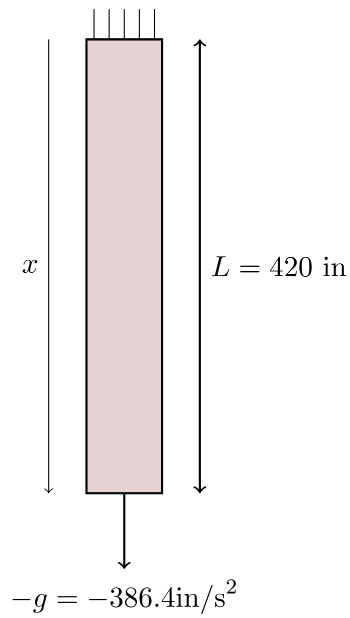
Weak forms and problem setup — Nonlinear power-law (strain-hardening) bar under gravity and line load

1 Governing Equations

1.1 Geometry and Boundary



A prismatic bar of length

$$L := 420 \text{ in}, \qquad \Omega := (0, L) \subset \mathbb{R},$$

is fixed/clamped at $x = 0$ end and stress–free at $x = L$, and $[x] = \text{in}$.
Here, $\sigma(x)$ denotes the axial (Cauchy) stress in the bar, with units of lb/in^2.

1.2 Material Properties and Parameters

Constants

$$\begin{aligned} A &= 10 \text{ in}^2, && \text{cross-sectional area,} \\ g &= 386.4 \text{ in/s}^2, && \text{gravitational acceleration.} \end{aligned}$$

Where, A, n and g are constants.

Simulation Parameters

$E(\mu)$ is the perturbed strength coefficient, defined by

$$E(\mu) = 77\,000 + \mu, \quad [E] = \text{lb/in}^2, \quad \mu \in (-1000,\, 1000).$$

$\rho(\rho_{\text{param}})$ is the perturbed mass density, defined by

$$\rho(\rho_{\text{param}}) = 0.28907 + \rho_{\text{param}}, \quad [\rho] = \text{lb/in}^3, \quad \rho_{\text{param}} \in (0,\, 0.5).$$

Where, μ and ρ_{param} are the simulation parameters.

1.3 Governing Laws

Let $u : [0, L] \to \mathbb{R}$ be the axial displacement ($[\ u\ ] = \text{in}$). Define the strain

$$\varepsilon = u', \quad [\varepsilon] = 1 \text{ (dimensionless)}.$$

The material follows the power-law

$$\sigma(\varepsilon) = E\,|\varepsilon|^{n-1}\,\varepsilon, \quad [\sigma] = \text{lb/in}^2,$$

where $n = 0.26$ is hardening exponent, and the internal axial force is

$$N(\varepsilon) = A\,\sigma(\varepsilon), \quad [N] = \text{lb}.$$

And let the distributive force be:

$$f(x) = 1\,000 + \rho A\,(L - x) \quad , [f] = \text{lb}, \qquad x \in \Omega.$$

1.4 Governing PDE

Let the space of admissible strong solutions be given by:

$$\mathcal{V}_{\text{strong}} := C^1([0, L]) \cap C^2((0, L)).$$

Find $u \in \mathcal{V}_{\text{strong}}$ satisfying

$$-\frac{d}{dx}\Big(EA\big|u'\big|^{n-1}u'\Big) = f(x), \qquad 0 < x < L, \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$u'(L) = 0. \tag{3}$$

Where, due to the Neumann boundary condition 3 which is $u'(L) = 0$, we require $u \in C^1([0, L])$, which includes the boundary($x = 0$ and $x = L$) and due to 1 we require $u \in C^2((0, L))$.

2 Weak Form

2.1 Variational Statement

Let V be the test function space:

$$V := \{v \in H^1(\Omega) \mid v(0) = 0\}.$$

Find $u \in V$ such that

$$R(u)[v] = 0 \qquad \forall v \in V,$$

with nonlinear residual functional

$$R(u)[v] := \int_0^L EA\,|u'|^{n-1}u'\,v'\,\mathrm{d}x \;-\; \int_0^L f(x)\,v\,\mathrm{d}x = 0 \quad \forall v \in V. \tag{4}$$

2.2 Jacobian (Fréchet Derivative)

For a perturbation $\delta u \in V$, for all $v \in V$:

$$J(u)[\delta u,\, v] = \int_0^L EA\,n\,|u'|^{n-1}\,\delta u'\,v'\,\mathrm{d}x. \tag{5}$$

References

[1] Kythe, P. K., Wei, D., & Okrouhlik, M. (2004). *An introduction to linear and nonlinear finite element analysis: a computational approach. Appl. Mech. Rev.*, 57(5), B25-B25. (1st ed., pp. 252–256)