Stat 509: Statistics for Engineers
Homework Assignment 5

1. The load required to degrade the structural integrity of a foundation is normally distributed with a mean of 22 tons and a standard deviation of 5 tons.

   (a) While at the site, a fellow engineer asks you the approximate probability that it takes between 17 and 27 tons to degrade the foundation. Being knowledgeable in statistics, what you are able to tell him off the top of your head?

   Based on the empirical rule, you tell him the probability is approximately 68%, as these two values correspond to loads within one standard deviation of the mean.

   (b) Verify that your estimate in (a) is accurate using R.

   pnorm(27,22,5)−pnorm(17,22,5) = pnorm(1)−pnorm(−1) = 0.6826895

   (c) A similar foundation is to be constructed so that the load required to degrade its integrity is again normally distributed with a standard deviation of 5 tons, but the improved foundation is required to only have a 20% chance to degrade under a 22 ton load or less. What must the mean of this distribution be so that the foundation is constructed to the given specifications? Use R.

   let Z be the standard normal distribution. qnorm(.2) yields the 0.2 quantile of Z. Recall, Z = X −μ
   σ , where X ∼ N(μ,σ). Thus,
   P(Z < qnorm(.2)) = 0.2
   =⇒ P μ 22−μ 5 < qnorm(.2)¶ = 0.2
   =⇒ μ = 22−5 ·qnorm(.2) = 26.20811

2. Using integrals, directly calculate the variance of the Weibull distribution with shape α and scale μ.

V(X) = E(X 2)−[E(X)]2
In class we calculated E(X) = μ·Γ(1+ 1 α).

$$E(X^2) = \int_0^\infty x^2 \cdot \frac{\alpha}{\mu}\left(\frac{x}{\mu}\right)^{\alpha-1} e^{-\left(\frac{x}{\mu}\right)^\alpha} dx$$

$$u = \left(\frac{x}{\mu}\right)^\alpha$$

$$du = \frac{\alpha}{\mu}\left(\frac{x}{\mu}\right)^{\alpha-1} dx$$

$$x = \mu \cdot u^{1/\alpha}$$

$$= \int_{(0/\mu)^\alpha}^{(\infty/\mu)^\alpha} \left[\mu \cdot u^{\frac{1}{\alpha}}\right]^2 e^{-u} du$$

$$= \mu^2 \int_0^\infty u^{\frac{2}{\alpha}} e^{-u} du$$

$$= \mu^2 \int_0^\infty u^{\left[\left(\frac{2}{\alpha}+1\right)-1\right]} e^{-u} du$$

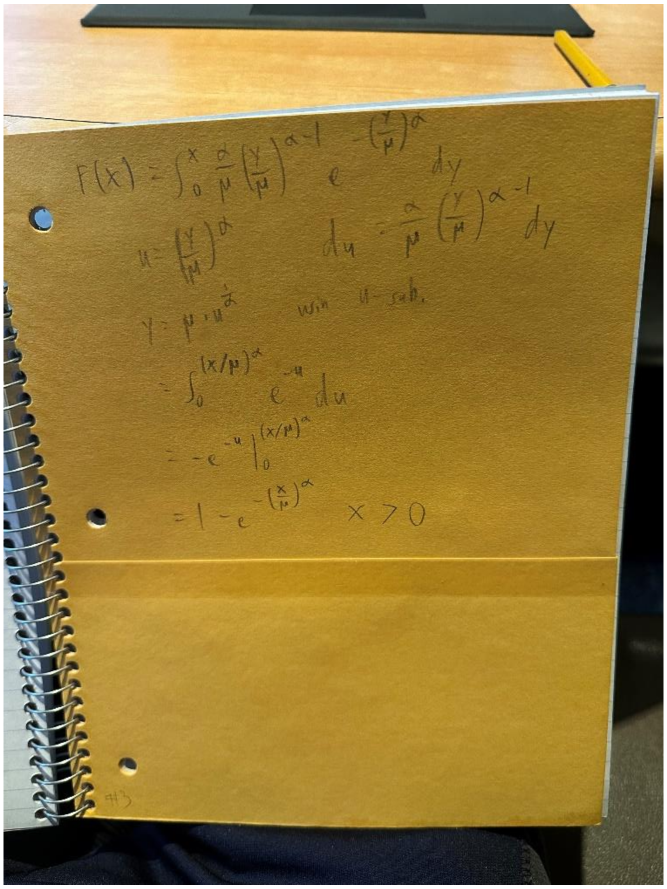
$$= \mu^2 \Gamma\left(1+\frac{2}{\alpha}\right)$$ using the def. of gamma function

$$\Rightarrow V(X) = \mu^2 \Gamma\left(1+\frac{2}{\alpha}\right) - \left[\mu \cdot \Gamma\left(1+\frac{1}{\alpha}\right)\right]^2$$

$$= \mu^2\left(\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right)$$

#2

3. Using integrals, directly calculate the CDF of the Weibull distribution with shape α and scale μ.



4. Let T represent the lifetime in years of a part which follows a Weibull distribution with shape 2 and scale 5. For (g) through (k), additionally provide the appropriate R code.

   (a) What is $f(t)$?

Answer below

   (b) What is $F(t)$?

Answer below

   (c) What is $S(t)$?

Answer below

   (d) What is $h(t)$?

Answer below

   (e) What is $\mathbf{E}(T)$? Make sure to simplify the gamma function in terms of pi.

Answer below

   (f) What is $\mathbf{V}(T)$? Make sure to simplify the gamma function in terms of pi.

Answer below

   (g) What is $P(T > 6)$?

P(T > 6) = 1−P(T ≤ 6) = 1−F(6) = S(6) = e − ¡ 6 5 ¢2 = 1−pweibull(6,2,5) = 0.2369278

   (h) What is $P(2 < T < 5)$?

P(2 < T < 5) = F(5)−F(2) = e − ¡ 2 5 ¢2 −e − ¡ 5 5 ¢2 = pweibull(5,2,5)−pweibull(2,2,5) = 0.4842643

   (i) What is $P(2 \le T < 5)$?

For a continuous distribution there is no probability mass at single points. Thus, P(2 ≤ T < 5) = P(2 < T < 5) = 0.4842643.

   (j) What is the 25^{th} percentile of T?

Answer below

   (k) What is the probability that the part is still functional after 8 years, given that it was still functional after

5 years?

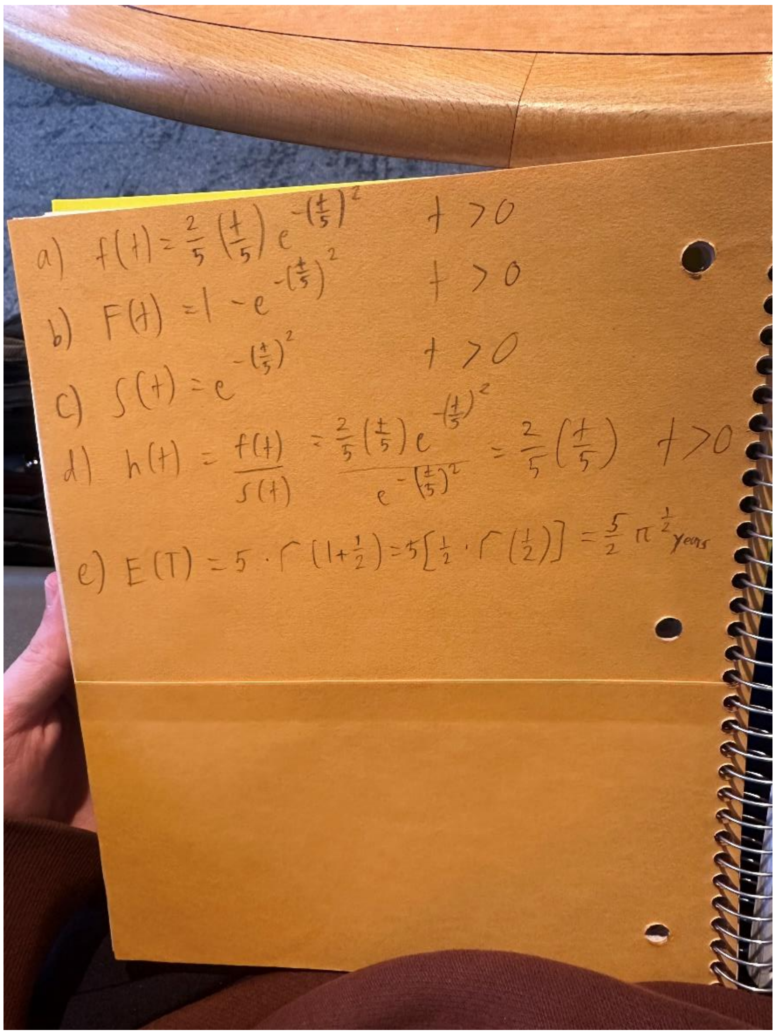
Answer below

(l) For this population of parts, does it appear that the part gets weaker or stronger as time passes? You must mathematically justify your answer.

To answer this question we need to take the derivative of the hazard function h(t) which we calculated in (d).

$\partial h(t)/\partial t = 2\ 25$

Because the derivative is positive, we know the hazard function is monotone increasing in t. As the hazard function gives us the rate of failure, the rate of failure is increasing with time and thus the parts appear to be getting weaker over time. In general, the derivative could still contain the variable t, in which case we could have to see for which values in the support is the derivative is positive, negative or 0.



j) $F(q^*) = 0{,}25$

$$1 - e^{-\left(\frac{q^*}{5}\right)^2} = 0{,}25$$

$$\Rightarrow q^* = 5\left[\ln(0{,}75^{-1})\right]^{1/2} = \text{qweibull}(0{,}25, 2, 5)$$
$$= 0{,}2101361$$

k) $$P(T > 8 \mid T > 5) = \frac{P(T > 8 \cap T > 5)}{P(T > 5)} = \frac{P(T > 8)}{P(T > 5)}$$

$$= \frac{S(8)}{S(5)} = \frac{e^{-\left(\frac{8}{5}\right)^2}}{e^{-\left(\frac{5}{5}\right)^2}} = \frac{1 - \text{pweibull}(8,2,5)}{1 - \text{pweibull}(5,2,5)} = 0{,}210136($$

d) $h(t) = \frac{f(t)}{S(t)} = \frac{\frac{2}{5}\left(\frac{t}{5}\right)e^{-\left(\frac{t}{5}\right)^2}}{e^{-\left(\frac{t}{5}\right)^2}} = \frac{2}{5}\left(\frac{t}{5}\right)$

e) $E(T) = 5 \cdot \Gamma\left(1+\frac{1}{2}\right) = 5\left[\frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)\right] = \frac{5}{2}\pi^{\frac{1}{2}}$

f) $V(T) = 5^2\left(\Gamma\left(1+\frac{2}{2}\right) - \left[\Gamma\left(1+\frac{1}{2}\right)\right]^2\right)$

$= 25\left(\Gamma(2) - \left[\Gamma\left(\frac{3}{2}\right)\right]^2\right)$

$= 25\left(1 - \frac{\pi}{4}\right) \text{years}^2$