Weak forms and problem setup — Linear heat conduction with non-affine parametric source on a star-shaped plate

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1 Governing Equations

1.1 Geometry and Boundary

Let $\Omega \subset \mathbb{R}^2$ be the star-shaped polygonal plate shown in the input mesh `star_outer10.msh`. Its boundary $\Gamma = \partial\Omega$ is held at a prescribed temperature

$$T = 0 \quad \text{on } \Gamma.$$

1.2 Material and Heat-Source Parameterization

The thermal conductivity is *constant* ($\kappa = 1.0$)

$$k(x;\kappa) = \kappa > 0, \qquad x \in \Omega,$$

while the volumetric heat source depends non-affinely on the parameter $\beta \in \mathbb{R}$:

$$q(x;\beta) = \sin\big(\beta(x_1 + x_2)\big), \qquad x = (x_1, x_2) \in \Omega.$$

Units: $[k] = \mathrm{W/(m\,K)}$, $[q] = \mathrm{W/m^3}$.

1.3 Governing PDE

Let

$$\mathcal{V}_{\text{strong}} := C^0(\overline{\Omega}) \cap C^2(\Omega).$$

Find $T \in \mathcal{V}_{\text{strong}}$ such that

$$-\nabla\cdot\big(k\,\nabla T\big) = q(x;\beta) \qquad \text{in } \Omega, \tag{1}$$

$$T = 0 \qquad \text{on } \Gamma. \tag{2}$$

2 Weak Form

2.1 Variational Statement

Let

$$V := \{w \in H^1(\Omega) \mid w|_\Gamma = 0\}.$$

Given (κ, β), find $T \in V$ such that

$$a(T, v; \kappa) = \ell(v; \beta) \qquad \forall v \in V,$$

where

$$a(T, v; \kappa) := \int_\Omega \kappa \, \nabla T \cdot \nabla v \, dx, \tag{3}$$

$$\ell(v; \beta) := \int_\Omega \sin(\beta(x_1 + x_2)) \, v \, dx. \tag{4}$$