SSR bistable switch

Bistable switch

$$\dot{y} = \frac{\alpha}{1+z^2} - y$$
$$\dot{z} = \frac{\beta}{1+y^2} - \gamma z$$

turn into quasipolynomial form

$$\begin{aligned}
\dot{y} &= y[\alpha c e - 1] \\
\dot{z} &= z[\beta d f - \gamma] \\
\dot{a} &= a[2z^2 \beta d f c - 2z^2 \gamma c] \\
\dot{b} &= b[2y^2 \alpha c e d - 2y^2 d] \\
\dot{c} &= c[-c\dot{a}] \\
\dot{d} &= d[-d\dot{b}] \\
\dot{e} &= e[-e\dot{y}] \\
\dot{f} &= f[-f\dot{z}]
\end{aligned}$$

$$\begin{aligned}
a &= 1+z^2 \\
b &= 1+y^2 \\
c &= a^{-1} \\
d &= b^{-1} \\
e &= y^{-1} \\
f &= z^{-1}
\end{aligned}$$

$$\boxed{\begin{aligned}
\Rightarrow \dot{y} &= y[\alpha c e - 1] \\
\dot{z} &= z[\beta d f - \gamma] \\
\dot{a} &= a[2\beta z d c - 2\gamma z^2 c] \\
\dot{b} &= b[2\alpha y c d - 2y^2 d] \\
\dot{c} &= c[-2\beta z d c + 2\gamma z^2 c] \\
\dot{d} &= d[-2\alpha y d c + 2y^2 d] \\
\dot{e} &= e[-\alpha c e + 1] \\
\dot{f} &= f[-\beta d f + \gamma]
\end{aligned}}$$

QP form:

$$\dot{x}_i = x_i\left[\lambda_i + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}\right]$$

m quasimonomials $\{ce, df, zdc, z^2c, ycd, y^2d\}$ (g, h, ℓ, m, n, p)

$[y\ z\ a\ b\ c\ d\ e\ f]$

$$A = \begin{bmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & \beta & 0 & 0 & 0 & 0 \\
0 & 0 & 2\beta & -2\gamma & 0 & 0 \\
0 & 0 & 0 & 0 & 2\alpha & -2 \\
0 & 0 & -2\beta & 2\gamma & 0 & 0 \\
0 & 0 & 0 & 0 & -2\alpha & 2 \\
-\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

$$\lambda = \begin{bmatrix} -1 \\ -\gamma \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \gamma \end{bmatrix}$$

turn into quadratic form

$$\boxed{\begin{aligned}
\dot{g} &= g[-2\beta\ell + 2\gamma m - \alpha g + 1] \\
\dot{h} &= h[-2\alpha n + 2p - \beta h + \gamma] \\
\dot{\ell} &= \ell[\beta h - \gamma - 2\beta\ell + 2\gamma m - 2\alpha n + 2p] \\
\dot{m} &= m[2\beta h - 2\gamma - 2\beta\ell + 2\gamma m] \\
\dot{n} &= n[\alpha g - 1 - 2\beta\ell + 2\gamma m - 2\alpha n + 2p] \\
\dot{p} &= p[2\alpha g - 2 - 2\alpha n + 2p]
\end{aligned}}$$

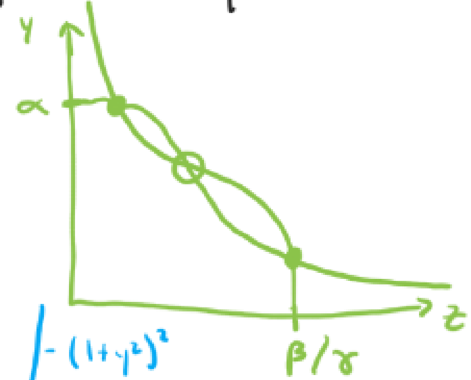
Quadratic gLV form

$$\dot{y}_i = y_i\left[\tilde{\lambda}_i + \sum_{j=1}^{m} \tilde{A}_{ij} y_j\right]$$

$$\tilde{\lambda}_i = \begin{bmatrix} 1 \\ \gamma \\ -\gamma \\ -2\gamma \\ -1 \\ -2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix}
-\alpha & 0 & -2\beta & 2\gamma & 0 & 0 \\
0 & -\beta & 0 & 0 & -2\alpha & 2 \\
0 & \beta & -2\beta & 2\gamma & -2\alpha & 2 \\
0 & 2\beta & -2\beta & 2\gamma & 0 & 0 \\
\alpha & 0 & -2\beta & 2\gamma & -2\alpha & 2 \\
2\alpha & 0 & 0 & 0 & -2\alpha & 2
\end{bmatrix}$$

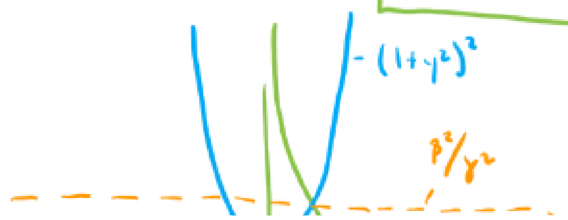
Are steady states preserved?

$$\dot{y} = \frac{\alpha}{1+z^2} - y$$
$$\dot{z} = \frac{\beta}{1+y^2} - \gamma z$$



$$y = \frac{\alpha}{1+z^2} \qquad z = \frac{\beta/\gamma}{1+y^2}$$

$$y + yz^2 = \alpha$$
$$\Rightarrow y\left(1 + \frac{(\beta/\gamma)^2}{(1+y^2)^2}\right) = \alpha$$
$$\Rightarrow y(1+y^2)^2 + y(\beta/\gamma)^2 = (1+y^2)^2\alpha$$
$$\Rightarrow y^5 - \alpha y^4 + 2y^3 - 2\alpha y^2 + (1+\beta^2/\gamma^2)y - \alpha = 0$$



$$(y-\alpha)(y^4+2y^2+1) = -\beta^2/\gamma^2\, y$$
$$(y-\alpha)(y^2+1)^2 = \frac{-\beta^2}{\gamma^2}\, y$$
$$\Rightarrow \left(\frac{\alpha}{y}-1\right)(y^2+1)^2 = \frac{\beta^2}{\gamma^2} \Rightarrow 0 < y^* < \alpha$$

$$\sim \left(\frac{\alpha}{y}-1\right)y^4 = \alpha y^3 - y^4 = y^3(\alpha - y)$$

$\{\underset{g}{ce},\ \underset{h}{df},\ \underset{\ell}{zdc},\ \underset{m}{z^2c},\ \underset{n}{ycd},\ \underset{p}{y^2d}\}$

$\frac{1/y}{1+z^2}$ $\frac{1/z}{1+y^2}$ $\frac{z}{(1+y^2)(1+z^2)}$ $\frac{z^2}{1+z^2}$ $\frac{y}{(1+y^2)(1+z^2)}$ $\frac{y^2}{1+y^2}$

(A)

$$\dot y = \frac{\alpha}{1+z^2} - y$$
$$\dot z = \frac{\beta}{1+y^2} - \gamma z$$

$$\boxed{\begin{aligned} a &= 1+z^2 \\ b &= 1+y^2 \\ c &= a^{-1} \\ d &= b^{-1} \\ e &= y^{-1} \\ f &= z^{-1} \end{aligned}}$$

(B)

$$\boxed{\begin{aligned} \Rightarrow \dot y &= y[\alpha c e - 1] \\ \dot z &= z[\beta d f - \gamma] \\ \dot a &= a[2\beta z d c - 2\gamma z^2 c] \\ \dot b &= b[2\alpha y c d - 2y^2 d] \\ \dot c &= c[-2\beta z d c + 2\gamma z^2 c] \\ \dot d &= d[-2\alpha y d c + 2y^2 d] \\ \dot e &= e[-\alpha c e + 1] \\ \dot f &= f[-\beta d f + \gamma] \end{aligned}}$$

(C)

$$\boxed{\begin{aligned} \dot g &= g[-2\beta\ell + 2\gamma m - \alpha g + 1] \\ \dot h &= h[-2\alpha n + 2p - \beta h + \gamma] \\ \dot \ell &= \ell[\beta h - \gamma - 2\beta\ell + 2\gamma m - 2\alpha n + 2p] \\ \dot m &= m[2\beta h - 2\gamma - 2\beta\ell + 2\gamma m] \\ \dot n &= n[\alpha g - 1 - 2\beta\ell + 2\gamma m - 2\alpha n + 2p] \\ \dot p &= p[2\alpha g - 2 - 2\alpha n + 2p] \end{aligned}}$$

Steady states in (A) satisfy $y = \frac{\alpha}{1+z^2}$ and $z = \frac{\beta/\gamma}{1+y^2}$

In (B): $\dot y = 0 \Rightarrow \alpha c e - 1 = 0 \Rightarrow \frac{\alpha/y}{1+z^2} = 1$

$\dot z = 0 \Rightarrow \beta d f - \gamma = 0 \Rightarrow \frac{\beta/z}{1+y^2} = \gamma$ Note $a, \dots, f > 0$

$\dot a, \dot c, \dot e \Rightarrow$ same as $\dot y = 0$

$\dot b, \dot d, \dot f \Rightarrow$ same as $\dot z = 0$

In (C): If $\dot a \to \dot f = 0$, then $\dot g \to \dot p = 0$ also $\left(\dot g = \frac{d}{dt}(abc) = \dot a \dots + \dot b \dots + \dot c\right)$

Therefore, if (A) has 2 stable steady states (y_1^*, z_1^*) and (y_2^*, z_2^*), then there are steady states $(y_1^*, z_1^*, a_1^*, b_1^*, c_1^*, d_1^*, e_1^*, f_1^*)$ and for $(a_2^*, \text{etc}\dots)$, and so there are steady states $(g_1^*, h_1^*, \ell_1^*, m_1^*, n_1^*, p_1^*)$ as well!

$g_1^* = c_1^* e_1^* = \frac{1/y_1^*}{1+z_1^{*2}}$, $h_1^* = $ etc.

How to convert from gLV variables back to y and z (physical)?

$\{\underset{g}{ce},\ \underset{h}{df},\ \underset{\ell}{zdc},\ \underset{m}{z^2c},\ \underset{n}{ycd},\ \underset{p}{y^2d}\} \Rightarrow \frac{n}{\ell} = \frac{y}{z} \Rightarrow y^3 = \frac{mnp}{\ell^2} \Rightarrow \boxed{y = \sqrt[3]{mnp/\ell^2}}$

$\frac{mp}{\ell} = y^2 z$ $\quad z^3 = \frac{m\ell p}{\dots}$

$$z^* = \frac{m\lambda p}{n^2} \qquad \boxed{z = \sqrt{m\lambda p}/n^2}$$

Next:

1. Is stability preserved from $\underset{(2D)}{op} \to \underset{(8D\ \text{quasipolynomial})}{qP} \to \underset{(6D\ gLV)}{gP}$?
2. What if we use

$$\dot{y} = \alpha - \frac{\alpha z^2}{1+z^2} - y$$
$$\dot{z} = \beta - \frac{\beta y^2}{1+y^2} - \gamma z$$

instead?

$a = 1+z^2$

$b = 1+y^2$

$c = a^{-1}$

$d = b^{-1}$

$$\dot{y} = \alpha - \alpha z^2 a^{-1} - y = \alpha - \alpha z^2 c - y$$
$$\dot{z} = \beta - \beta y^2 b^{-1} - \gamma z = \beta - \beta y^2 d - \gamma z$$
$$\dot{a} = 2z(\beta - \beta y^2 b^{-1} - \gamma z) = 2\beta z - 2\beta y^2 z d - 2\gamma z^2$$
$$\dot{b} = 2y(\alpha - \alpha z^2 a^{-1} - y) = 2\alpha y - 2\alpha y z^2 c - 2y^2$$
$$\dot{c} = -c^2\, 2z(\beta - \beta y^2 b^{-1} - \gamma z) = c^2(-2\beta z + 2\beta y^2 z d + 2\gamma z^2)$$
$$\dot{d} = -d^2\, 2y(\alpha - \alpha z^2 a^{-1} - y) = d^2(-2\alpha y + 2\alpha y z^2 c + 2y^2)$$
$$\dot{u} = -\alpha + y + \alpha z^2 c = -u + \alpha\left(\frac{\beta - v}{\gamma}\right)^2 c$$

$y = \alpha - u$ $\qquad z = \frac{\beta - v}{\gamma}$

$u = \alpha - y$ $\qquad v = \beta - \gamma z$

$\dot{u} = -\dot{y}$ $\qquad \dot{v} = -\gamma \dot{z}$

Can we change signs of the growth rate?

Consider $\dot{x} = -\mu x + M x^2$. Can we find $y = f(x)$ such that $\dot{y} = \mu y$ + other terms ?

E.g.: $y = x^2 \Rightarrow \dot{y} = 2x\dot{x} = -2\mu x^2 + 2Mx^3 = -2\mu y + 2M y^{3/2}$

$y = f(x) \Rightarrow \dot{y} = \frac{df(x)}{dx}\dot{x} = -\mu \frac{df(x)}{dx} x + M \frac{df(x)}{dx} x^2$

$\Rightarrow$ Need $\frac{df(x)}{dx} x =- y =- f(x) \Rightarrow \frac{df(x)}{dx} x + f(x) = 0$

$\Rightarrow y = x^{-1}$ $\qquad f(x) = x^{-1} \quad -x^{-2} x + x^{-1} = 0$

$\Rightarrow \dot{y} = -x^{-2}\dot{x} = \mu x^{-1} - M$

$= \mu y - M$

$y = x^{-1/2} \qquad \dot{y} = \frac{-1}{2} x^{-3/2}\dot{x} = \frac{\mu}{2} x^{-1/2} - \frac{M}{2} x^{1/2} = \frac{\mu}{2} y - \frac{M}{2}$

Attempt #2: Bistable switch $\rightarrow$ gLV w/ positive growth rates

original form:

$$\dot{y} = \alpha - \frac{\alpha z^2}{1+z^2} - y$$
$$\dot{z} = \beta - \frac{\beta y^2}{1+y^2} - \gamma z$$

Let $u = y^{-1}$, $v = z^{-1}$ $\Rightarrow$ $\dot{u} = -y^{-2}\dot{y} = -u^2\alpha + \frac{\alpha u^2 v^{-2}}{1+v^{-2}} + u$

$\dot{v} = -z^{-2}\dot{z} = -v^2\beta + \frac{\beta v^2 u^{-2}}{1+u^{-2}} + \gamma v$

steady state reducible form:

$$\dot{u} = u - \alpha u^2 + \frac{\alpha u^2}{1+v^2} = u\left[1 - \alpha u + \frac{\alpha u}{1+v^2}\right]$$
$$\dot{v} = \gamma v - \beta v^2 + \frac{\beta v^2}{1+u^2} = v\left[\gamma - \beta v + \frac{\beta v}{1+u^2}\right]$$

Let $a = \frac{1}{1+v^2} \Rightarrow \dot{a} = \frac{-2v\dot{v}}{(1+v^2)^2} = -2va^2\dot{v}$

$b = \frac{1}{1+u^2} \Rightarrow \dot{b} = -2u\dot{u}b^2$

quasipolynomial form:

$$\dot{u} = u\left[1 - \alpha u + \alpha u a\right]$$
$$\dot{v} = v\left[\gamma - \beta v + \beta v b\right]$$
$$\dot{a} = a\left[-2av^2\gamma + 2av^3\beta - 2av^3 b\beta\right]$$
$$\dot{b} = b\left[-2bu^2 + 2\alpha bu^3 - 2\alpha abu^3\right]$$

$c = ua \Rightarrow \dot{c} = \dot{u}a + u\dot{a}$
$d = vb$
$e = v^2 a \Rightarrow \dot{e} = 2va\dot{v} + v^2\dot{a}$
$f = v^3 a$ $\quad \dot{f} = 3v^2 a\dot{v} + v^3\dot{a}$
$g = v^3 ab$ $\quad \dot{g} = 3v^2 ab\dot{v} + v^3\dot{a}b + v^3 a\dot{b}$
$h = u^2 b$
$k = u^3 b$
$\ell = u^3 ab$

gLV form:

$$\dot{u} = u\left[1 - \alpha u + \alpha c\right]$$
$$\dot{v} = v\left[\gamma - \beta v + \beta d\right]$$
$$\dot{c} = c\left[1 - \alpha u + \alpha c - 2\gamma e + 2\beta f - 2\beta g\right]$$
$$\dot{d} = d\left[\gamma - \beta v + \beta d - 2h + 2\alpha k - 2\alpha\ell\right]$$
$$\dot{e} = e\left[2\gamma - 2\beta v + 2\beta d - 2\gamma e + 2\beta f - 2\beta g\right]$$
$$\dot{f} = f\left[3\gamma - 3\beta v + 3\beta d - 2\gamma e + 2\beta f - 2\beta g\right]$$
$$\dot{g} = g\left[3\gamma - 3\beta v + 3\beta d - 2\gamma e + 2\beta f - 2\beta g - 2h + 2\alpha k - 2\alpha\ell\right]$$
$$\dot{h} = h\left[2 - 2\alpha u + 2\alpha c - 2h + 2\alpha k - 2\alpha\ell\right]$$
$$\dot{k} = k\left[3 - 3\alpha u + 3\alpha c - 2h + 2\alpha k - 2\alpha\ell\right]$$
$$\dot{\ell} = k\left[3 - 3\alpha u + 3\alpha c - 2h + 2\alpha k - 2\alpha\ell - 2\gamma e + 2\beta f - 2\beta g\right]$$

$$\vec{\mu} = \begin{bmatrix} 1 \\ \gamma \\ 1 \\ \gamma \\ 2\gamma \\ 3\gamma \\ 3\gamma \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

$$M = \begin{array}{c} \begin{array}{cccccccccc} u & v & c & d & e & f & g & h & k & \ell \end{array} \\ \begin{bmatrix} -\alpha & & \alpha & & & & & & & \\ & -\beta & & \beta & & & & & & \\ -\alpha & & \alpha & & -2\gamma & 2\beta & -2\beta & & & \\ & -\beta & & \beta & & & & -2 & 2\alpha & -2\alpha \\ & -2\beta & & 2\beta & -2\gamma & 2\beta & -2\beta & & & \\ & -3\beta & & 3\beta & -2\gamma & 2\beta & -2\beta & & & \\ & -3\beta & & 3\beta & -2\gamma & 2\beta & -2\beta & -2 & 2\alpha & -2\alpha \\ -2\alpha & & 2\alpha & & & & & -2 & 2\alpha & -2\alpha \\ -3\alpha & & 3\alpha & & & & & -2 & 2\alpha & -2\alpha \\ -3\alpha & & 3\alpha & & -2\gamma & 2\beta & -2\beta & -2 & 2\alpha & -2\alpha \end{bmatrix} \end{array} \begin{array}{c} u \\ v \\ c \\ d \\ e \\ f \\ g \\ h \\ k \\ \ell \end{array}$$

$\frac{1}{bu^2 v} = \#$

$\frac{1}{au v^2} = \#$

$\frac{av}{bu} = \# \Rightarrow av \sim bu$

$\frac{z^2}{1+z^2} = a = \frac{1}{1+v^2}$

$\frac{y^2}{1+y^2} = b = \frac{1}{1+u^2}$

ess $\rightsquigarrow$ $y(1+z^2) = \alpha$, $\gamma z(1+y^2) = \beta$

$$\dot{y} = \alpha - \frac{\alpha z^2}{1+z^2} - y$$
$$\dot{z} = \beta - \frac{\beta y^2}{1+y^2} - \gamma z$$
$$\dot{u} = u\left[1 - \alpha u + \alpha u a\right]$$
$$\dot{v} = v\left[\gamma - \beta v + \beta v b\right]$$
$$\dot{a} = a\left[-2av^2\gamma + 2av^3\beta - 2av^3 b\beta\right]$$
$$\dot{b} = b\left[-2bu^2 + 2\alpha bu^3 - 2\alpha abu^3\right]$$

$$\dot{u} = u\left[1 - \alpha u + \alpha c\right]$$
$$\dot{v} = v\left[\gamma - \beta v + \beta d\right]$$
$$\dot{c} = c\left[1 - \alpha u + \alpha c - 2\gamma e + 2\beta f - 2\beta g\right]$$
$$\dot{d} = d\left[\gamma - \beta v + \beta d - 2h + 2\alpha k - 2\alpha\ell\right]$$
$$\dot{e} = e\left[2\gamma - 2\beta v + 2\beta d - 2\gamma e + 2\beta f - 2\beta g\right]$$
$$\dot{f} = f\left[3\gamma - 3\beta v + 3\beta d - 2\gamma e + 2\beta f - 2\beta g\right]$$
$$\dot{g} = g\left[3\gamma - 3\beta v + 3\beta d - 2\gamma e + 2\beta f - 2\beta g - 2h + 2\alpha k - 2\alpha\ell\right]$$
$$\dot{h} = h\left[2 - 2\alpha u + 2\alpha c - 2h + 2\alpha k - 2\alpha\ell\right]$$
$$\dot{k} = k\left[3 - 3\alpha u + 3\alpha c - 2h + 2\alpha k - 2\alpha\ell\right]$$
$$\dot{\ell} = k\left[3 - 3\alpha u + 3\alpha c - 2h + 2\alpha k - 2\alpha\ell - 2\gamma e + 2\beta f - 2\beta g\right]$$

u, v: $c = ua$, $d = vb$

$e = v^2 a$, $f = v^3 a$, $g = v^3 ab$

$h = u^2 b$

$$\dot{x}_1 = x_1\left[u - \alpha u^2 + \alpha u c\right]$$

$$x_1 = e^{(u-u_1^*)} - 1$$

$$\begin{bmatrix} k = u^3 b \\ \ell = u^3 ab \end{bmatrix}$$

$$x_1 = (u-u_1^*)(v-v_1^*)$$

$$M = \begin{array}{c} \\ \left[\begin{array}{cccccccccc} u & v & c & d & e & f & g & h & k & \ell \\ -\alpha & & \alpha & & & & & & & \\ & -\beta & & \beta & & & & & & \\ -\alpha & & \alpha & & -2\gamma & 2\beta & -2\beta & & & \\ & -\beta & & \beta & & & & -2 & 2\alpha & -2\alpha \\ & -2\beta & & 2\beta & -2\gamma & 2\beta & -2\beta & & & \\ & -3\beta & & 3\beta & -2\gamma & 2\beta & -2\beta & & & \\ & -3\beta & & 3\beta & -2\gamma & 2\beta & -2\beta & -2 & 2\alpha & -2\alpha \\ -2\alpha & & 2\alpha & & & & & -2 & 2\alpha & -2\alpha \\ -3\alpha & & 3\alpha & & & & & -2 & 2\alpha & -2\alpha \\ -3\alpha & & 3\alpha & & -2\gamma & 2\beta & -2\beta & -2 & 2\alpha & -2\alpha \end{array}\right] \end{array} \begin{array}{c} \\ u \\ v \\ c \\ d \\ e \\ f \\ g \\ h \\ k \\ \ell \end{array}$$

Problem: $M x_a = M x_b$

$$\bar{u} = \frac{1}{\alpha} + \bar{c} \qquad \bar{v} = \frac{\gamma}{\beta} + d \qquad \bar{c} = \frac{-1}{\alpha} + u + \frac{2\gamma\bar{e}}{\alpha} - \frac{2\beta\bar{f}}{\alpha} + \frac{2\beta\bar{g}}{\alpha}$$

$$[M x_a]_u = -\alpha(x_a)_u + \alpha(x_a)_c = -\alpha\left[\underbrace{x_{au}}_{=\frac{1}{\alpha}} - x_{ac}\right] = -1$$

$$\overset{\text{WHY}}{=} -\alpha(x_{bu} - x_{bc})$$

Attempt 3:

Attempt 3: Change basis so that (u_1^*, v_1^*) and $(u_2^*, v_2^*) \to (0, v^*)$, $(u^*, 0)$

original form:

$$\dot{y} = \alpha - \frac{\alpha z^2}{1+z^2} - y$$

$$\dot{z} = \beta - \frac{\beta y^2}{1+y^2} - \gamma z$$

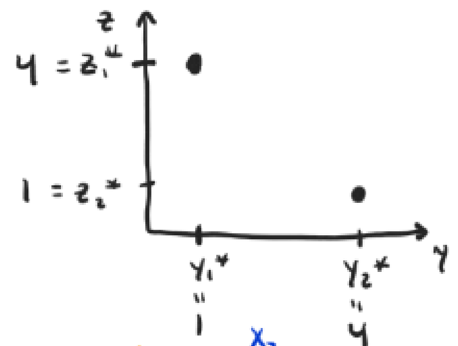
Let $u = y^{-1}$, $v = z^{-1}$ $\Rightarrow$ $\dot{u} = -y^{-2}\dot{y} = -u^2\alpha + \frac{\alpha u^2 v^{-2}}{1+v^{-2}} + u$

$$\dot{v} = -z^{-2}\dot{z} = -v^2\beta + \frac{\beta v^2 u^{-2}}{1+u^{-2}} + \gamma v$$

steady state reducible form:

$$\dot{u} = u - \alpha u^2 + \frac{\alpha u^2}{1+v^2} = u\left[1 - \alpha u + \frac{\alpha u}{1+v^2}\right]$$

$$\dot{v} = \gamma v - \beta v^2 + \frac{\beta v^2}{1+u^2} = v\left[\gamma - \beta v + \frac{\beta v}{1+u^2}\right]$$

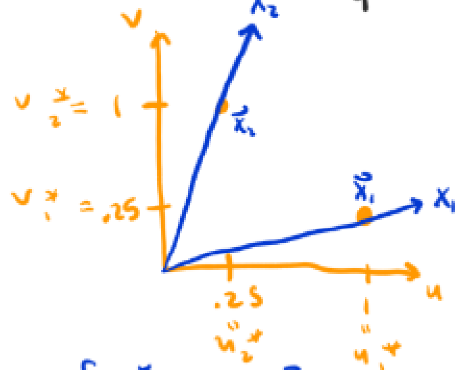


$$\hat{x}_1 = \frac{u_1^*\hat{u} + v_1^*\hat{v}}{\underbrace{\sqrt{u_1^{*2} + v_1^{*2}}}_{\equiv |x_1|}}, \qquad \hat{x}_2 = \frac{u_2^*\hat{u} + v_2^*\hat{v}}{\underbrace{\sqrt{u_2^{*2} + v_2^{*2}}}_{\equiv |x_2|}}$$

$$u\hat{u} + v\hat{v} = x_1\hat{x}_1 + x_2\hat{x}_2$$

$$\Rightarrow u = x_1\, \hat{x}_1 \cdot u + x_2\, \hat{x}_2 \cdot u = x_1\frac{u_1^*}{|x_1|} + x_2\frac{u_2^*}{|x_2|}$$

$$v = x_1\, \hat{x}_1 \cdot v + x_2\, x_2 \cdot v = x_1\frac{v_1^*}{|x_1|} + x_2\frac{v_2^*}{|x_2|}$$



$$\Rightarrow \frac{x_2}{|x_2|} = \frac{1}{u_2^*}\left[u - \frac{u_1^*}{|x_1|}x_1\right]$$

$$\frac{x_2}{|x_2|} = \frac{1}{v_2^*}\left[v - \frac{v_1^*}{|x_1|}x_1\right] \qquad \Rightarrow \frac{u}{u_2^*} - \frac{v}{v_2^*} = \frac{x_1}{|x_1|}\left[\frac{u_1^*}{u_2^*} - \frac{v_1^*}{v_2^*}\right]$$

$$\Rightarrow \frac{x_1}{|x_1|} = \frac{v_2^* u - u_2^* v}{u_1^* v_2^* - u_2^* v_1^*}$$

$$\Rightarrow \frac{x_2}{|x_2|} = \frac{1}{u_2^*}\left[u - u_1^*\left(\frac{v_2^* u - u_2^* v}{u_1^* v_2^* - v_1^* u_2^*}\right)\right]$$

$$= \frac{1}{u_2^*}\left[\frac{u_1^* u_2^* v - u_1^* \cancel{v_2^*} u + u_1^* \cancel{v_2^*} u - v_1^* u_2^* u}{u_1^* v_2^* - v_1^* u_2^*}\right]$$

$$\boxed{\Rightarrow \quad \begin{aligned} \frac{x_1}{|x_1|} &= \frac{v_2^* u - u_2^* v}{u_1^* v_2^* - v_1^* u_2^*} & \frac{\dot{x}_1}{|x_1|} &= \frac{v_2^* \dot{u} - u_2^* \dot{v}}{u_1^* v_2^* - v_1^* u_2^*} & u &= x_1 \frac{u_1^*}{|x_1|} + x_2 \frac{u_2^*}{|x_2|} \\ \frac{x_2}{|x_2|} &= \frac{-v_1^* u + u_1^* v}{u_1^* v_2^* - v_1^* u_2^*} & \frac{\dot{x}_2}{|x_2|} &= \frac{-v_1^* \dot{u} + u_1^* \dot{v}}{\underbrace{u_1^* v_2^* - v_1^* u_2^*}_{\equiv \phi}} & v &= x_1 \frac{v_1^*}{|x_1|} + x_2 \frac{v_2^*}{|x_2|} \end{aligned}}$$

$$\dot{u} = u - \alpha u^2 + \frac{\alpha u^2}{1+v^2} = u\left[1 - \alpha u + \frac{\alpha u}{1+v^2}\right]$$

$$\dot{v} = \gamma v - \beta v^2 + \frac{\beta v^2}{1+u^2} = v\left[\gamma - \beta v + \frac{\beta v}{1+u^2}\right]$$

$$\Rightarrow \frac{\dot{x}_1}{x_1} = \frac{1}{\phi}\left[v_2^*\left[\left(\frac{u_1^*}{|x_1|} x_1 + \frac{u_2^*}{|x_2|} x_2\right)\left(1 - \alpha\left(\frac{u_1^*}{|x_1|} x_1 + \frac{u_2^*}{|x_2|} x_2\right) + \frac{\alpha\left(\frac{u_1^*}{|x_1|} x_1 + \frac{u_2^*}{|x_2|} x_2\right)}{1 + \left(\frac{v_1^*}{|x_1|} x_1 + \frac{v_2^*}{|x_2|} x_2\right)^2}\right)\right.\right.$$

$$\left.\left. - u_2^*\left[\left(v_1^* x_1 + v_2^* x_2\right)\left(\gamma - \beta\left(v_1^* x_1 + v_2^* x_2\right) + \frac{\beta\left(v_1^* x_1 + v_2^* x_2\right)}{1 + \left(u_1^* x_1 + u_2^* x_2\right)^2}\right)\right]\right]$$

Set $x_1 = |x_1| x_1$, $x_2 = |x_2| x_2$

$$\phi \dot{x}_1 = x_1\left[u_1^* v_2^* - u_2^* v_1^* \gamma\right] + x_2(1-\gamma)(u_2^* v_2^*)$$
$$+ x_1^2\left[-\alpha u_1^{*2} v_2^* + \beta u_2^* v_1^{*2}\right] + x_2^2\left[-\alpha u_2^{*2} v_2^* + \beta u_2^* v_2^{*2}\right]$$
$$+ x_1 x_2 (2 u_2 v_2)(-\alpha u_1 + \beta v_1)$$
$$+ \frac{\alpha v_2 (u_1 x_1 + u_2 x_2)^2}{1 + (v_1 x_1 + v_2 x_2)^2} - \frac{\beta u_2 (v_1 x_1 + v_2 x_2)^2}{1 + (u_1 x_1 + u_2 x_2)^2}$$

Flip to quasipolynomial form

Check: @ $(x_1, x_2) = (1, 0)$, is $\dot{x}_1 = 0$?

$$\Rightarrow u_1 v_2 - u_2 v_1 \gamma - \alpha u_1^2 v_2 + \beta u_2 v_1^2 + \frac{\alpha u_1^2 v_2}{1 + v_1^2} - \frac{\beta u_2 v_1^2}{1 + u_1^2}$$
$$= u_1 v_2\left(1 - \alpha u_1 + \frac{\alpha u_1}{1 + v_1^2}\right) + u_2 v_1\left(-\gamma + \beta v_1 - \frac{\beta v_1}{1 + u_1^2}\right) = 0$$

$$\phi \dot{x}_1 = x_1\left[\overbrace{u_1 v_2 - u_2 v_1}^{J_0} + x_1\overbrace{\left(-\alpha u_1^2 v_2 + \beta u_2 v_1^2\right)}^{J_1} + x_2\overbrace{(2 u_2 v_2)(-\alpha u_1 + \beta v_1)}^{J_2}\right.$$
$$+ \frac{1}{1 + (v_1 x_1 + v_2 x_2)^2}\left[x_1\overbrace{(\alpha u_1^2 v_2)}^{J_3} + x_2\overbrace{(2\alpha u_1 u_2 v_2)}^{J_4}\right]$$
$$\left.+ \frac{1}{1 + (u_1 x_1 + u_2 x_2)^2}\left[x_1\overbrace{(-\beta u_2 v_1^2)}^{J_5} + x_2\overbrace{(-2\beta u_2 v_1 v_2)}^{J_6}\right]\right]$$
$$+ x_2\left[\overbrace{(1-\gamma)(u_2 v_2)}^{J_7} + x_2\overbrace{(u_2 v_2)(-\alpha u_2 + \beta v_2)}^{J_8}\right.$$
$$\left.+ \frac{x_2}{1 + (v_1 x_1 + v_2 x_2)^2}\overbrace{(\alpha u_2^2 v_2)}^{J_9} + \frac{x_2}{1 + (u_1 x_1 + v_2 x_2)^2}\overbrace{(-\beta u_2 v_2^2)}^{J_{10}}\right]$$

Call $b = \frac{1}{1 + (v_1 x_1 + v_2 x_2)^2}$, $a = \frac{1}{1 + (u_1 x_1 + u_2 x_2)}$

$$\Rightarrow \phi \dot{x}_1 = x_1\left[J_0 + J_1 x_1 + J_2 x_2 + J_3 x_1 b + J_4 x_2 b + J_5 x_1 a + J_6 x_2 a\right]$$

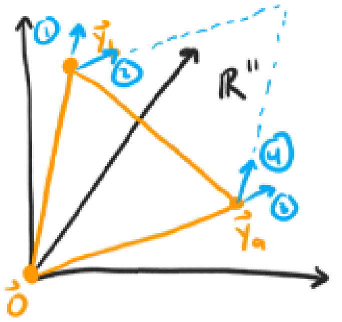
$$+x_2\left[J_7 + J_8 x_2 + J_9 x_2 b + J_{10} x_2 a\right]$$

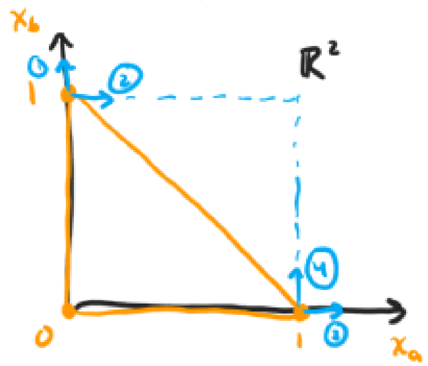
An alternative (geometric) approach

Until this point, we have been searching for new "reduced" variables that encode high-dimensional information in a 2D gLV model. SSR, as applied to a generic high-dimensional gLV system, has been shown to capture this spirit of dimensionality reduction via a particular change of variables. This change of basis was solely a function of the system's behavior at two steady states.

Can we invert this process? That is, based on behavior at the high-dimensional steady states, can we infer the proper change of variables that will preserve high-dimensional properties in the low dimensional system? The 2D gLV system has just 3 unconstrained degrees of freedom — can we determine what these should be? Can we apply a similar procedure to reduce non-gLV systems?

To approach this problem, let's begin with the previously studied 11D → 2D SSR. $\vec{y}_a, \vec{y}_b \in \mathbb{R}^{11}$ are f.p.s. $(1,0)$ and $(0,1) \in \mathbb{R}^2$ are f.p.s.





Can we draw a correspondence between the model's behavior along the infintesimal vectors ① ② ③ ④ in both models? Can pursuit of a correspondence lead to the same change of variables as in SSR?

ND: $\dot{y}_i = y_i\left[\rho_i + \sum_{j=1}^{N} K_{ij} y_j\right] = f(y_i)$ 2D: $\dot{x}_i = x_i\left[\mu_i + \sum_{j=1}^{2} M_{ij} x_j\right] = g(x_i)$

$\frac{d}{dt}(\vec{y}) = \vec{f}(\vec{y})$ $\frac{d}{dt}(\vec{x}) = \vec{g}(\vec{x})$

$\vec{y}_a$ and $\vec{y}_b$ are fixed points $\Rightarrow \vec{f}(\vec{y}_a) = \vec{f}(\vec{y}_b) = \vec{0}$.

$$\frac{d}{dt}\left(\vec{y}_a(1+\varepsilon)\right) = \vec{f}(\vec{y}_a(1+\varepsilon)) \approx \vec{f}(\vec{y}_a) + J\left[\vec{f}(\vec{y}_a)\right](\varepsilon\vec{y}_a)$$

$$= \begin{bmatrix} \rho_1 + K_{11} y_{a1} + \sum_j K_{1j} y_j & K_{12} y_1 & K_{13} y_1 & \cdots \end{bmatrix} \begin{bmatrix} \varepsilon y_{a1} \end{bmatrix}$$

$$= \begin{bmatrix} K_{11} y_{a1} & K_{12} y_{a1} & \cdots \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \varepsilon y_{a1} \\ \varepsilon y_{a2} \\ \vdots \end{bmatrix} + \begin{bmatrix} \rho_1 + \cancel{\sum_{j=1}^{N} K_{1j} y_{aj}}^{\,0} & 0 & 0 & \cdots \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \varepsilon y_{a1} \\ \varepsilon y_{a2} \\ \vdots \end{bmatrix}$$

$$= \varepsilon \begin{bmatrix} y_{a1} \sum_{i=1}^{N} K_{1i} y_{ai} \\ y_{a2} \sum_{i=1}^{N} K_{2i} y_{ai} \\ \vdots \end{bmatrix}.$$

We want to make a "composite" or "homogenized" x out of components of y, so that

$$x_a = \sum_i y_{ai} \qquad \vec{x}_a = (x_{a1}, 0)$$

$$\Rightarrow \frac{d}{dt}\left(\vec{x}_a (1+\varepsilon)\right)_a = \sum_i \frac{d}{dt}\left(\vec{y}_a (1+\varepsilon)\right)_i \qquad \vec{x}_b = (0, x_{b2})$$

$$\frac{d}{dt}\left(\vec{x}_a (1+\varepsilon)\right) = \begin{bmatrix} M_{11} x_1 & M_{12} x_1 \\ M_{21} x_2 & M_{22} x_2 \end{bmatrix}\Bigg|_{\vec{x} = x_a} \begin{pmatrix} \varepsilon x_{a1} \\ \varepsilon \cancel{x_2}^{\,0} \end{pmatrix} + \begin{bmatrix} \mu_1 + M_{11} x_1 + M_{11} x_2 & \\ 0 & \mu_2 + M_{21} x_1 + M_{22} x_2 \end{bmatrix} \begin{pmatrix} \varepsilon x_{a1} \\ \varepsilon \cancel{x_{a2}}^{\,0} \end{pmatrix}$$

$$= \varepsilon M_{11} x_{a1}^2$$

$$\Rightarrow M_{11} x_{a1}^2 = y_{a1} \sum_{i=1}^{N} K_{1i} y_{ai} + y_{a2} \sum_i K_{2i} y_{ai} + \cdots$$

$$\Rightarrow \boxed{M_{11} x_{a1}^2 = \vec{y}_a^{\,T} K \vec{y}_a}$$

Hence, by assuming x_a was the aggregate (sum) of y_a components, and by enforcing that the two systems behave identically to perturbations, we have solved for M_{11}. (I expect x_{a1} will be a free scaling parameter)

Next, lets investigate vector ②, and try to enforce once more that the system's behavior is consistent.

$$\frac{d}{dt}\left(\vec{y}_a + \varepsilon \vec{y}_b\right) = J\Big|_{\vec{y}_a} (\varepsilon \vec{y}_b)$$

$$= \begin{bmatrix} K_{11} y_{a1} & K_{12} y_{a1} & \cdots \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \varepsilon y_{b1} \\ \varepsilon y_{b2} \\ \vdots \end{bmatrix} + \begin{bmatrix} \rho_1 + \sum_i K_{1i} y_{ai} & 0 & 0 \\ 0 & \rho_2 + \sum_i K_{2i} y_{ai} & 0 \\ & & \end{bmatrix} \begin{bmatrix} \varepsilon y_{b1} \\ \varepsilon y_{b1} \\ \vdots \end{bmatrix}$$

$$= \varepsilon \begin{bmatrix} y_{a1} \sum_i K_{1i} y_{bi} \\ \vdots \\ \end{bmatrix}$$

$$\frac{d}{dt}\left(\vec{x}_a + \varepsilon \vec{x}_b\right) = \begin{bmatrix} \mu_1 + M_{11} x_1 + M_{11} x_1 + M_{12} x_2 & M_{12} x_1 \\ M_{21} x_2 & \mu_2 + M_{22} x_2 + M_{21} x_1 + M_{22} x_2 \end{bmatrix} \begin{bmatrix} \varepsilon \cancel{x_{b1}}^{\,0} \\ \varepsilon x_{b2} \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon M_{12} x_{a1} x_{b2} \\ \varepsilon x_{b2} (\mu_2 + M_{21} x_{a1}) \end{bmatrix}$$

As before, we want $\frac{d}{dt}(\vec{x}_a + \varepsilon \vec{x}_b)_a = \sum_i \frac{d}{dt}(\vec{y}_a + \varepsilon \vec{y}_b)_i$, and so

$$M_{12} x_{a1} x_{b2} = y_{a1} \sum_i K_{1i} y_{bi} + y_{a2} \sum_i K_{2i} y_{bi} + \cdots$$

$$\Rightarrow \boxed{M_{12} x_{a1} x_{b2} = \vec{y}_a^T K \vec{y}_b}$$

Similarly, interchanging a and b, we find

$$\boxed{M_{22} x_b^2 = \vec{y}_b^T K \vec{y}_b \quad \text{and} \quad M_{21} x_a x_b = \vec{y}_a^T K \vec{y}_b}$$

Then, the general trick is, for steady states a and b, force

$$\underbrace{\frac{d}{dt}(\vec{x}_\alpha + \varepsilon \vec{x}_\beta)_\alpha}_{\substack{\text{function of } M, \\ \text{s.s. condition gets } \mu \\ \text{(previously unknown)}}} = \underbrace{\sum_{i=1}^{N} \frac{d}{dt}(\vec{y}_\alpha + \varepsilon \vec{y}_\beta)_i}_{\substack{\text{function of arbitrary complex system} \\ \text{(known)}}}, \quad \text{for } \alpha, \beta \in \begin{matrix} (a,a), (a,b), \\ (b,a), (b,b) \end{matrix}$$

Note that this appears to be valid in general, for any system with steady states.