CWI Autumn School 2024: Numerical Methods for Bayesian Inverse Problems

Exercise Sheet 1

Exercise 1: Deconvolution. Let us consider a deconvolution problem and solve it numerically.

Fix $N \in \mathbb{N}$ and let X be the space of periodic sequences in $\mathbb{C}$, i.e.

$$X = \{(x_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \,:\, x_j = x_{j+N} \;\forall j \in \mathbb{Z}\}.$$

(This could arise, e.g., after discretising a continuous deconvolution problem on a uniform grid.) For $K \in X$ introduce the discrete convolution operator $T_K : X \to X$

$$(T_K f)_k = \sum_{j=0}^{N-1} K_{k-j} f_j \qquad \forall k \in \mathbb{Z}.$$

Furthermore denote by $\mathcal{F}_N : X \to X$ the discrete Fourier transform, i.e. for $f \in X$

$$(\mathcal{F}_N f)_k = \sum_{j=0}^{N-1} \exp\Big(-2\pi\mathrm{i}\frac{j\cdot k}{N}\Big) f_j \qquad \forall k \in \mathbb{Z}.$$

(a) Show that $\mathcal{F}_N^{-1} : X \to X$ is given through

$$(\mathcal{F}_N^{-1} f)_k = \frac{1}{N}\sum_{j=0}^{N-1} \exp\Big(2\pi\mathrm{i}\frac{j\cdot k}{N}\Big) f_j.$$

(b) Show that

$$(\mathcal{F}_N T_K f)_k = (\mathcal{F}_N K)_k (\mathcal{F}_N f)_k \qquad \forall k \in \mathbb{Z}.$$

(c) Show that if $(\mathcal{F}_N K)_k \neq 0$ for all $k \in \{0,\dots,N-1\}$ then T_K is invertible and

$$(T_K^{-1} f)_k = \sum_{j=0}^{N-1} L_{k-j} f_j \qquad \forall k \in \mathbb{Z},$$

where $L \in X$ satisfies $(\mathcal{F}_N L)_k = 1/(\mathcal{F}_N K)_k$ for $k \in \{0,\dots,N-1\}$.

HINT: Consider $e_n := \sqrt{N}\mathcal{F}_N^{-1} b_n$, where $b_n = (\delta_{jn})_{j=0}^{N-1}$ and find a spectral decomposition of T_K (with a suitable inner product on X).

(d) Implement a function `convolution(K,f)` that takes $(f_k)_{k=0}^{N-1}$ and returns $((T_k f)_k)_{k=0}^{N-1}$. Use the fast Fourier transform (FFT), e.g. the function `numpy.fft.fft` and its inverse `numpy.fft.ifft` in Python (or similar functions available in Matlab).

(e) To regularise the deconvolution problem we use truncation (or hard-thresholding), i.e. we cut off the high frequencies by replacing L with L^α such that $(\mathcal{F}_N L^\alpha)_k = g_\alpha((\mathcal{F}_N K)_k)$ with $g_\alpha(x) = 1/x$ if $|x| > \alpha$ and $g_\alpha(x) = 0$ otherwise.

The regularised inverse operator $T_{K,\alpha}^\dagger$ (replacing the ill-posed T_K^{-1}) is then defined as

$$(T_{K,\alpha}^\dagger f)_k = \sum_{j=0}^{N-1} L_{k-j}^\alpha f_j \qquad \forall k \in \mathbb{Z},$$

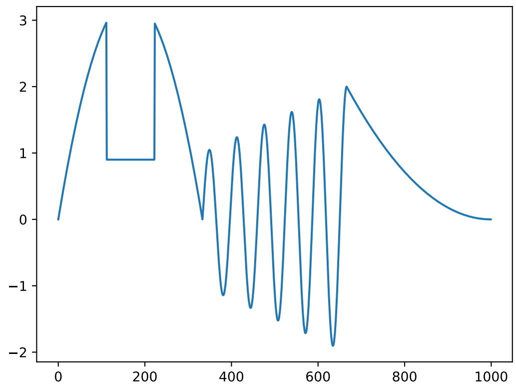
Implement a function `deconvolution(K,g,a)` that takes $(K_k)_{k=0}^{N-1}$, $(g_k)_{k=0}^{N-1}$ and α and returns $T_\alpha^\dagger g$.

(g) Test your code with the data $(f_k)_{k=0}^{N-1}$ provided in `signal.txt` (on Moodle) and use the following Gaussian kernel K:

```
import numpy as np
s = 100
gaussian = lambda x, s: np.exp(-s*x**2.)
K = gaussian(np.linspace(-1./2,1./2,N),s)
K = N*np.fft.ifftshift(K)/np.sum(K)
```

Plot f, $g = T_k f$ and the reconstructed signal $T_\alpha^\dagger g$ for $\alpha = 10^{-11}$.

Signal:



(h) Add noise to the signal via $g^\delta = g + \delta W$ with W being Gaussian noise, for example in Python `W=np.random.randn(N)`. Plot $T_\alpha^\dagger g^\delta$ for $\delta \in \{10^{-10}, 10^{-12}\}$. Explain what you observe.