Exercises for section 12.2

Ex 1. Let $f : A \to B$ be defined as $f = \{(1, a), (2, a), (3, a), (4, a)\}$. It is not injective because $f(1) = f(2) = a$. Also, it is not surjective because for all $x \in A$, $f(x) \neq b$.

Ex 2. Proposition: $f : (0, \infty) \to \mathbb{R}$ defined as $f(n) = ln(n)$ is bijective.

Proof.
We show that f is injective using the contrapositive approach. Suppose $a, b \in (0, \infty)$ and $f(a) = f(b)$. So $f(a) = f(b) \leftrightarrow ln(a) = ln(b) \leftrightarrow a = b$. Thus f is injective.

Let $b \in \mathbb{R}$ and observe that when $n = e^b \in (0, \infty)$ we have $f(n) = ln(e^b) = b$. Thus f is surjective.

□

Ex 3. Observe that $cos(0) = cos(2\pi) = 1$, thus the function is not injective. Also, observe that for all $a \in \mathbb{R}$, $cos(a) \neq 2$. But $2 \in \mathbb{R}$, thus the function is not surjective. The new domain and codomain would make the function surjective, but not injective.

Ex 4. f is not surjective because $(0, 0) \in \mathbb{Z}^2$, but $f(x) \neq (0, 0)$ for all $x \in \mathbb{Z}$. We proceed to show that f is injective.

Proposition: If $f : \mathbb{Z} \to \mathbb{Z}^2$ is defined as $f(n) = (2n, n + 3)$, then f is injective.

Proof. (Contrapositive.) Suppose $a, b \in \mathbb{Z}$ and $f(a) = f(b)$.
Because $f(a) = f(b)$, we have that $(2a, a+3) = (2b, b+3)$. Thus it must be the case that $2a = 2b \leftrightarrow a = b$. Hence $a = b$.

□

Ex 5. f is not surjective because $2 \in \mathbb{Z}$, but $f(x) \neq 2$ for all $x \in \mathbb{Z}$. We proceed to show that f is injective.

Proposition: If $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$, then f is injective.

Proof. (Contrapositive.) Suppose $a, b \in \mathbb{Z}$ and $f(a) = f(b)$.
Because $f(a) = f(b)$, we have that $2a + 1 = 2b + 1$. Then $2a + 1 = 2b + 1 \leftrightarrow a = b$. Hence $a = b$.

□

Ex 6. f is not injective because $f(4, 0) = f(0, -3) = 12$. We proceed to show that f is surjective.

Proposition: If $f : \mathbb{Z}^2 \to \mathbb{Z}$ is defined as $f(n, m) = 3n - 4m$, then f is surjective.

Proof. (Direct.) Suppose $b \in \mathbb{Z}$.
By Bezout's identity, we have that $3x + (-4)y = gcd(3, -4)$ for some integers x, y. So $3x + (-4)y = 1$, since $gcd(3, -4) = 1$. Then multiplying both sides by b we get $3bx - 4by = b$. Thus there exists integers $n = bx$ and $m = by$ such that $f(n, m) = 3n - 4m = b$.

□

Ex 7. f is not injective because $f(2,0) = f(0,-1) = 4$. f is not surjective because $f(n,m)$ is even for all integers n, m. So for an odd $b \in \mathbb{Z}$ it follows that $f(n,m) \neq b$.

Ex 8. Proposition $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m,n) = (m+n, 2m+n)$ is bijective.

Proof.
First we show that f is injective using the contrapositive approach. Suppose $f(a,b) = f(a',b')$ where $a, b, a', b' \in \mathbb{Z}$. So we have the following system of equations:

$$a + b = a' + b' \text{ and } 2a + b = 2a' + b'$$

Solving the first equation for a and substituing it into the second equation we get $2a + b = 2a' + b' \leftrightarrow 2(a' + b' - b) + b = 2a' + b' \leftrightarrow 2a' + 2b' - 2b + b = 2a' + b \leftrightarrow 2b' = 2b \leftrightarrow b' = b$. With this, we can simplify the first equation $a+b = a'+b' \leftrightarrow a = a'$. Thus $a = a'$ and $b = b'$, and consequently f is injective.

Then we show that f is surjective. Let $a, b \in \mathbb{Z}$. Observe that when $m = b - a, n = 2a - b$ we have $f(m,n) = (m+n, 2m+n) = ((b-a) + (2a-b), 2(b-a) + (2a-b)) = (a,b)$. Thus f is surjective.

□

Ex 9. Proposition $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined as $f(x) = \dfrac{5x+1}{x-2}$ is bijective.

Proof.
First we show that f is injective using the contrapositive approach. Suppose $f(a) = f(b)$ where $a, b \in \mathbb{R}, a, b \neq 2$. Then $f(a) = f(b) \leftrightarrow \dfrac{5a+1}{a-2} = \dfrac{5b+1}{b-2} \leftrightarrow 5ab - 10a + b - 2 = 5ab - 10b + a - 2 \leftrightarrow -11a = -11b \leftrightarrow a = b$. Thus $a = b$ and consequently f is injective.

Then we show that f is surjective. Let $b \in \mathbb{R}, b \neq 5$. Observe that when $x = \dfrac{2b+1}{b-5}, x \neq 2$ we have $f(x) = \dfrac{5x+1}{x-2} = \dfrac{5\left(\dfrac{2b+1}{b-5}\right)+1}{\left(\dfrac{2b+1}{b-5}\right)-2} = \dfrac{\left(\dfrac{10b+5}{b-5}\right)+\left(\dfrac{b-5}{b-5}\right)}{\left(\dfrac{2b+1}{b-5}\right)+\left(\dfrac{-2b+10}{b-5}\right)} = \dfrac{\left(\dfrac{11b}{b-5}\right)}{\left(\dfrac{11}{b-5}\right)} = \dfrac{11b}{11} = b$. Thus f is surjective.

□

Ex 10. **Proposition** $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined as $f(x) = \left(\dfrac{x+1}{x-1}\right)^3$ is bijective.

Proof. First we show that f is injective using the contrapositive approach. Suppose $a, b \in \mathbb{R}, a, b \neq 1$ and $f(a) = f(b)$. Then $f(a) = f(b) \leftrightarrow \left(\dfrac{a+1}{a-1}\right)^3 = \left(\dfrac{b+1}{b-1}\right)^3 \leftrightarrow \dfrac{a+1}{a-1} = \dfrac{b+1}{b-1} \leftrightarrow ab - a + b - 1 = ab - b + a - 1 \leftrightarrow -2a = -2b \leftrightarrow a = b$. Thus $a = b$ and consequently f is injective.

Then we show that f is surjective. Let $b \in \mathbb{R}, b \neq 1$. Observe that when $x = \dfrac{\sqrt[3]{b}+1}{\sqrt[3]{b}-1}, x \neq 1$ we have

$$f(x) = \left(\frac{x+1}{x-1}\right)^3 = \left(\frac{\frac{\sqrt[3]{b}+1}{\sqrt[3]{b}-1}+1}{\frac{\sqrt[3]{b}+1}{\sqrt[3]{b}-1}-1}\right)^3 = \left(\frac{\frac{\sqrt[3]{b}+1}{\sqrt[3]{b}-1}+\frac{\sqrt[3]{b}-1}{\sqrt[3]{b}-1}}{\frac{\sqrt[3]{b}+1}{\sqrt[3]{b}-1}-\frac{\sqrt[3]{b}-1}{\sqrt[3]{b}-1}}\right)^3 = \left(\frac{\frac{2\sqrt[3]{b}}{\sqrt[3]{b}-1}}{\frac{2}{\sqrt[3]{b}-1}}\right)^3 = \left(\frac{2\sqrt[3]{b}}{2}\right)^3 = \left(\sqrt[3]{b}\right)^3 = b$$

Thus f is surjective.

□

Ex 11. θ is not surjective because $0 \in \mathbb{Z}$, but for any $a \in \{0, 1\}, b \in \mathbb{N}$ we have $f(a, b) \neq 0$. We proceed to show that θ is injective.

Proof. (Contrapositive.) Suppose $a, a' \in \{0, 1\}, b, b' \in \mathbb{N}$, and $\theta(a, b) = \theta(a', b')$.
So we have that $\theta(a, b) = \theta(a', b') \leftrightarrow (-1)^a b = (-1)^{a'} b'$. We consider two cases.

Case 1. Suppose $a = 0$.
Then LHS is $(-1)^a b = b$ is positive. Thus RHS must also be positive, which means that $a' = 0$. So $a = a'$ which implies that $(-1)^a b = (-1)^{a'} b' \leftrightarrow b = b'$.

Case 2. Suppose $a = 1$.
Then LHS is $(-1)^a b = -b$ is negative. Thus RHS must also be negative, which means that $a' = 1$. So $a = a'$ which implies that $(-1)^a b = (-1)^{a'} b' \leftrightarrow b = b'$.

In any case we have that $a = a'$ and $b = b'$. Thus θ is injective.

□

Ex 12. Proposition: $\theta : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a,b) = a - 2ab + b$ is bijective.

Proof.
We show that θ is injective using the contrapositive approach. Suppose $a, a' \in \{0,1\}, b, b \in \mathbb{N}$, and $\theta(a,b) = \theta(a',b')$. So we have that $\theta(a,b) = \theta(a',b') \leftrightarrow a - 2ab + b = a' - 2a'b' + b'$. We consider two cases.

Case 1. Suppose $a = 0$.
Consider $a' = 1$, then $a - 2ab + b = a' - 2a'b' + b' \leftrightarrow b = 1 - b'$. Because $b' \in \mathbb{N}$ we have that $b \leq 0$. Thus we have have a contradiction, $b \leq 0$ and $b > 0$. So it must be the case that $a' = a = 0$ and consequently $a - 2ab + b = a' - 2a'b' + b' \leftrightarrow b = b'$.

Case 2. Suppose $a = 1$.
Consider $a' = 0$, then $a - 2ab + b = a' - 2a'b' + b' \leftrightarrow 1 - b = b'$. Because $b \in \mathbb{N}$ we have that $b' \leq 0$. Thus we have have a contradiction, $b' \leq 0$ and $b' > 0$. So it must be the case that $a' = a = 1$ and consequently $a - 2ab + b = a' - 2a'b' + b' \leftrightarrow b = b'$.

In any case we have that $a = a'$ and $b = b'$. Thus θ is injective.

Next we show that f is surjective. Let $a \in \{0,1\}, b \in \mathbb{N}$. As previously noted, we have $\theta(a,b) = b$ when $a = 0$ and $\theta(a,b) = 1 - b$ when $a = 1$. Then

$$Im(\theta) = Im(\theta(0,b)) \cup Im(\theta(1,b)) = \mathbb{N} \cup \{1 - x : x \in \mathbb{N}\} = \mathbb{N} \cup \{..., -3, -2, -1, 0\} = \mathbb{Z} = Cdm(\theta)$$

Where Im and Cdm denote image and codomain respectively. Thus f is surjective.

$\square$

Ex 13. f is neither injective or surjective.

When $x = 0$, $f(x,y) = (0,0)$ for all $y \in \mathbb{R}$. E.g. $f(0,1) = f(0,2) = (0,0)$, thus f is not injective.

Then observe that $(3,0) \in \mathbb{R} \times \mathbb{R}$, but $(3,0) \notin f$. Thus f is not surjective.

Ex 14. Let P denote the powerset. $\theta : P(\mathbb{Z}) \to P(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$ is bijective.

Proof.
We first show that θ is injective using the contrapositive approach. Suppose $X, Y \in P(\mathbb{Z})$ and $\theta(X) = \theta(Y)$. So $\theta(X) = \theta(Y) \leftrightarrow \overline{X} = \overline{Y} \leftrightarrow X = Y$. Thus θ is injective.

Then we show that θ is surjective. Suppose $Y \in P(\mathbb{Z})$. Observe then that when $X = \overline{Y} \in P(\mathbb{Z})$ we have $\theta(X) = \overline{X} = \overline{\overline{Y}} = Y$. Thus we have shown that for any set Y there exists a set X such that $\theta(X) = Y$. Thus θ is surjective.

$\square$

Ex 15. Let $X = \{A, B, C, D, E, F, G\}, Y = \{1, 2, 3, 4, 5, 6, 7\}$, and $f : X \to Y$.

Each element in the domain must point to one element in the codomain. Each element in X can mapped to any element in Y. So the number of f functions is $|Y|^{|X|} = 7^7 = 823543$.

For injective functions, it must be the case that each element in X points to a unique element in Y. We can only do this if the domain and codomain are of the same cardinality, as in this case. Thus there $|X|! = |Y|! = 7! = 5040$ number of f functions.

When domain and codomain cardinality are the same, all injective functions counted above are surjective. Moreover, these are the only surjective functions. Thus there are 5040 surjective f functions.

Thus there are 5040 bijective f functions.

Ex 16. Let $X = \{A, B, C, D, E\}, Y = \{1, 2, 3, 4, 5, 6, 7\}$, and $f : X \to Y$.

There are $|Y|^{|X|} = 7^5 = 16807$ f functions. There are $\dfrac{|Y|!}{(|Y| - |X|)!} = \dfrac{7!}{2!} = 2520$ injective f functions. Note that because $|X| < |Y|$, we cannot map each element in X uniquely to an element in Y. So by the pigeonhole principle there are 0 surjective f functions. Thus there are 0 bijective f functions.

Ex 17. Let $X = \{A, B, C, D, E, F, G\}, Y = \{1, 2\}$, and $f : X \to Y$.

There are $|Y|^{|X|} = 2^7 = 128$ f functions. By the pigeonhole principle, it follows that there are 0 injective functions. Of the 128 possible f functions, only two are not surjective. These two functions map all elements from X to one element in Y. Thus there are $128 - 2 = 126$ surjective f functions. Finally, because there are 0 injective functions, there are also 0 bijective f functions.

Ex 18. Proposition: $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = \dfrac{(-1)^n(2n-1)+1}{4}$ is bijective.

Proof.
First we show that f is injective using the contrapositive approach. Suppose $a, b \in$ and $f(a) = f(b)$. So we have $f(a) = f(b) \leftrightarrow \dfrac{(-1)^a(2a-1)+1}{4} = \dfrac{(-1)^b(2b-1)+1}{4} \leftrightarrow (-1)^a(2a-1) = (-1)^b(2b-1)$. Note that $(2a-1)$ and $(2b-1)$ are both positive. Thus it must be the case that $(-1)^a = (-1)^b$. So we can cancel that factor. Then $(-1)^a(2a-1) = (-1)^b(2b-1) \leftrightarrow 2a-1 = 2b-1 \leftrightarrow a = b$. Thus f is injective.

Then we show that f is surjective. Let $b \in \mathbb{Z}$. We consider two cases.

Case 1. Suppose b is a positive integer.

Observe that when $n = 2b \in \mathbb{N}$ we have $f(n) = \dfrac{(-1)^n(2n-1)+1}{4} = \dfrac{(-1)^{2b}(2(2b)-1)+1}{4} = \dfrac{4b}{4} = b$.

Case 2. Suppose b is a non-positive integer.

Observe that when $n = -2b+1 \in \mathbb{N}$ we have $f(n) = \dfrac{(-1)^n(2n-1)+1}{4} = \dfrac{(-1)^{-2b+1}(2(-2b+1)-1)+1}{4} = \dfrac{(-1)(-4b+2-1)+1}{4} = \dfrac{4b-2+1+1}{4} = b$.

Thus we have shown that for every integer z there exists an $n \in \mathbb{N}$ such that $f(n) = z$. So f is surjective. $\square$