Weak forms and problem setup — Heterogeneous linear elasticity in a two-material 3D block

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July 2025

1 Governing Equations

1.1 Geometry and Boundary

Let $\Omega \subset \mathbb{R}^3$ be the rectangular block:

$$\Omega := (0, L_x) \times (0, L_y) \times (0, L_z), \qquad L_x := 10,\ L_y := L_z := 1.$$

The boundary decomposition is:

$$\Gamma_D := \{x = 0\}, \quad \Gamma_N := \{x = L_x\}, \quad \Gamma_{\text{free}} := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N),$$

1.2 Material Parameterization

Partition at $x_{\text{int}} := L_x/2$:

$$\Omega_1 := \{x < x_{\text{int}}\}, \qquad \Omega_2 := \{x \geq x_{\text{int}}\}.$$

Given Young's modulus $E > 0$ and Poisson ratio $\nu \in (0, 0.5)$,

$$\lambda_0 := \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu_0 := \frac{E}{2(1+\nu)}.$$

Lamé coefficients:

$$(\lambda, \mu)(x; E, \nu) := \begin{cases} (50\lambda_0,\ 50\mu_0), & x \in \Omega_1, \\ (\lambda_0,\ \mu_0), & x \in \Omega_2. \end{cases}$$

Units: $[E] = \text{Pa},\ [\nu] = 1,\ [\lambda] = \text{Pa},\ [\mu] = \text{Pa},\ [t_N] = \text{Pa}.$

1.3 Governing PDE

Let

$$\mathcal{V}_{\text{strong}} := C^0(\overline{\Omega}; \mathbb{R}^3) \cap C^2(\Omega; \mathbb{R}^3).$$

Find $u \in \mathcal{V}_{\text{strong}}$ such that

$$-\nabla \cdot \sigma(u) = \mathbf{0} \qquad \text{in } \Omega, \tag{1}$$

$$u = \mathbf{0} \qquad \text{on } \Gamma_D, \tag{2}$$

$$\sigma(u)\, n = (0, -10^{-2}, 0) \qquad \text{on } \Gamma_N, \tag{3}$$

$$\sigma(u)\, n = \mathbf{0} \qquad \text{on } \Gamma_{\text{free}}. \tag{4}$$

Here,

$$\sigma(u) := 2\mu\,\varepsilon(u) + \lambda\ \mathrm{tr}(\varepsilon(u))\mathbf{I}, \qquad \varepsilon(u) := \tfrac{1}{2}(\nabla u + \nabla u^\top),$$

where $\mathbf{I}$ is the identity tensor.

2 Weak Form

2.1 Variational Statement

Let

$$V := \{v \in H^1(\Omega;\mathbb{R}^3) \mid v = 0 \text{ on } \Gamma_D\}.$$

Given (E,ν), find $u \in V$ such that

$$a(u,v;E,\nu) = \ell(v) \qquad \forall v \in V,$$

$$a(u,v;E,\nu) := \int_\Omega \big[2\mu\,\varepsilon(u) : \varepsilon(v) + \lambda\ \mathrm{tr}(\varepsilon(u))\ \mathrm{tr}(\varepsilon(v))\big]\,dx, \tag{5}$$

$$\ell(v) := \int_{\Gamma_N} t_N \cdot v\,ds. \tag{6}$$