252R 기초 데이터 분석 및 실습

3. Mathematics (1)

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Welcome To Industrial Engineering

고려대학교
KOREA UNIVERSITY

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252R 기초 데이터 분석 및 실습

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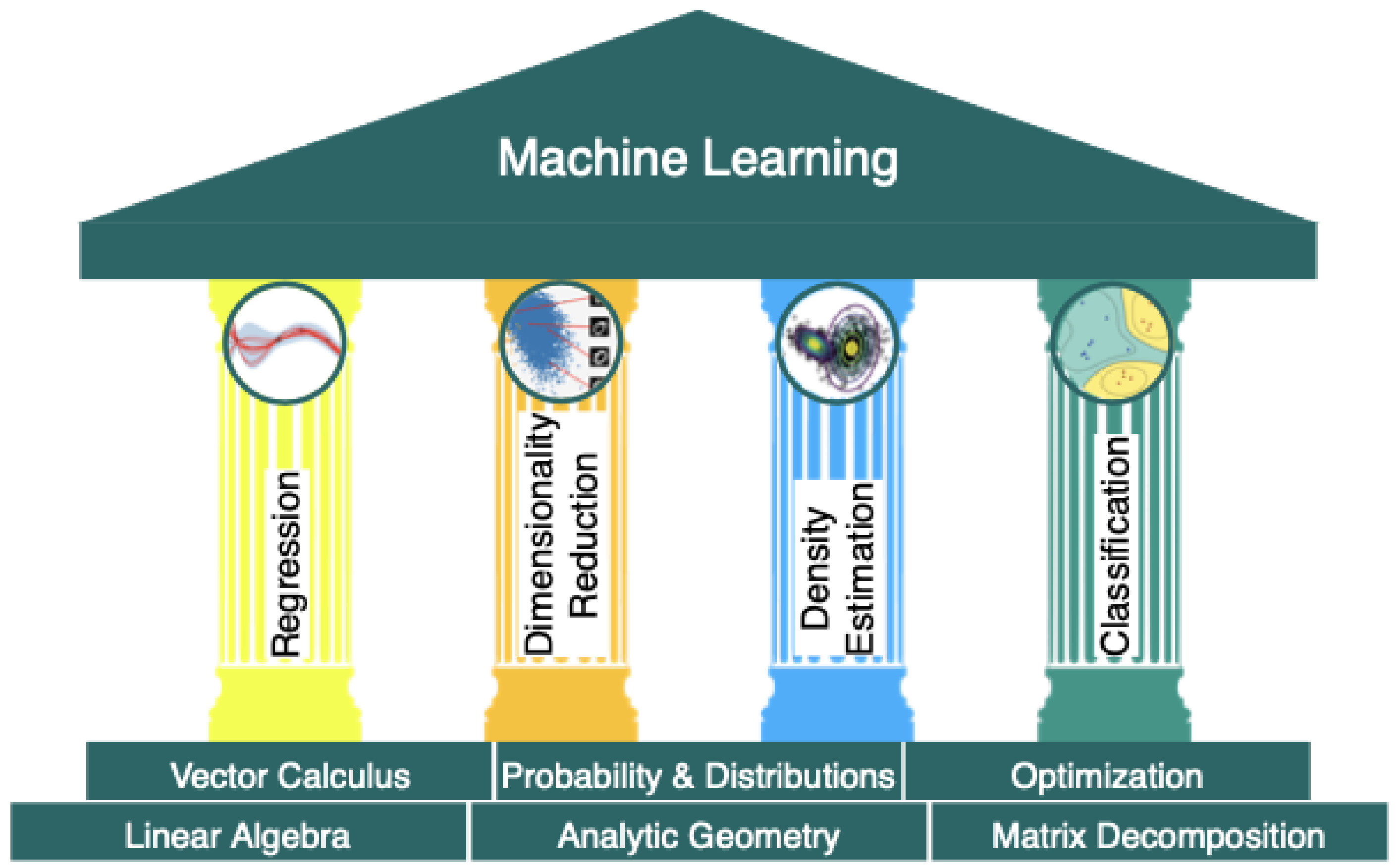
Intro

Mathematics (1)

- **머신러닝은 수학의 위에 세워져 있다.**
  - 데이터를 모델링하고, 학습하고, 예측하는 과정은 모두 수학적 도구 (선형대수, 미적분, 확률통계, 최적화) 에 기반한다.
- **단순 계산도구가 아닌 “언어”**
  - 데이터를 표현(행렬), 유사도 정의(내적), 모델 학습(최적화) 같은 것들을 통일적으로 설명하는 언어



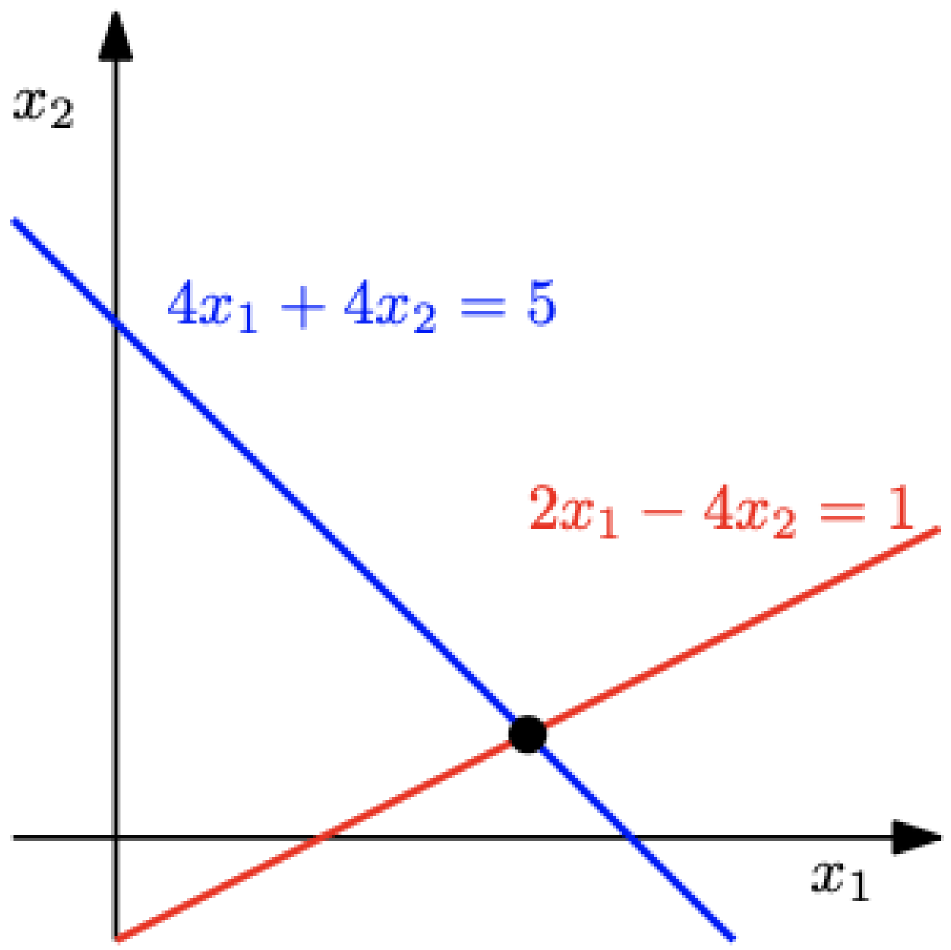
Intro

Mathematics (1)

- 연립방정식(선형시스템)을 행렬로 표현할 수 있다.
- 행렬의 기본 연산(덧셈, 곱셈, 전치)의 의미를 이해한다.
- 가우스 소거법으로 선형시스템의 해를 구할 수 있다.
- 선형 독립성의 정의와 직관적 의미 (: 중복 정보의 유무)를 설명할 수 있다.
- 기저와 차원의 개념을 이해하고, 주어진 벡터 집합의 차원을 판별할 수 있다.
- 데이터 분석·머신러닝에서 차원 개념이 왜 중요한지 설명할 수 있다.

System of Linear Equations

Mathematics (1)



$$4x_1 + 4x_2 = 5$$
$$2x_1 - 4x_2 = 1$$

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

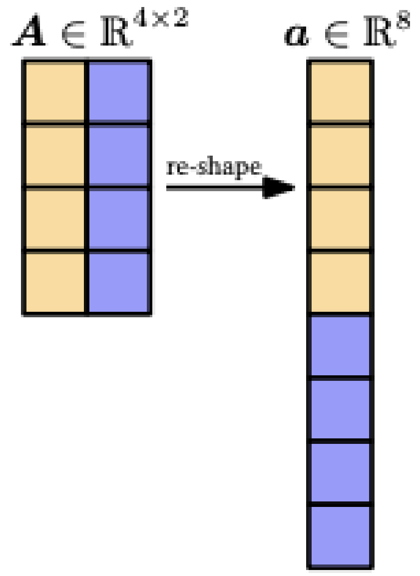
$$\iff \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

1. Unique Solution
2. No Solution
3. Infinitely Many Solution

Matrices

Mathematics (1)

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad a_{ij} \in \mathbb{R}.$$



$$\underbrace{\boldsymbol{A}}_{n\times k}\underbrace{\boldsymbol{B}}_{k\times m} = \underbrace{\boldsymbol{C}}_{n\times m}$$

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, k.$$

- *Associativity:*

$$\forall \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times p}, \boldsymbol{C} \in \mathbb{R}^{p\times q} : (\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$$

- *Distributivity:*

$$\begin{aligned} \forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{n\times p} : (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} &= \boldsymbol{AC} + \boldsymbol{BC} \\ \boldsymbol{A}(\boldsymbol{C}+\boldsymbol{D}) &= \boldsymbol{AC} + \boldsymbol{AD} \end{aligned}$$

- Multiplication with the identity matrix:

$$\forall \boldsymbol{A} \in \mathbb{R}^{m\times n} : \boldsymbol{I}_m \boldsymbol{A} = \boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{A}$$

Note that $\boldsymbol{I}_m \neq \boldsymbol{I}_n$ for $m \neq n$.

$$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{A}\boldsymbol{A}^{-1} &= \boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A} \\ (\boldsymbol{AB})^{-1} &= \boldsymbol{B}^{-1}\boldsymbol{A}^{-1} \\ (\boldsymbol{A}+\boldsymbol{B})^{-1} &\neq \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1} \\ (\boldsymbol{A}^\top)^\top &= \boldsymbol{A} \\ (\boldsymbol{A}+\boldsymbol{B})^\top &= \boldsymbol{A}^\top + \boldsymbol{B}^\top \\ (\boldsymbol{AB})^\top &= \boldsymbol{B}^\top \boldsymbol{A}^\top \end{aligned}$$

Solving Systems of Linear Equations

Mathematics (1)

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}.$$

$$\sum_{i=1}^{4} x_i \boldsymbol{c}_i = \boldsymbol{b}$$

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \left(\lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right) = \lambda_1 (8\boldsymbol{c}_1 + 2\boldsymbol{c}_2 - \boldsymbol{c}_3) = \boldsymbol{0}.$$

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \left(\lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix} \right) = \lambda_2 (-4\boldsymbol{c}_1 + 12\boldsymbol{c}_2 - \boldsymbol{c}_4) = \boldsymbol{0}$$

$$\Longrightarrow$$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 : \boldsymbol{x} = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\}.$$

General Solution

1. Find a particular solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.
2. Find all solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.
3. Combine the solutions from steps 1. and 2. to the general solution.

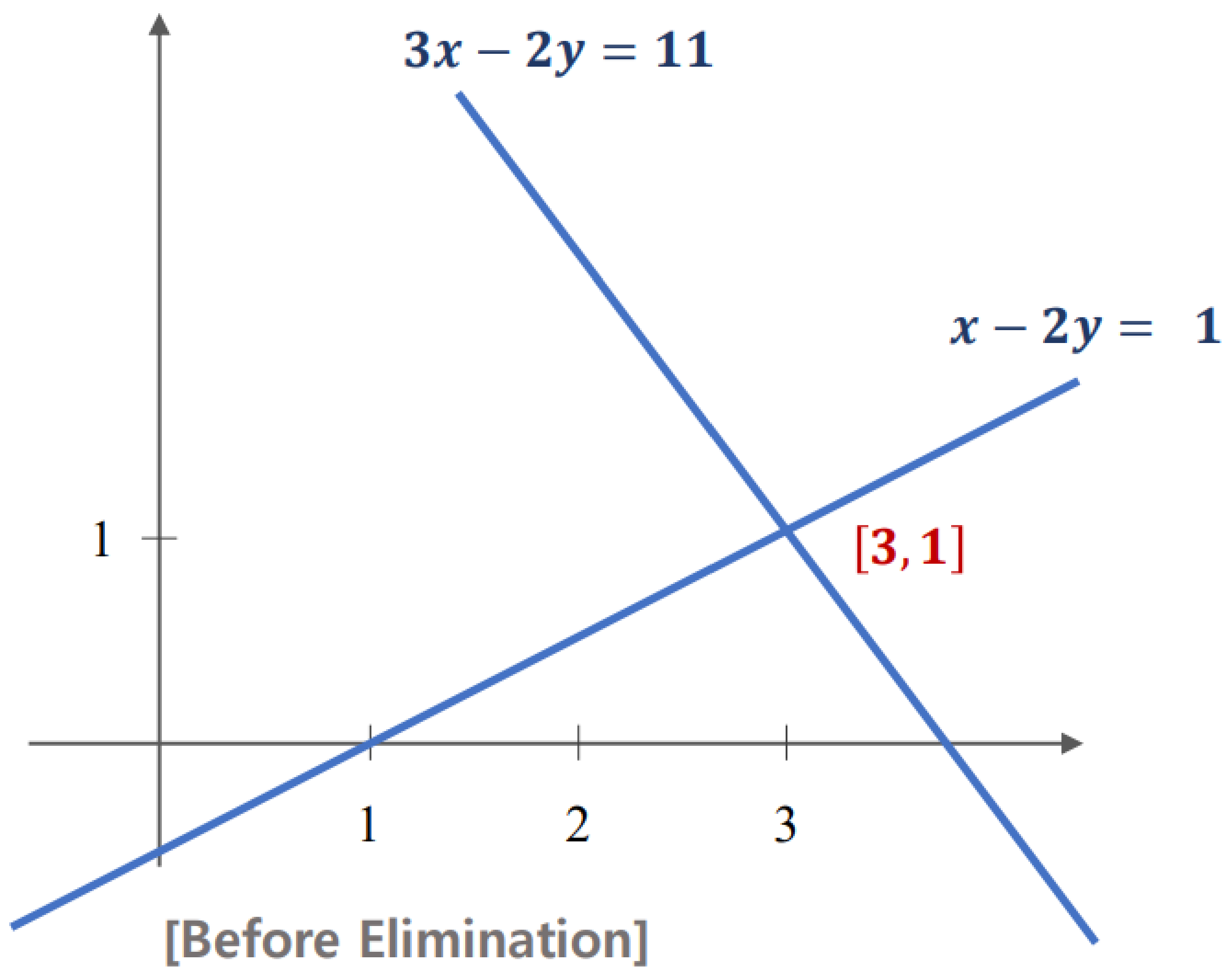
Solving Systems of Linear Equations

Mathematics (1)

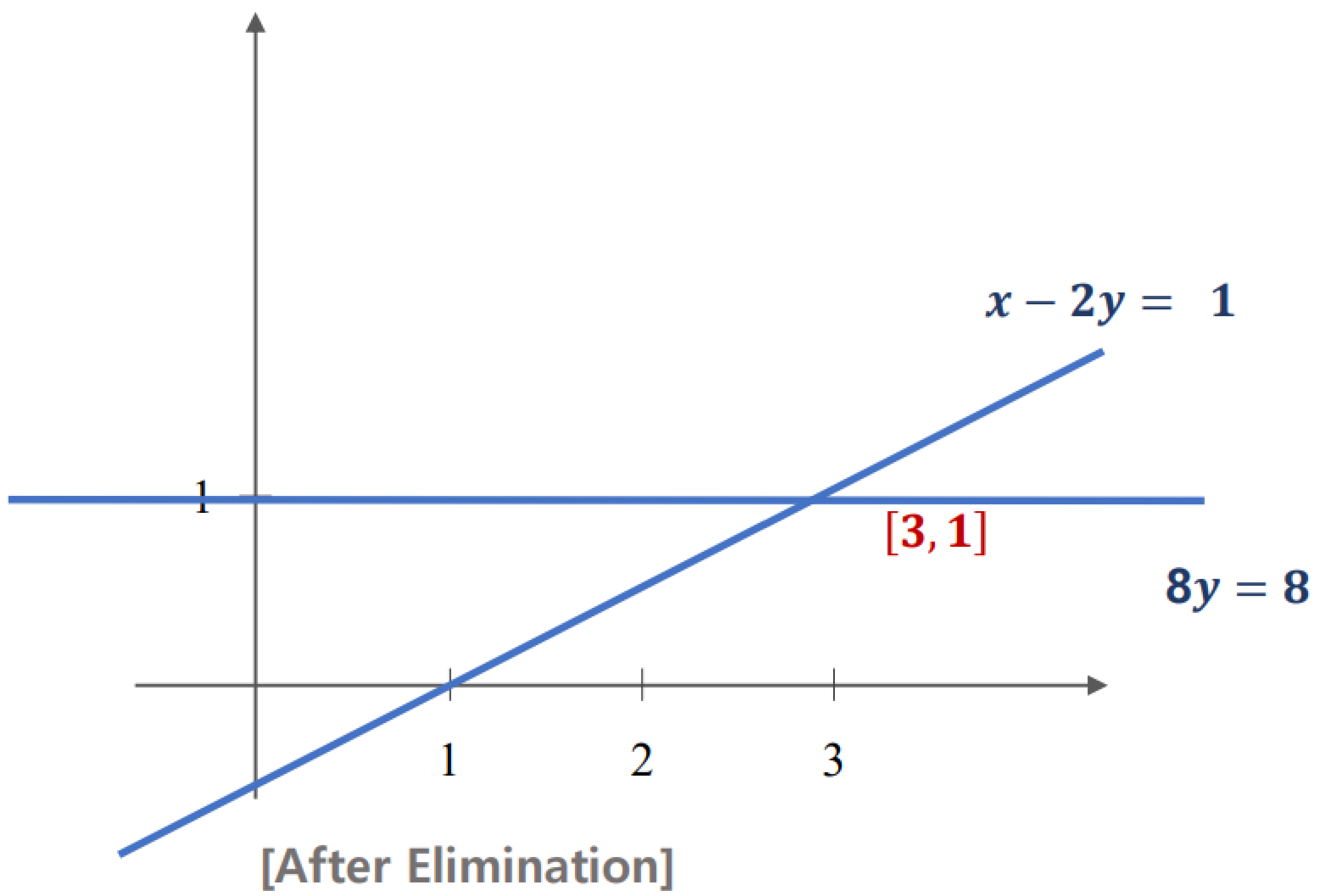
$$\begin{aligned} x - 2y &= \ 1 \\ 3x + 2y &= 11 \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$$

To eliminate x : Subtract a multiple (=3) of 3q. 1 from eq. 2



[Before Elimination]



[After Elimination]

Solving Systems of Linear Equations

Mathematics (1)

$$
\begin{array}{rcrcrcrcrcr}
-2x_1 & + & 4x_2 & - & 2x_3 & - & x_4 & + & 4x_5 & = & -3 \\
4x_1 & - & 8x_2 & + & 3x_3 & - & 3x_4 & + & x_5 & = & 2 \\
x_1 & - & 2x_2 & + & x_3 & - & x_4 & + & x_5 & = & 0 \\
x_1 & - & 2x_2 & & & - & 3x_4 & + & 4x_5 & = & a
\end{array}.
$$

Transform & Augmenting

$$
\left[\begin{array}{rrrrr|r}
-2 & 4 & -2 & -1 & 4 & -3 \\
4 & -8 & 3 & -3 & 1 & 2 \\
1 & -2 & 1 & -1 & 1 & 0 \\
1 & -2 & 0 & -3 & 4 & a
\end{array}\right]
\begin{array}{l}
\text{Swap with } R_3 \\
\\
\text{Swap with } R_1 \\
\\
\end{array}
$$

$$
\left[\begin{array}{rrrrr|r}
1 & -2 & 1 & -1 & 1 & 0 \\
4 & -8 & 3 & -3 & 1 & 2 \\
-2 & 4 & -2 & -1 & 4 & -3 \\
1 & -2 & 0 & -3 & 4 & a
\end{array}\right]
\begin{array}{l}
\\
-4R_1 \\
+2R_1 \\
-R_1
\end{array}
$$

REF

$$
\left[\begin{array}{rrrrr|r}
1 & -2 & 1 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 & -3 & 2 \\
0 & 0 & 0 & -3 & 6 & -3 \\
0 & 0 & -1 & -2 & 3 & a
\end{array}\right]
\begin{array}{l}
\\
\\
\\
-R_2 - R_3
\end{array}
$$

$$
\rightsquigarrow
\left[\begin{array}{rrrrr|r}
1 & -2 & 1 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 & -3 & 2 \\
0 & 0 & 0 & -3 & 6 & -3 \\
0 & 0 & 0 & 0 & 0 & a+1
\end{array}\right]
\begin{array}{l}
\\
\cdot(-1) \\
\cdot(-\frac{1}{3}) \\
\\
\end{array}
$$

$$
\rightsquigarrow
\left[\begin{array}{rrrrr|r}
1 & -2 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 3 & -2 \\
0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & a+1
\end{array}\right]
$$

Transform & Solve

$$
\begin{array}{rcrcrcrcrcr}
x_1 & - & 2x_2 & + & x_3 & - & x_4 & + & x_5 & = & 0 \\
& & & & x_3 & - & x_4 & + & 3x_5 & = & -2 \\
& & & & & & x_4 & - & 2x_5 & = & 1 \\
& & & & & & & & 0 & = & a+1
\end{array}.
$$

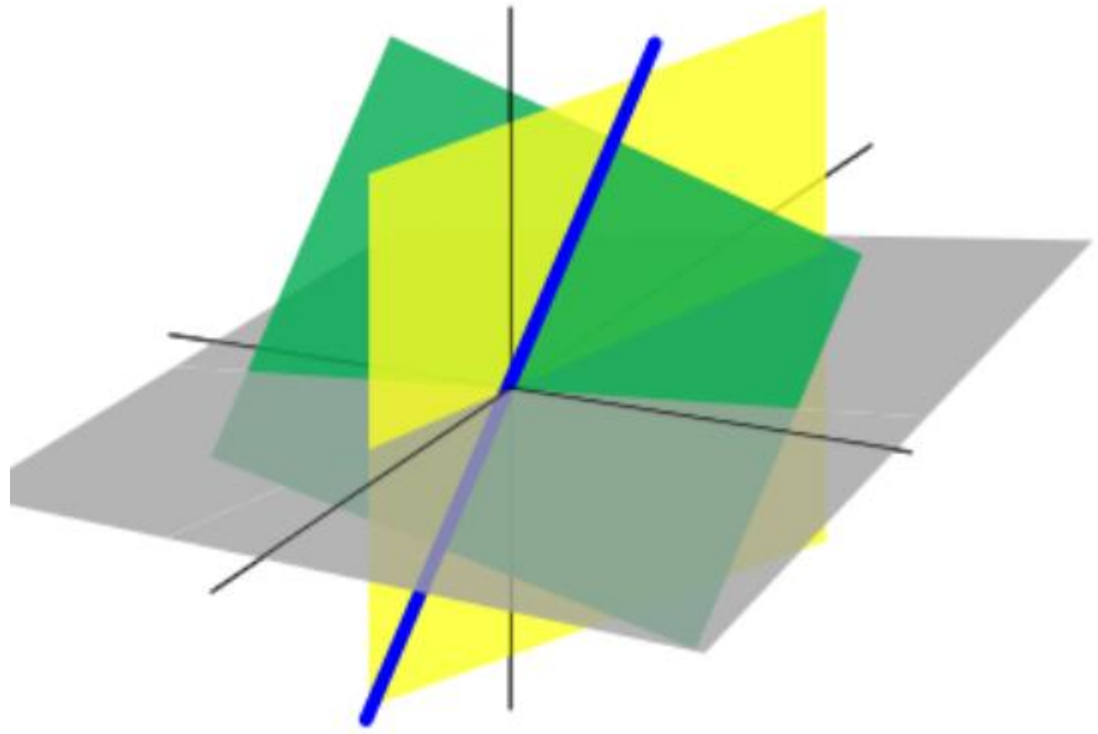
Only for $a = -1$ this system can be solved. A *particular solution* is

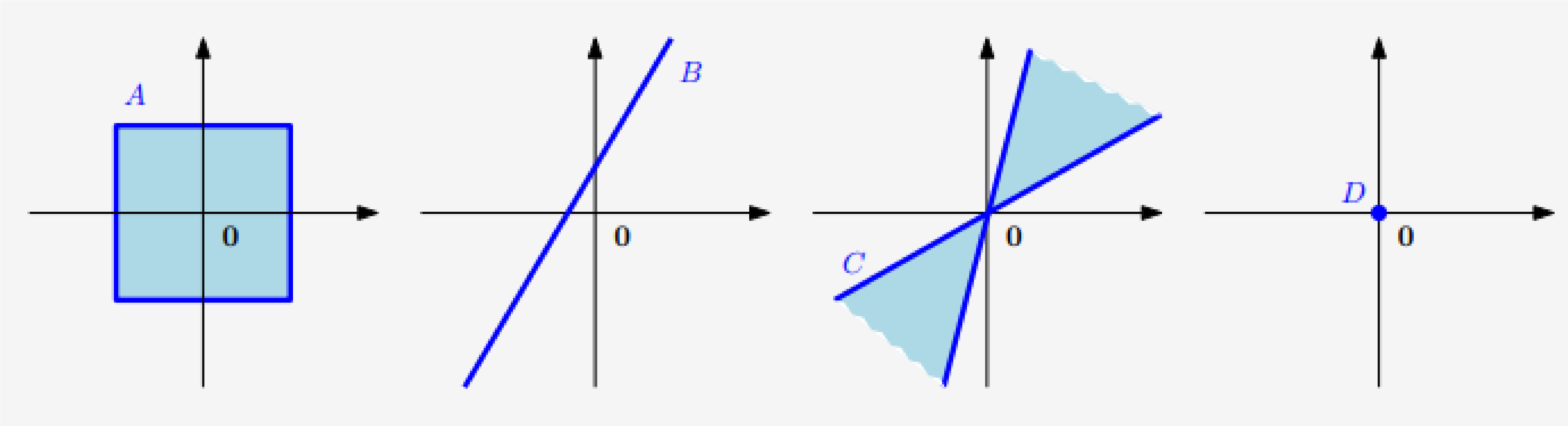
$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.
$$

Vector Subspace

Mathematics (1)

- Commutative law: $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$
- Distributive law (or Scalar multiplication): $c(\boldsymbol{v} + \boldsymbol{w}) = c\boldsymbol{v} + c\boldsymbol{w}$
- Unique Zero vector





Vector Subspace

Mathematics (1)

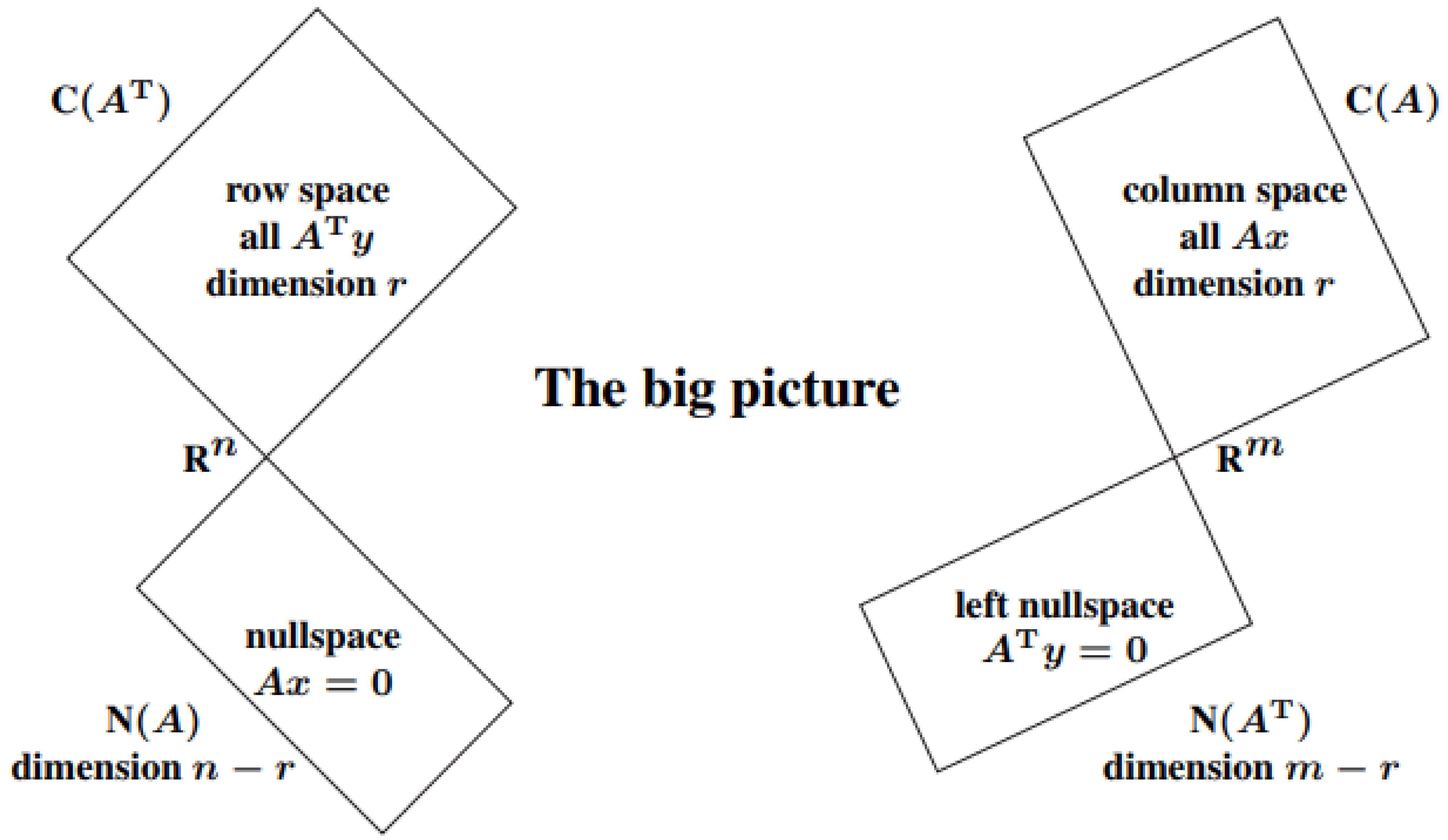
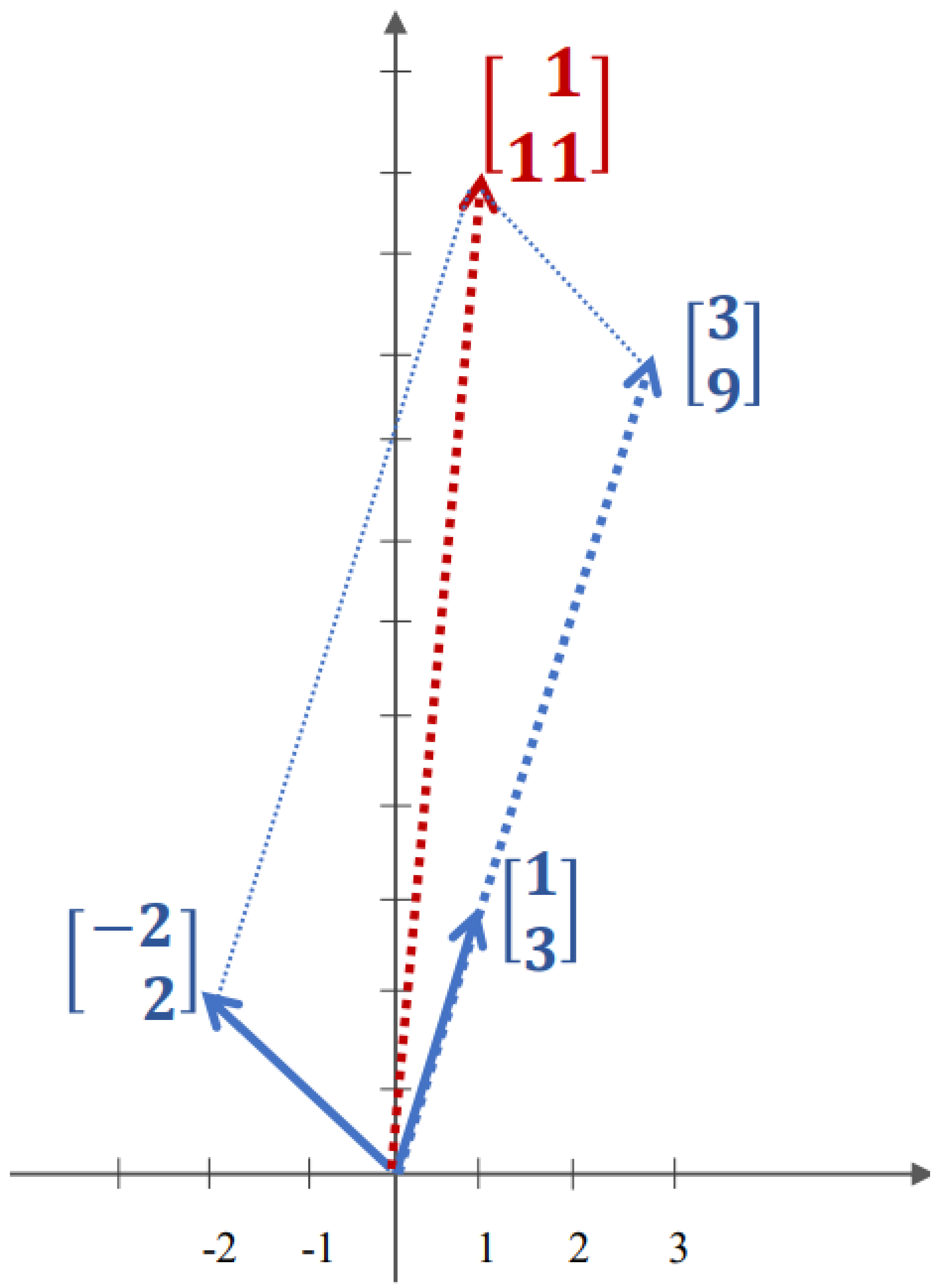
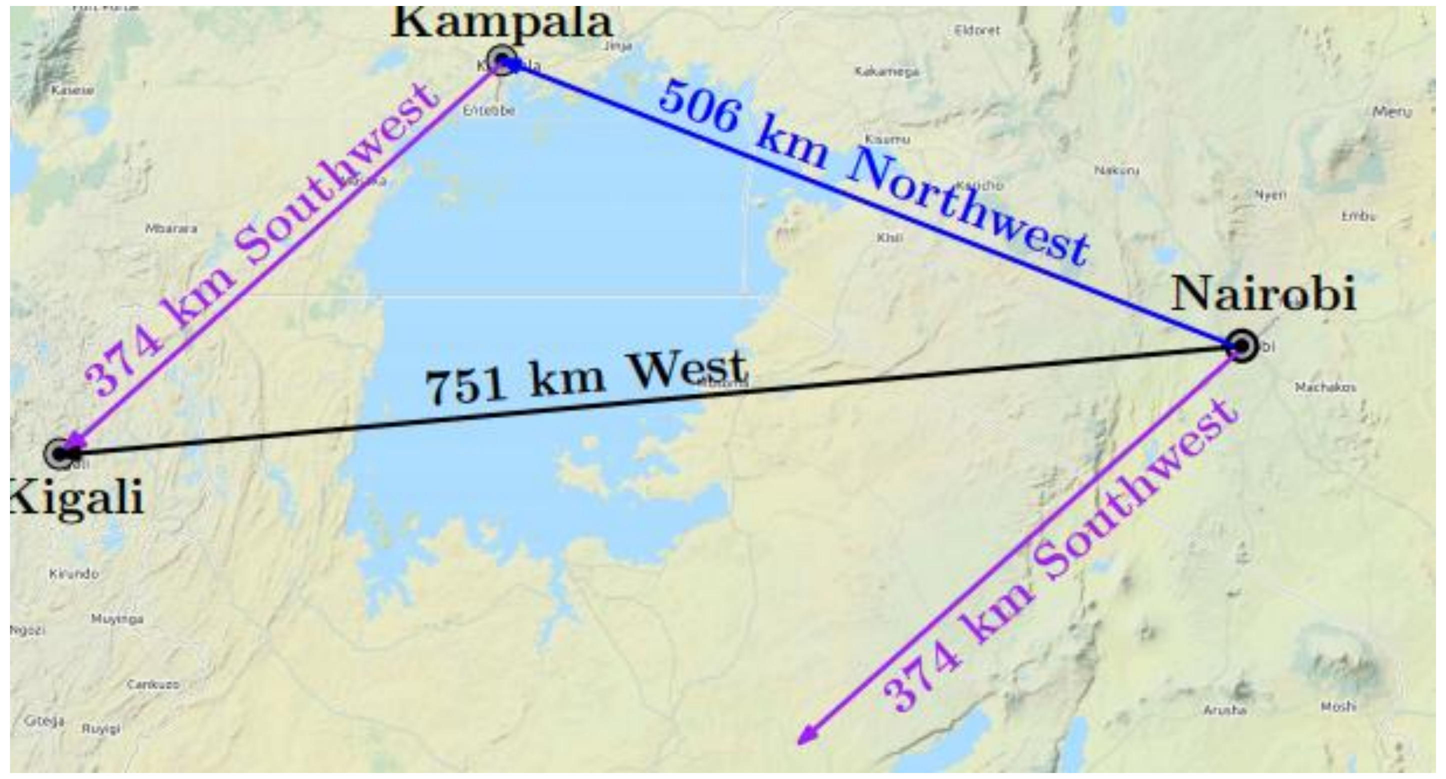


Figure 3.3: The dimensions of the Four Fundamental Subspaces (for R_0 and for A).

Linear Independence

Mathematics (1)





$$\mathbf{0} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$$

If and Only if all lambdas = 0
→ Independent!

$$\sum_{j=1}^{m} \psi_j \boldsymbol{x}_j = \sum_{j=1}^{m} \psi_j \boldsymbol{B}\boldsymbol{\lambda}_j = \boldsymbol{B}\sum_{j=1}^{m} \psi_j \boldsymbol{\lambda}_j \,.$$

$$\boldsymbol{v} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \in V$$

- **Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$.**
  Then, the following statements are equivalent:
  - $\mathcal{B}$ is a basis of V .
  - $\mathcal{B}$ is a minimal generating set.
  - $\mathcal{B}$ is a maximal linearly independent set of vectors in V ,
    i.e., adding any other vector to this set will make it linearly dependent.
  - Every vector $x \in V$ is a linear combination of vectors from $\mathcal{B}$, and every linear combination is unique, i.e., with $\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{b}_i = \sum_{i=1}^{k} \psi_i \boldsymbol{b}_i$

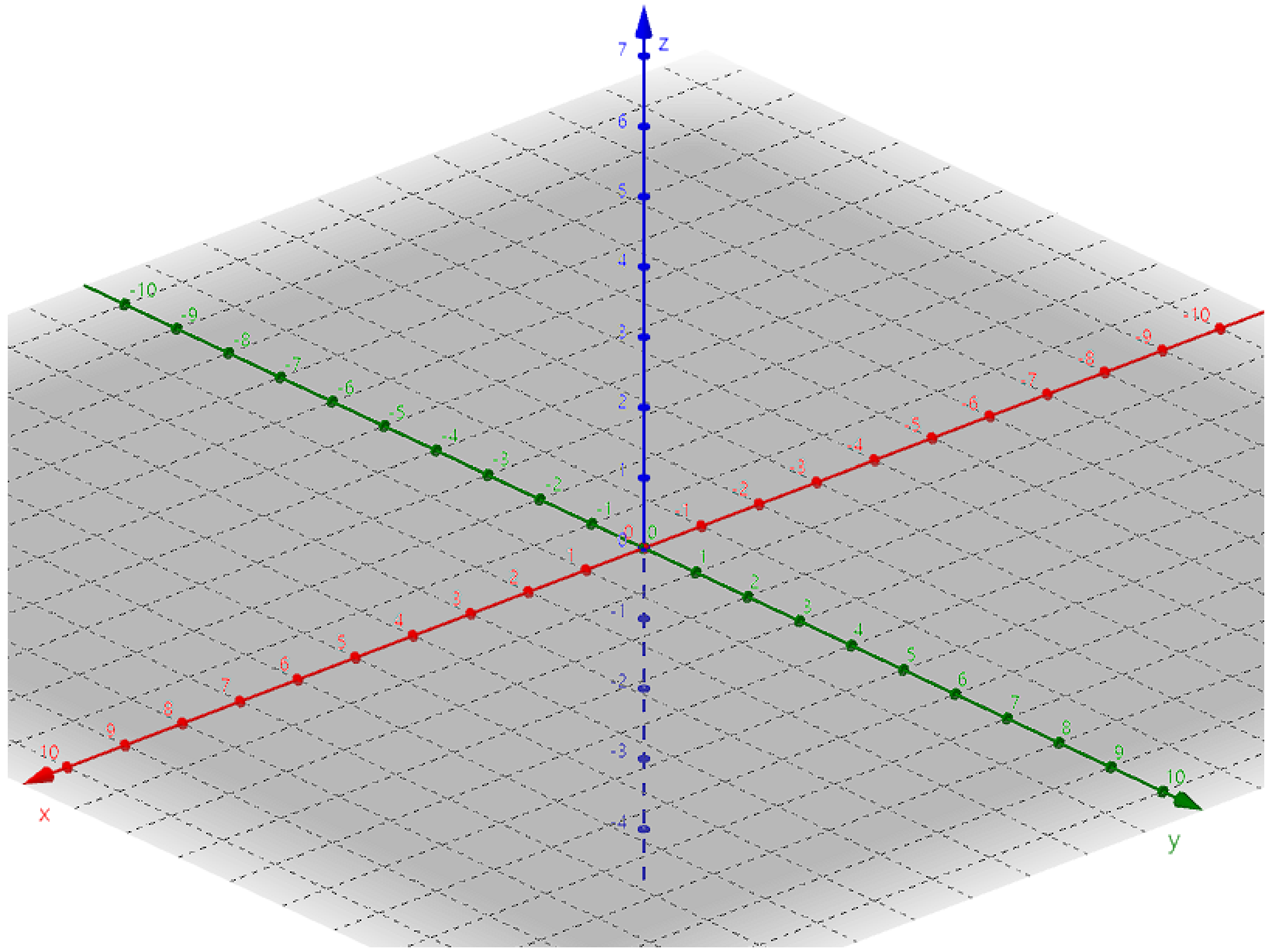
Basis and Dimension

Mathematics (1)

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}.$$

$$\mathcal{B}_1 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}, \mathcal{B}_2 = \left\{ \begin{bmatrix}0.5\\0.8\\0.4\end{bmatrix}, \begin{bmatrix}1.8\\0.3\\0.3\end{bmatrix}, \begin{bmatrix}-2.2\\-1.3\\3.5\end{bmatrix} \right\}.$$

Span a Space($\mathcal{R}^3$)



And, Dimension of Space

$\dim(V)$

$= N(\text{vectors in basis})$

QnA

감사합니다