Exercises for section 12.6

Ex 1.

$f([-3,5]) = \{x^2+3 : x \in [-3,5]\} = [3,28]$

$f^{-1}([12,19]) = \{x \in \mathbb{R} : f(x) \in [12,19]\} = [-4,-3] \cup [3,4]$

Ex 2.

$f(\{1,2,3\}) = \{f(x) : x \in \{1,2,3\}\} = \{3,8\}$

$f(\{4,5,6,7\}) = \{f(x) : x \in \{4,5,6,7\}\} = \{1,2,4,6\}$

$f(\emptyset) = \emptyset$

$f^{-1}(\{0,5,9\}) = \{x \in \{1,2,...,7\} : f(x) \in \{0,5,9\}\} = \emptyset$

$f^{-1}(\{0,3,5,9\}) = \{1,3\}$

Ex 3.

There are $\binom{7}{3} \cdot 4^4 = 8960$ different functions such that $|f^{-1}(\{3\})| = 3$.

Ex 4.

There are $\binom{8}{4} \cdot 6^4 = 90720$ different functions such that $|f^{-1}(\{2\})| = 90720$.

Ex 5. Proposition: If $f : A \to B$ and a subset $X \subseteq A$, then $X \subseteq f^{-1}(f(X))$.

Proof.
Suppose $x \in X$. Let $Y = f(X) = \{f(z) : z \in X\}$. So we have $f(x) \in Y$. Then by the preimage definition $f^{-1}(Y) = \{a \in A : f(a) \in Y\}$. Because $x \in A$ and $f(x) \in Y$, it follows that $x \in f^{-1}(Y)$. Thus we have shown that $X \subseteq f^{-1}(f(X))$.

□

Ex 6. Conjecture: If $f : A \to B$ and a subset $Y \subseteq B$, then $f(f^{-1}(Y)) = Y$.

We show the conjecture to be false. Suppose $f : \{1\} \to \{0,1\}$ defined as $f = \{(1,1)\}$. Then observe that when $Y = \{0\}$ we have $f(f^{-1}(Y)) = f(\emptyset) = \emptyset$. Thus $f(f^{-1}(Y)) \neq Y$.

Ex 7. Proposition: If $f : A \to B$ and subsets $W, X \subseteq A$, then $f(W \cap X) \subseteq f(W) \cap f(X)$.

Proof. Suppose $y \in f(W \cap X)$. Then there exists $x \in W$ and $x \in X$ such that $f(x) = y$. That also implies that $y \in f(W)$ and $y \in f(X)$. Thus $y \in f(W) \cap f(X)$ and consequently $f(W \cap X) \subseteq f(W) \cap f(X)$.

□

Ex 8. Conjecture: If $f : A \to B$ and subsets $W, X \subseteq A$, then $f(W \cap X) = f(W) \cap f(X)$.

We show the conjecture to be false. Let $f : \{1,2\} \to \{99\}$ defined as $f = \{(1,99),(2,99)\}$. Furthermore, let $W = X = \{1\} \subseteq \{1,2\}$. Then $f(W \cap X) = f(\emptyset) = \emptyset \neq \{99\} = \{99\} \cap \{99\} = f(W) \cap f(X)$.

Ex 9. Proposition: If $f : A \to B$ and subsets $W, X \subseteq A$, then $f(W \cup X) = f(W) \cup f(X)$.

Proof.
First we show that $f(W \cup X) \subseteq f(W) \cup f(X)$. Suppose $y \in f(W \cup X)$. By the definition of image, there exists an $x \in W$ or $x \in X$ such that $f(x) = y$. Thus $y \in f(W)$ or $y \in f(X)$, which implies $y \in f(W) \cup f(X)$.

Then we show that $f(W) \cup f(X) \subseteq f(W \cup X)$. Suppose $y \in f(W) \cup f(X)$. Thus there exists $x \in W$ or $x \in X$ such that $f(x) = y$. So $x \in W \cup X$ and consequently $y \in f(W \cup X)$.

Because $f(W \cup X) \subseteq f(W) \cup f(X)$ and $f(W) \cup f(X) \subseteq f(W \cup X)$, it follows that $f(W \cup X) = f(W) \cup f(X)$.

□

Ex 10. Proposition: If $f : A \to B$ and subsets $Y, Z \subseteq B$, then $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

Proof.
First we show that $f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$. Suppose $x \in f^{-1}(Y \cap Z)$. By the definition of preimage, we have that $y = f(x) \in Y \cap Z$. So $y \in Y$ and $y \in Z$. Thus $x \in f^{-1}(Y)$ and $x \in f^{-1}(Z)$ which means that $x \in f^{-1}(Y) \cap f^{-1}(Z)$.

Then we show that $f^{-1}(Y) \cap f^{-1}(Z) \subseteq f^{-1}(Y \cap Z)$. Suppose $x \in f^{-1}(Y) \cap f^{-1}(Z)$. So $x \in f^{-1}(Y)$ and $x \in f^{-1}(Z)$. Thus there exists $y = f(x)$ such that $y \in Y$ and $y \in Z$. So $y \in Y \cap Z$ and consequently $x \in f^{-1}(Y \cap Z)$.

Because $f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$ and $f^{-1}(Y) \cap f^{-1}(Z) \subseteq f^{-1}(Y \cap Z)$, it follows that $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

□

Ex 11. Proposition: If $f : A \to B$ and subsets $Y, Z \subseteq B$, then $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.

Proof.
First we show that $f^{-1}(Y \cup Z) \subseteq f^{-1}(Y) \cup f^{-1}(Z)$. Suppose $x \in f^{-1}(Y \cup Z)$. Then there exists $y = f(x)$ such that $y \in Y \cup Z$. So $y \in Y$ or $y \in Z$. Thus $x \in f^{-1}(Y)$ or $x \in f^{-1}(Z)$, which implies that $x \in f^{-1}(Y) \cup f^{-1}(Z)$.

Then we show that $f^{-1}(Y) \cup f^{-1}(Z) \subseteq f^{-1}(Y \cup Z)$. Suppose $x \in f^{-1}(Y) \cup f^{-1}(Z)$. Then there exists $y = f(x)$ such that $y \in Y$ or $y \in Z$. So $y \in Y \cup Z$ and consequently $x \in f^{-1}(Y \cup Z)$.

Because $f^{-1}(Y \cup Z) \subseteq f^{-1}(Y) \cup f^{-1}(Z)$ and $f^{-1}(Y) \cup f^{-1}(Z) \subseteq f^{-1}(Y \cup Z)$, it follows that $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.

□

Ex 12. Proposition: Let $f : A \to B$. f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

Proof.
We show that f is injective implies $X = f^{-1}(f(X))$. Suppose f is injective. By exercise 5 we have that $X \subseteq f^{-1}(f(X))$. Then by showing $f^{-1}(f(X)) \subseteq X$, it follows that $X = f^{-1}(f(X))$. Suppose $x \in f^{-1}(f(X))$. So $f(x) \in f(X)$ and since f is injective we have that $x \in X$. Thus $f^{-1}(f(X)) \subseteq X$.

Next we show that $X = f^{-1}(f(X))$ for all $X \subseteq A$ implies f is injective. Observe that when $|A| \leq 1$ it follows that f is injective. So henceforth, we only concern ourselves with $|A| > 1$. Suppose for the sake of contradiction that $X = f^{-1}(f(X))$ for all $X \subseteq A$ and f is not injective. Let X be defined such that $x \in X$, $y \in A$, $y \notin X$, $x \neq y$, and $f(x) = f(y)$. Then $f(x) = f(y) \in f(X)$ and consequently $x, y \in f^{-1}(f(X))$. But that leads to a contradiction as we have $y \notin X$, $y \in f^{-1}(f(X))$, and $X = f^{-1}(f(X))$. Thus f is injective.

□

Ex 12. Proposition: Let $f : A \to B$. f is surjective if and only if $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

Proof.
We show that f is surjective implies $Y = f(f^{-1}(Y))$. Suppose f is surjective. First we prove that $Y \subseteq f(f^{-1}(Y))$. Suppose $y \in Y$. By the surjective definition, there exists $x \in f^{-1}(Y)$ such that $y = f(x)$. Then $y \in f(f^{-1}(Y))$ and consequently we have established that $Y \subseteq f(f^{-1}(Y))$. Next we show that $f(f^{-1}(Y)) \subseteq Y$. Suppose $y \in f(f^{-1}(Y))$. Then there exists $x \in f^{-1}(Y)$ such that $y = f(x)$. Thus $y \in Y$ which means that $f(f^{-1}(Y)) \subseteq Y$.

Then we show that $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$ implies f is surjective. Suppose for the sake of contradiction that $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$ and f is not surjective. Let Y be defined such that $y \in Y$ and $f(x) \neq y$ for all $x \in A$. Thus $f(z) \neq y$ for all $z \in f^{-1}(Y)$ and consequently $y \notin f(f^{-1}(Y))$. But that leads to a contradiction as we $y \in Y$, $y \notin f(f^{-1}(Y))$ and $Y = f(f^{-1}(Y))$. Thus f is surjective.

□

Ex 13. Conjecture: If $f : A \to B$ and $X \subseteq A$, then $f(f^{-1}(f(X))) = f(X)$.

Proof.
First we show that $f(f^{-1}(f(X))) \subseteq f(X)$. Suppose $y \in f(f^{-1}(f(X)))$. So there exists $x \in f^{-1}(f(X))$ such that $f(x) = y$. Thus $y \in f(X)$.

Next we show that $f(X) \subseteq f(f^{-1}(f(X)))$. Suppose $y \in f(X)$. Then there exists $x \in f^{-1}(f(X))$ such that $f(x) = y$. Thus $y \in f(f^{-1}(f(X)))$.

Because $f(f^{-1}(f(X))) \subseteq f(X)$ and $f(X) \subseteq f(f^{-1}(f(X)))$, it follows that $f(f^{-1}(f(X))) = f(X)$.

□

Ex 14. Conjecture: If $f : A \to B$ and $Y \subseteq B$, then $f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y)$.

Proof.
First we show that $f^{-1}(f(f^{-1}(Y))) \subseteq f^{-1}(Y)$. Suppose $x \in f^{-1}(f(f^{-1}(Y)))$. Then we have that $f(x) \in f(f^{-1}(Y))$. Thus $x \in f^{-1}(Y)$.

Next we show that $f^{-1}(Y) \subseteq f^{-1}(f(f^{-1}(Y)))$. Suppose $x \in f^{-1}(Y)$. Then we have $f(x) \in f(f^{-1}(Y))$. Thus $x \in f^{-1}(f(f^{-1}(Y)))$.

Because $f^{-1}(f(f^{-1}(Y))) \subseteq f^{-1}(Y)$ and $f^{-1}(Y) \subseteq f^{-1}(f(f^{-1}(Y)))$, it follows that $f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y)$.

$\square$