Exercises for section 12.5

Ex 1. Proposition: $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 6 - n$ is bijective.

Proof.
First we show f is injective. Suppose $f(x) = f(y)$ where $x, y \in \mathbb{Z}$, then $f(x) = f(y) \leftrightarrow 6 - x = 6 - y \leftrightarrow x = y$. Thus f is injective.

Then we show that f is surjective. Suppose $x \in \mathbb{Z}$. Observe that when $y = 6 - x \in \mathbb{Z}$ we have $f(y) = 6 - y = 6 - (6 - x) = x$. Thus f is surjective.

□

Let $n = f(y) = 6 - y$, then $n = 6 - y \leftrightarrow y = 6 - n$. So $f^{-1}(n) = 6 - n$.

Ex 2.

Let $x = f(y)$, then $x = f(y) \leftrightarrow x = \dfrac{5y+1}{y-2} \leftrightarrow xy - 2x = 5y + 1 \leftrightarrow xy - 5y = 1 + 2x \leftrightarrow y = \dfrac{1+2x}{x-5}$. So $f^{-1}(x) = \dfrac{1+2x}{x-5}$.

Ex 3.Proposition: Let $B = \{2^n : n \in \mathbb{Z}\}$. Then $f : \mathbb{Z} \to B$ defined as $f(n) = 2^n$ is bijective.

Proof.
First we show that f is injective. Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{Z}$. Then $f(x) = f(y) \leftrightarrow 2^x = 2^y \leftrightarrow x = y$. Thus f is injective.

Then we show that f is surjective. Suppose $b \in B$. By the definition of B we have $b = 2^x$ for some $x \in \mathbb{Z}$. Thus $f(x) = 2^x = b$ and consequently f is surjective.

□

Let $x = f(y)$, then $x = f(y) \leftrightarrow x = 2^y \leftrightarrow y = log_2(x)$. So $f^{-1}(x) = log_2(x)$.

Ex 4.

Let $x = f(y)$, then $x = f(y) \leftrightarrow x = e^{y^3+1} \leftrightarrow ln(x) = (y^3+1)ln(e) \leftrightarrow y = \sqrt[3]{ln(x) - 1}$. So $f^{-1}(x) = \sqrt[3]{ln(x) - 1}$.

Ex 5.

Let $x = f(y)$, then $x = f(y) \leftrightarrow x = \pi y - e \leftrightarrow y = \dfrac{x+e}{\pi}$. So $f^{-1}(x) = \dfrac{x+e}{\pi}$.

Ex 6.

Let $(m, n) = f(x, y)$, then $(m, n) = f(x, y) \leftrightarrow (m, n) = (5x + 4y, 4x + 3y)$ yields the following system of linear equations

$$\begin{aligned} m &= 5x + 4y \\ n &= 4x + 3y \end{aligned}$$

Solving the system we get $y = 4m - 5n$ and $x = 4n - 3m$. So $f^{-1}(m, n) = (4n - 3m, 4m - 5n)$.

Ex 7. Proposition: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x, y) = ((x^2 + 1)y, x^3)$ is bijective.

Proof.
First we show that f is injective. Suppose $f(x, y) = f(x', y')$ for some $x, x', y, y' \in \mathbb{R}$. So $((x^2+1)y, x^3) = (((x')^2 + 1)y', (x')^3)$ yields the following equations

$$\begin{aligned}(x^2 + 1)y &= ((x')^2 + 1)y' \\ x^3 &= (x')^3\end{aligned}$$

The second equation is equivalent to $x = x'$, plugging that into the first we get $(x^2+1)y = ((x')^2+1)y' \leftrightarrow (x^2 + 1)y = (x^2 + 1)y' \leftrightarrow y = y'$. Thus we have that $x = x'$ and $y = y'$. So f is injective.

Then we show that f is surjective. Suppose $(a, b) \in \mathbb{R}^2$. Then when $x = \sqrt[3]{b}$ and $y = \dfrac{a}{b^{2/3} + 1}$ we have $f(x, y) = (a, b)$. Thus f is surjective.

$\square$

From the surjective part of the proof above, we have that $f^{-1}(x, y) = (\sqrt[3]{y}, \dfrac{x}{y^{2/3} + 1})$.

Ex 8.

We already showed θ to be bijective in exercise 14, section 12.2. Let $X = \theta(Y)$, then $X = \theta(Y) \leftrightarrow X = \overline{Y} \leftrightarrow Y = \overline{X}$. So $\theta^{-1}(X) = \overline{X}$.

Ex 9. Proposition: $f : \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x, y) = (y, 3xy)$ is bijective.

Proof.
First we show that f is injective. Suppose $f(a, b) = f(a', b')$ for some $a, a' \in \mathbb{R}, b, b' \in \mathbb{N}$. So we have $f(a, b) = f(a', b') \leftrightarrow (b, 3ab) = (b', 3a'b')$. From this it directly follows that $b = b'$. Plugging that into the second equation we get $3ab = 3a'b' \leftrightarrow 3ab = 3a'b \leftrightarrow a = a'$. TThus $a = a'$ and $b = b'$, which implies that f is injective.

Then we show that f is surjective. Suppose $(a, b) \in \mathbb{N} \times \mathbb{R}$. Then when $y = a$ and $x = \dfrac{b}{3a}$ we have $f(x, y) = (a, b)$. Thus f is surjective.

$\square$

From the surjective part of the proof above we have $f^{-1}(x, y) = (\dfrac{y}{3x}, x)$.

Ex 10.

The piecewise function below was derived through case-work.

$$f(x)^{-1} = \begin{cases} -2x + 1 & x \leq 0 \\ 2x & x > 0 \end{cases}$$