Weak forms and problem setup — Twisting of a Neo-Hookean Block

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1 Governing equations

1.1 Geometry and boundary

The reference configuration is

$$\Omega = (0, L_x) \times (0, L_y) \times (0, L_z) \subset \mathbb{R}^3, \quad L_x = 10,\ L_y = L_z = 1.$$

Partition the boundary:

$$\Gamma_L = \{x = 0\}, \quad \Gamma_R = \{x = L_x\}, \quad \Gamma_N = \partial\Omega \setminus (\Gamma_L \cup \Gamma_R).$$

1.2 Material model

Let $\mathbf{u} : \Omega \to \mathbb{R}^3$ be the displacement. Define

$$\mathsf{F} = \mathsf{I} + \nabla\mathbf{u}, \quad J = \det \mathsf{F}, \quad \mathsf{F}^{-T} = (\mathsf{F}^{-1})^T.$$

For Lamé parameters $\mu, \lambda > 0$, the first Piola–Kirchhoff stress is

$$\mathsf{P}(\mathsf{F}) = \mu\,\mathsf{F} + (\lambda \ln J - \mu)\,\mathsf{F}^{-T}.$$

1.3 Strong form

Seek $\mathbf{u}$ such that

$$\begin{aligned} -\nabla \cdot \mathsf{P}(\mathbf{u}) &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_L, \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma_R, \\ \mathsf{P}(\mathbf{u})\,\mathbf{n} &= 0 && \text{on } \Gamma_N, \end{aligned}$$

with prescribed displacement on Γ_R:

$$\mathbf{g}(x, y, z) = \begin{pmatrix} -0.1 \\ y(\cos\pi - 1) - z\sin\pi \\ y\sin\pi + z(\cos\pi - 1) \end{pmatrix}.$$

2 Weak form and derivations

Multiply the strong form by a test function $\mathbf{v} \in V$ and integrate over Ω:

$$0 = \int_\Omega (-\nabla \cdot \mathsf{P}) \cdot \mathbf{v}\, dx = -\int_{\partial\Omega} (\mathsf{P}\,\mathbf{n}) \cdot \mathbf{v}\, ds + \int_\Omega \mathsf{P} : \nabla \mathbf{v}\, dx.$$

On Γ_N we have $\mathsf{P}\,\mathbf{n} = 0$, and on $\Gamma_L \cup \Gamma_R$ the variation $\mathbf{v} = 0$, so the boundary integral vanishes. Hence the weak form:

$$\boxed{\int_\Omega \Big[\mu \operatorname{tr}(\mathsf{F}^T \nabla \mathbf{v}) + (\lambda \ln J - \mu) \operatorname{tr}(\mathsf{F}^{-T} \nabla \mathbf{v})\Big]\, dx = 0 \quad \forall \mathbf{v} \in V.}$$

Derivation of the weak form

Starting from $0 = \int_\Omega (-\nabla \cdot \mathsf{P}) \cdot \mathbf{v}\, dx$, apply Gauss' theorem and note vanishing boundary terms as above. Rewriting $\mathsf{P} : \nabla \mathbf{v} = \operatorname{tr}(\mathsf{P}^T \nabla \mathbf{v})$ and substituting the expression for $\mathsf{P}(\mathsf{F})$ yields the boxed form.

2.1 Jacobian (Fréchet derivative)

Define the residual functional

$$R(\mathbf{u})[\mathbf{v}] = \int_\Omega \mathsf{P}(\mathsf{F}(\mathbf{u})) : \nabla \mathbf{v}\, dx.$$

Its Fréchet derivative in direction $\delta \mathbf{u}$ is

$$DR(\mathbf{u})[\delta \mathbf{u}, \mathbf{v}] = \frac{d}{d\varepsilon} R(\mathbf{u} + \varepsilon\, \delta \mathbf{u})[\mathbf{v}]\Big|_{\varepsilon=0} = \int_\Omega (\delta \mathsf{P}) : \nabla \mathbf{v}\, dx.$$

We compute variations:

$$\delta \mathsf{F} = \nabla \delta \mathbf{u}, \quad \delta J = J \operatorname{tr}(\mathsf{F}^{-1} \nabla \delta \mathbf{u}), \quad \delta \ln J = \operatorname{tr}(\mathsf{F}^{-1} \nabla \delta \mathbf{u}), \quad \delta(\mathsf{F}^{-T}) = -\mathsf{F}^{-T} (\nabla \delta \mathbf{u}) \mathsf{F}^{-T}.$$

Thus

$$\begin{aligned} \delta \mathsf{P} &= \mu\, \delta \mathsf{F} + (\lambda \ln J - \mu)\, \delta(\mathsf{F}^{-T}) + \lambda\, (\delta \ln J)\, \mathsf{F}^{-T} \\ &= \mu\, \nabla \delta \mathbf{u} - (\lambda \ln J - \mu)\, \mathsf{F}^{-T} (\nabla \delta \mathbf{u}) \mathsf{F}^{-T} + \lambda \operatorname{tr}(\mathsf{F}^{-1} \nabla \delta \mathbf{u})\, \mathsf{F}^{-T}. \end{aligned}$$

Substitution gives the Jacobian bilinear form:

$$\boxed{DR(\mathbf{u})[\delta \mathbf{u}, \mathbf{v}] = \int_\Omega \Big[\mu \operatorname{tr}((\nabla \delta \mathbf{u})^T \nabla \mathbf{v}) - (\lambda \ln J - \mu) \operatorname{tr}((\nabla \delta \mathbf{u}\, \mathsf{F}^{-1})^T (\nabla \mathbf{v}\, \mathsf{F}^{-1})) + \lambda \operatorname{tr}(\mathsf{F}^{-1} \nabla \delta \mathbf{u}) \operatorname{tr}(\mathsf{F}^{-1} \nabla \mathbf{v})\Big]\, dx.}$$