Weak forms and problem setup — Twisting of a Neo-Hookean Block

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1 Governing Equations

1.1 Geometry and Boundary

Given a rectangular block:

$$\Omega := (0, L_x) \times (0, L_y) \times (0, L_z) \subset \mathbb{R}^3, \qquad L_x := 10,\ L_y := L_z := 1.$$

Faces:

$$\Gamma_L := \{x = 0\}, \quad \Gamma_R := \{x = L_x\}, \quad \Gamma_N := \partial\Omega \setminus (\Gamma_L \cup \Gamma_R).$$

1.2 Material Model (Compressible Neo-Hookean)

For Lamé parameters $(\mu, \lambda) > 0$:

$$F := I + \nabla u, \qquad J := \det(F), \qquad F^{-T} := (F^{-1})^\top,$$

$$\mathbf{P}(F) := \mu F + (\lambda \ln J - \mu) F^{-T}.$$

Units: $[\mu] = [\lambda] = \mathrm{Pa},\ [u] = \mathrm{m}$.

1.3 Governing PDE

Let,

$$\mathcal{V}_{\text{strong}} := C^0(\overline{\Omega}; \mathbb{R}^3) \cap C^2(\Omega; \mathbb{R}^3).$$

Find $P \in \mathcal{V}_{\text{strong}}$ such that:

$$-\nabla \cdot \mathbf{P}(u) = 0 \text{ in } \Omega, \qquad u|_{\Gamma_L} = 0, \qquad u|_{\Gamma_R} = g, \qquad \mathbf{P}(u)\, n|_{\Gamma_N} = 0.$$

$$g(x, y, z) := \big(-0.1,\ y(\cos\pi - 1) - z\sin\pi,\ y\sin\pi + z(\cos\pi - 1)\big)^\top.$$

2 Weak Form

2.1 Residual Functional

Let,

$$V := \big\{v \in H^1(\Omega; \mathbb{R}^3) \mid v|_{\Gamma_L} = 0,\ v|_{\Gamma_R} = 0\big\}.$$

Find $u \in V$ such that:

$$R(u)[v] = 0 \qquad \forall v \in V,$$

with nonlinear residual functional:

$$F := I + \nabla u, \qquad J := \det F, \qquad F^{-T} := (F^{-1})^{\top}.$$

$$R(u;\mu,\lambda)[v] = \int_{\Omega} \Big[\mu \ \mathrm{tr}\big(F^{\top}\nabla v\big) + (\lambda \ln J - \mu)\ \mathrm{tr}\big(F^{-T}\nabla v\big)\Big] dx.$$

2.2 Jacobian (Fréchet Derivative)

$$F := I + \nabla u, \qquad J := \det F,$$

$$J(u)[\delta u, v] := \frac{\partial}{\partial \varepsilon} R(u + \varepsilon \delta u)[v]\Big|_{\varepsilon=0}.$$

$$\delta F := \nabla \delta u,$$

$$\delta J = J\ \mathrm{tr}\big(F^{-1}\delta F\big) = J\ \mathrm{tr}\big(F^{-1}\nabla \delta u\big),$$

$$\delta \ln J = \frac{\delta J}{J} = \mathrm{tr}\big(F^{-1}\nabla \delta u\big),$$

$$\delta F^{-T} = -F^{-T}\delta F\, F^{-T} = -F^{-T}(\nabla \delta u)F^{-T}.$$

$$\begin{aligned}
\delta I &= \mu\, \delta\Big[\mathrm{tr}\big(F^{\top}\nabla v\big)\Big] + (\lambda \ln J - \mu)\, \delta\Big[\mathrm{tr}\big(F^{-T}\nabla v\big)\Big] + \lambda\, \delta(\ln J)\ \mathrm{tr}\big(F^{-T}\nabla v\big) \\
&= \mu\ \mathrm{tr}\big[(\nabla \delta u)^{\top}\nabla v\big] \\
&\quad - (\lambda \ln J - \mu)\ \mathrm{tr}\big[F^{-T}(\nabla \delta u)F^{-T}\nabla v\big] \\
&\quad + \lambda\ \mathrm{tr}\big(F^{-1}\nabla \delta u\big)\ \mathrm{tr}\big(F^{-T}\nabla v\big).
\end{aligned}$$

$$\mathrm{tr}\big[F^{-T}(\nabla \delta u)F^{-T}\nabla v\big] = \mathrm{tr}\big[(\nabla \delta u\, F^{-1})^{\top}(\nabla v\, F^{-1})\big].$$

$$J(u)[\delta u, v] = \int_{\Omega} \Big[\mu\ \mathrm{tr}\big((\nabla \delta u)^{\top}\nabla v\big) \ - \ (\lambda \ln J - \mu)\ \mathrm{tr}\big((\nabla \delta u\, F^{-1})^{\top}(\nabla v\, F^{-1})\big) \ + \ \lambda\ \mathrm{tr}\big(\nabla \delta u\, F^{-1}\big)\ \mathrm{tr}\big(\nabla v\, F^{-1}\big)\Big]\, \mathrm{d}x$$