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Problem Sheet 3 – Subset Simulation, FORM

Autumn School *Uncertainty Quantification for High-dimensional Problems*

Problem 1 (MH-MCMC in Subset Simulation)

Subset Simulation requires a method for generating samples from the conditional distribution $\boldsymbol{U}|F_{\ell-1}$, $\ell > 1$. In principle this can be done with Metropolis–Hastings Markov chain Monte Carlo (MH-MCMC), where we construct a Markov chain with stationary distribution (target distribution) $\boldsymbol{U}|F_{\ell-1}$.

For Metropolis–Hastings MCMC we require a proposal density $q(\cdot|\boldsymbol{u}^{(k-1)})$, calculate the acceptance probability α, and accept the candidate state as next state in the Markov chain with probability α, see Algorithm 2 in the lecture slides. For specific proposal densities the MH-MCMC algorithm in Subset Simulation simplifies.

Assume that $\boldsymbol{U}$ follows an n-variate standard normal distribution with pdf φ_n. Let $\varphi_n(\cdot|F_{\ell-1})$ denote the pdf of $\boldsymbol{U}|F_{\ell-1}$. Following (Papaioannou, Betz, et al., 2015) we choose as proposal density the n-variate normal density with mean $\rho\boldsymbol{u}^{(k-1)}$ and variance $(1-\rho^2)I_n$, where $\rho \in [0,1]$ denotes a correlation parameter, and $\boldsymbol{u}^{(k-1)}$ is the current state of the Markov chain.

Prove that for this choice of proposal density in MH-MCMC the acceptance probability of the candidate state $\boldsymbol{v}$ is

$$\alpha(\boldsymbol{u}^{(k-1)}, \boldsymbol{v}) = \mathbb{1}_{F_{\ell-1}}(\boldsymbol{v}).$$

Problem 2 (Calculation of most likely failure point in FORM)

Let $G\colon \mathbb{R}^n \to \mathbb{R}$ denote the limit-state function (LSF) and let $\boldsymbol{U} \sim N(0, I_n)$ follow the standard normal distribution. The so-called *most likely failure point* (MLFP) is defined as

$$\boldsymbol{u}^{MLFP} := \operatorname*{argmin}_{\boldsymbol{u}\in\mathbb{R}^n} \quad \frac{1}{2}\|\boldsymbol{u}\|_2^2, \quad \text{such that} \quad G(\boldsymbol{u}) = 0. \tag{1}$$

Hasofer and Lind (1974) determine the MLFP in (1) iteratively by a line search of the linearized LSF anchored at the current iterate $\boldsymbol{u}_k$ in the direction of $-\nabla G(\boldsymbol{u}_k)$.

Write down the iterative algorithm to approximate the MLFP according to Hasofer & Lind!