Bayesian shape inversion in acoustic scattering

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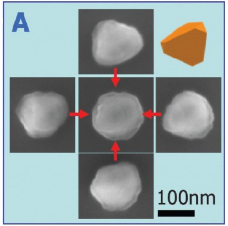
Uncertainty Quantification for High-Dimensional Problems
14 November 2024, CWI Amsterdam

Radboud University

Inverse scattering in applications



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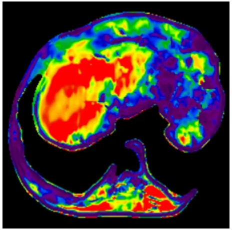

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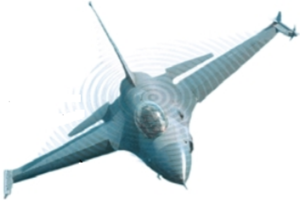
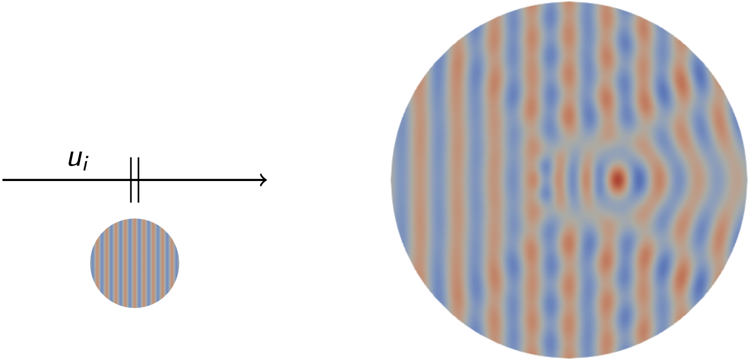


Borden, Inverse Problems (2002)

We consider reconstructing a **shape** in **time-harmonic** scattering.

Situation sketch



Goal: infer scatterer's shape from measurements of the scattered field.
Focus: effect of the frequency on the inversion result.

Bayesian shape inverse problem

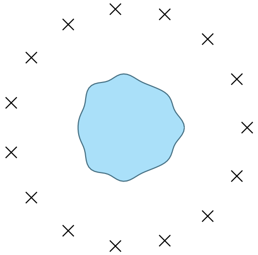
Assumptions

star-shaped scatterer

non-trapping regime [Moiola, Spence 2017]

finite dimensional measurements

additive noise $\boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \Sigma)$



Given a prior measure μ_0 on r, find the posterior $\mu^{\boldsymbol{\delta}}$ given the observations

$$\boldsymbol{\delta} = \mathcal{G}(r) + \boldsymbol{\eta},$$

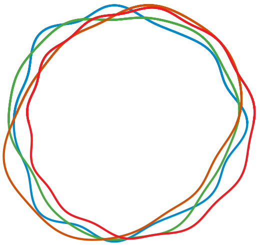
Plan

- Shape prior
- Forward model
- Frequency-explicit well-posedness
- Posterior sampling
- Numerical results

Shape prior

Key tool: random field

for:
domain deformations
normal displacement
boundary function
level set



We use the model:

$$r(\omega;\varphi) = r_0(\varphi) + \sum_{j=1}^{d} \beta_j Y_j(\omega)\psi_j(\varphi)$$

where $Y_j \sim \mathcal{U}([-1,1])$ independent, $\sum_{j\geq 1} |\beta_j| \leq \gamma_\beta \inf_\varphi r_0(\varphi)$.

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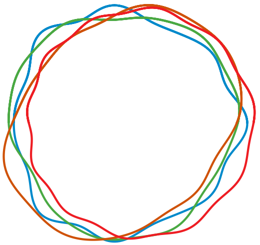

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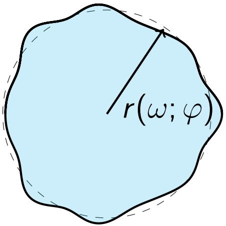
Properties of the shape prior

$$r(\omega;\varphi) = r_0(\varphi) + \sum_{j=1}^{d} \beta_j Y_j(\omega)\psi_j(\varphi)$$

Choices for $\{\psi_j\}_j$:

Laplace-Beltrami eigenfunctions [Church et al. 2020]

Localized supports - wavelets [van Harten, S. 2024]



Coefficients $\{\beta_j\}_j$:

asymptotic decay $\leftrightsquigarrow$ smoothness

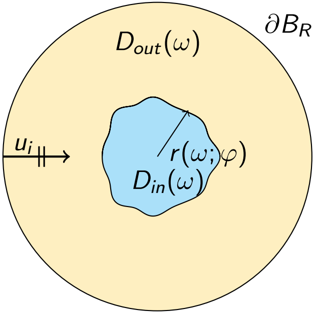
preasymptotic decay $\leftrightsquigarrow$ correlation length

Example:

$$r(\omega;\varphi) = r_0 + \frac{r_0}{4} \sum_{j=1}^{d/2} \frac{1}{1+\ell j^p} \left(Y_{2j-1}(\omega)\cos(j\varphi) + Y_{2j}(\omega)\sin(j\varphi)\right)$$

Helmholtz transmission problem

$$\begin{cases} -\alpha_{in}\Delta(u+u_i) - \kappa_0^2 n_{in}(u+u_i) = 0 \text{ in } D_{in}(\omega) \\ -\alpha_{out}\Delta(u+u_i) - \kappa_0^2 n_{out}(u+u_i) = 0 \text{ in } D_{out}(\omega) \\ + \text{ continuity conditions at interface} \\ + \text{ radiation condition on } u \text{ at } \partial B_R \end{cases}$$



Non-trapping assumption [Moiola, Spence 2019]: $\frac{n_{in}}{n_{out}} \leq \frac{\alpha_{in}}{\alpha_{out}}$

$$\boxed{\mathcal{G}(r) = \mathcal{O} \circ G(r), \text{ where } G(r) = u}$$

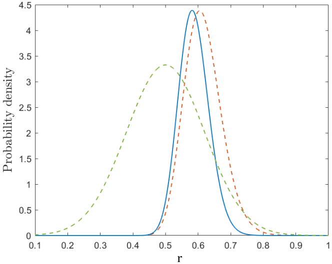
The Bayesian inverse problem

We look for probability distribution for $r = r(\omega, \varphi)$.

Bayes' rule

posterior $\propto$ likelihood $\cdot$ prior

$$\pi(r|\boldsymbol{\delta}) \propto L(\boldsymbol{\delta}|r) \cdot \pi_0(r)$$



Statistical model: $\boldsymbol{\delta} = \mathcal{G}(r) + \boldsymbol{\eta}$

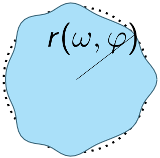
assuming $\boldsymbol{\eta} \sim \mathcal{N}(0, \Sigma) \Rightarrow L(\boldsymbol{\delta}|r) \propto \exp\left(-\frac{1}{2}\|\boldsymbol{\delta} - \mathcal{G}(r)\|_\Sigma^2\right)$

The Bayesian shape inverse problem

Prior

$$r(\omega;\varphi) = r_0(\varphi) + \sum_{j=1}^{d} \beta_j Y_j(\omega)\psi_j(\varphi)$$

with $Y_j \sim \mathcal{U}([-1,1])$ independent.



Frequency-explicit well-posedness [Kuijpers, S. 2024]

Existence and uniqueness

Posterior is absolutely continuous w.r.t. prior

Stability

Posterior depends continuously on the data with constant $\sim \kappa_0 R$

Bayesian inversion: frequency-explicit well-posedness

Theorem (Kuijpers, S. 2024)

Case 1: *r is μ_0-a.s. Lipschitz, $\alpha_{in} = \alpha_{out}$, $\frac{n_{in}}{n_{out}} < 1$ and $V = H^1(B_R)$*

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Then:

(i) $\mu^\delta \ll \mu_0$ with likelihood $L(\boldsymbol{\delta}|r) \propto \exp\left(-\frac{1}{2}\|\boldsymbol{\delta} - \mathcal{G}(r)\|_\Sigma^2\right)$

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(ii) for each $\gamma > 0$ s.t. $|\boldsymbol{\delta}|, |\boldsymbol{\delta}'| \le \gamma$,

$$d_{\mathrm{Hell}}(\mu^{\boldsymbol{\delta}}, \mu^{\boldsymbol{\delta}'}) \le C\|u_i\|_{H^1_{\kappa_0,\alpha,n}(B_R)}|\boldsymbol{\delta} - \boldsymbol{\delta}'| \sim (\kappa_0 R)|\boldsymbol{\delta} - \boldsymbol{\delta}'|$$

Main tools: shape calculus (i) and estimates from [Moiola, Spence 2019] (ii).

Frequency-explicit well-posedness: remarks

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Frequency dictates the lengthscale

Constants depend on the inverse problem setting (prior, measurements)

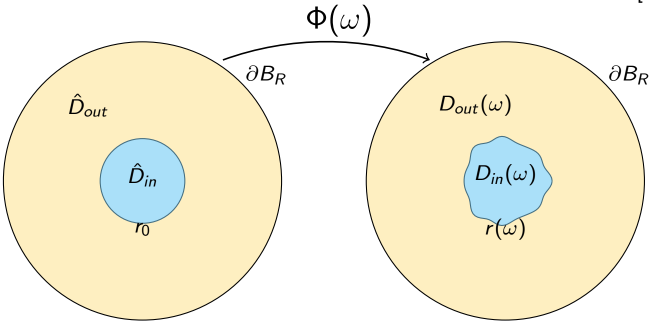
High frequency and/or *high contrast* worsen stability

For $\alpha_{in} = \alpha_{out}$, wider class of measurements allowed

Numerical realization: forward solver

Random PDE on *fixed* geometry $\leftrightsquigarrow$ *Fixed* PDE on *random* geometry

[Xiu, Tartakovsky 2006]



$$-\nabla\cdot(\alpha\nabla u) - \kappa_0^2 n u = 0 \text{ in } D_{in}(\omega) \cup D_{out}(\omega)$$

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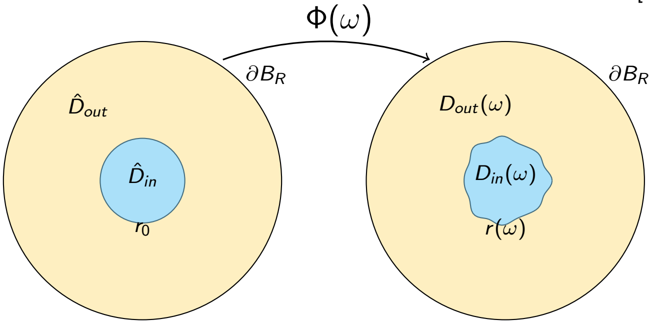


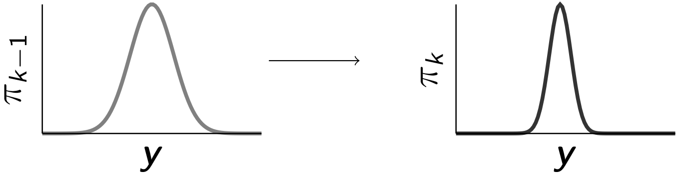
$$-\hat{\nabla}\cdot\left(\hat{\alpha}(\omega)\hat{\nabla}\hat{u}\right)-\kappa_0^2\hat{n}(\omega)\hat{u}=0 \text{ in } \hat{D}_{in}\cup\hat{D}_{out}$$

We solve on reference configuration with *h*-FEM and Perfectly Matched Layer

Numerical realization: SMC sampler

Approximate $\pi_k(\boldsymbol{y}|\boldsymbol{\delta}) \propto L(\boldsymbol{\delta}|\boldsymbol{y})^{\gamma_k}\pi_0(\boldsymbol{y})$ by $\pi_k(\boldsymbol{y}|\boldsymbol{\delta}) \approx \sum_{p=1}^{P} W_p^k \delta_{\boldsymbol{y}_p}(\boldsymbol{y})$ sequentially.

For $0 < \gamma_1 < \ldots < \gamma_K = 1$:



1. **Reweigh** (update distribution)
2. **Resample** (if needed, improve representation)
3. **MCMC** moves (reintroduce diversity)

Numerical realization: adaptivity in SMC

Selection of **inverse temperatures** [Latz et al. '18]:

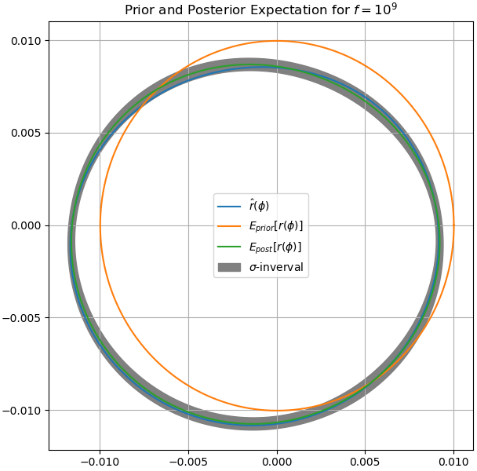
choose largest $\gamma_k \in (\gamma_{k-1}, 1]$ ensuring a given ESS (bisection algorithm).

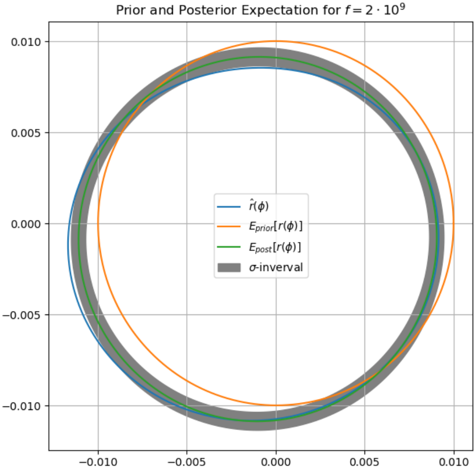
Variance in random walk Metropolis Hastings [Beskos et al. '15]:

$$\varepsilon_{j,k}^2 = \lambda_k \widehat{\mathrm{Var}}(Y_{j,k}), \quad j = 1, \ldots, J,$$

and λ_k adjusted according to acceptance ratio at $(k-1)$-th iteration.

Numerical results





Conclusions and extensions

UQ for time-harmonic wave propagation has numerical and statistical challenges

Recent PDE results allow to state **frequency-explicit** posterior stability

Stability constant **grows with frequency**

Similar results hold for **sound-soft** obstacle scattering

Possible extension to full **Maxwell equations**

Possible implications for **imaging**