

# Weak forms and problem setup — Nonlinear heat conduction in a two-material 1-D rod

## 1 Governing Equations

### 1.1 Geometry and Domains

We consider steady (time-independent) 1-D conduction through an inhomogeneous rod of length

$$L = 0.5 \text{ m}, \quad \Omega = (0, L).$$

The rod is partitioned at  $x_{\text{int}} = 0.4 \text{ m}$  into the sub-domains

$$\Omega_1 = (0, x_{\text{int}}), \quad \Omega_2 = [x_{\text{int}}, L).$$

### 1.2 Material and Heat Source Parameterization

Both the thermal conductivity  $k$  and the volumetric heat source  $q$  depend *non-linearly* on  $T$ :

$$k(x, T; \mu) = \begin{cases} k_1(T; \mu) := 16 + \mu + \frac{2150}{T - 73.15}, & x \in \Omega_1, \\ k_2(T; \mu) := 30 + \mu + 2.09 \times 10^{-2} T - 1.45 \times 10^{-5} T^2 + 7.67 \times 10^{-9} T^3, & x \in \Omega_2, \end{cases} \quad (1)$$

$$q(x, T; \beta) = \begin{cases} q_1(T; \beta) := 35\,000 + \beta + \frac{T}{10}, & x \in \Omega_1, \\ q_2(\beta) := 5\,000 + 10\beta, & x \in \Omega_2. \end{cases} \quad (2)$$

Units:  $[k] = \text{W}/(\text{m K})$ ,  $[q] = \text{W}/\text{m}^3$ .

### 1.3 Governing PDE

Define the classical solution space

$$V = \left\{ T \in C([0, L]) \cap C^1([0, L] \setminus \{x_{\text{int}}\}) \mid T|_{\Omega_i} \in C^2(\Omega_i), \ i = 1, 2 \right\}.$$

Find  $T \in V$  such that

$$-\frac{d}{dx} \left( k(x, T(x); \mu) T'(x) \right) = q(x, T(x); \beta) \quad \text{in } \Omega, \quad (3)$$

$$T'(0) = 0 \quad (\text{Neumann}), \quad (4)$$

$$T(L) = T_D = 573.15 \text{ K} \quad (\text{Dirichlet}). \quad (5)$$

(The dependence of  $k$  and  $q$  on  $T$  makes the problem nonlinear.)

## 2 Weak Form

### 2.1 Derivation

Let  $v$  be any test function with  $v(L) = 0$ . Multiplying (3) by  $v$ , integrating over each sub-domain, and applying integration by parts gives

$$\int_0^L k(x, T; \mu) T' v' dx = \int_0^L q(x, T; \beta) v dx.$$

### 2.2 Function Spaces

Introduce the Sobolev space

$$H^1(\Omega) = \{u \in L^2(\Omega) \mid u' \in L^2(\Omega)\}, \quad \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2.$$

Set

$$V := \{w \in H^1(\Omega) \mid w(L) = T_D\}, \quad V_0 := \{v \in H^1(\Omega) \mid v(L) = 0\}.$$

### 2.3 Abstract Weak Problem

Given parameters  $\boldsymbol{\mu} = (\mu, \beta)$ , find  $T \in V$  such that

$$a(T, v; \boldsymbol{\mu}) = \ell(T, v; \boldsymbol{\mu}) \quad \forall v \in V_0, \quad (6)$$

with nonlinear forms

$$a(T, v; \boldsymbol{\mu}) := \int_{\Omega} k(x, T; \mu) T' v' dx, \quad (7)$$

$$\ell(T, v; \boldsymbol{\mu}) := \int_{\Omega} q(x, T; \beta) v dx. \quad (8)$$

## 3 Discretized Forms

### 3.1 Finite-Element Spaces

Choose conforming subspaces  $V_h \subset V$  and  $V_{0,h} \subset V_0$  with nodal basis  $\{\phi_i\}_{i=1}^N$ . The discrete temperature reads

$$T_h(x) = \sum_{i=1}^N T_i \phi_i(x), \quad T_i = T_D \text{ on Dirichlet nodes.}$$

### 3.2 Nonlinear Stiffness Matrix and Load Vector

$$K_{ij}(T_h; \mu) := \int_{\Omega} k(x, T_h; \mu) \phi_i' \phi_j' dx, \quad (9)$$

$$f_i(T_h; \beta) := \int_{\Omega} q(x, T_h; \beta) \phi_i dx. \quad (10)$$

### 3.3 Discrete Nonlinear System

The nodal residual becomes

$$R_i(T; \mu, \beta) = \sum_{j=1}^N K_{ij}(T_h; \mu) T_j - f_i(T_h; \beta), \quad i = 1, \dots, N,$$

so the global system is  $\mathbf{R}(T; \mu, \beta) = \mathbf{0}$ .

Collecting all  $R_i$  entries gives the nonlinear vector equation

$$\mathbf{R}(T; \mu, \beta) = \mathbf{0}, \quad \mathbf{R}(T; \mu, \beta) := \mathbf{K}(T; \mu) T - \mathbf{f}(T; \beta).$$

At each Newton (or Picard) iteration we solve the linearized system

$$\mathbf{J}(T^{(m)}; \mu, \beta) \Delta T^{(m)} = -\mathbf{R}(T^{(m)}; \mu, \beta),$$

where  $\mathbf{J}(T; \mu, \beta)$  is the Jacobian matrix with entries  $J_{ij}$  defined in (11).

## 4 Residual and Jacobian

### 4.1 Continuous Residual Functional

For any  $v \in V_0$ ,

$$R(T; \mu, \beta)[v] = \int_{\Omega} k(x, T; \mu) T' v' dx - \int_{\Omega} q(x, T; \beta) v dx.$$

### 4.2 Jacobian (Fréchet Derivative)

For a perturbation  $\delta T \in V_0$ ,

$$J(T; \mu, \beta)[\delta T, v] = \int_{\Omega} k \delta T' v' + \frac{\partial k}{\partial T} \delta T T' v' - \frac{\partial q}{\partial T} \delta T v dx. \quad (11)$$

### 4.3 Finite-Element Jacobian Matrix

With basis  $\{\varphi_i\}$ ,

$$J_{ij} = \int_{\Omega} k \varphi_j' \varphi_i' + \frac{\partial k}{\partial T} T_h' \varphi_j - \frac{\partial q}{\partial T} \varphi_j \varphi_i dx.$$

*Closed-form derivatives:*

$$\frac{\partial k_1}{\partial T} = -\frac{2150}{(T - 73.15)^2}, \quad \frac{\partial k_2}{\partial T} = 2.09 \times 10^{-2} - 2(1.45 \times 10^{-5})T + 3(7.67 \times 10^{-9})T^2,$$

$$\frac{\partial q_1}{\partial T} = 0.1, \quad \frac{\partial q_2}{\partial T} = 0.$$