Weak forms and problem setup — Nonlinear power-law (strain-hardening) bar under gravity and line load

Ali Hamza Abidi Syed

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1 Governing Equations

1.1 Geometry and Boundary

A prismatic bar of length

$$L := 420 \text{ in}, \qquad \Omega := (0, L) \subset \mathbb{R},$$

is fixed at the top and stress–free at the bottom.

1.2 Material and Body-Force Data

Cross-section area	$A = 10 \text{ in}^2$	(constant)
Strength coefficient	$K = 77\,000 \text{ lb/in}^2$	(power-law)
Hardening exponent	$n = 0.26$	(dimensionless)
Density	$\rho = 0.28907 \text{ lb/in}^3$	
Applied line load	1 000 lb/in (tension, downward)	

Constitutive law (1-D power law):

$$\sigma(u') = KA\,\left|u'\right|^{n-1} u'.$$

Distributed body force:

$$f(x) = 1\,000 + \rho A\,(L - x) = 1\,000 + 2.8907\,(L - x) \quad [\text{lb/in}], \qquad x \in \Omega.$$

Units: $[K] = \text{lb/in}^2,\ [A] = \text{in}^2,\ [u] = \text{in},\ [f] = \text{lb/in}.$

1.3 Governing PDE

Let

$$\mathcal{V}_{\text{strong}} := C^1([0, L]) \cap C^2((0, L)).$$

Find $u \in \mathcal{V}_{\text{strong}}$ satisfying

$$-\frac{d}{dx}\Big(KA\left|u'\right|^{n-1} u'\Big) = f(x), \qquad 0 < x < L, \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$u'(L) = 0. \tag{3}$$

2 Weak Form

2.1 Variational Statement

Let

$$V := \{v \in H^1(\Omega) \mid v(0) = 0\}.$$

Find $u \in V$ such that

$$R(u)[v] = 0 \qquad \forall v \in V,$$

with nonlinear residual functional

$$R(u)[v] := \int_0^L KA\big|u'\big|^{n-1} u'\, v'\, dx - \int_0^L f(x)\, v\, dx.$$

2.2 Jacobian (Fréchet Derivative)

For a perturbation $\delta u \in V$, for all $v \in V$

$$J(u)[\delta u,\, v] = \int_0^L KA\, n\, \big|u'\big|^{n-1}\, \delta u'\, v'\, dx.$$