UNIVERSITÀ DEGLI STUDI DI MILANO

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RANK MONOTONICITY IN CENTRALITY MEASURES

Supervisor: Sebastiano VIGNA
Co-supervisor: Paolo BOLDI

Master thesis of:
Alessandro LUONGO
StudentID: 827057

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To those who devoted their time to me.

Jonathan Seagull discovered that boredom and fear and anger
are the reasons that a gull's life is so short,
and with those gone from his thought, he lived a long fine life indeed.
Richard Bach —*Jonathan Livingston Seagull*

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Chapter 1

Introduction

Da quella notte
è entrato il gatto
con una testa in bocca
e me l'ha messa
tra i miei circuiti
ed ha ripreso vita
ed è da allora
che ho la coscienza
perchè di notte sogno
di quella notte
che è entrato il gatto
con una testa in bocca.
—IL SOGNO DEL COMPUTER — Musica Per Bambini (Dei Nuovi Animali)

1.1 The problem of centrality in graphs

Graphs are used in science and engineering to model a vast variety of different systems. Some examples are: social choice theory [34], social networks [27], computation [18], neural networks in human brains [2], combinatorics [11], internet search engines, [31], papers co-authorship [25], autonomous systems across the internet [26], and so on. One of the most natural problem that arise dealing with graphs, is to find the most important node of a network, or more generally, to sort the nodes of the graph according the level of importance in the network. This operation is called ranking. To address this problem there have been suggested dozen of different approaches and centrality measures. A centrality measure is a function that assign at each node of a graph a certain score according to the importance of the node in the graph. Historically, this problem arose in social sciences around 1950, where graphs were used to model the dynamics of a social systems (for instance: "which person have more influence in the group", or "which is most powerful in a group of peers"). As usual, although the formulation of the problem is somehow simple and intuitive, the solution it is not. In fact, there is not a uniform agreement of what a central node should be. More generally, there is not a uniform agreement on how to define importance in this context. As noted in [3], the center of a star (something we all intuitively agree as most important point in a graph) is at the same time:

- the node with largest degree;
- the node that is closest to the other nodes (e.g., that has the smallest average distance to other nodes);
- the node through which most shortest paths pass;
- the node with the largest number of incoming paths of length k, for every k;
- the node that maximizes the dominant eigenvector of the graph matrix;
- the node with the highest probability in the stationary distribution of the natural random walk on the graph;

Each of these traits seems to match somehow the idea of centrality that we have in mind. Thus, during these decades, dozens of different centrality measures were proposed and debated. As Freeman said: "several measures are often only vaguely related to the intuitive ideas they purport to index, and many are so complex that it is difficult or impossible to discover what, if anything, they are measuring" [14]. For instance, even simple centrality measure, such as the Bavelas's index were criticized by Flament [13], who gave us some counterexamples: n-circuits and complete n-graphs which obviously differ greatly in centrality (the complete n-graph being highly centralized, the n-circuit hardly at all) but have the same Bavelas index. Moreover, the discovery that all homogeneous graph (graphs all of whose vertices are automorphic) have minimal Bavelas index, led Flament to think that Bavelas's index is a sort of measure of "disparity between the points of a graph": a measure which has nothing or little to do with centrality.

The gist behind the first works on graph centrality is that each measure of centrality is fundamentally *a proxy of some underlying network process or processes.* If the particular network process is irrelevant or unrealistic for a given network, then any measure of centrality based on that process will produce nonsense. For instance: applying an algorithm devised to give higher rank to nodes critical to the global flow of communication between all the nodes in the network *may* have anything to do with actual importance in the real network system. The underlying assumption that the communication occurs only over geodesic paths (i.e., shortest paths) and thus the routing of information is maximally efficient, my not be valid! This does *not* mean that centrality measures can not or should not be used, it is all the opposite: they should be used in an "exploratory manner", to gain some insight into the general structure and pattern of a network and to generate hypotheses towards the understanding of what processes might have generated that structure [29].

In the last decades, the field of Information Retrieval was born, and problem of centrality has been addressed from other points of view. This has allowed researcher to test old centrality measures on graphs of a totally different scale and structure, find new methods of evaluation of past measures, and create new ranking function. In this context, Boldi and Vigna [6] wrote three "axioms for centrality": *a collection of properties that a centrality measure must satisfy to be considered valid.* Moreover, the authors took some of the most important centrality measures known from the literature and evaluated them under the light of the axioms they proposed. These properties are called axioms because they are intuitive and somehow *sound*: they are general guidelines on the behavior that we intuitively expect from a centrality measure on very simple graphs. The first two axiom state how an algorithm should behave "globally" by changes of size and the density of a particular graphs. The third it is about the behavior of the score of a node, under the operation of arc addition. As suggested in [6], in this work we will study a fourth axiom: *rank monotonicity.* This axiom is about the behavior of the ranking of the graph under the action of arc addition. We analyze some of the most famous algorithms, trying to prove if they satisfy rank monotonicity, and if not, to give counterexamples.

1.2 Organization of the thesis

This work is organized in this way: In this chapter there is the introduction of this work, and an explanation of the formalism used in this work. In Chapter 2 there is an historical review of the previous works, with specific focus on the first axioms for centrality. In Chapter 3 there is a brief overview of the algorithm studied in this work. In Chapter 4 *rank monotonicity* is formally introduced and defined. In Chapter 5 the gist of this work: theorems about rank monotonicity on strongly connected graphs, generalization for the non-strongly connected case and theorems stating sufficient conditions for a spectral ranking to be rank monotone. There is also a simple proof of score monotonicity for closeness centrality and betweenness centrality in strongly connected graph. Finally, Chapter 6 concludes the thesis: I sum up the results of this work, and discuss possible future work, along with some open questions.

1.3 Formalism and Notation

Let G be a *directed* graph on a set N of n nodes (graph, nodes, and corresponding vector coordinates start from zero), and a set $A \subseteq N \times N$ of arcs. We write $x \to y$ when $(x, y) \in A$ and we call x and y the source and target of the arc. The transpose of a graph is obtained by reversing all arc directions (i.e., it has an arc $y \to x$ for every arc $x \to y$ of the original graph). A *symmetric graph* is a graph such that $e = (x, y) \in A$ iff $e' = (y, x) \in A$. Such graph is fixed by transposition, and can be identified with an *undirected* graph, that is, a graph whose arcs (usually called edges) are a subset of unordered pairs of nodes. The matrices representing these graphs are symmetric. As usual, the out degree $d^+(x)$ of a node x is the number of its successors, and the in degree $d^-(x)$ is the number of its predecessors. A *path* (of length k) is a sequence $x_0, x_1, \ldots, x_{k-1}$, where $x_j \to x_{j+1}$, $0 \leq j < k$. A path is a walk in which all vertices (except possibly the first and last) are not repeated. A *walk* (of length k) is a sequence $x_0, x_1, \ldots, x_{k-1}$, where $x_j \to x_{j+1}$ or $x_{j+1} \to x_j, 0 \leq j < k$. A walk of length s is formed by a sequence of s edges such that any two successive edges in the sequence share a vertex (node). The walk is also considered to include all the nodes incident to those edges, making it a sub graph. A walk can also be a list of adjacent nodes (if the graph is undirected). A *trail* is a walk where all edges are distinct. A trail might visit the same vertex twice, but only if it comes and goes from a different edge each time.

A *morphism* between graphs G and H with node sets N and P is a map $f : N \to P$ such that $x \to y \iff f(x) \to f(y)$. It is an *isomorphism* if f is a bijection. It is an *automorphism* if $G = H$.

A graph is *(strongly) connected* if there is a single (strongly) connected component, that is, for every choice of x and y there is a walk (path) from x to y. A strongly connected component is terminal if its nodes have no arc towards other components. A connected (strongly connected, respectively) component of a graph is a maximal subset in which every pair of nodes is connected by a walk (path, respectively).

A matrix is *irreducible* if and only if replacing non-zero entries in the matrix by ones, and viewing the matrix as the adjacency matrix of a directed graph such directed graph is strongly connected. Again, a matrix is irreducible if it is not similar via a permutation to a block upper triangular matrix (that has more than one block of positive size). A nonnegative square matrix A is called *primitive* if there is a k such that all the entries of A^k are positive. A sufficient condition for a matrix to be primitive, is for the matrix to be non-negative and irreducible matrix with a positive element on the main diagonal.

The distance $d(x, y)$ from x to y is the length of a shortest path from x to y, or ∞ if no such path exists. The nodes reachable from x are the nodes y such that $d(x, y) < \infty$. The nodes co-reachable from x are the nodes y such that $d(y, x) < \infty$. A node has trivial (co)reachable set if the latter contains only the node itself. The notation $\bar{A}$ where A is a non-negative matrix, will be used throughout the paper to denote the matrix obtained by l_1-normalizing the rows of A, that is, dividing each element of a row by the sum of the row (null rows are left unchanged). If there are no null rows, $\bar{A}$ is (row-)stochastic, that is, it is nonnegative and the row sums are all equal to one. (If the sum of more then one rows is less than 1, the matrix is said to be sub-stochastic).

1.3.1 Linear algebra and M-Matrices

We say that a square matrix is a *Z-matrix* if the off-diagonals elements are negative or zero. An *M-matrix* is a Z-matrix, which is of the form

$$kI - P$$

where P is a nonnegative real matrix, and k exceeds the spectral radius of P. An equivalent definition states that M-matrices are Z-matrices with the property that the real part of the dominant eigenvalue is positive. Most of the centrality measure based on linear algebra can be restated in terms of inverses of M-matrices. Inverses of M-matrices have been deeply studied in the field of applied linear algebra in the last decades because of their broad range of applications.

A matrix C is said to be *row diagonally dominant* if

$$|c_{ii}| \geq \sum_{k=0}^{k=n} |c_{ik}|.$$

C is said to be *pointwise row diagonally dominant* if

$$|c_{ii}| \geq |c_{ik}| \quad \forall k = 1 \dots n.$$

Both definitions can be made strict.

1.3.2 Sherman-Morrison formula

This formula allows us to express the inverse of a perturbed matrix as a function of the non perturbed matrix, up to the condition that the perturbation can be expressed as an outer product of vectors. The idea behind the Sherman-Morrison formula is to see how a low-rank perturbation affects the inverse. This formula has been used in numerical analysis, because it gives computational advantages compared to the calculation of the inverse of the perturbed matrix from scratch. Given two row vectors $\boldsymbol{u}, \boldsymbol{v}$ and a matrix A:

$$(A + \boldsymbol{u}^T \boldsymbol{v})^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{u}^T \boldsymbol{v} A^{-1}}{1 + \boldsymbol{v} A^{-1} \boldsymbol{u}^T}$$

This formula has been used widely in the literature, not only for numerical analysis, but to gain insight on matrix obtained by a perturbation, as made in [23] and [6].

1.3.3 Perron-Frobenius theorem

This theorem has been used widely in all the past literature in network science and graph centrality so it is useful to have a statement here:

Theorem. *Take an irreducible matrix T. Then:*

- *T has a positive (real) eigenvalue λ_{max} such that all other eigenvalues of T satisfy*
$$\mid \lambda \mid \leq \lambda_{max}.$$
- *λ_{max} has algebraic and geometric multiplicity one, and has an eigenvector $\boldsymbol{x}$ with $\boldsymbol{x} > 0$.*
- *λ_{max} is simple: right and left eigenspaces associated with the maximum eigenvalue are one dimensional.*
- *Any non-negative eigenvector is a multiple of $\boldsymbol{x}$.*

Proof is omitted, but can be found in [22].

1.3.4 Matrix-determinant lemma

This lemma have been used in one proof to show a property of a determinant, and therefore it is included.

Lemma 1. *If A is an invertible $n \times n$ matrix, and $\boldsymbol{u}, \boldsymbol{v}$ are two n-dimensional row vector, then:*

$$\det\bigl(A + \boldsymbol{u}^T \boldsymbol{v}\bigr) = \bigl(1 + \boldsymbol{v} A^{-1} \boldsymbol{u}^T\bigr) \det A.$$

More information can be found in [12].

Chapter 2

Previous Work

> Know how to solve every problem that has been solved.
>
> —Last lecture — Richard Feynman

2.1 Previous work

We now discuss some of the most important publications in this field. As we said before, since the study of graph centrality started with social sciences and sociometry, first centrality measures were called "sociometric statuses", or "centrality indexes" [19]. Sociologically, all these different indexes were meant to capture different aspects of human interaction. Even if the study of centrality index in a graph is a concept which is broader than the study of social networks, we can divide most of the index into two dichotomic views, closely related to our intuitive ideas of "importance" in a group of people. (This division can be found in [15].)

First, there are those who view a person as central if he or she is "close" to everyone else in the network. This view of centrality is motivated by the idea that a person who is close to others will have access to more information [24] [32] [33], have higher status [21] [19], have more power [10], [7], or have greater influence [16] than others. In this category falls also [4] and [30]. The other view is that people who are "central" should stand between others on the paths of communication [4] [14]. Such people indeed can facilitate or inhibit the communication of others and are, therefore, in a position to mediate the access of others to information, power, prestige, or influence.

We can also devise another twofold distinction on centrality measures. On one side we find *geometric* measures, relying on geodesic (i.e., shortest paths) between pairs of vertices. Note that some paths between nodes are discharged using this approach. On the other side, we have the so-called *spectral measures*, centralities meant to take into account all the possible paths in the graph. These measures are all very similar, because the concept of taking all the paths to a node is easily translated in the language of linear algebra: if A is the adjacency matrix of a graph, A^p contains in the entry of row i and column j the number of paths of length p from node i to node j.

2.2 History

2.2.1 1948 Bavelas — Mathematical model for group structure

In 1948, Bavelas [3] wanted to analyze the dynamics of groups, continuing the work of Kurt Lewin, a famous sociologist. In his work there is a strong topological footprint. Topology indeed offers a framework suitable to treat social phenomena such as "connection", "boundaries" and "neighbor" without using a metric. He was the first who applied concept as centrality to human communication. He tried to answer the question: *What is the optimal network for group performance?* In later years, he ran some experiment: he manipulated pattern of communication among the members of small groups, by controlling who could send messages to whom, and thus measured the impact of various patterns on group functioning and performance [4]. As we might expect today, researchers found that centralization (i.e., the extent to which one person served as a hub of communication) had a significant impact on individual and group functioning. The complexity of the task proved to be a critical moderating variable: centralization was beneficial when the task was simple and detrimental for complex tasks. A decentralized structure was also best when information was distributed unevenly among group members, or when the information was ambiguous. His work is formulated in terms of cells (which represent peers in a group) and geometrical distances between cells. He analyzed lower and upper bound of measure based on sum of distances of the cells in a structure. A more mathematical formulation of his theory can be found in the work of Sabidussi [32].

2.2.2 1949 Seeley — The net of reciprocal influence; a problem in treating sociometric data

Seeley in 1949 suggested to normalize the rows of a matrix, and use that matrix to calculate the left eigenvector as a centrality index with an iterative process. The matrix in consideration could be any real matrix, with each entry in row i and column j representing an *endorsement* or approval from i to j. This idea had nothing to do with graphs yet, and was a huge improvement compared to the centrality indices used in sociometry, which was mostly based on simple summations on the rows or the columns of the matrices considered. He noticed that previous indices of centrality did not take into consideration the weight of the peers who are expressing his endorsement. He noted that being endorsed or liked by someone who is being endorsed or liked a lot should be taken more into consideration than being liked by other peers whose endorsement is on average. His work went strangely unnoticed by the academic community, and was rediscovered later. His work it is expressed in terms of linear equations, and matrices are used only thanks to Cramer's rule for expressing the solutions of the unknowns. We will see more in detail his centrality in Chapter 3.

2.2.3 1952 Wei — The algebraic foundations of ranking theory

Wei's work proposed a solution to the problem of ranking teams [36]. He suggest to create a matrix M with $m_{ij} = 1$ if i beats j, $m_{ij} = 1/2$ for ties, and $m_{ij} = 0$ if i gets defeated by j. The next steps consist in using the previous matrix to do the following computation: starting by giving to each team uniform scores, each team sums to its current score the score of the team defeated. In case of draw, the team gets only half of the score of the team defeated. In matrix notation, we can consider the ranking induced by the following vector:

$$\lim_{k\to\infty} M^k \mathbf{1}^T$$

Using Perron-Frobenius theory one can see that (under suitable hypotheses) this vector converges to the right eigenvector. The work of Seeley and Wei, even if not directly related to graphs, shows that we can use the eigenvectors of a matrix in order to have a spectral ranking of the matrix itself. In matrix notation their centrality can be defined as:

$$\lambda \boldsymbol{r} = \boldsymbol{r} M$$

Where $\boldsymbol{r}$ is an eigenvector, M a matrix, and λ is its associated eigenvalue. As we know, all the possible eigenvector and eigenvalues satisfy this equation, but usually the eigenvector associated with the largest eigenvalue is preferred. We can speak of *the* spectral ranking associated with M if the eigenspace associated to the largest eigenvalue has unitary dimension. Even if the left eigenvector of M is the right eigenvector of the transposed matrix M^T, we still keep distinguishing them, because they carry two different semantic meaning: left eigenvector is used to denote endorsement, while right eigenvector is used to denote influence.

2.2.4 1953 Katz — A new status index derived from sociometric analysis

In 1953, the famous Katz index was published in "Psychometrika" [21]. His centrality takes in consideration all the possible path to a node, and not only the geodesics. The purpose of this paper is to suggest a new method of computing statuses, taking into account not only the number of direct "votes" received by each individual, but also the status of each individual who chooses the first. The matrix is constructed writing the connection between peers, and these connections are expressed as values in a matrix, where each row i represent a node (a person) and there is a 1 in column k if there is a link between i and k. If the connection between i and k it is not mutual the matrix is not symmetric. When considering the importance of a node x, each path from other nodes k is weighted by a factor proportional to his length. For convergency reasons, the weighting factor α should be less than $1/\rho(M)$, where $\rho(M)$ is the spectral radius of the adjacency matrix derived from the social graph. Katz noted that the element a^k_{ij} of the matrix A^k are the number of shortest path of length k from i to j, and the column sum of the matrix represent all the possible shortest path from all the nodes of the graph, that is, the number of direct choices made by the member of the groups to the person corresponding to that column. Katz thus wanted find the column sum of the following matrix:

$$T = \alpha C + \alpha^2 C^2 + \ldots + \alpha^k C^k = (I - \alpha C)^{-1} - I$$

which can be used to derive the following ranking:

$$k = \mathbf{1} \sum_{i=0}^{\infty} \alpha^i A^i$$

After some algebraic passages, Katz proved that:

$$k = \mathbf{1}(I - \alpha A)^{-1}$$

The assumptions underlying his the work are twofold: the first (common to all sociometric works) is that information used for calculating the centrality is accurate and therefore there exist certain links between humans. On the other side, where our information indicates no link, there is no communication, influence, or whatever else we are measuring. Secondly, we assume that each link independently has the same probability of being effective. It is evident that the effect of longer chains on the index will be greater for smaller values of $1/\alpha$. As one can imagine, the determination of the parameter α is the problem of this centrality measure itself. We know that it should be less than the reciprocal of the spectral radius, but Katz said that it is context dependent, in the sense that *either by investigation or omniscience, its value is known.* The work end with some numerical examples some consideration on the computation of the index and the convergence of the matrix.

2.2.5 1958 Berge — Théorie des graphes et ses applications

Six years later after Wei's work, Berge noticed how one can use Wei's ideas to measure the centrality of *any* directed graph. Wei's ideas can be applied indeed to the adjacency matrix of a graph. Unlike indegree centrality, which weights every contact equally, the eigenvector weights each node with the respective centrality of their peers. Eigenvector centrality can also be seen as a weighted sum of not only direct connections, but also indirect connections of every length. He mostly applied Wei's work to sociograms: directed graphs of reciprocal influence used in social science.

2.2.6 1965 Hubbell — An input-output approach to clique identification

Albeit he was working on clique identification, Hubbel in 1965 [19] suggested a new index with the explicit intention to keep in consideration the strength of the choice which has been made. His main contribution to the field was the introduction of the initial status of the system, also called boundary condition. This boundary condition is translated into an initial preference vector $\boldsymbol{v}$ in our recursive formulation.

$$\boldsymbol{r} = \boldsymbol{v} + \boldsymbol{r}M$$

The same formula can be written also as:

$$\boldsymbol{r} = \boldsymbol{v}(I - M)^{-1} = \boldsymbol{v}\sum_{k=0}^{\infty} M^k$$

Hubbell was the first to notice that the recursive (Seely) and pathwise (Katz) formulation of spectral ranking are the same thing.

2.2.7 1966 Sabidussi — The centrality index of a graph

Sabidussi [32] considered only finite undirected graphs. In 1966 he thought that one graph is more centralized than another is meaningful only relatively to a given point centrality function σ. This was the first attempt to axiomatize graph centrality. Sabidussi's opinion was that any claim that one index is better than another is meaningless unless one has defined what one means by saying that one graph is more centralized than another. So far, all authors who have dealt with this matter have been content to give a few examples and to appeal directly to intuition. As he said, *there is no doubt that an appeal to intuition must be made at some level, but it has to be made in a mathematically precise fashion.* He tried thereafter to give such a precise definition, and then to test to what extent the known indices satisfy the requirements of that definition. No centralities he examined survived the test, like Bavelas' and Bauchamp index. Unhappily, the consequences of his choices led to the impossibility of comparison of graphs of different size. He also introduced the idea of studying the changes in centrality during the evolution of a graph (i.e., while adding an arcs). He wrote a requirement weaker than score monotonicity: he required that if y has maximum score in the network, then it should have maximum score also after the addition of an arc towards y.

Let's define

$$D_i(x) = \{y \mid d(x, y) = i\},$$

where $d(x, y)$ is the distance between x and y. Then, the point centrality of a vertex x of a graph is:

$$s(x) = \sum_{i=1}^{\infty} i|D_i(x)|$$

It can also be stated as:

$$s(x) = \sum_{x \in G} d(x, z).$$

2.2.8 1973 Nieminen — On the centrality in directed graphs

Nieminen [30] in 1973 proposed a function for comparing (geometric) centralities of nodes in directed graphs. His *negative neighborhood function* for a node x in a graph G is defined as:

$$N_G^-(x, t) =| \{y | d(y, x) \leq t\} |$$

He required that if the neighborhood function of x dominates strictly that of y, then the centrality of x must be strictly larger than that of y. He noted that Bavelas's centrality does not satisfies some other of his axioms.

2.2.9 1979 Freeman — Centrality in social networks: conceptual clarification

In 1979, Freeman developed further betweenness centrality, a concept introduced by Anthonisse in [1]. To give an practical insight on how powerful is the idea of underlying the algorithm, consider this example based on a seminal paper by Mark Granovetter called "The strength of weak ties". He shown that most job seekers (who participated in the study) found their ultimate employment through a weak tie, that is, through an acquaintance, rather than a strong tie or a close friend. This can give us the idea that the information residing at either end of two strong ties is nearly identical, because those two peers exchange — somehow — more information. Thus, you and your friends are mostly aware of the same job opportunities. In contrast, weak ties synchronize their information more rarely, and thus serve as greater sources of novel information when such information is needed. That is, your acquaintances are more likely to know about jobs you have not already considered. Betweenness centrality is further formally explained in the next chapter.

2.2.10 1998 Kleinberg — Authoritative sources in a hyperlinked environment

In 1998 came HITS, developed by Kleinberg as a general case of Bonacich [7] centrality the same year of the more famous PageRank. Specifically addressed to the WWW, the paper contains a methodology for extracting relevant pages from a search query. There are two conceptual difficulties:

- For specific queries, we have a *Scarcity Problem*: there are very few pages that contain the required information, and it is often difficult to determine the identity of these pages.
- For broad-topic queries we have the *Abundance Problem*: The number of pages that could reasonably be returned as relevant is too large.

The model proposed is based on the relationship between pages (called authorities) for a topic and those pages that link many authorities, (called hubs). Given a relevant search query σ, they restrict the analysis on a subset Q_σ of all the pages containing the query string. Ideally, since Q_σ is still too broad, we would restrict to a sub-subset S_σ that:

1. is relatively small
2. is rich in relevant pages
3. contains most of the strongest authorities

For a parameter t (≈ 200), the top highest ranked pages out of the query σ are selected, let's call this set R_σ. Note that $R_\sigma \subset Q_\sigma$. Now we can use R_σ to produce the set of pages S_σ growing R_σ to include any page pointed to by a page in R_σ and vice-versa: any page that points to a page in R_σ. As next step, they proposed an algorithm to calculate in an iterative fashion the eigenvectors of the symmetric matrices AA^T and A^TA, where A is a matrix obtained following the previous steps.

2.2.11 1999 Page and Brin — Bringing order to the web

In 1999, the famous paper of Sergey Brin and Larry Page [31] were published. Around that years, centrality measures became interesting not only in social studies but also in computer science. Indeed, the first search engines were born, and most of them could not solve the problem of spam: external manipulation of search results using ad-hoc pages and links in order to change the importance of some pages in the web. PageRank was suggested as a solution to overcome the problem of spamming nodes (spam pages). A simple description of PageRank algorithm is the determination of the limiting distribution of a random walk on the Web-Grah. The resulting vector is thus a probability distribution over the web pages. More technical details will be given later in the next chapter. As others did, the authors tried to apply their centrality index to the graph of scientific journals, proven PageRank to be more effective than previous works in approximation of importance, as for instance indegree centrality.

2.2.12 2012 Chien — Link evolution: analysis and algorithms

The work of Chien, Dwork, Kumar, Simon and Sivakumar [9] provided the same definition of score monotonicity and rank monotonicity of [6]. They focused on the particular case of strongly connected graphs and the PageRank algorithm. This huge work suggest an iterative algorithm for calculating the PageRank of a page in incremental fashion. This is academically interesting and of high practical relevance because it allows to avoid to recompute the PageRank each time a new node or edge is added to the graph or a new link is found. They also proved PageRank to satisfy score monotonicity and rank monotonicity, albeit it has been proven only for strongly connected graph, and using the language of Markov chains.

PageRank's rank monotonicity

The question they addressed is the following: how does the stationary probability π_j of a state j changes if a transition probability p_{ij} of a regular Markov chain P is increased (and other transitions are decreased) to obtain a new regular Markov chain $\widetilde{P}$. They modeled changes in link structure of the WebGraph corresponding to addition of an error matrix E to P, such that $\bar{P} = P + E$ denotes the new Markov chain. Let $\widetilde{\pi}$ denote the stationary distribution of $\widetilde{P}$. Consider the special case in which E is all zeroes except in one row, say i, and in which only one entry is positive, say $e_{ij} > 0$. This corresponds to increasing the probability of a transition from state i to state j. Proving that Page Rank satisfies rank monotonicity means that if a link is added to page j, then j's *relative* popularity cannot decrease. In fact, the rank ordering of page j will outperform the score of every page it outperformed before the link was added.

They showed that for any states j and k of P , if $\pi_j \geq \pi_k$ and we increase the transition probability into j from some state i, then $\widetilde{\pi}_j \geq widetilde \pi_k$. If we increase the transition probability from state i to state j, then we must decrease the transition probability from i to some other states, w_0, w_1, $\ldots$, w_{k-1},w_y, since the sum of the probabilities must be 1.

They argued that there is no state k whose stationary probability starts out below that of j and whose final stationary probability exceeds that of j. Specifically they show how to express score of a node in terms of flow, and they showed how adding a node toward j, the flow changes in a way that the score of j can only increase. For any two states a, b, we define flow(a, b) as the probability that a random walk starting at a reaches b before returning to a.

Rank monotonicity Rank monotonicity is expressed as a property of the stationary distribution of regular Markov chains, and is captured in the following theorem.

Theorem. *Let P be a finite-state regular Markov chain. Let i, j be arbitrary states of P (not necessarily distinct), and let $w_1 ... w_y \neq j$ be arbitrary. Let $\tilde{P}$ be the Markov chain obtained by decreasing $p_{iw_1} \dots p_{iw_y}$ by amounts δ_0, δ_1, $\dots$, δ_{y-1},δ_y and increasing p_{ij} by $\sum_{l=1}^{y} \lambda_l$. Assume $\tilde{P}$ is regular. Let π and $\tilde{\pi}$ denote the stationary distribution of P and $\tilde{P}$, respectively. Then, for all states $k \neq j$, if $\pi_j > \pi_k$, then $\pi_j > \pi_k$. Moreover if $\pi_j = \pi_k \Rightarrow \tilde{\pi}_j \geq \tilde{\pi}_k$.*

Without stating the complete proof, here are some key passages:

This lemma says that there is a way to express the score of a node knowing only the score of another node and the reciprocal flows. Suppose you know the score of a node π_a and the reciprocal flow ($flow(a, b), flow(b, a)$), we can calculate the score of π_b. They used thus this lemma later in the proof. Indeed, the corollary state that modifying p, the score of the node receiving the arc can only increase.

Lemma. $\forall a, b \quad \pi_a = \pi_b * \frac{flow(b,a)}{flow(a,b)}$

There is a trivial corollary to this lemma, which gives a sufficient and necessary condition on flows, to show that the rank of a node π_j is bigger than π_k.

Corollary. $\forall a, b, \quad a \neq b, \pi_a > \pi_b$ *if and only if* $flow(a, b) < flow(b, a)$

The last part of their proof needs this last lemma, which state that if we add an arc towards j, the flow from j to other nodes can only increase or remain the same, and the flow from other nodes j can only decrease or remain the same.

Lemma. *For all k, $flow_P(j, k) \geq flow_{\tilde{P}}(j, k)$ and $flow_P(k, j) \leq flow_{\tilde{P}}(k, j)$*

This is the gist of the proof, and with some algebra, the complete proof follows smoothly.

2.2.13 2009 Rochat — Closeness centrality extended to unconnected graphs: the harmonic centrality index

Rochat in 2009 was the first to suggest a new centrality, called harmonic centrality (better defined in Chapter 3). The harmonic centrality of a node is defined as the sum of the inverted distances from the other nodes to the node considered. He compared the newly defined harmonic centrality with the similar closeness centrality. The comparison between different ranking has been made using the Spearman's correlation on the two ranking of randomly generated graphs. The advantage of switching the summation out of the denominator is avoiding cases where an infinite distance causes the fraction to be zero. Differences in ranking of specifically crafted graphs are discarded as unlikely to appear in social network analysis. In his work is emphasized that harmonic centrality, compared to closeness centrality, comes without any increase of computational complexity.

2.2.14 2009 Garg — Axiomatic Foundations of Centrality in Networks

In 2009, Garg wrote a different axiomatic approach to centrality measures in [17]. This is the last and yet more recent article that gave a strong axiomatic characterization of even three different centrality: degree, closeness and decay centrality. He focused only on undirected, unweighted and simple networks, but his work can be extended flawlessly to any kind of graph. According to his work, an axiomatization provide a better understanding of the mathematical structure of each centrality measure, and can add empirical evidence that centrality measures are correlated with each other. Moreover, degree, closeness and decay are established to be part of the same family of measure (i.e., they share a set of very similar axioms). A great part of his work use breadth-first search to segment nodes in the graphs in different layers. This approach is equivalent of saying that the centrality of a node i in G is the same as it would be in any BFS tree in G rooted at i. Any link in the graph that does not lie on a shortest path between i and some j can be discarded as far as i's centrality is concerned. Let $M(i, g)$ be the score of the node i given the arcs g of a graph G.

Degree centrality

This can be completely characterized by the following axioms:

1. **Isolation**: $N_g(i) = 0 \Rightarrow M(i, g) = 0$, with $N_g()$ being the neighborhood function of node i in a graph g. A node disconnected from the rest of the network should be assigned zero centrality.

2. **Symmetry**. If $i, j \in N$ are exchanged by an automorphism of the graph G then $M(i, g) = M(j, g)$.

3. **Additivity** Let K be the number of partition of the graph. For any $G_B(i) \in T_i$, and the corresponding subnetwork (partition of the graph), $P_{g_{B(i)}} =$

$g_1(i), \ldots, g_K(i)$,

- $M(i, g_B(i)) = \sum_{k=1}^{K} M(i, g_k(i))$.
- $M(i, gk(i)) = \sum_{jk \in T_k(i)} f(j^k)$ for some function $f()$.

We want that each nodes i derives its score independently and additively from each level. Moreover, with the second equation we want the measure depending on some generic function $f()$.

4. **Star Maximization**. Let G^* be a star graph of n nodes, with $n-1$ edges connecting $n-1$ nodes to a central node $c*$. This axiom says that in a star graph, the score of the nodes can be at most as high as c^*.

In this paper there is an equivalent definition of harmonic centrality under the name of closeness centrality. BFS is useful in this context because it contains exactly one geodesic between the root and every other node in G. The axioms used to characterize closeness centrality are similar to degree centrality.

2.2.15 2012 Kitti — Axioms for centrality scoring with principal eigenvectors

The axioms proposed in 2012 by Mitri Kitti in [22] are again a work deeply bounded with linear algebra and eigenvectors. Six axioms are there suggested, and thus some centrality measure are analyzed under the light of these axioms. The theoretical foundation behind his work are similar to Wei's work. In his work he modeled the process of decision-making, something very similar to a tournament. He does not refer to communication in a group, nor leadership measures. Thereafter, using Perron-Frobenius theorem, he state that the function we use to calculate eigenvectors satisfy this properties (and it is unique). He also suggested a "zero-sum" function based on the difference between right and left principal eigenvectors. This measure gives the net flow of walks to each node.

Axiomatization of principal eigenvector

In Section 3 there is the axiomatization of the principal eigenvector function. The purpose is to score the nodes of a directed graph, with the adjacency matrix assumed to be irreducible and nonnegative. Let $\mathcal{G}_n$ be the set of graphs of n nodes. Given a scoring function $F : \mathcal{G}_n \to R^n$ (for each n), Kitti state the following axioms:

- $F(A) \gg 0$.
  We want all scores to be positive. The reason behind this is that, since we

are dealing with an irreducible matrix A all nodes are connected to at least one other node. Hence, it makes sense that all nodes get positive scores.

- $F(\beta A) = F(A) \quad \forall \beta > 0$.
  We require that our function F is invariant if we multiply all the weights of the edges of the graph by the same constant. This property holds for all commonly used scoring methods.

- $F(A + \beta I) = F(A) \quad \forall \beta \geq 0$.
  With this requirement we want the score to be the same if we add to each node a self loop with equal weight for each node. This does not mean that auto-loops does not matter. What matter are the relative strengths of self-connection, not their absolute values.

- $F(A^k) = F(A) \quad \forall k \in \mathbb{N}$.
  The fourth axiom is the more subtle to understand. It requires that F is invariant to the strength of connections corresponding to any length of walks between the nodes. For example, in a voting context, a walk of length k between two nodes can be interpreted as an alternative being indirectly preferred to another.

- If $F(A) = F(B)$, then $AF(A) = \beta BF(B)$ for some $\beta > 0$.
  In the fifth axiom, we require that multiplying the scores of two matrices A and B with these matrices. This gives us $\boldsymbol{v}_1 = AF(A)$ and $\boldsymbol{v}_2 = BF(B)$. The axiom says that if the scores of two matrices are the same, the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are scalar multiples of each other.

- $\sum_i F_i(A) = 1$.
  The last axiom is the normalization of scores. Note that the normalization of scores is a different operation than the normalization of the original data, e.g., by forming a stochastic matrix from the initial adjacency matrix describing the binary relations of the nodes.

Thereafter, there is the main result of the paper: the $R()$ scoring function (principal right eigenvector) is the only scoring function that satisfies all of the six previous axiom.

Axiomatization of HITS

Fifth chapter is about the axiomatization of the HITS algorithm. This centrality assign a score of “authoritativeness” of a web page, based on the relation between web pages and “hubs”, pages that are good sources of link to other pages. HITS is thus a centrality measure meant to capture the authoritativeness of a web page.

The HITS method can be seen as a special case of the method introduced by Bonacich in [7] for assessing individual and group rankings simultaneously. Let us assume that there are m groups and n individuals who are the possible members of the groups. The $m \times n$ matrix A contains the weights associated with each individual in each group. In the Bonacich's algorithm the scores of individuals are given by $R(A^T A)$ and the scores of groups are given by $R(AA^T)$. Let the matrix M be a generic $m \times n$ matrix, and N be a generic $k \times n$ matrix. We define $M \boxplus N$ as:

$$\begin{pmatrix} M \\ N \end{pmatrix}$$

- (A1) $F(A \boxplus \beta P) = F(A)$ for any $\beta \geq 0$ and $n \times n$ permutation matrix P,
- (A2) $F(A^{2k}) = F(A) \quad \forall k = 1, 2, \ldots$
- (A3) Let A and B have the same dimensions. If $F(A) = F(B)$ then $AF(A) = \beta BF(B)$ for some $\beta > 0$.

The first axiom means that the scores are invariant for adding dummy groups, that is, adding groups corresponding to each individual with only that individual as the member. This means that the scores cannot be manipulated by adding identical groups for each individual. The second axiom says that we get the same score when we form an adjacency matrix $A^T A$ to individuals and consider the resulting walks of any fixed length. The product $A^T A$ means that the strength of the connection between two individuals is set equal to the sum of products of the weights in which the two individuals belong to each group. The last axiom says that the difference is that A and B should have the same dimensions. The interpretation of is that if the scores of individuals corresponding to two matrices with same dimensions are the same, then we get the same scores for the groups if we compute them by summing the weight-score products of all individuals belonging to each group and normalize the resulting vectors. This is indeed how the scores of the groups and individuals are related in the Bonacich-Kleinberg method.

The main result of Section 5 of [22] is stating that Bonacich-Kleinberg scoring function is the unique scoring function that satisfies the above axioms together. This scoring function has some additional properties, which are worth mentioning. First, the Bonacich-Kleinberg function is anonymous, in the sense that the order of individuals, (i.e., permutation of the rows) only change the order of the scores. It is also*symmetric* in the sense that if a permutation of rows keeps the matrix the same, the scores remain the same. In other words, symmetric individuals have the same scores.

2.3 Boldi 2012 — Axioms for Centrality

Last but not least, in 2012, Boldi and Vigna wrote "Axioms for Centrality" [6] on which part of this work is based upon. They suggested three property that a good centrality measure should comply with to be considered valid. This measure was proven to be the only one to satisfy all of the axioms. This status has been confirmed in this work. Axioms for Centrality are not meant to define what centrality is — like Shannon did with Entropy, but they represent a framework of *necessary* condition that an algorithm must satisfy to be considered a valid measure of centrality. Expressing some necessary conditions for centrality measures allow us to find which algorithms tend to reward nodes that are not central in the intuitive way we have defined. This approach is only meant to gain more knowledge on how algorithms behave, not for defining what centrality is. Before Axioms for Centrality was more difficult to compare centrality measure for various reasons. First, datasets were small (in social sciences datasets have a few dozen of nodes), and was not possible to extract sensible conclusion from them. Second, different algorithms were written trying to capture different aspect of what centrality is. As we know, the choice of the axioms is strongly subjective and arbitrary. However, we can use this to our advantage: the more axiom we have, the more we can understand the behavior of a centrality measure in a precise fashion. What is reasonable to expect from our set of axioms? What kind of axioms do we want? Assuming to work with centrality measure that are invariant by isomorphism (the score of the node must depend only on the structure of the graph) the authors required that:

- The axioms must be very clear
- The axioms must be exactly evaluable for the algorithms we consider (or most of them)
- The axioms should be formulated avoiding the trap of small or finite counterexample

Since the axiom must be exactly evaluable, (i.e., in algebraic closed form), the authors have chosen to express them on "simple" graph, to simplify calculations. More specifically, they chosen a set of vertex-transitive graphs: k-cliques and directed p-cycles. These of course are not the only possible kind of vertex transitive graph, but they exist for every k and p (this might not happen for more complicated structures, e.g., a cubic graph). In the article, eleven centrality measure has been studied under *changes of size, changes of (local) density, and arc addition.* The ratio between this choices is that we expect that nodes belonging to larger groups — when every other parameter is fixed — should be more important (size axiom), that nodes with a denser neighborhood (i.e., having more near

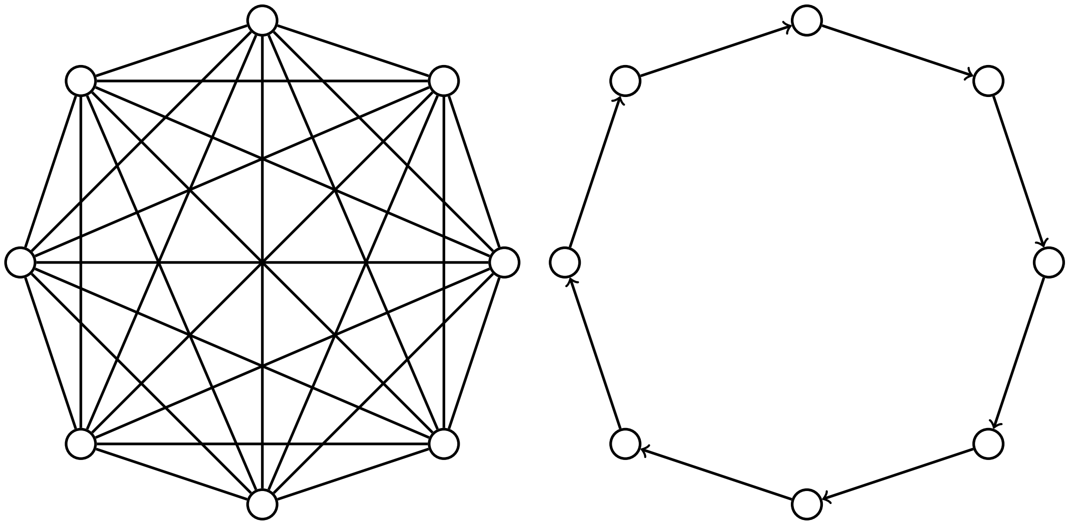


Figure 2.1: Graph for the Size Axiom: $S_{k,p}$

surrounding nodes)— when every other parameter is fixed — should also be more important (density axiom), and we also expect that adding an arc should increase the importance of the target node (score-monotonicity axiom).

The aim of this thesis is to study the behavior of some algorithms under the rank-monotonicity axiom, presented further in this work.

2.3.1 Size axiom

The first axioms is meant to capture the sensitivity of a measure to the change of size of a graph. Start by considering a disconnected graph composed by a k-clique and a p-cycle. Since all of our algorithms are invariant by isomorphism, all the nodes in the clique will have the same score, and we can argue the same for the nodes in the cycle.

Definition 2.3.1 (Size axiom)**.** *Consider the graph $S_{k,p}$ made by a k-clique and a directed p-cycle. A centrality measure satisfies the size axiom if for every k there is a P_k such that for all $p \geq P_k$ in $S_{k,p}$ the centrality of a node of the p-cycle is strictly larger than the centrality of a node of the k-clique, and if for every p there is a K_p such that for all $k \geq K_p$ in $S_{k,p}$ the centrality of a node of the k-clique is strictly larger than the centrality of a node of the p-cycle.*

What this axiom is saying, is that when the size of the cycle goes to infinity, we expect the nodes of the cycle to be more important than the nodes of the clique, since the clique does not change its size. We want a number P_k to exist such as for every clique with k nodes, all the cycle with more then P_k nodes have higher centrality that the nodes in the clique.

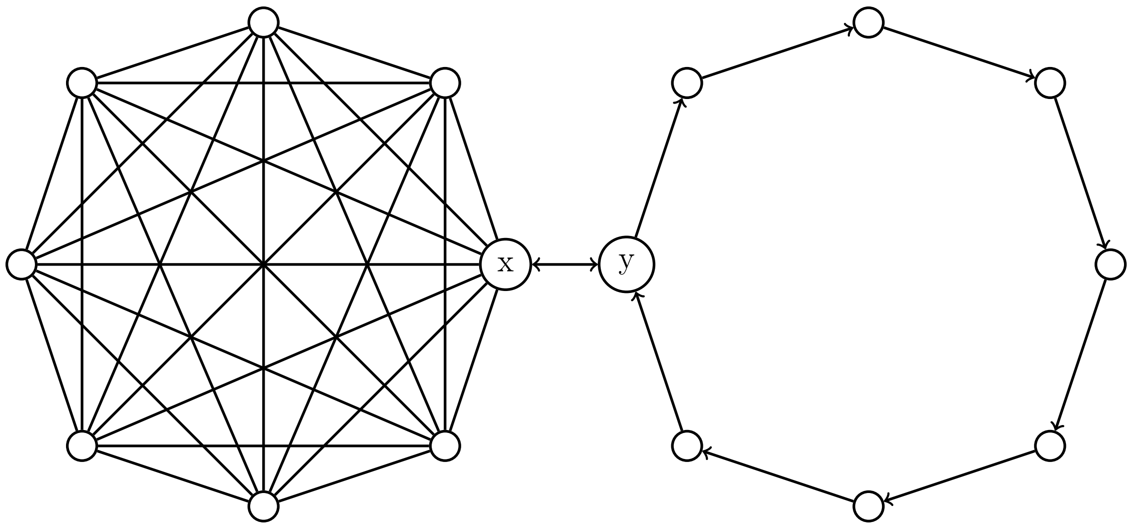


Figure 2.2: Graph for Density Axiom: $D_{k,p}$

It is clear that if $k \geq p$, nodes of the p-cycle would be have a lower score then the nodes of the k-clique, not only because the clique is more dense than the cycle, but also because it is formed by a larger number of elements. Since one might create some centrality measures that satisfy the size axiom for p and not for k, stating the axiom both in term of $k > P_k$ and $p > K_p$ gave the authors a finer granularity and avoids pathological cases. For instance, compare betweenness and indegree. With betweenness the centrality of each element of the clique will be always 0, no matter how large the clique is, and thus nodes in the clique can never have score higher than nodes in the cycle. On the other side, with indegree, no matter how big the cycle is, the centrality of an element of the cycle is always 1.

2.3.2 Density axiom

Defining what the behavior of an algorithm under the effect of a local changes of density is more complex than assessing changes of size.

It turns out that a good idea for capturing the change of density is a graph $D_{k,p}$ composed initially by a k-cycle and a p-cycle ($p, k \geq 3$) connected by a bidirectional arc (we call bridge) $x \leftrightarrow y$, where x is a node of the clique and y is a node of the cycle.

If we add arcs (thus increasing density) in the k-cycle (eventually turning the cycle into a clique) and leaving all the other parameter fixed, we expect the score of x to increase, because it is part of a denser community. This expectation is enclosed into the following axiom:

Definition 2.3.2 (Density Axiom). *Consider the graph $D_{k,p}$ made by a k-clique* *and a p-cycle ($p, k \geq 3$) connected by a bidirectional bridge $x \leftrightarrow y$, where x is a node of the clique and y is a node of the cycle. A centrality measure satisfies the density axiom if for $k = p$ the centrality of x is strictly larger than the centrality of y.*

When we have a k-clique and a p-cycle and $k = p$, we expect x, the node of the clique-part of the graph, to have an higher score than the nodes of the cycle. It's when $k \neq p$ that things start to become interesting. Thanks to the *watershed* we can understand different behaviors of the various centrality measures. Watershed is the value of k (expressed as a function of p) at which the centrality of x becomes larger than the centrality of y (if any). The watershed could give some insight as to how badly a measure can miss satisfying the density axiom. All the centrality measures studied in the paper that satisfy the density axiom have no watershed. For instance, with betweenness, one needs a clique whose size is quadratic in the size of the cycle before the node of the clique on the bridge becomes more important than the one on the cycle (compare this with closeness, where $k = p+1$ is sufficient).

2.3.3 Score-monotonicity axiom

Definition 2.3.3 (Score-Monotonicity Axiom)**.** *A centrality measure satisfies the score-monotonicity axiom if for every graph G and every pair of nodes x, y such that $x \not\to y$, when we add $x \to y$ to G the centrality of y increases.*

With this axiom, we want to express the simple intuition that the score of the node receiving a new arc should increase. Please note that this axiom does not say anything about the behavior of the scores of other nodes, nor about the effects of the addition of the arc on the final rank. The need to assess the changes in rank of the various centrality measure respect to the arc addition is captured in the fourth axiom, studied in this work.

2.3.4 Previous results

All their results are summarized in Table 2.1 where each column answer the question "does a given centrality measure satisfy the axiom?". We can notice that:

- Only harmonic centrality satisfy all three axioms.
- All spectral centrality measures are sensitive to density.
- Row-normalized spectral centrality measures (Seeley's index, PageRank and SALSA) are insensitive to size, whereas the remaining ones are only sensitive to the increase of k.

Centrality	Size	Density	Score monotonicity
Degree	only k	yes	yes
Harmonic	yes	yes	yes
Closeness	no	no	no
Lin	only k	no	no
Betweenness	only p	no	no
Dominant	only k	yes	no
Seeley	no	yes	no
Katz	only k	yes	no
PageRank	no	yes	yes
HITS	only k	yes	no
SALSA	no	yes	no

Table 2.1: Results summarized for the first three axioms

- Betweenness centrality is the only centrality sensitive to changes of p.
- All non-attenuated spectral measures are also non-monotone.
- All centralities satisfying the density axiom have no watershed: the axiom is satisfied for all $p, k \geq 3$.

Chapter 3

Centrality measures

What I cannot create I do not understand.

—Last lecture — Richard Feynman

In this chapter we review the most representative centrality measures in literature. We can cluster all the rankings into three main categories:

- Geometric measures
- Spectral measures
- Path-based measures

Geometric centralities are those measures assuming that importance is only function of distances. More precisely, a geometric centrality depends only on how many nodes exist at every distance. These are some of the oldest measures defined in the literature. Since each node is weighted only according to its distance with other nodes, the information of all the possible path between two nodes is discharged and not taken into account. In other words, with geometric centralities, the rank of node x is only function of the vector of distances.

$$D_x[i] = |\{k \mid d(k,x) = i\}|$$

Spectral measures rely on the isomorphism between directed graphs and some matrices derived from the graph. The idea is that the link structure of a graph can be translated into a matrix that indicates if there is a link between two nodes of the graph (adjacency matrix). The score is usually based on eigenvectors of this matrix. As is known, the normalized principal right eigenvector represent the asymptotic proportion of directed walks in the graph that being form each node of the graph cite. The left eigenvector on the other hand, tells the same information on walks terminating to each node [22]. Other centrality measures may be derived using normalizing or other techniques. Usually, when working with spectral measures, we assume working on (strongly) connected graphs; Otherwise the principal eigenvector may not be unique.

Path-based measures are devised for taking into account not only the shortest paths, but all the possible paths coming to a node while calculating its centrality. In this work we consider only one measure of such kind (betweenness), but is important to remark a concept very useful when dealing with spectral measures. It is well established that spectral measures can be though as path-based measures, thanks to the combinatorial interpretation of $\mathbf{1}A^k$. This vector associate at each node the number of paths of length k coming into the node. More specifically, the matrix A^k has as the component a_{ij} the number of paths of length k from i to j.

3.1 Indegree

This is the most intuitive centrality measure. The centrality of a node x is $d^-(x)$, simply the number of incoming arcs. It can be thought as a geometric measure, where you just count the nodes at distance one. Of course, the drawback of this simplicity relies in the disadvantages we have using this measure: In real-world application it is not sensible to use this measure in search engines due to the fact that it is easy to alter the score of web pages (i.e., spam). This centrality measure has an easy interpretation in the real world: it represent an election between n peers, where the arc from x to y stands for "x voted for y".

3.2 Closeness centrality

Closeness centrality, introduced by Bavelas in [3], define the score of a node x as the reciprocal of the sum of all the distances from the other nodes to x:

$$C(x) = \frac{1}{\sum_y d(y,x)}$$

The underlying idea behind this formula is that nodes with larger sum of distances are peripheral. Thus, taking the reciprocal, nodes with smaller denominator gain higher scores compared to nodes with bigger denominator. To make sense of this definition, we restrict its application only on strongly connected directed graphs and to undirected connected graphs, otherwise some of the distances in the graph will be ∞, resulting in score of 0 for all nodes which cannot be reached by the entire graph. We can define another flavor of closeness — that we call patched — more suitable for not strongly connected graphs.

$$C(x) = \frac{1}{\sum_{d(y,x)<\infty} d(y,x)}$$

This patched definition comes with a cost: nodes with small coreachable set will have a biased large centrality, and thus very distant nodes can bias the score.

3.3 Lin's centrality

This is a sort of double-patched version of closeness centrality which tries to address the bias introduces by nodes with small co-reachable set. The centrality of a node x with non-empty co-reachable set is:

$$L(x) = \frac{|\{y|d(y,x) < \infty\}|^2}{\sum_{d(y,x)<\infty} d(y,x)}$$

Nodes with an empty co-reachable set have centrality 1 by definition.

The rationale behind this definition is the following: first, we consider closeness not the inverse of a sum of distances, but rather the inverse of the average distance, which entails a first multiplication by the number of co-reachable nodes. This change normalizes closeness across the graph.

Now, however, we want nodes with a larger co-reachable set to be more important, given that the average distance is the same, so we multiply again by the number of co-reachable nodes.

3.4 Harmonic centrality

We define the harmonic mean as the value that substituted to the original $x_1..x_n$ values, leaves the sum of reciprocal unchanged. This is equivalent to say that harmonic mean is defined as:

$$H_{avg}(x_0, x_1, ..., x_n) = \frac{n}{\sum_{i=0}^{n} \frac{1}{x_i}}.$$

Harmonic centrality is the reciprocal of the denormalized harmonic mean, and is thus defined as:

$$H(x) = \sum_{y \neq x} \frac{1}{d(y,x)} = \sum_{d(y,x)<\infty, y \neq x} \frac{1}{d(y,x)}$$

This apparently simple trick (moving out the summation out of the denominator), solves the previous problem with closeness centrality: the presence of pairs of unreachable nodes. The idea of using harmonic mean in graph theory came at first by Marchiori and Latora in [27], who gave a new definition for the notion of "average shortest path". They suggested to replace the average distance with the harmonic mean of all distances. Indeed, in case a large number of pairs of nodes are not reachable, the average of finite, distances can be misleading: a graph might have a very low average distance while it is almost completely disconnected. For instance, take a look at Figure 3.1, were we have not $n(n-1)$ distances, but rather $2n$ distances to consider. This graph is almost completely disconnected, and nevertheless the average distance is $1/2$, which is very low: (there are 2 distances for each node, and one of them it's zero, the other is one.)

The harmonic mean of distances of a perfect matching is:

$$\frac{n(n-1)}{\sum_{x \in G} \sum_{y \in G, y \neq x} \frac{1}{d(y,x)}} = \frac{n(n-1)}{n} = (n-1)$$

.

That is: for every node there is exactly another node at a non-infinite, distance, and its distance is 1, so the sum of the inverse of all distances is n. This example shows us how the average distance is not a good indicator to measure how much a graph is connected or disconnected.

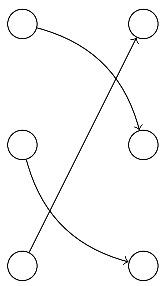


Figure 3.1: A perfect matching: a matching which matches all nodes of the graph. Every vertex of the graph is incident to exactly one vertex of the matching.

Essentially, harmonic centrality is a correction to closeness centrality that offers a way to take unreachable nodes into account by design without patching: infinite distances turn out to give zero contribution in the summation. It is also nice to notice, that each node gives a contribution that is independent of the dimension of the connected component. This allows us to compare nodes even if the graphs they belong to have different dimensions. This allows us to compare even graph of different dimension.

3.5 Betweenness

This centrality measures the probability that a random shortest path passes through a given node. σ_{yz} is the *number* of shortest paths going from y to z and $\sigma_{yz}(x)$ is the number of shortest paths going from y to z that pass through x, we define betweenness of x as:

$$B(x) = \sum_{y,z\neq x,\sigma_{yz}\neq 0} \frac{\sigma_{yz}(x)}{\sigma_{yz}}$$

If you think nodes as entities that communicate across arcs with other nodes, a node with large betweenness centrality has high impact on how items or information transit under the network, meaning that removing a node with high centrality modifies more the communication between nodes then a node with low centrality. This observation assume that the communication takes always the shortest path.

3.6 Left dominant eigenvector

This is the simplest spectral algorithm for centrality measure. We use the plain adjacency matrix and we iterate multiplication and normalization of a vector, starting from a uniform vector. We can think of left dominant eigenvector as a fixed point of iterated multiplication of a normalized vector (starting from giving each node the same score, that is, a uniform vector):

$$L(x) = \lim_{k\to\infty} \frac{\mathbf{1}A^k}{\|\mathbf{1}A^k\|}$$

3.7 Seeley's centrality

In 1949, Seeley "patched" left dominant eigenvector, applying l_1-normalization to the rows of the adjacency matrix representing the graph. If you think at the iterative algorithm for left eigenvector centrality, at each iteration each node gives to its successor all its own score. Seeley thought to redistribute equally the score of each node among successors, and this translates into a l_1-normalization of the row of the adjacency matrix. Since the normalized matrix is stochastic, it turns out that this approach is equivalent to studying the stationary distribution of the Markov chain defined by the natural random walk on the graph. Calculating Seeley's index on a symmetric graph it the same as finding the stationary distribution of a symmetric Markov chain and thus, Seeley's index collapses to a normalized version of indegree centrality. On not strongly connected graphs, the only nodes with non-zero score are those belonging to terminal components that are not formed by a single node of outdegree zero.

3.8 Katz's centrality

This algorithm was designed to compute the degree of influence of an actor in a social network [21]. This algorithm was one of the first to take into account not only the geodesic path coming into a node x, but also the total number of walks coming to a node. It can be expressed as:

$$\boldsymbol{k} = \mathbf{1}\sum_{i=0}^{\infty} \beta^i A^i$$

Or, thanks to the geometric series:

$$k = \mathbf{1}(I - \beta A)^{-1}$$

We notice however, that for the summation to be finite, the attenuation factor β must be smaller than $1/\lambda$, where λ is the dominant eigenvalue of the matrix A. A damping factor near zero reduces the influence of longer chains of votes, therefore a damping factor of zero results in a centrality that is equivalent to the degree centrality [20].When $\beta \to 1/\lambda$ we have the dominant eigenvector as a limiting case.

3.9 PageRank

The PageRank of a graph G is defined as this linear recurrence:

$$\boldsymbol{p} = \alpha \boldsymbol{p} \bar{A} + (I - \alpha)\boldsymbol{v}$$

where $\bar{A}$ is the l_1-row normalized adjacency matrix of the graph G, $\alpha \in [0, 1)$ is the *damping factor*, and $\boldsymbol{v}$ is the preference vector which is a probability distribution. This formula is not valid for general graphs: in presence of nodes with zero out degree (also known as dangling nodes), PageRank is not a probability distribution anymore, because of the presence of empty rows in the matrix. A solution is to patch the matrix A, replacing every null row with the preference vector itself. Replacing null rows by the preference vector is a strategy to virtually add arc from each dangling nodes to every other node in the graph. Since in case of dangling nodes the bigger eigenvalue could be smaller than one, another solution can be to perform a normalization of the eigenvector. PageRank can also be written as:

$$\boldsymbol{p} = (I - \alpha)\boldsymbol{v}(I - \alpha\bar{A})^{-1}$$

and, using the geometric series we can rewrite the previous equation into:

$$\boldsymbol{p} = (I - \alpha)\boldsymbol{v} \sum_{i=0}^{\infty} \alpha^i \bar{A}^i$$

An important observation we can make is that Katz's index and PageRank differ only by a constant factor and by the l_1 normalization applied to the matrix A. If the rows of A are patched, PageRank can also be interpreted as the stationary distribution of a Markov chain with the following transition matrix:

$$M = \alpha\bar{A} + (I - \alpha)\mathbf{1}^T\boldsymbol{v}$$

The $(I - \alpha)\mathbf{1}^T\boldsymbol{v}$ gives non zero probability for “random walker” in the node k to be in any other position in the graph, according to your preference vector. If the eigenvalue of $\bar{A}$ is 1 and $\alpha = 1$, this centrality measure collapse to Seeley's index. If the dominant eigenvalue is not unique (that is, the graph is not strongly connected) the limit of computation depends on the preference vector, i.e., where you place initially the random walkers).

3.10 SALSA

Given an l_1-normalized adjacency matrix of a graph G, this measure is as such defined. Starting with $a_0 = \mathbf{1}$ vector, one computes:

$$\boldsymbol{h}_{i+1} = \boldsymbol{a}_i \overline{A^T}$$
$$\boldsymbol{a}_{i+1} = \boldsymbol{h}_{i+1} \bar{A}$$

Luckily for us, SALSA centrality of a n-nodes graph can be calculated directly. First you have to compute the graph induced by the matrix $A^T A$ (which is symmetric). This graph has different component K. Two nodes x, y are adjacent in this graph if they have some common predecessor in the original graph. We call the component of a node x K_x. The score of each node can be expressed as:

$$S(x) = \frac{d^-(x)}{\sum_{k \in K_x} d^-(k)} \frac{|K_x|}{n}.$$

In other words, the SALSA score of a node is the ratio between its indegree and the sum of the indegrees of nodes in the same component, multiplied by the ratio between the component size and n.

Chapter 4

Rank monotonicity

Axioms appear so contrived
And science, the paragon of devious crafts
I struggle to assimilate this fact:
No wisdom brings solace.

—NO WISDOM BRINGS SOLACE — Sybreed (God Is An Automation)

4.1 Rank-monotonicity axiom

Adding an arc between two nodes can have multiple outcome: the rank of the node receiving the arc can increase, but since the addition of a node can modify the score of other nodes, also rank of other nodes can change. It is not surprising then, that after an axiom on the behavior of the score that we want to say something about the behavior of the rank, and as expected some attempts has been already made in literature. The first was Sabidussi [32], with a weak statements: he requires that if a node y has maximum score in the network, then it should have maximum score also after the addition of an arc towards y. Very recently, Brandes, Kosub and Nick on [8] (although in a German article), defined a centrality "radial" if the score of y is non-decreasing: the addition of an arc $x \to y$ does not decrease the rank of y, in the sense that when we add an arc towards y, nodes with a score smaller than or equal to y continue to have this property. It is weaker because it does not take into account the case when a node z with a lower score than y before the addition of the node, ends up having a score equal than that of y. Thus we will use the formulation of Chien in [9]. They stated that "when we add an arc towards y, nodes with a score smaller than y must continue to have a score smaller that of y, while nodes with a score equal to y must get a score that is smaller than or equal to that of y."

Definition 4.1.1 (Rank-monotonicity axiom). *A centrality measure satisfies the* rank-monotonicity axiom *if for every graph G and every pair of nodes x, y such that $x \not\to y$, when we add $x \to y$ to G the following happens:*

- *if the score of z was strictly smaller than the score of y, this remains true after adding $x \to y$;*
- *if the score of z was smaller than or equal to the score of y, this remains true after adding $x \to y$.*

We can go further, and state a *stricter* definition of monotonicity:

Definition 4.1.2 (Strict–rank-monotonicity axiom). *A centrality measure satisfies the* strict–rank-monotonicity axiom *if for every graph G and every pair of nodes x, y such that $x \not\to y$, when we add $x \to y$ to G the following happens:*

- *if the score of z was strictly smaller than the score of y, this remains true after adding $x \to y$;*
- *if the score of z was equal to the score of y, z becomes smaller than y.*

When we think about rank monotonicity, we should focus *not* on the behavior of the node y receiving the arc. Rank-monotonicity is a property about *nodes with* *a lower rank than* y, and their relationship with y. Of course, a centrality measure satisfying strict rank monotonicity satisfies rank monotonicity *a fortiori*.

We can try to use the fact that a centrality fails to satisfy score monotonicity leaving the score of the node receiving the arc unchanged, to prove how the same centrality fails to satisfy strict rank monotonicity. Moreover, if we find some way to decrease the score of a node (as in some counterexample to score-monotonicity in [6]) leaving the score of other nodes unchanged, we can use that as a counterexample to rank monotonicity as well. Note also that if a measure is not score monotone it can still be strict rank monotone. Imagine a centrality measure that decrease the score of all the nodes of the graph which are not receiving the arc. A graph with n nodes and 0 arcs will have uniform score of $n(n-1)/2$ for each node. A complete graph will have uniform score of 0. This measure is (never) score monotone, but still it's strict rank monotone.

One could be tempted to merge score and rank monotonicity in a single axioms, but avoiding this choice can give us finer granularity in the evaluation of the centrality measures. For instance, some centrality measures (specially based on spectral theory like Seeley and PageRank) tend to lump all the scores on dangling nodes or nodes belonging to strongly connected components, even if the node receiving the arc increase its score. Some examples of graphs measured with a centrality index that reward with more score dangling nodes than the node receiving the arc can be found in Chapter 5. Note that a sufficient condition for rank monotonicity is the node receiving the arc to gain more score than the other nodes. This is *just* a sufficient condition. As we will see, there are centrality measure that reward more nodes not receiving the arc, but still satisfy rank-monotonicity axioms. This property is captured in the following lemma:

Lemma 2 (Sufficiency Lemma)**.** *If for every graph G and every pair of node x, y, such that $x \not\to y$, when we add the arc $x \to y$ the node y has the biggest increase in score then the other nodes, then the centrality measure satisfy strict rank monotonicity.*

Chapter 5

Proofs and Counterexamples

For your information, I could be wrong
On the other hand, if I remember correctly
What you see is what you get or words to that effect
In any case, read between the lines, tell it like this
In other words, don't believe everything you read
Know what I mean?

—A CHAOTIC REALIZATION OF NOTHING YET MISUNDERSTOOD— Spastik Ink (Ink Compatible)

In this chapter we report all results about score and rank monotonicity of the centrality measures appearing in [6]. For score monotonicity, we provide some new theorems and counterexamples for the strongly connected case, which was not treated there. All results about rank monotonicity are new.

5.1 Indegree

The case of indegree is particularly simple, because the effect of adding an arc $x \to y$ is completely local, and consists of increasing the score of y by one. The following theorems are thus trivial:

Theorem 3. *Indegree satisfies strict rank monotonicity on all graphs.*

Proof. When we add an edge $x \to y$ the score of y increases by 1 and it is the only score modified. Therefore the rank can only increase, and thus indegree satisfies strict rank monotonicity. □

Corollary 3.1. *Indegree satisfies rank monotonicity on all graphs.*

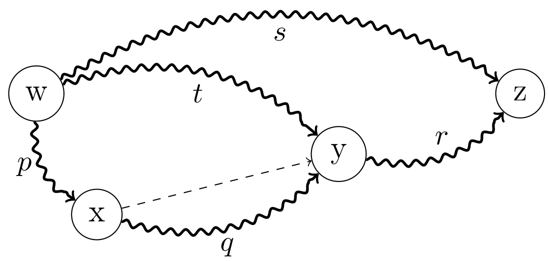
These results remains valid for strongly connected graphs.

5.2 Harmonic

We know from [6] that harmonic centrality satisfies score monotonicity. Here we are going to prove that it satisfies rank monotonicity, too.

Theorem 4. *Harmonic centrality satisfies strict rank monotonicity on all graphs.*

Proof. Consider a graph G, and the graph G' obtained by adding the arc $x \to y$ to G. In the following picture we represent the lengths (which might be infinite) of some of the shortests paths between a generic node w, the nodes x and y, and a generic node z.



We denote with $d(x, y)$ distances in G, and with $d'(x, y)$ distances in G'. To prove strict rank monotonicity, we assume first

$$\sum_{v \neq z} \frac{1}{d(v, z)} \leq \sum_{v \neq y} \frac{1}{d(v, y)}.$$

Note that $d(y, z) = d'(y, z)$ (a shortest path from y to z cannot pass through x and y) and $d'(z, y) \leq d(z, y)$. Thus,

$$\frac{1}{d'(y, z)} + \sum_{v \neq y,z} \frac{1}{d(v, z)} \leq \sum_{v \neq y,z} \frac{1}{d(v, y)} + \frac{1}{d'(z, y)}.$$

Consider the set $U = \{v \neq y, z \mid d(v, z) = d'(v, z)\}$. We have obviously

$$\frac{1}{d'(y, z)} + \sum_{v \in U} \frac{1}{d'(v, z)} + \sum_{v \notin U \cup \{y,z\}} \frac{1}{d(v, z)} \leq \sum_{v \notin U \cup \{y,z\}} \frac{1}{d(v, y)} + \sum_{v \in U} \frac{1}{d'(v, y)} + \frac{1}{d'(z, y)}.$$

We are left to analyze nodes $w \notin U \cup \{y, z\}$, for which $d'(w, z) < d(w, z)$, that is, $s > p + 1 + r$ (which implies $p, r < \infty$). Note that in this case $t > p + 1$, as otherwise $s > p + 1 + r \geq t + r$, contradicting the triangular inequality $s \leq t + r$. We conclude that

$$\frac{1}{d'(w, z)} - \frac{1}{d(w, z)} = \frac{1}{p + 1 + r} - \frac{1}{s} < \frac{1}{p + 1} - \frac{1}{t} = \frac{1}{d'(w, y)} - \frac{1}{d(w, y)},$$

since when $s, t < \infty$

$$\frac{1}{p + 1 + r} - \frac{1}{s} - \left(\frac{1}{p + 1} - \frac{1}{t}\right) = \frac{p + 1 - p - 1 - r}{(p + 1 + r)(p + 1)} + \frac{s - t}{st} < -\frac{r}{st} + \frac{r}{st} = 0,$$

and the remaining cases are trivial. Finally, since $d(x, z) \leq q + r$ we have either $d(x, z) = d'(x, z)$ or

$$\frac{1}{d'(x, z)} - \frac{1}{d(x, z)} \leq \frac{1}{r + 1} - \frac{1}{r + q} < 1 - \frac{1}{q} = \frac{1}{d'(x, y)} - \frac{1}{d(x, y)}.$$

Thus,

$$\sum_{v \neq z} \frac{1}{d'(v, z)} < \sum_{v \neq y} \frac{1}{d'(v, y)}.$$

□

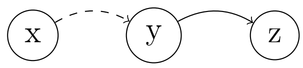
Corollary 4.1. *Harmonic centrality satisfies rank monotonicity on all graphs.*

5.3 Closeness

We know from [6] that closeness violates score monotonicity. Here, we prove that, however, this is no longer true on strongly connected graphs. Then, we show that similarly it violates rank monotonicity on general graphs, but it satisfies rank monotonicity on strongly connected graphs (albeit not strictly). For this paragraph $S(x)$ is the closeness centrality of the node x, with the subscript denoting the graphs we are referring to.

Theorem 5. *Closeness centrality does not satisfy rank monotonicity on all graphs.*

Proof. Consider the following graph G:



Here all node have score zero, either for lack on incoming arcs, or because of ∞ at the denominator. If we add the node $x \to y$ we have $S_{G'}(z) = \frac{1}{1+2}$ and $S_{G'}(y) = 0$. Node z has thus now a higher rank then y. □

As an immediate consequence, strict rank monotonicity is violated.

Corollary 5.1. *Closeness centrality does not satisfy strict rank monotonicity on all graphs.*

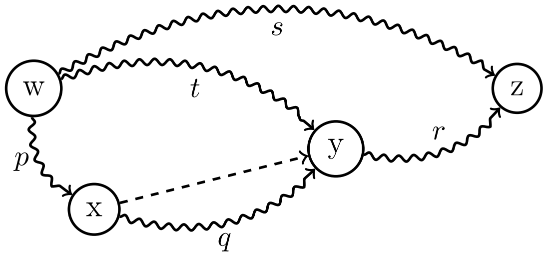
5.3.1 Strongly connected graphs

Rank Monotonicity

This is a neat example where a centrality gains some additional properties if we restrict it's usage on some particular cases. Indeed, if we apply closeness centrality only to strongly connected graphs, rank monotonicity and score monotonicity are satisfied.

Theorem 6. *Closeness centrality satisfies rank monotonicity on strongly connected graphs.*

Proof. Consider a strongly graph G, and the graph G' obtained by adding the arc $x \to y$ to G. In the following picture we represent the (finite) lengths of some of the shortest paths between a generic node w, the nodes x and y, and a generic node z.



We denote with $d(x,y)$ distances in G, and with $d'(x,y)$ distances in G'. To prove rank monotonicity, we assume first

$$\frac{1}{\sum_{v\neq z} d(v,z)} \leq \frac{1}{\sum_{v\neq y} d(v,y)}.$$

Which is equivalent of saying that:

$$\sum_{v\neq z} d(v,z) \geq \sum_{v\neq y} d(v,y).$$

Note that $d(y,z) = d'(y,z)$ (a shortest path from y to z cannot pass through x and y) and $d'(z,y) \leq d(z,y)$. Thus,

$$\sum_{v\neq y,z} d(v,z) + d'(y,z) \geq \sum_{v\neq y,z} d(v,y) + d'(z,y).$$

Consider the set $U = \{v \neq y, z \mid d(v,z) = d'(v,z)\}$. We have obviously

$$\sum_{v\notin U\cup\{z\}} d(v,z) + \sum_{v\in U} d'(v,z) + d'(y,z) \geq \sum_{v\notin U\cup\{y,z\}} d(v,y) + \sum_{v\in U} d'(v,y) + d'(z,y).$$

We are left to analyze nodes $w \notin U \cup \{y,z\}$, for which $d'(w,z) < d(w,z)$, that is, $s > p+1+r$. Note that in this case $t > p+1$, as otherwise $s > p+1+r \geq t+r$, contradicting the triangular inequality $s \leq t+r$.

We conclude that

$$d'(w,z) - d(w,z) = p+1+r-s \geq p+1-t = d'(w,y) - d(w,y),$$

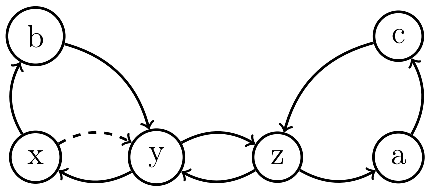
Thus,

$$\frac{1}{\sum_{v\neq z} d'(v,z)} \leq \frac{1}{\sum_{v\neq y} d'(v,y)}.$$

□

Theorem 7. *Closeness centrality does not satisfy strict rank monotonicity on strongly connected graphs.*

Proof. We need to exhibit a counterexample to strict rank monotonicity. Take the strongly connected graph in the following figure:



First, notice how the arcs $y \to x$ and $z \to a$ are only needed to make the graph strongly connected. We show how two nodes (y and z) are ex-aequo in the graph, and remains ex-aequo after we add the arc $x \to y$. We call G the original graph and G' the graph obtained adding $x \to y$ to G. Since y and z are symmetric in G, we can say that:

$$S_G(y) = \frac{1}{d(b,y) + d(x,y) + d(z,y) + d(c,y) + d(a,y)} = \frac{1}{1+2+1+2+3} = \frac{1}{9}$$

$$S_G(z) = \frac{1}{d(b,z) + d(x,z) + d(y,z) + d(c,z) + d(a,z)} = \frac{1}{2+3+1+1+2} = \frac{1}{9}.$$

Now we move to the scores of nodes y and z in G', and we see that the decrease in score has been uniform:

$$S_{G'}(y) = \frac{1}{1+1+1+2+3} = \frac{1}{8} = \frac{1}{2+2+1+1+2} = S_{G'}(z).$$

In other words, the only distance we have decreased equally is in the co-reachable set of both y and z, since they were ex-aequo (thanks to the symmetry of the graph G), they stay ex-aequo.

□

Score monotonicity

Since the graph is strongly connected, the distance between every node exists and it is finite, and thus adding an arc between two connected nodes can only decrease distances between them. If we focus on the denominator, we note that a sum can increase only in two cases: when we add more elements or when we increase the elements in the sum. Since adding an arc does not have any of these effects, but it decrease one or more distances, score monotonicity is satisfied.

Theorem 8. *Closeness centrality satisfies score monotonicity on strongly connected graphs.*

Proof. Let G be a graph such that x is not connected directly to y, and G' the same graph with the arc $x \to y$. Respectively we denote d_G and $d_{G'}$ the distance function. Our aim is to prove prove that:

$$S_{G'}(y) > S_G(y).$$

Where:

$$S_{G'}(y) = \frac{1}{\sum_{u \in G', u \neq x} d_{G'}(u, y) + 1}$$

and

$$S_G(y) = \frac{1}{\sum_{u \in G, u \neq x} d_G(u, y) + d_G(x, y)}.$$

Since $1 < d_G(x, y)$ in the formula above, we have to prove that:

$$\sum_{u \in G', u \neq x} d_{G'}(u, y) \leq \sum_{u \in G, u \neq x} d_G(u, y).$$

The previous inequality holds because adding a node we can only decrease already finite distances from other nodes to y with a path passing through x.

$\square$

5.4 Patched closeness

In this section we show how patching closeness centrality will not suffice to solve the problem with lack of connectivity of the graph, making closeness centrality behave counter-intuitively and thus violates rank monotonicity. For this paragraph $S(x)$ is the (patched) closeness centrality of the node x, with the subscript denoting the graphs we are referring to.

Theorem 9. *Patched closeness centrality does not satisfy rank monotonicity on all graphs.*

Proof. Take into consideration the following graph G:

Before adding the dashed arc, the centrality of nodes y and z are the both 1. After we added the arc $x \to y$, we see that $S_{G'}(y) = \frac{1}{1+1}$ and $S_{G'}(z) = \frac{1}{1}$, so $S_{G'}(z) \geq S_{G'}(y)$: an ex-aequo has gained a higher score than the node receiving the arc. $\square$

Corollary 9.1. *Patched closeness centrality does not satisfy strict rank monotonicity on all graphs.*

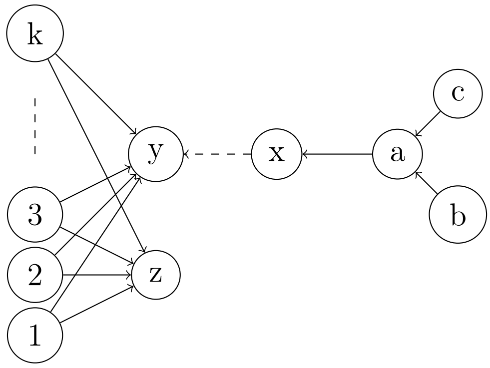


Figure 5.1: A counterexample showing that Lin's index fails to satisfy rank monotonicity

On strongly connected graphs patched closeness centrality is the same as original closeness, so no further results are necessary.

5.5 Lin's centrality

Since Lin's index is very similar to the patched version of closeness centrality, we expect it fails to satisfy rank monotonicity as well. Indeed we obtain a counterexample to rank monotonicity simply by modifying the counterexample for score monotonicity in [6].

Theorem 10. *Lin's index does not satisfy rank monotonicity on all graphs.*

Proof. We can extend the example in [6]. Consider the graph in Figure 5.1. We aim to prove that an ex-aequo of y get an higher rank than y, the node receiving the edge, thanks to the fact that the node y decrease its score.

Before we add the edge $x \to y$, for $k > 2$, the score of y is: $\frac{(k+1)^2}{k}$ and so is the score of z. After adding an the arc $x \to y$, the centrality of y becomes $(k+5)^2/(k+9)$, and the value of z remains unchanged. Therefore we find that for $k > 3$:

$$\frac{(k+5)^2}{k+9} < \frac{(k+1)^2}{k}.$$

For $k > 3$ the new value of y is smaller than z, thus rank monotonicity is violated. $\square$

Corollary 10.1. *Lin's index does not satisfy strict rank monotonicity on all graphs.*

If the graph is strongly connected, Lin's index is equivalent — up to a scaling factor — to closeness centrality. Therefore no further proofs are needed.

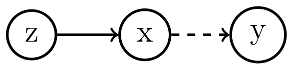
Corollary 10.2. *Lin's index satisfy score monotonicity on strongly connected graphs.*

5.6 Betweenness

We know already from [6] that betweenness centrality does not satisfies score monotonicity. A trivial counterexample shows that it violates rank monotonicity. For strongly connected graphs we show (with a single counterexample) that it fails to satisfies score monotonicity and rank monotonicity as well. The counterexample works also if the graph considered is symmetric.

Theorem 11. *Betweenness centrality does not satisfy rank monotonicity on all graphs.*

Proof. Given the following graph G



we can see that every node has zero score. When we add the arc $x \to y$ only the rank of x increase, thanks to the shortest path from $z \to y$, contrary to what we expect. So rank monotonicity is violated: there are no shortest path passing through y (so its score is 0) and the only node whose score has increased is x. □

Corollary 11.1. *Betweenness centrality does not satisfy strict rank monotonicity on all graphs.*

5.6.1 Strongly connected graphs

As we shall see, betweenness in fact violates rank monotonicity even on strongly connected graphs.

Lemma 12. *Shortest path between nodes of a clique does not contribute to the centrality of any node outside the clique*

That is, if you are counting the shortest path that pass through a node x that is not part of a clique κ, you can avoid to count the shortest path between elements of κ.

Theorem 13. *Betweenness centrality does not satisfy rank monotonicity on strongly connected graphs.*

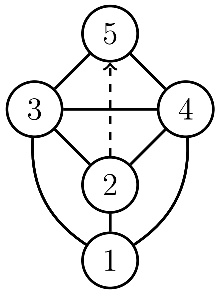


Figure 5.2: Strongly connected — Betweenness centrality, counterexample to score and rank monotonicity.

Node	Score in G	Score in G'
1	0	0
2	0	1/3
3	2	4/3
4	2	4/3
5	0	0

Table 5.1: Scores for betweenness centrality in graph G

Proof. Take the graph G in figure 5.2, and consider adding a directed arc $2 \to 5$, leading to graph G'. In G the centrality of each node is showed in the table 5.1. There are three ex-aequo nodes in G, $(1, 2, 5)$. If we add the arc $2 \to 5$ (and we call that graph G', we do not increase the number of shortest path passing through the node 5, therefore its score remains 0. Instead, there is a new shortest path from 1 to 5 passing through 2 and therefore the score of 2 increases. □

The counterexample in Figure 5.2 works in some sense also for undirected graphs, as adding an undirected edge connecting 2 and 5 increases the number of path passing through 2 even more. However, it is in practice delicate to state the rank monotonicity axiom for undirected graphs, as we expect modifications to the ranks of *two* nodes, so it is difficult to state which should be their final relationship. Certainly, starting from an ex-aequo and ending up with different scores does not seem to be a healthy behavior.

Corollary 13.1. *Betweenness centrality does not satisfy strict rank monotonicity on strongly connected graphs.*

Moreover, since in our example the node 5 receiving the arc does not change its score, we have a counterexample to score monotonicity.

Corollary 13.2. *Betweenness centrality does not satisfy score monotonicity on strongly connected graphs.*

5.7 SALSA

The counterexample to score monotonicity in [6] already suffice to prove rank monotonicity. We instead add some nodes to the graph, making it strongly connected. In this single counterexample we disprove rank monotonicity, and score monotonicity as well, even on strongly connected graphs. Before that, we state the following lemma.

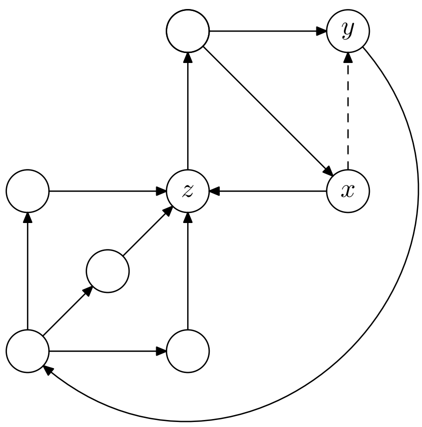
Lemma 14 (Singleton's lemma)**.** *Every node in a graph whose associated component in the symmetric graph $A^T A$ is a singleton, share the same score of $1/n$.*

Proof. If a node is a singleton the first ratio will be always 1 ($= \frac{d^-(x)}{d^-(x)}$), while the second will always be $1/n$. □

Thanks to the fact that the connected components in the graph $A^T A$ are created "locally" (to discover your neighborhood in $A^T A$ you just need to check whether your parents have other children), one can connect all the node in the original graph G making it strongly connected.

Theorem 15. *SALSA does not satisfy rank monotonicity on strongly connected graphs.*

Proof. Consider the following graph where we are going to add the arc $x \to y$ (dashed in the picture):



Before adding the arc, the indegree of y is 1, and its component consist only of another node (x). SALSA centrality of y is $(1/2)(2/8) = 1/8$. The indegree of z is 4, and its score is $(4/4)(1/8) = 1/8$. After adding an arc $x \to y$, the indegree of y becomes 2, but now its component consists of three nodes: $\{y, z, x\}$. The sum of indegrees within the component is $2 + 1 + 4 = 7$, hence the centrality of y goes down to $(2/7)(3/8) = 3/28$. On the other side, the score of the node z

grows. Before adding the arc, z has 4 incoming arcs, and its component (which is $\{z\}$ itself) has thus the same amount of incoming arcs. After adding the arc, the incoming arcs of z remains 4, the total amount of incoming arcs in the component goes to 7, and the number of element of the component is now 3. Thus we have a score of $(4/7)(3/8) = 3/14$. We can note that the node y, receiving an arc, has become smaller than one of its ex-aequo, and it has lost some score at the same time. □

Since the node y decrease its score when we add the arc $x \to y$, we have a counterexample to score monotonicity as well.

Corollary 15.1. *SALSA does not satisfy score monotonicity on strongly connected graphs.*

5.8 Damped spectral rankings

Rank monotonicity of PageRank was proved by Chien, Dwork, Kumar, Simon and Sivakumar [9]. Their proof works for a generic *regular* Markov chain: in the case of PageRank this condition is true, for instance, if the preference vector is strictly positive or if the graph is strongly connected.

Here we lift almost all hypothesis on the underlying graph and the preference vector. In fact, we prove rank monotonicity for certain updates of a generic *damped spectral ranking* [35], given by

$$\boldsymbol{r} = \boldsymbol{v} \sum_{n \geq 0} (\alpha M)^n = \boldsymbol{v}\bigl(1 - \alpha M\bigr)^{-1},$$

where M is a nonnegative matrix, $0 \leq \alpha < 1/\rho(M)$ is a *damping factor*, and $\boldsymbol{v}$ is a nonnegative *preference vector*.

In order to prove our result, we first need the following:

Lemma 16. *Let M be a nonnegative matrix, $0 \leq \alpha < 1/\rho(M)$ a damping factor and $\boldsymbol{v}$ a nonnegative preference vector. Let*

$$\boldsymbol{r} = \boldsymbol{v} \sum_{n \geq 0} (\alpha M)^n$$

be the associated damped spectral ranking and let $C = \bigl(1 - \alpha M\bigr)^{-1}$. Then, given y and z such that $c_{yz} > 0$ and letting $q = c_{yy}/c_{yz}$, we have $c_{wy} \leq q c_{wz}$ for all w. In particular, if $r_y \neq 0$

- *if $r_z \leq r_y$, then $c_{yz} \leq c_{yy}$;*

- *if $r_z < r_y$, then $c_{yz} < c_{yy}$.*

Proof. The first claim is a restatement of the known property [37] that for all y, z and w

$$c_{wz} \geq \frac{c_{wy}c_{yz}}{c_{yy}},$$

so

$$qc_{wz} \geq c_{wy}.$$

Note now that if $c_{yy} < c_{yz}$, then $q < 1$, and

$$r_y = \sum_w v_w c_{wy} < \sum_w v_w c_{wz} = r_z,$$

which proves the first item (the strict inequality is due to the assumption $r_y \neq 0$). If $c_{yy} \leq c_{yz}$, then $q \leq 1$, and the second item follows similarly. □

Note that the hypothesis on r_y is necessary: consider the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

whose spectral radius is the golden ratio $\bigl(1+\sqrt{5}\bigr)/2$; for $\alpha = 3/5$ we have

$$\sum_{n\geq 0}(\alpha M)^n = \begin{pmatrix} 25 & 15 & 0 \\ 15 & 10 & 0 \\ 0 & 0 & \frac{5}{2} \end{pmatrix}.$$

If we consider the preference vector $\langle 0,0,1\rangle$, the associated spectral ranking will be $\langle 0,0,5/2\rangle$. If you let z and y be the first and second node, respectively, we have $c_{yz} = 15 > 0$, $0 = r_y \leq r_z = 0$ but $c_{yz} = 15 > 10 = c_{yy}$, showing that Lemma 16 would not be true if the requirement $r_y \neq 0$ was dropped.

We can finally prove our main theorem:

Theorem 17. *Let M and M' be two nonnegative matrices, such that $M' - M = \boldsymbol{\chi}_x^T\boldsymbol{\delta}$ (i.e., the matrices differ only on the x-th row, and $\boldsymbol{\delta}$ is the corresponding row difference). Let also $\boldsymbol{v}$ be a nonnegative preference vector and $0 \leq \alpha < \min\left(1/\rho(M), 1/\rho(M')\right)$; let $\boldsymbol{r}$ and $\boldsymbol{r}'$ be the damped spectral rankings associated to M and M' respectively. Assume further that:*

1. *there is exactly one y such that $\delta_y > 0$;*

2. *$0 \neq r_y$*

3. $r_y \le r'_y$.

Then, if $r_z \le r_y$ *we have* $r'_z - r_z \le r'_y - r_y$. *As a consequence,* $r_z \le r_y$ *implies* $r'_z \le r'_y$, *whereas* $r_z < r_y$ *implies* $r'_z < r'_y$.

Proof. In this proof, as in the Lemma, we let $C = \bigl(1 - \alpha M\bigr)^{-1}$. First of all, we note that given the hypotheses both $1 - \alpha M$ and $1 - \alpha M'$ are M-matrices, so they both have positive determinants. Since M' is obtained from M by a rank-one correction ($M' = M + \boldsymbol{\chi}_x^T \boldsymbol{\delta}$), applying the matrix determinant lemma we have

$$\det\bigl(1 - \alpha M'\bigr) = \det\bigl(1 - \alpha M - \alpha \boldsymbol{\chi}_x^T \boldsymbol{\delta}\bigr) = \bigl(1 - \alpha \boldsymbol{\delta}(1 - \alpha M)^{-1} \boldsymbol{\chi}_x^T\bigr) \det(1 - \alpha M).$$

We conclude that necessarily

$$1 - \alpha \boldsymbol{\delta}(1 - \alpha M)^{-1} \boldsymbol{\chi}_x^T > 0. \tag{5.2}$$

We now use the Sherman–Morrison formula to write down the inverse of $1-\alpha M'$ as a function of $1 - \alpha M$. More precisely,

$$\begin{aligned}\bigl(1 - \alpha M'\bigr)^{-1} = \Bigl(1 - \alpha\bigl(M + \boldsymbol{\chi}_x^T \boldsymbol{\delta}\bigr)\Bigr)^{-1} &= \bigl(1 - \alpha M - \alpha \boldsymbol{\chi}_x^T \boldsymbol{\delta}\bigr)^{-1} \\ &= \bigl(1 - \alpha M\bigr)^{-1} + \frac{\bigl(1 - \alpha M\bigr)^{-1} \alpha \boldsymbol{\chi}_x^T \boldsymbol{\delta} \bigl(1 - \alpha M\bigr)^{-1}}{1 - \alpha \boldsymbol{\delta}\bigl(1 - \alpha M\bigr)^{-1} \boldsymbol{\chi}_x^T}.\end{aligned}$$

We now multiply by the preference vector $\boldsymbol{v}$, obtaining the explicit spectral-rank correction:

$$\begin{aligned}\boldsymbol{r}' = \boldsymbol{v}\bigl(1 - \alpha M'\bigr)^{-1} &= \boldsymbol{v}\bigl(1 - \alpha M\bigr)^{-1} + \boldsymbol{v}\frac{\bigl(1 - \alpha M\bigr)^{-1} \alpha \boldsymbol{\chi}_x^T \boldsymbol{\delta} \bigl(1 - \alpha M\bigr)^{-1}}{1 - \alpha \boldsymbol{\delta}\bigl(1 - \alpha M\bigr)^{-1} \boldsymbol{\chi}_x^T} \\ &= \boldsymbol{r} + \frac{\alpha \boldsymbol{r} \boldsymbol{\chi}_x^T \boldsymbol{\delta} \bigl(1 - \alpha M\bigr)^{-1}}{1 - \alpha \boldsymbol{\delta}\bigl(1 - \alpha M\bigr)^{-1} \boldsymbol{\chi}_x^T} = \boldsymbol{r} + \frac{\alpha r_x}{1 - \alpha \boldsymbol{\delta}\bigl(1 - \alpha M\bigr)^{-1} \boldsymbol{\chi}_x^T} \boldsymbol{\delta}\bigl(1 - \alpha M\bigr)^{-1}.\end{aligned}$$

The case $r_x = 0$ is obvious. Thus, let us assume that $r_x > 0$. By (5.2), we can gather all the scalar values appearing in the second summand into a single positive constant κ and just write

$$\boldsymbol{r}' - \boldsymbol{r} = \kappa \boldsymbol{\delta}\bigl(1 - \alpha M\bigr)^{-1}.$$

Note that if

$$\Bigl[\boldsymbol{\delta}\bigl(1 - \alpha M\bigr)^{-1}\Bigr]_z \le 0$$

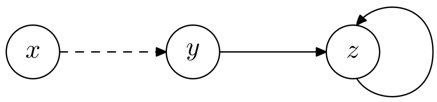


Figure 5.3: A counterexample for the last hypotheses of Theorem 17.

the thesis is trivial by the hypothesis $r_y \leq r'_y$. This holds true, in particular, if $c_{yz} = 0$, as in that case

$$\Big[\boldsymbol{\delta}\big(1 - \alpha M\big)^{-1}\Big]_z = -\sum_{w \neq y} |\delta_w| c_{wz} \leq 0.$$

If $c_{yz} > 0$, since $r_y \neq 0$ we know from Lemma 16 that $q = c_{yy}/c_{yz} \geq 1$, and for all w we have $c_{wy} \leq q c_{wz}$. It follows that

$$\begin{aligned}\Big[\boldsymbol{\delta}\big(1 - \alpha M\big)^{-1}\Big]_y &= \delta_y c_{yy} - \sum_{w \neq y} |\delta_w| c_{wy} \geq \delta_y q c_{yz} - \sum_{w \neq y} q |\delta_w| c_{wz} \\ &= q\Big(\delta_y c_{yz} - \sum_{w \neq y} |\delta_w| c_{wz}\Big) = q\Big[\boldsymbol{\delta}\big(1 - \alpha M\big)^{-1}\Big]_z \geq \Big[\boldsymbol{\delta}\big(1 - \alpha M\big)^{-1}\Big]_z .\end{aligned}$$

This completes the proof. □

We remark that no hypothesis can be weakened. The update vector must increase a single coordinate to model an increase of importance of y alone.

The condition $r_y \neq 0$ cannot be weakened, as the counterexample shown in Figure 5.3 proves. The spectral ranking $\boldsymbol{r}$ induced by the adjacency matrix with preference vector $\langle 1, 0, 0\rangle$, without the dotted arrow, has $r_y = r_z = 0$. If we add the dotted arrow, though, the score vector becomes $\boldsymbol{r'} = \langle 1, \alpha, \alpha^2/(1 - \alpha)\rangle$, and $r'_z - r_z = r'_z = \alpha^2/(1 - \alpha)$ is larger than $r'_y - r_y = r'_y = \alpha$ for $\alpha > 1/2$.

This shows, in particular, that all known variants of PageRank (strongly preferential, weakly preferential, pseudoranks, etc. [5]) and Katz's index cannot be proved to satisfy rank monotonicity for an arbitrary preference vector without additional hypotheses (e.g., that all scores are positive).

Finally, the condition $r_y \leq r'_y$ cannot be eliminated. Consider once again the matrix M of (5.1), and its spectral rank with $\alpha = 3/5$ and preference vector $\langle 1, 1, 4\rangle$, which is $\langle 40, 25, 10\rangle$. If we update the second row using the vector $\langle -1, 1, 0\rangle$ the new scores will be $\langle 5/2, 25/4, 10\rangle$ contradict the thesis.

Note, however, that we can actually prove the condition under mild assumptions on M and $\boldsymbol{\delta}$:

Theorem 18. *Condition (3) of Theorem 17 can be substituted by the following two hypotheses (that imply it)*

1. *$1-\alpha M$ is (strictly) diagonally dominant*

2. $\sum_z \delta_z \geq 0$.

Moreover, in the strict case, $r_y < r'_y$, provided that $r_x > 0$.

Proof. The proof follows the lines of the proof of Theorem 17, noting again that for $r_x = 0$ the statement trivializes. However, once we get the update vector $\boldsymbol{\delta}$ we now note that being $1-\alpha M$ diagonally dominant, the (nonnegative) inverse $C = (1-\alpha M)^{-1}$ has the property that the entries c_{ii} on the diagonal are (strictly) larger than off-diagonal entries C_{ki} on the same column [28, Remark 3.3]. Thus,

$$\Big[\boldsymbol{\delta}(1-\alpha M)^{-1}\Big]_y = \delta_y c_{yy} - \sum_{z\neq y} \big|\delta_z\big| c_{zy} \geq \delta_y c_{yy} - \sum_{z\neq y} \big|\delta_z\big| c_{yy} \geq 0.$$

In the strict case, if there is at least one index $z \neq y$ such that $\big|\delta_z\big| c_{zy} \neq 0$, then the first inequality is strict; otherwise, the second inequality is strict (because $\delta_y > 0$ and $c_{yy} > 0$). □

Finally, we can prove strict rank monotonicity in the presence of score monotonicity:

Theorem 19. *Let M and M' be two nonnegative matrices as in Theorem 17 and let $\boldsymbol{r}$ and $\boldsymbol{r}'$ be the damped spectral rankings associated to M and M' respectively. Assume further that:*

1. *there is exactly one y such that $\delta_y > 0$;*

2. $r_x, r_y \neq 0$

3. $r_y < r'_y$.

Then, if $r_z \leq r_y$ we have $r'_z - r_z < r'_y - r_y$, and in particular $r'_z < r'_y$.

The proof is the same as that of Theorem 17: the additional hypotheses makes it possible to make the relevant inequalities strict.

Corollary 19.1. *PageRank satisfies the strict rank-monotonicity axiom, for any graph, damping factor and preference vector, provided all scores are nonzero. The latter condition is always true if the preference vector is everywhere nonzero or if the graph is strongly connected.*

Proof. Consider two nodes x and y of a graph G such that there is no arc from x to y, and let d be the outdegree of x. Given the normalized matrix $\bar{A}$ of G, and the normalized matrix $\bar{A}'$ of the graph G' obtained by adding to G the arc $x \to y$, we have

$$\bar{A}' - \bar{A} = \boldsymbol{\chi}_x^T \boldsymbol{\delta},$$

where $\boldsymbol{\delta}$ is the difference between the rows corresponding to x in $\bar{A}$ and $\bar{A}'$, which contains $-1/d(d+1)$ in the positions corresponding to the successors of x in G, and $1/(d+1)$ in the position corresponding to y (note that if $d = 0$, we have just the latter entry), so we can apply Theorem 19. The hypothesis $r_y < r'_y$ is always verified by Theorem 18. □

Corollary 19.2. *Katz's index satisfies the strict rank-monotonicity axiom, for any graph, attenuation factor and preference vector, provided all scores are nonzero. The latter condition is always true if the preference vector is everywhere nonzero or if the graph is strongly connected.*

Proof. Consider two nodes x and y of a graph G such that there is no arc from x to y. Given the matrix A of G, and the matrix A' of the graph G' obtained by adding to G the arc $x \to y$, we have

$$A' - A = \boldsymbol{\chi}_x^T \boldsymbol{\chi}_y,$$

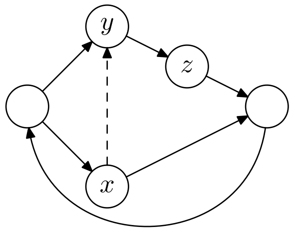
and we can apply Theorem 17. The hypothesis $r_y < r'_y$ is trivially verified, as the only nonzero entry of $\boldsymbol{\delta} = \boldsymbol{\chi}_y$ is the positive one. □

5.9 Non-damped spectral rankings

The case of the rank monotonicity for the dominant eigenvector remains open. Seeley's index, instead, is based on a matrix which, if the graph is strongly connected, yields a regular Markov chain. Thus we can derive rank monotonicity for Seeley's index on strongly connected graphs from the work of [9]. We can moreover prove that strict rank monotonicity cannot be attained:

Theorem 20. *Seeley's centrality does not satisfy strict rank monotonicity on strongly connected graphs.*

Proof. Take the graph in the following graph. You can note that it is almost symmetric, and by breaking the symmetry we can observe the following behavior: two ex-aequo nodes remains ex-aequo after one of them (y) receives an arc ($x \to y$).



Normalizing the left eigenvector, we can see that the score of x, y, z is $1/7$. After adding the arc, x loses score and goes to $1/8$, and y and z both get score of $3/16$. To get an explanatory insight on this, think about the (left) eigenvalue equation $xA = \lambda x$ and its iterative process to compute the eingenvectors. Since z has only one incoming arc from a node (y), the component of the matrix a_{zy} is 1. We can write then $x_y = a_{zy}x_z$. Adding the arc from $x \to y$ does not change this part of the matrix, and the two nodes remains ex-aequo, and thus strict rank monotonicity is violated.

□

5.10 Roundup

The results of this chapter are summarized in Table 5.2 for the general case, and Table 5.3 for strongly connected graphs. As we can see, out of the geometrical and path based measures, only harmonic centrality and indegree satisfies rank monotonicity. Both measure satisfy Lemma 2.

Centrality	Rank monotonicity	Strict rank monotonicity
Indegree	yes	yes
Harmonic	yes	yes
Closeness	no	no
Closeness (patched)	no	no
Lin's index	no	no
Betweenness	no	no
Dominant	?	?
Seeley	?	no
Katz	yes	yes
PageRank	yes	yes
SALSA	no	no

Table 5.2: Summarized results for rank monotonicity.

For the spectral measure the Theorem 17 has been widely used to prove rank monotonicity. We also noticed that strong connectivity of a graph can often imply rank monotonicity in cases when rank monotonicity is not valid generally (closeness centrality, patched closeness centrality, Lin's index). All the geometric centralities which satisfies score monotonicity, satisfies rank monotonicity as well. In the results collected up to this point, it's almost always true that if a centrality does not satisfy score monotonicity and fail to satisfy density or size axioms, it almost certainly does not satisfy rank monotonicity as well. The only exception is Katz, for which the size axiom is only partially satisfied. Closeness and harmonic centrality, which are very similar from a mathematical point of view, are also the most different seen under the light of our axioms. Score monotonicity has been proved for closeness centrality and Lin's index on strongly connected graphs. On the other side, betweenness centrality and SALSA does not satisfy score monotonicity at all, even on strongly connected graphs.

5.10.1 Results for strongly connected graphs

Here are summarized the results of a few centrality measure on strongly connected graphs.

Centrality	Rank monotonicity	Strict rank monotonicity
Degree	yes	yes
Harmonic	yes	yes
Closeness	yes	no
Lin's Index	yes	no
Betweenness	no	no
Dominant	?	?
Seeley	yes	no
Katz	yes	yes
PageRank	yes	yes
SALSA	no	no

Table 5.3: Summarized results for rank monotonicity on strongly connected graphs.

Here we have only two example of measure (closeness centrality and Lin's index) which performs better — according to rank monotonicity — on strongly connected graphs. Indeed, on strongly connected graphs, they also satisfy score monotonicity. Betweenness centrality fails to satisfy score monotonicity on strongly connected graphs, and we were able to craft a single counterexample for rank and score monotonicity.

Chapter 6

Conclusions

You are once again surrounded by a brilliant white light.
Allow the light to lead you away from your past and into this lifetime.
As the light dissipates you will slowly fade back into consciousness,
remembering all you have learned.
When I tell you to open your eyes you will return to the present,
feeling peaceful and refresh.
Open your eyes Nicholas.

—FINALLY FREE — Dream Theater (Scenes From A Memory)

	Monotonicity				Other axioms (see [6])	
	General		Strongly connected			
Centrality	Score	Rank	Score	Rank	Size	Density
Degree	yes	yes*	yes	yes*	only k	yes
Harmonic	yes	yes*	yes	yes*	yes	yes
Closeness	no	no	yes	yes	no	no
Lin	no	no	yes	yes	only k	no
Betweenness	no	no	no	no	only p	no
Dominant	no	?	?	?	only k	yes
Seeley	no	?	yes	yes	no	yes
Katz	yes	yes*	yes	yes*	only k	yes
PageRank	yes	yes*	yes	yes*	no	yes
SALSA	no	no	no	no	no	yes

Table 6.1: A summarization of all known results. "yes*" means that the strict version of rank monotonicity is satisfied.

6.1 Conclusions

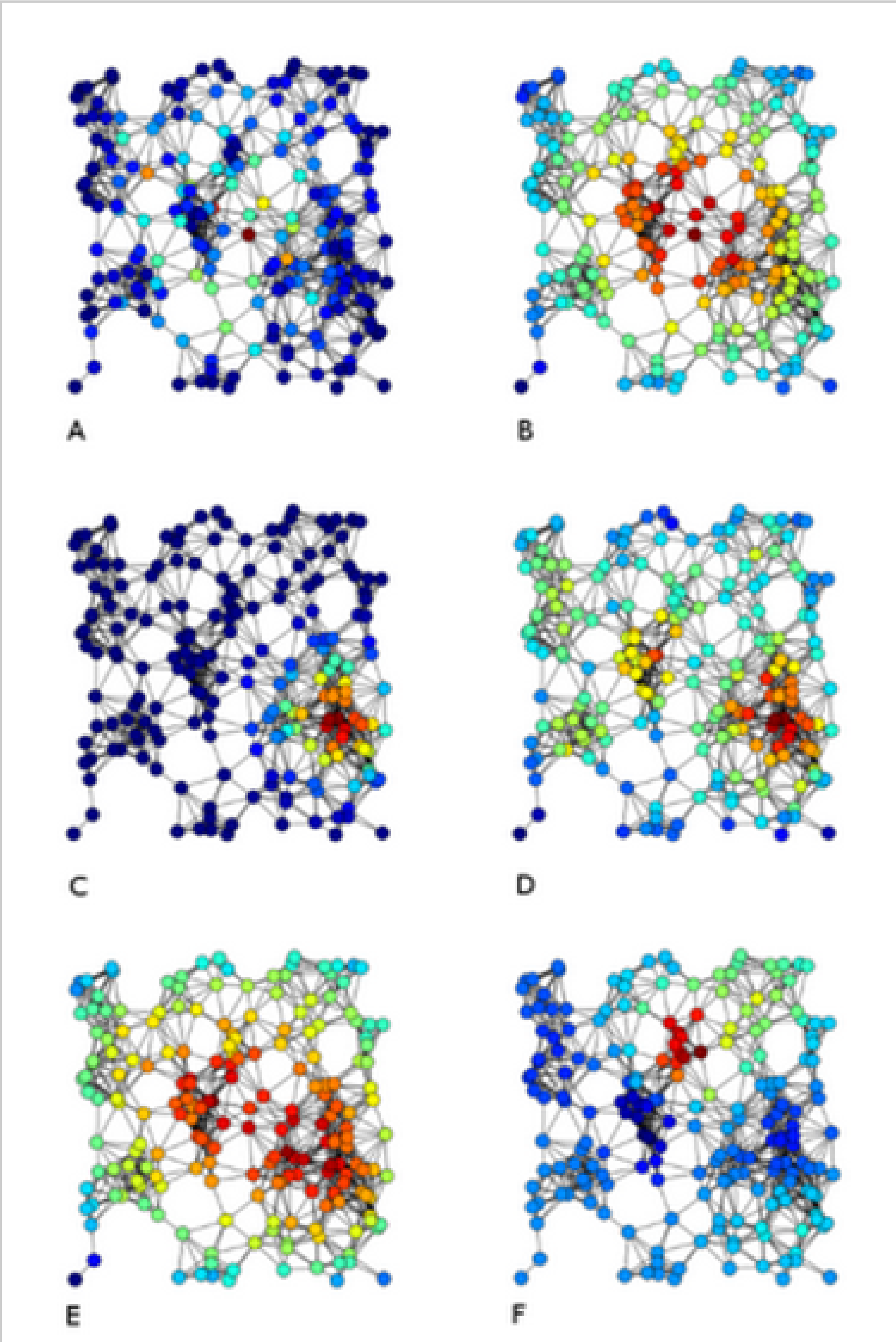
In this work, multiple centrality measures has been studied under the light of rank monotonicity, a property we expect a good centrality measure should satisfy and that complements score monotonicity. This property is about the behavior of the centrality measures under the operation of arc addition: specifically, we don't want the rank of a node z to improve respect to y after the addition of an arc $x \to y$. The main result of this work is 17, which characterizes spectral centrality measures that satisfy rank monotonicity. Harmonic centrality has confirmed to be the only centrality measure that satisfy all the axioms for centrality (Theorem 4). Score monotonicity for SALSA, betweenness, closeness centrality and Lin's index has been proved or disproved both on general and on strongly connected graphs.

In Table 6.1 we summarize the state of the art for centrality measures under examination. In each cell you can see if the centrality satisfies the axiom in the respective column. This table can be interpreted as a decision-tree (see Figure 6.2) in order to chose a centrality measure that satisfy or not a set of axioms.

6.1.1 Future work

We present here a few possible extensions of this work:

- Studying the behavior of other algorithms under the light of the axioms



Examples of A) Betweenness centrality, B) Closeness centrality, C) Eigenvector centrality, D) Degree centrality, E) Harmonic centrality and F) Katz centrality of the same graph.

Figure 6.1: A graphical color-based display of basic centrality measures.

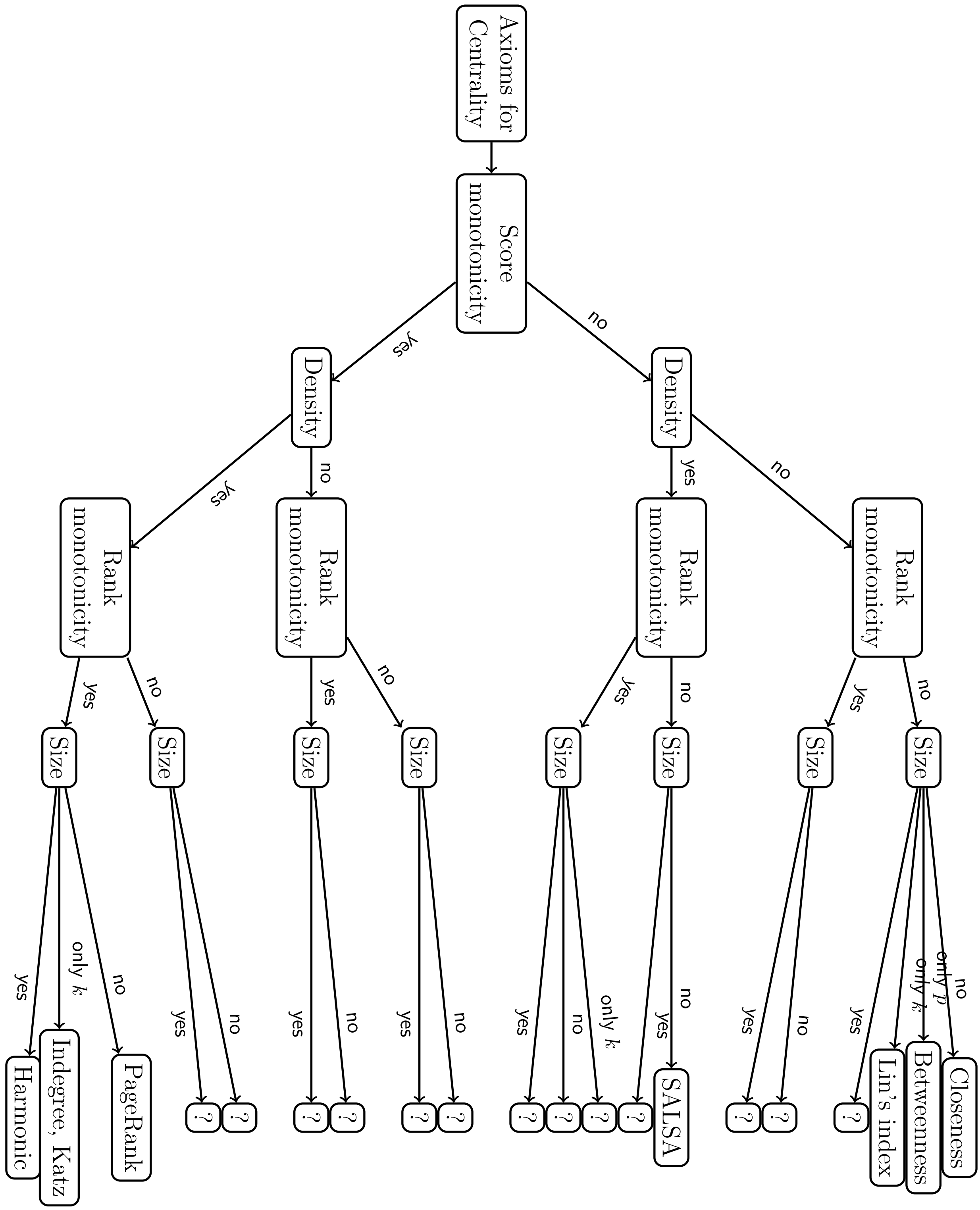


Figure 6.2: Decision tree for Axioms for Centrality

discussed in this paper.

- Find a fully axiomatic characterization of harmonic centrality on directed graph, generalizing [17].
- Find, if any, the implication of harmonic centrality in clique formation.
- Patch betweenness centrality, and take into consideration in the score of the node x also the shortest path arriving to the node x. Which axioms does this new centrality satisfy?
- Can we measure how much distant are two centrality and see if there is any connection with the axioms?
- We know that the spectral radius of a matrix is monotone in the matrix entries. Given a adjacency matrix M, which is the rank-one perturbation I can make, in order to increase at most the spectral radius of the matrix?

6.2 Acknowledgements

I particularly thank my supervisors for precious help and hints they gave me. I immensely thank Giorgio Riccobono, Bernard Kapidani, and Fabrizio Romano Genovese for the patience and the time spent talking on this thesis, and for the precious discussion about mathematics and science I had the pleasure to have in this years of study. I also thank Yasmine Masotti for her support.

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... after all, we're all alike.