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The random timestep Euler method and its continuous dynamics

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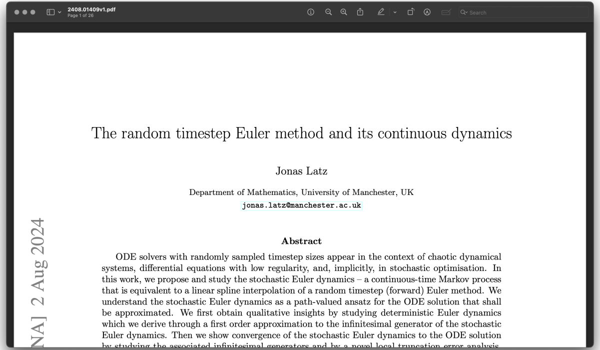

The random timestep Euler method and its continuous dynamics

Jonas Latz

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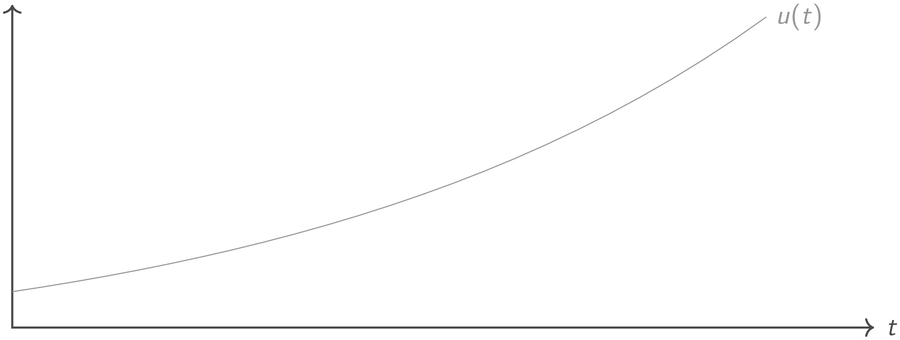
Abstract

ODE solvers with randomly sampled timestep sizes appear in the context of chaotic dynamical systems, differential equations with low regularity, and, implicitly, in stochastic optimisation. In this work, we propose and study the stochastic Euler dynamics – a continuous-time Markov process that is equivalent to a linear spline interpolation of a random timestep (forward) Euler method. We understand the stochastic Euler dynamics as a path-valued ansatz for the ODE solution that shall be approximated. We first obtain qualitative insights by studying deterministic Euler dynamics which we derive through a first order approximation to the infinitesimal generator of the stochastic Euler dynamics. Then we show convergence of the stochastic Euler dynamics to the ODE solution by studying the associated infinitesimal generators and by a novel local truncation error analysis

funding: Isaac Newton Institute, Cambridge / EPSRC

Idea

Consider some initial value problem on $X := \mathbb{R}^d$: $$u' = f(u), \qquad u(0) = u_0.$$



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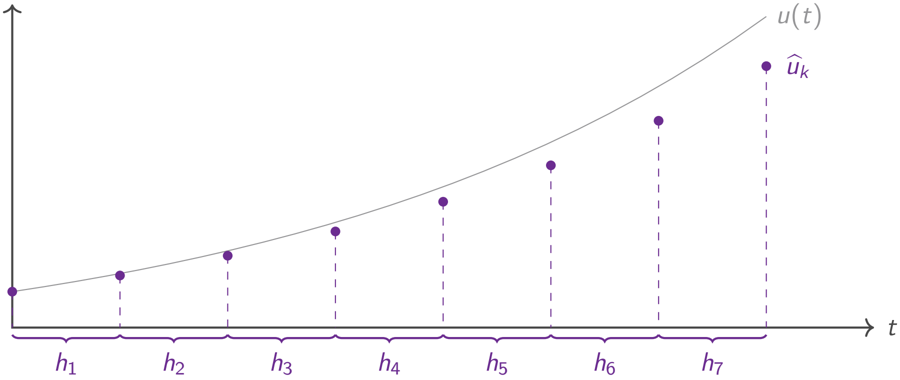
The Euler method: $$\widehat{u}_k = \widehat{u}_{k-1} + h_k f(\widehat{u}_{k-1}), \qquad \widehat{u}_0 = u_0.$$



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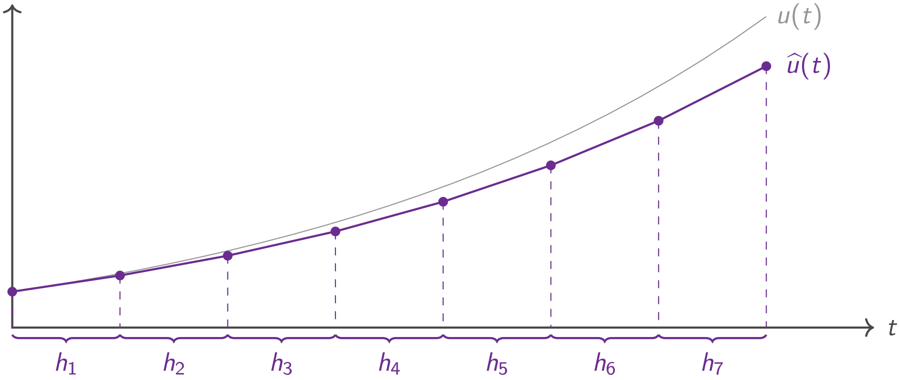

Idea

Linear interpolation

- Approximate the path $(u(t))_{t\geq 0}$ by linearly interpolating pairwise points obtained from the Euler method and obtain the linear interpolant:

$$\widehat{u}(t) := \frac{t_K - t}{t_K - t_{K-1}}\widehat{u}_{K-1} + \frac{t - t_{K-1}}{t_K - t_{K-1}}\widehat{u}_K \qquad (t \in [t_{K-1}, t_K], K \in \mathbb{N}).$$

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This talk:

- The linear interpolant has a lot of structure in case we randomise the timestep sizes
- Analyse the convergence/approximation properties of this linear interpolant and its stability

Overview

The random timestep Euler method

Convergence and accuracy

Stability

Conclusions

The random timestep Euler method

We discretise the ODE $u' = f(u), u(0) = u_0$ with the random timestep Euler method:

$$\widehat{V}_k = \widehat{V}_{k-1} + H_k f(\widehat{V}_{k-1}), \qquad \widehat{V}_0 = u_0$$

with exponentially distributed stepsizes $H_1, H_2, \ldots \sim \mathrm{Exp}(h^{-1})$ i.i.d. and a stepsize parameter $\mathbb{E}[H_1] = h > 0$. We also define the jump times $T_k := \sum_{i=1}^k H_i$ $(k \in \mathbb{N})$ and $T_0 := 0$.

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Random timestep ODE solvers are used if...

- ...f is of very low regularity (i.e., no point evaluations) [Eisenmann+al. 2019] [Jentzen+Neuenkirch 2009]
- ...$(u(t))_{t \geq 0}$ is chaotic and its stationary behaviour shall be analysed [Abdulle+Garegnani 2020]
- ...in stochastic optimisation **[Jin+L.+Liu+Schönlieb 2023] [L. 2020]**

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Randomised methods have become quite popular in numerical analysis, although not so much when it comes to ODEs [Chen+al. 2024] [Halko+al. 2011] [Robbins+Monro 1951] [Stuart+Teckentrup 2018]

Back to the initial example

Consider some initial value problem on $X := \mathbb{R}^n$: $$u' = f(u), \qquad u(0) = u_0.$$

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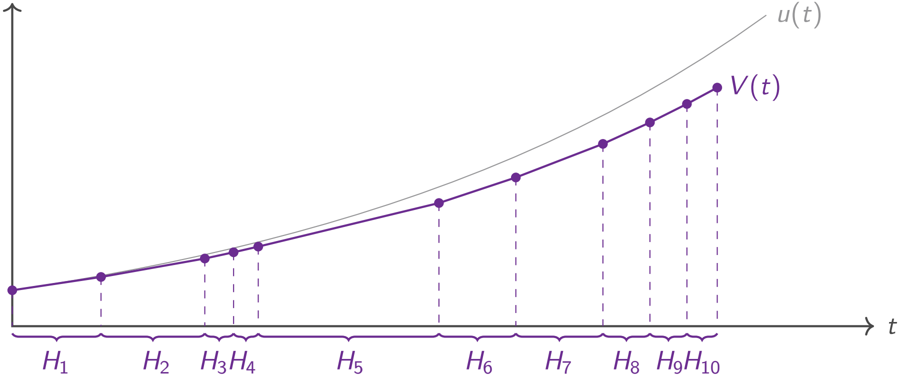

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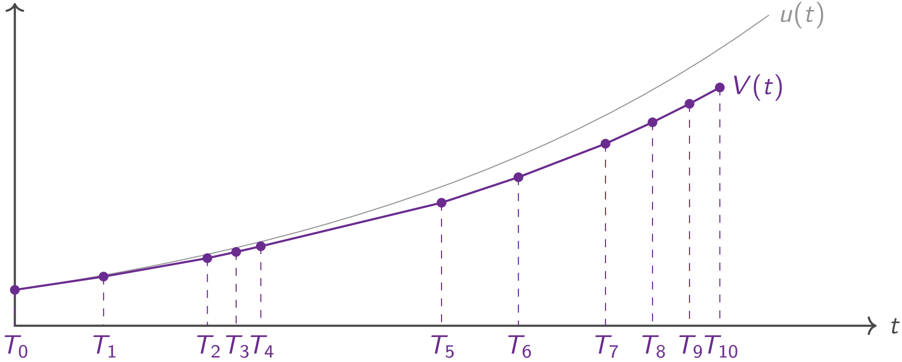

Finding structure in piecewise linear interpolation

Idea: Write the linear interpolation $(V(t))_{t\geq 0}$ of the random timestep Euler trajectory $(\widehat{V}_k)_{k=0}^{\infty}$ as an ODE with jumps after exponential waiting times.

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Then, however, we do not need $(\widehat{V}_k)_{k=0}^{\infty}$ at all, we can just update $(\overline{V}(t))_{t\geq 0}$ from $(V(t))_{t\geq 0}$:

$$\overline{V}(T_k) = V(T_k-)$$

The stochastic Euler dynamics

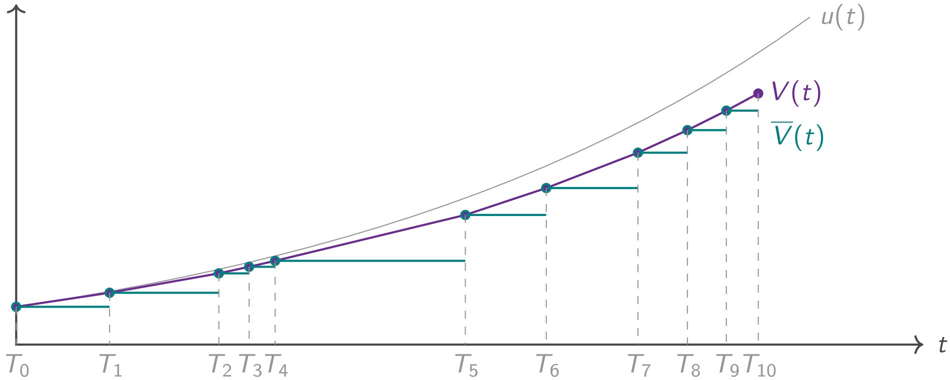
The linear interpolation of $(\widehat{V}_k)_{k=0}^{\infty}$ satisfies the following ODE with (random) jumps:

$$
\begin{aligned}
V'(t) &= f(\overline{V}(t)) && (t \in (T_{k-1}, T_k], k \in \mathbb{N}) \\
\overline{V}'(t) &= 0 && (t \in (T_{k-1}, T_k), k \in \mathbb{N}) \\
\overline{V}(T_k) &= V(T_k-) && (k \in \mathbb{N}) \\
V(0) &= \overline{V}(0) = u_0,
\end{aligned}
$$

where, still, $T_K := \sum_{k=1}^{K} H_k$ $(K \in \mathbb{N})$ and $T_0 := 0$ are the jump times.

We refer to $(V(t), \overline{V}(t))_{t \geq 0}$ (or just $(V(t))_{t \geq 0}$) as stochastic Euler dynamics and to $(\overline{V}(t))_{t \geq 0}$ as companion process.

The stochastic Euler dynamics



$V(t)$ is the linear interpolant of the random timestep Euler method

$\overline{V}(t)$ remembers the value of $V(t)$ at the last jump time

The stochastic Euler dynamics

Due to the memorylessness of the exponential distribution, one can show that $(V(t), \overline{V}(t))_{t \geq 0}$ forms a continuous-time Markov process, more specifically it is a

- piecewise-deterministic Markov process [Davis 1984]

  i.e. a piecewise-ODE with random switches/jumps

- Feller process, [Davis 1993]

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Indeed, we can compute the infinitesimal generator of $(V(t), \overline{V}(t))_{t \geq 0}$ as

$$\mathcal{A}_h \varphi(v, \overline{v}) = \langle f(\overline{v}), \nabla_v \varphi(v, \overline{v}) \rangle + \frac{1}{h}(\varphi(v, v) - \varphi(v, \overline{v})) \qquad (v, \overline{v} \in X)$$

for appropriate test functions $\varphi : X^2 \to \mathbb{R}$ and use it to, e.g., compute $\mathbb{P}((V(t), \overline{V}(t)) \in \cdot)$ $(t \geq 0)$.

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Convergence

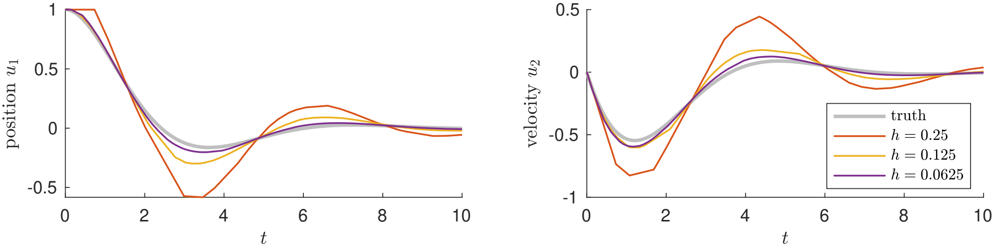

Figure: *The trajectory of an underdamped harmonic oscillator $u_1' = u_2, u_2' = -u_1 - u_2$ with initial value $u(0) = (1, 0)^T$ and realisations of the corresponding stochastic Euler dynamics $(V(t))_{t\geq 0}$ for $h \in \{0.25, 0.125, 0.0625\}$. We show the ground truth $(u(t))_{t\geq 0}$ and one realisation of $(V(t))_{t\geq 0}$ for each of the stepsize parameters. The stochastic Euler dynamics approaches the ODE solution as h decreases.*

Does $(V(t))_{t\geq 0}$ converge to $(u(t))_{t\geq 0}$ as $h \downarrow 0$? In which sense?

Convergence

Theorem 3.4. [L. 2024]

Let $f : X \to X$ be Lipschitz continuous and let $(u(t))_{t\geq 0}$ and $(V(t), \overline{V}(t))_{t\geq 0}$ be the associated ODE solution and stochastic Euler dynamics with initial values $u_0 \in X$, respectively. Moreover, we assume that we have access to the jump time process $(\overline{T}(t))_{t\geq 0}$. Then,

$$(V(t))_{t\geq 0} \Rightarrow (u(t))_{t\geq 0} \ (h \downarrow 0).$$

- Convergence in probability/weak convergence on the probability space and convergence w.r.t. a weighted uniform metric in time:

$$d_C((x(t))_{t\geq 0}, (y(t))_{t\geq 0}) = \int_0^\infty \exp(-t) \min\left\{1, \sup_{s\leq t} \|x(s) - y(s)\|\right\} \mathrm{d}t \quad (x, y \in C^0[0,\infty)).$$

- Theorem 3.4 shows convergence of the complete path $(V(t))_{t\geq 0}$, but in a weighted metric, so it rather shows uniform convergence on all bounded intervals $[0, T]$.

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- $(\overline{T}(t))_{t\geq 0}$ is the Markov jump process with jump times $(T_k)_{k=1}^{\infty}$ and $\overline{T}(T_k) = T_k$ $(k \in \mathbb{N}_0)$.
- The statement doesn't indicate a convergence rate
- Proof is based on the perturbed test function theory by Kushner [Kushner 1990]
  - Show tightness of a truncated version of the process $(V(t))_{t\geq 0}$
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- ODE limits of continuous-time Markov process are usual in, e.g., chemical reaction networks

[Darling+Norris 2008]

Accuracy

Convergence statement without rate is not as useful for computations – what can we do?

- In ODE timestepping, we often compute truncation errors, e.g. for the Euler method

$$\|\widehat{u}_1 - u(h_1)\| = \|u_0 + h_1 f(u_0) - u(h_1)\| = O(h_1^2; h_1 \downarrow 0).$$

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- Computing $\mathbb{E}[\|u(\varepsilon) - V(\varepsilon)\|^2]^{1/2}$, is non-trivial, as we do not know how many jumps have occurred in $(V(t))_{t\geq 0}$ on the interval $[0, \varepsilon]$, which leaves us with no expression for for $V(\varepsilon)$

Accuracy

Idea: Define a Poisson process $(K(t))_{t\geq 0}$ that has the same jump times as $(V(t))_{t\geq 0}$. Then, $K(t)$ counts the number of jumps up to $t \geq 0$ and allows us to compute

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Theorem 3.6. [L. 2024]

Let $(V(t), \overline{V}(t))_{t\geq 0}$ be the stochastic Euler dynamics corresponding to $u' = Au$, $u(0) = u_0$, with $A \in \mathbb{R}^{d\times d}$, and let $(K(t))_{t\geq 0}$ be the associated Poisson process. Then,

(i) $\mathbb{E}[\|V(\varepsilon) - u(\varepsilon)\|^2 | K(\varepsilon) = k]^{1/2} = O(\varepsilon^2; \varepsilon \downarrow 0)$ $(k \in \mathbb{N}_0)$,

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- Would expect a similar result for f nonlinear, but the proof technique should be rather different

Accuracy: Numerical Experiment

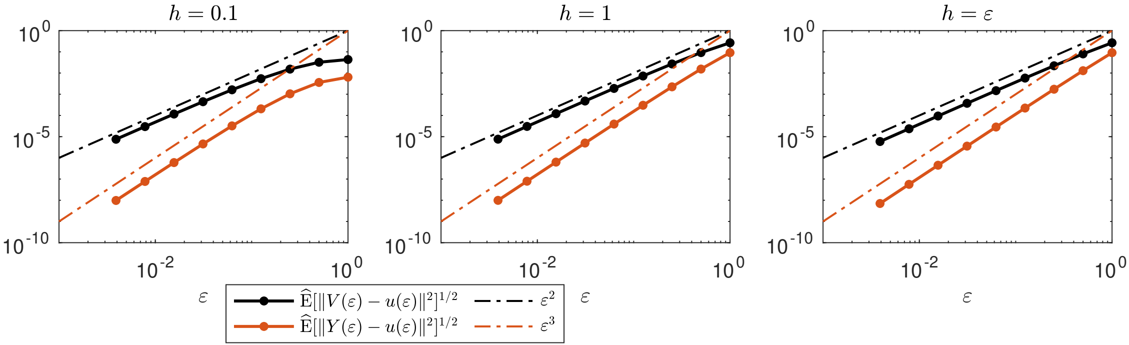

Figure: *Estimated root mean square truncation error for the stochastic Euler dynamics corresponding (black) to $u' = -u$, $u(0) = 1$ using 10^5 samples; the orange lines refer to the second order stochastic Euler dynamics that we don't discuss during this talk.*

Stability

The random timestep Euler method

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Stability

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Stability

Consider $u' = -Au$ with $u(0) = u_0$ and A being symmetric positive semi-definite.

- What are the assumptions on the stepsize parameter $h = \mathbb{E}[H_1]$ that will lead to a stable stochastic Euler dynamics $(V(t), \overline{V}(t))_{t \geq 0}$?
- What is the speed of convergence of $V(t) \to 0$ $(t \to \infty)$?

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Idea: Apply Foster–Lyapunov criteria to the infinitesimal generator [Meyn+Tweedie 1993]

- Find a 'norm-like' function L, with $\mathcal{A}_h L \leq -\kappa L$ for some $\kappa > 0$, then apply Grönwall's Lemma
- For the stochastic Euler dynamics in $d = 1$ dimension, $L(v, \overline{v}) = v^2 + c(v - \overline{v})^2$ for a $c > 0$ works
- Go from $d = 1$ to $d > 1$ by diagonalising A

Stability

Theorem 4.3. [L. 2024]

Let $A \in \mathbb{R}^{d\times d}$ be symmetric positive definite and let $(V(t), \overline{V}(t))_{t\geq 0}$ be the stochastic Euler dynamics corresponding to $u' = -Au$. Let $\lambda_1 \leq \cdots \leq \lambda_d$ be the eigenvalues of A, let h be chosen such that $\lambda_d h < 1$, and let $\kappa' \in (0, \min\{2\lambda_1, 1/(2h)\})$. Then,

$$\mathbb{E}\left[\|V(t)\|^2\right] + c_3'\mathbb{E}[\|V(t) - \overline{V}(t)\|^2] \leq \exp(-\kappa' t)\|u_0\|^2,$$

with $c_3' := \min\{1/\max\{(1/h-\kappa)/\kappa, (\kappa - 1/h + \lambda_\ell)/\lambda_\ell\} : \ell \in \{1, \ldots, d\}\}$.

Stability

Theorem 4.3. [L. 2024]

Let $A \in \mathbb{R}^{d\times d}$ be symmetric positive definite and let $(V(t), \overline{V}(t))_{t\geq 0}$ be the stochastic Euler dynamics corresponding to $u' = -Au$. Let $\lambda_1 \leq \cdots \leq \lambda_d$ be the eigenvalues of A, let h be chosen such that $\lambda_d h < 1$, and let $\kappa' \in (0, \min\{2\lambda_1, 1/(2h)\})$. Then,

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with $c_3' := \min\{1/\max\{(1/h-\kappa)/\kappa, (\kappa - 1/h + \lambda_\ell)/\lambda_\ell\} : \ell \in \{1,\ldots,d\}\}$.

- We certainly need $\lambda_d h < 1$, which is stronger than stability in the Euler method
- We know that $\|u(t)\|^2 \leq \exp(-2\lambda_1 t)\|u_0\|^2$, we have the same bound above, if $h \leq 1/(4\lambda_1)$
- Theory does not contain complex-valued eigenvalues/non-diagonalisable matrices for the moment

Stability: Numerical Experiment

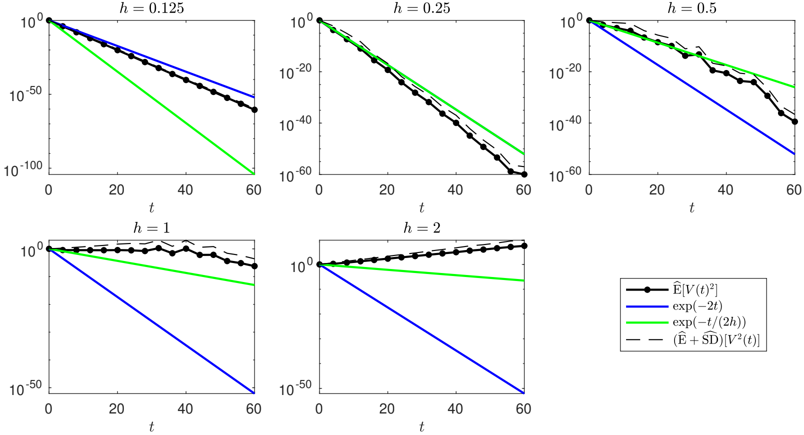

Figure: *Stochastic Euler dynamics* $(V(t))_{t\geq 0}$ *approximating* $u' = -u$, $u(0) = u_0$ *for* $h = 2^{-3}, \ldots, 2^{1}$; *sample means and standard deviations are estimated using* 10^6 *independent samples.*

Overview

The random timestep Euler method

Convergence and accuracy

Stability

Conclusions

Conclusions

further results

- Also study deterministic Euler dynamics, an ODE arising from an approximation of $\mathcal{A}_h$
- Brief study of certain second order stochastic Euler dynamics
- Some experiments on an underdamped harmonic oscillator (A not positive definite)

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take-home messages

- Stochastic Euler dynamics are an exciting framework for the analysis of (randomised) ODE solvers
- Use infinitesimal generators/first-order PDEs to analyse ODE solvers
- The theory of classical ODE solvers is partially recovered, partially quite different

A lot of open questions remain!

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Open positions@UoM:

Professor of Applied Mathematics (general)

Professor of Applied Mathematics (interface between applied maths and AI, ML, data science)

Senior Lecturer/Reader in Computational Statistics (=Associate Professor)

Lecturer in Numerical Analysis and Data Science (=Assistant Professor)

PhD position in UKRI-CDT ‘Image segmentation in radioastronomy with physical models on graphs’ with JL and Prof. Anna Scaife

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