Exercises 10-18.

Ex 10. Proposition: S_n: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, for all $n \geq 0$.

Proof. (Induction)
Basis step.
Suppose $n = 0$, then $\sum_{i=1}^{n} i = \sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2} = \frac{n(n+1)}{2}$. Thus S_0 is true.

We now show that if S_n is true, then S_{n+1} is true.
Inductive step. Suppose S_n is true and $n \geq 0$.
Then observe

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) && (1)\\
&= \frac{n(n+1)}{2} + (n+1) && \text{(By our inductive hypothesis.)} \quad (2)\\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} && (3)\\
&= \frac{n(n+1) + 2(n+1)}{2} && (4)\\
&= \frac{(n+1)((n+1)+1)}{2} && (5)
\end{aligned}$$

Thus S_{n+1} is true. By mathematical induction it follows that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \geq 0$.

□

Ex 11. Proposition: S_n: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n \geq 0$.

Proof. (Induction)
Basis step.
Suppose $n = 0$, then $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{0} i^2 = 0 = \frac{0 \cdot 1 \cdot 1}{6} = \frac{n(n+1)(2n+1)}{6}$. Thus S_0 is true.
We now show that if S_n is true, then S_{n+1} is true.
Inductive step. Suppose S_n is true and $n \geq 0$.
Then observe

$$\begin{align}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \tag{6}\\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \text{(By our inductive hypothesis.)} \tag{7}\\
&= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \tag{8}\\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \tag{9}\\
&= \frac{(n(2n+1) + 6(n+1))(n+1)}{6} \tag{10}\\
&= \frac{(2n^2 + 7n + 6)(n+1)}{6} \tag{11}\\
&= \frac{(2n+3)(n+2)(n+1)}{6} \tag{12}\\
&= \frac{(2(n+1)+1)((n+1)+1)(n+1)}{6} \tag{13}
\end{align}$$

Thus S_{n+1} is true. By mathematical induction it follows that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 0$.

□

Ex 12. Proposition: S_n: $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, for all $n \geq 0$.

Proof. (Induction)
Basis step.
Suppose $n = 0$, then $\sum_{i=1}^{n} i^3 = \sum_{i=1}^{0} i^3 = 0 = \frac{0 \cdot 1}{4} = \frac{n^2(n+1)^2}{4}$. Thus S_0 is true.
We now show that if S_n is true, then S_{n+1} is true.
Inductive step. Suppose S_n is true and $n \geq 0$.
Then observe

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 && (14)\\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 && \text{(By our inductive hypothesis.)} \quad (15)\\
&= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} && (16)\\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} && (17)\\
&= \frac{(n^2 + 4(n+1))(n+1)^2}{4} && (18)\\
&= \frac{(n^2 + 4n + 4)(n+1)^2}{4} && (19)\\
&= \frac{(n+2)^2(n+1)^2}{4} && (20)\\
&= \frac{((n+1)+1)^2(n+1)^2}{4} && (21)
\end{aligned}$$

Thus S_{n+1} is true. By mathematical induction it follows that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all $n \geq 0$.

$\square$

Ex 13. Proposition: S_n: $\displaystyle\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$, for all $n \geq 0$.

Proof. (Induction)
Basis step.
Suppose $n = 0$, then $\displaystyle\sum_{i=1}^{n} i(i+1)(i+2) = \sum_{i=1}^{0} i(i+1)(i+2) = 0 = \frac{0 \cdot 1 \cdot 2 \cdot 3}{4} = \frac{n(n+1)(n+2)(n+3)}{4}$.
Thus S_0 is true.

We now show that if S_n is true, then S_{n+1} is true.
Inductive step. Suppose S_n is true and $n \geq 0$.
Then observe

$$
\begin{aligned}
\sum_{i=1}^{n+1} i(i+1)(i+2) &= \sum_{i=1}^{n} i(i+1)(i+2) + (n+1)(n+2)(n+3) && (22)\\
&= \frac{n(n+1)(n+2)(n+3)}{4} + (n+1)(n+2)(n+3) \qquad \text{(By our supposition.)} && (23)\\
&= \frac{n(n+1)(n+2)(n+3)}{4} + \frac{4(n+1)(n+2)(n+3)}{4} && (24)\\
&= \frac{n(n+1)(n+2)(n+3) + 4(n+1)(n+2)(n+3)}{4} && (25)\\
&= \frac{(n+1)(n+2)(n+3)(n+4)}{4} && (26)\\
&= \frac{(n+1)((n+1)+1)((n+1)+2)((n+1)+3)}{4} && (27)
\end{aligned}
$$

Thus S_{n+1} is true. By mathematical induction it follows that $\displaystyle\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for all $n \geq 0$.

□

Ex 14. Proposition: S_n: $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ where $a \neq 1$, for all $n \geq 1$.

Proof. (Induction)
Basis step.
Suppose $n = 1$, then $\sum_{i=0}^{n} a^i = a^0 + a^1 = 1 + a = \frac{(a+1)(a-1)}{(a-1)} = \frac{a^2-1}{a-1} = \frac{a^{n+1}-1}{a-1}$. Thus S_1 is true.

We now show that if S_n is true, then S_{n+1} is true.
Inductive step. Suppose S_n is true and $n \geq 1$.
Then observe

$$\begin{aligned}
\sum_{i=0}^{n+1} a^i &= \sum_{i=0}^{n} a^i + a^{n+1} && (28)\\
&= \frac{a^{n+1}-1}{a-1} + a^{n+1} && \text{(By inductive hypothesis.)} \quad (29)\\
&= \frac{a^{n+1}-1}{a-1} + \frac{a^{n+1}(a-1)}{a-1} && (30)\\
&= \frac{a^{n+1}-1+a^{n+1}(a-1)}{a-1} && (31)\\
&= \frac{((a-1)+1)a^{n+1}-1}{a-1} && (32)\\
&= \frac{(a)a^{n+1}-1}{a-1} && (33)\\
&= \frac{a^{n+2}-1}{a-1} && (34)\\
&= \frac{a^{(n+1)+1}-1}{a-1} && (35)
\end{aligned}$$

Thus S_{n+1} is true. By mathematical induction it follows that $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ where $a \neq 1$, for all $n \geq 1$.

□

Ex 15. Proposition: S_n: $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$, for all $n \geq 1$.

Proof. (Induction)
Basis step.
Suppose $n = 1$, then $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{n}{n+1}$. Thus S_1 is true.
We now show that if S_n is true, then S_{n+1} is true.
Inductive step. Suppose S_n is true and $n \geq 1$.
Then observe

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \tag{36}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \qquad \text{(By inductive hypothesis.)} \tag{37}$$

$$= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \tag{38}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)} \tag{39}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)} \tag{40}$$

$$= \frac{(n+1)}{((n+1)+1)} \tag{41}$$

Thus S_{n+1} is true. By mathematical induction it follows that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \geq 1$.

$\square$

Ex 16. Proposition: S_n: 3 divides $n^3 + 2n$, for all $n \geq 0$.

Proof. (Induction)
Basis step.
Suppose $n = 0$, then $n^3 + 2n = 0^3 + 2 \cdot 0 = 0 + 0 = 0$. Because $3|0$, it follows that S_0 is true.

We now show that if S_n is true, then S_{n+1} is true.
Inductive step. Suppose S_n is true and $n \geq 0$.
Note that $n^3 + 2n = 3x$ for some integer x, by our inductive hypothesis. Then observe

$$\begin{aligned}
(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 && (42)\\
&= n^3 + 3n^2 + 5n + 3 && (43)\\
&= n^3 + 2n + 3n^2 + 3n + 3 && (44)\\
&= 3x + 3n^2 + 3n + 3 \qquad \text{(By our inductive hypothesis.)} && (45)\\
&= 3(x + n^2 + n + 1) && (46)
\end{aligned}$$

Thus S_{n+1} is true. By mathematical induction it follows that 3 divides $n^3 + 2n$ for all $n \geq 0$.

$\square$

Ex 17. Proposition: A tree with n nodes has exactly $n-1$ edges.

Proof. (Induction)
Basis step.
Suppose we have a tree with one node, then $n = 1$. A tree with one node has zero edges. Also $n-1 = 1-1 = 0$. Thus the proposition holds for one node.

We now show that if the proposition is true for a tree with n nodes, then it is true for $n+1$ nodes.
Inductive step. Suppose the proposition is true for n nodes and $n \geq 1$.
Note that when we add a new node to a tree, we add one new edge. Suppose we have n nodes with $n-1$ edges. After adding one node to it we have $n+1$ nodes and $n = (n+1)-1$ edges.
Thus the proposition is true for $n+1$ nodes. By mathematical induction it follows that a tree with n nodes has exactly $n-1$ edges.

□

Ex 18. Proposition: S_n: $\displaystyle\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$, for all $n \in \mathbb{N}$

Proof. (Induction)
Basis step.
Suppose $n = 1$, then $\displaystyle\sum_{i=1}^{n} i^3 = \sum_{i=1}^{1} i^3 = 1^3 = 1^2 = \left(\sum_{i=1}^{1} i\right)^2 = \left(\sum_{i=1}^{n} i\right)^2$. Thus S_1 is true.
We now show that if S_n is true, then S_{n+1} is true.
Inductive step. S_n is true and $n \geq 1$
Then observe

$$\begin{align}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \tag{47}\\
&= \left(\sum_{i=1}^{n} i\right)^2 + (n+1)^3 && \text{(By our inductive hypothesis.)} \tag{48}\\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 && \text{(By def. triangle number.)} \tag{49}\\
&= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} \tag{50}\\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \tag{51}\\
&= \frac{(n^2 + 4(n+1))(n+1)^2}{4} \tag{52}\\
&= \frac{(n+2)^2(n+1)^2}{4} \tag{53}\\
&= \left(\frac{(n+2)(n+1)}{2}\right)^2 \tag{54}\\
&= \left(\sum_{i=1}^{n+1} i\right)^2 && \text{(By def. triangle number.)} \tag{55}
\end{align}$$

Thus S_{n+1} is true. By mathematical induction it follows that $\displaystyle\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$.

$\square$