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Solution Sheet 3 – Subset Simulation, FORM

Autumn School *Uncertainty Quantification for High-dimensional Problems*

Problem 1 (MH-MCMC in Subset Simulation)

Subset Simulation requires a method for generating samples from the conditional distribution $\boldsymbol{U}|F_{\ell-1}$, $\ell > 1$. In principle this can be done with Metropolis–Hastings Markov chain Monte Carlo (MH-MCMC), where we construct a Markov chain with stationary distribution (target distribution) $\boldsymbol{U}|F_{\ell-1}$.

For Metropolis–Hastings MCMC we require a proposal density $q(\cdot|\boldsymbol{u}^{(k-1)})$, calculate the acceptance probability α, and accept the candidate state as next state in the Markov chain with probability α, see Algorithm 2 in the lecture slides. For specific proposal densities the MH-MCMC algorithm in Subset Simulation simplifies.

Assume that $\boldsymbol{U}$ follows an n-variate standard normal distribution with pdf φ_n. Let $\varphi_n(\cdot|F_{\ell-1})$ denote the pdf of $\boldsymbol{U}|F_{\ell-1}$. Following (Papaioannou, Betz, et al., 2015) we choose as proposal density the n-variate normal density with mean $\rho\boldsymbol{u}^{(k-1)}$ and variance $(1-\rho^2)I_n$, where $\rho \in [0,1]$ denotes a correlation parameter, and $\boldsymbol{u}^{(k-1)}$ is the current state of the Markov chain.

Prove that for this choice of proposal density in MH-MCMC the acceptance probability of the candidate state $\boldsymbol{v}$ is

$$\alpha(\boldsymbol{u}^{(k-1)}, \boldsymbol{v}) = \mathbb{1}_{F_{\ell-1}}(\boldsymbol{v}).$$

Solution

Observe that

$$\varphi_n(\boldsymbol{u}|F_{\ell-1}) = \varphi_n(\boldsymbol{u})\mathbb{1}_{F_{\ell-1}}(\boldsymbol{u})/\mathbb{P}(F_{\ell-1}).$$

Moreover,

$$q(\boldsymbol{v}|\boldsymbol{u}^{(k-1)}) \propto \exp(-(\boldsymbol{v}-\rho\boldsymbol{u}^{(k-1)})^\top(\boldsymbol{v}-\rho\boldsymbol{u}^{(k-1)})/2(1-\rho^2))$$

and

$$q(\boldsymbol{u}^{(k-1)}|\boldsymbol{v}) \propto \exp(-(\boldsymbol{u}^{(k-1)}-\rho\boldsymbol{v})^\top(\boldsymbol{u}^{(k-1)}-\rho\boldsymbol{v})/2(1-\rho^2)).$$

Looking at Algorithm 2 we thus obtain the ratio

$$\begin{aligned}
r(\boldsymbol{u}^{(k-1)}, \boldsymbol{v}) &= \frac{\varphi(\boldsymbol{v}|F_{\ell-1})}{\varphi(\boldsymbol{u}^{(k-1)}|F_{\ell-1})} \frac{q(\boldsymbol{u}^{(k-1)}|\boldsymbol{v})}{q(\boldsymbol{v}|\boldsymbol{u}^{(k-1)})} \\
&= \frac{\varphi(\boldsymbol{v})}{\varphi(\boldsymbol{u}^{(k-1)})} \frac{\mathbb{1}_{F_{\ell-1}}(\boldsymbol{v})}{\mathbb{1}_{F_{\ell-1}}(\boldsymbol{u}^{(k-1)})} \frac{q(\boldsymbol{u}^{(k-1)}|\boldsymbol{v})}{q(\boldsymbol{v}|\boldsymbol{u}^{(k-1)})} \\
&= \frac{\exp(-\boldsymbol{v}^\top \boldsymbol{v}/2)}{\exp(-\boldsymbol{u}^{(k-1)^\top} \boldsymbol{u}^{(k-1)}/2)} \frac{\exp(-(\boldsymbol{u}^{(k-1)} - \rho \boldsymbol{v})^\top (\boldsymbol{u}^{(k-1)} - \rho \boldsymbol{v})/2(1-\rho^2))}{\exp(-(\boldsymbol{v} - \rho \boldsymbol{x}^{(k-1)})^\top (\boldsymbol{v} - \rho \boldsymbol{u}^{(k-1)})/2(1-\rho^2))} \mathbb{1}_{F_{\ell-1}}(\boldsymbol{v}) \\
&= \frac{\exp(-\boldsymbol{v}^\top \boldsymbol{v}/2)}{\exp(-\boldsymbol{u}^{(k-1)^\top} \boldsymbol{x}^{(k-1)}/2)} \frac{\exp(-\boldsymbol{u}^{(k-1)^\top} \boldsymbol{u}^{(k-1)}/2)}{\exp(-\boldsymbol{v}^\top \boldsymbol{v}/2)} \mathbb{1}_{F_{\ell-1}}(\boldsymbol{v}) \\
&= \mathbb{1}_{F_{\ell-1}}(\boldsymbol{v}).
\end{aligned}$$

Hence the acceptance probability of the candidate state $\boldsymbol{v}$ is

$$\alpha(\boldsymbol{x}^{(k-1)}, \boldsymbol{v}) = \min\{1, \mathbb{1}_{F_{\ell-1}}(\boldsymbol{v})\} = \mathbb{1}_{F_{\ell-1}}(\boldsymbol{v}).$$

This means that we accept candidates $\boldsymbol{v} \in F_{\ell-1}$ and reject candidates $\boldsymbol{v} \notin F_{\ell-1}$. With another proposal density it can happen that we reject candidates $\boldsymbol{v} \in F_{\ell-1}$ which are actually in $F_{\ell-1}$. This is not desirable.

Problem 2 (Calculation of most likely failure point in FORM)

Let $G\colon \mathbb{R}^n \to \mathbb{R}$ denote the limit-state function (LSF) and let $\boldsymbol{U} \sim N(0, I_n)$ follow the standard normal distribution. The so-called *most likely failure point* (MLFP) is defined as

$$\boldsymbol{u}^{MLFP} := \underset{\boldsymbol{u} \in \mathbb{R}^n}{\operatorname{argmin}} \ \frac{1}{2}\|\boldsymbol{u}\|_2^2, \quad \text{such that} \quad G(\boldsymbol{u}) = 0. \tag{1}$$

Hasofer and Lind (1974) determine the MLFP in (1) iteratively by a line search of the linearized LSF anchored at the current iterate $\boldsymbol{u}_k$ in the direction of $-\nabla G(\boldsymbol{u}_k)$.

Write down the iterative algorithm to approximate the MLFP according to Hasofer & Lind!

Solution

Assume that we have an approximation $\boldsymbol{u}_k \in \mathbb{R}^n$ and that $\nabla G(\boldsymbol{u}_k) \neq \boldsymbol{0}$. The next iterate $\boldsymbol{u}_{k+1} = \lambda \nabla G(\boldsymbol{u}_k)$ for some $\lambda \neq 0$. That is, the direction vector of $\boldsymbol{u}_{k+1}$ is parallel to the direction of the gradient of the LSF G at the current iterate. Next we linearize the LSF around the current iterate and use $G(\boldsymbol{u}_{k+1}) = 0$. We obtain

$$\underbrace{G(\boldsymbol{u}_{k+1})}_{=0} = G(\boldsymbol{u}_k) + \nabla G(\boldsymbol{u}_k)^\top (\underbrace{\boldsymbol{u}_{k+1}}_{=\lambda \nabla G(\boldsymbol{u}_k)} - \boldsymbol{u}_k).$$

This gives

$$\lambda = \frac{\nabla G(\boldsymbol{u}_k)^\top \boldsymbol{u}_k - G(\boldsymbol{u}_k)}{\|\nabla G(\boldsymbol{u}_k)\|_2^2}$$

and thus the next iterate is

$$\boldsymbol{u}_{k+1} = \frac{\nabla G(\boldsymbol{u}_k)^\top \boldsymbol{u}_k - G(\boldsymbol{u}_k)}{\|\nabla G(\boldsymbol{u}_k)\|_2^2} \nabla G(\boldsymbol{u}_k).$$

The MLFP iteration is summarized in Algorithm 1.

Algorithm 1 Calculation of $\boldsymbol{u}^{MLFP}$ (Hasofer & Lind, 1974)

```
Input: initial guess u_0, tolerance δ > 0
k = 0
while |G(u_k)| > δ do
    α_k = (∇G(u_k)^T u_k − G(u_k)) / ||∇G(u_k)||_2^2
    u_{k+1} = α_k ∇G(u_k)
    Set k = k + 1
end while
Output: x_k
```