Remarks on the Numerical analysis of basins of attraction using complex variables

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IV CONFERENCE ON DYNAMICS, CONTROL, AND APPLICATIONS TO APPLIED ENGINEERING AND LIFE SCIENCE AND
I WORKSHOP ON NONLINEAR DYNAMICS, ENERGY PRODUCTION, RENEWABLE ENERGY SOURCES, ENERGY TRANSFER AND ENERGY HARVESTING

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Outline

- If we consider complex variables ($z \in \mathbb{C}$)
- Chaotic systems with real variables are widely studied.
- Chaotic systems with complex variables is a more recent issue for nonlinear dynamic analysis
- Its applications range from: physics, quantum mechanics, astrophysics, dynamics of high energy accelerators, etc. [1] [2].

[1] Marshall, Delmar, and J. C. Sprott. "Simple driven chaotic oscillators with complex variables." Chaos: An Interdisciplinary Journal of Nonlinear Science 19.1 (2009).

[2] Mahmoud, G.M., Mahmoud, E.E. Phase and antiphase synchronization of two identical hyperchaotic complex nonlinear systems. Nonlinear Dyn 61, 141–152 (2010).

Mathematical Modeling

- Considering the variables $z \in$ described by $z = x + yi$, $\dot{z} = \dot{x} + i\dot{y}$ and $\bar{z} = x - yi$, therefore, an oscillator can be described as follows:

$$\dot{z} + f(z, \bar{z}) = A\mathrm{e}^{\mathrm{i}\omega \mathrm{t}} \tag{1}$$

- In this way, rewriting Eq. 1, we have:

$$\begin{aligned} \dot{x} &= -\Re(f(z, \bar{z})) + Acos(\omega t) \\ \dot{y} &= -\Im(f(z, \bar{z})) + Asin(\omega t) \end{aligned} \tag{2}$$

- Whereas:

$$f(z, \bar{z}) = (z^2 - \bar{z}^2)z + \bar{z} \tag{3}$$

Therefore, we have:

$$\begin{aligned} \dot{x} &= 4xy^2 - x + Acos(\omega t) \\ \dot{y} &= -4x^2y - y + Asin(\omega t) \end{aligned} \tag{4}$$

Uncertainty Coefficient

- This is related to the sensitivity of the final state of the phase space trajectories.
  - An exponent close to 1 means that the basin of attraction has smooth contours;
  - An exponent close to 0 means that the attraction basin is fractalized (sieved attraction basins).
- The uncertainty coefficient defines the uncertainty of the initial conditions of the system, which can be attracted by two or more attractors.

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Uncertainty Coefficient and entropy of basins of attraction

- A measure to quantify the uncertainty of basins of attraction;
- Provides an efficient condition for the existence of fractal limits, when the entropy is greater than $log(2)$ the basin is fractal;
- The basin of attraction is considered with a phase space coloring where each initial condition corresponds to a color referring to an attractor[1].
- To calculate entropy, we define a ball of radius ε, the proportion of the colors of each ball ε defines the probability P_j associated with each color[2]. The entropy of basins of attraction is defined as the mean Gibbs entropy (S_i) which is given by:

$$S_i = -\sum_{j=1}^{N_a} P_j log(P_j) \tag{5}$$

Where N_a is the number of different colors of the attraction basin,

[1] Daza, A., Wagemakers, A., & Sanjuán, M. A. (2022). Classifying basins of attraction using the basin entropy. Chaos, Solitons & Fractals, 159, 112112.

[2] Daza, Alvar, et al. "Basin entropy: a new tool to analyze uncertainty in dynamical systems." Scientific reports 6.1 (2016): 31416.

Numerical Results

- For numerical analyzes we used the 4th order Runge and Kutta method, with an integration step $h = 0.001$ and a total time of $10^6[s]$
- The transient time of 40% of the total time;
- $\omega = 1.0$
- $A \in [0.5, 1.0]$
- Attraction basins a grid $x^0 \in [-1, 1] \times y^0 \in [-1, 1]$ of 800×800 points;

Mathematical Modeling

$$
\begin{aligned}
\dot{x} &= 4xy^2 - x + Acos(\omega t) \\
\dot{y} &= -4x^2y - y + Asin(\omega t)
\end{aligned}
\qquad (6)
$$

Numerical Results

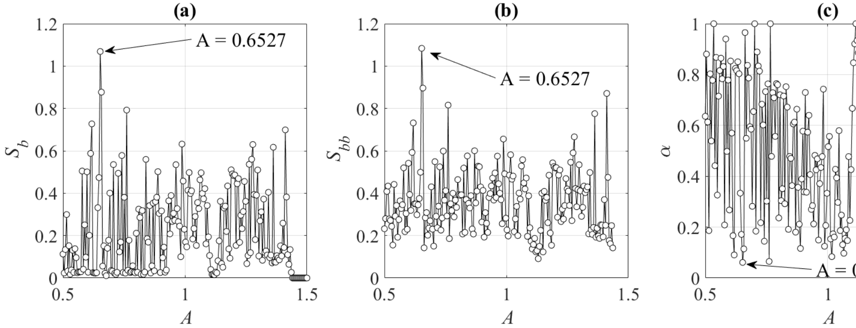
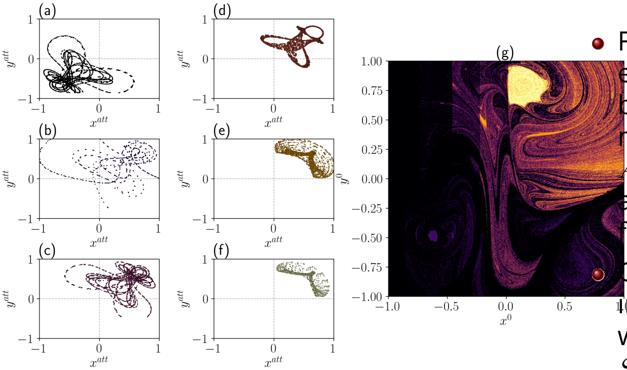


Figure: (a) Entropy basin, (b) Entropy basin boundery and (c) uncertain coefficient



- For $A = 0.6527$ the entropy of the attraction basins and the edge are maximum, with $A = 0.6527 > log(2)$ the attraction basin has a fractal behavior
- Containing 6 represented in fig. 2 (a)-(f) attractors, with $S_b = 1.069$, $S_{bb} = 1.084$ and $\alpha = 0.061$.

Figure: (a)-(f) attractors e (g) Basin of attraction.

Numerical Results

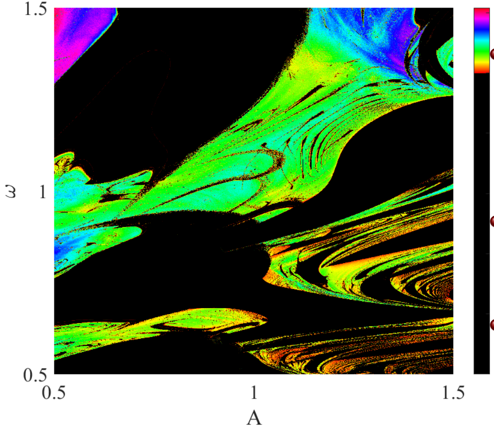
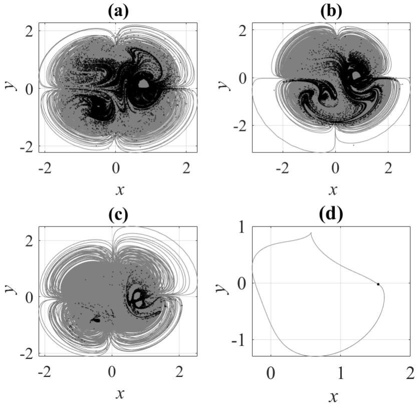


Figure: Lyapunov Exponent for $A \in [0.5, 1.0] \times \omega \in [0.5, 1.0]$

- Considering the initial condition $(0, 0)$ we calculate the Lyapunov Exponent (λ) considering the following intervals $\omega \in [0.5, 1.0] \times A \in [0.5, 1.0]$
- Region in black $(\lambda < 0)$ regions in which the system has orbits with periodic behavior.
- Region between xxx and red $(\lambda > 0)$ region in which the system has orbits with chaotic behavior.



- (a) $A = 0.6527$, chaotic behavior
- (b) $A = 1.0$, chaotic behavior
- (c) $A = 1.1$, chaotic behavior
- (d) $A = 1.3$, periodic behavior.

Figure: Phase maps (Gray line) and Poincare maps (Black dots for$\omega = 1.0$. (a)$A = 0.6527$,(b) $A = 1.0$,(c) $A = 1.1$ and (d) $A = 1.3$

Conclusion

- The use of the entropy of attraction basins and the uncertainty coefficient corroborates the determination of attraction basins with screening in oscillatory systems in complex variables.
- Such analyzes also allow determining the behavior of attraction basins in electromechanical systems.
- The dynamic analyzes with the Maximum Lyapunov Exponent (λ) are the regions for the parametric analysis $(A \times \omega)$ the chaos regions and the periodic regions.
- Phase and Poincaré Maps showed the behavior of trajectories in phase space.
- For future work, determine the parameter space for entropy and the uncertainty coefficient to determine the sieve regions of the attraction basins.

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References

1. Daza, Alvar, et al. ”Basin entropy: a new tool to analyze uncertainty in dynamical systems.” Scientific reports 6.1 (2016): 31416.
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3. Puy, Andreu, et al. ”A test for fractal boundaries based on the basin entropy.” Communications in Nonlinear Science and Numerical Simulation 95 (2021): 105588.
4. Grebogi, Celso, et al. ”Final state sensitivity: an obstruction to predictability.” Physics Letters A 99.9 (1983): 415-418.
5. McDonald, Steven W., et al. ”Fractal basin boundaries.” Physica D: Nonlinear Phenomena 17.2 (1985): 125-153.

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