Chapter 1.1 - Exercises

Ex 1. i) Proposition: If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof. (Direct)
Suppose $ax = a$ for some number $a \neq 0$. Then $ax = a \implies x = a \times a^{-1} = 1$. □

Ex 1. ii) Proposition: $x^2 - y^2 = (x - y)(x + y)$.

Proof. (Direct)
Observe that $(x + y)(x - y) = x^2 - xy + xy - y^2 = x^2 - y^2$. □

Ex 1. iii) Proposition: If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof. (Direct)
Given $x^2 = y^2$. Because $x^2 = |x|^2$, it follows that $\sqrt{|x|^2} = \sqrt{y^2} \implies |x| = y$. Thus $x = y$ or $x = -y$. □

Ex 1. iv) Proposition: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Proof. (Direct)
Observe that $(x - y)(x^2 + xy + y^2) = x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 = x^3 - y^3$. □

Ex 1. v) Proposition: $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + ... + xy^{n-2} + y^{n-1})$.

Proof. (Direct)
Observe that

$$\begin{aligned}(x - y)(x^{n-1} + x^{n-2}y + ... + xy^{n-2} + y^{n-1}) = \\ x(x^{n-1} + x^{n-2}y + ... + xy^{n-2} + y^{n-1}) - y(x^{n-1} + x^{n-2}y + ... + xy^{n-2} + y^{n-1}) = \\ (x^n + x^{n-1}y + ... + x^2y^{n-2} + xy^{n-1}) - (x^{n-1}y + x^{n-2}y^2 + ... + xy^{n-1} + y^n) = \\ x^n + (x^{n-1}y - x^{n-1}y) + (x^{n-2}y^2 - x^{n-2}y^2) + ... + (x^2y^{n-2} - x^2y^{n-2}) + (xy^{n-1} - xy^{n-1}) - y^n = \\ x^n - y^n\end{aligned}$$

□

Ex 1. vi) Proposition: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

Proof. (Direct)
Observe that $(x + y)(x^2 - xy + y^2) = x^3 - x^2y + xy^2 + x^2y - xy^2 + y^3 = x^3 + y^3$. □

Ex 2. Because $x = y$ implies $x - y = 0$, we can not divide both sides of $(x + y)(x - y) = y(x - y)$ with $x - y = 0$.

Ex 3. i) Proposition: If $b, c \neq 0$, then $\dfrac{a}{b} = \dfrac{ac}{bc}$.

Proof. (Direct)
Given $b, c \neq 0$. Then $\frac{a}{b} = \frac{ac}{bc}$ implies $abc = abc$. □

Ex 3. ii) Proposition: If $b, d \neq 0$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

Proof. (Direct)
Given $b, d \neq 0$. Observe that $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$. □

Ex 3. iii) Proposition: If $a, b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$.

Proof.
TODO

□

Ex 3. iv) Proposition: If $b, d \neq 0$, then $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{db}$.

Proof.
TODO

□

Ex 3. v) Proposition: If $b, c, d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.

Proof.
TODO

□

Ex 3. vi) Proposition: If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

Proof.
TODO

□