Department of Mathematics
Technical University of Munich
Elisabeth Ullmann

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Problem Sheet 2 – Cross-entropy method

Autumn School *Uncertainty Quantification for High-Dimensional Problems*

Problem 1 (Cross-entropy method with Gaussian densities)

Consider the cross-entropy (CE) method with a family of n-variate Gaussian densities, that are parameterized by their mean vector $\boldsymbol{\mu} \in \mathbb{R}^n$ and invertible covariance matrix $\Sigma \in \mathbb{R}^{n\times n}_{sym}$. That is, the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\mu}, \mathrm{vec}(\Sigma))^\top$ and

$$g(\boldsymbol{u};\boldsymbol{\theta}) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}(\boldsymbol{u}-\boldsymbol{\mu})^\top \Sigma^{-1}(\boldsymbol{u}-\boldsymbol{\mu})\right), \quad \boldsymbol{u} \in \mathbb{R}^n.$$

Let $\boldsymbol{u}_j^{(i)}$, $i = 1, \ldots, N$ denote samples of $g(\cdot;\boldsymbol{\theta}_j)$. Following the lecture notes we define the objective function

$$J(\boldsymbol{\theta};\gamma_j,\boldsymbol{\theta}_j) := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{G\le\gamma_j\}}(\boldsymbol{u}_j^{(i)}) W(\boldsymbol{u}_j^{(i)};\boldsymbol{\theta}_0,\boldsymbol{\theta}_j) \ln g(\boldsymbol{u}_j^{(i)};\boldsymbol{\theta}),$$

where $W(\boldsymbol{u};\boldsymbol{\theta}_0,\boldsymbol{\theta}_j) = g(\boldsymbol{u};\boldsymbol{\theta}_0)/g(\boldsymbol{u};\boldsymbol{\theta}_j)$. Let

$$H_j^{(i)} := \mathbb{1}_{\{G\le\gamma_j\}}(\boldsymbol{u}_j^{(i)}) W(\boldsymbol{u}_j^{(i)};\boldsymbol{\theta}_0,\boldsymbol{\theta}_j), \quad i = 1, \ldots, N.$$

Show that the solution of the optimization problem

$$J(\boldsymbol{\theta};\gamma_j,\boldsymbol{\theta}_j) \to \max_{\boldsymbol{\theta}}!$$

is given by

$$\boldsymbol{\mu}_{opt} = \frac{\sum_{i=1}^N H_j^{(i)} \boldsymbol{u}_j^{(i)}}{\sum_{i=1}^N H_j^{(i)}},$$
$$\Sigma_{opt} = \frac{\sum_{i=1}^N H_j^{(i)} (\boldsymbol{u}_j^{(i)} - \boldsymbol{\mu}_{opt})(\boldsymbol{u}_j^{(i)} - \boldsymbol{\mu}_{opt})^\top}{\sum_{i=1}^N H_j^{(i)}}.$$

Hint. It holds

$$\frac{\partial \det(\Sigma)}{\partial \Sigma} = \det(\Sigma)\Sigma^{-\top}.$$

Problem 2 (Implementation of CE method)

For $\boldsymbol{u} \in \mathbb{R}^2$ consider the limit-state function $G(\boldsymbol{u}) = -\min(u_1, u_2) + a$, where $a > 0$. Suppose that the nominal density f is the bivariate standard normal density. In this case $P_f = \Phi(-a)^2$.

The Matlab file `CE_Gaussian_family.m` implements the CE method with a family of Gaussian densities parameterized by their mean value $\boldsymbol{\mu} \in \mathbb{R}^2$ and covariance matrix $\Sigma \in \mathbb{R}^{2\times 2}$.

a) In which lines do we use the analytical updates for $\boldsymbol{\mu}_{opt}$ and Σ_{opt} derived in Problem 1?

b) Compare the estimate for P_f obtained with standard Monte Carlo and the CE method for different values of a! What do you conclude?