Recitation 11

Graph Contraction and MSTs

11.1 Announcements

- *SegmentLab* has been released, and is due **Monday afternoon**. It's worth 135 points.
- *Midterm 2* is on **Friday, November 11**.

11.2 Contraction

In the textbook, we presented an algorithm for counting the number of connected components in a graph:

Algorithm 11.1. *(Algorithm 17.22 in the textbook.)*

countComponents (V, E) =
  if $|E| = 0$ **then** $|V|$ **else**
  let
    (V', P) = *starPartition* (V, E)
    $E' = \{(P[u], P[v]) : (u, v) \in E \mid P[u] \neq P[v]\}$
  in
    countComponents (V', E')
  end

with `starPartition` implemented as follows:

Algorithm 11.2. *(Algorithm 17.15 in the textbook.)*

starPartition (V, E) =
let
  $TH = \{(u, v) \in E \mid \neg heads(u) \wedge heads(v)\}$
  $P = \bigcup_{(u,v) \in TH} \{u \mapsto v\}$
  $V' = V \setminus domain(P)$
  $P' = \{u \mapsto u : u \in V'\}$
in
  $(V', P' \cup P)$
end

Now, suppose we implemented star partitioning for enumerated graphs as follows:

```
val enumStarPartition : (int * int) Seq.t * int -> int Seq.t
```

Specifically, given a graph represented as a sequence of edges E where every vertex is labeled $0 \leq v < n$, (`enumStarPartition` (E, n)) returns a mapping P where $P[v]$ is the super-vertex containing v. (If v was a star center or was unable to contract, then $P[v] = v$.)

Task 11.3. *Implement a function* `enumCountComponents` *which counts the number of components of an enumerated graph. It should take in a graph represented as* (E, n) *and use* `enumStarPartition` *internally.*

A direct but *incorrect* translation of the original code might look like this:

```
1 fun incorrectCountComponents (E, n) =
2   if |E| = 0 then n else
3   let
4     val P = enumStarPartition (E, n)
5     val E′ = ⟨(P[u], P[v]) : (u, v) ∈ E | P[u] ≠ P[v]⟩
6   in
7     incorrectCountComponents (E′, n)
8   end
```

The problem with this code is that it doesn't actually count the number of connected components, despite performing the contraction correctly. This is because we never modify the value n.

A first step in fixing the issue is to add a line after line 5 which counts the number of distinct vertices in E'. Specifically, we use P to identify which vertices no longer exist, filter them out, then simply take the length of the resulting sequence:

```
val n′ = |⟨v : 0 ≤ v < n | P[v] = v⟩|
```

We could then pass n' in to the recursive call rather than n. However, we now notice an even bigger problem: *not all vertices in* E' *are labeled* $0 \leq v < n'$.

What we really need to do is construct a new labeling within the range $[0, n')$. We can do so by marking each each contracted vertex with a 0 and each remaining vertex with a 1 and running a $+$-scan. This determines a sequence P' which maps each remaining vertex to a unique label in the range $[0, n')$. This step also conveniently calculates n'. At the end of the round, when we promote edges by relabeling their endpoints, we have to further relabel them according to P'. The code is as follows.

Algorithm 11.4. *Counting connected components in an enumerated graph.*

```
fun enumCountComponents (E, n) =
  if |E| = 0 then n else
  let
    val P = enumStarPartition (E, n)
    fun isAlive v = if P[v] = v then 1 else 0
    val (P′, n′) = Seq.scan + 0 ⟨isAlive(v) : 0 ≤ v < n⟩
    val E′ = ⟨(P′[P[u]], P′[P[v]]) : (u, v) ∈ E | P[u] ≠ P[v]⟩
  in
    enumCountComponents (E′, n′)
  end
```

11.2.1 Cost Bounds

Task 11.5. *Recall that a **forest** is a collection of trees. What are the work and span of* `enumCountComponents` *when applied to a forest? Assume that* (`enumStarPartition` (E, n)) *requires* $O(n + |E|)$ *work and* $O(\log n)$ *span.*

Line 6 of `enumCountComponents` clearly requires $O(n)$ work and $O(\log n)$ span. Line 7 is just a `map` followed by a `filter`, and therefore requires $O(m)$ work and $O(\log n)$ span. But how do n and m change, round-to-round?

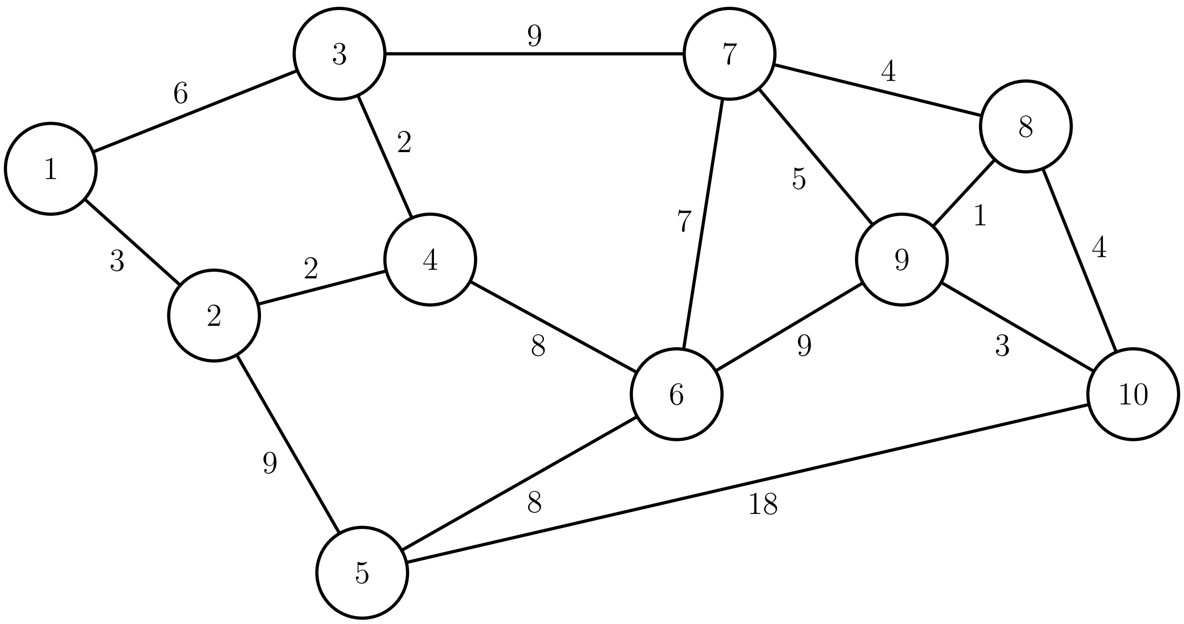
Regarding n, we recall that star-partitioning removes at least $n/4$ vertices in expectation, and therefore we expect the number of vertices to decrease geometrically.

For *general* graphs, we can't say that m decreases geometrically. However, a tree has $n - 1$ edges, and therefore m is initially upper bounded by $n - 1$. Furthermore, on each round, exactly one edge is deleted for every vertex which is deleted. Therefore, for forests and trees, m *decreases geometrically* during contraction. Therefore the total work and span of this algorithm for an input forest of n vertices are $O(n)$ and $O(\log^2 n)$, respectively.

11.3 Borůvka's Algorithm

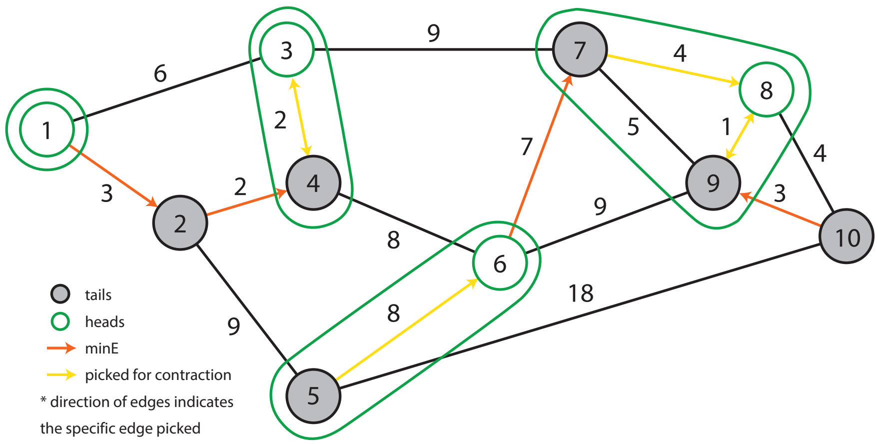
The textbook describes two versions of Borůvka's algorithm: one which performs tree contraction at each round, and another which performs a single round of star contraction at each round. We will be using the latter, since it has better overall span ($O(\log^2 n)$ rather than $O(\log^3 n)$).

Task 11.6. *Run Borůvka's algorithm on the following graph. Draw the graph at each round, and identify which edges are MST edges. Use the coin flips specified.*

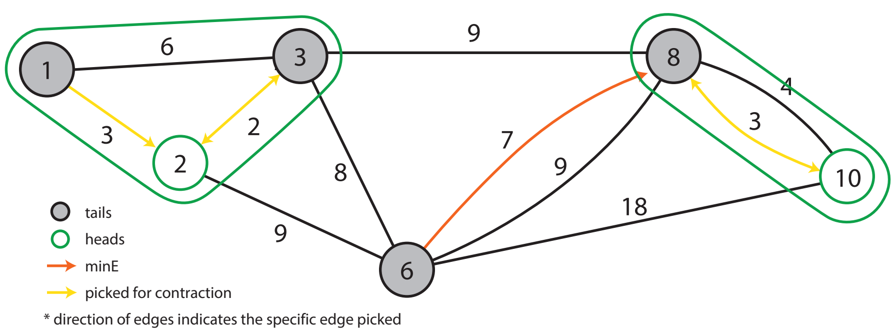


Round	Vertices 1	2	3	4	5	6	7	8	9	10
0	H	T	H	T	T	H	T	H	T	T
1	T	H	T			T		T		H
2		T				H				T

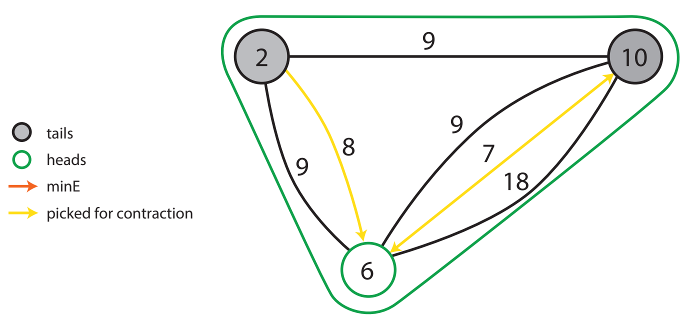
Round 0:



Round 1:



Round 2:



11.4 Additional Exercises

Exercise 11.7. *In graph theory, an* ***independent set*** *is a set of vertices for which no two vertices are neighbors of one another. The* ***maximal independent set*** *(MIS) problem is defined as follows:*

> *For a graph* (V, E)*, find an independent set* $I \subseteq V$ *such that for all* $v \in (V \setminus I)$*,* $I \cup \{v\}$ *is not an independent set.*[a]

Design an efficient parallel algorithm based on graph contraction which solves the MIS problem.

[a] The condition that we cannot extend such an independent set I with another vertex is what makes it "maximal." There is a closely related problem called **maximum independent set** where you find the largest possible I. However, this problem turns out to be NP-hard!

.