Assignment 1 (MATLAB)

Phys/Biol 319, Prof. Paul Wiseman

Due February 1st, 2021 5 pm (online submission on MyCourses)

`Include your m-files with your assignment. We will not run your code, so include the output (graphs, etc) in the assignment.`

1 Matrix Limits

Consider the following $n \times n$ matrix,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & \dots & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & \dots & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & \dots & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & \dots & 0 & 1/2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1/2 & 0 & \dots & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & \dots & 1/2 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

which can be written in short-hand as,

$$\begin{cases} a_{m,m+3} = a_{m+3,m} = 1/2, & \text{for } m \leq n-3 \\ a_{n-2,1} = a_{1,n-2} = 1/2, & \\ a_{n-1,2} = a_{2,n-1} = 1/2, & \\ a_{n,3} = a_{3,n} = 1/2, & \\ a_{m,n} = 0, & \text{otherwise} \end{cases}$$

This matrix has some special properties that we will examine.

(a) Construct the matrix A in `MATLAB` for $n = 10$. **Hint:** use the built-in functions `eye` and `circshift` to do this.

(b) Plot (i) the maximum and (ii) the second lowest number in the matrix A^N as a function of N (n is still 10). Plot both curves on the same figure.

(c) Deduce the limits,

$$\lim_{N\to\infty} A^{2N} \quad \text{and} \quad \lim_{N\to\infty} A^{2N+1} \tag{2}$$

for $n = 10$. You may use the same format presented in equation 1 given above or use index notation to write your answer.

(d) Deduce *only the non-zero values* in the limiting matrix,

$$\lim_{N\to\infty} A^{N} \tag{3}$$

for **any** value of n. **Hint:** play around with the limiting behaviour for different values of n to see a pattern.

2 Estimating the area of a circle

Assume for a moment that we don't know how to calculate the area of a circle. Using dimensional analysis, we can determine that

$$A \propto r^2 \tag{4}$$

but we don't yet know the value of the proportionality constant (from experience, we can call it π). In this problem we will cleverly determine the value of π through the use of uniform distributions generated in MATLAB.

Consider the geometry shown in figure 1 below. A circle of radius 1 (and therefore area π) is centered at the origin and a square of side length 2 is superimposed. The idea is that we generate a uniform distribution of points inside of the boundaries of the square. The ratio of these points which fall inside of the circle to the total number of points generated should be equal to the ratio of the area of the square over the area of the circle. This can be written as so:

$$\frac{\text{area of square}}{\text{area of circle}} = \frac{4}{\pi} = \frac{\text{generated points which fall inside of the square}}{\text{generated points which fall inside of the circle}} \tag{5}$$

(a) First, randomly generate N points (x, y) which follow a uniform distribution inside of the square.

(b) iterate through each of the points generated. For each point, check if it falls inside of the bounds of the circle and keep a tally. You can keep a tally of points in the circle by initiating a variable c and increment it by 1 every time a point satisfies your condition.

(c) Once you have the tally of points generated inside of the circle, use equation 5 to determine π.

`Give your estimate of` π `for the following total number of points in the distribution:` $N = 100, 1000, 10000$.

`Generate scatter plots for each case and give distinct colors to the points which fall on either side of the boundary of the circle.`

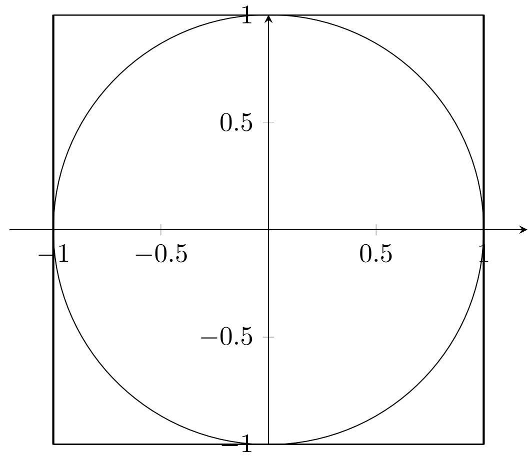

Figure 1: Unit circle at the origin with a 2×2 square superimposed.

3 Counting Photons

A detector is used to record photons coming from a weak light source. Each time a photon hits the detector, the time is recorded. A sample dataset of these times (in seconds) is provided in the file `times.mat`.

(a) Plot a histogram of the times between two consecutive photons (these are not the times provided). Use the `histogram` command to accomplish this task.

We would like to make the area of this histogram equal to 1, so that we can think of it as a probability distribution. In order to do this, pass the `'Normalization'` option to `histogram` with the value `'pdf'`.

The histogram follows the well-known *exponential* probability distribution which has the form,

$$f(t; k) = ke^{-kt} \tag{6}$$

where k can be interpreted as the mean number of photons detected per second, and t is time. Try to find k by overlaying the expression above with different values of k to see which one seems to fit the data best.

For this question, include your fit overlaid with the histogram along with the value of k you found.

(b) We would now like to measure the probability of finding x photons during a large time interval $T = 5$ s. To do that, we will split T into N smaller intervals of length dt (such that $N * dt = T$). It can be shown that the probability to observe a **single** photon during a time interval of length dt is given by kdt as long as the approximation $kdt << 1$ holds. For our purposes, we can choose $dt = 0.01$ s, which should be small enough.

For each of these N segments, toss a weighted coin which has a probability of success given by kdt. IF the coin toss is successful, we record the event because it means we have detected a photon. We will repeat this $N = 500$ times to cover the 5 s interval. Because the photon emission process is random, different runs of this experiment should produce different photon counts. Thus, to get an idea of the probability distribution, we will run the experiment M times where M is large (you can take $M = 5000$ for example).

Plot a histogram of your counts and use the `'probability'` option as your `'Normalization'`. This histogram is well approximated by the *Poisson* probability distribution (although the true distribution is binomial) given by,

$$P(X = x; \lambda) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad \text{where } x = 0, 1, 2, ... \tag{7}$$

Here λ can be interpreted as the mean number of photons detected in a 5 s interval. Using the k you found in part *(a)*, what is λ? Check that this is the case by superimposing the expression above with the appropriate value of λ.

For this question, include your fit overlaid with the histogram along with the value of λ you found.

Note: the difference between the normalization in parts (a) and (b) is due to the difference between the *discrete* and *continuous* probability distributions. The former type of distribution can be defined on the integers (i.e. a countable set), whereas the latter can be defined on the reals (i.e. an uncountable set).