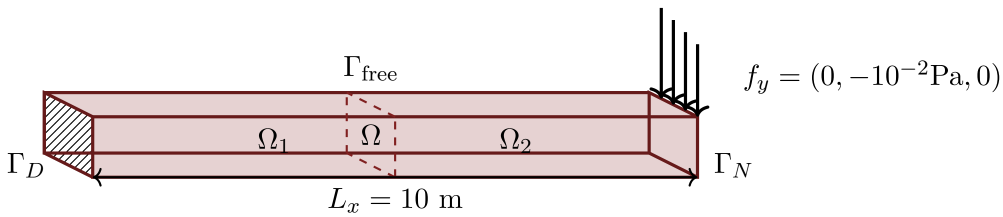
Weak forms and problem setup — Heterogeneous linear elasticity in a two-material 3D block

1 Governing Equations

1.1 Geometry and Boundary



Let $\Omega \subset \mathbb{R}^3$ be the rectangular block:

$$\Omega := (0, L_x) \times (0, L_y) \times (0, L_z), \qquad L_x := 10\text{m},\ L_y := L_z := 1\text{m}.$$

The domain Ω is partitioned at $x_{\text{int}} := L_x/2$ as:

$$\Omega_1 := \{x < x_{\text{int}}\}, \qquad \Omega_2 := \{x \geq x_{\text{int}}\}.$$

Where, $\Omega = \Omega_1 \cup \Omega_2$.
The boundary decomposition is:

$$\Gamma_D := \{x = 0\}, \quad \Gamma_N := \{x = L_x\}, \quad \Gamma_{\text{free}} := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N),$$

Where, $\partial\Omega$ is the boundary of the domain Ω and define the closure of Ω as follows: $\overline{\Omega} = \Omega \cup \partial\Omega$.

1.2 Material Parameterization

Simulation Parameters

Given Young's modulus $E \in (2, 10)$ and Poisson ratio $\nu \in (0.2, 0.4)$,

$$\lambda_0 := \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu_0 := \frac{E}{2(1+\nu)}.$$

Where λ_0 and μ_0 are the Lamé-parameters. Then we define the Lamé-parameters for our two different materials in the rod, at Ω_1 and Ω_2 respectively, as follows:

$$(\lambda, \mu)(x; E, \nu) := \begin{cases} (50\lambda_0,\ 50\mu_0), & x \in \Omega_1, \\ (\lambda_0,\ \mu_0), & x \in \Omega_2. \end{cases}$$

Units: $[E] = \text{Pa},\ [\nu] = 1,\ [\lambda] = \text{Pa},\ [\mu] = \text{Pa}.$
Where, the simulation parameters are: E and ν.

1.3 Governing Laws

Let $\mathbf{u} : \Omega \to \mathbb{R}^3$ be the displacement field ($[\mathbf{u}] = \mathrm{m}$).
The displacement gradient and the small strain are:

$$\nabla \mathbf{u} \in \mathbb{R}^{3\times 3}, \qquad [\nabla \mathbf{u}] = 1, \qquad \varepsilon(\mathbf{u}) := \tfrac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^\top\right), \qquad [\varepsilon] = 1.$$

For an isotropic linear elastic solid, the Cauchy stress is:

$$\sigma(\mathbf{u}) = 2\mu\, \varepsilon(\mathbf{u}) + \lambda \operatorname{tr}\big(\varepsilon(\mathbf{u})\big)\, \mathsf{I}, \qquad [\sigma] = \mathrm{Pa}, \quad [\lambda] = [\mu] = \mathrm{Pa},$$

with Lamé-parameters (λ, μ) and I as the identity tensor.
The traction on a boundary with unit normal $\mathbf{n}$ is:

$$-\mathbf{f}_y := \sigma(\mathbf{u})\, \mathbf{n}, \qquad [\mathbf{f}_y] = \mathrm{Pa}, \qquad [\mathbf{n}] = 1.$$

Static equilibrium (no body forces) requires:

$$-\nabla\cdot \sigma(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega, \qquad [\nabla \cdot \sigma] = \mathrm{N}/\mathrm{m}^3.$$

1.4 Governing PDE

Let the space of admissible strong solutions be given by:

$$\mathcal{V}_{\text{strong}} := C^0(\overline{\Omega}) \cap C^2(\Omega).$$

Find $\mathbf{u} \in \mathcal{V}_{\text{strong}}$ such that

$$\begin{aligned}
-\nabla \cdot \sigma(\mathbf{u}) &= \mathbf{0} && \text{in } \Omega, && (1)\\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D, && (2)\\
\sigma(\mathbf{u})\, \mathbf{n} = -\mathbf{f}_y &= (0, -10^{-2}, 0) && \text{on } \Gamma_N, && (3)\\
\sigma(\mathbf{u})\, \mathbf{n} &= \mathbf{0} && \text{on } \Gamma_{\text{free}}. && (4)
\end{aligned}$$

Here,

$$\sigma(\mathbf{u}) := 2\mu\, \varepsilon(\mathbf{u}) + \lambda \operatorname{tr}(\varepsilon(\mathbf{u}))\, \mathsf{I}, \qquad \varepsilon(\mathbf{u}) := \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top),$$

where I is the identity tensor.

2 Weak Form

2.1 Variational Statement

Let V be the test function space:

$$V := \{\mathbf{v} \in H^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \Gamma_D\}.$$

Given (E, ν), find $\mathbf{u} \in V$ such that

$$a(\mathbf{u}, \mathbf{v}; E, \nu) = \ell(\mathbf{v}) \qquad \forall \mathbf{v} \in V,$$

$$a(\mathbf{u}, \mathbf{v}; E, \nu) := \int_\Omega \left[2\mu\, \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + \lambda \operatorname{tr}(\varepsilon(\mathbf{u})) \operatorname{tr}(\varepsilon(\mathbf{v}))\right] dx, \tag{5}$$

$$\ell(\mathbf{v}) := \int_{\Gamma_N} -\mathbf{f}_y \cdot \mathbf{v}\, ds. \tag{6}$$

References

[1] Choi, Youngsoo, Arrighi, William J., Copeland, Dylan M., Anderson, Robert W., & Oxberry, Geoffrey M. (2019, October 17). *libROM*. [Computer software]. https://github.com/LLNL/libROM. https://doi.org/10.11578/dc.20190408.3; Global pROM for linear elasticity https://www.librom.net/examples.html Accessed: 28 July, 2025