Weak forms and problem setup — Nonlinear heat conduction in a two-material 1-D rod

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1 Governing Equations

1.1 Geometry and Domains

We consider steady (time–independent) 1-D conduction through an inhomogeneous rod of length

$$L = 0.5 \text{ m}, \qquad \Omega = (0, L).$$

The rod is partitioned at $x_{\text{int}} = 0.4$ m into the sub-domains

$$\Omega_1 = (0, x_{\text{int}}), \qquad \Omega_2 = [x_{\text{int}}, L).$$

1.2 Material and Heat Source Parameterization

Both the thermal conductivity k and the volumetric heat source q depend *non-linearly* on T:

$$k(x,T;\mu) = \begin{cases} k_1(T;\mu) := 16 + \mu + \dfrac{2150}{T - 73.15}, & x \in \Omega_1, \\ k_2(T;\mu) := 30 + \mu \; + \; 2.09 \times 10^{-2}\, T \\ \qquad\qquad\qquad\quad - \; 1.45 \times 10^{-5}\, T^2 & x \in \Omega_2, \\ \qquad\qquad\qquad\quad + \; 7.67 \times 10^{-9}\, T^3, \end{cases} \tag{1}$$

$$q(x,T;\beta) = \begin{cases} q_1(T;\beta) := 35\,000 + \beta + \dfrac{T}{10}, & x \in \Omega_1, \\ q_2(\beta) := 5\,000 + 10\,\beta, & x \in \Omega_2. \end{cases} \tag{2}$$

Units: $[k] = \mathrm{W/(m\,K)}$, $[q] = \mathrm{W/m^3}$.

1.3 Governing PDE

Define the classical solution space

$$V = \Big\{ T \in C\big([0,L]\big) \cap C^1\big([0,L] \setminus \{x_{\text{int}}\}\big) \mid T|_{\Omega_i} \in C^2(\Omega_i),\ i = 1,2 \Big\}.$$

Find $T \in V$ such that

$$-\frac{d}{dx}\Big(k\big(x, T(x); \mu\big)\, T'(x) \Big) = q\big(x, T(x); \beta\big) \qquad \text{in } \Omega, \tag{3}$$

$$T'(0) = 0 \qquad \text{(Neumann)}, \tag{4}$$

$$T(L) = T_D = 573.15 \text{ K} \qquad \text{(Dirichlet)}. \tag{5}$$

(The dependence of k and q on T makes the problem nonlinear.)

2 Weak Form

2.1 Derivation

Let v be any test function with $v(L) = 0$. Multiplying (3) by v, integrating over each sub-domain, and applying integration by parts gives

$$\int_0^L k\big(x, T; \mu\big)\, T'\, v'\, dx \;=\; \int_0^L q\big(x, T; \beta\big)\, v\, dx.$$

2.2 Function Spaces

Introduce the Sobolev space

$$H^1(\Omega) = \big\{ u \in L^2(\Omega) \;\big|\; u' \in L^2(\Omega) \big\}, \qquad \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2.$$

Set

$$V := \{\, w \in H^1(\Omega) \mid w(L) = T_D \}, \qquad V_0 := \{\, v \in H^1(\Omega) \mid v(L) = 0 \}.$$

2.3 Abstract Weak Problem

Given parameters $\boldsymbol{\mu} = (\mu, \beta)$, find $T \in V$ such that

$$a(T, v; \boldsymbol{\mu}) = \ell(T, v; \boldsymbol{\mu}) \quad \forall v \in V_0, \tag{6}$$

with nonlinear forms

$$a(T, v; \boldsymbol{\mu}) := \int_\Omega k\big(x, T; \mu\big)\, T'\, v'\, dx, \tag{7}$$

$$\ell(T, v; \boldsymbol{\mu}) := \int_\Omega q\big(x, T; \beta\big)\, v\, dx. \tag{8}$$

3 Discretized Forms

3.1 Finite-Element Spaces

Choose conforming subspaces $V_h \subset V$ and $V_{0,h} \subset V_0$ with nodal basis $\{\phi_i\}_{i=1}^N$. The discrete temperature reads

$$T_h(x) = \sum_{i=1}^N T_i \phi_i(x), \qquad T_i = T_D \text{ on Dirichlet nodes.}$$

3.2 Nonlinear Stiffness Matrix and Load Vector

$$K_{ij}\big(T_h; \mu\big) := \int_\Omega k\big(x, T_h; \mu\big)\phi_i' \phi_j'\, dx, \tag{9}$$

$$f_i\big(T_h; \beta\big) := \int_\Omega q\big(x, T_h; \beta\big)\phi_i\, dx. \tag{10}$$

3.3 Discrete Nonlinear System

The nodal residual becomes

$$R_i(T;\mu,\beta) = \sum_{j=1}^{N} K_{ij}\big(T_h;\mu\big)T_j \;-\; f_i\big(T_h;\beta\big), \qquad i = 1,\dots,N,$$

so the global system is $\mathbf{R}(T;\mu,\beta) = \mathbf{0}$.

Collecting all R_i entries gives the nonlinear vector equation

$$\mathbf{R}(T;\mu,\beta) = \mathbf{0}, \qquad \mathbf{R}(T;\mu,\beta) := \mathbf{K}\big(T;\mu\big)\,T - \mathbf{f}\big(T;\beta\big).$$

At each Newton (or Picard) iteration we solve the linearized system

$$\mathbf{J}\big(T^{(m)};\mu,\beta\big)\,\Delta T^{(m)} = -\mathbf{R}\big(T^{(m)};\mu,\beta\big),$$

where $\mathbf{J}(T;\mu,\beta)$ is the Jacobian matrix with entries J_{ij} defined in (11).

4 Residual and Jacobian

4.1 Continuous Residual Functional

For any $v \in V_0$,

$$R(T;\mu,\beta)[v] = \int_\Omega k\big(x,T;\mu\big)\,T'\,v'\,dx - \int_\Omega q\big(x,T;\beta\big)\,v\,dx.$$

4.2 Jacobian (Fréchet Derivative)

For a perturbation $\delta T \in V_0$,

$$J(T;\mu,\beta)[\delta T, v] = \int_\Omega k\,\delta T'\,v' + \frac{\partial k}{\partial T}\,\delta T\,T'\,v' - \frac{\partial q}{\partial T}\,\delta T\,v\,dx. \tag{11}$$

4.3 Finite-Element Jacobian Matrix

With basis $\{\varphi_i\}$,

$$J_{ij} = \int_\Omega k\,\varphi_j'\varphi_i' + \frac{\partial k}{\partial T}\,T_h'\,\varphi_j\varphi_i' - \frac{\partial q}{\partial T}\,\varphi_j\varphi_i\,dx.$$

Closed-form derivatives:

$$\frac{\partial k_1}{\partial T} = -\frac{2150}{(T-73.15)^2}, \qquad \frac{\partial k_2}{\partial T} = 2.09\times10^{-2} - 2(1.45\times10^{-5})T + 3(7.67\times10^{-9})T^2,$$

$$\frac{\partial q_1}{\partial T} = 0.1, \qquad \frac{\partial q_2}{\partial T} = 0.$$

4.4 Newton–Picard Solution Procedure

Given (μ, β), tolerance $\varepsilon > 0$, and initial guess $T^{(0)}$:

1. Assemble stiffness and load $K^{(m)}$, $f^{(m)}$ and assemble the residual:

$$R^{(m)} = K^{(m)} T^{(m)} - f^{(m)}$$

2. Assemble the Jacobian:

$$J^{(m)} = \mathbf{J}\big(T^{(m)}; \mu, \beta\big)$$

.

3. Solve $J^{(m)} \Delta T^{(m)} = -R^{(m)}$ on the free dofs(without the dirichlet nodes).
4. Update $T^{(m+1)} = T^{(m)} + \lambda \Delta T^{(m)}$ with $\alpha \in (0, 1]$(with most cases being $\alpha = 1$).
5. If $\|R^{(m)}\|_2 < \varepsilon$, return $T^{(m)}$ while enforcing dirichlet condition.