Math 70
Hw 02
February 8, 2018

B1,B2,B3,B4,B5

B1

a) Let $n \geq 3$ and $j \in J_n$. Find a one-to-one and onto function $f : S_{n-1} \to \{p \in S_n; p(1) = 1\}$

b) Let $n \geq 3$ and $i, j \in J_n$. Find a one-to-one and onto function $f : \{p(x) \neq j$ for $x \neq i\} \to S_{n-1}$.

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B2

a) Let A be an $n \times n$ invertible matrix. Prove that if the entries A and A^{-1} are integer numbers then $det(A) \in \{1, -1\}$.

b) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that rotates $\pi/2$ radians the plane $\{(x, y, z); z = 0\}$ and satisfies $L(0, 0, 1) = (0, 0, 5)$. Let A be the matrix of L with respect to the canonical basis. Prove that $det(A) > 0$. What if $\pi/2$ is replaced by an angle $\theta \geq [0, \pi/2]$ and 5 by a nonzero real number α?

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B3 Let A be an invertible matrix.

a) Use Cramer's rule to prove that $B = (b_{ij}) = A^{-1}$ then

$$b_{i1} = \frac{C_{i1}}{det(A)}$$

where C_{i1} is the matrix resulting from replacing the i-th column of A by the vector $(1, 0, \ldots, 0)$.

b) State the generalization of part a for b_{ij}.

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B4

a) Let A be an $n \times n$ matrix. Suppose there exists an invertible matrix P such that $P^{-1}AP$ is a triangular matrix (t_{ij}). Prove that $\lim_{k\to+\infty}(det(A))^k = 0$ if $|t_{11} \cdot t_{22} \cdots t_{nn}| < 1$.

b) State a sufficient condition for $\lim_{k\to\infty}(det(A))^k = +\infty$

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B5

a) Construct a function $f : [0, \pi] \to M_{2\times 2}$ (the vector space of 2×2 matrices) such that $f(0) = I, f(\pi) = -I$ and $det(f(t)) > 0$ for all $t \in [0, \pi]$. Hint: Consider rotations.

b) Let

$$A_k = \begin{bmatrix} f_k & g_k \\ u_k & v_k \end{bmatrix}$$

define a sequence of 2×2 matrices. Prove that if $\lim_{k\to\infty} f_k = \lim_{k\to\infty} g_k = \lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = 0$ then $\lim_{k\to\infty} det(A_k) = 0$

c) State the generalization of part b to $n \times n$ matrices.

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