqrt{\kappa}\right) \cdot \left\{\ln\left[1 - \left(k/x^1\right)\right] - \ln\left[1 + \left(k/x^1\right)\right]\right\}. \tag{8.47}$$

The spherically symmetric metrics from (8.45i), (8.27i), and (8.35) and the Theorem 8.2.2 can be summarized as

$$(\mathrm{d}s_0)^2 = \left[\frac{1 + \left(k/x^1\right)}{1 - (k/x^1)}\right]^2 \cdot \left[1 - \left(k/x^1\right)^2\right]^2 \cdot \left\{\left(\mathrm{d}x^1\right)^2 + \left(x^1\right)^2 \right.$$
$$\left. \times \left[\left(\mathrm{d}x^2\right)^2 + \left(\sin x^2\right)^2 \cdot \left(\mathrm{d}x^3\right)^2\right]\right\} - \left[\frac{1 - \left(k/x^1\right)}{1 + (k/x^1)}\right]^2 \cdot \left(\mathrm{d}x^4\right)^2. \quad (8.48\text{i})$$

$$\overset{\circ}{g}_{\alpha\beta}(\cdot\cdot) \cdot (\mathrm{d}x^\alpha) \cdot \left(\mathrm{d}x^\beta\right) = \left[1 - \left(k/x^1\right)^2\right]^2 \cdot \left\{\left(\mathrm{d}x^1\right)^2 + \left(x^1\right)^2 \right.$$
$$\left. \times \left[\left(\mathrm{d}x^2\right)^2 + \left(\sin x^2\right)^2 \cdot \left(\mathrm{d}x^3\right)^2\right]\right\}, \quad (8.48\text{ii})$$

$$(\mathrm{d}s)^2 = \left[V\left(x^1\right)\right]^2 \cdot \left[1 - \left(k/x^1\right)^2\right]^2 \cdot \left\{\left(\mathrm{d}x^1\right)^2 + \left(x^1\right)^2 \right.$$
$$\left. \times \left[\left(\mathrm{d}x^2\right)^2 + \left(\sin x^2\right)^2 \cdot \left(\mathrm{d}x^3\right)^2\right]\right\} - \left[V\left(x^1\right)\right]^{-2} \cdot \left(\mathrm{d}x^4\right)^2. \quad (8.48\text{iii})$$

The function $V\left(x^1\right)$, by (8.37) and (8.48ii), satisfies the second order, semilinear, ordinary differential equation:

$$\partial_1\partial_1 V + \left(\frac{2}{x^1}\right) \cdot \left[1 + \frac{(k)^2}{\left(x^1\right)^2 - (k)^2}\right] \cdot \partial_1 V$$
$$+ \left(2e^2/\kappa\right) \cdot \left\{\left[1 - \left(k/x^1\right)^2\right]^2 \cdot \ln\left[\frac{x^1 - k}{x^1 + k}\right]\right\}^2 \cdot \left[V\left(x^1\right)\right]^3 = 0. \quad (8.49)$$

Now, we make the following coordinate transformation[3]:

$$x = \ln\left[\left(x^1 + k\right) / \left(x^1 - k\right)\right], \quad x^1 > k > 0, \quad (8.50\text{i})$$

$$x^1 = k \cdot \coth\left(x/2\right), \quad x > 0, \quad (8.50\text{ii})$$

$$\widehat{V}(x) := V\left(x^1\right), \quad (8.50\text{iii})$$

$$\frac{\mathrm{d}V\left(x^1\right)}{\mathrm{d}x^1} = -(2/k) \cdot \left[\sinh\left(x/2\right)\right]^2 \cdot \frac{\mathrm{d}\widehat{V}(x)}{\mathrm{d}x}. \quad (8.50\text{iv})$$

[3] In the sequel $x \in \mathbb{R}$ and $x \notin \mathbb{R}^4$.

With the help of (8.50i–iv), the ordinary differential equation (8.49) reduces to

$$(2/k)^2 \cdot \frac{\mathrm{d}^2\widehat{V}(x)}{\mathrm{d}x^2} + \left(2e^2/\kappa\right) \cdot (2)^4 \cdot (x)^2 \cdot (\operatorname{cosech} x)^4 \cdot \left[\widehat{V}(x)\right]^3 = 0. \tag{8.51}$$

Defining

$$\begin{aligned} U(x) &:= (2k) \cdot \widehat{V}(x), \\ U'(x) &:= \frac{\mathrm{d}U(x)}{\mathrm{d}x} = (2k) \cdot \frac{\mathrm{d}\widehat{V}(x)}{\mathrm{d}x}, \end{aligned} \tag{8.52}$$

the ordinary differential equation (8.51) reduces to

$$\begin{aligned} &U''(x) + \left(2e^2/\kappa\right) \cdot \left[x \cdot (\operatorname{cosech} x)^2\right]^2 \cdot [U(x)]^3 = 0, \\ \text{or,} \quad &U''(x) + \left(\tfrac{e^2}{4\pi}\right) \cdot \left[x \cdot (\operatorname{cosech} x)^2\right]^2 \cdot [U(x)]^3 = 0. \end{aligned} \tag{8.53}$$

The corresponding metric of (8.48ii), associated electrostatic potential, and the wave function of (8.14i) go over into[4]

$$\begin{aligned} (\mathrm{d}s^2) = \; &[U(x)]^2 \cdot \Big\{(\operatorname{cosech} x)^4 \cdot (\mathrm{d}x)^2 + (\operatorname{cosech} x)^2 \\ &\times \left[\left(\mathrm{d}x^2\right)^2 + \left(\sin x^2\right)^2 \cdot \left(\mathrm{d}x^3\right)^2\right]\Big\} - [U(x)]^{-2} \cdot \left(2k \cdot \mathrm{d}x^4\right)^2 \\ =: \; &[U(x)]^2 \cdot \Big\{(\operatorname{cosech} x)^4 \cdot (\mathrm{d}x)^2 + (\operatorname{cosech} x)^2 \\ &\times \left[(\mathrm{d}\theta)^2 + (\sin\theta)^2 \cdot (\mathrm{d}\varphi)^2\right]\Big\} - [U(x)]^{-2} \cdot (\mathrm{d}t)^2, \end{aligned} \tag{8.54i}$$

$$\widehat{\mathcal{A}}(x) = (e)^{-1} \cdot \left\{E - m \cdot [U(x)]^{-1}\right\}, \tag{8.54ii}$$

$$\widehat{\Psi}(\cdot\cdot) = \left(1/\sqrt{\kappa}\right) \cdot x \cdot \mathrm{e}^{-\mathrm{i}Ex^4}, \tag{8.54iii}$$

$$\overline{\widehat{\Psi}}(\cdot\cdot) = \left(1/\sqrt{\kappa}\right) \cdot x \cdot \mathrm{e}^{\mathrm{i}Ex^4}. \tag{8.54iv}$$

In quantum mechanics, the wave function representing a localizable particle is *square-integrable*. In general relativity, the appropriate square-integrability *emerges from the existence of the invariant integral representing the total charge* of a material source. It is furnished by (8.2iii) and (8.54iii,iv) as

[4]The complete coordinate transformation is provided by (8.50i) together with $(\widehat{x}^2, \widehat{x}^3) = (\widehat{\theta}, \widehat{\phi}) = (\theta, \phi)$, $\widehat{x}^4 = 2k \cdot x^4 = t$. (However, we have dropped hats on the coordinates subsequently.)

$$
\begin{aligned}
Q &:= -\int\limits_{\mathbf{D}} J_4\left(\mathbf{x}, x^4\right) \cdot n^4\left(\mathbf{x}, x^4\right) \cdot \sqrt{g^{\#}} \cdot \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3 \\
&= -ie \cdot \int\limits_{\mathbf{D}} \left[\overline{\Psi}(\cdot\cdot) \cdot D_4\Psi - \Psi(\cdot\cdot) \cdot \overline{D}_4\overline{\Psi}\right] \cdot n^4(\cdot\cdot) \cdot \sqrt{g^{\#}} \cdot \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3,
\end{aligned}
\tag{8.55}
$$

$$
\text{where,}\quad g^{\#}_{\alpha\beta}(\cdot\cdot)\mathrm{d}x^\alpha \mathrm{d}x^\beta = [U(\cdot\cdot)]^2 \cdot \Big\{[\operatorname{cosech}(x)]^4 \cdot (\mathrm{d}x)^2 + [\operatorname{cosech}(x)]^2 \left[(\mathrm{d}x^2)^2 + (\sin(x^2))^2 \cdot (\mathrm{d}x^3)^2\right]\Big\}.
$$

(Here, $n^4(\mathbf{x},\ x^4) = |U(\mathbf{x})|$.)

Let us calculate the right-hand side of the above equation, utilizing (8.54i–iv) and (8.17iii), we obtain

$$
\begin{aligned}
Q &= (ie) \cdot (2i) \cdot \int\limits_{\mathbf{D}} |\Psi(\cdot\cdot)|^2 \cdot [E - e\,\mathcal{A}(\cdot\cdot)] \cdot [U(\cdot\cdot)]^4 \cdot \sqrt{\overset{\circ}{g}} \cdot \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3 \\
&= -2(e \cdot m) \cdot \int\limits_{\mathbf{D}} |\Psi(\cdot\cdot)|^2 \cdot [U(\cdot\cdot)]^3 \cdot \sqrt{\overset{\circ}{g}} \cdot \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3.
\end{aligned}
$$

Using (8.54i) and (8.54iii), we deduce that

$$
Q = -2 \cdot (e \cdot m/\kappa) \cdot (4\pi) \cdot \int\limits_{0_+}^{\infty} (x)^2 \cdot [U(\cdot\cdot)]^3 \cdot (\operatorname{cosech} x)^4 \cdot \mathrm{d}x,
$$

for $\quad U(\cdot\cdot) > 0.$

Now we use the differential equation (8.53) to derive that

$$
\begin{aligned}
Q &= \left[(8\pi) \cdot (e \cdot m)/\left(2e^2\right)\right] \cdot \int\limits_{0_+}^{\infty} U''(x)\, \mathrm{d}x \\
&= \frac{4\pi m}{e} \left\{\lim_{L\to\infty} \left[U'(L)\right] - U'(0_+)\right\}.
\end{aligned}
$$

Due to physical considerations, we set the total charge as[5]

---

[5]We can set $Q =: e$ for the case $U(\cdot\cdot) < 0$.

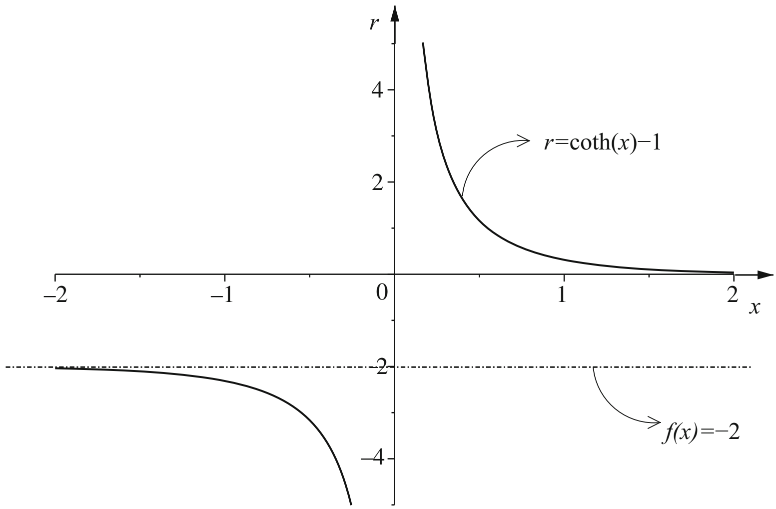

Fig. 8.3 The graph of the function $r = \coth x - 1 > 0$

$$Q =: -e = \frac{4\pi m}{e}\left\{\lim_{L\to\infty}\left[U'(L)\right] - U'(0_+)\right\}. \tag{8.56}$$

The equation above is a consequence of the square-integrability of the wave function in the E–M–K–G equations.

Now, we make another coordinate transformation in the following:

$$r = \coth x - 1, \qquad x > 0, \tag{8.57i}$$

$$\frac{dr}{dx} = -(\operatorname{cosech} x)^2 < 0, \tag{8.57ii}$$

$$x = (1/2)\cdot \ln|(r+2)/r|, \qquad r > 0, \tag{8.57iii}$$

$$(\operatorname{cosech} x)^2 = r\cdot(r+2). \tag{8.57iv}$$

(See Fig. 8.3 depicting the function $r = \coth x - 1$.)

The metric (8.54i) and the ordinary differential equation (8.53) go over into

$$u(r) := U(x), \tag{8.58i}$$

$$(ds)^2 = [u(r)]^2\cdot\left\{(dr)^2 + r\cdot(r+2)\cdot\left[(d\theta)^2 + (\sin\theta)^2\cdot(d\varphi)^2\right]\right\} - [u(r)]^{-2}\cdot(dt)^2, \tag{8.58ii}$$

$$\overset{\circ}{g}_{\alpha\beta}(\cdot\cdot)\cdot(dx^\alpha)\cdot(dx^\beta) = (dr^2) + r\cdot(r+2)\cdot\left[(d\theta)^2 + (\sin\theta)^2\cdot d\varphi^2\right], \tag{8.58iii}$$

$$u''(r) + 2(r+1) \cdot [r \cdot (r+2)]^{-1} \cdot u'(r) + e^2/(4\pi)$$
$$\times \left[(1/2) \cdot \ln|(r+2)/r|\right]^2 \cdot [u(r)]^3 = 0. \tag{8.58iv}$$

The three-dimensional metric $\overset{\circ}{g}_{\alpha\beta}(\cdot\cdot) \cdot (\mathrm{d}x^\alpha) \cdot (\mathrm{d}x^\beta)$ inherent in (8.58iii) can be expressed as[6]

$$\overset{\circ}{g}_{\alpha\beta}(\cdot\cdot) \cdot (\mathrm{d}x^\alpha) \cdot (\mathrm{d}x^\beta) = (\mathrm{d}r)^2 + r^2 \cdot [1 + (2/r)] \cdot \left[(\mathrm{d}\theta)^2 + (\sin\theta)^2 \cdot (\mathrm{d}\varphi)^2\right]. \tag{8.59}$$

In the limit $r \to \infty$, the metric above asymptotically approaches a spherical polar chart for *the* (*flat*) *Euclidean geometry*. Therefore, in the limit $r \to \infty$, the four-dimensional metric in (8.58ii) asymptotically approaches *a conformastat electrovac, or, charged-dust metric* of the W–M–P–D class. (See (4.71) and (4.97).) Thus, the modification of the conformastat metric in (8.58ii) must be due to the presence of the localizable complex wave field. Using the coordinate transformation in (8.57i–iv), and assuming the $\lim\limits_{r\to 0+} u'(r)$ exists, we conclude that

$$U'(x) = -r \cdot (r+2) \cdot u'(r), \tag{8.60i}$$

$$\lim_{L\to\infty} U'(L)) = -\lim_{r\to 0_+} \left[r \cdot (r+2) \cdot u'(r)\right] = 0. \tag{8.60ii}$$

Substituting (8.60ii) into (8.56), we determine that the total charge is given by

$$Q = -e = -\frac{4\pi m}{e} U'(0_+),$$

or

$$U'(0_+) = \frac{e^2}{4\pi m} = m = \pm \frac{e}{\sqrt{4\pi}}. \tag{8.61}$$

(We choose the positive sign in the sequel.)

The coordinate chart discussed in (8.57i–iv) makes it simpler to analyze the physical properties of the solutions. However, the corresponding ordinary differential equation (8.58iv) is mathematically more complicated and difficult to explore rigorously. Therefore, for mathematical analysis, we revert back to the coordinate chart in (8.53) and (8.54i). To investigate the second order, ordinary differential equation (8.53), we need *another initial condition* besides the one in (8.61). We assume $U \in C^2([0,\infty) \subset \mathbb{R}; \mathbb{R})$, so that $\lim\limits_{x\to 0_+} U''(x)$ exists. Therefore, by the differential equation (8.53), and L'Hospital's rule, we must have

---

[6]Note that (8.59) is *different* from the corresponding equation in (8.54i).

$$\left| \lim_{x \to 0_+} \left\{ \left[x \cdot (\operatorname{cosech} x)^2 \right]^2 \cdot [U(x)]^3 \right\} \right| = \text{ a positive constant,} \tag{8.62i}$$

or,
$$\lim_{x \to 0_+} U(x) = U(0_+) = 0. \tag{8.62ii}$$

Now we are in a position to pose the initial-value problem for the ordinary differential equation (8.53) as the following:

$$U \in C^2 ([0, \infty) \subset \mathbb{R}; \mathbb{R}) \quad \text{ and } \quad U(x) > 0, \tag{8.63i}$$

$$U''(x) + (e^2/4\pi) \cdot \left[x \cdot (\operatorname{cosech} x)^2 \right]^2 \cdot [U(x)]^3 = 0, \tag{8.63ii}$$

$$U(0) = 0, \tag{8.63iii}$$

$$U'(0) = m = e/\sqrt{4\pi}. \tag{8.63iv}$$

Remark: We note that for every solution of (8.63i–iv), there exists another solution $\widehat{U}(x) := -U(x) < 0$, with $\widehat{U}(0) = 0$, and $\widehat{U}'(0) = -m = -\left(e/\sqrt{4\pi} \right)$.

We also notice that the function defined by

$$y(x) := \sqrt{1/4\pi} \cdot e \cdot U(x), \tag{8.64i}$$

satisfies the differential equation

$$y''(x) + \left[x \cdot (\operatorname{cosech} x)^2 \right]^2 \cdot [y(x)]^3 = 0. \tag{8.64ii}$$

Next we shall investigate the initial value problem for the ordinary differential equation (8.64ii) with the notation

$$p(x) := \left[x \cdot (\operatorname{cosech} x)^2 \right]^2, \tag{8.65i}$$

$$y''(x) + p(x) \cdot [y(x)]^3 = 0. \tag{8.65ii}$$

A pertinent theorem will be stated before continuing.

Theorem 8.3.1. *Let the ordinary differential equation* (8.65ii) *hold in the interval* $0 < x < \infty$. *Moreover, let the initial conditions be given by*

$$y(0) = 0 \quad \textit{and} \quad y'(0) = a = \textit{ constant.} \tag{8.66}$$

Then, there exists a unique solution of the differential equation (8.65ii) *with the initial conditions* (8.66).

For the proof, the choice of $a \geq 0$ suffices. By the use of the Banach space $C^0([0, x_0) \subset \mathbb{R}; \mathbb{R})$ with the norm defined to be $\| \vec{\mathbf{u}}(x) \| := \max_{0 \leq x \leq x_0} |u(x)|$, the preceding theorem was proved in [60]. We shall now state a corollary.

Corollary 8.3.2. *Let the constant be chosen as* $a \geq 0$. *Then, for* $0 \leq x < \infty$, *the unique solution* $y(x; a)$ *of the initial-value problem* (8.65ii) *and* (8.66) *must satisfy the weak inequality:*

$$|y(x; a)| \leq |a| \cdot x = a \cdot x. \tag{8.67}$$

Proof of the corollary above was also given in [60].

Now, we are in a position to pose a *nonlinear eigenvalue problem for a theoretical fine-structure constant*. (See [60].) We call this quantity a theoretical fine-structure constant or a fine-structure parameter due to the exact similarity of the constant of the same name in electrodynamics. Here, we are including electrostatic and gravitational self-interactions. We shall explore the possible, theoretical values that this analogous parameter must possess in static, spherically symmetric, localized systems.

Before proceding, we have to add *a third condition* for $U(x)$, in addition to the two initial values in (8.63iii) and (8.63iv). We add as the third condition the equation (8.60ii), yielding $\lim_{L \to \infty} U'(L) = 0$. Thus, we summarize *the nonlinear eigenvalue problem for a theoretical fine-structure constant* as:

$$U \in C^2([0, \infty) \subset \mathbb{R}; \mathbb{R}) \quad \text{and} \quad U(x) > 0, \tag{8.68i}$$

$$U''(x) + (e^2/4\pi) \cdot \left[x \cdot (\operatorname{cosech} x)^2\right]^2 \cdot [U(x)]^3 = 0, \tag{8.68ii}$$

$$U(0) = 0, \tag{8.68iii}$$

$$U'(0) = m = e/\sqrt{4\pi}, \tag{8.68iv}$$

$$\lim_{L \to \infty} U'(L) = 0. \tag{8.68v}$$

The eigenvalue $(e^2/4\pi)$ is what we refer to as *the fine-structure parameter*. We are seeking here *the theoretical values* of this parameter from the exact solutions of (8.68i–v). (For those who have studied electrodynamics, *the experimental value* of the *electrodynamic* fine-structure constant is known to be $(e^2/4\pi) = 1/137$.)

In terms of the functions $y(x)$ and $p(x)$, defined in (8.64i) and (8.65i), respectively, (8.68i–v) go over into:

$$y \in C^2([0, \infty) \subset \mathbb{R}; \mathbb{R}) \quad \text{and} \quad y(x) > 0, \tag{8.69i}$$

$$y''(x) + p(x) \cdot [y(x)]^3 = 0, \tag{8.69ii}$$

$$y(0) = 0, \tag{8.69iii}$$

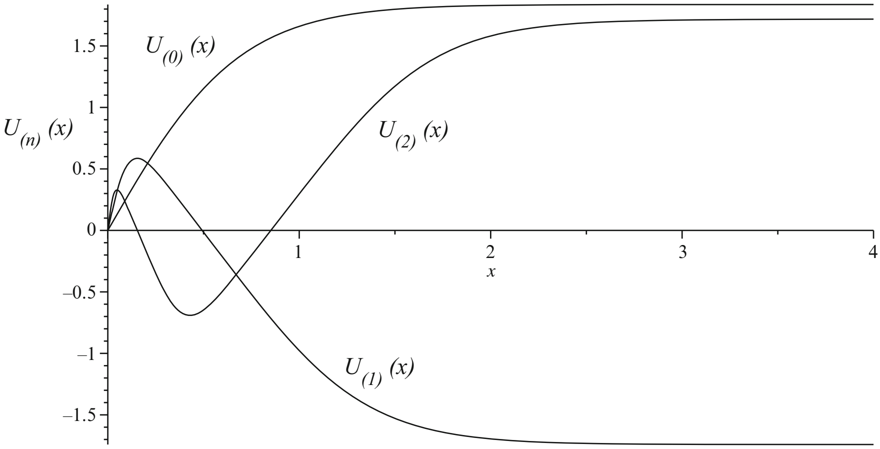

Fig. 8.4 Graphs of the eigenfunctions $U_{(0)}(x)$, $U_{(1)}(x)$ and $U_{(2)}(x)$

$$y'(0) = e \cdot m/\sqrt{4\pi} = e^2/(4\pi), \tag{8.69iv}$$

$$\lim_{L\to\infty} y'(L) = 0. \tag{8.69v}$$

Now, we shall state *the existence theorem* on the nonlinear eigenvalues $e_{(n)}$ (for the parameter e) and the corresponding eigenfunctions $U_{(n)}(x)$ for (8.68i–v).

Theorem 8.3.3. *Consider the nonlinear eigenvalue problem of* (8.68i–v). *For each nonnegative integer* $n \geq 0$, *there exists a constant* $e = e_{(n)}$ *and a corresponding eigenfunction* $U(x) = U_{(n)}(x)$ *which has exactly* n *zeros in* $(0, \infty) \subset \mathbb{R}$ *and satisfies* (8.68i–v).

(For the proof, see [60].) We depict the first three eigenfunctions $U_{(0)}(x)$, $U_{(1)}(x)$, and $U_{(2)}(x)$ in Fig. 8.4.

Now we shall describe very briefly the numerical method to compute the nonlinear eigenvalue $e = e_{(0)}$ and the corresponding eigenfunction $U_{(0)}(x)$. We consider (8.69i–v) in the interval $[0, \infty) \subset \mathbb{R}$. By Theorem 8.3.3, there exists a unique solution $y_{(\infty)}(x)$ of the problem such that

$$y_{(\infty)}(0) = 0 = \lim_{L\to\infty} y'_{(\infty)}(L). \tag{8.70}$$

This solution must also satisfy the following differential equation, *the integro-differential equation, and the integral equation* respectively:

$$y''_{(\infty)}(x) = -p(x) \cdot \left[y_{(\infty)}(x)\right]^3, \tag{8.71i}$$

$$y'_{(\infty)}(x) = \int_x^\infty [p(t)] \cdot \left[y_{(\infty)}(t)\right]^3 \cdot \mathrm{d}t, \tag{8.71ii}$$

$$y_{(\infty)}(x) = \int_0^x t \cdot [p(t)] \cdot \left[y_{(\infty)}(t)\right]^3 \cdot \mathrm{d}t + x \cdot \int_x^\infty [p(t)] \cdot \left[y_{(\infty)}(t)\right]^3 \cdot \mathrm{d}t. \quad (8.71\text{iii})$$

Putting

$$y_{(\infty)}(x) =: x \cdot \beta(x), \quad (8.72)$$

we find that $\beta(x)$ satisfies the integral equation

$$\beta(x) = (x)^{-1} \cdot \int_0^x (t)^4 \cdot [p(t)] \cdot [\beta(t)]^3 \cdot \mathrm{d}t + \int_x^\infty (t)^3 \cdot [p(t)] \cdot [\beta(t)]^3 \cdot \mathrm{d}t. \quad (8.73)$$

Now, consider *the alternate nonlinear eigenvalue problem* (see [94]) of the integral equation:

$$\lambda \cdot v(x) := (x)^{-1} \cdot \int_0^x (t)^4 \cdot [p(t)] \cdot [v(t)]^3 \cdot \mathrm{d}t + \int_x^\infty (t)^3 \cdot [p(t)] \cdot [v(t)]^3 \cdot \mathrm{d}t, \quad (8.74\text{i})$$

and $$\int_0^\infty (t)^4 \cdot [p(t)] \cdot [v(t)]^3 \cdot \mathrm{d}t = \lambda. \quad (8.74\text{ii})$$

This problem can be solved by *successive approximation* as follows. Beginning with a function $v_{(0)}(x) = (x)^{-1} + 0(x^{-2})$ as $x \to \infty$, functions $v_{(j)}(x)$ with same properties are defined recursively as

$$\lambda_{(j+1)} \cdot v_{(j+1)}(x) := (x)^{-1} \cdot \int_0^x (t)^4 \cdot [p(t)] \cdot \left[v_{(j)}(t)\right]^3 \cdot \mathrm{d}t + \int_x^\infty (t)^3 \cdot [p(t)] \cdot \left[v_{(j)}(t)\right]^3 \cdot \mathrm{d}t, \quad (8.75\text{i})$$

with $$\lambda_{(j+1)} := \int_0^\infty (t)^4 \cdot [p(t)] \cdot \left[v_{(j)}(t)\right]^3 \cdot \mathrm{d}t. \quad (8.75\text{ii})$$

This procedure, when carried out numerically on a computer, proved to be convergent on the pair $(\lambda, v(x))$ of the nonlinear eigenvalue problem of (8.74i,ii).

Recall from (8.69i–v) that

$$y''_{(0)}(x) = -[p(x)] \cdot \left[y_{(0)}(x)\right]^3, \tag{8.76i}$$

$$y_{(0)}(0) = 0, \tag{8.76ii}$$

$$\lim_{L\to\infty} y'_0(L) = 0, \tag{8.76iii}$$

$$y'_{(0)}(0) = \left[e_{(0)}\right]^2 / 4\pi. \tag{8.76iv}$$

Now, we define the solution as

$$y_{(0)}(x) := y_{(\infty)}(x) = x \cdot \beta(x) = \lambda \cdot [x \cdot v(x)]\,. \tag{8.77}$$

With help of (8.70), (8.71i–iv), (8.73), and (8.76i–iv), the equation (8.77) follows. Thus, from (8.64i) and (8.77) we deduce that

$$U_{(0)}(x) = \sqrt{4\pi} \cdot \left(e_{(0)}\right)^{-1} \cdot y_{(0)}(x) = \sqrt{4\pi} \cdot (e_0)^{-1} \cdot x \cdot \beta(x). \tag{8.78}$$

The graph of the eigenfunction $U_{(0)}(x)$ was already plotted in the Fig. 8.4. *The corresponding theoretical eigenvalue of the fine-structure parameter and the charge*, from numerical analysis via computer, turns out to be

$$\left(e_{(0)}\right)^2 / 4\pi \approx 7.28, \tag{8.79i}$$

$$e_{(0)}/\sqrt{4\pi} \approx 2.70. \tag{8.79ii}$$

$U_{(2)}(x)$ was also plotted in Fig. 8.4. This function has two isolated zeros in the open interval $(0, \infty) \subset \mathbb{R}$. It is nonnegative from the origin to the first zero at $r = r_{(1)}$. It takes negative values in the open interval $\left(r_{(1)}, r_{(2)}\right)$ between the first and the second zero. Then, the function takes positive values, approaching asymptotically to a horizontal straight line.

The eigenfunction $U_{(j)}(x)$ has j zeros at $r = r_{(1)}, \ldots, r_{(j)}$ in the open interval $(0, \infty) \subset \mathbb{R}$. After the jth zero, the function $U_{(j)}(x)$ approaches asymptotically to a horizontal straight line.

Now, we shall explore geometrical and physical properties of the space–time universe associated with the nonlinear eigenvalue $e_{(0)}$ and the corresponding eigenfunction $U_{(0)}(x)$. We shall use the coordinate chart of (8.58i,ii). Thus, we express

$$u_{(0)}(r) := U_{(0)}(x), \tag{8.80i}$$

$$\begin{aligned}(\mathrm{d}s)^2 = {} & \left[u_{(0)}(r)\right]^2 \cdot \left\{(\mathrm{d}r)^2 + r \cdot (r+2) \cdot \left[(\mathrm{d}\theta)^2 + (\sin\theta)^2 \cdot (\mathrm{d}\varphi)^2\right]\right\} \\ & - \left[u_{(0)}(r)\right]^{-2} \cdot (\mathrm{d}t)^2. \end{aligned} \tag{8.80ii}$$

The $t = $ constant hypersurface representing the spatial universe from (8.80ii) is provided by the metric

$$
\begin{aligned}
(\mathrm{d}l)^2 &:= \left[u_{(0)}(r)\right]^2 \cdot \left\{(\mathrm{d}r)^2 + r\cdot(r+2)\cdot\left[(\mathrm{d}\theta)^2 + (\sin\theta)^2\cdot(\mathrm{d}\varphi)^2\right]\right\} \\
&=: g^{\#}_{\alpha\beta}(\cdot\cdot)\cdot(\mathrm{d}x^{\alpha})\cdot\left(\mathrm{d}x^{\beta}\right),
\end{aligned} \tag{8.81i}
$$

$$
\sqrt{g^{\#}} = \left[u_{(0)}(r)\right]^3 \cdot r\cdot(r+2)\cdot\sin\theta > 0, \tag{8.81ii}
$$

$$
\int_{\mathbf{D}} \sqrt{g^{\#}}\cdot \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 = (4\pi)\cdot\left\{\lim_{L\to\infty}\int_{0_+}^{L}[u_0(r)]^3\cdot r\cdot(r+2)\cdot\mathrm{d}r\right\} \longrightarrow \infty. \tag{8.81iii}
$$

The radial distance is defined by

$$
\mathsf{R}(r) := \int_{0_+}^{r}\left[u_{(0)}(w)\right]\cdot\mathrm{d}w, \tag{8.82i}
$$

$$
\lim_{r\to\infty}\mathsf{R}(r) \longrightarrow \infty. \tag{8.82ii}
$$

The ratio of the circumference divided by the radial distance is given by

$$
\text{Ratio} := \frac{(2\pi)\cdot u_{(0)}(r)\cdot[r\cdot(r+2)]^{1/2}}{\mathsf{R}(r)}. \tag{8.83}
$$

The ratio above starts with "infinite slope," decaying monotonically to zero. (Compare and contrast with the Schwarzschild case in the Example 3.1.2.)

The area of a 2-sphere at the radial distance $\mathsf{R}(r)$ is furnished from (8.81i) as:

$$
\text{Spherical area} := (4\pi)\cdot\left[u_{(0)}(r)\right]^2\cdot r\cdot(r+2). \tag{8.84}
$$

The spherical area above starts from zero increases asymptotically to *a (positive) constant*.

Now, we investigate the properties of *the four-dimensional universe* with the metric (8.80ii). We deduce that

$$
\sqrt{-g} = \left[u_{(0)}(r)\right]^2\cdot[r\cdot(r+2)]\cdot\sin\theta > 0, \tag{8.85i}
$$

$$
\begin{aligned}
&\int_{\mathbf{D}}\sqrt{-g}\cdot\mathrm{d}x^1\cdot\mathrm{d}x^2\cdot\mathrm{d}x^3\cdot\mathrm{d}x^4 \\
&\quad= \lim_{L\to\infty}\cdot\lim_{t\to\infty}\left\{(4\pi)\cdot\int_{0_+}^{L}\int_{t_0}^{t}\left[u_{(0)}(r)\right]^2\cdot r\cdot(r+2)\cdot\mathrm{d}r\cdot\mathrm{d}t\right\} \longrightarrow \infty.
\end{aligned} \tag{8.85ii}
$$

Thus, the total, invariant volume is *unbounded and the universe is open*.

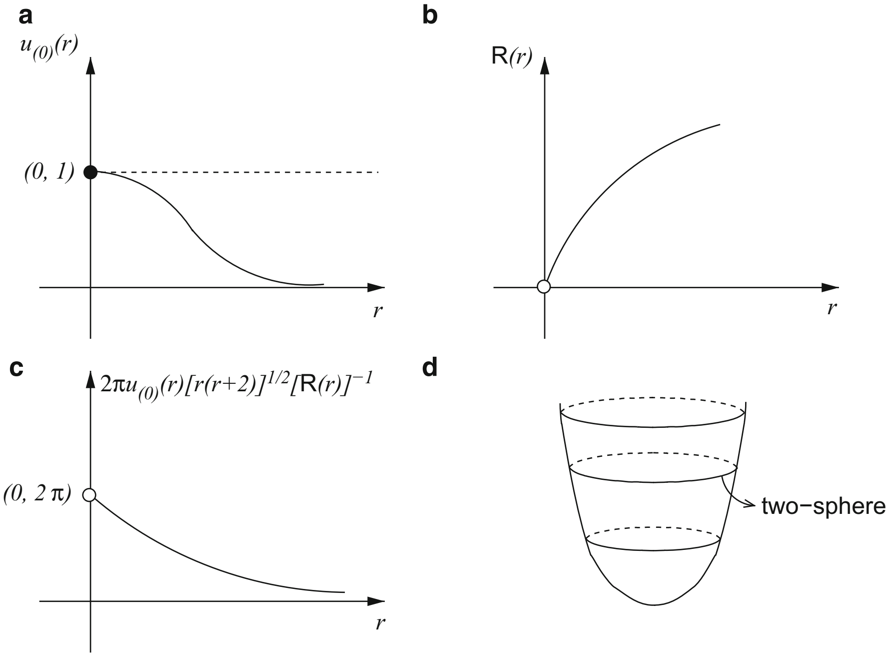

Fig. 8.5 (**a**) Qualitative graph of eigenfunction $u_{(0)}(r)$. (**b**) Qualitative plot of the radial distance $\mathsf{R}(r)$ versus r. (**c**) Qualitative plot of the ratio of circumference divided by radial distance. (**d**) Qualitative two-dimensional projection of the three-dimensional, spherically symmetric geometry

With informations coded in (8.81i), (8.81iii), (8.82ii), (8.83), and (8.84), we interpret qualitatively the spatial geometry associated with the non-linear eigenvalue $e_{(0)}$ and the corresponding eigenfunction $u_{(0)}(r)$. The Fig. 8.5a–d depict various aspects of the spherically symmetric geometry.

Now, let us work out *the radial, null geodesics* from the metric in (8.80ii). These are provided by solutions of the ordinary differential equations:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left[u_{(0)}(r)\right]^{-4} > 0, \tag{8.86i}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left[u_{(0)}(r)\right]^{-2}. \tag{8.86ii}$$

The slope of the outgoing, radial null geodesic increases monotonically toward the "infinite slope."

Now we shall graphically furnish the radial null geodesics of (8.86i,ii) qualitatively in Fig. 8.6.

Now, as a comparison of the properties of the various eigenfunctions, we shall investigate the eigenfunction $U_{(2)}(x)$ also shown in the graph of Fig. 8.4.

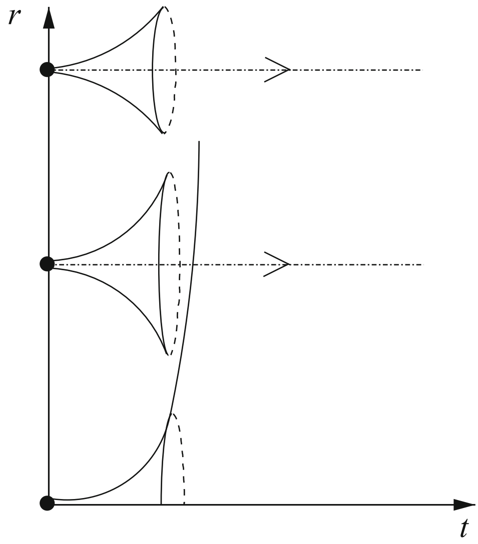

Fig. 8.6 Qualitative plots of three null cones representing radial, null geodesics

(The interested reader can follow the analysis here and above to study the eigenfunction $U_{(1)}(x)$.) We shall examine the corresponding eigenfunction $u_{(2)}(r)$. The graphs analogous to Fig. 8.5a–d (which illustrate properties for $u_{(0)}(r)$) will be provided in Fig. 8.7a–d which depict the spherically symmetric geometry inherited from the eigenfunction $u_{(2)}(r)$.

Remark: The three-dimensional, spherically symmetric geometry depicted in the Fig. 8.7d consists of *three constituent pieces* joint together (like in *a sausage*). The corresponding third *Betti number* is characterized by *two*.

Now, we shall calculate the nonzero, orthonormal components of the curvature tensor for the metric in (8.60ii). These are furnished in the following:

$$R_{(2)(3)(2)(3)}(\cdot\cdot) = -\Big\{\left(u^{-2}\cdot u'\right)^2 + 2\left[(r+1)/(r\cdot(r+2))\right]\cdot u^{-3}\cdot u' + u^{-2}\cdot[r\cdot(r+2)]^{-2}\Big\}, \tag{8.87i}$$

$$R_{(1)(2)(1)(2)}(\cdot\cdot) \equiv R_{(1)(3)(1)(3)}(\cdot\cdot) = -\Big\{u^{-3}\cdot u'' - \left(u^{-2}\cdot u'\right)^2 + [(r+1)/(r\cdot(r+2))]\cdot u^{-3}\cdot u' - [r\cdot(r+2)]^{-2}\cdot u^{-2}\Big\}, \tag{8.87ii}$$

$$R_{(1)(4)(1)(4)}(\cdot\cdot) = -u^{-3}\cdot u'' + 3\cdot\left(u^{-2}\cdot u'\right)^2, \tag{8.87iii}$$

$$R_{(2)(4)(2)(4)}(\cdot\cdot) \equiv R_{(3)(4)(3)(4)}(\cdot\cdot) = -\Big\{\left(u^{-2}\cdot u'\right)^2 + [(r+1)/(r\cdot(r+2))]\cdot u^{-3}\cdot u'\Big\}. \tag{8.87iv}$$

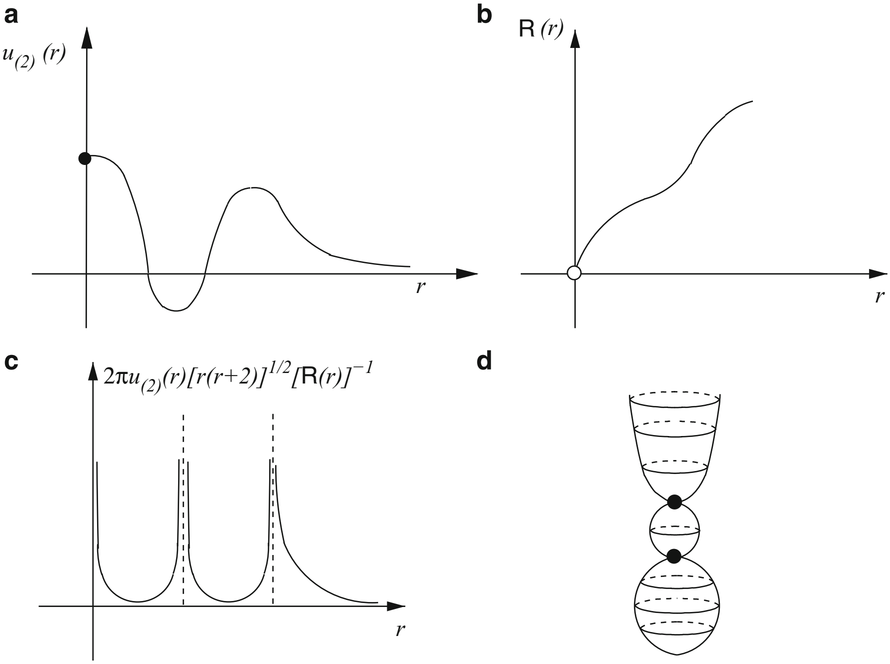

Fig. 8.7 (**a**) Qualitative graph of the eigenfunction $u_{(2)}(r)$. (**b**) Qualitative plot of the radial distance $\mathsf{R}(r) := \int_{0_+}^{r} \left|u_{(2)}(w)\right| \cdot dw$. (**c**) Qualitative plot of the ratio of circumference divided by radial distance. (**d**) Qualitative, two-dimensional projection of the three-dimensional, spherically symmetric geometry

Remarks: (i) For every eigenfunction $u_{(j)}(r)$, $j \in \{0, 1, 2, \ldots\}$, the orthonormal components of the curvature tensor are provided by (8.87i–iv) with modifications from $u(r) \longrightarrow u_{(j)}(r)$.

(ii) For the first eigenfunction $u_{(0)}(r)$, the corresponding orthonormal components of the curvature tensor from (8.87i–iv) are all of class $C^3((0, \infty) \subset \mathbb{R}; \mathbb{R})$. Moreover, in the limiting process, $\lim_{r \to 0_+} \left[R_{(a)(b)(c)(d)}(\cdot\cdot)\right] \equiv 0$.

(iii) For *every other eigenfunction* $u_{(j)}(r)$, the orthonormal components of the curvature tensor are all of class piecewise $C^3_p\left((0, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{j-1}, r_j) \subset \mathbb{R}; \mathbb{R}\right)$, but the components of the curvature tensor in (8.87i–iv) have singularities at the zeroes of $u_{(j)}(r)$. These singularities are analogous to the following: Consider the double cone $(x^1)^2 + (x^2)^2 - (x^3)^2 = 0$ embedded in Euclidean space $\mathbb{E}_3$. (Compare with Figs. 1.13 and 2.2.) This hypersurface (which is actually a topological variety) has one curvature singularity at the vertex where the two cones meet. That is why we have included the generalized differential manifold characterized by Fig. 8.7d.

Now, let us go back to spherically symmetric field equations (8.44i–vi), (8.45i,ii), (8.46), (8.47), (8.48i–iii), (8.49), (8.50i–iv), (8.53), (8.54i–iv), (8.57i–iv), and

(8.58i–iii). These equations involve *only electrogravitational self-interactions and no external fields*. The corresponding eigenfunctions $u_{(j)}(r)$ are depicted in Figs. 8.5a–d and 8.7a–d for the scenarios when $j = 0$ and $j = 2$. We shall now make some remarks about the physical properties of these solutions in the following:

Remarks: (i) The wave-mechanical models of a charged, scalar particle under consideration here are *self-consistent and complete*. The field equations used in this chapter emerge from the coupling of (a) Einstein's gravitational equations, (b) Maxwell's electromagnetic field equations, and (c) the (relativistic) Klein–Gordon wave equation.

(ii) In a spherically symmetric space–time domain, rigorous techniques of solving *nonlinear eigenvalue problems* are employed. The existence of *a denumerably infinite number of* (*theoretical*) *eigenvalues* for the charge parameter e, the "fine-structure parameter" $(e^2/4\pi)$, and the (proper) mass parameter m is established.

(iii) The "equilibrium condition" $e = \pm\sqrt{\kappa/2} \cdot m$ implies that *a spectrum of particles* can exist with masses of the order of the *Planck mass* ($\sqrt{\hbar \cdot c/G} \approx 2.17 \times 10^{-5}$ gm in c.g.s. units.) Thus, these particles are remarkably similar to conjectured particles called *geons* [262]. Interestingly, *the spectrum is discrete*, a result that is qualitatively similar to the result found in elementary particle physics.

(iv) The lowest, theoretical eigenvalue of the positive charge parameter is given by $e_{(0)} = 9.57$. It is worthwhile to compare this value with the experimentally measured magnitude of the elementary charge (the charge of an electron), $|e_{(\mathrm{e})}| = 0.303$. The ratio $\left[e_{(0)}/|e_{(\mathrm{e})}|\right] = 31.59$.

Below we provide Table 8.1 displaying the properties (in the fundamental units specified by $\hbar = c = G = 1$, $\kappa = 8\pi$) of the first five condensates (which may represent gravitationally bound charged "particles") out of the denumerably infinite number. These values are obtained from the graphs of the eigenfunctions $U_{(j)}(x)$ as depicted for a few of them in Fig. 8.4.

Table 8.1 Physical quantities associated with the first five eigenfunctions of $U_{(j)}(x)$.

	Dimensionless charge parameter	Dimensionless mass parameter	Eigenvalue of the fine-structure parameter
Condensate 0	$\lvert e_{(0)}\rvert = 9.57$	$m_{(0)} = 2.70$	$(e_{(0)})^2/4\pi = 7.28$
Condensate 1	$\lvert e_{(1)}\rvert = 25.13$	$m_{(1)} = 7.09$	$(e_{(1)})^2/4\pi = 50.26$
Condensate 2	$\lvert e_{(2)}\rvert = 44.66$	$m_{(2)} = 12.60$	$(e_{(2)})^2/4\pi = 158.76$
Condensate 3	$\lvert e_{(3)}\rvert = 67.53$	$m_{(3)} = 19.05$	$(e_{(3)})^2/4\pi = 362.90$
Condensate 4	$\lvert e_{(4)}\rvert = 93.26$	$m_{(4)} = 26.31$	$(e_{(4)})^2/4\pi = 692.21$

Appendix 1
Variational Derivation of Differential Equations

Let us recall the definition of a *critical point* for a function f belonging to the class $C^2\left((a,b) \subset \mathbb{R}; \mathbb{R}\right)$. Critical points are furnished by the roots of the equation $f'(x) := \frac{\mathrm{d}f(x)}{\mathrm{d}x} = 0$ in the interval (a,b). *Local (or relative) extrema* are usually given by a critical point x_0 such that $f''(x_0) \neq 0$. In case $f'(x_0) = 0$ and $f''(x_0) > 0$, the function f has *a local (or relative) minimum*. Similarly, for a critical point x_0 with $f''(x_0) < 0$, f has *a local (or relative) maximum*. In case a critical point is neither a maximum nor a minimum, it is called *a stationary point (or a point of inflection)*. We will now provide some simple examples. (1) Consider $f(x) := 1$ for $x \in (-1,1)$. Every point in the interval $(-1,1)$ is a critical point. (2) Consider $f(x) := x$ in the open interval $(-2,2)$. There exist *no critical points* in the open interval. However, if we extend the function into *the closed interval* $[-2,2]$, there exists an extremum at the end points (*which are not critical points*). (3) The function $f(x) := (x)^3$, for $x \in \mathbb{R}$, has a stationary point (or a point of inflection) at the origin. (4) The periodic function $f(x) := \sin x,\ x \in \mathbb{R}$, has denumerably infinite number of local extrema, but no stationary points. (See, e.g., [32].)

Now, we generalize these concepts to a function of N real variables $x \equiv (x^1, \ldots, x^N)$. We assume that $f \in C^2(D \subset \mathbb{R}^N; \mathbb{R})$. A local minimum x_0 is defined by the property $f(x) \geq f(x_0)$ for all x in the neighborhood $N_\delta(x_0)$. Similarly, a local maximum is defined by the property $f(x) \leq f(x_0)$ for all x in the neighborhood. Consider f as a 0-form (see Sect. 1.2). The critical points are provided by the roots of the 1-form equation:

$$\begin{aligned} \mathrm{d}f(x) &= \tilde{\mathbf{0}}(x), \\ \partial_i f &= 0, \qquad i \in \{1, 2, \ldots, N\}. \end{aligned} \tag{A1.1}$$

A local (strong) minimum at x_0 is implied by the inequality

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left(x^i - x_0^i\right)\left(x^j - x_0^j\right)\left[\frac{\partial^2 f(x)}{\partial x^i \partial x^j}\right]_{|x_0} > 0. \tag{A1.2}$$

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4,

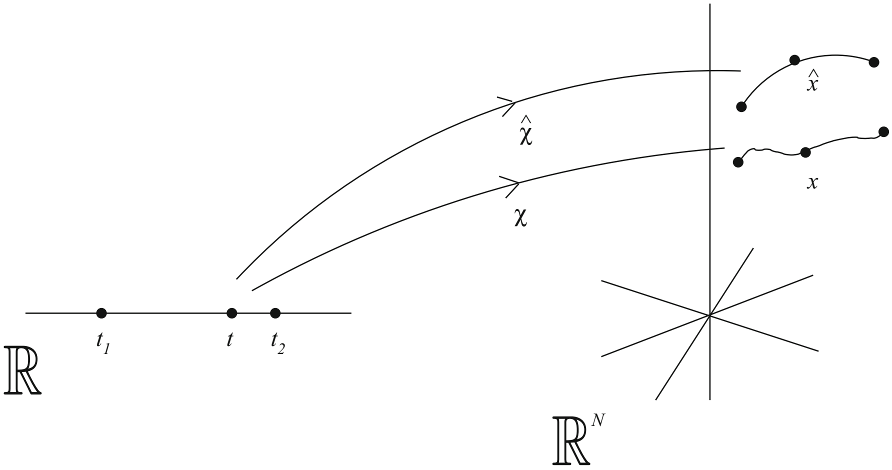

Fig. A1.1 Two twice-differentiable parametrized curves into $\mathbb{R}^N$

Similarly, a local (strong) maximum is guaranteed by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left(x^i - x_0^i\right)\left(x^j - x_0^j\right)\left[\frac{\partial^2 f(x)}{\partial x^i \partial x^j}\right]_{|x_0} < 0. \tag{A1.3}$$

Example A1.1. Consider the quadratic polynomial $f(x) := \sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij} x^i x^j$ for $x \in \mathbb{R}^N$. There exists *one critical point* at the origin $x_0 := (0, 0, \ldots, 0)$. Now,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} x^i x^j \left[\frac{\partial^2 f(x)}{\partial x^i \partial x^j}\right]_{|(0,0,\ldots,0)} = 2\sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij} x^i x^j > 0.$$

Therefore, the origin is a local, as well as a global (strong), minimum. □

Consider a parametrized curve $\mathcal{X}$ of class $C^2\left([t_1, t_2] \subset \mathbb{R}; \mathbb{R}^N\right)$, as furnished in equation (1.19). *A Lagrangian function* $L : [t_1, t_2] \times D_N \times D'_N \longrightarrow \mathbb{R}$ is such that $L(t; x; u)$ is twice differentiable. The *action function* (or *action functional*) J is a mapping from the domain set $C^2\left([t_1, t_2] \subset \mathbb{R}; \mathbb{R}^N\right)$ into the range set $\mathbb{R}$ and is given by:

$$J(\mathcal{X}) := \int_{t_1}^{t_2} [L(t; x; u)]_{|x^i = \mathcal{X}^i(t),\, u^i = \frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}} \cdot \mathrm{d}t. \tag{A1.4}$$

We assume that the function J is of class C^2, thus, totally differentiable in the abstract sense. (See (1.13).)

We consider a "slightly varied differentiable curve" $\widehat{\mathcal{X}} \in C^2\big([t_1, t_2] \subset \mathbb{R}; \mathbb{R}^N\big)$. (See Fig. A1.1.)

Two nearby curves are specified by the equations

$$\begin{aligned} x^i &= \mathcal{X}^i(t), \\ \widehat{x}^i &= \widehat{\mathcal{X}}^i(t) := \mathcal{X}^i(t) + \varepsilon h^i(t), \\ t &\in [t_1, t_2] \subset \mathbb{R}. \end{aligned} \tag{A1.5}$$

Here, $|\varepsilon| > 0$ is a small positive number and $h^i(t)$ is arbitrary twice-differentiable function. The "slight variation" of the action function J in (A1.4) is furnished by

$$\begin{aligned} \Delta J(\mathcal{X}) &:= J\left(\widehat{\mathcal{X}}\right) - J(\mathcal{X}) \\ &= \varepsilon \int_{t_1}^{t_2} \left\{ h^i(t) \cdot \left[\frac{\partial L(\cdot\cdot)}{\partial x^i}\right]_{|..} + \frac{\mathrm{d}h^i(t)}{\mathrm{d}t} \cdot \left[\frac{\partial L(\cdot\cdot)}{\partial u^i}\right]_{|..} \right\} \mathrm{d}t + 0(\varepsilon^2) \\ &= \varepsilon \int_{t_1}^{t_2} \left\{ h^i(t) \cdot \left[\frac{\partial L(\cdot\cdot)}{\partial x^i}\right]_{|..} \right\} \mathrm{d}t + \varepsilon \left\{ h^i(t) \cdot \left[\frac{\partial L(\cdot\cdot)}{\partial u^i}\right]_{|..} \right\} \Bigg|_{t_1}^{t_2} \\ &\quad - \varepsilon \int_{t_1}^{t_2} \left\{ h^i(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial L(\cdot\cdot)}{\partial u^i}\right]_{|..} \right\} \mathrm{d}t + 0(\varepsilon^2). \end{aligned} \tag{A1.6}$$

The *variational derivative* of the function J is given by

$$\begin{aligned} \frac{\delta J(\mathcal{X})}{\delta \mathcal{X}} &:= \lim_{\varepsilon \to 0} \left\{ \frac{\Delta J(\mathcal{X})}{\varepsilon} \right\} \\ &= \int_{t_1}^{t_2} \left\{ \left[\frac{\partial L(\cdot\cdot)}{\partial x^i}\right]_{|..} - \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial L(\cdot\cdot)}{\partial u^i}\right]_{|..} \right\} \cdot h^i(t)\, \mathrm{d}t \\ &\quad + \left\{ \left[\frac{\partial L(\cdot\cdot)}{\partial u^i}\right]_{|..} \cdot h^i(t) \right\} \Bigg|_{t_1}^{t_2} + 0\,. \end{aligned} \tag{A1.7}$$

The critical functions or critical curves $\mathcal{X}_{(0)}$ are defined by the solutions of the equation $\frac{\delta J(\mathcal{X})}{\delta \mathcal{X}} = 0$. We can prove from (A1.7) (with some more work involving the Dubois-Reymond lemma [224]), that critical functions $\mathcal{X}_{(0)}$ must satisfy N *Euler–Lagrange equations*:

$$\frac{\partial L(\cdot\cdot)}{\partial x^i}_{|..} - \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \left[\frac{\partial L(\cdot\cdot)}{\partial u^i}\right]_{|..} \right\} = 0. \tag{A1.8}$$

Moreover, we must impose *variationally permissible boundary conditions* (consisting partly of prescribed conditions and partly of variationally natural boundary conditions [159]) so that the equations

$$\left\{ \left[\frac{\partial L(\cdot\cdot)}{\partial u^i} \right]_{|\cdot\cdot} \cdot\, h^i(t) \right\} \Bigg|_{t_1}^{t_2} = \left\{ \left[\frac{\partial L(\cdot\cdot)}{\partial u^i} \right]_{|\cdot\cdot} \cdot\, h^i(t) \right\}_{|t_2} - \left\{ \left[\frac{\partial L(\cdot\cdot)}{\partial u^i} \right]_{|\cdot\cdot} \cdot\, h^i(t) \right\}_{|t_1} = 0. \tag{A1.9}$$

must hold.

Usually, $2N$ Dirichlet boundary conditions

$$\mathcal{X}^i(t_1) = x^i_{(1)} = \text{ prescribed consts.}, \quad \mathcal{X}^i(t_2) = x^i_{(2)} = \text{ prescribed consts.}$$

are chosen. Such conditions imply, by (A1.5), that $h^i(t_1) = h^i(t_2) \equiv 0$. Therefore, (A1.9) will be validated. However, there are *many other variationally permissible boundary conditions* which imply (A1.9).

Example A1.2. Consider the Newtonian mechanics of a free particle in a Cartesian coordinate chart. The Lagrangian is furnished by

$$\begin{aligned} L(v) &:= \frac{m}{2}\, \delta_{\alpha\beta}\, v^\alpha v^\beta, \quad m > 0, \\ \frac{\partial L(v)}{\partial v^\alpha} &= m\, \delta_{\alpha\beta}\, v^\beta, \\ \frac{\partial^2 L(v)}{\partial v^\alpha\, \partial v^\beta} &= m\, \delta_{\alpha\beta}. \end{aligned} \tag{A1.10}$$

The Euler–Lagrange equations (A1.8), with a slight change of notation and fixed boundary conditions $\mathcal{X}(t_1) = x_{(1)}$ and $\mathcal{X}(t_2) = x_{(2)}$, yield the unique critical curve

$$\mathcal{X}^\alpha_{(0)}(t) = \left[\frac{x^\alpha_{(2)} - x^\alpha_{(1)}}{t_2 - t_1} \right](t - t_1) + x^\alpha_{(1)}. \tag{A1.11}$$

This is a portion of a straight line or geodesic. Substituting the above solution into the finite variation in (A1.6), we derive that

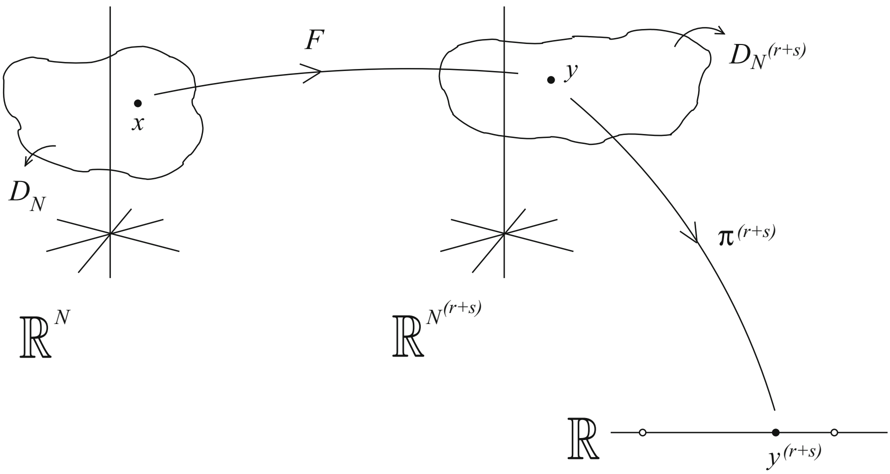

Fig. A1.2 The mappings corresponding to a tensor field $y^{(r+s)} = \phi^{(r+s)}(x)$

$$\begin{aligned} &J\left(\mathcal{X}_{(0)} + \varepsilon h\right) - J\left(\mathcal{X}_{(0)}\right) \\ &\quad = \varepsilon m \int_{t_1}^{t_2} \delta_{\alpha\beta} \left[\frac{\mathrm{d}h^\alpha(t)}{\mathrm{d}t}\ \frac{\mathrm{d}\mathcal{X}^\beta_{(0)}(t)}{\mathrm{d}t}\right] \mathrm{d}t + \frac{1}{2}\cdot\varepsilon^2 m \int_{t_1}^{t_2} \delta_{\alpha\beta} \frac{\mathrm{d}h^\alpha(t)}{\mathrm{d}t}\ \frac{\mathrm{d}h^\beta(t)}{\mathrm{d}t}\,\mathrm{d}t + 0 \\ &\quad = 0 + \frac{1}{2}\cdot\varepsilon^2 m \int_{t_1}^{t_2} \delta_{\alpha\beta}\ \frac{\mathrm{d}h^\alpha(t)}{\mathrm{d}t}\ \frac{\mathrm{d}h^\beta(t)}{\mathrm{d}t}\,\mathrm{d}t\ >\ 0. \end{aligned} \tag{A1.12}$$

Therefore, the critical curve $\mathcal{X}_{(0)}$, which is a part of a geodesic in the Euclidean space, constitutes *a strong minimum* compared to other neighboring smooth curves. □

Remarks. (i) Local, strong minima provide *stable equilibrium configurations*.
(ii) *Timelike geodesics* in a pseudo-Riemannian space–time manifold constitute *strong*, *local maxima* !

Now we shall consider $(r + s)$th order, differentiable tensor fields in an N-dimensional manifold M_N, as discussed in (1.30). (More sophisticated treatments of tensor fields employ the concepts of tensor bundles [38,56].) For the sake of brevity, we denote an $(r+s)$th order tensor field by the symbol $\phi^{(r+s)}(x)$ (in this appendix). Various mappings, which are necessary for the variational formalism, are shown in Fig. A1.2.

According to the mapping diagram,

$$y^{(r+s)} = \pi^{(r+s)} \circ F(x) =: \phi^{(r+s)}(x),$$
$$x \in \overline{D}_N \subset \mathbb{R}^N. \tag{A1.13}$$

The Lagrangian function $\mathcal{L} : \overline{D}_N \times \mathbb{R}^{(N)^{r+s}} \times \mathbb{R}^{(N)^{r+s+1}} \longrightarrow \mathbb{R}$ is such that $\mathcal{L}(x; y^{(r+s)}; y^{(r+s)}_{i})$ is twice differentiable. *The action function (or functional)* is defined by

$$J(F) := \int\limits_{\overline{D}_N} \mathcal{L}\left(x; y^{(r+s)}; y^{(r+s)}_{i}\right)\Big|_{y^{(r+s)}=\phi^{(r+s)}(x),\ y^{(r+s)}_{i}=\partial_i \phi^{(r+s)}} \cdot \mathrm{d}^N x. \tag{A1.14}$$

A “slightly varied function” is defined by

$$\widehat{y}^{(r+s)} = \pi^{(r+s)} \circ \widehat{F}(x) = \phi^{(r+s)}(x) + \varepsilon h^{(r+s)}(x), \tag{A1.15}$$

where $|\varepsilon|$ is a small positive number. The variation in the totally differentiable action function is furnished by

$$\begin{aligned} \Delta J(F) &:= J\big(\widehat{F}\big) - J(F) \\ &= \varepsilon \int\limits_{\overline{D}_N} \left\{ h^{(r+s)}(x) \cdot \left[\frac{\partial \mathcal{L}(\cdot\cdot)}{\partial y^{(r+s)}}\right]\Big|_{\cdot\cdot} + \partial_i h^{r+s} \cdot \left[\frac{\partial \mathcal{L}(\cdot\cdot)}{\partial y^{r+s}_{i}}\right]\Big|_{\cdot\cdot} \right\} \mathrm{d}^N x + 0(\varepsilon^2). \end{aligned} \tag{A1.16}$$

Here, indices $(r+s)$ are to be *automatically summed appropriately*.

Now, we introduce the notion of *a total-partial derivative* by

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x^i}\left[f(x; y^{(r+s)}; y^{(r+s)}_{j})\right]\Big|_{\cdot\cdot} \\ &:= \left[\frac{\partial f(\cdot\cdot)}{\partial x^i}\right]\Big|_{\cdot\cdot} + \partial_i \phi^{(r+s)} \cdot \left[\frac{\partial f(\cdot\cdot)}{\partial y^{(r+s)}}\right]\Big|_{\cdot\cdot} + \partial_i \partial_j \phi^{(r+s)} \cdot \left[\frac{\partial f(\cdot\cdot)}{\partial y^{(r+s)}_{j}}\right]\Big|_{\cdot\cdot}. \end{aligned} \tag{A1.17}$$

By (A1.17), we can express (A1.16) as

$$\begin{aligned} \Delta J(F) = \varepsilon \int\limits_{\overline{D}_N} &\left\{ h^{(r+s)}(x) \cdot \left[\frac{\partial \mathcal{L}(\cdot\cdot)}{\partial y^{(r+s)}}\right]\Big|_{\cdot\cdot} \right. \\ &\left. + \left(\frac{\mathrm{d}}{\mathrm{d}x^i}\left[h^{(r+s)}(x) \cdot \frac{\partial \mathcal{L}(\cdot\cdot)}{\partial y^{(r+s)}_{i}}\Big|_{\cdot\cdot}\right] - h^{(r+s)}(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x^i}\left[\frac{\partial \mathcal{L}(\cdot\cdot)}{\partial y^{r+s}_{i}}\right]\Big|_{\cdot\cdot} \right) \right\} \mathrm{d}^N x + 0(\varepsilon^2). \end{aligned} \tag{A1.18}$$

The variational derivative is provided by

$$
\begin{aligned}
\frac{\delta J(F)}{\delta F} &:= \lim_{\varepsilon\to 0}\left[\frac{\Delta J(F)}{\varepsilon}\right] \\
&= \int\limits_{\overline{D}_N}\left\{h^{(r+s)}(x)\left(\left[\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y^{(r+s)}}\right]_{|\cdot\cdot} - \frac{\mathrm{d}}{\mathrm{d}x^i}\left[\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y^{r+s}_{i}}\right]_{|\cdot\cdot}\right)\right\}\mathrm{d}^N x \\
&\quad + \int\limits_{\partial D_N}\left\{h^{(r+s)}(x)\cdot\left[\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y^{(r+s)}_{i}}\right]_{|\cdot\cdot}\right\} n_i\,\mathrm{d}^{N-1}x.
\end{aligned}
\tag{A1.19}
$$

(We have used *Gauss' theorem* in (1.155) for the last term in (A1.19).) *The critical or stationary functions* $\phi^{(r+s)}_{(0)}(x)$ are given by the solution of the equations $\frac{\delta J(F)}{\delta F} = 0$. We can derive, from (A1.19) [159], that critical functions must satisfy equations

$$
\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y^{(r+s)}}_{|\cdot\cdot} - \frac{\mathrm{d}}{\mathrm{d}x^i}\left[\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y^{(r+s)}_{i}}\right]_{|\cdot\cdot} = 0, \tag{A1.20i}
$$

and

$$
\int\limits_{\partial D_N}\left\{h^{(r+s)}(x)\cdot\left[\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y^{(r+s)}_{i}}\right]_{|\cdot\cdot}\right\} n_i(x)\,\mathrm{d}^{N-1}x = 0. \tag{A1.20ii}
$$

Here, (A1.20i) yields Euler–Lagrange equations for critical tensor fields, whereas (A1.20ii) provides *the variationally admissible boundary conditions*.

Example A1.3. Consider a background pseudo-Riemannian space–time and a scalar field $y = \phi(x)$ for $x \in D_4 \subset \mathbb{R}^4$. The Lagrangian $\mathcal{L}$, which is *a scalar density*, is furnished by

$$
\begin{aligned}
\sqrt{-g(x)}\cdot L(x; y; y_i) = \mathcal{L}(x; y; y_i) &:= -\frac{\sqrt{-g(x)}}{2}\cdot g^{ij}(x)\,y_i\,y_j, \\
\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y} \equiv 0, \quad \frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y_k} &= -\sqrt{-g(x)}\cdot g^{kj}(x)\,y_j.
\end{aligned}
\tag{A1.21}
$$

Euler–Lagrange equations (A1.21), using (A1.17), yield

$$
\begin{aligned}
0 - \frac{\mathrm{d}}{\mathrm{d}x^k}\left[\frac{\partial\mathcal{L}(\cdot\cdot)}{\partial y_k}\right]_{|\cdot\cdot} &= \partial_j\phi\cdot\partial_k\left[\sqrt{-g}\cdot g^{kj}\right] + \partial_j\partial_k\phi\cdot\sqrt{-g}\cdot g^{kj} \\
&= \sqrt{-g}\left[g^{kj}(x)\nabla_k\nabla_j\phi\right] = \sqrt{-g}\,\Box\phi = 0, \\
\text{or}\quad \Box\phi = 0.
\end{aligned}
\tag{A1.22}
$$

Thus, the critical functions $\phi_{(0)}(x)$ satisfy *the wave equation* in the prescribed curved space–time.

Consider the wave equation (A1.22) in a domain $D_4 := \mathbf{D}_3 \times (0,T)$, $0 < T < T_1$. Let the prescribed metric functions $g_{ij}(x)$ be real-analytic and $g^{44}(x) < 0$ in the domain D_4. Consider the following initial value problem for the wave equation:

$$\phi_{(0)}(\mathbf{x},0) = p(\mathbf{x}), \quad \partial_4\phi_{(0)}{}_{|(\mathbf{x},0)} = q(\mathbf{x}), \quad \mathbf{x} \in \mathbf{D}_3.$$

Here, the prescribed functions $p(\mathbf{x})$ and $q(\mathbf{x})$ are assumed to be real-analytic. This initial value problem, by the Cauchy-Kowaleski Theorem 2.4.7, admits a unique, real-analytic solution $\phi_{(0)}(x)$ for a small positive number T. The "varied function" $\phi_{(0)}(x)+\varepsilon h(x)$, which respects the initial values, must have $h(\mathbf{x},0) = 0 = \partial_4 h_{|(\mathbf{x},0)}$. However, other than being real-analytic, there are no restrictions on $h(x)$ for $0 < x^4 < T$. Thus, the integral in (A1.20ii),

$$-\int\limits_{\partial D_4} h(x)\cdot g^{ij}(x)\cdot \partial_j\phi \cdot n_i(x)\,\sqrt{-g}\,\mathrm{d}^3x \not\equiv 0.$$

Therefore, *in general, initial value problems are not variationally admissible* (due to the fact that on the final time hypersurface, the function $h(x)$ is completely arbitrary). □

Now, we shall investigate the variational derivation of the gravitational field equations. However, this process demands *a slightly different mathematical approach*. We explain the methodology by first exploring *the following toy model*.

Example A1.4. Consider the flat space–time M_4 and a Minkowskian coordinate chart. Therefore, the metric tensor field is given by $g_{ij}(x) = d_{ij} =$ consts., and $\sqrt{-g}(x) \equiv 1$. We explore a Lagrangian function

$$\begin{aligned}
&\mathcal{L}\left(x; y, \gamma^i; y_i, \gamma^i{}_j\right) := \left[d^{ij}\,\partial_i\partial_j\,W(x)\right]\cdot y - \gamma^i{}_i,\\
&\mathcal{L}\left(x; y, \gamma^i; y_i, \gamma^i{}_j\right)_{\Big|\substack{y=\phi(x),\ y_i=\partial_i\phi\\ \gamma^i=\Gamma^i(x),\ \gamma^i{}_j=\partial_j\Gamma^i}} = (\Box\, W)\cdot\phi(x) - \partial_i\Gamma^i.
\end{aligned} \tag{A1.23}$$

The slightly varied functions are denoted by $\widehat{y} = \phi(x) + \varepsilon h(x)$ and $\widehat{\gamma}^i = \Gamma^i(x) + \varepsilon h^i(x)$. By (A1.16), we obtain that

$$\begin{aligned}
\frac{\Delta J(F)}{\varepsilon} &= \int\limits_{D_4}\left[\Box\, W\cdot h(x)\right]\mathrm{d}^4x - \int\limits_{D_4}\partial_i h^i\,\mathrm{d}^4x\\
&= \int\limits_{D_4}\left[\Box\, W\cdot h(x)\right]\mathrm{d}^4x - \int\limits_{\partial D_4} h^i(x)\,n_i\,\mathrm{d}^3x.
\end{aligned}$$

Therefore, the critical functions, obeying $\frac{\delta J(F)}{\delta F} = 0$, must satisfy:

$$\Box\, W(x) = 0,$$

$$\text{and,} \quad \int_{\partial D_4} h^i(x)\, n_i\, \mathrm{d}^3x = 0. \tag{A1.24}$$

Here, the coefficient function $W(x)$ need not satisfy any boundary condition and the critical functions $\Gamma^i_{(0)}(x)$ need not satisfy any differential equation! $\Box$

Now, we shall take up the case of general relativity. Recall that in (2.146i–iii) for relativistic mechanics, position variables and momentum variables are treated on the same footing. Similarly, in what is known as *the Hilbert-Palatini approach of variation*, metric functions $y^{ij} = g^{ij}(x)$ and connection coefficients $\gamma^k{}_{ij} = \left\{ {k \atop ij} \right\}$ are *both treated as independent variables*. We choose the following invariant Lagrangian function $L(\cdot\cdot)$[1]:

$$\begin{aligned} &L\left(x; y^{ij}, \gamma^k{}_{ij}; y^{ij}{}_k, \gamma^k{}_{ijl}\right) \\ &\quad := y^{ij}\left[\gamma^k{}_{kij} - \gamma^k{}_{ijk} - \gamma^l{}_{lk}\gamma^k{}_{ij} + \gamma^l{}_{ik}\gamma^k{}_{lj}\right] \\ &\quad =: y^{ij}\rho_{ij}\left(\gamma^c{}_{ab};\ \gamma^c{}_{abd}\right), \end{aligned}$$

$$\begin{aligned} L(\cdots)\Big|_{\substack{y^{ij}=g^{ij}(x),\ y^{ij}{}_k=\partial_k g^{ij}, \\ \gamma^k{}_{ij}=\left\{ {k \atop ij} \right\},\ \gamma^k{}_{ijl}=\partial_l\left\{ {k \atop ij} \right\}}} &= g^{ij}(x)\cdot\left[\partial_j \left\{ {k \atop i\ j} \right\} - \partial_k \left\{ {k \atop i\ j} \right\} - \left\{ {l \atop l\ k} \right\}\cdot\left\{ {k \atop i\ j} \right\}\right. \\ &\qquad \left. + \left\{ {l \atop i\ k} \right\}\cdot\left\{ {k \atop l\ j} \right\}\right]; \end{aligned}$$

$$\begin{aligned} \mathcal{L}(\cdot\cdot)_{|\cdot\cdot} = \sqrt{-g}(x)\, g^{ij}(x)\cdot\Bigg[\partial_j \left\{ {k \atop i\ j} \right\} - \partial_k \left\{ {k \atop i\ j} \right\} \\ - \left\{ {l \atop l\ k} \right\}\cdot\left\{ {k \atop i\ j} \right\} + \left\{ {l \atop i\ k} \right\}\cdot\left\{ {k \atop l\ j} \right\}\Bigg]. \end{aligned} \tag{A1.25}$$

From the equations above, we can deduce the following partial derivatives:

$$\left[y \cdot y^{ab}\right]\cdot y_{ac} = y\cdot\delta^b{}_c, \quad \frac{\partial(y)}{\partial y_{ab}} = y\cdot y^{ab},$$

$$\frac{\partial\left(\sqrt{-y}\right)}{\partial y_{ab}} = +\frac{1}{2}\sqrt{-y}\cdot y^{ab}, \quad \frac{\partial\left(\sqrt{-y}\right)}{\partial y^{ab}} = -\frac{1}{2}\sqrt{-y}\cdot y_{ab},$$

[1] *Caution*: Here, $\gamma^c{}_{ab}$ is *not* related to Ricci rotation coefficients as such.

$$\frac{\partial}{\partial y^{ab}}\left(\sqrt{-y}\cdot y^{ij}\right)=\sqrt{-y}\left[\delta^{i}_{\ a}\cdot\delta^{j}_{\ b}-\frac{1}{2}\,y_{ab}y^{ij}\right];$$

$$\frac{\partial\rho_{ij}(\cdot\cdot)}{\partial\gamma^{a}_{\ bc}}=-\delta^{b}_{\ a}\cdot\gamma^{c}_{\ ij}-\delta^{b}_{\ i}\cdot\delta^{c}_{\ j}\cdot\gamma^{l}_{\ la}+\delta^{b}_{\ i}\cdot\gamma^{c}_{\ aj}+\delta^{c}_{\ j}\cdot\gamma^{b}_{\ ia},$$

$$\frac{\partial\rho_{ij}(\cdot\cdot)}{\partial\gamma^{a}_{\ bcd}}=\delta^{b}_{\ a}\cdot\delta^{c}_{\ i}\cdot\delta^{d}_{\ j}-\delta^{d}_{\ a}\cdot\delta^{b}_{\ i}\cdot\delta^{c}_{\ j}. \tag{A1.26}$$

We denote the slightly varied functions by $\widehat{y}^{ij}=g^{ij}(x)+\varepsilon h^{ij}(x)$ and $\widehat{\gamma}^{k}_{\ ij}=\left\{ {k \atop ij} \right\}+\varepsilon h^{k}_{\ ij}(x)$. (Note that $h^{ij}(x)$ and $h^{k}_{\ ij}(x)$ are $(2+0)$th order and $(1+2)$th order *tensor fields*, respectively, even though $\left\{ {k \atop ij} \right\}$ is not!)[2] Using (A1.16) for the Lagrangian in (A1.25), we derive that

$$\begin{aligned}\frac{\Delta J(F)}{\varepsilon}=&\int_{D_4}\left\{\left[\rho_{ij}(\cdot\cdot)\cdot\frac{\partial\left(\sqrt{-y}\cdot y^{ij}\right)}{\partial y^{ab}}\right]_{|\cdot\cdot}\cdot h^{ab}(x)\right\}\mathrm{d}^4x\\&+\int_{D_4}\sqrt{-g}(x)\cdot g^{ij}(x)\cdot\left\{\left[\frac{\partial\rho_{ij}(\cdot\cdot)}{\partial\gamma^{a}_{\ bc}}\right]_{|\cdot\cdot}\cdot h^{a}_{\ bc}(x)\right.\\&\left.+\left[\frac{\partial\rho_{ij}(\cdot\cdot)}{\partial\gamma^{a}_{\ bcd}}\right]_{|\cdot\cdot}\cdot\partial_d h^{a}_{\ bc}\right\}\mathrm{d}^4x+0(\varepsilon).\end{aligned} \tag{A1.27}$$

Therefore, for critical functions, (A1.25)–(A1.27), and a shortcut notation in regard to $G_{ab}(x)$ yield

$$\begin{aligned}0=&\int_{D_4}G_{ab}(x)\cdot h^{ab}(x)\cdot\sqrt{-g}(x)\,\mathrm{d}^4x\\&+\int_{D_4}\nabla_j\left[g^{ji}(x)\cdot h^{k}_{\ ki}(x)-g^{ki}(x)\cdot h^{j}_{\ ki}(x)\right]\cdot\mathrm{d}^4v.\end{aligned} \tag{A1.28}$$

Thus, the critical functions $g^{ij}_{(0)}(x)$ and $\left\{ {k \atop ij} \right\}$ must satisfy

$$G_{ab}(x)=0 \quad \text{in} \quad D_4, \tag{A1.29i}$$

---

[2]The popular way of writing one of the variations in (A1.26) is to put $\delta\left(\sqrt{|g|}\,g^{ij}\right)=\sqrt{|g|}\left[\delta g^{ij}-\frac{1}{2}\cdot g_{kl}\cdot g^{ij}\cdot\delta g^{kl}\right]$. (This equation holds in any Riemannian or pseudo-Riemannian manifold.)

and (via the divergence theorem)

$$\int_{\partial D_4} \left[g^{ij}(x) \cdot h^{k}{}_{ki}(x) - g^{ki}(x) \cdot h^{j}{}_{ki}(x) \right] n_j \, \mathrm{d}^3 v = 0. \tag{A1.29ii}$$

The variables $\sqrt{-y}\, y^{ij}$ and γ^{k}_{lij} are analogous to the variables y and $\gamma^{i}{}_{j}$ respectively of Example A1.4. Equations (A1.29i) and (A1.29ii) are exactly analogous to the equations in (A1.4).

Equation (A1.29i) is obviously equivalent to the vacuum equations of (2.160i). Moreover, (A1.29ii) provides implicitly the variationally admissible boundary conditions for the vacuum equations.

There is *an unsatisfactory aspect in the derivation of (A1.29i,ii)*. In that process, we have varied the metric tensor and connection coefficients independently. However, the variational derivation *does not reveal the actual relationship between these two*. We can rectify this logical gap by augmenting the Lagrangian in (A1.25) with *Lagrange multipliers* $\lambda^{ij}{}_{k}$ to incorporate the *required constraints*. Defining the unique entries $\left[y_{ij} \right] := \left[y^{ij} \right]^{-1}$, we furnish the augmented Lagrangian $\tilde{L}$ as the following function of $(4)^{16}$ variables (without assuming any symmetry):

$$\begin{aligned}
&\tilde{L}\left(x; y^{ij}, \gamma^{k}{}_{ij}, \lambda^{ij}{}_{k}; y^{ij}{}_{k}, \gamma^{k}{}_{ijl}\right) \\
&\quad := y^{ij} \left[\gamma^{k}{}_{kij} - \gamma^{k}{}_{ijk} - \gamma^{l}{}_{lk} \cdot \gamma^{k}{}_{ij} + \gamma^{l}{}_{ik} \cdot \gamma^{k}{}_{lj} \right] \\
&\qquad + \lambda^{ij}{}_{k} \left[\gamma^{k}{}_{ij} - \frac{1}{2} y^{kl} \left(y_{jli} + y_{lij} - y_{ijl} \right) \right], \\
&\tilde{L}(\cdot\cdot)\Big|_{\substack{y^{ij} = g^{ij}(x),\ y^{ij}{}_{k} = \partial_k g^{ij},\ \lambda^{ij}{}_{k} = \Lambda^{ij}{}_{k}(x), \\ \gamma^{k}{}_{ij} = \left\{ \begin{smallmatrix} k \\ i\,j \end{smallmatrix} \right\}(x),\ \gamma^{k}{}_{ijl} = \partial_l \left\{ \begin{smallmatrix} k \\ i\,j \end{smallmatrix} \right\}}} \\
&\quad = g^{ij}(x) \left[\partial_j \begin{Bmatrix} k \\ k\,i \end{Bmatrix} - \partial_k \begin{Bmatrix} k \\ i\,j \end{Bmatrix} - \begin{Bmatrix} l \\ l\,k \end{Bmatrix}(x) \cdot \begin{Bmatrix} k \\ i\,j \end{Bmatrix}(x) \right. \\
&\qquad \left. + \begin{Bmatrix} l \\ i\,k \end{Bmatrix}(x) \cdot \begin{Bmatrix} k \\ l\,j \end{Bmatrix}(x) \right] \\
&\qquad + \Lambda^{ij}{}_{k}(x) \left[\begin{Bmatrix} k \\ i\,j \end{Bmatrix}(x) - \frac{1}{2} g^{kl}(x) \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} \right) \right], \\
&\tilde{\mathcal{L}}(\cdot\cdot)_{|..} := \sqrt{-y}\, \tilde{L}(\cdot\cdot)_{|..}\,.
\end{aligned} \tag{A1.30}$$

The Euler–Lagrange equation $\frac{\partial \tilde{\mathcal{L}}}{\partial (\lambda^{ij}{}_{k})}\Big|_{..} - 0 = 0$ yield the Christoffel symbols in terms of metric tensor components.

Now, the boundary term in (A1.29ii) is related to *the trace of the exterior curvature* of the hypersurface ∂D_4. To show this, consider the projection operator of p. 191, viz.,

$$\mathcal{P}^i{}_j(x) := \delta^i{}_j - \varepsilon(n)\, n^i(x) n_j(x).$$

We define the *extended extrinsic curvature* [56, 126] by

$$\chi_{ij}(x) := -\frac{1}{2}\mathcal{P}^k_i \mathcal{P}^l{}_j \left(\nabla_k n_l + \nabla_l n_k\right), \tag{A1.31i}$$

$$K_{\mu\nu}(u) = \partial_\mu \xi^i \cdot \partial_\nu \xi^j \cdot \chi_{ij}\left(\xi(u)\right). \tag{A1.31ii}$$

(Here, the extrinsic curvature, $K_{\mu\nu}(u)$, was defined in (1.234). The vector ξ^i was defined in the same section.) We can derive *the trace* from (A1.31ii) as

$$\chi^i{}_i(..) = -\nabla_i n^i \equiv -\nabla^i n_i. \tag{A1.32}$$

The variation of $\chi^i{}_i(..)$ is denoted by (*Caution*: Note that ε is *different* from $\varepsilon(n)$.):

$$\hat{\chi}^i{}_i(..) - \chi^i{}_i(..) =: \varepsilon h(..). \tag{A1.33}$$

Now, (A1.32) yields *two* expressions:

$$-\chi^i{}_i(..) = \partial_i n^i + \left\{ {i \atop j\ i} \right\} n^j, \tag{A1.34i}$$

$$-\chi^i{}_i(..) = g^{ij}(..)\left[\partial_j n_i - \left\{ {k \atop i\ j} \right\} n_k\right]. \tag{A1.34ii}$$

For a fixed boundary ∂D_4, the variations of n_i are exactly zero. In consideration of the boundary term in (A1.29ii), *only* the variation of $\left\{ {k \atop ij} \right\}$ is allowed. The variation of $g_{ij}(..)$ and $n^i(..)$ is set to zero for this boundary term. Therefore, (A1.34i,ii) provide

$$-\varepsilon h(..) = 0 + \varepsilon h^i{}_{ij}(..) n^j(..), \tag{A1.35i}$$

$$-\varepsilon h(..) = g^{ij}(..)\left[0 - \varepsilon h^k{}_{ij}(..) n_k(..)\right]. \tag{A1.35ii}$$

Adding the two equations above, we obtain

$$-\,2h(..) = \left[g^{jk}(..) h^i{}_{ij}(..) - g^{ij}(..) h^k{}_{ij}(..)\right] n_k(..). \tag{A1.36}$$

Therefore, the boundary term in (A1.29ii) is furnished by

$$\int_{\partial D_4} \left[g^{ij}(x) h^k_{\ ki}(x) - g^{ki}(x) h^j_{\ ki}(x) \right] n_j \, \mathrm{d}^3 v = -2 \int_{\partial D_4} h(x) \, \mathrm{d}^3 v. \tag{A1.37}$$

A popular way to write the above is to express it as

$$\int_{\partial D_4} \left[g^{ij}(x) \delta \begin{Bmatrix} k \\ k\,i \end{Bmatrix} - g^{ki}(x) \delta \begin{Bmatrix} j \\ k\,i \end{Bmatrix} \right] n_j \, \mathrm{d}^3 v = -2 \int_{\partial D_4} \delta \left[\chi^i_{\ i}(x) \right] \mathrm{d}^3 v. \tag{A1.38}$$

(We have *tacitly assumed here that* $\delta g^{ij}(x) \equiv 0$ on the boundary.)[3]

Example A1.5. The invariant Lagrangians (A1.25) and (A1.30) contain second-order derivatives of the metric tensor components. In general, for Lagrangians involving second-order derivatives of the metric, we would have obtained *fourth-order field equations* as the corresponding Euler–Lagrange equations. However, for the Lagrangian density $\sqrt{-g(x)}\,R(x)$, the higher order terms are transformable into a boundary term in (A1.29ii) and thus do not contribute to the field equations. Einstein investigated a Lagrangian [84] which has only first-order derivatives of the metric tensor components. It is furnished by

$$\begin{aligned}
\mathcal{L}(\cdot\cdot) &:= \sqrt{-y}\, y^{ij} \left[\gamma^l_{\ ik} \cdot \gamma^k_{\ lj} - \gamma^l_{\ lk} \cdot \gamma^k_{\ ij} \right] \\
&\equiv Y^{ij} \cdot F_{ij}\left(Y^{kl}_{\ \ m} \right), \\
Y^{ij} &:= \sqrt{-y}\, y^{ij}, \\
\mathcal{L}(\cdot\cdot)&\Big|_{y^{ij} = g^{ij}(x),\ \gamma^k_{\ ij} = \begin{Bmatrix} k \\ ij \end{Bmatrix}} \\
&= \sqrt{-g(x)} \cdot g^{ij}(x) \cdot \left[\begin{Bmatrix} l \\ i\,k \end{Bmatrix} \cdot \begin{Bmatrix} k \\ l\,j \end{Bmatrix} - \begin{Bmatrix} l \\ l\,k \end{Bmatrix} \cdot \begin{Bmatrix} k \\ i\,j \end{Bmatrix} \right].
\end{aligned} \tag{A1.39}$$

Note that the above Lagrangian is only *a part of* $\sqrt{-g(x)}\,R(x)$*, and it is not a scalar density* (of weight $+1$). We can compute the partial derivatives as

$$\begin{aligned}
\frac{\partial \mathcal{L}(\cdot\cdot)}{\partial Y^{ij}}&\Big|_{Y^{ij} = \sqrt{-g}\, g^{ij},\ Y^{ij}_{\ \ k} = \partial_k(\sqrt{-g}\, g^{ij})} \\
&= \begin{Bmatrix} l \\ l\,k \end{Bmatrix} \cdot \begin{Bmatrix} k \\ i\,j \end{Bmatrix} - \begin{Bmatrix} l \\ i\,k \end{Bmatrix} \cdot \begin{Bmatrix} k \\ l\,j \end{Bmatrix}, \\
\frac{\partial \mathcal{L}(\cdot\cdot)}{\partial Y^{ij}_{\ \ k}}\Big|_{\cdot\cdot} &= \delta^k_{\ i} \begin{Bmatrix} l \\ j\,l \end{Bmatrix} - \begin{Bmatrix} k \\ i\,j \end{Bmatrix}.
\end{aligned} \tag{A1.40}$$

---

[3]In *the metric variation approach*, it is assumed that the connection is the metric connection and that only the metric is to be varied. In this scenario, the boundary term is much more complicated. The general case (i.e., *not assuming that* $\delta g_{ij}(x) \equiv 0$ *on the boundary*) yields $-2\int_{\partial D_4} \delta \left[\chi^i_{\ i}(x)\right] \mathrm{d}^3 v + \int_{\partial D_4} \nabla_i \left[\mathcal{P}^{ij}(x) n^k \delta g_{jk}(x)\right] \mathrm{d}^3 v - \int_{\partial D_4} g^{ik}(x) \nabla_i n^j \delta g_{jk}(x)\, \mathrm{d}^3 v$ for the boundary term.

(Consult [84].) The corresponding variational equations (A1.20i,ii) yield

$$\frac{\partial \mathcal{L}(\cdot\cdot)}{\partial Y^{ij}}\Big|_{\cdot\cdot} - \frac{\mathrm{d}}{\mathrm{d}x^k}\left[\frac{\partial \mathcal{L}(\cdot\cdot)}{\partial Y^{ij}_{\ \ k}}\right]\Big|_{\cdot\cdot} = -R_{ij}(x) = 0, \tag{A1.41i}$$

$$\int\limits_{\partial D_4}\left[h^{kj}(x)\begin{Bmatrix} l \\ j\ l \end{Bmatrix} - h^{ij}(x)\begin{Bmatrix} k \\ i\ j \end{Bmatrix}\right] n_k\,\mathrm{d}^3x = 0. \tag{A1.41ii}$$

Thus, (A1.41i,ii) provide the gravitational field equations and variationally admissible boundary conditions outside material sources. □

Example A1.6. We shall now derive the *Arnowitt–Deser–Misner (ADM) action integral* [7], [184]. Assuming $g_{44}(x) < 0$, we express the metric tensor in the following equations:

$$\begin{aligned}
\mathbf{g}_{\cdot\cdot}(x) &= g_{\alpha\beta}(x)\left[\mathrm{d}x^\alpha + N^\alpha(x)\,\mathrm{d}x^4\right] \otimes \left[\mathrm{d}x^\beta + N^\beta(x)\,\mathrm{d}x^4\right] \\
&\qquad\qquad\qquad\qquad - [N(x)]^2\,\mathrm{d}x^4 \otimes \mathrm{d}x^4\,; \\
\left[g_{ij}\right] &= \left[\begin{array}{c|c} g_{\alpha\beta} & g_{\mu\nu}N^\nu \\ \hline g_{\mu\nu}N^\nu & g_{\rho\sigma}N^\rho N^\sigma - N^2 \end{array}\right], \\
\left[g^{ij}\right] &= \left[\begin{array}{c|c} g^{\alpha\beta} - N^{-2}N^\alpha N^\beta & N^{-2}N^\mu \\ \hline N^{-2}N^\mu & -N^{-2} \end{array}\right]; \\
\sqrt{-\det\left[g_{ij}\right]} &= N\,\sqrt{\det\left[g_{\alpha\beta}\right]}\,.
\end{aligned} \tag{A1.42}$$

Suppose that the space–time locally admits a one-parameter family of three-dimensional spacelike hypersurfaces. On the spacelike hypersurface characterized by $x^4 = T$, the intrinsic metric is furnished as

$$\begin{aligned}
\mathbf{g}_{\cdot\cdot}(\mathbf{x}, T) &= g_{\alpha\beta}(\mathbf{x}, T)\,\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta \\
&=: \overline{g}_{\alpha\beta}(\mathbf{x}, T)\,\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta.
\end{aligned} \tag{A1.43}$$

There is a slight notational difference between this equation and (1.223) because we choose here *one member of the infinitely many hypersurfaces*. We show the ADM decomposition schematically in Fig. A1.3.

Gauss' equation (1.242i) for a hypersurface yields in this example, with the (usual) notation $\overline{A}^\alpha(\cdot\cdot) := \overline{g}^{\alpha\beta}(\cdot\cdot)\,A_\beta(\cdot\cdot)$,

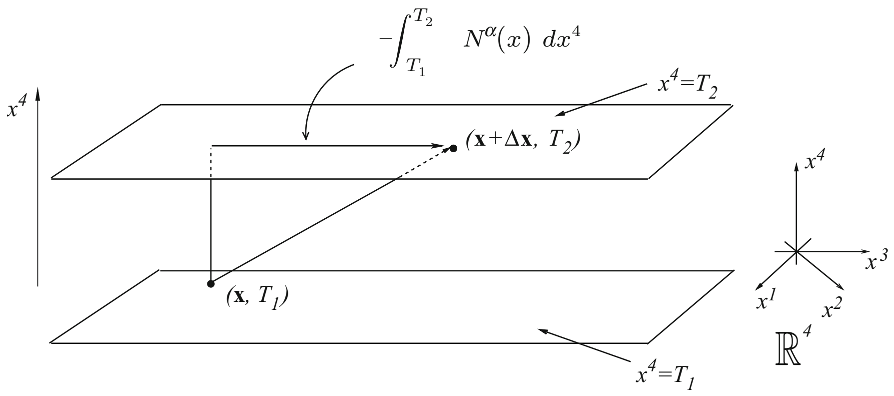

Fig. A1.3 Two representative spacelike hypersurfaces in an ADM decomposition of space–time

$$\overline{R}_{\sigma\mu\nu\lambda}(\mathbf{x},T) = K_{\mu\nu}(\mathbf{x},T)K_{\sigma\lambda}(\mathbf{x},T) - K_{\mu\lambda}(\mathbf{x},T)K_{\sigma\nu}(\mathbf{x},T) + R_{\sigma\mu\nu\lambda}(\mathbf{x},T),$$

$$R^{\sigma}{}_{\mu\nu\lambda}(\mathbf{x},T) = \overline{R}^{\sigma}{}_{\mu\nu\lambda}(\mathbf{x},T) + K_{\mu\lambda}(\cdot\cdot)\cdot\overline{K}^{\sigma}{}_{\nu}(\cdot\cdot) - K_{\mu\nu}(\cdot\cdot)\cdot\overline{K}^{\sigma}{}_{\lambda}(\cdot\cdot),$$

$$\begin{aligned} R(\mathbf{x},T) &= R^{ij}{}_{ji}(\mathbf{x},T) = R^{\alpha\beta}{}_{\beta\alpha}(\mathbf{x},T) + 2R^{\alpha 4}{}_{4\alpha}(\mathbf{x},T) \\ &= \overline{R}(\mathbf{x},T) + \overline{K}^{\alpha}{}_{\beta}(\cdot\cdot)\cdot\overline{K}^{\beta}{}_{\alpha}(\cdot\cdot) - \left[\overline{K}^{\alpha}{}_{\alpha}(\cdot\cdot)\right]^2 + 2R^{\alpha 4}{}_{4\alpha}(\cdot\cdot). \end{aligned} \tag{A1.44}$$

The last term in (A1.44) can be expressed as a divergence term analogous to $\nabla_j A^j$ [184]. Therefore, the Hilbert action integral goes over into

$$\begin{aligned} \int_{\overline{D}_4} R(x)\,\sqrt{-g}(x)\,\mathrm{d}^4x &= \int_{\overline{D}_4} \left\{\overline{R}(\mathbf{x},T) + \overline{K}^{\alpha}{}_{\beta}(\cdot\cdot)\cdot\overline{K}^{\beta}{}_{\alpha}(\cdot\cdot) - \left[\overline{K}^{\alpha}{}_{\alpha}(\cdot\cdot)\right]^2\right\} \\ &\quad \cdot N(\mathbf{x},T)\,\sqrt{g(\mathbf{x},T)}\cdot\mathrm{d}^3\mathbf{x}\,dT + (\text{boundary term}). \end{aligned} \tag{A1.45}$$

The above action integral is useful in several approaches to *quantum gravity* [10, 199, 221, 246]. □

Finally, we would like to make the following two mathematical comments. Firstly, consider *a nonconstant Lagrangian density function* $\mathcal{L}\left(y_{ij}; y_{ijk}\right)$ such that the action functional is given by

$$\begin{aligned} J(F) &:= \int_{\overline{D}_4} \mathcal{L}\left(y_{ij}; y_{ijk}\right)_{|\cdot\cdot}\, \mathrm{d}^4x, \\ y_{ij} &= \pi_{ij}\circ F(x) = g_{ij}(x), \\ y_{ijk} &= \pi_{ijk}\circ F'(x) = \partial_k g_{ij}. \end{aligned} \tag{A1.46}$$

It can be rigorously proved [171] that *such an integral cannot be tensorially invariant*. That is why, in Example A1.5, we had to deal with *a noninvariant action integral*.

The second comment is about a Lagrangian *scalar density* of the type $\mathcal{L}\left(y_{ij}; y_{ijk}; y_{ijkl}\right)$. It can be rigorously proved [170, 171], that in a four-dimensional manifold, the most general Lagrangian density (made up of curvature invariants), which yields *second-order Euler–Lagrange equations*, must be of the form

$$\begin{aligned}\mathcal{L}(\cdot\cdot)_{|..} := {} & c_{(1)}\sqrt{-g(x)}\, R(x) + c_{(2)}\sqrt{-g(x)} \\ & + c_{(3)}\, \varepsilon^{ijkl}\, R^{mn}{}_{ij}(x) \cdot R_{mnkl}(x) + c_{(4)}\sqrt{-g(x)} \\ & \cdot \left[(R(x))^2 - 4R^{j}_{\ i}(x) \cdot R^{i}_{\ j}(x) + R^{kl}{}_{ij}(x) \cdot R^{ij}{}_{kl}(x)\right]. \qquad \text{(A1.47)}\end{aligned}$$

Here, $c_{(1)}, c_{(2)}, c_{(3)}, c_{(4)}$ are four arbitrary constants with just *one* constraint $[c_{(1)}]^2 + [c_{(3)}]^2 + [c_{(4)}]^2 > 0$. The first term is simply proportional to the Einstein-Hilbert Lagrangian density, giving rise to the equations of motion of general relativity (2.161i). The second term gives rise to a cosmological constant (proportional to $c_{(2)}$) in the equations of motion (see (2.158)). The third term has been shown to yield trivial equations of motion in four dimensions [157]. Finally, each term in the last expression in (A1.47) produces *fourth-order equations*. However, *their combination yields second-order equations* and, in four dimensions, its integral is simply a topological number proportional to the Euler characteristic. This expression is commonly referred to as the *Gauss-Bonnet term*.

Outside this appendix, *we shall refrain from adding subscripts* (0) *to critical functions* $\phi^{(r+s)}_{(0)}(x)$ satisfying the Euler–Lagrange equations (A1.20i).

Appendix 2
Partial Differential Equations

In general relativity, a considerable amount of effort is expended in attempting to solve partial differential equations. Therefore, we are including here *an extremely brief review* of the subject. (For extensive study, we suggest [43, 77, 94].)

Consider *the simplest partial differential equation* known to mankind:

$$u = U(x, y), \quad (x, y) \in D \subset \mathbb{R}^2;$$

$$\partial_x u = \frac{\partial U(x, y)}{\partial x} = 0. \tag{A2.1}$$

The general solution of equation above is furnished by

$$u = U(x, y) = f(y), \quad y_1 < y < y_2. \tag{A2.2}$$

Here, $f(y)$ is *an arbitrary differentiable function*. Equation (A2.2) comprises of *infinitely many solutions*, and it is aptly called the *most general solution* of the partial differential equation (p.d.e.) (A2.1).

Remarks. (i) The function $f(y)$ in (A2.2) may be misconstrued as a function of *a single variable*. Strictly speaking, it is *a shortcut* notation for a function $F(c, y)$ of two variables such that *the first variable is restricted to a constant* c.

(ii) The function $f(y)$ can be chosen to be *discontinuous in the interval* (y_1, y_2), and still it will satisfy the p.d.e. (A2.1) *exactly*!

(iii) In case the domain of validity $D \subset \mathbb{R}^2$ in (A2.1) is *nonconvex*, the set of solutions in (A2.2) is *not the most general*. (See [112] for *counter-examples*.)

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Now, let us consider the two-dimensional (one space and one time) wave equation given by the second order, linear p.d.e.:

$$w = W(x,t), \quad (x,t) \in D \subset \mathbb{R}^2,$$
$$(x,0) \subset D \text{ for } x_1 < x < x_2\,;$$
$$\partial_x \partial_x w - \partial_t \partial_t w = \frac{\partial^2 W(x,t)}{\partial x^2} - \frac{\partial^2 W(x,t)}{\partial t^2} = 0. \tag{A2.3}$$

The most general solution of the equation above is furnished by $W(x,t) = f(x-t) + g(x+t)$. Here, $f, g \in C^2(D \subset \mathbb{R}^2; \mathbb{R})$, but otherwise *arbitrary*. Thus, there are infinitely many solutions of the p.d.e. (A2.3). *A class of general solutions* containing infinitely many solutions is provided by $W(x,t) = x - t + g(x+t),\ g \in C^2(D \subset \mathbb{R}^2; \mathbb{R})$. *A particular solution* $W(x,t) = x - t + \exp(x+t)$ solves *uniquely* the initial value problem $W(x,0) = x + e^x,\ \frac{\partial W(x,t)}{\partial t}_{|t=0} = e^x - 1$.

Let us employ *a doubly null coordinate system* $u = x - t,\ v = x + t$. (Compare with the Example 2.1.17.) The p.d.e. in (A2.3) goes over into

$$w = \widehat{W}(u,v) := W(x,y), \quad (u,v) \in \widehat{D} \subset \mathbb{R}^2\,;$$
$$\frac{\partial^2 \widehat{W}(u,v)}{\partial u \partial v} = 0. \tag{A2.4}$$

The most general solution is provided by

$$w = \widehat{W}(u,v) = f(u) + g(v), \quad \text{where } f, g \in C^2(\widehat{D} \subset \mathbb{R}^2; \mathbb{R}).$$

However, we notice that the condition of C^2 differentiability on $f(u)$ can be completely relaxed. *Discontinuous functions* $f(u)$ can yield *exact solutions* $\widehat{W}(u,v) = f(u) + g(v)$ for the p.d.e. (A2.4), *but not for the p.d.e.* $\frac{\partial^2 \widehat{W}(u,v)}{\partial v\, \partial u}$. Physically speaking, such solutions yield *shock waves not revealed in the usual* $x - t$ *coordinate system*. However, the definition of a solution can be generalized to a *weak solution* [43], which can extract rigorously discontinuous solutions in *any coordinate system*. Instead of the p.d.e. (A2.3), a weak solution has to satisfy *the integral condition*:

$$\int_D W(x,t)(\partial_x \partial_x f - \partial_t \partial_t f)\, dx\, dt = 0, \tag{A2.5}$$

for every C^∞-function f with *compact support* in D. (Such functions are called *distributions* [270].)

The second-order p.d.e. (A2.3) *is exactly equivalent to the following linear, first order system*:

$$\partial_x w = P_x(x,t),$$
$$\partial_t w = P_t(x,t),$$
$$\partial_x P_x - \partial_t P_t = 0,$$
$$\partial_t P_x - \partial_x P_t = 0. \tag{A2.6}$$

The last of the above equations is *the integrability condition*. (We have already discussed a system of $(N-1)$, first-order, linear p.d.e.s in (1.236).)

In a general system of p.d.e.s in $D \subset \mathbb{R}^N$, the highest derivatives that occur determine *the order of the system*. In a nonlinear system, the highest power (or exponent) of highest derivatives, is called *the degree of the system*.

An N-dimensional first-order p.d.e. is called a *linear equation* if it has the form:

$$\sum_{i=1}^{N} \Gamma^i(x)\,\partial_i w + \Gamma(x) w = h(x), \quad x \in D \subset \mathbb{R}^N. \tag{A2.7}$$

Here, $\Gamma^i(x)$, $\Gamma(x)$, and $h(x)$ are prescribed continuous functions. (For a *homogeneous p.d.e.*, $h(x) \equiv 0$.)

An N-dimensional first-order equation is called a *semilinear p.d.e.* provided it is given by

$$\sum_{i=1}^{N} \Gamma^i(x)\,\partial_i w + h(w, x) = 0. \tag{A2.8}$$

A first-order p.d.e. is called *quasilinear p.d.e.* if it has the form:

$$\sum_{i=1}^{N} \Gamma^i(w, x)\,\partial_i w + h(w, x) = 0. \tag{A2.9}$$

A second-order p.d.e. in an N-dimensional domain is called a *linear p.d.e.* provided it is furnished by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma^{ij}(x) \cdot \partial_i \partial_j w + \sum_{i=1}^{N} \Gamma^i(x)\,\partial_i w + \Gamma(x) w = h(x). \tag{A2.10}$$

(The linear equation (A2.10) is homogeneous provided $h(x) \equiv 0$.)

A second-order p.d.e. is called *a semilinear p.d.e.* provided it is expressible as

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma^{ij}(x) \cdot \partial_i \partial_j w + h(\partial_i w, w, x) = 0. \tag{A2.11}$$

A second-order p.d.e. is called *a quasilinear p.d.e.* if it has the form:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \Gamma^{ij}(\partial_k w, x) \cdot \partial_i \partial_j w + h(\partial_i w, w, x) = 0. \tag{A2.12}$$

A system of p.d.e.s in an N-dimensional domain is a collection of *coupled p.d.e.s*.

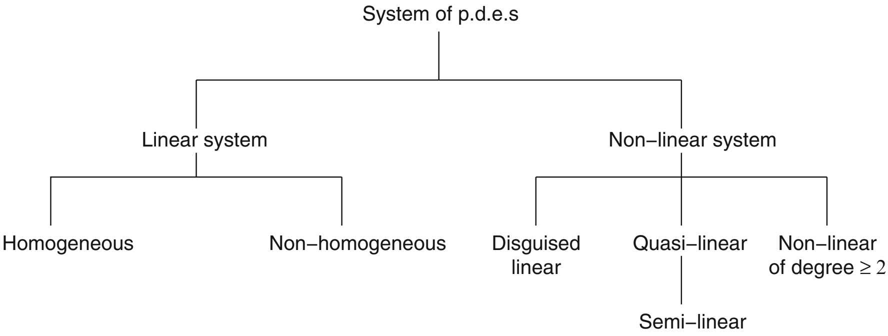

Fig. A2.1 Classification diagram of p.d.e.s

We classify systems of p.d.e.s in an N-dimensional domain in Fig. A2.1.

We shall now *sketch very briefly some of the solution procedures* for each of the type of systems mentioned in the figure.

Linear systems with constant coefficients are soluble usually by employing Fourier series, Fourier integrals, or Laplace integrals. Special classes of linear p.d.e.s admit *separable solutions* of the type $W(x) = \prod_{i=1}^{N} W_{(i)}(x^i)$.

Consider *a single quasilinear first-order p.d.e.* (A2.9). *Lagrange's solution method* involves $(N+1)$ *characteristic ordinary differential equations* (o.d.e.s):

$$\frac{\mathrm{d}x^1}{\Gamma^1(w,x)} = \frac{\mathrm{d}x^2}{\Gamma^2(w,x)} = \cdots = \frac{\mathrm{d}x^N}{\Gamma^N(w,x)} = -\frac{\mathrm{d}w}{h(w,x)} = \mathrm{d}t,$$

or,
$$\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t} = \Gamma^i(w,x)_{|..}\,, \qquad \frac{\mathrm{d}\mathcal{W}(t)}{\mathrm{d}t} = -h(w,x)_{|..}\,,$$

$$\mathcal{W}(t) := W\left[\mathcal{X}(t)\right] \equiv W[\mathcal{X}^1(t), \ldots, \mathcal{X}^N(t)]. \tag{A2.13}$$

The general solution of (A2.13) consists of $(N+1)$ arbitrary constants. Alternatively, the solution curve for (A2.13) may be furnished by the intersections of the following differentiable hypersurfaces:

$$\phi_{(i)}(x;w) = c_{(i)} = \text{const.},$$

$$\frac{\partial\left(\phi_{(1)}, \ldots, \phi_{(N)}\right)}{\partial(x^1, \ldots, x^N)} \neq 0. \tag{A2.14}$$

Consider *an arbitrary nonconstant function* $F \in C^1(D_{(\phi)} \subset \mathbb{R}^N; \mathbb{R})$. The most general solution of the quasilinear p.d.e. in (A2.9) is *implicitly provided by*

$$F\left[\phi_{(1)}(x;w), \ldots, \phi_{(N)}(x;w)\right] = c = \text{const.} \tag{A2.15}$$

Remark. Lagrange's method is valid for a linear or a semilinear p.d.e. also.

Example A2.1. Consider the following first-order, quasilinear p.d.e.:

$$w = W(x), \quad x \in D \subset \mathbb{R}^2;$$
$$w(\partial_1 w + \partial_2 w) - (w)^3 = 0.$$

The characteristic o.d.e.s (A2.13) reduce to

$$\frac{\mathrm{d}\mathcal{X}^1(t)}{\mathrm{d}t} = \mathcal{W}(t) = \frac{\mathrm{d}\mathcal{X}^2(t)}{\mathrm{d}t}, \quad \frac{\mathrm{d}\mathcal{W}(t)}{\mathrm{d}t} = [\mathcal{W}(t)]^3,$$
$$\frac{\mathrm{d}\mathcal{X}^1(t)}{\mathrm{d}t} - [\mathcal{W}(t)]^{-2} \cdot \frac{\mathrm{d}\mathcal{W}(t)}{\mathrm{d}t} \equiv 0.$$

Solving the above equations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{X}^1(t) - \mathcal{X}^2(t)\right] \equiv 0,$$
$$\phi_{(1)}(x; w) := x^1 - x^2 = c_{(1)},$$
$$\phi_{(2)}(x; w) := x^1 + (w)^{-1} = c_{(2)},$$
$$\frac{\partial\left(\phi_{(1)}, \phi_{(2)}\right)}{\partial(x^1, x^2)} = 1.$$

Therefore, by (A2.14), the most general solution of the p.d.e. is *implicitly provided by*

$$F\left(x^1 - x^2,\ x^1 + (w)^{-1}\right) = c = \text{const.}$$

(However, there is *an additional singular solution*, $W(x^1, x^2) \equiv 0$.) In case $\frac{\partial F(\phi_{(1)},\phi_{(2)})}{\partial \phi_{(2)}} \neq 0$, the above yields, by the implicit function theorem [32],

$$x^1 + (w)^{-1} = g(x^1 - x^2; c),$$

or, $$w = W(x^1, x^2) = \left[g(x^1 - x^2; c) - x^1\right]^{-1},$$

or, $$w(\partial_1 w + \partial_2 w) - w^3 = [\cdot\cdot]^{-3} \cdot [(-g' + 1) + g'] - [\cdot\cdot]^{-3} \equiv 0.$$

Here, g is *an arbitrary differentiable function*. The solution above furnishes a general class of *infinitely many explicit solutions*. (The domain of validity D must *avoid* the curve given by $g(x^1 - x^2; c) - x^1 = 0$.)

The quasilinear p.d.e. of this example is *a disguised linear equation*. By the transformation of the variable $s := -(w)^{-1}$, the p.d.e. is transformed into the linear equation:

$$\partial_1 s + \partial_2 s - 1 = 0.$$

The characteristic o.d.e.s are

$$\frac{\mathrm{d}\mathcal{X}^1(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{X}^2(t)}{\mathrm{d}t} = 1 = \frac{\mathrm{d}\mathcal{S}(t)}{\mathrm{d}t}\,.$$

The most general solution is implicitly provided by (A2.15) as

$$F\left(x^1 - x^2,\, x^1 - s\right) = F\left(x^1 - x^2,\, x^1 + (w)^{-1}\right) = c.$$

Here, F is an arbitrary, nonconstant function of class C^1. □

Now we consider *the nonlinear first-order p.d.e.*:

$$\begin{aligned}
&G\left(x^1, \ldots, x^N; w; p_{(1)}, \ldots, p_{(N)}\right) = 0,\\
&p_{(i)}{}_{|..} := \partial_i w = \frac{\partial W(x)}{\partial x^i}, \qquad D \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N.
\end{aligned} \tag{A2.16}$$

Here, G is assumed to be a nonconstant differentiable function such that at least one of $p_{(i)}$ occurs as $[p_{(i)}]^{n_{(i)}}$, $n_{(i)} \geq 2$. The associated $(2N+1)$ *characteristic o.d.e.s* are furnished by [43, 94]

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t} &= \frac{\partial G(\cdot\cdot)}{\partial p_{(i)}}\Big|_{..}\,,\\
\frac{\mathrm{d}\mathcal{P}_i(t)}{\mathrm{d}t} &= -\frac{\partial G(\cdot\cdot)}{\partial x^i}\Big|_{..} - \left[p_{(i)}\frac{\partial G(\cdot\cdot)}{\partial w}\right]_{|..}\,,\\
\frac{\mathrm{d}\mathcal{W}(t)}{\mathrm{d}t} &= \sum_{i=1}^{N}\left[p_{(i)}\frac{\partial G(\cdot\cdot)}{\partial p_{(i)}}\right]_{|..}\,.
\end{aligned} \tag{A2.17}$$

Note that along each of the characteristic curves,

$$\begin{aligned}
\frac{\mathrm{d}[G(\cdot\cdot)]}{\mathrm{d}t}\Big|_{..} &= \frac{\partial G(\cdot\cdot)}{\partial x^i}\Big|_{..}\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t} + \frac{\partial G(\cdot\cdot)}{\partial p_{(i)}}\Big|_{..}\frac{\mathrm{d}\mathcal{P}_i(t)}{\mathrm{d}t}\\
&\quad + \frac{\partial G(\cdot\cdot)}{\partial w}\Big|_{..}\frac{\mathrm{d}\mathcal{W}(t)}{\mathrm{d}t} \equiv 0.
\end{aligned} \tag{A2.18}$$

Therefore, the solution hypersurface for the p.d.e. (A2.16) is spanned by the congruence of characteristic curves.

Example A2.2. Consider the (truly nonlinear) p.d.e.:

$$\begin{aligned}
&G\left(x^1, x^2; w; p_{(1)}, p_{(2)}\right) := p_{(1)} \cdot p_{(2)} - w = 0,\\
&\partial_1 w \cdot \partial_2 w = w,\\
&W(0, x^2) := (x^2)^2.
\end{aligned}$$

(This is an initial value problem of a nonlinear p.d.e.) The characteristic o.d.e.s (A2.17) reduce to

$$\frac{\mathrm{d}\mathcal{X}^1(t)}{\mathrm{d}t} = \mathcal{P}_{(2)}(t), \qquad \frac{\mathrm{d}\mathcal{X}^2(t)}{\mathrm{d}t} = \mathcal{P}_{(1)}(t),$$

$$\frac{\mathrm{d}\mathcal{P}_{(1)}(t)}{\mathrm{d}t} = \mathcal{P}_{(1)}(t), \qquad \frac{\mathrm{d}\mathcal{P}_{(2)}(t)}{\mathrm{d}t} = \mathcal{P}_{(2)}(t), \qquad \frac{\mathrm{d}\mathcal{W}(t)}{\mathrm{d}t} = 2\mathcal{P}_{(1)}(t)\cdot\mathcal{P}_{(2)}(t).$$

The general solutions of the system of o.d.e.s are given by

$$\begin{aligned}
&\mathcal{P}_{(1)}(t) = c_{(1)}\mathrm{e}^t, \qquad \mathcal{P}_{(2)}(t) = c_{(2)}\mathrm{e}^t,\\
&\mathcal{X}^1(t) = c_{(2)}\mathrm{e}^t + c_{(3)}, \qquad \mathcal{X}^2(t) = c_{(1)}\mathrm{e}^t + c_{(4)},\\
&\mathcal{W}(t) = c_{(1)}c_{(2)}\mathrm{e}^{2t} + c_{(5)}.
\end{aligned}$$

Here, $c_{(1)}, \ldots, c_{(5)}$ are arbitrary constants of integration with $c_{(2)} \neq 0$. Inserting initial values $\mathcal{X}^1(0) = 0,\ \mathcal{W}(0) = W\left[\mathcal{X}^1(0),\ \mathcal{X}^2(0)\right] = \left[\mathcal{X}^2(0)\right]^2$, etc., we finally obtain

$$\left(\mathcal{X}^1(t), \mathcal{X}^2(t)\right) = \left(c_{(2)}\cdot(\mathrm{e}^t - 1), (4)^{-1}c_{(2)}\cdot(\mathrm{e}^t + 1)\right),$$

$$W\left[\mathcal{X}^1(t), \mathcal{X}^2(t)\right] = \mathcal{W}(t) = \left[(c_{(2)}/2)\mathrm{e}^t\right]^2 = \left[\frac{\mathcal{X}^1(t) + 4\mathcal{X}^2(t)}{4}\right]^2 .$$

From equation above, we conclude that

$$w = W(x^1, x^2) = \left[\frac{x^1 + 4x^2}{4}\right]^2$$

is the unique solution of the present p.d.e. under the imposed initial values. □

Example A2.3. Consider a domain of pseudo-Riemannian space–time and the relativistic *Hamilton–Jacobi equation* (for $m > 0$) furnished by

$$G\left(x^1, x^2, x^3, x^4; s; p_1, p_2, p_3, p_4\right) := (2m)^{-1}\left[g^{ij}(x)p_i p_j + m^2\right] = 0, \tag{A2.19i}$$

$$g^{ij}(x)\cdot\frac{\partial S(x)}{\partial x^i}\cdot\frac{\partial S(x)}{\partial x^j} + m^2 = 0 \tag{A2.19ii}$$

or

$$g^{ij}(x)\cdot\nabla_i S\cdot\nabla_j S + m^2 = 0. \tag{A2.19iii}$$

Compare (A2.19i) with the constraint of the mass shell in (2.33). (*Caution*: Here the parameter s is *not the proper time parameter* of (2.146i–iii).)

From (A2.19i), we deduce that

$$\frac{\partial G(\cdot\cdot)}{\partial p_i} = (m)^{-1} \cdot g^{ij}(x) \cdot p_j, \qquad \frac{\partial G(\cdot\cdot)}{\partial x^i} = (2m)^{-1} \cdot \left[\partial_i g^{kj}\right] \cdot p_k p_j,$$

$$\frac{\partial G(\cdot\cdot)}{\partial s} \equiv 0.$$

Therefore, the characteristic equations (A2.17) yield

$$\begin{aligned} \frac{d\mathcal{X}^i(t)}{dt} &= (m)^{-1} \cdot g^{ij}(x) \cdot p_j{}_{|..}, \\ \frac{d\mathcal{P}_i(t)}{dt} &= -(2m)^{-1} \cdot \left[\partial_i g^{jk}\right] \cdot p_j p_k{}_{|..}, \\ \frac{d\mathcal{S}(t)}{dt} &= m^{-1} \cdot g^{ij}(x) \cdot p_i \cdot p_j{}_{|..} = -m. \end{aligned} \tag{A2.20}$$

Identifying the parameter t with *the proper time parameter*, we have just re-derived *the relativistic canonical equations* (2.146i–iii) for *a timelike geodesic* in space–time. □

Example A2.4. Consider the (truly) nonlinear, first-order p.d.e.:

$$G\left(x^1, x^2; w; p_{(1)}, p_{(2)}\right) := p_{(1)} \cdot p_{(2)} - x^1 = 0,$$

or,

$$\partial_1 w \cdot \partial_2 w - x^1 = 0, \qquad x = (x^1,\, x^2) \in D \subset \mathbb{R}^2.$$

We shall solve the p.d.e. by *a different technique*. Note that the nonzero *Hessian* is furnished by

$$\det\left[\frac{\partial^2 G(\cdot\cdot)}{\partial p_{(i)} \partial p_{(j)}}\right] = -1 \neq 0.$$

We make *a Legendre transformation* (as we have done in (2.138)), by

$$p_{(i)} = \frac{\partial W(x)}{\partial x^i},$$

$$\frac{\partial}{\partial x^k}\left[x^i\, p_{(i)} - W(x)\right] = \delta^i{}_k\, p_{(i)} - \frac{\partial W(x)}{\partial x^k} \equiv 0,$$

$$\sigma\left(p_{(1)}, p_{(2)}\right) := x^i\, p_{(i)} - W(x^1, x^2),$$

$$\frac{\partial\sigma(\cdot\cdot)}{\partial p_{(i)}} = x^i,$$

$$W(x^1, x^2) = x^i\, p_{(i)} - \sigma(p_{(1)}, p_{(2)}).$$

The p.d.e. under consideration reduces to

$$\frac{\partial\sigma(\cdot\cdot)}{\partial p_{(1)}} = p_{(1)} \cdot p_{(2)}.$$

The most general solution of the above is provided by

$$\sigma(p_{(1)}, p_{(2)}) = (1/2)\left(p_{(1)}\right)^2 \cdot p_{(2)} + h\left(p_{(2)}\right).$$

Here, $h(p_{(2)})$ is *an arbitrary differentiable function*. Thus, the most general solution of the original p.d.e. is *implicitly given by*

$$x^1 = \frac{\partial\sigma(\cdot\cdot)}{\partial p_{(1)}} = p_{(1)} \cdot p_{(2)},$$

$$x^2 = \frac{\partial\sigma(\cdot\cdot)}{\partial p_{(2)}} = (1/2)\left(p_{(1)}\right)^2 + h'\left(p_{(2)}\right),$$

$$\begin{aligned} W(x^1, x^2) &= \frac{\partial\sigma(\cdot\cdot)}{\partial p_{(1)}} \cdot p_{(1)} + \frac{\partial\sigma(\cdot\cdot)}{\partial p_{(2)}} \cdot p_{(2)} - \sigma(\cdot\cdot) \\ &= \left(p_{(1)}\right)^2 \cdot p_{(2)} - h\left(p_{(2)}\right) + p_{(2)} \cdot h'\left(p_{(2)}\right). \end{aligned}$$

Choosing $h\left(p_{(2)}\right) \equiv 0$, we obtain *a particular solution* $W(x^1, x^2) = x^1 \cdot \sqrt{2x^2}$ for $x^2 > 0$. □

Remark. The technique of Legendre transformations is often employed in attempts at the canonical quantization of gravitational fields [10, 221, 246].

We shall now investigate second-order p.d.e.s in $\mathbb{R}^2$. We start with a second-order, semilinear p.d.e. (in a two-dimensional domain) expressed as

$$\begin{aligned} &\sum_{i=1}^{2}\sum_{j=1}^{2} \Gamma^{ij}(x)\, \partial_i \partial_j w + h(x; w; \partial_i w) = 0, \\ &\Gamma^{11}(x)\, \partial_1\partial_1 w + 2\Gamma^{12}(x)\, \partial_1\partial_2 w + \Gamma^{22}(x)\, \partial_2\partial_2 w + h(x; w; \partial_i w) = 0, \\ &\left[\Gamma^{11}(x)\right]^2 + \left[\Gamma^{12}(x)\right]^2 + \left[\Gamma^{22}(x)\right]^2 > 0 \\ &x \in D \subset \mathbb{R}^2. \end{aligned} \tag{A2.21}$$

The coefficients $\Gamma^{ij}(x)$ are of class C^1. The classification of the p.d.e. (A2.21) is based on the determinant

$$\Delta(x) := \det\left[\Gamma^{ij}(x)\right]. \tag{A2.22}$$

The p.d.e. (A2.21) is said to be

I. *Hyperbolic p.d.e.* in D, provided $\Delta(x) < 0$
II. *Elliptic p.d.e.* in D, provided $\Delta(x) > 0$
III. *Parabolic p.d.e.* in D, provided $\Delta(x) = 0$

Example A2.5. Consider *the Tricomi's p.d.e.* [43]:

$$\partial_1 \partial_1 w + x^1 \cdot \partial_2 \partial_2 w = 0.$$

The p.d.e. is *elliptic in the half-plane* $x^1 > 0$, and it is *hyperbolic in the other half-plane* $x^1 < 0$! □

Now, we would like to simplify the p.d.e. (A2.21) for the purpose of solving it. We explore a possible coordinate transformation of class C^2 and (1.107i) to derive the following equations:

$$\widehat{x}^i = \widehat{X}^i(x) = \widehat{X}^i(x^1, x^2), \tag{A2.23i}$$

$$w = W(x) = \widehat{W}(\widehat{x}), \tag{A2.23ii}$$

$$\widehat{\Gamma}^{ij}(\widehat{x}) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial \widehat{X}^i(x)}{\partial x^k} \cdot \frac{\partial \widehat{X}^j(x)}{\partial x^l} \cdot \Gamma^{kl}(x), \tag{A2.23iii}$$

$$\widehat{\Delta}(\widehat{x}) = \left[\frac{\partial(\widehat{x}^1, \widehat{x}^2)}{\partial(x^1, x^2)}\right]^2 \cdot \Delta(x), \tag{A2.23iv}$$

$$\operatorname{sgn}\left[\widehat{\Delta}(\hat{x})\right] = \operatorname{sgn}\left[\Delta(x)\right]. \tag{A2.23v}$$

By (A2.23ii) and (A2.23iv), it is clear that *the classification of a p.d.e. remains intact under a coordinate transformation*.

The p.d.e. (A2.21) goes over into

$$\widehat{\Gamma}^{11}(\widehat{x})\, \widehat{\partial}_1 \widehat{\partial}_1 w + 2\widehat{\Gamma}^{12}(\widehat{x})\, \widehat{\partial}_1 \widehat{\partial}_2 w + \widehat{\Gamma}^{22}(\widehat{x})\, \widehat{\partial}_2 \widehat{\partial}_2 w + \cdots = 0, \tag{A2.24i}$$

$$\widehat{\Gamma}^{11}(\widehat{x}) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial \widehat{X}^1(\cdot\cdot)}{\partial x^k} \cdot \frac{\partial \widehat{X}^1(\cdot\cdot)}{\partial x^l} \cdot \Gamma^{kl}(x), \tag{A2.24ii}$$

$$\widehat{\Gamma}^{22}(\widehat{x}) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial \widehat{X}^2(\cdot\cdot)}{\partial x^k} \cdot \frac{\partial \widehat{X}^2(\cdot\cdot)}{\partial x^l} \cdot \Gamma^{kl}(x), \tag{A2.24iii}$$

$$\widehat{\Gamma}^{12}(\widehat{x}) = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial \widehat{X}^1(\cdot\cdot)}{\partial x^k} \cdot \frac{\partial \widehat{X}^2(\cdot\cdot)}{\partial x^l} \cdot \Gamma^{kl}(x) \equiv \widehat{\Gamma}^{21}(\widehat{x}). \tag{A2.24iv}$$

The *characteristic surface* $\phi(x)$ over $D \subset \mathbb{R}^2$ of p.d.e. (A2.21) is defined by the first-order, nonlinear p.d.e. of degree two:

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \Gamma^{ij}(x) \cdot \frac{\partial \phi(x)}{\partial x^i} \frac{\partial \phi(x)}{\partial x^j} = 0. \tag{A2.25}$$

In case *both the coordinate functions* $\widehat{X}^1(x)$ and $\widehat{X}^2(x)$ satisfy the characteristic p.d.e. (A2.25), the original p.d.e. (A2.21) reduces to

$$\widehat{\Gamma}^{12}(\widehat{x}) \cdot \widehat{\partial}_1 \widehat{\partial}_2 w + \cdots = 0,$$

$$\text{or,} \quad \frac{\partial^2 \widehat{W}(\widehat{x}^1, \widehat{x}^2)}{\partial \widehat{x}^1 \partial \widehat{x}^2} + \cdots = 0. \tag{A2.26}$$

The *normal forms* of the various p.d.e.s are listed below:

I(i). *Hyperbolic p.d.e.*:

$$\frac{\partial^2 \widehat{W}(\widehat{x}^1, \widehat{x}^2)}{\partial \widehat{x}^1 \partial \widehat{x}^2} + \widehat{h}(\cdot\cdot) = 0. \tag{A2.27i}$$

I(ii). *Hyperbolic p.d.e.*:

$$\frac{\partial^2 \widehat{W}(\widehat{x}^1, \widehat{x}^2)}{(\partial \widehat{x}^1)^2} - \frac{\partial^2 \widehat{W}(\widehat{x}^1, \widehat{x}^2)}{(\partial \widehat{x}^2)^2} + \widehat{h}(\cdot\cdot) = 0. \tag{A2.27ii}$$

II. *Elliptic p.d.e.*:

$$\frac{\partial^2 \widehat{W}(\widehat{x}^1, \widehat{x}^2)}{(\partial \widehat{x}^1)^2} + \frac{\partial^2 \widehat{W}(\widehat{x}^1, \widehat{x}^2)}{(\partial \widehat{x}^2)^2} + \widehat{h}(\cdot\cdot) = 0. \tag{A2.27iii}$$

III. *Parabolic p.d.e.*:

$$\frac{\partial^2 \widehat{W}(\widehat{x}^1, \widehat{x}^2)}{(\partial \widehat{x}^1)^2} + \widehat{h}\left(\widehat{x}^1, \widehat{x}^2; w; \widehat{\partial}_1 w, \widehat{\partial}_2 w\right) = 0. \tag{A2.27iv}$$

Example A2.6. Consider the homogeneous, linear, second-order p.d.e.:

$$\begin{aligned}
&\partial_1 \partial_1 w - \left(c^{-2}\right) \cdot \partial_2 \partial_2 w + \partial_1 w - \left(c^{-1}\right) \cdot \partial_2 w = 0, \\
&\Delta(x) = -\left(c^{-2}\right) < 0.
\end{aligned} \tag{A2.28}$$

The parameter c is assumed to be nonzero, and the p.d.e. (A2.28) is obviously hyperbolic.

The characteristic surface of p.d.e. (A2.28) is governed by the nonlinear first-order p.d.e.:

$$(\partial_1 \phi)^2 - \left(c^{-2}\right) \cdot (\partial_2 \phi)^2 = 0, \tag{A2.29i}$$

$$G\left(x^1, x^2; \phi; p_{(1)}, p_{(2)}\right) := (1/2)\left[\left(p_{(1)}\right)^2 - \left(c^{-2}\right) \cdot \left(p_{(2)}\right)^2\right] = 0. \tag{A2.29ii}$$

The characteristic curves for equations above (which are *bicharacteristic curves* for the p.d.e. (A2.28)) are furnished by (A2.17) as

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{X}^1(t)}{\mathrm{d}t} = \frac{\partial G(\cdot\cdot)}{\partial p_{(1)}}\Big|_{\cdot\cdot} = p_{(1)}|_{\cdot\cdot}\,, \quad \frac{\mathrm{d}\mathcal{X}^2(t)}{\mathrm{d}t} = -c^{-2} \cdot p_{(2)}|_{\cdot\cdot}\,, \\
&\frac{\mathrm{d}\mathcal{P}_{(1)}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{P}_{(2)}(t)}{\mathrm{d}t} \equiv 0.
\end{aligned} \tag{A2.30}$$

The general solutions of (A2.30) and (A2.29ii) are given by

$$\begin{aligned}
&\mathcal{P}_{(1)}(t) = c_{(1)}, \quad \mathcal{P}_{(2)}(t) = c_{(2)}, \quad c_{(1)}^2 = \left(c^{-2}\right) \cdot c_{(2)}^2, \;\; c_{(1)} = \pm c^{-1} \cdot c_{(2)}, \\
&\mathcal{X}^1(t) = -c^{-1} c_{(2)} t + c_{(3)} = c\, \mathcal{X}^2(t) + \widehat{c}_{(3)}, \\
&\text{or else,} \quad \mathcal{X}^1(t) = c^{-1} c_{(2)} t + c_{(3)} = -c\, \mathcal{X}^2(t) + \widehat{c}_{(4)}, \\
&\text{thus,} \quad \phi_{(1)}(x) := x^1 - c x^2 = \widehat{c}_{(3)}, \quad \phi_{(2)}(x) := x^1 + c x^2 = \widehat{c}_{(4)}.
\end{aligned} \tag{A2.31}$$

Making a coordinate transformation

$$\begin{aligned}
&\widehat{x}^1 = \phi_{(1)}(x) = x^1 - c x^2, \quad \widehat{x}^2 = \phi_{(2)}(x) = x^1 + c x^2, \\
&w = W(x) = \widehat{W}\left(\widehat{x}\right),
\end{aligned}$$

the p.d.e. (A2.28) reduces to

$$\widehat{\partial}_1 \left[\widehat{\partial}_2 w + (1/2) w \right] = 0. \tag{A2.32}$$

The above second-order p.d.e. is equivalent to a pair of first-order equations

$$\begin{aligned} &\widehat{\partial}_2 w + (1/2) w = \widehat{g}\,(\widehat{x})\,, \\ &\widehat{\partial}_1 \widehat{g} = 0. \end{aligned} \tag{A2.33}$$

(Compare the above equations with (A2.6).) *The most general solution* of (A2.33) and (A2.28), putting $\widehat{g}(\widehat{x}^2) = \mathrm{e}^{-\widehat{x}^2/2} g'(\widehat{x}^2)$, is provided by

$$\begin{aligned} w = \widehat{W}\,(\widehat{x}) &= \mathrm{e}^{-\widehat{x}^2/2} g\left(\widehat{x}^2\right) + \mathrm{e}^{-(\widehat{x}^2/2)} \cdot f\left(\widehat{x}^1\right) \\ &= \mathrm{e}^{-\left(x^1 + c x^2\right)/2} g\left(x^1 + c x^2\right) + \mathrm{e}^{-(x^1 + c x^2)/2} \cdot f\left(x^1 - c x^2\right) = W(x). \end{aligned} \tag{A2.34}$$

Here, f and g are of class C^2 and *otherwise arbitrary*.

Consider now *a related nonhomogeneous, linear p.d.e.:*

$$\partial_1 \partial_1 w - \left(c^{-2}\right) \cdot \partial_2 \partial_2 w + \partial_1 w - \left(c^{-1}\right) \cdot \partial_2 w = 4. \tag{A2.35}$$

A particular solution of this equation is given by

$$w_{(p)} = 2\left(x^1 - c x^2\right).$$

Therefore, denoting solution (A2.34) of the homogeneous equation by $w_{(h)}$, *the most general solution* of (A2.35) is furnished by *the superposition*:

$$\begin{aligned} w = w_{(p)} + w_{(h)} &= 2\left(x^1 - c x^2\right) + \mathrm{e}^{-\left(x^1 + c x^2\right)/2} g\left(x^1 + c x^2\right) \\ &+ \mathrm{e}^{-(x^1 + c x^2)/2} \cdot f\left(x^1 - c x^2\right). \end{aligned}$$

□

Example A2.7. Consider *the elliptic Liouville equation* [191]:

$$\Delta w \equiv \nabla^2 w = \partial_1 \partial_1 w + \partial_2 \partial_2 w = 4 \mathrm{e}^{2w}, \quad x \in D \subset \mathbb{R}^2. \tag{A2.36}$$

The above is a semilinear, second-order, elliptic p.d.e. in *the normal form*.

The corresponding *characteristic surface*, as given by (A2.25), reduces to the p.d.e.

$$(\partial_1 \phi)^2 + (\partial_2 \phi)^2 = 0. \tag{A2.37}$$

Obviously, only real-valued solutions[1] of (A2.37) are provided by *constant-valued* functions $\phi(x)$! Therefore, nondegenerate, *characteristic curves*, analogous to the preceding hyperbolic example, *do not exist* in an elliptic case. However, there is an interesting device for treating elliptic equations by *complex conjugate coordinates* (which are *formally analogous to the characteristic coordinates* of hyperbolic cases). (See [191].) □

Let us introduce complex conjugate coordinates and complex derivatives by the following equations:

$$\begin{aligned}
&\zeta := x^1 + i x^2, \quad \overline{\zeta} := x^1 - i x^2, \\
&x^1 = \mathrm{Re}\,(\zeta) = (1/2)\left(\zeta + \overline{\zeta}\right), \quad x^2 = \mathrm{Im}\,(\zeta) = (1/2i)\left(\zeta - \overline{\zeta}\right), \\
&\widehat{F}\left(\zeta, \overline{\zeta}\right) := F\left(x^1, x^2\right), \\
&\partial_\zeta \widehat{F} = \frac{\partial}{\partial \zeta}\widehat{F}\left(\zeta, \overline{\zeta}\right) := (1/2)\left[\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right] F\left(x^1, x^2\right), \\
&\partial_{\overline{\zeta}} \widehat{F} = \frac{\partial}{\partial \overline{\zeta}}\widehat{F}\left(\zeta, \overline{\zeta}\right) := (1/2)\left[\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right] F\left(x^1, x^2\right), \\
&\Delta F \equiv \nabla^2 F = \left[\left(\frac{\partial}{\partial x^1}\right)^2 + \left(\frac{\partial}{\partial x^2}\right)^2\right] F\left(x^1, x^2\right) = 4 \cdot \frac{\partial^2 \widehat{F}\left(\zeta, \overline{\zeta}\right)}{\partial \overline{\zeta} \partial \zeta}\,.
\end{aligned} \tag{A2.38}$$

A *holomorphic function* $f(\zeta)$ and a *conjugate holomorphic function* $\overline{g(\zeta)}$ satisfy respectively the p.d.e.s:

$$\frac{\partial f(\zeta)}{\partial \overline{\zeta}} = (1/2)\left[\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right]\left[\mathrm{Re}\,\left(f\left(x^1 + i x^2\right)\right) + \mathrm{i}\,\mathrm{Im}\,\left(f\left(x^1 + i x^2\right)\right)\right] = 0, \tag{A2.39i}$$

$$\frac{\partial \overline{g(\zeta)}}{\partial \zeta} = (1/2)\left[\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right]\left[\mathrm{Re}\,\left(\overline{g\left(x^1 + i x^2\right)}\right) - \mathrm{i}\,\mathrm{Im}\left(\overline{g\left(x^1 + i x^2\right)}\right)\right] = 0. \tag{A2.39ii}$$

(The complex equation (A2.39i) is exactly equivalent to *the Cauchy-Riemann equations*.)

[1] A nonlinear (real) differential equation *of degree greater than one* may or may not admit any solution. The p.d.e. $(\partial_1\phi)^2 + (\partial_2\phi)^2 + \cosh[\phi(x)] = 0$ does not admit any real-valued solution function $\phi(x)$. However, the p.d.e. $(\partial_1\phi)^2 + (\partial_2\phi)^2 + \cosh[\phi(x) - \sqrt{2}] = 1$ admits *a single, particular solution* $\phi(x^1, x^2) = \sqrt{2}$.

Consider the real-valued *harmonic function* $h\left(x^1, x^2\right) = \widehat{h}(\zeta, \overline{\zeta})$ satisfying

$$\begin{aligned} &\Delta h \equiv \nabla^2 h = \partial_1 \partial_1 h + \partial_2 \partial_2 h = 0, \\ \text{or,} \quad &\partial_\zeta \partial_{\overline{\zeta}} \widehat{h} = 0. \end{aligned} \tag{A2.40}$$

It can be proved [191] that *the most general solution* of (A2.40) is given by

$$\begin{aligned} \widehat{h}\left(\zeta, \overline{\zeta}\right) &= (1/2)\left[f(\zeta) + \overline{f(\zeta)}\right], \\ h\left(x^1, x^2\right) &= \operatorname{Re}\left[f\left(x^1 + i x^2\right)\right]. \end{aligned} \tag{A2.41}$$

Here, $f(\zeta)$ is *an arbitrary holomorphic function* in the domain of consideration.

Now let us go back to the Liouville equation (A2.36) again. *The most general solution* of this equation is provided by [191]

$$\begin{aligned} w = W\left(x^1, x^2\right) &= \widehat{W}\left(\zeta, \overline{\zeta}\right) = \log\left[\frac{|f'(\zeta)|}{1 - |f(\zeta)|^2}\right], \\ D &:= \left\{\zeta \in \mathbb{C} : \ f'(\zeta) \neq 0, |f(\zeta)| < 1\right\}. \end{aligned} \tag{A2.42}$$

Here, $f'(\zeta) \neq 0$, and $|f(\zeta)| < 1$, but the holomorphic function $f(\zeta)$ is *otherwise arbitrary*.

Now, we consider *second-order*, *semilinear p.d.e.s* in an N-dimensional domain. Such p.d.e.s need *not be tensor field equations*. (That is why we suspended the summation convention in preceding discussions!) Let us now revert back to the usual summation convention for tensorial, as well as *nontensorial* p.d.e.s. Recall that the semilinear, second-order p.d.e. (A2.11), reinstating the summation convention, can be expressed as

$$\Gamma^{ij}(x) \cdot \partial_i \partial_j w + h\,(x; w; \partial_i w) = 0\,; \qquad x \in D \subset \mathbb{R}^N. \tag{A2.43}$$

Here, the coefficients $\Gamma^{ji}(x) \equiv \Gamma^{ij}(x)$ are continuous functions such that *at least one of them is nonzero*. Therefore, the symmetric matrix $\left[\Gamma^{ij}(x_0)\right]$ has N real (usual) eigenvalues so that at least *one* of them is nonzero. The eigenvalues can be always arranged in the order:

$$\begin{aligned} &\lambda_{(1)}(x_0) > 0, \ldots, \lambda_{(p)}(x_0) > 0\,; \quad \lambda_{(p+1)}(x_0) < 0, \ldots, \lambda_{(p+n)}(x_0) < 0\,; \\ &\lambda_{(p+n+1)}(x_0) = \cdots = \lambda_{(p+n+\nu)}(x_0) = 0. \end{aligned} \tag{A2.44}$$

Here, $p + n + \nu = N$, $r := p + n$ is the *rank* of the matrix $\left[\Gamma^{ij}(x_0)\right]$, and $\nu = N - r$ is *the nullity of the matrix*. Thus, $\nu > 0$ if and only if the matrix $\left[\Gamma^{ij}(x_0)\right]$

is *singular*. (Without loss of generality, we can always arrange $p - n \geq 0$. Compare with the metric equation (1.90).) According to *Sylvester's law* of inertia [177, 240], the numbers p, n, ν *remain invariant* under a (real) coordinate transformation of the type (A2.23iii) in (A2.43). Thus, it is logical to choose the classification of the general semilinear p.d.e. (A2.43) according to the following criteria:

I.	*Elliptic p.d.e.*	if $\nu = 0 = n, \quad p > 0$
II.	*Hyperbolic p.d.e.*	if $\nu = 0, \quad n = 1, \quad p = N - 1$
III.	*Ultrahyperbolic p.d.e.*	if $\nu = 0, \quad 1 < n < p < N - 1$
IV.	*Parabolic p.d.e.*	if $\nu > 0$

Note that the $(N - 1)$-dimensional *characteristic hypersurface* of equation (A2.43) is governed by the first-order, nonlinear p.d.e.:

$$\Gamma^{ij}(x) \cdot \partial_i \phi \cdot \partial_j \phi = 0. \tag{A2.45}$$

Example A2.8. Consider the following semilinear, second-order p.d.e.:

$$\partial_1 \partial_1 w + \partial_2 \partial_2 w - \partial_3 \partial_3 w + \left[\frac{2w}{1 - w^2} \right] \cdot \left[(\partial_1 w)^2 + (\partial_2 w)^2 - (\partial_3 w)^2 \right] = 0\,;$$

$$x \in D \subset \mathbb{R}^3\,; \quad |w| < 1. \tag{A2.46}$$

In this case, $N = 3, \ p = 2, \ n = 1, \ \nu = 0,$ and $r = 3$. Therefore, the p.d.e. is *hyperbolic* in the domain of consideration. *The corresponding characteristic surface* is furnished by

$$(\partial_1 \phi)^2 + (\partial_2 \phi)^2 - (\partial_3 \phi)^2 = 0.$$

The bicharacteristic curves are (null) straight lines in the three-dimensional domain.

Now, we make the following transformation:

$$g(w) := \frac{1}{2} \ln \left| \frac{1 + w}{1 - w} \right|, \quad w = \tanh(g), \quad |w| < 1,$$

$$W\left(x^1, x^2, x^3\right) := \tanh\left[g\left(x^1, x^2, x^3\right)\right].$$

The p.d.e. (A2.46) reduces to *the linear p.d.e.*:

$$\partial_1 \partial_1 g + \partial_2 \partial_2 g - \partial_3 \partial_3 g = 0. \tag{A2.47}$$

Thus, the original p.d.e. (A2.46) is *a disguised linear p.d.e.*

A class of general solutions of (A2.47) and (A2.46) is provided by

$$g\left(x^1, x^2, x^3\right) = \int\limits_{\mathbb{R}^2} f\left(k_1 x^1 + k_2 x^2 + \sqrt{k_1^2 + k_2^2} \cdot x^3\right) \mathrm{d}k_1 \, \mathrm{d}k_2,$$

$$W\left(x^1, x^2, x^3\right) = \tanh\left[\int\limits_{\mathbb{R}^2} f\left(k_1 x^1 + k_2 x^2 + \sqrt{k_1^2 + k_2^2} \cdot x^3\right) \mathrm{d}k_1 \, \mathrm{d}k_2\right].$$

Here, f is a twice-differentiable function such that *the integrals converge uniformly*. The function f is *otherwise arbitrary*. (Uniform convergences are needed to commute differentiations and the integration [32].) □

Let us go back to the semilinear, second-order p.d.e. (A2.43). It is equivalent to *the first-order system*:

$$\begin{aligned} &\partial_i w = p_{(i)}, \\ &\partial_j p_{(i)} = \partial_i p_{(j)}, \\ &\Gamma^{ij}(x)\, \partial_i p_{(j)} + h\big(x; w; p_{(i)}\big) = 0. \end{aligned} \tag{A2.48}$$

(Compare (A2.6) and (A2.33).)

In general, a system of semilinear second-order p.d.e.s (and possibly another system of first-order p.d.e.s) can be expressed equivalently as *a single first order system*:

$$\Gamma^{i}_{AB}(x)\, \partial_i w^B + h_A(x; w) = 0. \tag{A2.49}$$

Here, the summation convention is *also carried on capital Roman indices* which take values from $\{1, 2, \ldots, d\}$.

We construct the *characteristic matrix* [43] by the following:

$$\underset{d \times d}{[\Gamma(\cdot\cdot)]} \equiv [\Gamma_{AB}(x)] := \left[\Gamma^{i}_{AB}(x)\, \partial_i \phi\right]. \tag{A2.50}$$

The characteristic hypersurface is furnished by the first-order p.d.e.:

$$\det[\Gamma(\cdot\cdot)] = \det\left[\Gamma^{i}_{AB}(x)\, \partial_i \phi\right] = 0. \tag{A2.51}$$

Example A2.9. Consider Maxwell's equations for electromagnetic fields in flat space–time. (See (2.54i–iv).) With the notation

$$\left(w^1, w^2, w^3\right) := \left(E^1, E^2, E^3\right), \quad \left(w^4, w^5, w^6\right) := \left(H^1, H^2, H^3\right),$$

six of the Maxwell's equations can be expressed as

$$\begin{aligned}
\partial_3 w^5 - \partial_2 w^6 + \partial_4 w^1 &= 0,\\
\partial_1 w^6 - \partial_3 w^4 + \partial_4 w^2 &= 0,\\
\partial_2 w^4 - \partial_1 w^5 + \partial_4 w^3 &= 0,\\
\partial_3 w^2 - \partial_2 w^3 - \partial_4 w^4 &= 0,\\
\partial_1 w^3 - \partial_3 w^1 - \partial_4 w^5 &= 0,\\
\partial_2 w^1 - \partial_1 w^2 - \partial_4 w^6 &= 0.
\end{aligned} \tag{A2.52}$$

By (A2.50) and (A2.51), we derive

$$\underset{6\times 6}{[\Gamma]} = \begin{bmatrix}
\partial_4\phi & 0 & 0 & 0 & \partial_3\phi & -\partial_2\phi\\
0 & \partial_4\phi & 0 & -\partial_3\phi & 0 & \partial_1\phi\\
0 & 0 & \partial_4\phi & \partial_2\phi & -\partial_1\phi & 0\\
0 & \partial_3\phi & -\partial_2\phi & -\partial_4\phi & 0 & 0\\
-\partial_3\phi & 0 & \partial_1\phi & 0 & -\partial_4\phi & 0\\
\partial_2\phi & -\partial_1\phi & 0 & 0 & 0 & -\partial_4\phi
\end{bmatrix}, \tag{A2.53i}$$

$$\det[\Gamma] = -(\partial_4\phi)^2 \cdot \left[(\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 - (\partial_4\phi)^2\right]^2. \tag{A2.53ii}$$

It is clear that for $\partial_4\phi \neq 0$, the system of p.d.e.s in (A2.52) is *hyperbolic*. □

(We have obtained classifications of p.d.e.s in general relativity on page 202.)

Now, we shall very briefly touch upon the topic of *nonunique (or chaotic) solutions*. Consider a system of o.d.e.s:

$$\begin{aligned}
&\frac{d\mathcal{X}^i(t)}{dt} = F^i(t;x)_{|x^j=\mathcal{X}^j(t)},\\
&D := \left\{(t;x) \in \mathbb{R}\times\mathbb{R}^N : |t-t_{(0)}| < A,\ \|x - x_{(0)}\| < B\right\},\\
&\|x - x_{(0)}\|^2 := \delta_{ij}\cdot\left[x^i - x^i_{(0)}\right]\cdot\left[x^j - x^j_{(0)}\right].
\end{aligned} \tag{A2.54}$$

(Compare the equation above with (1.75).) Suppose that the functions $F^i(\cdot\cdot)$ are continuous over $\overline{D} := D \cup \partial D$. However, the entries of *the Jacobian matrix*

$$\begin{bmatrix}
\partial_1 F^1 & \partial_2 F^1 & \cdots & \partial_N F^1\\
\vdots & \vdots & & \vdots\\
\partial_1 F^N & \partial_2 F^N & \cdots & \partial_N F^N
\end{bmatrix}$$

are continuous in D, but *not continuous on* ∂D. Then, the initial value problem

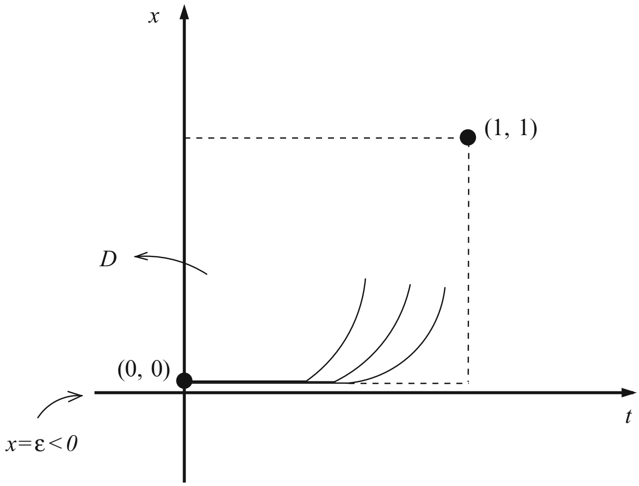

Fig. A2.2 Graphs of nonunique solutions

$$\mathcal{X}^i\left(t_{(0)}\right) = x^i_{(0)}, \quad \frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\Big|_{t=t_0} = c^i_{(0)},$$

has *nonunique* (*or chaotic*) *solutions*[2] [151].

Example A2.10. Consider *the single o.d.e.* :

$$\frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t} = F(t;x)_{|..} := \sqrt{\mathcal{X}(t)}\,,$$

$$D = \{(t,x) \in \mathbb{R}\times\mathbb{R} : 0 < t < 1,\ 0 < x < 1\}.$$

The function $F(t;x) = \sqrt{x}$ is continuous over $\overline{D}$. However, $\frac{\partial F(\cdot\cdot)}{\partial x} = \frac{1}{2\sqrt{x}}$ *is not continuous on* ∂D. Therefore, the solution for the specific I.V.P.

$$\mathcal{X}(0) = 0, \quad \frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t}\Big|_{t=0} = 0$$

has (infinitely) many answers! We furnish these solutions explicitly in the following:

$$\mathcal{X}(t) := \begin{cases} 0 \;\text{ for }\; t \le k < 1\,; \\ (1/4)(t-k)^2 \;\text{ for }\; 0 < k < t. \end{cases}$$

The functions $\mathcal{X}(t)$ are shown graphically in the Fig. A2.2.

[2]Moreover, *bifurcation points* will satisfy simultaneous equations: $F^i(t;x) = 0$, and $\det\left[\frac{\partial F^i(\cdot\cdot)}{\partial x^j}\right] = 0$. (See [138].)

Appendix 3
Canonical Forms of Matrices

We shall start this appendix with some very simple models. Consider 2×2 matrices with real entries (or elements). (See [5].)

Example A3.1. Consider the symmetric matrix $[S] := \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. The characteristic polynomial is given by $p(\lambda) := \det[S - \lambda I] = \lambda^2 - 6\lambda + 8$. Therefore, *the usual eigenvalues* are $\lambda_{(1)} = 4$ and $\lambda_{(2)} = 2$. The corresponding eigenvectors are

$$[\vec{\mathbf{e}}_{(1)}] = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad [\vec{\mathbf{e}}_{(2)}] = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

These column vectors are orthonormal in *the usual Euclidean sense*. We construct a matrix $[P]$ (with help of eigenvectors) as

$$[P] := \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

By the similarity transformation

$$[P]^{-1}[S][P] = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix},$$

and the symmetric matrix is *diagonalized*. □

Example A3.2. Consider the symmetric matrix $[S] := \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. The usual eigenvalue $\lambda_{(1)} = \sqrt{2}$ has *the multiplicity* 2. Moreover, *every nonzero* 2×1 *column vector is an eigenvector*. □

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4,

Example A3.3. Consider the matrix $[M] := \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$. The (usual) characteristic polynomial is furnished by $p(\lambda) = (\lambda + 1)^2$. The eigenvalue $\lambda_{(1)} = -1$ is of *multiplicity* 2. The eigenvectors are of the form $\begin{bmatrix} t \\ t \end{bmatrix}$, $t \neq 0$. Therefore, there is *only one eigendirection* and the matrix is *nondiagonalizable.* (This matrix has a nonlinear or *nonsimple elementary divisor* $E_{(2)}(\lambda) = (\lambda + 1)^2$ according to (A3.3).) However, consider the matrix

$$[P] := \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}.$$

By the similarity transformation

$$[P]^{-1}[M][P] = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus, the matrix $[M]$ is reducible to an *upper triangular form*. Moreover, the triangular form is *a diagonal matrix plus a nilpotent matrix*. □

Example A3.4. Consider *the antisymmetric matrix* $[A] := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The characteristic polynomial $p(\lambda) = \lambda^2 + 1$. Thus, there exist *no real eigenvalues*. Consequently, there are *no (real) eigenvectors*. However, extending into the (algebraically closed) complex field, we have *two complex-conjugate eigenvalues* $\lambda_{(1)} = i$, $\lambda_{(2)} = -i = \overline{\lambda}_{(1)}$. Consider the complex matrix $[P] := \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. The similarity transformation $[P]^{-1}[A][P]$ reduces $[A]$ into the complex diagonal matrix $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. □

Example A3.5. Consider the 2×2 matrix generated by the Lorentz metric (in Example 1.3.2) $[D] := \left[d_{(i)(j)}\right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. *The Lorentz-invariant characteristic polynomial* of a matrix $[M]$ is defined by $p^{\#}(\lambda) := \det[M - \lambda D]$. *The invariant eigenvalues* are furnished by the roots of $p^{\#}(\lambda) = 0$. (Compare with (1.213).) A symmetric matrix can be expressed as $[S] = \begin{bmatrix} 2a & a+b \\ a+b & 2b \end{bmatrix}$. *The corresponding invariant eigenvalues* are given by $p^{\#}(\lambda) = -[\lambda - (a-b)]^2 = 0$. There exists one (real) invariant eigenvalue $\lambda_{(1)} = (a - b)$ of multiplicity 2. There exists one eigendirection $\begin{bmatrix} t \\ -t \end{bmatrix}$ $(t \neq 0)$. The Lorentz-invariant separations (of (1.86)) are provided by $[t, -t]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} t \\ -t \end{bmatrix} \equiv 0$. Thus, the eigendirection is along *a double* *null vector*. Moreover, this matrix is associated with nonsimple elementary divisor $E_{(2)}(\lambda) = (\lambda - a + b)^2$ according to (A3.3). □

Example A3.6. Consider *the symmetric matrix* $[S] := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (This happens to be one of the Pauli matrices.) *The usual eigenvalues* are given by $\lambda_{(1)} = 1$, $\lambda_{(2)} = -1$. The Lorentz-invariant eigenvalues are furnished by $p^{\#}(\lambda) = -(\lambda^2 + 1) = 0$. Therefore, we have *complex-conjugate* invariant eigenvalues $\lambda_{(1)} = i,\ \lambda_{(2)} = -i = \overline{\lambda}_{(1)}$. Therefore, *a symmetric matrix*, with real entries, can have *complex, Lorentz-invariant eigenvalues!* □

Now we shall investigate the canonical classification of an $N \times N$ matrix $[M]$ with real entries $M_{(a)(b)}$. The (usual) characteristic polynomial is furnished by

$$p(\lambda) := \det\left[M_{(a)(b)} - \lambda\,\delta_{(a)(b)}\right] = (-\lambda)^N + c_{(1)}\lambda^{N-1} + \cdots + c_{(N)} = 0,$$

$$\text{or,} \qquad \det\left[\delta^{(a)(c)} M_{(c)(b)} - \lambda\,\delta^{(a)}{}_{(b)}\right] = p(\lambda) = 0.$$

The equation above admits $2s \geq 0$ complex roots and $N - 2s \geq 0$ real roots [21].

Now consider *the same real* $N \times N$ *matrix* $[M]$, together with an $N \times N$ metric matrix $[D]$ of (1.90) (*which is not positive definite*). The corresponding *invariant, characteristic polynomial equation* is provided by

$$p^{\#}(\lambda) := \det\left[M_{(a)(b)} - \lambda\,d_{(a)(b)}\right] = (-\lambda)^N + c^{\#}_{(1)}\lambda^{N-1} + \cdots + c^{\#}_{(N)} = 0,$$

$$\text{or,} \qquad \det\left[d^{(a)(c)} M_{(c)(b)} - \lambda\,\delta^{(a)}{}_{(b)}\right] = 0.$$

Suppose that the above equation admits $2s^{\#} > 0$ complex roots and $N - 2s^{\#} > 0$ real roots. *These invariant roots must differ from the roots of the usual characteristic equation* $p(\lambda) = 0$. (Compare with Example A3.6.) However, the subsequent general theorems involving *any real* $N \times N$ *matrix* $[M] = [M^i{}_j]$ *apply to both distinct cases* $p(\lambda) = 0$ *and* $p^{\#}(\lambda) = 0$. In the case where there exist *only real roots* of $\det[M^i{}_j - \lambda\,\delta^i{}_j] = 0$ as eigenvalues, the *Jordan decomposition theorem* is stated as follows:

Theorem A3.7. Let $[M] = [M^i{}_j]$ be an $N \times N$ matrix with real entries and real eigenvalues as roots of $\det[M^i{}_j - \lambda\,\delta^i{}_j] = 0$. Then the matrix can be transformed by a similarity transformation into the following block diagonal form:

$$[M]_{(J)} = \begin{bmatrix} \boxed{A_{(1)}} & & & 0 \\ & \boxed{A_{(2)}} & & \\ & & \ddots & \\ 0 & & & \boxed{A_{(k)}} \end{bmatrix}. \tag{A3.1}$$

Here, $\left[A_{(l)}\right]$, for $l \in \{1, \ldots, k\}$, is an $n_{(l)} \times n_{(l)}$ matrix given by

$$\left[A_{(l)}\right] := \begin{bmatrix} \boxed{J_{(l)}^{(1)}} & & & 0 \\ & \boxed{J_{(l)}^{(2)}} & & \\ & & \ddots & \\ 0 & & & \boxed{J_{(l)}^{(q_l)}} \end{bmatrix}, \tag{A3.2i}$$

$$\underset{n_{(l)}^{(i)} \times n_{(l)}^{(i)}}{\left[\; J_{(l)}^{(i)} \;\right]} := \begin{bmatrix} \lambda_{(l)} & 1 & 0 \cdots 0 \\ 0 & \lambda_{(l)} & 1 \cdots 0 \\ \vdots & \vdots \ddots & \ddots \;\; \vdots \\ \vdots & \vdots & \ddots \;\; 1 \\ 0 & 0 & 0 \;\; \lambda_{(l)} \end{bmatrix}, \tag{A3.2ii}$$

$$i \in \{1, \ldots, q_l\}, \quad \sum_{i=1}^{q_l} n_{(l)}^{(i)} =: n_{(l)}, \quad \sum_{l=1}^{k} n_{(l)} = N.$$

Proof of the above theorem is available in [133, 177].

Remarks. (i) It is usually assumed that $n_{(l)}^{(1)} \geq n_{(l)}^{(2)} \geq \cdots \geq n_{(l)}^{(q_l)} \geq 1$.

(ii) In the case $n_{(l)}^{(i)} = 1$, the corresponding 1×1 matrix $\left[J_{(l)}^{(i)}\right]$ has *the single entry* $\lambda_{(l)}$. It stands for *a diagonal element*.

(iii) Some authors write $\left[J_{(l)}^{(i)}\right]$ as *a lower triangular form.*

(iv) In relativity theory, after solving $p^{\#}(\lambda) = 0$, many authors use the inequalities in the reversed order as

$$n_{(l)}^{(q_l)} \geq \cdots \geq n_{(l)}^{(2)} \geq n_{(l)}^{(1)} \geq 1.$$

(v) The *Segre characteristic* of the matrix $[M]_{(J)}$ is denoted ([90, 210]) by the symbol $[(n_{(1)}^{(1)}, n_{(1)}^{(2)}, \ldots), (n_{(2)}^{(1)}, n_{(2)}^{(2)}, \ldots), \ldots]$.

The matrices $[M]_{(J)}$ in (A3.1) and (A3.2i,ii) define a hierarchy of *elementary divisors* [113]. It is provided by the following string of equations:

$$E_{(N)}(\lambda) := \left(\lambda - \lambda_{(1)}\right)^{n_{(1)}^{(1)}} \cdot \left(\lambda - \lambda_{(2)}\right)^{n_{(2)}^{(1)}} \cdots ,$$

$$E_{(N-1)}(\lambda) := \left(\lambda - \lambda_{(1)}\right)^{n_{(1)}^{(2)}} \cdot \left(\lambda - \lambda_{(2)}\right)^{n_{(2)}^{(2)}} \cdots ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

(A3.3)

Example A3.8. We shall discuss an example from the general theory of relativity. Consider the orthonormal components $t_{(a)(b)} := T_{(a)(b)}(x_0) = t_{(b)(a)}$ of the energy–momentum–stress tensor in (2.45) and (2.161ii) (at a particular event $x_0 \in D \subset \mathbb{R}^4$). *The usual eigenvalues* of the symmetric matrix $\left[t_{(a)(b)}\right]$ are all real, and the matrix is always diagonalizable. However, *the Lorentz invariant eigenvalues* of $\left[t_{(a)(b)}\right]$ are, physically speaking, much more relevant. Since $\det[t_{(a)(b)} - \lambda\, d_{(a)(b)}] = 0 \Leftrightarrow \det[d^{(a)(c)} t_{(c)(b)} - \lambda\, \delta^{(a)}_{(b)}] = 0$, these invariant eigenvalues are identical with the usual eigenvalues of the 4×4 matrix $[\Theta] := \left[d^{(a)(c)} t_{(c)(b)}\right]$ *which need not be symmetric*. (See Examples A3.5 and A3.6.) *Assuming that the invariant eigenvalues are all real*, we classify the allowable types of canonical forms of $[\Theta]$ in the following:

I: Type-I$_{(a)}$:

$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)} & 0 \\ 0 & 0 & 0 & -\lambda_{(4)} \end{bmatrix}. \tag{A3.4}$$

Here, if $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}, -\lambda_{(4)}$ are *all distinct*. The Segre characteristic in (A3.4) is specified by $[1, 1, 1, 1]$.

Type-I$_{(b)}$: If $\lambda_{(1)} = \lambda_{(2)}$ and $\lambda_{(1)}, \lambda_{(3)}, -\lambda_{(4)}$ are distinct. The Segre characteristic is specified by $[(1, 1),\ 1, 1]$.

Type-I$_{(c)}$: If $\lambda_{(1)} = \lambda_{(2)},\ \lambda_{(3)} = -\lambda_{(4)}$ and $\lambda_{(1)} \neq \lambda_{(3)}$. The Segre characteristic is provided by $[(1, 1), (1, 1)]$.

Type-I$_{(d)}$: If $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)}$ and $\lambda_{(1)} \neq -\lambda_{(4)}$. The Segre characteristic is furnished by $[(1, 1, 1), 1]$.

Type-I$_{(e)}$: If $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)} = -\lambda_{(4)}$. The Segre characteristic is specified by $[(1, 1, 1, 1)]$.

II: Type-II$_{(a)}$:

$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)} & 1 \\ 0 & 0 & 0 & \lambda_{(3)} \end{bmatrix}. \tag{A3.5}$$

If $\lambda_{(1)}, \lambda_{(2)}$, and $\lambda_{(3)}$ are all distinct. The Segre characteristic is given by $[1, 1, 2]$.

Type-II$_{(b)}$: If $\lambda_{(1)} = \lambda_{(2)}$ and $\lambda_{(1)} \neq \lambda_{(3)}$ in (A3.5). The Segre characteristic is provided by $[(1, 1), 2]$.

Type-II$_{(c)}$: Here, $\lambda_{(1)} \neq \lambda_{(2)}$ and $\lambda_{(2)} = \lambda_{(3)}$. The Segre characteristic is furnished by $[1, (1, 2)]$.

Type-II$_{(d)}$: If $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)}$. The Segre characteristic is specified by $[(1, 1, 2)]$.

III: Type-III$_{(a)}$:

$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 1 & 0 & 0 \\ 0 & \lambda_{(1)} & 0 & 0 \\ 0 & 0 & \lambda_{(2)} & 1 \\ 0 & 0 & 0 & \lambda_{(2)} \end{bmatrix}. \tag{A3.6}$$

Here, we assume $\lambda_{(1)} \neq \lambda_{(2)}$. The Segre characteristic is $[2, 2]$.

Type-III$_{(b)}$: If $\lambda_{(1)} = \lambda_{(2)}$ in (A3.6). The Segre characteristic is provided by $[(2, 2)]$.

IV: Type-IV$_{(a)}$:

$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 1 & 0 \\ 0 & 0 & \lambda_{(2)} & 1 \\ 0 & 0 & 0 & \lambda_{(2)} \end{bmatrix}. \tag{A3.7}$$

Here, we assume that $\lambda_{(1)} \neq \lambda_{(2)}$. The Segre characteristic is $[1, 3]$.

Type-IV$_{(b)}$: Here, we assume that $\lambda_{(1)} = \lambda_{(2)}$. The corresponding Segre characteristic is $[(1, 3)]$.

V:
$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 1 & 0 & 0 \\ 0 & \lambda_{(1)} & 1 & 0 \\ 0 & 0 & \lambda_{(1)} & 1 \\ 0 & 0 & 0 & \lambda_{(1)} \end{bmatrix}. \tag{A3.8}$$

The Segre characteristic is furnished by [4]. The corresponding eigenvectors are *quadrupole null vectors along a single null eigendirection*.

Since *the signature of the Lorentz metric* $[d_{(a)(b)}]$ is $+2$ and $t_{(b)(a)} = t_{(a)(b)}$, the 14 cases in the equations above in this example *contain all possible Segre characteristics* of the energy–momentum–stress tensor (*with real Lorentz-invariant eigenvalues*). (See [90, 210].) Furthermore, it can be noted that an eigenvector associated with *a nonsimple, elementary divisor is null.* □

Now we shall deal with a real $N \times N$ matrix $[M] = [M^i{}_j]$ which possesses only $2s = N$ *complex conjugate roots* of the characteristic equation. (Thus, in this case, there are *no real roots*.) The following theorem elaborates the canonical form of such a matrix:

Theorem A3.9. Let $[M]$ be a real $N \times N$ matrix such that the (usual) characteristic equation admits $2s = N$ complex conjugate roots $\lambda_{(l)} = a_{(l)} + i b_{(l)}$, $\overline{\lambda}_{(l)} = a_{(l)} - i b_{(l)}$, $l \in \{1, \ldots, s\}$. Then, the Jordan canonical form of the matrix is furnished by:

$$[M]_{(J)} = \begin{bmatrix} \boxed{B_{(1)}} & & & \\ & \boxed{B_{(2)}} & & \\ & & \ddots & \\ & & & \boxed{B_{(r)}} \end{bmatrix},$$

$$\underset{(2n_{(l)} \times 2n_{(l)})}{\left[B_{(l)}\right]} := \begin{bmatrix} \boxed{C^{(1)}_{(l)}} & & & \\ & \boxed{C^{(2)}_{(l)}} & & \\ & & \ddots & \\ & & & \boxed{C^{(s_l)}_{(l)}} \end{bmatrix},$$

$$\underset{\left(2n_{(l)}^{(i)} \times 2n_{(l)}^{(i)}\right)}{\left[C_{(l)}^{(i)}\right]} := \begin{bmatrix} \boxed{\begin{matrix} a_{(l)} & b_{(l)} \\ -b_{(l)} & a_{(l)} \end{matrix}} & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & & & \\ & & \boxed{\begin{matrix} a_{(l)} & b_{(l)} \\ -b_{(l)} & a_{(l)} \end{matrix}} & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \\ & & & \ddots & \ddots \\ & & & \ddots & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \\ & & & & \boxed{\begin{matrix} a_{(l)} & b_{(l)} \\ -b_{(l)} & a_{(l)} \end{matrix}} \end{bmatrix},$$

$$i \in \{1, \ldots, s_l\}, \quad \sum_{i=1}^{s_l} n_{(l)}^{(i)} =: n_{(l)}, \quad \sum_{l=1}^{r} n_{(l)} = s = (N/2). \tag{A3.9}$$

The proof of this theorem can be found in [133, 177]. The Segre characteristic of the matrix $[M]_{(J)}$, in (A3.9) is given by

$$\left[\left(n_{(1)}^{(1)}, n_{(1)}^{(2)}, \ldots\right), \overline{\left(n_{(1)}^{(1)}, n_{(1)}^{(2)}, \ldots\right)}; \left(n_{(2)}^{(1)}, n_{(2)}^{(2)}, \ldots\right), \overline{\left(n_{(2)}^{(1)}, n_{(2)}^{(2)}, \ldots\right)}; \ldots\right].$$

Finally, we consider an $N \times N$ real matrix $[M]$ which admits *both real and complex conjugate (usual) eigenvalues*.

The canonical or block diagonal form is provided by

$$[M]_{(J)} = \left[\begin{array}{cccc|cccc} \boxed{A_{(1)}} & & & & & & & \\ & \boxed{A_{(2)}} & & & & & & \\ & & \ddots & & & & & \\ & & & \boxed{A_{(k)}} & & & & \\ \hline & & & & \boxed{B_{(1)}} & & & \\ & & & & & \boxed{B_{(2)}} & & \\ & & & & & & \ddots & \\ & & & & & & & \boxed{B_{(r)}} \end{array}\right]. \tag{A3.10}$$

Here, matrices $\left[A_{(l)}\right]$ and $\left[B_{(l)}\right]$ are furnished by (A3.2i) and (A3.9), respectively.

Example A3.10. Consider the 4×4 matrix

$$[M] := \begin{bmatrix} -1/2 & -1/2 & 0 & 0 \\ 1/2 & -3/2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

The usual characteristic polynomial equation is furnished by

$$p(\lambda) = (\lambda + 1)^2 \cdot (\lambda^2 - 4\lambda + 5) = (\lambda + 1)^2 \cdot (\lambda - 2 - i) \cdot (\lambda - 2 + i) = 0.$$

Thus, the distinct eigenvalues are

$$\lambda_{(1)} = -1, \quad a_{(1)} + i b_{(1)} = 2 + i, \quad a_{(1)} - i b_{(1)} = 2 - i$$

with multiplicities 2, 1, and 1, respectively. Therefore, the Jordan canonical form is given by the block diagonal form

$$[M]_{(J)} = \left[\begin{array}{c|c} A_{(1)} & \\ \hline & B_{(1)} \end{array}\right] =: \left[\begin{array}{cc|cc} -1 & 1 & & \\ 0 & -1 & & \\ \hline & & 2 & 1 \\ & & -1 & 2 \end{array}\right].$$

The Segre characteristic is $\left[2; 1, \overline{1}\right]$. The elementary divisor for this matrix is $E_{(4)}(\lambda) = (\lambda + 1)^2 \cdot (\lambda - 2 - i) \cdot (\lambda - 2 + i)$. □

Example A3.11. Consider a domain of the space–time manifold and orthonormal components of the energy–momentum–stress tensor field provided by [126]:

$$\left[T_{(a)(b)}(x)\right] := \begin{bmatrix} \Lambda_{(1)}(x) & 0 & 0 & 0 \\ 0 & \Lambda_{(2)}(x) & 0 & 0 \\ 0 & 0 & \Sigma(x) & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{A3.11i}$$

$$\left[t_{(a)(b)}\right] := \left[T_{(a)(b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \sigma & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{A3.11ii}$$

$$[\Theta] := \left[d^{(a)(c)}t_{(c)(b)}\right] = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \sigma & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \neq [\Theta]^T . \tag{A3.11iii}$$

Four eigenvalues (invariant values for the matrix in (A3.11ii) or usual values of the matrix in (A3.11iii)) are furnished by $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)} = (1/2)\left[\sigma + \sqrt{\sigma^2 - 4}\right]$, and $\lambda_{(4)} = (1/2)\left[\sigma - \sqrt{\sigma^2 - 4}\right]$. Here, $\lambda_{(1)}$ and $\lambda_{(2)}$ are real and $\lambda_{(3)}, \lambda_{(4)}$ are real provided $|\sigma| \geq 2$. However, the eigenvectors in this case can be spacelike! Therefore, for the case $\lambda_{(1)} \neq \lambda_{(2)}$ and $|\sigma| > 2$, we have the Segre characteristic $[1, 1, 1, 1]$. In the case $\lambda_{(1)} \neq \lambda_{(2)}$ and $|\sigma| = 2$, the Segre characteristic is $[1, 1, (1, 1)]$. But for $\lambda_{(1)} \neq \lambda_{(2)}$ and $|\sigma| < 2$, the Segre characteristic is $\left[1, 1; 1, \overline{1}\right]$. Moreover, there are corresponding Segre characteristics for the coincidence $\lambda_{(1)} = \lambda_{(2)}$. This example illustrates that the energy–momentum–stress tensor of *an exotic material can change Segre characteristics from domain to domain!* □

Example A3.12. In Example A3.8, we considered *invariant eigenvalues* of the symmetric energy–momentum–stress matrix $[t_{(a)(b)}] := [T_{(a)(b)}(x_0)]$. In that example, we restricted the classification only to the various cases of *real, invariant eigenvalues*. In this example, we extend the mathematical investigation to cases of *complex, invariant eigenvalues* of $[t_{(a)(b)}]$. (*Recall that these eigenvalues are identical to the usual eigenvalues of the nonsymmetric matrix* $[\Theta] = [\Theta^{(a)}{}_{(b)}] := [d^{(a)(c)}t_{(c)(b)}]$.)

The Jordan canonical forms of various types follow from (A3.9). The type VI is furnished below:

VI: Type-VI$_{(a)}$:

$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & a_{(1)} & b_{(1)} \\ 0 & 0 & -b_{(1)} & a_{(1)} \end{bmatrix} . \tag{A3.12}$$

Here, we assume that real eigenvalues $\lambda_{(1)} \neq \lambda_{(2)}$ and the real number $b_{(1)} \neq 0$. The corresponding Segre characteristic is provided by $[1,\ 1;\ 1,\ \overline{1}]$.
Type-VI$_{(b)}$: If $\lambda_{(1)} = \lambda_{(2)}$ and $b_{(1)} \neq 0$, the Segre characteristic is given by $[(1,\ 1);\ 1,\ \overline{1}]$.

VII: Type-VII$_{(a)}$:

$$[\Theta]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 1 & 0 & 0 \\ 0 & \lambda_{(1)} & 0 & 0 \\ 0 & 0 & a_{(1)} & b_{(1)} \\ 0 & 0 & -b_{(1)} & a_{(1)} \end{bmatrix} . \tag{A3.13}$$

Here, we take the real number $b_{(1)} \neq 0$. The Segre characteristic is furnished by $[2; 1, \overline{1}]$.

VIII: Type-VIII$_{(a)}$:

$$[\Theta]_{(J)} = \begin{bmatrix} a_{(1)} & b_{(1)} & 0 & 0 \\ -b_{(1)} & a_{(1)} & 0 & 0 \\ 0 & 0 & a_{(2)} & b_{(2)} \\ 0 & 0 & -b_{(2)} & a_{(2)} \end{bmatrix}. \tag{A3.14}$$

Here, we have taken $b_{(1)} \neq 0$, $b_{(2)} \neq b_{(1)}$, and $b_{(2)} \neq 0$. The Segre characteristic is $[1, \overline{1}; 1, \overline{1}]$.

Type-VIII$_{(b)}$: Here, we assume that real numbers $a_{(2)} = a_{(1)}$ and $b_{(2)} = b_{(1)} \neq 0$. The corresponding Segre characteristic is $[(1, \overline{1}; 1, \overline{1})]$. □

Every type of $[\Theta]_{(J)}$ *exhibited in the previous example* (involving energy–momentum–stress tensor components $T_{(a)(b)}(x_0)$) *violates energy conditions in (2.190)–(2.192)*. Therefore, in the usual conditions of the macroscopic universe, these types of energy–momentum–stress tensor components are *usually deemed unphysical*. However, in the quantum realm, it may be possible to violate energy conditions. An example of this is an effect known as the *Casimir effect* where quantum fields subject to certain boundary conditions may violate energy conditions (see, e.g., [236, 254] and references therein). Exotic solutions in general relativity usually possess energy condition violation, as may be seen in Appendix 6 and Sect. 5.3, as well as [68] where energy condition violation is shown to result inside a black hole in a certain class of gravitational collapse models.

Appendix 4
Conformally Flat Space–Times and "the Fifth Force"

Let us recall the criterion for conformal flatness of a Riemannian (or a pseudo-Riemannian) manifold discussed in Theorem 1.3.33. We reiterate the same theorem with different notations in the following theorem.

Theorem A4.1. Let $D \subset \mathbb{R}^N$ (with $N > 3$) be a domain corresponding to an open subset of an N-dimensional Riemannian (or a pseudo-Riemannian) manifold of differentiability class C^3. Then, the domain $D \subset \mathbb{R}^N$ is conformally flat if and only if the tensor field $C^i{}_{jkl}(x)\frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l = \mathbf{O}^{\cdot}{}_{\cdots}(x)$ in D.

Proof of the above theorem is provided in [56, 90, 171].

Consider a conformally flat domain of a pseudo-Riemannian (or a Riemannian) manifold M_N. It is endowed with a coordinate chart such that

$$g_{ij}(x) = \exp\left[2\Lambda(x)\right] \cdot d_{ij}. \tag{A4.1}$$

We consider another chart, intersecting the preceding one, such that

$$\widehat{g}_{ij}(\widehat{x}) = \exp\left[2\widehat{\mu}(\widehat{x})\right] \cdot d_{ij}. \tag{A4.2}$$

Let us consider the following set of coordinate transformations furnished by

$$\widehat{x}^i = \lambda \cdot x^i, \quad \lambda \neq 0, \tag{A4.3i}$$

$$\widehat{x}^i = x^i + c^i, \tag{A4.3ii}$$

$$\widehat{x}^i = l^i{}_j\, x^i, \quad d_{ij} \cdot l^i{}_k \cdot l^j{}_m = d_{jm}, \tag{A4.3iii}$$

$$\widehat{x}^i = \left[x^i / \left(d_{jk} \cdot x^j x^k\right)\right], \tag{A4.3iv}$$

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$$\widehat{x}^i = \left[x^i - b^i\left(x_l x^l\right)\right] \cdot \left[1 - 2(b_j \cdot x^j) + (b_k \cdot b^k) \cdot (x_m \cdot x^m)\right]^{-1}. \qquad \text{(A4.3v)}$$

Here, λ, c^i, $l^i{}_j$, and b^i are constant-valued parameters. Moreover, $x_i := d_{ij} x^j$, and $x_i x^i$ is *assumed to be nonzero*.

The set of coordinate transformations presented above constitutes *a group*. It is called the *conformal group* $\mathcal{C}(p, n; \mathbb{R})$ (with $p + n = N$). This group involves $(1/2) \cdot (N+1) \cdot (N+2)$ independent parameters. The following theorem provides the importance of this group in regard to conformally flat manifolds.

Theorem A4.2. A conformally flat metric $\mathbf{g}_{..}(x) = \exp[2\nu(x)] \cdot \mathbf{d}_{..}$ goes over into another conformally flat metric $\widehat{\mathbf{g}}_{..}(\widehat{x}) = \exp[2\widehat{\mu}(\widehat{x})] \cdot \mathbf{d}_{..}$ for $N \geq 3$ if and only if the coordinate transformation belongs to the conformal group $\mathcal{C}(p, n; \mathbb{R})$.

Partial proof of the above theorem is provided in [56].

We shall now state *Willmore's theorem* [266] for a conformally flat domain of a pseudo-Riemannian (or a Riemannian) manifold.

Theorem A4.3. Let a domain $D \subset \mathbb{R}^N$ with $N > 3$, corresponding to a domain of a pseudo-Riemannian (or a Riemannian) manifold of differentiability class C^3, admit a tensor field $\mathbf{S}_{..}(x)$ of differentiability class C^2. Moreover, let the following tensor field equations hold:

$$\begin{aligned} R_{hijk}(x) = {} & g_{hk}(x) \cdot S_{ij}(x) + g_{ij}(x) \cdot S_{hk}(x) \\ & - g_{hj}(x) \cdot S_{ik}(x) - g_{ik}(x) \cdot S_{hj}(x). \end{aligned} \qquad \text{(A4.4)}$$

(i) The conformal tensor $\mathbf{C}^{\cdot}{}_{...}(x) = \mathbf{O}^{\cdot}{}_{...}(x)$ in D. (ii) Moreover, $S_{ji}(x) \equiv S_{ij}(x)$ in D.

Proof of the above theorem is skipped here. However, the next theorem due to Das [51] *incorporates the proof of the preceding theorem.*

Theorem A4.4. Let the curvature tensor $\mathbf{R}^{\cdot}{}_{...}(x)$ of a pseudo-Riemannian (or a Riemannian) manifold of differentiability class C^3 and dimension $N > 3$ admit a twice-differentiable tensor field $\mathbf{T}_{..}(x)$ in the domain of consideration. Moreover, let the following tensor field equations hold:

$$\begin{aligned} R_{lijk}(x) = {} & \kappa \cdot (N-2)^{-1} \cdot \Big\{\big[g_{lj}(x) \cdot T_{ik}(x) + g_{ik}(x) \cdot T_{lj}(x) \\ & - g_{lk}(x) \cdot T_{ij}(x) - g_{ij}(x) \cdot T_{lk}(x)\big] \\ & + 2(N-1)^{-1} \cdot T(x) \cdot \big[g_{lk}(x) \cdot g_{ij}(x) - g_{lj}(x) \cdot g_{ik}(x)\big]\Big\}, \\ T(x) := {} & g^{ij}(x) \cdot T_{ij}(x). \end{aligned} \qquad \text{(A4.5)}$$

(Here, κ is an arbitrary non-zero constant). Then,

$$\text{(i)}\quad G_{ij}(x) = -\kappa \cdot T_{ij}(x), \tag{A4.6i}$$

$$\text{(ii)}\quad T_{ji}(x) \equiv T_{ij}(x)\,; \quad \nabla_i T^{ij} = 0, \tag{A4.6ii}$$

$$\text{(iii)}\quad \nabla_k T_{ij} - \nabla_j T_{ik} + (N-1)^{-1} \cdot \left[g_{ik}(x) \cdot \nabla_j T - g_{ij}(x) \cdot \nabla_k T \right] = 0, \tag{A4.6iii}$$

$$\text{(iv)}\quad C^{\,l}_{\ ijk}(x) \equiv 0, \tag{A4.6iv}$$

(v) Moreover,

$$R_{ijk}(x) := (N-2)^{-1} \cdot (N-3) \cdot \nabla_l C^{\,l}_{\ ijk} = 0. \tag{A4.6v}$$

Proof. (i) By the single contraction in (A4.5), it follows that

$$R_{ij}(x) = \kappa \cdot \left[(N-2)^{-1} \cdot T(x) \cdot g_{ij}(x) - T_{ij}(x) \right], \tag{A4.7i}$$

$$R(x) = 2\kappa \cdot (N-2)^{-1} \cdot T(x), \tag{A4.7ii}$$

$$G_{ij}(x) = -\kappa \cdot T_{ij}(x). \tag{A4.7iii}$$

(ii) By the algebraic symmetry of Einstein's tensor and the differential identities $\nabla_i G^{ij} \equiv 0$, (A4.6ii) follow.
(iii) By *the first contracted Bianchi's identities* (1.150i), (A4.6ii), and (A4.5), the proof of (A4.6iii) follows.
(iv) Substituting $R_{lijk}(x)$ from (A4.5), using (A4.7i,ii), (1.169i) yields (A4.6iv).
(v) The covariant differentiation of $C^{\,l}_{\ ijk}(x)$ in (1.169i), together with (A4.6iv), implies (A4.6v).

■

In case we would like to apply the preceding theorem to general relativity, we must choose $p = 3$, $n = 1$, and $N = 4$ for the pseudo-Riemannian space–time manifold. The theorem, that is suited to general relativity is stated and proved in the following:

Theorem A4.5. Let the curvature tensor $\mathbf{R}^{\cdot}_{\ \cdots}(x)$ of the pseudo-Riemannian space–time manifold of differentiability class C^3 admit a twice differentiable tensor field $\mathbf{T}_{\cdot\cdot}(x)$ in the domain $D \subset \mathbb{R}^4$ of consideration. Moreover, let the following tensor field equation hold:

$$R_{lijk}(x) = (\kappa/2)\cdot\Big\{\big[g_{lj}(x)\cdot T_{ik}(x) + g_{ik}(x)\cdot T_{lj}(x) - g_{lk}(x)\cdot T_{ij}(x) - g_{ij}(x)\cdot T_{lk}(x)\big] + (2/3)\cdot T(x)\cdot\big[g_{lk}(x)\cdot g_{ij}(x) - g_{lj}(x)\cdot g_{ik}(x)\big]\Big\}. \quad \text{(A4.8)}$$

Then, the following implications for $g_{ij} = \phi^2\cdot d_{ij}$, $\Box\phi := d^{ij}\partial_i\partial_j\phi$, and $\widetilde{T}_{(0)} := d^{ij}\widetilde{T}_{ij}$ hold true:

(i) $\partial_i\partial_j\phi - d_{ij}\cdot\Box\phi - 2\phi^{-1}\left[\partial_i\phi\cdot\partial_j\phi - (1/4)\, d_{ij} d^{kl}\partial_k\phi\cdot\partial_l\phi\right]$

$$= -(\kappa/2)\cdot T_{ij}(x)\cdot\phi(x). \quad \text{(A4.9)}$$

(ii) $R(x) = 6\phi^{-3}\cdot\Box\phi.$ (A4.10)

(iii) $R_{ij}(x) = 2\phi^{-1}\cdot\partial_i\partial_j\phi + d_{ij}\phi^{-1}\cdot\Box\phi - 4\phi^{-2}\cdot\partial_i\phi\cdot\partial_j\phi$ (A4.11)

$$+\,\phi^{-2} d_{ij} d^{kl}\partial_k\phi\cdot\partial_l\phi = -\kappa\,\widetilde{T}_{ij} =: -\kappa\left[T_{ij} - (1/2)\, d_{ij} d^{kl} T_{kl}\right].$$

(iv) $\Box\phi + (\kappa/6)\cdot\widetilde{T}_{(0)}(x)\cdot\phi(x) = 0.$ (A4.12)

(v) Field equations (A4.11) are covariant under the conformal group $\mathcal{C}(3,1;\mathbb{R})$ involving 15 independent parameters.

Proof. (i) The proof follows by noting that $g_{ij}(x) = [\phi(x)]^2\cdot d_{ij}$ and inserting this metric into the usual expression for $R_{ij}(x)$. Then, using (A4.7iii) and (1.149ii), the following equations emerge:

$$G_{ij}(x) = 2\phi^{-1}\cdot\partial_i\partial_j\phi - 4\phi^{-2}\cdot\partial_i\phi\cdot\partial_j\phi - d_{ij}\cdot\left[2\phi^{-1}\cdot\Box\phi - \phi^{-2}\cdot d^{kl}\partial_k\phi\cdot\partial_l\phi\right] = -\kappa\cdot T_{ij}(x). \quad \text{(A4.13)}$$

Next, from Einstein’s field equations (A4.7iii), (A4.9) is derived.

(ii) Double contraction of (A4.13) leads the (A4.10).

(iii) By the equality $R_{ij}(x) = G_{ij}(x) + (1/2)\cdot g_{ij}(x)\cdot R(x)$, and (A4.9), (A4.10) and definitions

$$\widetilde{T}_{ij}(x) := T_{ij}(x) - (1/2) g_{ij}(x)\cdot T^k_{\ k}(x), \quad \text{(A4.14i)}$$

$$\widetilde{T}_{(0)}(x) := d^{ij}\widetilde{T}_{ij}(x) = -T_{(0)}(x) =: -d^{ij}\, T_{ij}(x), \quad \text{(A4.14ii)}$$

(A4.11) is deduced.

(iv) Double contractions of equations (A4.11) (with d^{ij}) lead to (A4.12).

(v) This statement can be proved by using Theorem A4.2.

■

Remarks. (i) It is clear from (A4.5) and (A4.8) that $T_{ij}(x) \equiv 0$ implies that $R_{lijk}(x) \equiv 0$. In other words, mathematically, the support of $\mathbf{R}_{....}(x) \equiv$ the support of $\mathbf{T}_{..}(x)$ in conformally flat space–times. Therefore, the class of gravitational phenomena governed by field equation (A4.8) or (A4.9) is *nontrivial only in the presence of material sources*. Outside material sources, this class of gravitational forces just "switches off"! That is why we interpret gravitational force, arising from field equation (A4.9) as the "*fifth force*".[1] (See [101, 216].) This effect can equivalently be seen utilizing (1.169i) and field equation (2.163i). The trace of the latter implies that $R(x) = 0$ and $R_{ij}(x) = 0$ when $T_{ij}(x) = 0$ holds. Then, condition (A4.6iv) for conformally flat space–times for $N > 3$, when used in (1.169i), implies that $R_{ijkl}(x) = 0$ whenever $T_{ij}(x) = 0$.

(ii) Field equation (A4.9), governing *the "fifth force", does not admit general covariance*. However, this equation *does admit covariance under the 15-parameter conformal group* $\mathcal{C}(3, 1; \mathbb{R})$.

(iii) The system of field equations (A4.9) is *highly overdetermined*. (See [52].)

In spite of the fact that the system of field equations is overdetermined, many exact solutions of the system *do exist*. In fact, many of these exact solutions turn out to be *extremely important* for understanding of the cosmological universe. We shall furnish some of these exact solutions in the following examples.

Example A4.6. In this example, the following choice is made:

$$T_{ij}(x) := -(3K_0/\kappa) \cdot g_{ij}(x),$$
$$T(x) = -12 \cdot \kappa^{-1} \cdot K_0. \tag{A4.15}$$

(Here, K_0 is a constant.)

Substituting (A4.15) into (A4.8), we obtain

$$R_{hijk}(x) = K_0 \cdot \left[g_{hj}(x) \cdot g_{ki}(x) - g_{hk}(x) \cdot g_{ji}(x)\right]. \tag{A4.16}$$

Therefore, by (1.164i), the metric is that of a *space–time of constant curvature*. Such a space–time is also called the de Sitter universe (for $K_0 > 0$) and anti-de Sitter universe (for $K_0 < 0$). (See #1 of Exercise 6.1. Also see [126].)

By Theorem 1.3.30, we can transform this metric locally to the conformally flat form:

$$\mathrm{d}s^2 = \left[1 + (K_0/4) \cdot \left(d_{kl} \cdot x^k x^l\right)\right]^{-2} \cdot d_{ij} \cdot \mathrm{d}x^i\, \mathrm{d}x^j. \tag{A4.17}$$

By the equation $g_{ij}(x) = [\phi(x)]^2 \cdot d_{ij}$, we obtain in this example

$$\phi(x) = \left[1 + (K_0/4) \cdot \left(d_{ij} \cdot x^i x^j\right)\right]^{-1}. \tag{A4.18}$$

Thus, an explicit expression for $\phi(x)$ is furnished. □

[1] It should be stressed here that this is nomenclature only. This phenomenon is still a special class of gravitational effect, and *not due to some new force*.

The next example deals with F–L–R–W cosmological metrics of (6.7i) and (6.41). *These are extremely important metrics* in regard to the proper understanding of the cosmological universe.

Example A4.7. In this example, we choose the energy–momentum–stress tensor for a perfect fluid from Theorem 6.1.3 and (6.46). It is provided by

$$T_{ij}(x) = \left[\rho(x) + p(x)\right] \cdot U_i(x) \cdot U_j(x) + p(x) \cdot g_{ij}(x), \tag{A4.19i}$$

$$U^i(x) \cdot U_i(x) \equiv -1. \tag{A4.19ii}$$

Recall the F–L–R–W metric from (6.41) as

$$\begin{aligned} &\mathrm{d}s^2 = \left[a(x^4)\right]^2 \cdot \left[1 + (k_0/4) \cdot \delta_{\alpha\beta} \cdot x^\alpha x^\beta\right]^{-2} \cdot \delta_{\mu\nu} \cdot \mathrm{d}x^\mu \mathrm{d}x^\nu - \left(\mathrm{d}x^4\right)^2, \\ &D := \left\{x : \mathbf{x} \in \mathbf{D} \subset \mathbb{R}^3;\ x^4 > 0\right\}, \\ &k_0 \in \{0, 1, -1\}. \end{aligned} \tag{A4.20}$$

The corresponding fluid velocity components, in the comoving frame (derivable from field equations $\mathcal{E}^\alpha_{\ 4}(\cdot\cdot) = 0$) are given by

$$U^\alpha(x) \equiv 0, \quad U^4(x) \equiv 1. \tag{A4.21}$$

Einstein's field equations from (6.47i–iii) are furnished by

$$\mathcal{E}^1_{\ 1}(\cdot\cdot) \equiv \mathcal{E}^2_{\ 2}(\cdot\cdot) \equiv \mathcal{E}^3_{\ 3}(\cdot\cdot) = \left[\frac{2\ddot{a}}{a} + \frac{(\dot{a})^2 + k_0}{a^2}\right] + \kappa\, p(x^4) = 0, \tag{A4.22i}$$

$$\mathcal{E}^4_{\ 4}(\cdot\cdot) = 3\left[\frac{(\dot{a})^2 + k_0}{a^2}\right] - \kappa\, \rho(x^4) = 0, \tag{A4.22ii}$$

$$\mathcal{T}^4(\cdot\cdot) = \dot{\rho} + 3\left(\frac{\dot{a}}{a}\right) \cdot (\rho + p) = 0, \tag{A4.22iii}$$

$$\dot{a} := \frac{\mathrm{d}a(x^4)}{\mathrm{d}x^4}, \quad \text{etc.} \tag{A4.22iv}$$

In the case of $k_0 = 0$, the metric from (A4.20) goes over into

$$\mathrm{d}s^2 = [a(x^4)]^2 \cdot \left[\delta_{\alpha\beta} \cdot \mathrm{d}x^\alpha \mathrm{d}x^\beta\right] - (\mathrm{d}x^4)^2. \tag{A4.23}$$

(Consult Example 7.2.2.) Now we make a transformation,

$$\begin{aligned}
\widehat{x}^{\alpha} &= x^{\alpha}, \\
\widehat{x}^{4} &= \int \frac{\mathrm{d}x^4}{a(x^4)}, \\
\widehat{a}(\widehat{x}^4) &:= a(x^4) > 0.
\end{aligned} \tag{A4.24}$$

The metric in (A4.23) yields the conformally flat form:

$$\begin{aligned}
\mathrm{d}s^2 &= [\widehat{a}(\widehat{x}^4)]^2 \cdot \left[\delta_{\alpha\beta} \cdot \mathrm{d}\widehat{x}^{\alpha}\mathrm{d}\widehat{x}^{\beta} - (\mathrm{d}\widehat{x}^4)^2\right], \\
&=: [\widehat{\phi}(\widehat{x})]^2 \cdot d_{ij}\, \mathrm{d}\widehat{x}^{i} \cdot \mathrm{d}\widehat{x}^{j}.
\end{aligned} \tag{A4.25}$$

Therefore, the corresponding $\widehat{\phi}(\widehat{x})$ field is explicitly furnished by

$$\widehat{\phi}(\widehat{x}) = \widehat{a}(\widehat{x}^4). \tag{A4.26}$$

For the case of $k_0 = 1$, (6.7i), (A4.20), and the answer to # 2i of Exercise 6.1 yield

$$\mathrm{d}s^2 = \left[\widehat{a}(\widehat{x}^4)\right]^2 \cdot \left\{(\mathrm{d}\widehat{\chi})^2 + \sin^2\widehat{\chi} \cdot \left[(\mathrm{d}\widehat{\theta})^2 + \sin^2\theta \cdot (\mathrm{d}\widehat{\phi})^2\right] - (\mathrm{d}\widehat{x}^4)^2\right\}. \tag{A4.27}$$

Now we make another coordinate transformation by

$$\begin{aligned}
r^{\#} &= \frac{2\sin\widehat{\chi}}{\cos\widehat{\chi} + \cos\widehat{x}^4}, \\
\widehat{x}^4 &= \frac{2\sin\widehat{x}^4}{\cos\widehat{\chi} + \cos\widehat{x}^4}, \\
\left(\theta^{\#}, \phi^{\#}\right) &= \left(\widehat{\theta}, \widehat{\phi}\right), \\
\cos\widehat{\chi} &+ \cos\widehat{x}^4 \neq 0; \\
a^{\#}(x^{\#4}) &:= \widehat{a}(\widehat{x}^4) > 0.
\end{aligned} \tag{A4.28}$$

The corresponding conformally flat metric from (A4.27) emerges as

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{4} \cdot \left[a^{\#}(x^{\#4})\right]^2 \cdot \left[\cos\chi^{\#} + \cos x^{\#4}\right]^2 \\
&\quad \times \left\{(\mathrm{d}^{\#})^2 + (r^{\#})^2 \cdot \left[(\mathrm{d}\theta^{\#})^2 + \sin^2\theta^{\#} \cdot (\mathrm{d}\phi^{\#})^2\right] - (\mathrm{d}x^{\#4})^2\right\} \\
&=: [\phi^{\#}(x^{\#})]^2 \cdot \left\{(\mathrm{d}r^{\#})^2 + (r^{\#})^2 \cdot \left[(\mathrm{d}\theta^{\#})^2 + \sin^2\theta^{\#} \cdot (\mathrm{d}\phi^{\#})^2\right] - (\mathrm{d}x^{\#4})^2\right\}.
\end{aligned} \tag{A4.29}$$

Therefore, the corresponding field $\phi^{\#}(x^{\#})$ is analytically furnished by

$$\phi^{\#}(x^{\#}) = \frac{1}{2} \cdot \left[a^{\#}(x^{\#4})\right] \cdot \left[\cos \chi^{\#} + \cos x^{\#4}\right]. \tag{A4.30}$$

Thus, in an important model of cosmology, an analytic expression for the $\phi^{\#}(x^{\#})$ field is explicitly provided. $\square$

Appendix 5
Linearized Theory and Gravitational Waves

In this appendix *we briefly review* the *linearized theory of gravitation* and discuss a class of solutions known as *gravitational waves*. There are currently several major experiments in progress which hope to detect gravitational waves in the near future. If they are successful, a new arena, known as *gravitational wave astronomy*, will potentially yield insight into many interesting astrophysical phenomena. The subject of gravitational waves and the detection of gravitational waves is vast, and we can only touch upon the subject here. The interested reader is referred to [33, 103, 172, 230], and references therein.

Although wave solutions also exist in the full nonlinear theory, as mentioned in Chap. 7, we shall study here the more physically relevant scenario where the gravitational field is weak. In such a case the metric may be written as a perturbation about the flat space–time metric:

$$g_{ij}(x) = d_{ij} + \varepsilon h_{ij}(x) + \mathcal{O}(\varepsilon^2)\,, \tag{A5.1}$$

with ε, a small parameter such that terms of order ε^2 or higher *may be ignored*.[1] (We will use equalities below instead of approximate equalities, and it is understood that this implies "equal to order ε.") The flat space–time metric is known as *the background metric*. Of course, one could also perturb a non-flat solution to the Einstein field equations (in fact, many interesting phenomena do indeed consider non-flat backgrounds); however, (A5.1) will suffice for our purpose.

In the linear approximation, the Einstein equations are computed utilizing metric (A5.1), and terms to order ε are retained. The conjugate metric tensor components $g^{ij}(x)$ from (A5.1) are furnished by

[1](1) Here, $h_{ij}(x)$ will be used to denote the perturbation of the metric. This notation should not be confused with the one of $h_{ij}(x)$ in Appendix 1, where it represented the variation of the metric tensor components.
(2) Weak gravitational fields, up to *an arbitrary order* $\mathcal{O}(\varepsilon^n)$, have been investigated in [64].

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4,

$$g^{ij}(x) = d^{ij} - \frac{\varepsilon}{2} \cdot \left(d^{ik} \cdot d^{jl} + d^{jk} \cdot d^{il}\right) \cdot h_{kl}(x). \tag{A5.2}$$

Therefore, we obtain by raising indices,

$$\begin{aligned} h^{kl}(x) &= g^{km}(x) \cdot g^{ln}(x) \cdot h_{mn}(x) \\ &= \Big\{ d^{km} \cdot d^{ln} - \frac{\varepsilon}{2} \cdot \left[d^{ln} \cdot \left(d^{kp} \cdot d^{mq} + d^{mp} \cdot d^{kq}\right) \right. \\ &\qquad \left. + \, d^{km} \cdot \left(d^{lp} \cdot d^{nq} + d^{np} \cdot d^{lq}\right)\right] \cdot h_{pq}(x) \Big\} \cdot h_{mn}(x), \end{aligned} \tag{A5.3i}$$

$$\begin{aligned} d_{ij} h^{ij}(x) &=: h^{i}{}_{i}(x) \\ &= \left\{ d^{ij} - \frac{\varepsilon}{2} \cdot \left(d^{ik} \cdot d^{jl} + d^{jk} \cdot d^{il}\right) \cdot h_{kl}(x) \right\} \cdot h_{ij}(x), \end{aligned} \tag{A5.3ii}$$

$$\det\left[g_{ij}(x)\right] = -\left[1 + \varepsilon h^{k}_{k}(x)\right]. \tag{A5.3iii}$$

Further, we shall require the Christoffel symbols, which are given as

$$\left\{ {i \atop j\ k} \right\} = \frac{\varepsilon}{2} d^{il} \left[\partial_k h_{lj}(x) + \partial_j h_{lk}(x) - \partial_l h_{jk}(x)\right]. \tag{A5.4}$$

In addition, to form the Einstein tensor in linearized theory, we shall require the Riemann tensor and its contractions to order ε which may be computed from (A5.4) as:

$$\begin{aligned} R^{i}{}_{jkl}(x) &= \partial_k \left\{ {i \atop l\ j} \right\} - \partial_l \left\{ {i \atop k\ j} \right\} \\ &= \frac{\varepsilon}{2} \left[\partial_k \partial_j h^{i}{}_{l}(x) + \partial_l \partial^i h_{jk}(x) - \partial_k \partial^i h_{jl}(x) - \partial_l \partial_j h^{i}{}_{k}(x)\right], \end{aligned} \tag{A5.5i}$$

$$R_{ij}(x) = \frac{\varepsilon}{2} \left[\partial_i \partial_j h^{k}_{k}(x) + \partial_k \partial^k h_{ij}(x) - \partial^k \partial_i h_{jk}(x) - \partial_k \partial_j h^{k}_{i}(x)\right], \tag{A5.5ii}$$

$$R(x) = \varepsilon \left[\partial^k \partial_k h^{i}{}_{i}(x) - \partial_k \partial^i h^{k}_{i}(x)\right]. \tag{A5.5iii}$$

(Compare the above with (2.172).) From these, the Einstein tensor is furnished as

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2} g_{ij} R \\ &= \frac{\varepsilon}{2} \left[\partial_i \partial_j h^{k}_{k}(x) + \partial_k \partial^k h_{ij}(x) + d_{ij} \partial_k \partial^l h^{k}_{l}(x) \right. \\ &\qquad \left. - d_{ij} \partial^k \partial_k h^{l}{}_{l}(x) - \partial^k \partial_i h_{jk}(x) - \partial_k \partial_j h^{k}_{i}(x)\right]. \end{aligned} \tag{A5.6}$$

At this point it is worthwhile noting that if we introduce the *trace-reversed perturbation tensor*, defined by

$$\overline{h}_{ij}(x) := h_{ij}(x) - \frac{1}{2} d_{ij} h^k_{\ k}(x), \tag{A5.7}$$

the expression (A5.6) simplifies greatly:

$$G_{ij} = \frac{\varepsilon}{2}\Big[\partial_k\partial^k \overline{h}_{ij}(x) + d_{ij}\partial_k\partial^l \overline{h}^k_{\ l}(x) - \partial^k\partial_i \overline{h}_{jk}(x) - \partial_k\partial_j \overline{h}^k_{\ i}(x)\Big]. \tag{A5.8}$$

One may simplify this expression even further by an appropriate choice of gauge (which is equivalent to a coordinate transformation in general relativity theory). From (A5.8) it can be seen that considerable simplification would occur if we could choose a coordinate system such that

$$\partial^k \overline{h}_{kj}(x) = 0\ . \tag{A5.9}$$

The condition in (A5.9) is known as the *Lorentz gauge condition* (also known as the *harmonic*, *deDonder*, or *Fock-deDonder gauge condition*). Establishing such a condition is always possible under an infinitesimal (and differentiable) coordinate transformation of the form $x^i \to x'^i = x^i + \varepsilon\xi^i(x)$. Under such a transformation, the metric perturbation becomes

$$h_{ij}(x) \to h'_{ij}(x) = h_{ij}(x) - \partial_i \xi_j(x) - \partial_j \xi_i(x), \tag{A5.10i}$$

$$\overline{h}_{ij}(x) \to \overline{h}'_{ij}(x) = h'_{ij}(x) - \frac{1}{2} d_{ij} h'^k_{\ k}(x) = \overline{h}_{ij}(x) - \partial_i \xi_j(x) - \partial_j \xi_i(x) + d_{ij}\partial_k \xi^k(x). \tag{A5.10ii}$$

(Note that $\varepsilon\xi^i(x)$ is of order ε, and therefore, its index is raised or lowered with the background metric.)

By taking the four-divergence of (A5.10ii) it can be seen that $\overline{h}'_{ij}(x)$ respects the Lorentz gauge condition if

$$\partial_k\partial^k \xi_i(x) = \partial^j \overline{h}_{ji}(x). \tag{A5.11}$$

Therefore, under this gauge condition, the Einstein tensor may be written as

$$G_{ij} = \frac{\varepsilon}{2}\partial_k\partial^k \overline{h}'_{ij}(x)\,,$$

so that the linearized Einstein field equations read

$$\varepsilon\partial_k\partial^k \overline{h}'_{ij}(x) = -2\kappa\varepsilon T_{ij}(x). \tag{A5.12}$$

Note that since the background space-time is flat, $T_{ij}(x)$ is of order ε.

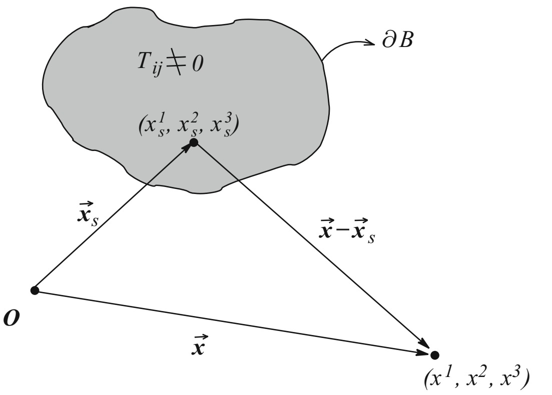

Fig. A5.1 An illustration of the quantities in (A5.13) in the three-dimensional spatial submanifold. The coordinates x_s, known as *the source points*, span the entire source (*shaded region*). O represents an arbitrary origin of the coordinate system

Remarks. (i) From physics, (A5.12) is the equation for *a spin-two* tensor field propagating on flat space–time with a source $T_{ij}(x)$ [98]. *Gravitons* (hypothetical quanta mediating gravitational interactions) should therefore carry a spin of two in the corresponding quantum theory.

(ii) In the case where $T_{ij}(x) = 0$, (A5.12) is simply the wave equation for a second-rank tensor field $\overline{h}'_{ij}(x)$ with speed equal to unity (the speed of light, or massless particles). Therefore, in the linearized theory, it can be argued that gravitational interactions propagate at the speed of light.

If $T_{ij}(x)$ is known and of compact support, one may find a solution to (A5.12) utilizing the flat space–time retarded Green function for the wave operator:

$$\overline{h}'_{ij}(\vec{x}, x^4) = \frac{\kappa}{2\pi} \int_B \frac{T_{ij}(\vec{x}, x^4 - |\vec{x} - \vec{x}_s|)}{|\vec{x} - \vec{x}_s|} \, dx_s^1 dx_s^2 dx_s^3 \,, \tag{A5.13}$$

where the domain of integration, B, is over the source (see Fig. A5.1).

For the remainder of this appendix, we concentrate on the case where $T_{ij}(x) = 0$ (i.e., away from sources). In this scenario, (A5.12) reduces to the wave equation:

$$\partial_k \partial^k \overline{h}'_{ij}(x) = 0. \tag{A5.14}$$

The Lorentz gauge condition is actually a class of gauges, as $\xi_i(x)$ is not completely fixed. Within the Lorentz gauge it is possible to add to $\xi_i(x)$ a vector field, $\zeta_i(x)$, such that the condition

$$\partial_k \partial^k \zeta_i(x) = 0 \tag{A5.15}$$

holds, as this will not spoil (A5.11). The Lorentz gauge condition, along with the conditions (A5.15), allows us to reduce the number of components of $\overline{h}_{ji}(x)$ (*omitting primes* from now on) from ten to two. At this stage, all coordinate freedom has been exhausted, and the system of equations contains *two physical* (i.e., not arising from coordinate artifacts which do not affect the curvature) *degrees of freedom*.

For physical considerations, it is convenient to choose $\xi_4(x)$ such that $\overline{h}^k{}_k = 0$ and choose $\xi_\mu(x)$ such that $\overline{h}_{4\mu}(x) = 0$. This choice constitutes what is known as the *traceless-transverse gauge*. Note that in this coordinate system there is no distinction between $h_{ij}(x)$ and $\overline{h}_{ij}(x)$. We therefore use the notation ${}^{\mathrm{TT}}h_{ij}(x)$ to denote metric perturbations in the traceless-transverse gauge below.

A class of important solutions to (A5.14) is comprised of a superposition of plane waves. Let us consider a single wave

$$ {}^{\mathrm{TT}}h_{ij}(x) = \mathrm{Re}\left[A_{ij} \cdot \mathrm{e}^{\mathrm{i}k_l x^l}\right], \tag{A5.16} $$

with A_{ij}, the components of a symmetric, constant complex tensor (called *the polarization tensor*) and k^l, the components of a null vector (called *the wave vector*). The traceless condition implies that $A^i{}_i = 0$, and the condition ${}^{TT}h_{4\mu}(x) = 0$ implies $A_{4\mu} = 0$. Without loss of generality, let us consider the wave in (A5.16) traveling in the x^3 direction. The wave vector then possesses the form $[k^i] = [0, 0, \omega, \omega]$ with ω representing *the frequency of the wave*. The conditions (A5.9) and $A_{4\mu} = 0$ dictate that $A_{i3} = 0$ and $A_{44} = 0$. Assembling all the information we have gathered on the tensor, we see that the two degrees of freedom mentioned previously are encoded in the two components $A_{11} = -A_{22}$ and $A_{12} = A_{21}$ (the first equality arising from the traceless condition and the second equality from the symmetric property of $h_{ij}(x)$).

To determine the physical effects of these gravitational waves, let us consider two nearby test particles, located at $x^2 = x^3 = 0$ in the space–time of a gravitational wave. Consider a vector connecting these two particles, whose components, at some initial time, t_0, are furnished by $[\eta^i]_{|t_0} = [\epsilon, 0, 0, 0]$. To the linear order considered in this appendix, the geodesic deviation equations (1.191) yield, after some calculation:

$$ \frac{\partial^2 \eta^1(x)}{(\partial x^4)^2} = \frac{\epsilon}{2} \frac{\partial^2}{(\partial x^4)^2} \left[{}^{\mathrm{TT}}h_{11}(x)\right], \tag{A5.17i} $$

$$ \frac{\partial^2 \eta^2(x)}{(\partial x^4)^2} = \frac{\epsilon}{2} \frac{\partial^2}{(\partial x^4)^2} \left[{}^{\mathrm{TT}}h_{12}(x)\right]. \tag{A5.17ii} $$

(Note that since the Riemann tensor is already of order ε, the components of the 4-velocity ($\xi'^{\,i}$ in the notation of (1.191)) are given by $\delta^i{}_4$.) On the other hand, if two

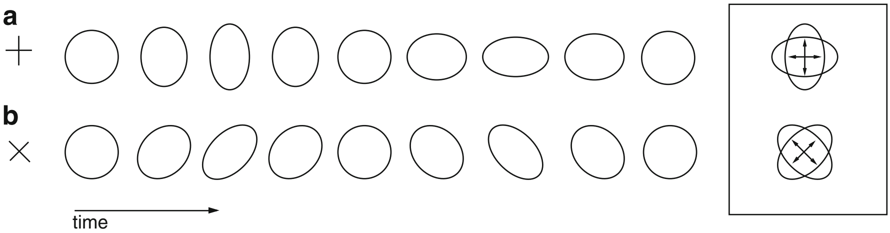

Fig. A5.2 The $+$ (*top*) and $\times$ (*bottom*) polarizations of gravitational waves. A loop of string is deformed as shown over time as a gravitational wave passes out of the page. Inset: a superposition of the two most extreme deformations of the string for the $+$ and $\times$ polarizations

particles are located at $x^1 = x^3 = 0$ and the initial separation vector components are given by $[\eta^i]_{|t_0} = [0, \epsilon, 0, 0]$, then the separation vector satisfies the equations:

$$\frac{\partial^2 \eta^2(x)}{(\partial x^4)^2} = \frac{\epsilon}{2} \frac{\partial^2}{(\partial x^4)^2} \left[{}^{\mathrm{TT}}h_{22}(x) \right], \tag{A5.18i}$$

$$\frac{\partial^2 \eta^1(x)}{(\partial x^4)^2} = \frac{\epsilon}{2} \frac{\partial^2}{(\partial x^4)^2} \left[{}^{\mathrm{TT}}h_{12}(x) \right]. \tag{A5.18ii}$$

From the form of ${}^{\mathrm{TT}}h_{ij}(x)$ in (A5.16), it is obvious that the geodesic separation vector components, $\eta^i(x)$, are oscillatory. There exist two *polarization states* of gravitational waves. One state affects particle motion, as shown in Fig. A5.2a and is characterized by ${}^{\mathrm{TT}}h_{11}(x) = -{}^{\mathrm{TT}}h_{22}(x) \neq 0$ and ${}^{\mathrm{TT}}h_{12}(x) = 0 = {}^{\mathrm{TT}}h_{21}(x)$. This state is known as the $+$-state. The other state is characterized by ${}^{\mathrm{TT}}h_{12}(x) = {}^{\mathrm{TT}}h_{21}(x) \neq 0$ and ${}^{\mathrm{TT}}h_{22}(x) = 0 = -{}^{\mathrm{TT}}h_{11}(x)$ and is known as the $\times$-state. This motion is depicted in Fig. A5.2b. These two states correspond to two independent polarizations since A_{11} (or equivalently $A_{22} = -A_{11}$) and A_{12} ($= A_{21}$) are independent of each other. The two polarizations are orthogonal to each other in the sense that $\overline{A^{ij}}_{|+} A_{ij\,|\times} = 0$ (the bar here representing complex conjugation).

There are currently a number of gravitational wave detectors in operation throughout the world, with several others in the advanced planning and design stages. The largest land-based detector, Laser Interferometer Gravitational Wave Observatory (LIGO), consists of two interferometers: one in Hanford, Washington, and one in Livingston, Louisiana, which have characteristic lengths of approximately 4 km. The LISA detector (Laser Interferometer Space Antenna), originally scheduled for launch in the next decade, would consist of three orbiting satellites, each separated by approximately *5 million kilometers*. These mammoth devices possess the sensitivity to detect gravitational waves from various astrophysical sources (see Fig. A5.3) with amplitudes as small as 10^{-9} cm. (See, for example, [182].) The current status of the LISA project is under assessment.

We have treated here only the very simplest type of gravitational wave, as a more in-depth treatment is beyond the scope of this text. A more thorough review would

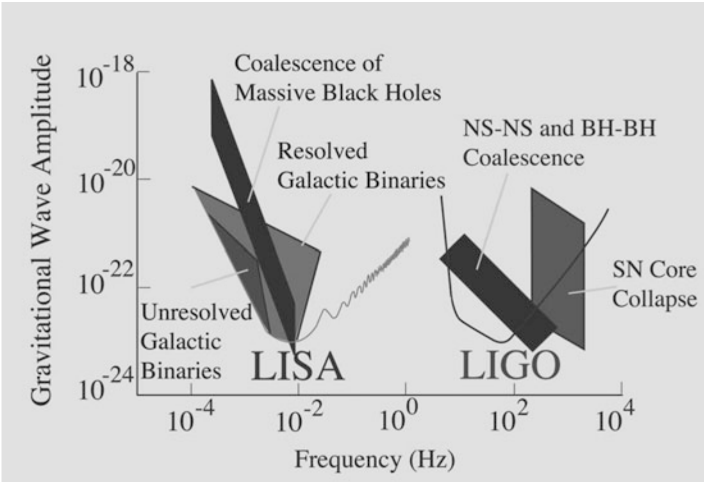

Fig. A5.3 The sensitivity of the LISA and LIGO detectors. The *dark regions* indicate the likely amplitudes (vertical axis, denoting change in length divided by mean length of detector) and frequencies (horizontal axis, in cycles per second) of astrophysical sources of gravitational waves. The approximately "U"-shaped lines indicate the extreme sensitivity levels of the LISA (*left*) and LIGO (*right*) detectors. BH = black hole, NS = neutron star, SN = supernova (Figure courtesy of NASA)

include solutions with sources ($T_{ij}(x) \neq 0$, such as many of the phenomena in Fig. A5.3), which are astrophysically relevant, as well as estimations of the power emitted via gravitational wave emission for various astrophysical processes such as black hole collisions and binary neutron star orbits. (In closing this appendix we note that the 1993 Nobel Prize in Physics was awarded to Russell A. Hulse and Joseph H. Taylor, Jr., for their discovery of a binary pulsar system which is observed to be losing energy at a rate in agreement with gravitational wave calculations [137].)

Appendix 6
Exotic Solutions: Wormholes, Warp-Drives, and Time Machines

This appendix deals with *several interesting classes of exotic solutions* to the gravitational field equations. The purpose of studying such solutions is several-fold: On the one hand, they are interesting in their own right, as they illustrate just how rich the arena of solutions to the field equations can be. There are no analogs of these solutions in pregeneral relativistic theories of gravity. It is therefore of great interest in gravitational field research to study just what is possible within general relativity theory under extreme situations. Secondly, there is pedagogical value in studying such solutions. As we will show below, these solutions concretely illustrate the utility of many of the topics and techniques discussed in this text in a very simple and clear fashion. Finally, the types of solutions mentioned here actually have application in specialized physical topics within general relativity theory. Wormholes, for example, provide a simplistic picture of "space–time foam," a potential model for the vacuum in theories of quantum gravity, depicted in Fig. A6.1. (Whether this topology is allowed to fluctuate is not clear at the moment, as some arguments against topology change are indifferent to whether one is considering classical or quantum effects [126].) An in-depth review of exotic solutions of the type discussed here may be found in [168].

A6.1 Wormholes

The first exotic solution we will discuss is the wormhole. This type of object has already been mentioned previously. (See Fig. 2.17c.) Qualitatively, the wormhole represents a handle (or shortcut) in space–time, which introduces a nontrivial topology. The most often studied topology is $R^2 \times S^2$. The wormhole handle may connect two otherwise disconnected universes (*an interuniverse* wormhole) or else connect two regions of the same universe (*an intrauniverse* wormhole). The two cases are depicted in Fig. A6.2. It should be noted that the near-throat region (the "narrowest" section of the handle) has an uncanny resemblance to the manifold depicted in Fig. 3.1 and, in fact, the part of the Schwarzschild manifold depicted in

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4,

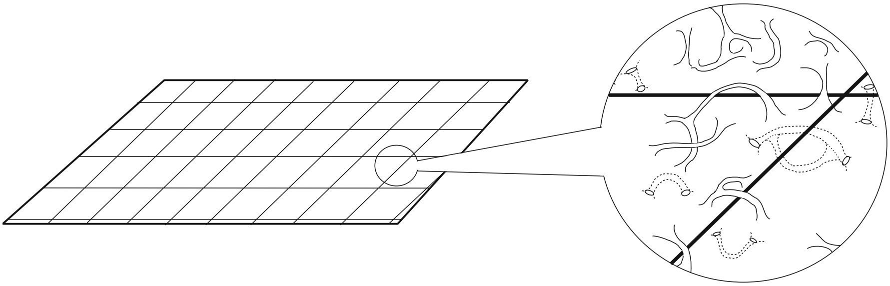

Fig. A6.1 A possible picture for the space–time foam. Space-time that seems smooth on large scales (*left*) may actually be endowed with a sea of nontrivial topologies (represented by handles on the *right*) due to quantum gravity effects. (Note that, as discussed in the main text, this topology is *not necessarily* changing.) One of the simplest models for such a handle is the wormhole

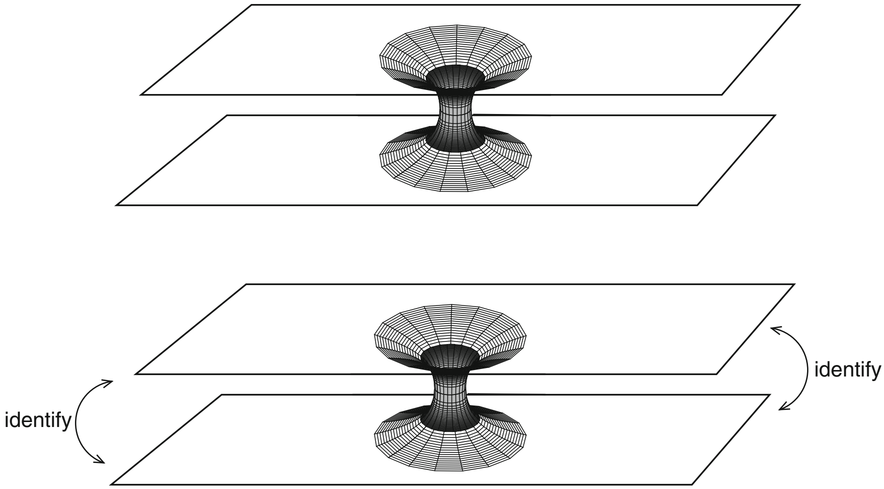

Fig. A6.2 A qualitative representation of an interuniverse wormhole (*top*) and an intra-universe wormhole (*bottom*). In the second scenario, the wormhole could provide a shortcut to otherwise distant parts of the universe

Fig. 3.1 can actually represent one half of a type of wormhole known as an *Einstein-Rosen bridge*. The shading in the throat region here represents the presence of some matter field, which is the source of the gravitational field supporting the wormhole.

Let us now quantify the geometry represented in Fig. A6.2 . We shall be limiting our attention to the case where the wormhole exhibits spherical symmetry and, for simplicity, we will assume the space–time is static. In such a case, we may take the line element to be that of (3.1) and utilize the field equations and conservation equations given by (3.44i–vii). Note that here, r_1 is equal to the coordinate radius of the throat. To avoid confusion, below we shall use coordinate indices (r, θ, φ, t) *instead of* $(1, 2, 3, 4)$ as was used in Chap. 3.

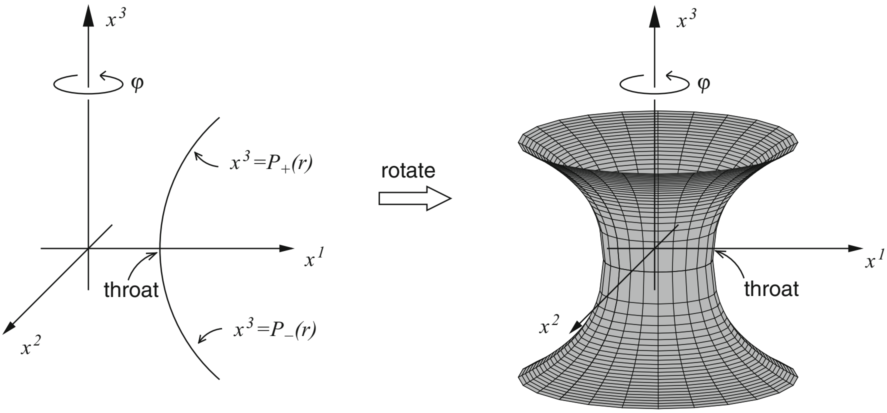

Fig. A6.3 *Left*: A cross-section of the wormhole profile curve near the throat region. *Right*: The wormhole is generated by rotating the profile curve about the x^3-axis

Since we have a specific geometry in mind, the g or mixed methods of solving the field equations will be most useful here. To mathematically construct the wormhole, we consider Fig. A6.3 which illustrates a cross-section or profile curve of a two-dimensional section, t = constant, $\theta = \pi/2$, of the wormhole near the throat (see also Fig. 1.18). The wormhole itself is constructed via the creation of a surface of revolution when one rotates this profile curve about the x^3-axis. The surface of revolution may be parameterized as follows:

$$x^\alpha_{|..} = \xi^\alpha(r,\varphi),\ \left(\xi^1(r,\varphi),\ \xi^2(r,\varphi),\ \xi^3(r,\varphi)\right) := (r\cos\varphi,\ r\sin\varphi,\ P(r)), \quad \text{(A6.1)}$$

where $r = \sqrt{(x^1)^2 + (x^2)^2}$ and φ is shown in Fig. A6.3. Therefore, the induced metric on the surface of revolution is given by

$$\mathrm{d}\sigma^2_{|..} = \left[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\right]_{|..} = \left\{1 + [\partial_r P(r)]^2\right\} \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2. \quad \text{(A6.2)}$$

Note that the corresponding four-dimensional space–time metric, (3.1), may now be written as

$$\mathrm{d}s^2_\pm = \left\{1 + [\partial_r P_\pm(r)]^2\right\} \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2(\theta)\,\mathrm{d}\varphi^2 - \mathrm{e}^{\gamma_\pm(r)}\,\mathrm{d}t^2, \quad \text{(A6.3)}$$

with coordinate ranges

$$r_1 < r < r_2,\ \ 0 < \theta < \pi,\ \ -\pi < \varphi < \pi,\ \ -\infty < t < \infty.$$

For the case of a wormhole throat, the coordinate chart of (A6.3) is *insufficient to describe the entire wormhole*. Instead, two spherical charts are required, one for the upper half of the wormhole (the "+" domain) and one for the lower half (the "−" domain). (See [74, 75, 107, 185, 254] for various charts which cover both the + and − domains.) Hence, we use the $\pm$ subscript in the above line element.

The constant m_0 in (3.45i) can be set by the requirement that $\partial_r P_\pm(r) \to \pm\infty$ as $r \to r_1$ or, equivalently, that (3.45i) vanish in the limit $r \to r_1$. This condition implies that the constant $2m_0 = r_1$.

We are now in a position to analyze the geometry. For now we limit the analysis to the upper half of the profile curve (the "+" region) in Fig. A6.3 since the bottom half is easily obtained once we have the upper half (for example, via reflection symmetry through the $x^1 - x^2$ plane if one wants a wormhole symmetric on either side of the throat, although this is not required). It should also be noted that $r = 0$ is *not* part of the manifold, unlike in the case of stars studied in earlier chapters. Therefore, divergences as r tends to zero are not an issue here. From the profile curve in Fig. A6.3, it may be seen that a function $P(r)$ is required with the following properties:

1. The derivative, $\partial_r P(r)$, must be "infinite" at the throat and finite elsewhere.
2. The derivative, $\partial_r P_+(r)$, must be positive at least near the throat (negative for the "−" domain).
3. The function $P_+(r)$ must possess a negative second derivative at least somewhere near the throat region (positive for the "−" domain).
4. Since there will be a cutoff at some $r = r_2$, where the solution is joined to vacuum (Schwarzschild or Kottler solution), no specification needs to be made as r tends to infinity.

From now on, we often *drop* the explicit r dependence in functions for brevity.

In terms of the function $P(r)$, the field equations read

$$\frac{(\partial_r P)^2 - (\partial_r \gamma)\, r}{\left[1 + (\partial_r P)^2\right] r^2} = -\kappa T^r_{\ r}, \tag{A6.4i}$$

$$\begin{aligned}
&-\frac{1}{4r}\left[1 + (\partial_r P)^2\right]^{-2} \Big\{-4\,(\partial_r P)\cdot(\partial_r\partial_r P) + 2\,(\partial_r\gamma) + 2\,(\partial_r\gamma)\cdot(\partial_r P)^2 \\
&+r\,(\partial_r\gamma)^2 + 2\,r\,(\partial_r\partial_r\gamma)\cdot(\partial_r P)^2 + 2\,r\,(\partial_r\partial_r\gamma) + r\,(\partial_r\gamma)^2\cdot(\partial_r P)^2 \\
&-2\,r\,(\partial_r P)\cdot(\partial_r\partial_r P)\cdot(\partial_r\gamma)\Big\} = -\kappa T^\theta_{\ \theta} \equiv -\kappa T^\varphi_{\ \varphi},
\end{aligned} \tag{A6.4ii}$$

$$\frac{(\partial_r P)\left[(\partial_r P)^3 + 2\,(\partial_r\partial_r P)\, r + \partial_r P\right]}{r^2\left[1 + (\partial_r P)^2\right]^2} = -\kappa T^t_{\ t}, \tag{A6.4iii}$$

where it is understood that these equations must hold in both the $+$ and $-$ domain.

A natural question to ask at this stage is what are the properties a matter field must possess in order to support such a wormhole space–time? Analyzing (A6.4iii) by performing a careful limit as $r \to r_1$, and taking into account the properties of $P(r)$ listed above, we see that $G^t{}_t(r_1) = \kappa\rho(r_1) \leq \frac{1}{r_1^2}$, $\rho(r)$ being the local energy density as measured by static observers. (We are assuming here that the stress-tensor components are nonsingular.) To make the arguments more physical, we shall employ a mixed method as opposed to the purely geometric g-method, and therefore we will treat $T^r_r(r)$ as a function that is to be prescribed.

The orthonormal Riemann tensor components are furnished by

$$R_{(t)(r)(t)(r)\pm} = \frac{1}{2}\left(1 - \frac{b_\pm}{r}\right)\left[\partial_r\partial_r\gamma_\pm + \frac{1}{2}(\partial_r\gamma_\pm)^2 - \frac{1}{2}\partial_r\gamma_\pm\frac{\partial_r b_\pm - \frac{b_\pm}{r}}{r - b_\pm}\right] , \tag{A6.5i}$$

$$R_{(t)(\theta)(t)(\theta)\pm} = \frac{1}{2r}\partial_r\gamma_\pm\left(1 - \frac{b_\pm}{r}\right) = R_{(t)(\varphi)(t)(\varphi)\pm} , \tag{A6.5ii}$$

$$R_{(r)(\theta)(r)(\theta)\pm} = \frac{1}{2r^2}\left[\partial_r b_\pm - \frac{b_\pm}{r}\right] = R_{(r)(\varphi)(r)(\varphi)\pm} , \tag{A6.5iii}$$

$$R_{(\theta)(\varphi)(\theta)(\varphi)\pm} = \frac{b_\pm}{r^3} , \tag{A6.5iv}$$

where we use the notation $b_\pm := r\frac{(\partial_r P_\pm)^2}{1+(\partial_r P_\pm)^2}$ to simplify the expressions.

To continue the analysis, it is noted that to make the the orthonormal Riemann components finite in this scenario, $\partial_r\gamma(r)$ must be finite. (See [74] for full details as to why this renders all components finite.) From the field equations, it can be shown that this quantity is given by

$$\partial_r\gamma_\pm = r\left[\kappa T^r{}_{r\pm} + \frac{1}{r^2}\right]\left[1 + (\partial_r P_\pm)^2\right] - \frac{1}{r}, \tag{A6.6}$$

from which it may be seen that a prescription of T^r_r compatible with $\kappa T^r_r(r_1) = \kappa p_r(r_1) = -\frac{1}{r_1^2}$ must be applied. Therefore, *a negative pressure* (i.e., *a tension*) must be present at the wormhole throat in order to support it.

Finally, we consider the combination

$$\kappa\left(\rho_\pm + p_{r\,\pm}\right) = G^t{}_{t\,\pm} - G^r{}_{r\,\pm} = -\frac{e^{\gamma_\pm}}{r}\,\partial_r\left\{e^{-\gamma_\pm}\left[1 + (\partial_r P_\pm)^2\right]^{-1}\right\} \tag{A6.7}$$

in the immediate vicinity of the wormhole throat. This quantity is directly related to the energy conditions discussed in Chap. 2. From the discussions above, it can be seen that at the throat, nonnegativity of this combination can (barely) be met (i.e., $\rho(r_1) + p_r(r_1) \leq 0$). However, as one moves away from the throat, note that the quantity in braces in (A6.7) must go from a value of zero at the throat (where $\partial_r P \to \infty$) to a positive value away from the throat. Therefore, around the throat region, the derivative of the expression in braces is positive and thus (A6.7) *is negative in a neighborhood of the throat*. That is, a static wormhole throat *violates the weak/null and strong energy conditions!* (See, e.g., [146].)

The reader interested in this topic is referred to Visser's book on the subject of Lorentzian wormholes [254].

A6.2 Warp-Drive Space–Times

We consider here briefly another interesting class of solutions to the field equations known as *warp-drive* space-times. The first such solution was put forward by Miguel Alcubierre in 1994 [4], and since then much analysis has been done on these types of solutions. We will present here the original Alcubierre warp-drive.

The basic idea is simple. One wishes to construct a space–time in which an object (popularly, a spaceship is chosen) can be propelled between two points, such that the travel time (as measured by both the proper time of the object and external observers) is much less than the distance separating the points divided by the speed of light. Ideally, one also wishes to demand that the interval of proper time of the traveling observers is the same as the proper time interval for static external observers (so that the two observers age by the same amount!). Before beginning the analysis, it should also be noted that *no object will be traveling faster than the local speed of light*.

It is useful to employ the ADM formalism here (see Appendix 1) utilizing the metric (A1.42). An ADM form of the metric is useful, as a space–time domain constructed from the ADM formalism will not contain any closed causal curves (to be discussed in the next section) unless one introduces a topological identification. The original warp-drive used the following quantities:

$$N(x) = 1\,, \tag{A6.8i}$$

$$N^{\alpha}(x) = -{}_s v(x^4)\, f({}_s r(x))\, \delta^{\alpha}{}_1\,, \tag{A6.8ii}$$

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta}\,, \tag{A6.8iii}$$

where we are employing a Cartesian-like coordinate system in the three-dimensional spatial hypersurface. The line element therefore is the following:

$$\mathrm{d}s^2 = \left[\mathrm{d}x^1 - {}_s v(x^4)\, f({}_s r(x))\, \mathrm{d}x^4\right]^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 - \left(\mathrm{d}x^4\right)^2. \tag{A6.9}$$

Here, ${}_s v(x^4) := \frac{\mathrm{d}\,{}_s\mathcal{X}^1(x^4)}{\mathrm{d}x^4}$ represents the coordinate velocity of the spaceship and ${}_s r(x) := \left[\left(x^1 - {}_s\mathcal{X}^1(x^4)\right)^2 + (x^2)^2 + (x^3)^2\right]^{1/2}$. The coordinate position of the

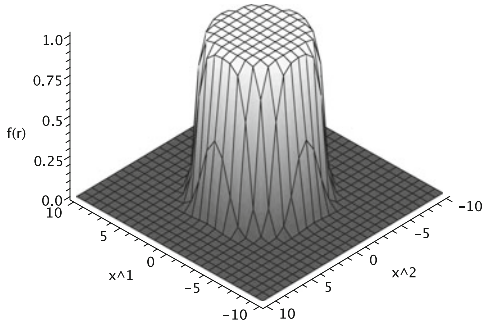

Fig. A6.4 A "top-hat" function for the warp-drive space–time with one direction (x^3) suppressed. The center of the ship is located at the center of the top hat, corresponding to ${}_sr = 0$

ship is $x^1 =_s \mathcal{X}^1(x^4)$, $x^2 = 0$, $x^3 = 0$ (note that we are assuming spatial motion in the x^1 direction only) and therefore ${}_sr(x)$ represents the coordinate distance from the ship. For the system to function as desired, the function $f({}_sr(x))$ should approximate a "top-hat" function. In the original warp-drive solution, the following function was chosen (the explicit x dependence is henceforth dropped):

$$f({}_sr) := \frac{\tanh\left[\sigma\left({}_sr + R\right)\right] - \tanh\left[\sigma\left({}_sr - R\right)\right]}{2\tanh\left(\sigma R\right)}\,, \tag{A6.10}$$

with $R > 0$ and $\sigma \gg 0$ but otherwise arbitrary constants. This function is displayed in Fig. A6.4 with the x^3 coordinate suppressed.

This space–time possesses several interesting properties. Note from (A6.9) that $x^4 =$ constant hypersurfaces are always flat. Also, for distances ${}_sr \gg R^2$, the full four-dimensional space–time is flat, as is the space–time anywhere where the first two derivatives of $f({}_sr)$ are zero. As well, it is not difficult to show that observers possessing 4-velocity components $u^i = [{}_sv f({}_sr), 0, 0, 1]$ are traveling along *geodesics*. As a concrete example, consider an observer located at $x^1_{\mathcal{O}} = {}_s\mathcal{X}^1(x^4)$, $x^2_{\mathcal{O}} = x^3_{\mathcal{O}} = 0$ and therefore traveling with the ship. It may immediately be seen that, for such an observer, the invariant infinitesimal line element (A6.9) yields the condition $\mathrm{d}s^2_{\mathcal{O}} = -(\mathrm{d}x^4)^2_{\mathcal{O}}$. (Note that $f({}_sr) = 1$ in the immediate vicinity of the ship.) Recalling that $\mathrm{d}s^2$ along a world line yields minus the infinitesimal proper time squared along that world line tells us that the lapse of proper time for an observer traveling with the ship is equal to the lapse of proper time for a static observer located at some distance $> R$ from the ship, for whom $\mathrm{d}s^2 = -(\mathrm{d}x^4)^2$ also holds. Therefore, there is *no relative time dilation between such observers*. (We are assuming that the function $f({}_sr)$ is very close to a true top-hat function, which is a good approximation as long as σ is large.)

Qualitatively, the top hat, often called the warp bubble, is "propelled" through the external space–time with a coordinate velocity ${}_sv$, and any object enclosed in the bubble moves along with it. This velocity can be very large, the important point being that objects inside the bubble always possess 4-velocities that are timelike. This emphasizes an important point in general relativity that the structure of the light cones is *a local phenomenon*. As long as 4-velocities remain within their local light cone, the velocity is timelike or subluminal (in the local sense).

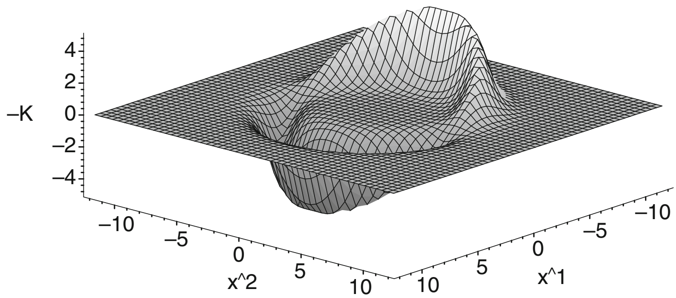

Fig. A6.5 The expansion of spatial volume elements, (A6.12), for the warp-drive space–time with the x^3 coordinate suppressed. Note that, in this model, there is contraction of volume elements in front of the ship and an expansion of volume elements behind the ship. The ship, however, is located in a region with no expansion nor compression

To glean further properties of this space–time, let us next consider the extrinsic curvature components, $K^{\alpha}_{\ \beta}$, of the spatial slices. A small amount of calculation reveals that

$$K^{\alpha}_{\ \beta} = -{}_{s}v \frac{\mathrm{d}\, f({}_{s}r)}{\mathrm{d}\, {}_{s}r} \frac{\partial\, {}_{s}r}{\partial x^1} \delta^{\alpha}_{\ 1} \delta^{1}_{\ \beta} + \text{off-diagonal terms.} \tag{A6.11}$$

The negative trace of this quantity yields an expression for the expansion of volume elements in this space:

$$-K^{\alpha}_{\ \alpha} = {}_{s}v \frac{\mathrm{d}\, f({}_{s}r)}{\mathrm{d}\, {}_{s}r} \left[\frac{x^1 - {}_{s}\mathcal{X}^1(x^4)}{{}_{s}r} \right]. \tag{A6.12}$$

This function is plotted (with the x^3 coordinate suppressed) in Fig. A6.5.

It should be stressed that the contraction of volume elements in front of the spaceship is simply a “by-product” phenomenon due to the type of metric chosen. The space ship does *not* travel at apparent superluminal speed by moving through the compressed space. The space ship is always located inside the warp bubble, where the space is approximately flat.

Finally, we briefly analyze the properties of the matter field required to support a warp bubble such as the one described here. The energy density, as measured by observers with 4-velocity components $u_i = [0, 0, 0, -1]$ (geodesic observers), is given by

$$\begin{aligned} T_{ij}(x)u^i(x)\,u^j(x) &= -\frac{1}{\kappa} G_{ij}(x)u^i(x)\,u^j(x) \\ &= -\frac{1}{4\kappa}({}_{s}v)^2 \left[\left(\frac{\partial f({}_{s}r)}{\partial x^2} \right)^2 + \left(\frac{\partial f({}_{s}r)}{\partial x^3} \right)^2 \right]. \end{aligned} \tag{A6.13}$$

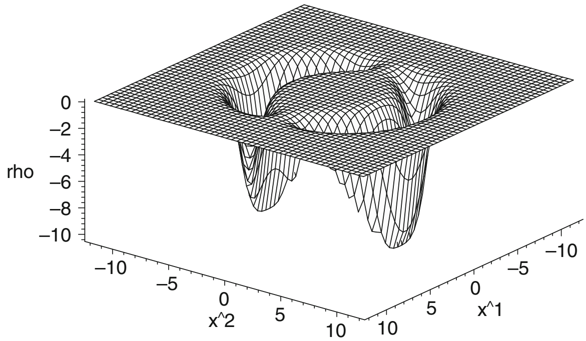

Fig. A6.6 A plot of the distribution of negative energy density in a plane ($x^3 = 0$) containing the ship for a warp-drive space–time

From this expression, it can be noted that *a negative energy density* is required in order to generate this exotic geometry. The distribution of this negative energy density is plotted in Fig. A6.6. Again, energy conditions are violated.

A6.3 Time Machines

We conclude our survey of exotic solutions by considering some space-times which contain closed causal (null or timelike) or closed timelike curves (CTCs). We have already briefly touched upon such phenomena when discussing the Kerr space–time. We shall present a few more such solutions here.

CTCs are associated with space-times which possess some property which allows a continuous timelike curve to intersect with itself. That is, some observer may travel into his or her own past. (These definitions may be extended to include null curves as well.) Two qualitative examples of CTCs are displayed in Fig. A6.7, where "forward light cones" have been drawn to indicate the timelike nature of the curves. It should be noted that the space-times in the figures may possess an everywhere nonvanishing vector field which is timelike and continuous (and thus may be *time-orientable*). Also, locally, one can define a "past" and a "future" in the figures. However, *globally* this is not possible, as a forward-pointing timelike vector tangent to a timelike curve may be Fermi–Walker transported along the curve in the forward direction (see Fig. 2.13), only to wind up in the past of its point of origin.

Causality violation refers to the violation of the premise that cause precedes effect, as would be possible in a space–time with CTCs. We define the causality violating region of a space–time point p, $\mathcal{C}(p)$, as

$$\mathcal{C}(p) := \mathcal{C}^{-}(p) \cap \mathcal{C}^{+}(p),$$

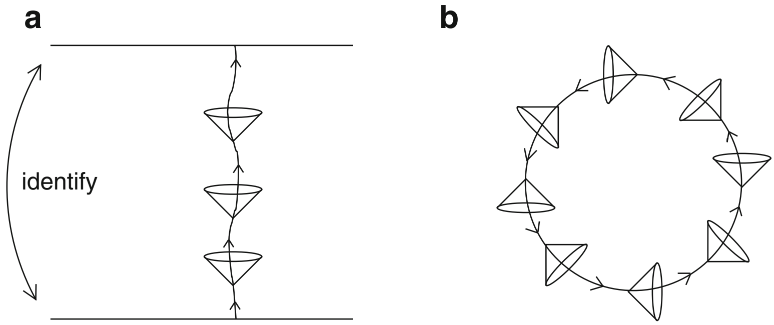

Fig. A6.7 Two examples of closed timelike curves. In (**a**) the closure of the timelike curve is introduced by topological identification. In (**b**) the time coordinate is periodic

where $\mathcal{C}^-(p)$ and $\mathcal{C}^+(p)$ represent the causal past and causal future of p, respectively (see Fig. 2.2). For causally well-behaved space-times, this intersection is empty for all p. The causality violating region of a space–time M is

$$\mathcal{C}(M) = \bigcup_{p \in M} \mathcal{C}(p),$$

and a space–time whose $\mathcal{C}(M) = \emptyset$ is known as a *causal space–time*.

It is interesting to note that the source of causality violation in many space-times with closed timelike curves is the presence of large angular momentum. Several of the examples below shall reveal this phenomenon, and it will become clear that the coupling of a time coordinate with an angular coordinate (caused by the presence of rotation) can cause the angular coordinate to become timelike, and thus generate causality violating regions of the type in part b) of Fig. A6.7.

We consider first the anti-de Sitter (AdS) space–time (see Exercise 10 of Sect. 2.3). AdS space–time is a constant (negative) curvature solution to the field equations which is supported by a negative cosmological constant, Λ. The solution may be obtained from the induced metric on the hyper-hyperboloid defined by

$$-U^2 - V^2 + X^2 + Y^2 + Z^2 = -\frac{1}{2|\Lambda|}, \tag{A6.14}$$

embedded in the five-dimensional flat space admitting a line element of

$$\mathrm{d}s_5^2 = -\mathrm{d}U^2 - \mathrm{d}V^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2. \tag{A6.15}$$

We show a schematic of this embedding (with 2 dimensions suppressed) in Fig. A6.8. (See Exercise #1iii of Sect. 6.1 for a discussion of the analogous diagram for de Sitter space–time.)

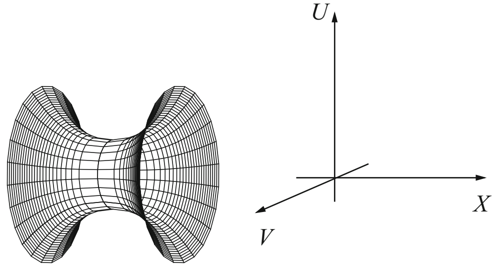

Fig. A6.8 The embedding of anti-de Sitter space–time in a five-dimensional flat "space–time" which possesses two timelike coordinates, U and V (two dimensions suppressed)

The topology of the AdS manifold is therefore $S^1 \times R^3$. Referring to Fig. A6.8, it can be shown that, for example, curves represented by circles orbiting around the X-axis on the hyperbolic manifold are everywhere timelike. This space–time therefore exhibits closed timelike curves of the type displayed in Fig. A6.7a). Usually, when dealing with AdS space–time, one deals with *the universal covering space* of anti-de Sitter space–time, which involves unwrapping the S^1 part of the topology, thus yielding R^4 topology.

Next, in our brief review of causality-violating solutions, we present the *Gödel universe* [116]. The Gödel universe represents a rotating universe supported by a negative cosmological constant, Λ, and a dust matter source and possessing R^4 topology. The rotation is somewhat peculiar in the sense that all points in this universe may be seen as equivalent. That is, there is no preferred point about which the Gödel universe is rotating. (See [126] for details.)

In one popular coordinate system, the Gödel universe metric admits a line element of

$$\mathrm{d}s^2 = \frac{2}{|\Lambda|}\left[\mathrm{d}\varrho^2 + \mathrm{d}z^2 + \left(\sinh^2\varrho - \sinh^4\varrho\right)\mathrm{d}\varphi^2 + 2\sqrt{2}\sinh^2\varrho\,\mathrm{d}\varphi\,\mathrm{d}t - \mathrm{d}t^2\right], \tag{A6.16}$$

with $\Lambda < 0$, $-\infty < t < \infty$, $0 < \varrho < \infty$, $-\infty < z < \infty$ and $-\pi < \varphi < \pi$.

One can see from (A6.16), like in the case of the Kerr geometry, that the t and φ coordinates are coupled. Furthermore, note that at $\varrho = \varrho_* := \ln(1+\sqrt{2})$, the metric component $g_{\varphi\varphi}$ switches sign. This is one indication that the φ coordinate changes character from a spacelike nature to a timelike nature.[1] The φ coordinate is truly an angular coordinate, that is, it is periodic. Therefore, for values of $\varrho > \ln(1+\sqrt{2})$, this "angle" is timelike. Although not a geodesic, an observer in the Gödel space–time at $\varrho > \varrho_*$ could travel along a trajectory in the φ direction and meet

[1] One has to be cautious when applying this loose criterion. A more rigorous discussion is based on the eigenvalue structure of the metric tensor as, for example, presented in various preceding chapters.

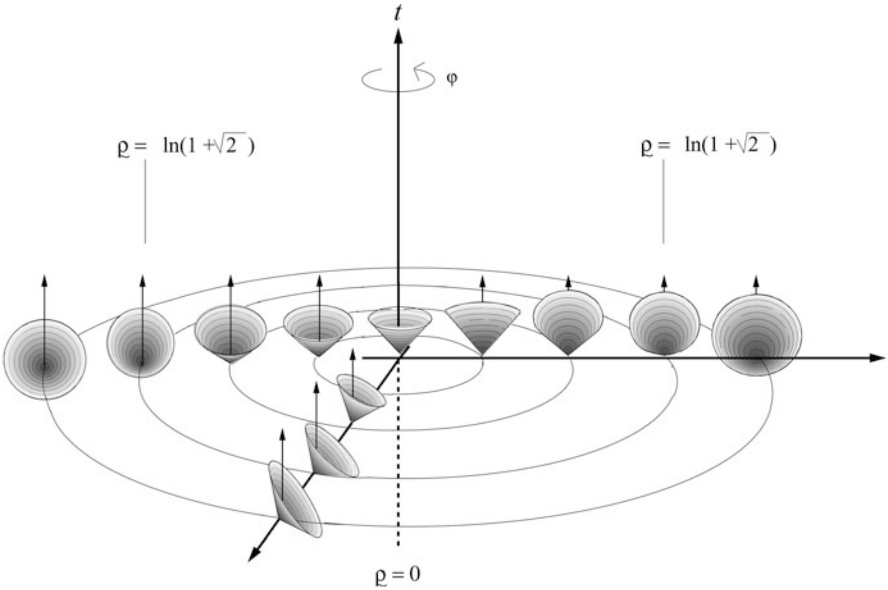

Fig. A6.9 The light-cone structure about an axis $\varrho = 0$ in the Gödel space–time. On the *left*, the light cones tip forward, and on the *right*, they tip backward. Note that at $\varrho = \ln(1+\sqrt{2})$ the light cones are sufficiently tipped over that the φ direction is null. At greater ϱ, the φ direction is timelike, indicating the presence of a closed timelike curve

up with him/herself after one complete "revolution"! This type of causal structure is similar to that depicted in Fig. A6.7b. A figure of this peculiar light-cone structure is displayed in Fig. A6.9 (also see figure caption).

The final solution we discuss is the *Van Stockum space–time* [251]. This solution to the field equations was one of the first to exhibit the behavior of CTCs. Physically, the solution describes an infinitely long cylinder of rotating dust with a vacuum exterior, the angular momentum preventing the gravitational collapse of the dust. The interior solution admits the following line element in corotating cylindrical coordinates:

$$\mathrm{d}s^2 = \mathrm{e}^{-\omega^2\varrho^2}\left(\mathrm{d}\varrho^2 + \mathrm{d}z^2\right) + \varrho^2(\mathrm{d}\varphi)^2 - \left(\mathrm{d}t - \omega\varrho^2\,\mathrm{d}\varphi\right)^2\,, \tag{A6.17}$$

with $-\infty < t < \infty,\ 0 < \varrho < \varrho_0,\ -\infty < z < \infty$ and $-\pi < \varphi < \pi$.

Here, ω represents the (constant) angular velocity of the dust cylinder. Again notice that the φ and t coordinates are coupled and that for $\omega\varrho > 1$ the periodic φ coordinate becomes timelike. One could argue that by placing the boundary of the dust cylinder at some $\varrho = \varrho_0 < \frac{1}{\omega}$, the problem of CTCs would be cured, as the above interior metric is no longer valid at $\omega\varrho = 1$. However, it has been shown that if $\omega\varrho_0 > \frac{1}{2}$ CTCs still appear in the vacuum exterior [248]. But causality violation is avoided if $\omega\varrho_0 \leq \frac{1}{2}$, although there may exist other pathologies. As can be seen here, strong rotation also causes the light cones to tip and a large enough value of ω will cause CTCs to occur somewhere in the space–time. This is analogous to the situation shown in Fig. A6.7b.

Before closing this section, we mention that the wormholes described earlier in this appendix may also be used as a time machine. One process involves the time dilation effect when two mouths of a wormhole move with respect to each other. The interested reader is referred to [247].

Appendix 7
Gravitational Instantons

Let us go back to Tricomi's p.d.e. of Example A2.5. It is given by

$$\partial_1 \partial_1 w + x^1 \cdot \partial_2 \partial_2 w = 0.$$

The partial differential equation above originates in the theory of gas dynamics [43]. The p.d.e. *is elliptic for the half-plane* $x^1 > 0$*, but hyperbolic for the other half-plane* $x^1 < 0$. Thus, this equation motivates us to consider mathematically the following metric in order to introduce the subject of this appendix:

$$\mathbf{g}^{\cdot\cdot}\left(x^1, x^2\right) := \left(\frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^1}\right) + x^1 \cdot \left(\frac{\partial}{\partial x^2} \otimes \frac{\partial}{\partial x^2}\right), \tag{A7.1i}$$

$$\mathbf{g}_{\cdot\cdot}\left(x^1, x^2\right) = \left(\mathrm{d}x^1 \otimes \mathrm{d}x^1\right) + \left(x^1\right)^{-1} \cdot \left(\mathrm{d}x^2 \otimes \mathrm{d}x^2\right), \tag{A7.1ii}$$

$$D_2 := \left\{\left(x^1, x^2\right) : x^1 > 0,\ x^2 \in \mathbb{R}\right\} \cup \left\{\left(x^1, x^2\right) : x^1 < 0,\ x^2 \in \mathbb{R}\right\}. \tag{A7.1iii}$$

The metric above is positive definite (or Riemannian) for $x^1 > 0$, but pseudo-Riemannian (of *signature zero*) for $x^1 < 0$. (A7.1iii) yields a union of apparently *two disconnected two-dimensional differential manifolds*.

Let us consider *another toy model* of a four-dimensional metric furnished by:

$$\mathbf{g}_{\cdot\cdot}(x) := \left(x^1\right)^4 \cdot \delta_{\alpha\beta} \cdot \left(\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta\right) - \left(x^1\right)^{-2} \cdot \left(x^4\right)^3 \cdot \left(\mathrm{d}x^4 \otimes \mathrm{d}x^4\right),$$

$$\mathrm{d}s^2 = \left(x^1\right)^4 \cdot \delta_{\alpha\beta} \cdot \left(\mathrm{d}x^\alpha \mathrm{d}x^\beta\right) - \left(x^1\right)^{-2} \cdot \left(x^4\right)^3 \cdot \left(\mathrm{d}x^4\right)^2,$$

$$D := \left\{x : x^1 > 0, x^2 \in \mathbb{R}, x^3 \in \mathbb{R}, x^4 > 0\right\} \cup \left\{x : x^1 > 0, x^2 \in \mathbb{R}, x^3 \in \mathbb{R}, x^4 < 0\right\}. \tag{A7.2}$$

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In case $x^4 > 0$, the above metric is transformable to the Kasner metric of Example 4.2.1. However, in case $x^4 < 0$, the metric yields *a Ricci-flat, positive definite metric*. Two distinct sectors of the manifold have a common three-dimensional, positive definite hypersurface with the metric:

$$\overline{\mathbf{g}}_{..}(\mathbf{x}) := \left(x^1\right)^4 \cdot \delta_{\alpha\beta} \cdot \left(\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta\right),$$

$$\mathbf{D} := \left\{\mathbf{x} : \; x^1 > 0, \; x^2 \in \mathbb{R}, \; x^3 \in \mathbb{R}\right\}.$$

The toy model in (A7.2) goes *beyond the traditional definition of either a Riemannian or else a pseudo-Riemannian manifold*. The common boundary $\mathbf{D}$ can be called an *instanton-horizon*.

The static, vacuum, space–time metrics can generate positive definite, Ricci-flat metrics as in (A7.2).

At this point, we should mention that a gravitational instanton metric *is not synonymous with a Riemannian metric*. That is, the presence of a positive definite metric is necessary but not sufficient to guarantee that the solution is a gravitational instanton.[1]

Theorem A7.1. Let the function $w(\mathbf{x})$ and the positive definite, three-dimensional metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ be of class C^3 in $\mathbf{D} \subset \mathbb{R}^3$. Moreover, let

$$\mathbf{g}_{..}(x) := \mathrm{e}^{-2w(\mathbf{x})} \cdot \overset{\circ}{\mathbf{g}}_{..}(\mathbf{x}) - \mathrm{e}^{2w(\mathbf{x})} \left(\mathrm{d}x^4 \otimes \mathrm{d}x^4\right)$$

satisfy vacuum field equations $\mathbf{R}_{..}(x) = \mathbf{O}_{..}(x)$ in $D := \mathbf{D} \times \mathbb{R}$. Then, the four-dimensional, positive definite metric

$$\widehat{\mathbf{g}}_{..}(x) := \mathrm{e}^{-2w(\mathbf{x})} \cdot \overset{\circ}{\mathbf{g}}_{..}(\mathbf{x}) + \mathrm{e}^{2w(\mathbf{x})} \cdot \left(\mathrm{d}x^4 \otimes \mathrm{d}x^4\right)$$

is Ricci flat in $\mathbf{D} \times \mathbb{R}$.

Proof. For both of the metrics $\mathbf{g}_{..}(x)$ and $\widehat{\mathbf{g}}_{..}(x)$, conditions of Ricci flatness reduce to *the same equations* (4.41i,ii). ■

However, such an easy transition to the positive definite status is *not possible from stationary, vacuum metrics* (*which are not static*). Let us explore this situation in more detail. Consider the "stationary," positive definite, four-dimensional metric:

$$\mathbf{g}_{..}(x) := \mathrm{e}^{-2u(\mathbf{x})} \cdot \overset{\circ}{\mathbf{g}}_{..}(\mathbf{x}) + \mathrm{e}^{2u(\mathbf{x})} \cdot \left[w_\alpha(\mathbf{x}) \cdot \mathrm{d}x^\alpha + \mathrm{d}x^4\right] \otimes \left[w_\beta(\mathbf{x}) \cdot \mathrm{d}x^\beta + \mathrm{d}x^4\right];$$

$$x = (\mathbf{x}, \, x^4) \in \mathbf{D} \times \mathbb{R}. \tag{A7.3}$$

[1] A gravitational instanton metric is a C^3 class of vacuum metrics in a "space–time" with a positive definite $+4$ signature which is geodesically complete. Often in cases of interest, they possess a self-dual or anti-self dual Riemann tensor (to be discussed shortly).

(Obviously, the metric above admits the Killing vector $\frac{\partial}{\partial x^4}$. Needless to say, the word timelike is undefined in this context.) Some of the Ricci-flat conditions imply the existence of potential $\mathcal{N}(\mathbf{x})$ such that

$$\partial_\alpha \mathcal{N} = (1/2) \cdot \mathrm{e}^{4u} \cdot \overset{\circ}{\eta}_\alpha{}^{\beta\gamma} \cdot \left(\partial_\gamma w_\beta - \partial_\beta w_\gamma\right). \tag{A7.4}$$

The function $\mathcal{N}(\mathbf{x})$ is called the *NUT potential*.

The remaining conditions of Ricci flatness reduce to the following p.d.e.s:

$$\overset{\circ}{R}_{\mu\nu}(\mathbf{x}) + 2\left(\partial_\mu u \cdot \partial_\nu u\right) - (1/2) \cdot \mathrm{e}^{-4u} \cdot \partial_\mu \mathcal{N} \cdot \partial_\nu \mathcal{N} = 0, \tag{A7.5i}$$

$$\overset{\circ}{\nabla}{}^2 u - (1/2) \cdot \mathrm{e}^{-4u} \cdot \overset{\circ}{g}{}^{\mu\nu} \cdot \partial_\mu \mathcal{N} \cdot \partial_\nu \mathcal{N} = 0, \tag{A7.5ii}$$

$$\overset{\circ}{\nabla}{}^2 \mathcal{N} - 4\, \overset{\circ}{g}{}^{\mu\nu} \cdot \partial_\mu u \cdot \partial_\nu \mathcal{N} = 0. \tag{A7.5iii}$$

Now we shall derive some conclusions about potentials $u(\mathbf{x})$ and $\mathcal{N}(\mathbf{x})$.

Theorem A7.2. Let field equations (A7.5ii,iii) hold in a star-shaped domain $\mathbf{D} \subset \mathbb{R}^3$.

(i) Moreover, let $\mathcal{N}(\mathbf{x})$ be continuous up to and on the boundary $\partial\mathbf{D}$. In case $\mathcal{N}(\mathbf{x})$ attains the extremum at an interior point $\mathbf{x}_0 \in \mathbf{D}$, the metric (A7.3) is transformable to the "static form."

(ii) Let the function $u(\mathbf{x})$, satisfying (A7.5ii) attain the maximum at an interior point $\mathbf{x}_0 \in \mathbf{D}$. Then, the four-dimensional curvature tensor $\mathbf{R}_{....}(x) \equiv \mathbf{O}_{....}(x)$ for all $x \in \mathbf{D} \times \mathbb{R}$.

Proof. (i) Use Theorems 4.2.2 and 4.2.3 for the function $\mathcal{N}(\mathbf{x})$.

(ii) Note that $\overset{\circ}{\nabla}{}^2 u = (1/2) \cdot \mathrm{e}^{-4u} \cdot \overset{\circ}{g}{}^{\mu\nu} \cdot \partial_\mu \mathcal{N} \cdot \partial_\nu \mathcal{N} \geq 0$. Use Hopf's Theorem 4.2.2 for the function $u(\mathbf{x})$.

■

Now *assume a functional relationship*

$$\mathrm{e}^{2u(\mathbf{x})} = f\,[\mathcal{N}(\mathbf{x})] > 0,$$

$$f' := \frac{\mathrm{d}\, f[\mathcal{N}]}{\mathrm{d}\mathcal{N}} \neq 0. \tag{A7.6}$$

Substituting (A7.6) into (A7.5ii,iii), subtracting (A7.5iii) from (A7.5ii), and assuming $\partial_\alpha \mathcal{N} \not\equiv 0$, we deduce that

$$\frac{f''}{f'} + \frac{f'}{f} - \frac{1}{f f'} = 0. \tag{A7.7}$$

(Compare and contrast the equation above with (4.156.)) The general solution of (A7.7) is furnished by

$$\begin{aligned} \{f[\mathcal{N}]\}^2 &= c_0 + 2c_1\mathcal{N} + (\mathcal{N})^2, \\ e^{4u(\mathbf{x})} &= \left[\mathcal{N}(\mathbf{x}) + c_1\right]^2 + \left(c_0 - c_1^2\right) > 0. \end{aligned} \tag{A7.8}$$

Here, c_0 and c_1 are constants of integration. We now make special choices

$$\begin{aligned} c_0 &= c_1 = 0, \\ e^{2u(\mathbf{x})} &= \mathcal{N}(\mathbf{x}) > 0. \end{aligned} \tag{A7.9}$$

Field equations (A7.5i) yield

$$\overset{\circ}{R}_{\mu\nu}(\mathbf{x}) = 0. \tag{A7.10}$$

Thus, the three-dimensional metric $\overset{\circ}{\mathbf{g}}_{..}(\mathbf{x})$ *is (flat) Euclidean*. Choosing a Cartesian chart for $\mathbf{D} \subset \mathbb{R}^3$, (A7.5ii,iii) boil down to the Euclidean Laplace's equation:

$$\nabla^2 \left\{[\mathcal{N}(\mathbf{x})]^{-1}\right\} := \delta^{\mu\nu} \cdot \partial_\mu \partial_\nu \left\{[\mathcal{N}(\mathbf{x})]^{-1}\right\} = 0. \tag{A7.11}$$

Equations (A7.4) reduce to

$$\begin{aligned} \partial_\beta w_\alpha - \partial_\alpha w_\beta &= -\varepsilon_{\alpha\beta\gamma} \cdot \partial_\gamma \left\{[\mathcal{N}(\mathbf{x})]^{-1}\right\}, \\ \text{or,} \quad \vec{\nabla} \times \vec{\mathbf{w}}(\mathbf{x}) &= \vec{\nabla} \cdot \left\{[\mathcal{N}(\mathbf{x})]^{-1}\right\}. \\ \text{or,} \quad \operatorname{curl} \vec{\mathbf{w}} &= \operatorname{grad}\left\{[\mathcal{N}(\mathbf{x})]^{-1}\right\}. \end{aligned} \tag{A7.12}$$

The "stationary metric" reduces to

$$ds^2 = [\mathcal{N}(\mathbf{x})]^{-1} \cdot \delta_{\mu\nu} \cdot (dx^\mu dx^\nu) + [\mathcal{N}(\mathbf{x})] \cdot \left[w_\alpha(\mathbf{x})\, dx^\alpha + dx^4\right]^2. \tag{A7.13}$$

The class of exact solutions given by (A7.13), (A7.11), and (A7.12) was discovered by Kloster, Som, and Das (K–S–D in short) in 1974 [149]. Hawking rediscovered the same metric in 1977 [125]. We shall call the metric in (A7.13) as the *H–K–S–D metric*.

Now recall *the Hodge-dual mapping* in (1.113). It implies for the curvature tensor

$${}^*R_{ijkl}(x) = (1/2) \cdot \eta_{rskl}(x) \cdot R_{ij}{}^{rs}(x). \tag{A7.14}$$

In case

$${}^*R_{ijkl}(x) \equiv R_{ijkl}(x), \tag{A7.15}$$

the curvature tensor $\mathbf{R}_{....}(x)$ is called *self-dual*. On the other hand, if

$$^{*}R_{ijkl}(x) \equiv -(1/2) \cdot \eta_{rskl}(x) \cdot R_{ij}{}^{rs}(x) = -R_{ijkl}(x), \tag{A7.16}$$

the curvature tensor is called *anti-self-dual*. Both (A7.15) and (A7.16) can be combined into

$$^{*}R_{ijkl}(x) \equiv \varepsilon \cdot R_{ijkl}(x), \qquad \varepsilon = \pm 1. \tag{A7.17}$$

It can be proved that in case of the H–K–S–D metric of (A7.13), the condition in (A7.17) is *identically satisfied* by the use of (A7.9) [110].

Instantons are regular, finite-action solutions of the field equations in a four-dimensional, positive definite, flat (or curved) "space–time". For Yang-Mills type of gauge field theories, the construction of Atiyah, Drinfield, Hitchin, and Manin [11] allows one to compute all regular, self-dual (or anti-self-dual) solutions. However, in the case of gravitation, the situation is more complicated. *We define gravitational instantons as positive definite metrics of class C^3, which are Ricci flat and geodesically complete*. There exist gravitational instantons which are neither self-dual, nor anti-self-dual. A class of these are provided by the "static metrics" of Theorem A7.1. Fortunately, H–K–S–D metrics supply infinitely many self-dual (or anti-self-dual) gravitational instantons which are useful for some path-integral approaches to quantum gravity.

Example A7.3. Consider *the multicenter H–K–S–D gravitational instantons* furnished by the function

$$[\mathcal{N}(\mathbf{x})]^{-1} = a + \sum_{A=1}^{N} \frac{2m_A}{\| \mathbf{x} - \mathbf{x}_A \|} > 0. \tag{A7.18}$$

Here, $a \geq 0$ and $m_A > 0$ are constants. (Compare with the pseudo-Riemannian Example 4.2.8.) If $m_A = m$ for all $A \in \{1, \ldots, N\}$ and if x^4 is *a periodic variable* with the fundamental period in $0 \leq x^4 \leq (\kappa\, m/N)$, then singularities at $\mathbf{x} = \mathbf{x}_A$ are *removable*. Such solutions qualify as gravitational instantons. In case $a = 1$ and $N = 1$, the metric is called the self-dual *Taub-NUT metric* [126]. In case $a = 0$ and $N = 2$, we obtain the *Eguchi-Hansen gravitational instantons* [85]. □

Appendix 8
Computational Symbolic Algebra Calculations

In this appendix, we present some calculations relevant to general relativity utilizing popular symbolic algebra computer programs. Calculations in general relativity are notoriously lengthy and complicated, and therefore, computational symbolic algebra programs are very useful in this field. Two popular symbolic algebra programs are *Maple™* and *Mathematica™* , both of which we concentrate on here. For the sake of brevity, *we do not discuss special packages*, available for both programs, which are designed to supplement the ability of these programs to do general relativistic calculations. (For example, we do not discuss the use of the excellent add-on package available for both Maple™ and Mathematica™ called GRTensorII, which is available at http://grtensor.phy.queensu.ca/).

A8.1 Sample Maple™ Work Sheet

This work sheet deals with quantities related to the vacuum Schwarzschild metric:

```
> #MAPLE worksheet involving the vacuum Schwarzschild
  metric.

> restart: #The restart command clears all set
  variables.
> with(tensor): #This loads Maple's tensor package.

> coord:=[r, theta, phi, t]; #The coord command
  defines˜the four coordinates.
```

$$coord := [r, \theta, \phi, t]$$

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```
> g_compts:=array(symmetric, sparse, 1..4, 1..4);
  #create a 4x4 sparse array, to be used for metric
  components. The symmetric command means that if
  you have non-diagonal components, you only have
  to enter one of the symmetric pair, the other will
  automatically be assumed.
```

$$g\_compts := \mathrm{array}(symmetric,\ sparse,\ 1..4,\ 1..4,\ [])$$

```
> g_compts[1,1]:=1/(1-2*m/r); #This is g_{11}.
```

$$g\_compts_{1,1} := \frac{1}{1-\frac{2\,m}{r}}$$

```
> g_compts[2,2]:=r^2; #This is g_{22}.
```

$$g\_compts_{2,2} := r^2$$

```
> g_compts[3,3]:=r^2*(sin(theta))^2; #This is g_{33}.
```

$$g\_compts_{3,3} := r^2\sin(\theta)^2$$

```
> g_compts[4,4]:=-1/g_compts[1,1]; #This is g_{44}.
```

$$g\_compts_{4,4} := -1+\frac{2\,m}{r}$$

```
> g:=create([-1,-1], eval(g_compts)); #This command
  creates the metric. The [-1,-1] indicates covariant
  components on both indices.
```

$$g := \mathrm{table}([index\_char = [-1,\ -1],$$

$$compts = \begin{bmatrix} \frac{1}{1-\frac{2\,m}{r}} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & -1+\frac{2\,m}{r} \end{bmatrix}])$$

```
> ginv:=invert(g,'detg'); #Creates the inverse
  metric.
```

$$ginv := \mathrm{table}([index\_char = [1,\ 1],$$

$$compts = \begin{bmatrix} -\frac{-r+2\,m}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2\sin(\theta)^2} & 0 \\ 0 & 0 & 0 & \frac{r}{-r+2\,m} \end{bmatrix}])$$

```
> D1g:=d1metric(g,coord): #Creates first partial
  derivatives of the metric. Note the colon (instead
  of semi-colon). This suppresses the output, which
  can be lengthy.

> D2g:=d2metric(D1g,coord): #Creates the second
  partial derivatives of the metric.

> Cf1:=Christoffel1(D1g): #Creates Christoffel
  symbols of the first kind.

> Cf2:=Christoffel2(ginv,Cf1): #Creates Christoffel
  symbols of the second kind.

> displayGR(Christoffel2, Cf2); #Display only the
  non-zero components of the Christoffel symbols
  of the second kind. The {a,bc} indicates that the
  first index is ''contravariant'' and the second
  and third indices are ''covariant''.
```

The Christoffel Symbols of the Second Kind

$$non-zero\ components\ :$$

$$\{1,11\} = \frac{m}{(-r+2\,m)\,r}$$

$$\{1,22\} = -r+2\,m$$

$$\{1,33\} = (-r+2\,m)\sin(\theta)^2$$

$$\{1,44\} = -\frac{(-r+2\,m)\,m}{r^3}$$

$$\{2,12\} = \frac{1}{r}$$

$$\{2,33\} = -\sin(\theta)\cos(\theta)$$

$$\{3,13\} = \frac{1}{r}$$

$$\{3,23\} = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\{4,14\} = -\frac{m}{(-r+2\,m)\,r}$$

```
> RMN:=Riemann(ginv,D2g,Cf1): #Creates the Riemann
  curvature tensor.

> displayGR(Riemann,RMN); #Display only the non-zero
  components of the Riemann tensor.
```

The Riemann Tensor

$$non-zero\ components\ :$$

$$R1212 = \frac{m}{-r + 2\,m}$$

$$R1313 = \frac{m\sin(\theta)^2}{-r + 2\,m}$$

$$R1414 = -\frac{2\,m}{r^3}$$

$$R2323 = 2\,r\,m\sin(\theta)^2$$

$$R2424 = -\frac{(-r + 2\,m)\,m}{r^2}$$

$$R3434 = -\frac{(-r + 2\,m)\,m\sin(\theta)^2}{r^2}$$

$$character : [-1, -1, -1, -1]$$

```
> RICCI:=Ricci(ginv,RMN); #Create the Ricci tensor.
  Maple contracts the first and fourth index of the
  Riemann tensor in order to create the Ricci tensor,
  compatible with the conventions in this book. It
  is zero here since we are dealing with a vacuum
  solution.
```

$$RICCI := \text{table}([index\_char = [-1, -1], compts = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}])$$

```
> RS:=Ricciscalar(ginv,RICCI); #The Ricci scalar.
```

$$RS := \text{table}([index\_char = [], compts = 0])$$

```
> upRMN:=raise(ginv, RMN, 1,2,3,4): #Calculates the
  contravariant Riemann tensor by raising the first,
  second, third and fourth index using the inverse
  metric.
```

```
> Kret:=prod(RMN,upRMN,[1,1],[2,2],[3,3],[4,4]);
  #Creates the Kretschmann scalar by multiplying
  the covariant and contravariant Riemann tensor and
  contracting first index with first, second with
  second, third with third and fourth with fourth.
```

$$Kret := \text{table}([index\_char = [], compts = \frac{48\,m^2}{r^6}])$$

```
> Estn:=Einstein(g,RICCI,RS): #Calculates the
  covariant Einstein tensor.
```

```
> mixeinst:=raise(ginv,Estn,1): #Raises the first
  index of the covariant Einstein tensor, creating
  the mixed-character Einstein tensor.
```

```
> displayGR(Einstein,mixeinst); #Displays only
  the non-zero components of the mixed-character
  Einstein tensor. There are no non-zero
  components. Caution must be applied when using
  this command to display this tensor's components;
  only components where the row-index is less than
  or equal to the column-index will be displayed.
```

$$\textit{The Einstein Tensor}$$
$$non-zero\ components\ :$$
$$None$$
$$character\ :\ [1,\ -1]$$

A8.2 Sample Mathematica™ Notebook

This notebook deals with quantities related to the vacuum Schwarzschild metric. It is based on a notebook which originally supplemented the text *Gravity: An introduction to Einstein's general relativity* by Hartle [124].

`Mathematica notebook involving the vacuum Schwarzschild metric.`

`This next command clears various variables that might be set. Note that in` *`Mathematica`*`, you need to press shift - enter after you are finished typing a cell.`

Clear[coord, metric, ginv, riemann, ricci, rs, einstein, christoffel, *r*, theta, phi, *t*]

`This next command sets the coordinates to be used.`

coord = {*r*, theta, phi, *t*}

Out[2]={*r*, theta, phi, *t*}

`This next command creates the metric.`

metric = {{1/(1 − 2 ∗ *m*/*r*), 0, 0, 0}, {0, *r*^2, 0, 0}, {0, 0, *r*^2 ∗ (Sin[theta])^2, 0}, {0, 0, 0, −(1 − 2 ∗ *m*/*r*)}}

Out[3]= $\left\{\left\{\frac{1}{1-\frac{2m}{r}}, 0, 0, 0\right\}, \left\{0, r^2, 0, 0\right\}, \left\{0, 0, r^2\text{Sin[theta]}^2, 0\right\}, \left\{0, 0, 0, -1+\frac{2m}{r}\right\}\right\}$

This next command will display the metric in matrix form.

metric//MatrixForm

Out[4]//MatrixForm= $\begin{pmatrix} \frac{1}{1-\frac{2m}{r}} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\text{Sin[theta]}^2 & 0 \\ 0 & 0 & 0 & -1+\frac{2m}{r} \end{pmatrix}$

This next command will create the inverse metric.

ginv1 = Inverse[metric]

Out[5]= $\left\{\left\{\frac{2mr^3\text{Sin[theta]}^2 - r^4\text{Sin[theta]}^2}{\frac{2mr^3\text{Sin[theta]}^2}{1-\frac{2m}{r}} - \frac{r^4\text{Sin[theta]}^2}{1-\frac{2m}{r}}}, 0, 0, 0\right\}, \left\{0, \frac{\frac{2mr\text{Sin[theta]}^2}{1-\frac{2m}{r}} - \frac{r^2\text{Sin[theta]}^2}{1-\frac{2m}{r}}}{\frac{2mr^3\text{Sin[theta]}^2}{1-\frac{2m}{r}} - \frac{r^4\text{Sin[theta]}^2}{1-\frac{2m}{r}}}, 0, 0\right\},\right.$
$\left.\left\{0, 0, \frac{\frac{2mr}{1-\frac{2m}{r}} - \frac{r^2}{1-\frac{2m}{r}}}{\frac{2mr^3\text{Sin[theta]}^2}{1-\frac{2m}{r}} - \frac{r^4\text{Sin[theta]}^2}{1-\frac{2m}{r}}}, 0\right\}, \left\{0, 0, 0, \frac{r^4\text{Sin[theta]}^2}{\left(1-\frac{2m}{r}\right)\left(\frac{2mr^3\text{Sin[theta]}^2}{1-\frac{2m}{r}} - \frac{r^4\text{Sin[theta]}^2}{1-\frac{2m}{r}}\right)}\right\}\right\}$

The last output was messy. Let's simplify it a bit with the Simplify command.

ginv = Simplify[ginv1]

Out[6]= $\left\{\left\{1-\frac{2m}{r}, 0, 0, 0\right\}, \left\{0, \frac{1}{r^2}, 0, 0\right\}, \left\{0, 0, \frac{\text{Csc[theta]}^2}{r^2}, 0\right\}, \left\{0, 0, 0, \frac{r}{2m-r}\right\}\right\}$

We can also display the inverse metric in matrix form.

ginv//MatrixForm

Out[7]//MatrixForm= $\begin{pmatrix} 1-\frac{2m}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{\text{Csc[theta]}^2}{r^2} & 0 \\ 0 & 0 & 0 & \frac{r}{2m-r} \end{pmatrix}$

We next calculate the Christoffel symbols of the second kind. The triplets such as $\{i, 1, 4\}$ in the braces correspond to the index (i) and the range of the index (1 to 4). The operator D[metric[[s, j]], coord[[k]]] calculates the partial derivative of the $s-j$ metric component with respect to the coordinate k.

christoffel = Simplify[Table[(1/2) ∗ Sum[(ginv[[i, h]]) ∗ (D[metric[[h, j]], coord[[k]]] + D[metric[[h, k]], coord[[j]]] − D[metric[[j, k]], coord[[h]]]), {h, 1, 4}], {i, 1, 4}, {j, 1, 4}, {k, 1, 4}]]

Out[8]= $\left\{\left\{\left\{\frac{m}{2mr-r^2},0,0,0\right\},\{0,2m-r,0,0\},\left\{0,0,(2m-r)\mathrm{Sin}[\mathrm{theta}]^2,0\right\},\right.\right.$ $\left.\left.\left\{0,0,0,\frac{m(-2m+r)}{r^3}\right\}\right\},\left\{\left\{0,\frac{1}{r},0,0\right\},\left\{\frac{1}{r},0,0,0\right\},\{0,0,-\mathrm{Cos}[\mathrm{theta}]\mathrm{Sin}[\mathrm{theta}],0\},\right.\right.$ $\left.\{0,0,0,0\}\right\},\left\{\left\{0,0,\frac{1}{r},0\right\},\{0,0,\mathrm{Cot}[\mathrm{theta}],0\},\left\{\frac{1}{r},\mathrm{Cot}[\mathrm{theta}],0,0\right\},\{0,0,0,0\}\right\},$ $\left.\left\{\left\{0,0,0,-\frac{m}{2mr-r^2}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{-\frac{m}{2mr-r^2},0,0,0\right\}\right\}\right\}$

Next the Riemann tensor (first index contravariant, second through fourth index covariant) is created. ***Note that this differs from the one calculated in the Maple™ worksheet which was completely covariant in character***. The riemann:= statement supresses the output.

riemann:=riemann = Simplify[Table[
D[christoffel[[i, j, l]], coord[[k]]] − D[christoffel[[i, j, k]], coord[[l]]]+
Sum[christoffel[[s, j, l]]christoffel[[i, k, s]] − christoffel[[s, j, k]]christoffel[[i, l, s]],
{s, 1, 4}], {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}]]

This next command will display only the non - zero Riemann components. ***Note that these differ from the ones calculated in the Maple™ worksheet which was completely covariant in character***. The output from the first command is supressed.

riemannarray:=Table[If[UnsameQ[riemann[[i, j, k, l]], 0], {ToString[R[i, j, k, l]],
riemann[[i, j, k, l]]}], {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, k − 1}]

TableForm[Partition[DeleteCases[Flatten[riemannarray], Null], 2]]

Out[11]//TableForm=

R[1, 2, 2, 1]	$\frac{m}{r}$
R[1, 3, 3, 1]	$\frac{m\mathrm{Sin}[\mathrm{theta}]^2}{r}$
R[1, 4, 4, 1]	$\frac{2m(-2m+r)}{r^4}$
R[2, 1, 2, 1]	$\frac{m}{(2m-r)r^2}$
R[2, 3, 3, 2]	$-\frac{2m\mathrm{Sin}[\mathrm{theta}]^2}{r}$
R[2, 4, 4, 2]	$\frac{m(2m-r)}{r^4}$
R[3, 1, 3, 1]	$\frac{m}{(2m-r)r^2}$
R[3, 2, 3, 2]	$\frac{2m}{r}$
R[3, 4, 4, 3]	$\frac{m(2m-r)}{r^4}$
R[4, 1, 4, 1]	$-\frac{2m}{(2m-r)r^2}$
R[4, 2, 4, 2]	$-\frac{m}{r}$
R[4, 3, 4, 3]	$-\frac{m\mathrm{Sin}[\mathrm{theta}]^2}{r}$

This next command creates the Ricci tensor. Note the summation over the index i.

ricci:=ricci = Simplify[Table[Sum[riemann[[i, j, l, i]], {i, 1, 4}], {j, 1, 4}, {l, 1, 4}]]

```
Next we display only the non - zero components of the
Ricci tensor. There are no non-zero components.
```

ricciarray:=Table[If[UnsameQ[ricci[[j, l]], 0], {ToString[R[j, l]], ricci[[j, l]]}], {j, 1, 4}, {l, 1, j}]

TableForm[Partition[DeleteCases[Flatten[ricciarray], Null], 2]]
Out[14]//TableForm =
{}

```
We next calculate the Ricci curvature scalar.
```

RS = Sum[ginv[[i, j]]ricci[[i, j]], {i, 1, 4}, {j, 1, 4}]
Out[15]=0

```
We next calculate the Einstein tensor, by explicitly
creating it with the Ricci tensor, the Ricci scalar and
the metric.
```

einstein:=einstein = ricci − (1/2)RS ∗ metric

```
Display only the non - zero components of the Einstein
tensor. There are no non-zero components, indicating a
vacuum solution.
```

einsteinarray:=Table[If[UnsameQ[einstein[[j, l]], 0], {ToString[G[j, l]], einstein[[j, l]]}], {j, 1, 4}, {l, 1, j}]

TableForm[Partition[DeleteCases[Flatten[einsteinarray], Null], 2]]
Out[18]//TableForm={}

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