Exercises for Section 11.2

Ex 1. R is reflexive, symmetric, and transitive.

Ex 2.
R is not reflexive because $(a, a) \notin R$.
R is not symmetric because $(a, b) \in R$, but $(b, a) \notin R$.
R is transitive.

Ex 3.
R is not reflexive because $(a, a) \notin R$.
R is not symmetric because $(a, b) \in R$, but $(b, a) \notin R$.
R is not transitive because $(c, b) \in R$ and $(b, c) \in R$, but $(c, c) \notin R$.

Ex 4. R is reflexive, symmetric, and transitive.

Ex 5.
R is not reflexive because $(1, 1) \notin R$.
R is symmetric and transitive.

Ex 6.
This is the equality relation on $\mathbb{Z}$.
It is reflexive, symmetric and transitive.

Ex 7.
The 16 different relations on $A = \{a, b\}$:

- 1 - $\emptyset$. This relation is symmetric and transitive, but not reflexive.
- 1 - $\{(a, b), (b, a)\}$. This relation is symmetric, but not reflexive and not transitive.
- 2 - $\{(a, a), (b, b), (a, b), (b, a)\}, \{(a, a), (b, b)\}$. These relations are reflexive, symmetric and transitive.
- 2 - $\{(a, a)\}, \{(b, b)\}$. These relations are symmetric and transitive, but not reflexive.
- 2 - $\{(a, b)\}, \{(b, a)\}$. These relations are transitive, but not reflexive and not symmetric.
- 4 - $\{(a, a), (a, b)\}, \{(a, a), (b, a)\}, \{(b, b), (a, b)\}, \{(b, b), (b, a)\}$. These relations are transitive, but not reflexive and not symmetric.
- 2 - $\{(a, a), (b, b), (a, b)\}, \{(a, a), (b, b), (b, a)\}$. These relations are reflexive and transitive, but not symmetric.
- 2 - $\{(a, a), (a, b), (b, a)\}, \{(b, b), (a, b), (b, a)\}$. These relations are symmetric, but not reflexive and not transitive.

Ex 8.
$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x - y| < 1\}$. This relation is the equality relation on $\mathbb{Z}$. The relation is reflexive, symmetric, and transitive.

Ex 9.
$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 2|x - y\}$. This relation is the $\equiv \pmod 2$ relation on $\mathbb{Z}$. It is reflexive, symmetric and transitive.

Ex 10.
R is symmetric and transitive, but not reflexive. Suppose $x \in A$. R is not reflexive because $(x, x) \notin R$.

Ex 11.
R is reflexive, symmetric, and transitive.

Ex 12. Proposition: The $|$ (division) relation on $\mathbb{Z}$ is reflexive and transitive.

Proof.
The division relation is clearly reflexive as every integer is divisable by itself.

We now use the direct approach to show that the relation is transitive. Suppose $x|y$ and $y|z$ where $x, y, z \in \mathbb{Z}$. So we have that $y = xa$ and $z = yb$ for some integers a, b, by defintion of divisibility. Plugging y in $z = yb$, we get $z = x(ab)$. Thus $x|z$.

□

Ex 13. Proposition: If $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$, then R is reflexive, symmetric and transitive.

Proof.
First we show that the relation is reflexive. Note that $x - x = 0 \in \mathbb{Z}$ for all $x \in R$. Thus the relation is reflexive.

Next we show that the relation is symmetric. Let $(x, y) \in R$. Then $x - y = z$ and $z \in \mathbb{Z}$, by definition of the relation. Multiply both sides by -1, we get $y - x = -z$. Because z is an integer, it follows that $-z$ is an integer. Therefore $(y, x) \in R$ and consequently R is symmetric.

Finally we show that the relation is transitive. Let $(x, y) \in R$ and $(y, z) \in R$. Then $x - y = a \in \mathbb{Z}$ and $y - z = b \in \mathbb{Z}$. Plugging the second equation into the first, we get $x - y = a \leftrightarrow x - (b + z) = a \leftrightarrow x - z = a + b$. The sum of two integers is an integer, so $a + b$ is an integer and thus $(x, z) \in R$. Hence the relation is transitive.

□

Ex 14. **Proposition:** If R is a symmetric and transitive relation on set A, and there is an element $a \in A$ for which $(a, x) \in R$ for every $x \in A$, then R is reflexive.

Proof. (Direct.)
Because $(a, x) \in R$ and R is symmetric, it follows that $(x, a) \in R$. Then by the definition of transitive property, since $(x, a) \in R$ and $(a, x) \in R$, it must be the case that $(x, x) \in R$. Thus R is reflexive.

□

Ex 15. The proposition is false. Let $|A| = 1$ and $R = \emptyset$ be a relation on A. Then R is symmetric and transitive, but not reflexive.

Ex 16. **Proposition:** If R is the relation $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 \equiv y^2 \pmod 4\}$, then it is reflexive, symmetric and transitive.

Proof. (Direct.)
First we show that the relation is reflexive. Suppose $z \in \mathbb{Z}$. Then $z^2 - z^2 = 0$ and $4|0$ implies that $z^2 \equiv z^2 \pmod 4$. Thus $(z, z) \in R$.

Next we show the symmetric property. Suppose $(a, b) \in R$. Then by definition of modular congruence, it follows that $4|a^2 - b^2$. Thus $a^2 - b^2 = 4x$ for some integer x. Then we multiply both sides by -1, so $a^2 - b^2 = 4x \leftrightarrow (-1)(a^2 - b^2) = 4x(-1) \leftrightarrow (b^2 - a^2) = 4(-x)$. Thus $4|b^2 - a^2$ and consequently $(b, a) \in R$. Thus the relation is symmetric.

Finally, we show the transitive property. Suppose $(a, b)\ inR$ and $(b, c) \in R$. Then it follows that $a^2 - b^2 = 4i$ and $b^2 - c^2 = 4j$ for some integers i, j. Adding the second equation to the first, we get $a^2 - b^2 + (b^2 - c^2) = 4i + 4j \leftrightarrow a^2 - c^2 = 4(i + j)$. Thus $4|a^2 - c^2$ and consequently $(a, c) \in R$. Thus the relation is transitive.

□

Ex 17. The relation is reflexive and symmetric, but not transitive. It is not transitive because $(1, 2) \in R$ and $(2, 3) \in R$, but $(1, 3) \notin R$.

Ex 18. A reflexive, symmetric, but not transitive relation:
Let $R = \{(a, b) \in \mathbb{Z}^2 : a = b\} \cup \{(1, 2), (2, 1), (2, 3), (3, 2)\}$ be a relation on $\mathbb{Z}$.

A reflexive, but not symmetric and not transitive relation:
Let $R = \{(a, b) \in \mathbb{Z}^2 : a = b\} \cup \{(0, 1), (1, 2)\}$ be a relation on $\mathbb{Z}$.

A symmetric and transitive, but not reflexive relation:
Let $R = \{(a, b) \in \mathbb{Z}^2 : a = 1, a = b\} = \{(1, 1)\}$ be a relation on $\mathbb{Z}$.