Exercises for Section 11.5

Ex 1. Addition and multiplication tables for $\mathbb{Z}_2$

+	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

*	[0]	[1]
[0]	[0]	[0]
[1]	[0]	[1]

Ex 2. Addition and multiplication tables for $\mathbb{Z}_3$

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

*	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Ex 3. Addition and multiplication tables for $\mathbb{Z}_4$

+	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[0]	[1]	[1]
[3]	[3]	[0]	[1]	[2]

*	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

Ex 4. Addition and multiplication tables for $\mathbb{Z}_6$

+	**[0]**	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[0]
[2]	[2]	[3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

*	**[0]**	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

Ex 5.Proposition: If $[a], [b] \in \mathbb{Z}_5$ and $[a] \cdot [b] = [0]$, then $[a] = [0]$ or $[b] = [0]$.

Proof. (Contradiction.) Suppose $[a], [b] \in \mathbb{Z}_5$, $[a] \cdot [b] = [0]$, $[a] \neq [0]$, and $[b] \neq [0]$.
$[a] \neq [0]$ implies that $a = 5x + r$ for some integers x, r where $1 \leq r \leq 4$. Likewise, $[b] \neq [0]$ implies that $b = 5y + c$ for some integers y, c where $1 \leq c \leq 4$. Then $[a] \cdot [b] = [0]$ implies that 5 divides $a \cdot b = (5x + r)(5y + c) = 25xy + 5xc + 5yr + cr$. Thus cr must be divisable by 5. But this contradicts Euclid's lemma. Therefore it must be the case that $[a] = [0]$ or $[b] = [0]$.

□

Ex 6. For $\mathbb{Z}_6$, note the counter-example for $\mathbb{Z}_6$: $[2] \cdot [3] = [2 \cdot 3] = [6] = [0]$. For $\mathbb{Z}_7$, the statement is true. A proof almost identical to the one above could be used to show the statement to be true.

Ex 7.
a) $[8] + [8] = [8 + 8] = [16] = [7]$
b) $[24] + [11] = [24 + 11] = [35] = [8]$
c) $[21] \cdot [15] = [21 \cdot 15] = [315] = [0]$
d) $[8] \cdot [8] = [8 \cdot 8] = [64] = [1]$

Ex 8. Proposition: If $[a], [b] \in \mathbb{Z}_n$, $[a] = [a']$, and $[b] = [b']$, then $[a] + [b] = [a'] + [b']$.

Proof. (Direct). Suppose $[a], [b] \in \mathbb{Z}_n$, $[a] = [a']$, and $[b] = [b']$.
Because $[a] = [a']$, it follows that n divides $a - a'$, so $a - a' = nx$ for some integer x. Likewise, $[b] = [b']$ implies that n divides $b - b'$, so $b - b' = ny$ for some integer y. Then $a + b = (nx + a') + (ny + b') \leftrightarrow (a + b) - (a' + b') = n(x + y) \leftrightarrow a + b \equiv a' + b' \pmod{n}$. Therefore $[a] + [b] = [a'] + [b']$.

□