Exercises for Section 11.3

Ex 1.
$[1] = \{1\}$
$[2] = [3] = \{2, 3\}$
$[4] = [5] = [6] = \{4, 5, 6\}$

Ex 2.
$R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (d, a), (b, c), (c, b), (e, d), (d, e), (a, e), (e, a)\}$.

Ex 3.
$R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (d, a), (b, c), (c, b)\}$.

Ex 4.
R has one equivalence class: $\{a, b, c, d, e\}$.

Ex 5.
The two equivalence relations on A are $\{(a, a), (b, b)\}$ and $\{(a, a), (a, b), (b, a), (b, b)\}$.

Ex 6.
The five different equivalence relations on A are:
$\{(a, a), (b, b), (c, c)\}$
$\{(a, a), (b, b), (c, c), (a, b), (b, a)\}$
$\{(a, a), (b, b), (c, c), (b, c), (b, c)\}$
$\{(a, a), (b, b), (c, c), (a, c), (c, a)\}$
$\{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b), (a, c), (c, a)\}$

Ex 7.
Lemma 1: Let $x, y \in \mathbb{Z}$. x and y have the same parity if and only if $3x - 5y$ is even.

Proof.
First we show that if x and y have the same parity, then $3x - 5y$ is even. We consider two cases.
Case 1. Suppose x and y are even.
Then $x = 2a$ and $y = 2b$ for some integers a, b. Thus $3x - 5y = 3(2a) - 5(2b) = 2(3a - 5b)$ is even.
Case 2. Suppose x and y are odd.
Then $x = 2a + 1$ and $y = 2b + 1$ for some integers a, b. Thus $3x - 5y = 3(2a + 1) - 5(2b + 1) = 6a + 3 - 10b - 5 = 6a - 10b - 2 = 2(3a - 5b - 1)$ is even.

Secondly we show that if $3x - 5y$ is even, then x and y have the same parity. We use the contrapositive technique with two cases.
Case 1. Suppose x is even and y is odd.
Then $x = 2a$ and $y = 2b + 1$ for some integers a, b. Thus $3x - 5y = 3(2a) - 5(2b + 1) = 6a - 10b - 5 = 2(3a - 5b - 2) - 1$ is odd.
Case 2. Suppose x is odd and y is even.
Then $x = 2a + 1$ and $y = 2b$ for some integers a, b. Thus $3x - 5y = 3(2a + 1) - 5(2b) = 6a + 3 - 10b = 2(3a - 5b + 1) + 1$ is odd.

Therefore x and y have the same parity if and only if $3x - 5y$ is even.

$\square$

Proposition: If $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 2|(3x - 5y)\}$, then R is an equivalence relation.

Proof.
The reflexive, symmetric, and transitive properties of R follow from lemma 1.

$\square$

R has two equivalence classes, the set of odd and the set of even integers.

Ex 8. Proposition: If $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 + y^2 \text{ is even}\}$, then R is an equivalence relation.

Proof.
First we show that R is reflexive. Let $a \in \mathbb{Z}$. Suppose a is even, so $a = 2b$ for some integer b. Then $a^2 + a^2 = 4b^2 + 4b^2 = 2(4b^2)$ is even. Likewise, suppose a is odd, then $a = 2b + 1$ for some integer b. So $a^2 + a^2 = (2b + 1)^2 + (2b + 1)^2 = 2(4b^2 + 4b + 1)$ is even. In either case we have that $a \in R$.

Secondly we show that R is symmetric. Suppose $(x, y) \in R$. Then $x^2 + y^2 = 2a$ for some integer a. By the commutative property of addittion, it follows that $x^2 + y^2 = y^2 + x^2 = 2a$. Thus $(y, x) \in R$.

Finally we show that R is transitive. Suppose $(x, y), (y, z) \in R$. So $x^2 + y^2 = 2a$ and $y^2 + z^2 = 2b$ for some integers a, b. After subtracting the latter equation from the former, we get $x^2 + y^2 - (y^2 + z^2) = 2a - 2b \leftrightarrow x^2 - z^2 = 2(a - b)$. Thus $x^2 - z^2$ is even which implies that $x^2 + z^2$ is even. Therefore $(x, z) \in R$.

$\square$

R has two equivalence classes, the set of odd and the set of even integers.

Ex 9. Proposition: Suppose $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 4|(x + 3y)\}$, then R is an equivalence relation on $\mathbb{Z}$.

Proof.
First we show that R is reflexive. Suppose $a \in \mathbb{Z}$, then $a+3a = 4a$ which is divisable by 4. Thus $(a, a) \in R$.

Secondly we show that R is symmetric. Suppose $(x, y) \in R$, so $x + 3y = 4a$ for some integer a. Multiply by 3, we get $3x + 9y = 12a \leftrightarrow 3x + y = 12 - 8y = 4(3 - 2y)$ which is divisable by 4. Thus $(y, x) \in R$.

Finally we show that R is transitive. Suppose $(x, y), (y, z) \in R$. Then $x+3y = 4a$ and $y+3z = 4b$ for some integers a, b. Add the latter equation to the former, we get $x+3y+(y+3z) = 4a+4b \leftrightarrow x+3z = 4(a+b-y)$. Thus $(x, z) \in R$.

$\square$

Equivalence classes:
$[0] = \{x \in \mathbb{Z} : (x, 0) \in R\} = \{x \in \mathbb{Z} : 4|x\} = \{..., -8, -4, 0, 4, 8, ...\}$.
$[1] = \{x \in \mathbb{Z} : (x, 1) \in R\} = \{x \in \mathbb{Z} : x \equiv 1 \pmod 4\} = \{..., -7, -3, 1, 5, 9, ...\}$.
$[2] = \{x \in \mathbb{Z} : (x, 2) \in R\} = \{x \in \mathbb{Z} : x \equiv 2 \pmod 4\} = \{..., -2, 2, 6, 10, ...\}$.
$[3] = \{x \in \mathbb{Z} : (x, 3) \in R\} = \{x \in \mathbb{Z} : x \equiv 3 \pmod 4\} = \{..., -5, -1, 3, 7, 11, ...\}$.

Ex 10. Proposition: Suppose R and S are two equivalence relations on a set A. $R \cap S$ is also an equivalence relation.

Proof.
First we show that $R \cap S$ is reflexive. Because R and S are equivalence relations, it follows that $(a, a) \in R$ and $(a, a) \in S$ for all $a \in A$. Thus $(a, a) \in R \cap S$.

Then we show that $R \cap S$ is symmetric. Suppose $(a, b) \in R \cap S$. Then $(a, b) \in R$ and $(a, b) \in S$, by definition of set intersection. Because R and S are equivalence relations, it follows that $(b, a) \in R$ and $(b, a) \in S$. Thus $(b, a) \in R \cap S$.

Finally we show that $R \cap S$ is transitive. Suppose $(a, b), (b, c) \in R \cap S$. Then $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$, by definition of set intersection. Because R and S are equivalence relations, it follows that $(a, c) \in R$ and $(a, c) \in S$. Thus $(a, c) \in R \cap S$.

□

Ex 11. Disproof: Consider the equivalence relation $R = \mathbb{Z} \times \mathbb{Z}$ on $\mathbb{Z}$. Then R has one equivalence class, namely $[0] = \mathbb{Z}$.

Ex 12. Disproof: Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ and $S = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$ be equivalence relations on $A = \{1, 2, 3\}$. Then $R \cup S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$ is not an equivalence relation on A because $R \cup S$ is not transitive. It is not transitive because $(1, 2), (2, 3) \in R \cup S$, but $(1, 3) \notin R \cup S$.

Ex 13. There are $\frac{|A|}{m}$ equivalence classes of size m. There are m^2 pairs corresponding to each class. Thus $|R| = \frac{|A|}{m} \cdot m^2 = m|A|$.

Ex 14. Proposition: S is an equivalence relation and $R \subseteq S$.

Proof.

Because R is reflexive, it follows that $(x, x) \in R$ for all $x \in A$. So $(x, x) \in S$, by definition of S. Thus S is reflexive.

We show that S is symmetric. Suppose $(x, y) \in S$. That means that there exists n such that

$$(x, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, y) \in R$$

Because R is symmetric, it is must also be that the case that

$$(x_1, x), (x_2, x_1), ..., (x_n, x_{n-1}), (y, x_n) \in R$$

Thus $(y, x) \in S$ and consequently S is symmetric.

We show that S is transitive. Suppose $(x, y) \in S$ and $(y, z) \in S$. Then, by definition S, there exists $n_1, n_2 \in \mathbb{N}$ such that

$$(x, x_1), (x_1, x_2), ..., (x_{n_1-1}, x_{n_1}), (x_{n_1}, y) \in R$$
$$(y, y_1), (y_1, y_2), ..., (y_{n_2-1}, y_{n_2}), (y_{n_2}, z) \in R$$

In other words, there is a path from x to z in R. So there exists

$$(x, x_1), (x_1, x_2), ..., (x_{n_1-1}, x_{n_1}), (x_{n_1}, y), (y, y_1), (y_1, y_2), ..., (y_{n_2-1}, y_{n_2}), (y_{n_2}, z) \in R$$

Thus $(x, z) \in S$ and consequently S is transitive.

$\square$

Note: we have skipped showing $R \subseteq S$ and that S is the unique smallest equivalence relation on A containing R.

Ex 15. An equivalence relation R on set A with 4 equivalence classes is a graph with 4 connected components. The number of different equivalence relations S on A such that $R \subseteq S$ is then the number of ways to merge these 4 connected components. This is equivalent to the Bell number; the number of ways to partition a finite set. Thus the answer is $B_4 = 15$ where B_4 denotes the 4-th bell number. Below we enumerate the 15 different partitions.

Let $CC = \{1, 2, 3, 4\}$ denote the set of connected components of R. Then the 15 partitions are:

1. $\{\{1\}, \{2\}, \{3\}, \{4\}\}$
2. $\{\{1, 2\}, \{3\}, \{4\}\}$
3. $\{\{1, 3\}, \{2\}, \{4\}\}$
4. $\{\{1, 4\}, \{2\}, \{3\}\}$

5. $\{\{2,3\},\{1\},\{4\}\}$
6. $\{\{2,4\},\{1\},\{3\}\}$
7. $\{\{3,4\},\{1\},\{2\}\}$
8. $\{\{1,2\},\{3,4\}\}$
9. $\{\{1,3\},\{2,4\}\}$
10. $\{\{1,4\},\{2,3\}\}$
11. $\{\{1\},\{2,3,4\}\}$
12. $\{\{2\},\{1,3,4\}\}$
13. $\{\{3\},\{1,2,4\}\}$
14. $\{\{4\},\{1,2,3\}\}$
15. $\{\{1,2,3,4\}\}$

Ex 16. Proposition: Relation $=$ defined on page 213 is transitive.

Proof. (Direct.) Suppose $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$.

So we have that $ad = bc$ and $cf = de$. Substituing the former into the latter yields $\left(\frac{ad}{b}\right) f = de \leftrightarrow af = be \leftrightarrow \frac{a}{b} = \frac{e}{f}$. Thus the relation $=$ is transitive.

$\square$