Exercises 7-31.

Ex 7. a) True. $2^{n+1} = 2 \cdot 2^n$, but when $c = 3$ we have that $2 \cdot 2^n \leq 3 \cdot 2^n = c \cdot 2^n$ for all $n \in \mathbb{N}$.

Ex 7. b) False. Observe that $2^{2n} = 2^n \cdot 2^n$. There exists no c and $n_0 \in \mathbb{N}$ such that $2^n \geq c$ for all $n_0 \geq 0$. This conclusion is clear by the fact that $2^n \geq c$ where $n \geq c$ and $n \geq n_0$.

Ex 8. a) Note that $f(n) = log(n^2) = 2log(n)$. Then for all $n \geq b^5$ where b is the base of the logarithm

$$\begin{aligned} 2log(n) &\leq 3log(n) + 15 = c_1(log(n) + 5) && \text{when } c_1 = 3 \\ 2log(n) &\geq log(n) + 5 = c_2(log(n) + 5) && \text{when } c_2 = 1 \end{aligned}$$

Thus $f(n) \in \Theta(log(n) + 5)$.

Ex 8. b) We show that $f(n) \in \Omega(g(n))$. Because $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{1}{2} \cdot \lim_{n\to\infty} \frac{\sqrt{n}}{log(n)} = \infty$, it follows that there exists positive constants c and n_0 such that $\frac{f(n)}{g(n)} \geq c$ for all $n \geq n_0$. This is equivalent to the big-Omega definition.

Ex 8. c) Observe that when $c = 1$ and $n \geq b$ where b is the base of the logarithm, it follows that $f(n) = log(n)log(n) \geq log(n) = c \cdot g(n)$. Thus $f(n) \in \Omega(g(n))$.

Ex 8. d) We show that $f(n) \in \Omega(g(n))$. Because $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n}{log^2(n)} = \infty$, it follows that there exists positive constants c and n_0 such that $\frac{f(n)}{g(n)} \geq c$ for all $n \geq n_0$. This is equivalent to the big-Omega definition.

Ex 8. e) We show that $f(n) \in \Omega(g(n))$. Because $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{nlog(n) + n}{log(n)} = \lim_{n\to\infty} n + \frac{n}{log(n)} = \infty$, it follows that there exists positive constants c and n_0 such that $\frac{f(n)}{g(n)} \geq c$ for all $n \geq n_0$. This is equivalent to the big-Omega definition.

Ex 8. f) Note that both $f(n)$ and $g(n)$ are constant functions. In other words, they do not depend on n. Then let $d = log(10) > 0$ and observe that

$$\begin{aligned} 10 &\geq 1 = \frac{1}{d} \cdot d = c_1 \cdot d && \text{where } c_1 = \frac{1}{d} \\ 10 &\leq 11 = \frac{11}{d} \cdot d = c_2 \cdot d && \text{where } c_2 = \frac{11}{d} \end{aligned}$$

Thus $f(n) \in \Theta(g(n))$.

Ex 8. g) We show that $f(n) \in \Omega(g(n))$. Because $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{2^n}{10n^2} = \infty$, it follows that there exists positive constants c and n_0 such that $\frac{f(n)}{g(n)} \geq c$ for all $n \geq n_0$. This is equivalent to the big-Omega definition.

Ex 8. h) Observe that $c \cdot g(n) = 3^n \geq 2^n$ when $c = 1$ for all positive n. Thus $f(n) \in O(g(n))$.

Ex 9. a) Observe that $c \cdot f(n) = \frac{n^2 - n}{2} \geq 6n = g(n)$ when $c = 1$ for all $n \geq 13$. Thus $g(n) \in O(f(n))$.

Ex 9. b) Observe that $c \cdot g(n) = 100n^2 \geq n + 2\sqrt{n} = f(n)$ when $c = 100$ for all $n \geq 1$.
Thus $f(n) \in O(g(n))$.

Ex 9. c) We show that $f(n) \in O(g(n))$. Because $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{nlog(n)}{n\frac{\sqrt{n}}{2}} = \lim_{n\to\infty} \frac{log(n)}{\frac{\sqrt{n}}{2}} = 0$, it follows that there exists positive constants c and n_0 such that $\frac{f(n)}{g(n)} \leq c$ for all $n \geq n_0$. This is equivalent to the big-O definition.

Ex 9. d) Observe that $f(n) = n + log(n) \geq n \geq \sqrt{n} = g(n)$ for all positive n. Thus $g(n) \in O(f(n))$.

Ex 9. e) We show that $g(n) \in O(f(n))$. Because $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{log(n) + 1}{2(log(n))^2} = \lim_{n\to\infty} \frac{1}{2log(n)} + \frac{1}{2(log(n))^2} =$ 0, it follows that there exists positive constants c and n_0 such that $\frac{g(n)}{f(n)} \leq c$ for all $n \geq n_0$. This is equivalent to the big-O definition.

Ex 9. f) We show that $f(n) \in O(g(n))$. Because $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{4nlog(n) + n}{\frac{n^2 - n}{2}} = 0$, it follows that there exists positive c and n_0 such that $\frac{f(n)}{g(n)} \leq c$ for all $n \geq n_0$. This is equivalent to the big-O definition.

Ex 10. Proposition: $n^3 - 3n^2 - n + 1 \in \Theta(n^3)$.

Proof. (Direct.)
Because

$$\lim_{n\to\infty} \frac{n^3 - 3n^2 - n + 1}{n^3} =$$
$$\lim_{n\to\infty} 1 - \frac{3}{n} - \frac{1}{n^2} + \frac{1}{n^3} = 1$$

, it follows that there exists positive constants c_1, c_2, and n_0 such that $\frac{n^3 - 3n^2 - n + 1}{n^3} \geq c_1$ for all $n \geq n_0$ and $\frac{n^3 - 3n^2 - n + 1}{n^3} \leq c_2$ for all $n \geq n_0$. These two statements are respectively equivalent to the big-Omega and big-O definition. Thus we have shown $n^3 - 3n^2 - n + 1 \in \Omega(n^3)$ and $n^3 - 3n^2 - n + 1 \in O(n^3)$. Consequently $n^3 - 3n^2 - n + 1 \in \Theta(n^3)$.

□

Ex 11. Proposition: $n^2 \in O(2^n)$.

Proof. (Direct.)
Because $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$, it follows that there exists positive c and n_0 such that $\frac{n^2}{2^n} \leq c$ for all $n \geq n_0$. This is equivalent to the big-O definition. Thus $n^2 \in O(2^n)$.

□

Ex 12. a) $c = 1.5$

Ex 12. b) $c = 2$

Ex 12. c) $c = 2$

Ex 13. Proposition: If $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, then $f_1(n) + f_2(n) \in O(g_1(n) + g_2(n))$.

Proof. (Direct.) Suppose $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$.

By our supposition it follows that there exists positive constants c_1, c_2, n_1, n_2 such that $f_1(n) \leq c_1 \cdot g_1(n)$ for all $n \geq n_1$ and $f_2(n) \leq c_2 \cdot g_2(n)$ for all $n \geq n_2$. Then let $c_3 = max(c_1, c_2)$ and $n_3 = max(n_1, n_2)$. So $f_1(n) \leq c_3 \cdot g_1(n)$ and $f_2(n) \leq c_3 \cdot g_2(n)$ for all $n \geq n_3$. After adding the second inequality to the first, we have that $f_1(n) + f_2(n) \leq c_3 \cdot (g_1(n) + g_2(n))$ for all $n \geq n_3$. Consequently $f_1(n) + f_2(n) \in O(g_1(n) + g_2(n))$.
□

Ex 14. Proposition: If $f_1(n) \in \Omega(g_1(n))$ and $f_2(n) \in \Omega(g_2(n))$, then $f_1(n) + f_2(n) \in \Omega(g_1(n) + g_2(n))$.

Proof. (Direct.) Suppose $f_1(n) \in \Omega(g_1(n))$ and $f_2(n) \in \Omega(g_2(n))$.

By our supposition it follows that there exists positive constants c_1, c_2, n_1, n_2 such that $f_1(n) \geq c_1 \cdot g_1(n)$ for all $n \geq n_1$ and $f_2(n) \geq c_2 \cdot g_2(n)$ for all $n \geq n_2$. Then let $c_3 = min(c_1, c_2)$ and $n_3 = max(n_1, n_2)$. So $f_1(n) \geq c_3 \cdot g_1(n)$ and $f_2(n) \geq c_3 \cdot g_2(n)$ for all $n \geq n_3$. After adding the second inequality to the first, we have that $f_1(n)+f_2(n) \geq c_3 \cdot (g_1(n)+g_2(n))$ for all $n \geq n_3$. Consequently $f_1(n)+f_2(n) \in \Omega(g_1(n)+g_2(n))$. □

Ex 15. Proposition: If $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, then $f_1(n) \cdot f_2(n) \in O(g_1(n) \cdot g_2(n))$.

Proof. (Direct.) Suppose $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$.

By our supposition it follows that there exists positive constants c_1, c_2, n_1, n_2 such that $f_1(n) \leq c_1 \cdot g_1(n)$ for all $n \geq n_1$ and $f_2(n) \leq c_2 \cdot g_2(n)$ for all $n \geq n_2$. Then let $c_3 = max(c_1, c_2)$ and $n_3 = max(n_1, n_2)$. So $f_1(n) \leq c_3 \cdot g_1(n)$ and $f_2(n) \leq c_3 \cdot g_2(n)$ for all $n \geq n_3$. After multiplying the second inequality with the first, we have that $f_1(n) \cdot f_2(n) \leq (c_3)^2 \cdot (g_1(n) \cdot g_2(n))$ for all $n \geq n_3$. Consequently $f_1(n) \cdot f_2(n) \in O(g_1(n) \cdot g_2(n))$.

□

Ex 16. Proposition: For all $k \geq 1$ and all sets of constants $\{a_k, a_{k-1}, a_{k-2}, ..., a_1, a_0\} \in \mathbb{R}$, $a_k n^k + a_{k-1}n^{k-1} + a_{k-2}n^{k-2} + ... + a_1 n + a_0 \in O(n^k)$.

Proof. (Direct.)

Because

$$\lim_{n\to\infty} \frac{a_k n^k + a_{k-1}n^{k-1} + a_{k-2}n^{k-2} + ... + a_1 n + a_0}{n^k} =$$
$$\lim_{n\to\infty} a_k + \frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + ... + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k} = a_k$$

, it follows that there exists positive constants c and n_0 such that $\frac{a_k n^k + a_{k-1}n^{k-1} + a_{k-2}n^{k-2} + ... + a_1 n + a_0}{n^k} \leq c$ for all $n \geq n_0$. This is equivalent to the big-O definition. Thus $a_k n^k + a_{k-1}n^{k-1} + a_{k-2}n^{k-2} + ... + a_1 n + a_0 \in O(n^k)$.

□

Ex 17. Proposition: $(n + a)^b = \Theta(n^b)$ for any real constants a and b, $b > 0$.

Proof. (Direct.)

Observe that when we multiply $(n + a)^b$ with $\frac{n^b}{n^b}$ we get $(n + a)^b \cdot \frac{n^b}{n^b} = \frac{(n + a)^b}{n^b} \cdot n^b = \left(1 + \frac{a}{n}\right)^b \cdot n^b$.

Then $\lim_{n\to\infty} \frac{\left(1 + \frac{a}{n}\right)^b n^b}{n^b} = \lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^b = 1$ implies that $(n + a)^b = \Theta(n^b)$.

□

Ex 18.
$lg(lg(n)) < ln(n) = lg(n) < (lg(n))^2 < \sqrt{n} < n < nlog(n) < n^{1+\varepsilon} < n^2 + lg(n) = n^2 < n^3 < n - n^3 + 7n^5 < 2^n = 2^{n-1} < e^n < n!$

Ex 19.
$6 < log(log(n)) < log(n) = ln(n) < (log(n))^2 < n^{1/3} + log(n) < \sqrt{n} < \dfrac{n}{log(n)} < n < nlog(n) < n^2 + log(n) = n^2 < n^3 < n - n^3 + 7n^5 < \left(\dfrac{1}{3}\right)^n < \left(\dfrac{3}{2}\right)^n < 2^n < n!$

Ex 20. a) Let $f(n) = n$ and $g(n) = n^2$. Then $f(n) \in o(g(n))$, but $f(n) \notin \Theta(g(n))$.

Ex 20. b) None

Ex 20. c) None

Ex 20. d) Let $f(n) = n^2$ and $g(n) = n$. Then $f(n) \in \Omega(g(n))$, but $f(n) \notin O(g(n))$.

Ex 21. a) True b) False c) True d) False e) True f) True g) False

Ex 22. a) $f(n) \in \Omega(g(n))$ b) $f(n) \in O(g(n))$ c) $f(n) \in \Omega(g(n))$

Ex 23. a) Yes. $O(n^2)$ in worst case does not preclude $O(n)$ for some other case.

Ex 23. b) Technicaly yes. Big-O is an upper bound, if an algorithm always runs in constant time, we can technically say that worst case is $O(2^n)$. This would be a "loose" upper bound and not a often not a useful one.

Ex 23. c) Yes, the lower bound of n^2 (theta specifices lower and upper bound) only applies for the worst-case input.

Ex 23. d) No, this conclusion follows from the definition of theta. The worst-case input has a lower bound of n^2.

Ex 23. e) Yes.

Ex 24. a) No b) No c) Yes d) Yes. All answers based on the definitions of Big-O and Big-Omega.

Ex 25. a) $g(n) = f(n)$ b) $g(n) = n$ c) $g(n) = log(n)$ d) $g(n) = log(n!)$

Ex 26. $f_2 < f_1 < f_4 < f_3$

Ex 27. $f_1 < f_2 < f_3 < f_4$

Ex 28. a) $g(n) = n^4$ b) $g(n) = 4^n$ c) $g(n) = 5^n$

Ex 29. a) True b) True c) True

Ex 30. a) $g(n) = 4^n$ b) $g(n) = nlogn$ c) $g(n) = log(n)^{10}$ d) $g(n) = n^100$

Ex 31.
a) A is O and o of B.
b) A is 0 and o of B.
c) None
d) A is O and o of B.
e) A is Ω and ω of B.
f) A is O and o of B.