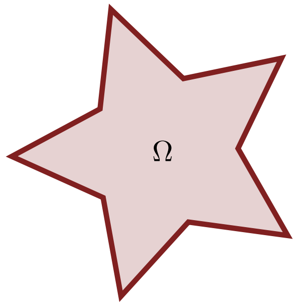
# Weak forms and problem setup — Linear heat conduction with non-affine parametric source on a star-shaped plate

# 1 Governing Equations

## 1.1 Geometry and Boundary




Let $\Omega \subset \mathbb{R}^2$ be the star-shaped polygonal plate shown in the figure above; on its boundary $\Gamma = \partial\Omega$ we impose $T = 0$.

## 1.2 Material and Heat-Source Parameterization

### Constants

$$k(x;\kappa) = \kappa, \quad \kappa = 1.0, \quad [k] = \mathrm{W/(m\,K)}, \quad \text{(thermal conductivity)}.$$

Where, $\kappa$ is a constant parameter.

### Simulation Parameter

$$q(x;\beta) = \sin\bigl(\beta\,(x_1+x_2)\bigr), \quad \beta \in (0.2,\,6), \quad [q] = \mathrm{W/m^3}, \quad x = (x_1, x_2) \in \Omega, \quad \text{(volumetric heat source)}.$$

Where, $\beta$ is the simulation parameter.

## 1.3 Governing Laws

Let $T : \Omega \to \mathbb{R}$ ($[T] = \mathrm{K}$) be the temperature field and define the variation in temperature as:

$$\nabla T = (\partial_{x_1} T, \partial_{x_2} T), \quad ([\nabla T] = \mathrm{K/m}).$$

By Fourier's law the heat flux

$$\mathbf{q} = -\,k\,\nabla T, \quad [\mathbf{q}] = \mathrm{W/m^2},$$

and energy conservation (no storage) gives

$$-\nabla\cdot\mathbf{q} = q(x;\beta), \quad [q] = \mathrm{W/m^3}.$$

Hence the steady-state conduction equation,

$$-\nabla\cdot\bigl(k\,\nabla T\bigr) = q(x;\beta).$$

### 1.4 Governing PDE

Let the space of admissible strong solutions be given by:

$$\mathcal{V}_{\text{strong}} := C^0(\Omega \cup \Gamma) \cap C^2(\Omega).$$

Find $T \in \mathcal{V}_{\text{strong}}$ such that

$$-\nabla \cdot \big(k \, \nabla T\big) = q(x; \beta) \qquad \text{in } \Omega, \tag{1}$$

$$T = 0 \qquad \text{on } \Gamma. \tag{2}$$

Where $T \in C^0(\Omega \cup \Gamma)$ due to the dirichlet boundary condition 2 on the boundary $\Gamma$ and $T \in C^2(\Omega)$ due to the strong form 1.

## 2 Weak Form

### 2.1 Variational Statement

Let $V$ be the test function space:

$$V := \{v \in H^1(\Omega) \mid v|_\Gamma = 0\}.$$

Given $(\kappa, \beta)$, find $T \in V$ such that

$$a(T, v; \kappa) = \ell(v; \beta) \qquad \forall v \in V,$$

where

$$a(T, v; \kappa) := \int_\Omega \kappa \, \nabla T \cdot \nabla v \, dx, \tag{3}$$

$$\ell(v; \beta) := \int_\Omega \sin(\beta(x_1 + x_2)) \, v \, dx. \tag{4}$$

## References

[1] Choi, Youngsoo, Arrighi, William J., Copeland, Dylan M., Anderson, Robert W., & Oxberry, Geoffrey M. (2019, October 17). *libROM*. [Computer software]. https://github.com/LLNL/libROM. https://doi.org/10.11578/dc.20190408.3; Global pROM for Poisson problem example. https://www.librom.net/examples.html Accessed: 28 July, 2025