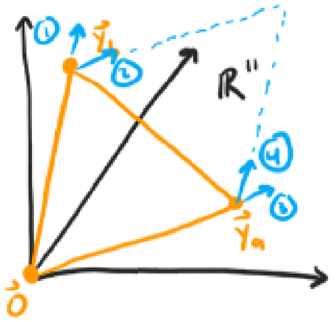
$+x_2[J_7 + J_8 x_2 + J_9 x_2 b + J_{10} x_2 a]$

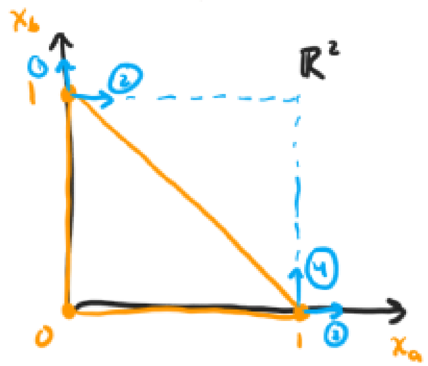
An alternative (geometric) approach

Until this point, we have been searching for new "reduced" variables that encode high-dimensional information in a 2D gLV model. SSR, as applied to a generic high-dimensional gLV system, has been shown to capture this spirit of dimensionality reduction via a particular change of variables. This change of basis was solely a function of the system's behavior at two steady states.

Can we invert this process? That is, based on behavior at the high-dimensional steady states, can we infer the proper change of variables that will preserve high-dimensional properties in the low dimensional system? The 2D gLV system has just 3 unconstrained degrees of freedom — can we determine what these should be? Can we apply a similar procedure to reduce non-gLV systems?

To approach this problem, let's begin with the previously studied 11D → 2D SSR. $\vec{y}_a, \vec{y}_b \in \mathbb{R}^{11}$ are f.p.s. $(1,0)$ and $(0,1) \in \mathbb{R}^2$ are f.p.s.





Can we draw a correspondence between the model's behavior along the infintesimal vectors 1 2 3 4 in both models? Can pursuit of a correspance lead to the same change of variables as in SSR?

ND: $\dot{y}_i = y_i\left[\rho_i + \sum_{j=1}^{N} K_{ij} y_j\right] = f(y_i)$ 2D: $\dot{x}_i = x_i\left[\mu_i + \sum_{j=1}^{2} M_{ij} x_j\right] = g(x_i)$

$\frac{d}{dt}(\vec{y}) = \vec{f}(\vec{y})$ $\frac{d}{dt}(\vec{x}) = \vec{g}(\vec{x})$

$\vec{y}_a$ and $\vec{y}_b$ are fixed points $\Rightarrow \vec{f}(\vec{y}_a) = \vec{f}(\vec{y}_b) = \vec{0}$.

$$\frac{d}{dt}\left(\vec{y}_a(1+\varepsilon)\right) = \vec{f}\left(\vec{y}_a(1+\varepsilon)\right) \approx \vec{f}(\vec{y}_a) + J\left[\vec{f}(\vec{y}_a)\right](\varepsilon\vec{y}_a)$$

$$= \begin{bmatrix} \rho_1 + K_{11}y_{a1} + \sum_j K_{1j} y_j & K_{12}y_{a1} & K_{13}y_{a1} & \cdots \end{bmatrix}\begin{bmatrix} \varepsilon y_{a1} \end{bmatrix}$$

$$\begin{bmatrix} \\ \end{bmatrix} \begin{bmatrix} \varepsilon y_{a2} \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} K_{11} y_{a1} & K_{12} y_{a1} & \cdots \\ & & \end{bmatrix} \begin{bmatrix} \varepsilon y_{a1} \\ \varepsilon y_{a2} \\ \vdots \end{bmatrix} + \begin{bmatrix} \cancel{\rho_1 + \sum_{j=1}^{N} K_{1j} y_{aj}}^{\,0} & 0 & 0 & \cdots \\ & & & \end{bmatrix} \begin{bmatrix} \varepsilon y_{a1} \\ \varepsilon y_{a2} \\ \vdots \end{bmatrix}$$

$$= \varepsilon \begin{bmatrix} y_{a1} \sum_{i=1}^{N} K_{1i} y_{ai} \\ y_{a2} \sum_{i=1}^{N} K_{2i} y_{ai} \\ \vdots \end{bmatrix}.$$

We want to make a "composite" or "homogenized" x out of components of y, so that

$$x_a = \sum_i y_{ai} \qquad \vec{x}_a = (x_{a1}, 0)$$

$$\Rightarrow \frac{d}{dt}\Big(\vec{x}_a (1+\varepsilon)\Big)_a = \sum_i \frac{d}{dt}\Big(\vec{y}_a (1+\varepsilon)\Big)_i \qquad \vec{x}_b = (0, x_{b2})$$

$$\frac{d}{dt}\big(\vec{x}_a (1+\varepsilon)\big) = \begin{bmatrix} M_{11} x_1 & M_{12} x_1 \\ M_{21} x_2 & M_{22} x_2 \end{bmatrix}\Bigg|_{\vec{x} = x_a} \begin{pmatrix} \varepsilon x_{a1} \\ \varepsilon \cancel{x_2}^{\,0} \end{pmatrix} + \begin{bmatrix} \mu_1 + M_{11} x_1 + M_{11} x_2 & \\ 0 & \mu_2 + M_{21} x_1 + M_{22} x_2 \end{bmatrix} \begin{bmatrix} \varepsilon x_{a1} \\ \varepsilon \cancel{x_{a2}}^{\,0} \end{bmatrix}$$

$$= \varepsilon M_{11} x_{a1}^2$$

$$\Rightarrow M_{11} x_{a1}^2 = y_{a1} \sum_{i=1}^{N} K_{1i} y_{ai} + y_{a2} \sum_i K_{2i} y_{ai} + \cdots$$

$$\Rightarrow \boxed{M_{11} x_{a1}^2 = \vec{y}_a^{\,T} K \vec{y}_a}$$

Hence, by assuming x_a was the aggregate (sum) of y_a components, and by enforcing that the two systems behave identically to perturbations, we have solved for M_{11}. (I expect x_{a1} will be a free scaling parameter)

Next, let's investigate vector ②, and try to enforce once more that the system's behavior is consistent.

$$\frac{d}{dt}\left(\vec{y}_a + \varepsilon \vec{y}_b\right) = J\Big|_{\vec{y}_a} (\varepsilon \vec{y}_b)$$

$$= \begin{bmatrix} K_{11} y_{a1} & K_{12} y_{a1} & \cdots \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \varepsilon y_{b1} \\ \varepsilon y_{b2} \\ \vdots \end{bmatrix} + \begin{bmatrix} \rho_1 + \sum_i K_{1i} y_{ai} & 0 & 0 \\ 0 & \rho_2 + \sum_i K_{2i} y_{ai} & 0 \\ & & \end{bmatrix} \begin{bmatrix} \varepsilon y_{b1} \\ \varepsilon y_{b1} \\ \vdots \end{bmatrix}$$

$$= \varepsilon \begin{bmatrix} y_{a1} \sum_i K_{1i} y_{bi} \\ \vdots \end{bmatrix}$$

$$\frac{d}{dt}\left(\vec{x}_a + \varepsilon \vec{x}_b\right) = \begin{bmatrix} \mu_1 + M_{11} x_1 + M_{11} x_1 + M_{12} x_2 & M_{12} x_1 \\ M_{21} x_2 & \mu_2 + M_{22} x_2 + M_{21} x_1 + M_{22} x_2 \end{bmatrix} \begin{bmatrix} \varepsilon \cancel{x_{b1}}^{\,0} \\ \varepsilon x_{b2} \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon M_{12} x_{a1} x_{b2} \\ \varepsilon x_{b2} (\mu_2 + M_{21} x_{a1}) \end{bmatrix}$$

As before, we want $\frac{d}{dt}(\vec{x}_a + \varepsilon \vec{x}_b)_a = \sum_i \frac{d}{dt}(\vec{y}_a + \varepsilon \vec{y}_b)_i$, and so

$$M_{12} x_{a1} x_{b2} = y_{a1} \sum_i K_{1i} y_{bi} + y_{a2} \sum_i K_{2i} y_{bi} + \cdots$$

$$\Rightarrow \boxed{M_{12} x_{a1} x_{b2} = \vec{y}_a^T K \vec{y}_b}$$

Similarly, interchanging a and b, we find

$$\boxed{M_{22} x_b^2 = \vec{y}_b^T K \vec{y}_b \quad \text{and} \quad M_{21} x_a x_b = \vec{y}_a^T K \vec{y}_b}$$

Then, the general trick is, for steady states a and b, force

$$\underbrace{\frac{d}{dt}(\vec{x}_\alpha + \varepsilon \vec{x}_\beta)_\alpha}_{\substack{\text{function of } M, \\ \text{s.s. condition gets } \mu \\ \text{(previously unknown)}}} = \underbrace{\sum_{i=1}^{N} \frac{d}{dt}(\vec{y}_\alpha + \varepsilon \vec{y}_\beta)_i}_{\substack{\text{function of arbitrary complex system} \\ \text{(known)}}}, \quad \text{for } \alpha, \beta \in \begin{matrix} (a,a), (a,b), \\ (b,a), (b,b) \end{matrix}$$

Note that this appears to be valid in general, for any system with steady states.