Week 5: Graph Algorithms

Week 5

1/88

In This Lecture …

- Directed graphs, weighted graphs ([S] 19.1-19.3, 20-20.1)
- More graph algorithms ([S] 20.2-20.4, 21-21.3, 22.1-22.2)

Nerdy Things You Should Know

2/88

Consider the following scenario …

- you're sitting in a lab
- you're looking at some code like `'/^s?[0-9]{7}$/'`
- you want to ask a question about the code
- but you're not sure how to refer to the ^ char
- and you don't want to sound clueless

Fear not! This is … **How to speak #@*%$! Ascii**

From
blog.codinghorror.com/ascii-pronunciation-rules-for-programmers/

... Nerdy Things You Should Know

3/88

Symbol	Common Name	Silliest Name
&		
*		
"		
^		
@		
!		
#		
%		

Directed Graphs

Directed Graphs (Digraphs)

5/88

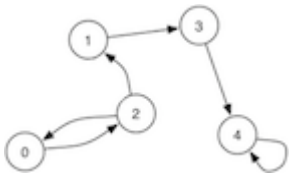
In our previous discussion of graphs:

- an edge indicates a relationship between two vertices

- an edge indicates nothing more than a relationship

In many real-world applications of graphs:

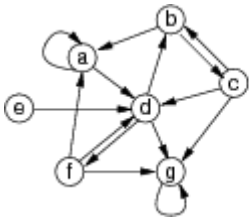
- edges are directional ($v \rightarrow w \neq w \rightarrow v$)



- edges have a *weight* (cost to go from $v \rightarrow w$)

... Directed Graphs (Digraphs) 6/88

Example digraph and adjacency matrix representation:



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Undirectional ⇒ symmetric matrix
Directional ⇒ non-symmetric matrix

Maximum #edges in a digraph with V vertices: V^2

... Directed Graphs (Digraphs) 7/88

Terminology for digraphs …

Directed path: sequence of $n \geq 2$ vertices $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n$

- where $(v_i,v_{i+1}) \in edges(G)$ for all v_i,v_{i+1} in sequence
- if $v_1 = v_n$, we have a *directed cycle*

Reachability: w is reachable from v if ∃ directed path $v,\ldots,w$

Digraph Applications 8/88

Potential application areas:

Domain	**Vertex**	**Edge**
Web	web page	hyperlink
scheduling	task	precedence
chess	board position	legal move
science	journal article	citation

dynamic data	malloc'd object	pointer
programs	function	function call
`make`	file	dependency

... Digraph Applications 9/88

Problems to solve on digraphs:

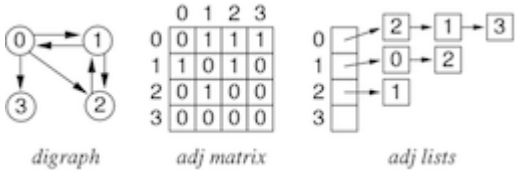
- is there a directed path from *s* to *t*? (transitive closure)
- what is the shortest path from *s* to *t*? (shortest path)
- are all vertices mutually reachable? (strong connectivity)
- how to organise a set of tasks? (topological sort)
- which web pages are "important"? (PageRank)
- how to build a web crawler? (graph traversal)

Digraph Representation 10/88

Similar set of choices as for undirectional graphs:

- array of edges (directed)
- vertex-indexed adjacency matrix (non-symmetric)
- vertex-indexed adjacency lists

V vertices identified by *0 .. V-1*



Reachability

Transitive Closure 12/88

Given a digraph *G* it is potentially useful to know

- is vertex *t* reachable from vertex *s*?

Example applications:

- can I complete a schedule from the current state?
- is a malloc'd object being referenced by any pointer?

Transitive closure: a -> b, b-> c then a-> c

Informally, transitive closure (TC) gives you a set of places one can get to from a starting place

How to compute transitive closure?

... Transitive Closure 13/88

One possibility of implementing *reachable()*:

- implement it via `hasPath(G,s,t)` (itself implemented by DFS or BFS algorithm)
- feasible if *reachable(G,s,t)* is infrequent operation

What if we have an algorithm that frequently needs to check reachability?

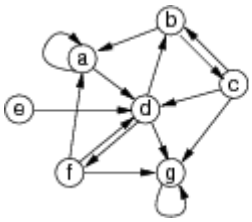
Would be very convenient/efficient to have:

```
reachable(G,s,t):
|  return G.tc[s][t]   // transitive closure matrix
```

Of course, if V is *very* large, then this is not feasible.

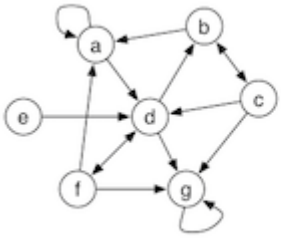
Exercise #1: Transitive Closure Matrix 14/88

Which reachable s .. t exist in the following graph?



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Transitive closure of example graph:



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

adjacency matrix

	a	b	c	d	e	f	g
a	1	1	1	1	0	1	1
b	1	1	1	1	0	1	1
c	1	1	1	1	0	1	1
d	1	1	1	1	0	1	1
e	1	1	1	1	0	1	1
f	1	1	1	1	0	1	1
g	0	0	0	0	0	0	1

reachability matrix

... Transitive Closure 16/88

Goal: produce a matrix of reachability values

- if *tc[s][t]* is 1, then t is reachable from s
- if *tc[s][t]* is 0, then t is not reachable from s

So, how to create this matrix?

Observation:

$\forall i,s,t \in$ vertices(G):
$(s,i) \in$ edges(G) and $(i,t) \in$ edges(G) $\Rightarrow$ $tc[s][t] = 1$

`tc[s][t]=1` if there is a path from s to t of length 2 $(s \rightarrow i \rightarrow t)$

... Transitive Closure 17/88

If we implement the above as:

```
make tc[][] a copy of edges[][]
for all i∈vertices(G) do
   for all s∈vertices(G) do
      for all t∈vertices(G) do
         if tc[s][i]=1 and tc[i][t]=1 then
            tc[s][t]=1
         end if
      end for
   end for
end for
```

then we get an algorithm to convert `edges` into a *tc*

This is known as *Warshall's algorithm*

... Transitive Closure

How it works …

After iteration 1, `tc[s][t]` is 1 if

- either $s \rightarrow t$ exists or $s \rightarrow 0 \rightarrow t$ exists

After iteration 2, `tc[s][t]` is 1 if any of the following exist

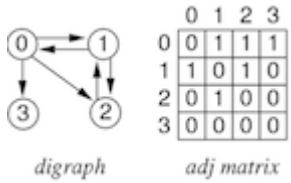
- $s \rightarrow t$ or $s \rightarrow 0 \rightarrow t$ or $s \rightarrow 1 \rightarrow t$ or $s \rightarrow 0 \rightarrow 1 \rightarrow t$ or $s \rightarrow 1 \rightarrow 0 \rightarrow t$

Etc. … so after the V^{th} iteration, `tc[s][t]` is 1 if

- there is any directed path in the graph from *s* to *t*

Exercise #2: Transitive Closure

Trace Warshall's algorithm on the following graph:



digraph

	0	1	2	3
0	0	1	1	1
1	1	0	1	0
2	0	1	0	0
3	0	0	0	0

adj matrix

1st iteration `i=0`:

tc	[0]	[1]	[2]	[3]
[0]	0	1	1	1
[1]	1	1	1	1
[2]	0	1	0	0
[3]	0	0	0	0

2nd iteration `i=1`:

tc	[0]	[1]	[2]	[3]
[0]	1	1	1	1

[1]	1	1	1	1
[2]	1	1	1	1
[3]	0	0	0	0

3rd iteration `i=2`: unchanged

4th iteration `i=3`: unchanged

... Transitive Closure 21/88

Cost analysis:

- storage: additional V^2 items (each item may be 1 bit)
- computation of transitive closure: $O(V^3)$
- computation of `reachable()`: $O(1)$ after having generated `tc[][]`

Amortisation: would need many calls to `reachable()` to justify other costs

Alternative: use DFS in each call to `reachable()`
Cost analysis:

- storage: cost of queue and set during reachable
- computation of `reachable()`: cost of DFS = $O(V^2)$ (for adjacency matrix)

Digraph Traversal 22/88

Same algorithms as for undirected graphs:

```
depthFirst(v):
```

1. mark v as visited
2. for each (v,w)∈*edges*(G) do
   if w has not been visited then
   `depthFirst(w)`

```
breadth-first(v):
```

1. enqueue v
2. while queue not empty do
   dequeue v
   if v not already visited then
   mark v as visited
   enqueue each vertex w adjacent to v

Example: Web Crawling 23/88

Goal: visit every page on the web

Solution: breadth-first search with "implicit" graph

```
webCrawl(startingURL):
|  mark startingURL as alreadySeen
|  enqueue(Q,startingURL)
|  while Q is not empty do
```

```
|  |  nextPage=dequeue(Q)
|  |  visit nextPage
|  |  for each hyperLink on nextPage do
|  |  |  if hyperLink not alreadySeen then
|  |  |     mark hyperLink as alreadySeen
|  |  |     enqueue(Q,hyperLink)
|  |  |  end if
|  |  end for
|  end while
```

`visit` scans page and collects e.g. keywords and links

Weighted Graphs

Weighted Graphs 25/88

Graphs so far have considered

- edge = an association between two vertices/nodes
- may be a precedence in the association (directed)

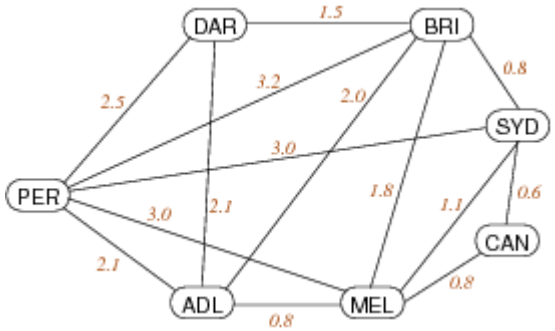
Some applications require us to consider

- a *cost* or *weight* of an association
- modelled by assigning values to edges (e.g. positive reals)

Weights can be used in both directed and undirected graphs.

... Weighted Graphs 26/88

Example: major airline flight routes in Australia



Representation: edge = direct flight; weight = approx flying time (hours)

... Weighted Graphs 27/88

Weights lead to minimisation-type questions, e.g.

1. Cheapest way to connect all vertices?

- a.k.a. *minimum spanning tree* problem
- assumes: edges are weighted and undirected

2. Cheapest way to get from *A* to *B*?

- a.k.a *shortest path* problem

- assumes: edge weights positive, directed or undirected

Exercise #3: Implementing a Route Finder

If we represent a street map as a graph

- what are the vertices?
- what are the edges?
- are edges directional?
- what are the weights?
- are the weights fixed?

Weighted Graph Representation

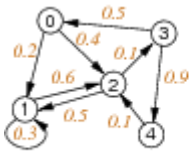
Weights can easily be added to:

- adjacency matrix representation (0/1 → int or float)
- adjacency lists representation (add int/float to list node)

Both representations work whether edges are directed or not.

... Weighted Graph Representation

Adjacency matrix representation with weights:



Weighted Digraph

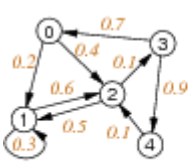
	0	1	2	3	4
0	·	0.2	0.4	·	·
1	·	0.3	0.6	·	·
2	·	0.5	·	0.1	·
3	0.5	·	·	·	0.9
4	·	·	0.1	·	·

Adjacency Matrix

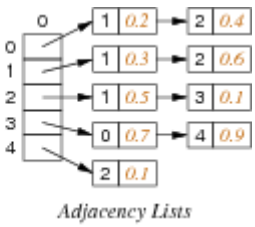
Note: need distinguished value to indicate "no edge".

... Weighted Graph Representation

Adjacency lists representation with weights:



Weighted Digraph



Adjacency Lists

Note: if undirected, each edge appears twice with same weight

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... Weighted Graph Representation

Sample adjacency matrix implementation in C requires minimal changes to previous Graph ADT:

WGraph.h

```
// edges are pairs of vertices (end-points) plus positive weight
typedef struct Edge {
```

```
   Vertex v;
   Vertex w;
   int    weight;
} Edge;

// returns weight, or 0 if vertices not adjacent
int adjacent(Graph, Vertex, Vertex);
```

... Weighted Graph Representation

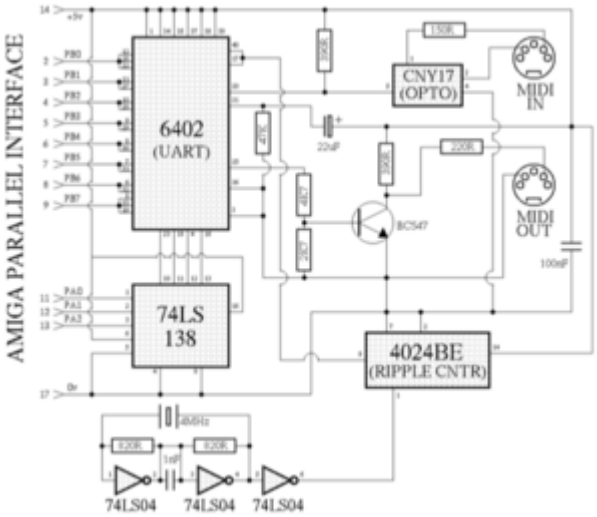
WGraph.c

```
typedef struct GraphRep {
   int **edges;  // adjacency matrix storing positive weights
                 // 0 if nodes not adjacent
   int   nV;     // #vertices
   int   nE;     // #edges
} GraphRep;


void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));
   if (g->edges[e.v][e.w] == 0) {  // edge e not in graph
      g->edges[e.v][e.w] = e.weight;
      g->nE++;
   }
}

int adjacent(Graph g, Vertex v, Vertex w) {
   assert(g != NULL && validV(g,v) && validV(g,w));
   return g->edges[v][w];
}
```

Minimum Spanning Trees

Exercise #4: Minimising Wires in Circuits

Electronic circuit designs often need to make the pins of several components electrically equivalent by wiring them together.



To interconnect a set of *n* pins we can use an arrangement of *n-1* wires each connecting two pins.

What kind of algorithm would …

- help us find the arrangement with the least amount of wire?

Minimum Spanning Trees

Reminder: *Spanning tree ST* of graph *G=(V,E)*

- *spanning* = all vertices, *tree* = no cycles
  - *ST* is a subgraph of *G* (*G'=(V,E')* where $E' \subseteq E$)
  - *ST* is *connected* and *acyclic*

Minimum spanning tree MST of graph *G*

- *MST* is a spanning tree of *G*
- sum of edge weights is no larger than any other ST

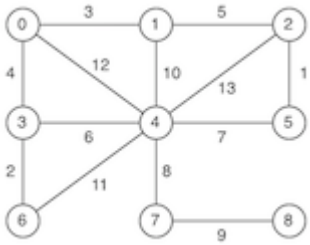
Applications: Computer networks, Electrical grids, Transportation networks …

Problem: how to (efficiently) find MST for graph *G*?

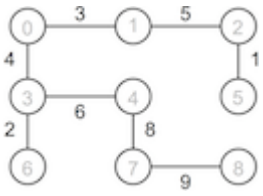
NB: MST may not be unique (e.g. all edges have same weight ⇒ every ST is MST)

... Minimum Spanning Trees

Example:



An MST …



... Minimum Spanning Trees

Brute force solution:

```
findMST(G):
|  Input  graph G
|  Output a minimum spanning tree of G
|
|  bestCost=∞
|  for all spanning trees t of G do
|  |  if cost(t)<bestCost then
|  |     bestTree=t
|  |     bestCost=cost(t)
|  |  end if
|  end for
|  return bestTree
```

Example of *generate-and-test* algorithm.

Not useful because #spanning trees is potentially large (e.g. n^{n-2} for a complete graph with n vertices)

... Minimum Spanning Trees 39/88

Simplifying assumption:

- edges in *G* are not directed (MST for digraphs is harder)

Kruskal's Algorithm 40/88

One approach to computing MST for graph *G* with *V* nodes:

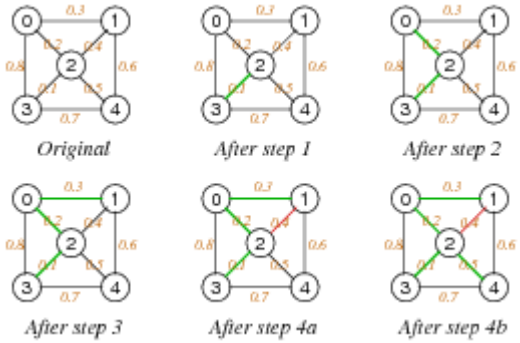
1. start with empty MST
2. consider edges in increasing weight order
   - add edge if it does not form a cycle in MST
3. repeat until *V-1* edges are added

Critical operations:

- iterating over edges in weight order
- checking for cycles in a graph

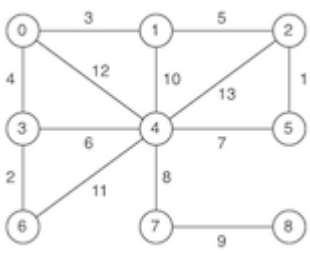
... Kruskal's Algorithm 41/88

Execution trace of Kruskal's algorithm:

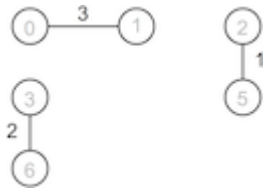


Exercise #5: Kruskal's Algorithm 42/88

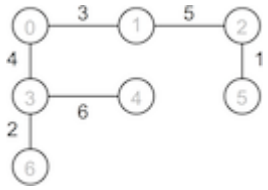
Show how Kruskal's algorithm produces an MST on:



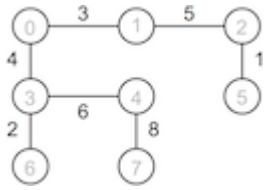
After 3rd iteration:



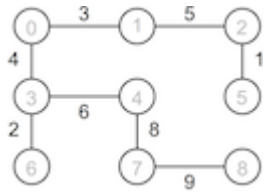
After 6th iteration:



After 7th iteration:



After 8th iteration (*V*-1=8 edges added):



... Kruskal's Algorithm 44/88

Pseudocode:

```
KruskalMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
|  sort edges(G) by weight
|  for each e∈sortedEdgeList do
|  |  MST = MST ∪ {e}
|  |  if MST has a cyle then
|  |     MST = MST \ {e}
|  |  end if
|  |  if MST has n-1 edges then
|  |     return MST
|  |  end if
|  end for
```

... Kruskal's Algorithm 45/88

Rough time complexity analysis …

- sorting edge list is *O(E·log E)*
- at least *V* iterations over sorted edges
- on each iteration …
    - getting next lowest cost edge is *O(1)*

- checking whether adding it forms a cycle: cost = ??

Possibilities for cycle checking:

- use DFS … too expensive?
- could use *Union-Find data structure* (see Sedgewick Ch.1)

Prim's Algorithm

Another approach to computing MST for graph *G=(V,E)*:

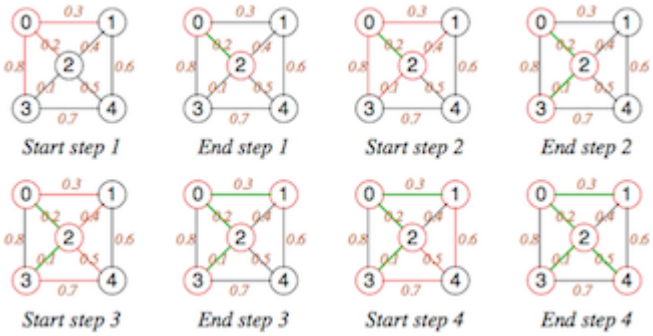
1. start from any vertex *v* and empty MST
2. choose edge not already in MST to add to MST
   - must be incident on a vertex *s* already connected to *v* in MST
   - must be incident on a vertex *t* not already connected to *v* in MST
   - must have minimal weight of all such edges
3. repeat until MST covers all vertices

Critical operations:

- checking for vertex being connected in a graph
- finding min weight edge in a set of edges

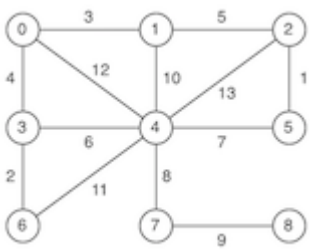
... Prim's Algorithm

Execution trace of Prim's algorithm (starting at *s*=0):



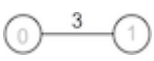
Exercise #6: Prim's Algorithm

Show how Prim's algorithm produces an MST on:

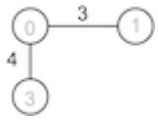


Start from vertex 0

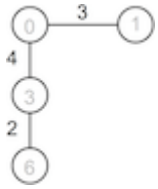
After 1st iteration:



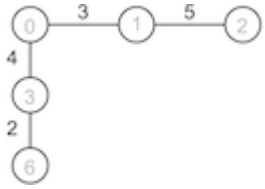
After 2nd iteration:



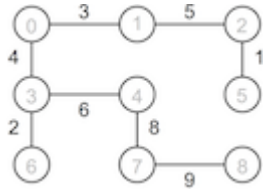
After 3rd iteration:



After 4th iteration:



After 8th iteration (all vertices covered):



... Prim's Algorithm 50/88

Pseudocode:

```
PrimMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
|  usedV={0}
|  unusedE=edges(g)
|  while |usedV|<n do
|  |  find e=(s,t,w)∈unusedE such that {
|  |     s∈usedV, t∉usedV and w is min weight of all such edges
|  |  }
|  |  MST = MST ∪ {e}
|  |  usedV = usedV ∪ {t}
|  |  unusedE = unusedE \ {e}
|  end while
|  return MST
```

Critical operation: finding best edge

... Prim's Algorithm 51/88

Rough time complexity analysis …

- *V* iterations of outer loop

- in each iteration …
  - find min edge with set of edges is *O(E)* ⇒ *O(V·E)* overall
  - find min edge with *priority queue* is *O(log E)* ⇒ *O(V·log E)* overall

---

Sidetrack: Priority Queues 52/88

Some applications of queues require

- items processed in order of "priority"
- rather than in order of entry (FIFO — first in, first out)

Priority Queues (*PQueues*) provide this via:

- `join`: insert item into PQueue with an associated priority (replacing `enqueue`)
- `leave`: remove item with highest priority (replacing `dequeue`)

Time complexity for naive implementation of a PQueue containing *N* items …

- *O(1)* for `join` *O(N)* for `leave`

Most efficient implementation ("heap") …

- *O(log N)* for `join`, `leave`

Other MST Algorithms 53/88

Boruvka's algorithm … complexity *O(E·log V)*

- the oldest MST algorithm
- start with *V* separate components
- join components using min cost links
- continue until only a single component

Karger, Klein, and Tarjan … complexity *O(E)*

- based on Boruvka, but non-deterministic
- randomly selects subset of edges to consider
- for the keen, here's the paper describing the algorithm

Shortest Path

Shortest Path 55/88

Path = sequence of edges in graph G $p = (v_0,v_1), (v_1,v_2), \ldots, (v_{m-1},v_m)$

cost(path) = sum of edge weights along path

Shortest path between vertices *s* and *t*

- a simple path *p(s,t)* where *s = first(p), t = last(p)*
- no other simple path *q(s,t)* has *cost(q) < cost(p)*

Assumptions: weighted digraph, no negative weights.

Finding shortest path between two given nodes known as *source-target* SP problem

Variations: *single-source* SP, *all-pairs* SP

Applications: navigation, routing in data networks, …

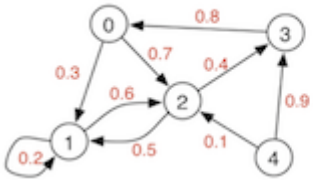
Single-source Shortest Path (SSSP) 56/88

Given: weighted digraph G, source vertex s

Result: shortest paths from s to all other vertices

- `dist[]` V-indexed array of cost of shortest path from s
- `pred[]` V-indexed array of predecessor in shortest path from s

Example:



	0	1	2	3	4
dist	0	0.3	0.7	1.1	inf
pred	-	0	0	2	-

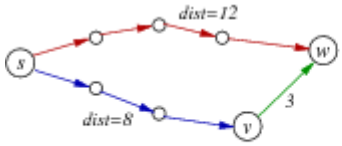
Shortest paths from s=0

Edge Relaxation 57/88

Assume: `dist[]` and `pred[]` as above (but containing data for shortest paths *discovered so far*)

`dist[v]` is length of shortest known path from s to v
`dist[w]` is length of shortest known path from s to w

Relaxation updates data for w if we find a shorter path from s to w:




```
dist[v]=8, dist[w]=12
pred[v]=?, pred[w]=?
```


```
dist[v]=8, dist[w]=11
pred[v]=?, pred[w]=v
```

Relaxation along edge e=`(v,w,weight)`:

- if **`dist[v]+weight < dist[w]`** then
  update `dist[w]:=dist[v]+weight` and `pred[w]:=v`

Dijkstra's Algorithm 58/88

One approach to solving single-source shortest path problem …

Data: G, s, `dist[]`, `pred[]` and

- *vSet*: set of vertices whose shortest path from s is unknown

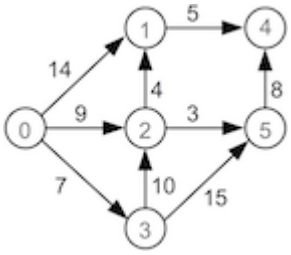
Algorithm:

```
dist[]  // array of cost of shortest path from s
pred[]  // array of predecessor in shortest path from s

dijkstraSSSP(G,source):
|  Input graph G, source node
```

```
|
|  initialise dist[] to all ∞, except dist[source]=0
|  initialise pred[] to all -1
|  vSet=all vertices of G
|  while vSet≠∅ do
|  |  find s∈vSet with minimum dist[s]
|  |  for each (s,t,w)∈edges(G) do
|  |     relax along (s,t,w)
|  |  end for
|  |  vSet=vSet\{s}
|  end while
```

Exercise #7: Dijkstra's Algorithm 59/88

Show how Dijkstra's algorithm runs on (source node = 0):



	[0]	[1]	[2]	[3]	[4]	[5]
dist	**0**	∞	∞	∞	∞	∞
pred	**–**	**–**	**–**	**–**	**–**	**–**

---

	[0]	[1]	[2]	[3]	[4]	[5]
dist	**0**	∞	∞	∞	∞	∞
pred	**–**	**–**	**–**	**–**	**–**	**–**

dist	**0**	**14**	**9**	**7**	∞	∞
pred	**–**	**0**	**0**	**0**	**–**	**–**

dist	**0**	**14**	**9**	**7**	∞	**22**
pred	**–**	**0**	**0**	**0**	**–**	**3**

dist	**0**	**13**	**9**	**7**	∞	**12**
pred	**–**	**2**	**0**	**0**	**–**	**2**

dist	**0**	**13**	**9**	**7**	**20**	**12**
pred	**–**	**2**	**0**	**0**	**5**	**2**

dist	**0**	**13**	**9**	**7**	**18**	**12**
pred	**–**	**2**	**0**	**0**	**1**	**2**

... Dijkstra's Algorithm

Time complexity analysis …

Each edge needs to be considered once ⇒ *O(E)*.

Outer loop has *O(V)* iterations.

Implementing "**find** `s∈vSet` **with** `minimum dist[s]`"

1. try all `s∈vSet` ⇒ cost = *O(V)* ⇒ overall cost = $O(E + V^2) = O(V^2)$
2. using a PQueue to implement extracting minimum
   - can improve overall cost to $O(E + V \cdot log\ V)$ (for best-known implementation)

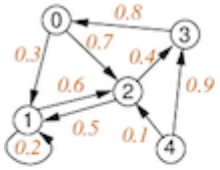
All-pair Shortest Path (APSP)

Given: weighted digraph *G*

Result: shortest paths between all pairs of vertices

- `dist[][]` *V*×*V*-indexed matrix of cost of shortest path from v_{row} to v_{col}
- `path[][]` *V*×*V*-indexed matrix of next node in shortest path from v_{row} to v_{col}

Example:



Weighted Digraph

V	0	1	2	3	4	
0	0	0.3	0.7	1.1	inf	dist
1	1.8	0	0.6	1.0	inf	
2	1.2	0.5	0	0.4	inf	
3	0.8	1.1	1.5	0	inf	
4	1.3	0.6	0.1	0.5	0	

0	–	1	2	2	–	path
1	2	–	2	2	–	
2	3	1	–	3	–	
3	0	0	0	–	–	
4	2	2	2	2	–	

Shortest paths between all vertices

e.g., 0 to 3, 1 to 0

Floyd's Algorithm

One approach to solving all-pair shortest path problem…

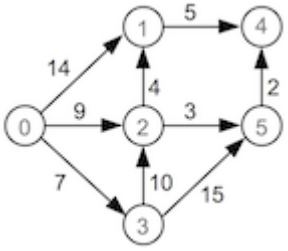
Data: *G*, `dist[][]`, `path[][]` Algorithm:

```
dist[][]  // array of cost of shortest path from s to t
path[][]  // array of next node after s on shortest path from s to t

floydAPSP(G):
|  Input graph G
|
|  initialise dist[s][t]=0 for each s=t
|                       =w for each (s,t,w)∈edges(G)
|                       =∞ otherwise
|  initialise path[s][t]=t for each (s,t,w)∈edges(G)
|                       =-1 otherwise
|  for all i∈vertices(G) do
|  |  for all s∈vertices(G) do
```

```
|  |  |  for all t∈vertices(G) do
|  |  |  |  if dist[s][i]+dist[i][t] < dist[s][t] then
|  |  |  |     dist[s][t]=dist[s][i]+dist[i][t]
|  |  |  |     path[s][t]=path[s][i]
|  |  |  |  end if
|  |  |  end for
|  |  end for
|  end for
```

Exercise #8: Floyd's Algorithm 64/88

Show how Floyd's algorithm runs on:



dist	[0]	[1]	[2]	[3]	[4]	[5]	path	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**0**	**14**	**9**	**7**			[0]		**1**	**2**	**3**		
[1]		**0**			**5**		[1]					**4**	
[2]		**4**	**0**			**3**	[2]		**1**				**5**
[3]			**10**	**0**		**15**	[3]			**2**			**5**
[4]					**0**		[4]						
[5]					**2**	**0**	[5]					**4**	

After 1st iteration `i=0`: unchanged After 2nd iteration `i=1`:

dist	[0]	[1]	[2]	[3]	[4]	[5]	path	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**0**	**14**	**9**	**7**	**19**	∞	[0]	–	**1**	**2**	**3**	**1**	–
[1]	∞	**0**	∞	∞	**5**	∞	[1]	–	–	–	–	**4**	–
[2]	∞	**4**	**0**	∞	**9**	**3**	[2]	–	**1**	–	–	**1**	**5**
[3]	∞	∞	**10**	**0**	∞	**15**	[3]	–	–	**2**	–	–	**5**
[4]	∞	∞	∞	∞	**0**	∞	[4]	–	–	–	–	–	–
[5]	∞	∞	∞	∞	**2**	**0**	[5]	–	–	–	–	**4**	–

After 3rd iteration `i=2`:

dist	[0]	[1]	[2]	[3]	[4]	[5]	path	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**0**	**13**	**9**	**7**	**18**	**12**	[0]	–	**2**	**2**	**3**	**2**	**2**
[1]	∞	**0**	∞	∞	**5**	∞	[1]	–	–	–	–	**4**	–

[2]	∞	4	0	∞	9	3	[2]	–	1	–	–	1	5
[3]	∞	14	10	0	19	13	[3]	–	2	2	–	2	2
[4]	∞	∞	∞	∞	0	∞	[4]	–	–	–	–	–	–
[5]	∞	∞	∞	∞	2	0	[5]	–	–	–	–	4	–

After 4th iteration `i=3`: unchanged After 5th iteration `i=4`: unchanged After 6th iteration `i=5`:

dist	[0]	[1]	[2]	[3]	[4]	[5]	path	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	13	9	7	14	12	[0]	–	2	2	3	2	2
[1]	∞	0	∞	∞	5	∞	[1]	–	–	–	–	4	–
[2]	∞	4	0	∞	5	3	[2]	–	1	–	–	5	5
[3]	∞	14	10	0	15	13	[3]	–	2	2	–	2	2
[4]	∞	∞	∞	∞	0	∞	[4]	–	–	–	–	–	–
[5]	∞	∞	∞	∞	2	0	[5]	–	–	–	–	4	–

... Floyd's Algorithm 66/88

Why Floyd's algorithm is correct:

A shortest path from s to t using only nodes from $\{0,\ldots,i\}$ is the shorter of

- a shortest path from s to t using only nodes from $\{0,\ldots,i\text{-}1\}$
- a shortest path from s to i using only nodes from $\{0,\ldots,i\text{-}1\}$
  plus a shortest path from i to t using only nodes from $\{0,\ldots,i\text{-}1\}$

Also known as Floyd-Warshall algorithm (can you see why?)

... Floyd's Algorithm 67/88

Cost analysis …

- initialising `dist[][]`, `path[][]` $\Rightarrow O(E)$
- V iterations to update `dist[][]`, `path[][]` $\Rightarrow O(V^3)$

Time complexity of Floyd's algorithm: $O(V^3)$ (same as Warshall's algorithm for transitive closure)

Digraph Applications

PageRank 69/88

Goal: determine which Web pages are "important"

Approach: ignore page contents; focus on hyperlinks

- treat Web as graph: page = vertex, hyperlink = directed edge
- pages with many incoming hyperlinks are important
- need to compute "incoming degree" for vertices

Problem: the Web is a *very* large graph

- approx. 10^{14} pages, 10^{15} hyperlinks

Assume for the moment that we could build a graph …

Most frequent operation in algorithm "Does edge (v,w) exist?"

... PageRank 70/88

Simple PageRank algorithm:

```
PageRank(myPage):
|  rank=0
|  for each page in the Web do
|  |  if linkExists(page,myPage) then
|  |     rank=rank+1
|  |  end if
|  end for
```

Note: requires *inbound* link check

... PageRank 71/88

V = # pages in Web, E = # hyperlinks in Web

Costs for computing PageRank for each representation:

Representation	linkExists(v,w)	Cost
Adjacency *matrix*	`edge[v][w]`	1
Adjacency *lists*	`inLL(list[v],w)`	$\cong E/V$

Not feasible …

- adjacency matrix … $V \cong 10^{14} \Rightarrow$ matrix has 10^{28} cells
- adjacency list … V lists, each with ≅10 hyperlinks $\Rightarrow 10^{15}$ list nodes

So how to really do it?

... PageRank 72/88

Approach: the random web surfer

- if we randomly follow links in the web …
- … more likely to re-discover pages with many inbound links

```
curr=random page, prev=null
for a long time do
```

```
|  if curr not in array rank[] then
|     rank[curr]=0
|  end if
|  rank[curr]=rank[curr]+1
|  if random(0,100)<85 then            // with 85% chance ...
|     prev=curr
|     curr=choose hyperlink from curr  // ... crawl on
|  else
|     curr=random page                 // avoid getting stuck
|     prev=null
|  end if
end for
```

Could be accomplished while we crawl web to build search index

---

Exercise #9: Implementing Facebook 73/88

Facebook could be considered as a giant "social graph"

- what are the vertices?
- what are the edges?
- are edges directional?

What kind of algorithm would …

- help us find people that you might like to "befriend"?

Network Flow

This topic is out of the scope for this term.

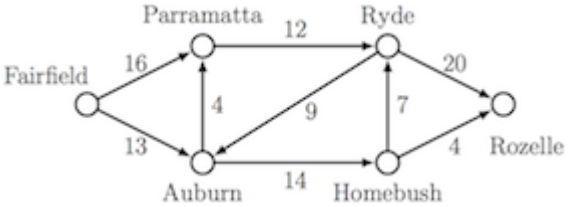
This is not going to be examined or assessed.

We will leave the slides for your reference.

Exercise #10: Merchandise Distribution 75/88

Lucky Cricket Company …

- produces cricket balls in Fairfield
- has a warehouse in Rozelle that stocks them
- ships them from factory to warehouse by leasing space on trucks with limited capacity:



What kind of algorithm would …

- help us find the maximum number of crates that can be shipped from Fairfield to Rozelle per day?

Flow Networks 76/88

Flow network …

- weighted graph *G=(V,E)*
- distinct nodes *s∈V* (*source*), *t∈V* (*sink*)

Edge weights denote *capacities* Applications:

- Distribution networks, e.g.
  - source: oil field
  - sink: refinery
  - edges: pipes
- Traffic flow

... Flow Networks 77/88

Flow in a network *G=(V,E)* … nonnegative *f(v,w)* for all vertices *v,w∈V* such that

- *f(v,w)≤capacity* for each edge *e=(v,w,capacity) ∈ E*
- *f(v,w)=0* if no edge between *v* and *w*
- total flow *into* a vertex = total flow *out of* a vertex:

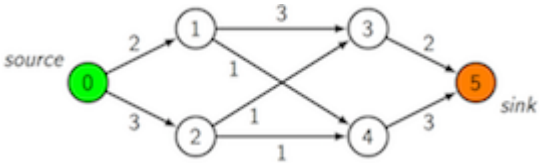
$$\sum_{x \in V} f(x, v) = \sum_{y \in V} f(v, y) \quad \text{for all } v \in V \setminus \{s,t\}$$

Maximum flow … no other flow from *s* to *t* has larger value

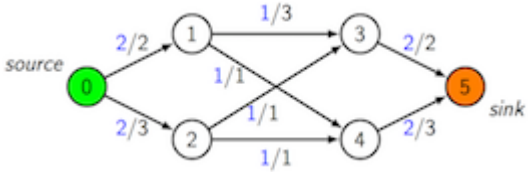
---

... Flow Networks 78/88

Example:



A (maximum) flow …

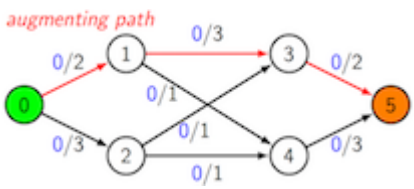


Augmenting Paths 79/88

Assume … *f(v,w)* contains current flow

Augmenting path: any path from source *s* to sink *t* that can currently take more flow

Example:



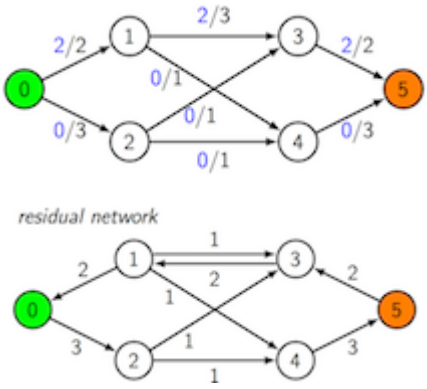
Residual Network

Assume … flow network $G=(V,E)$ and flow $f(v,w)$

Residual network (V,E'):

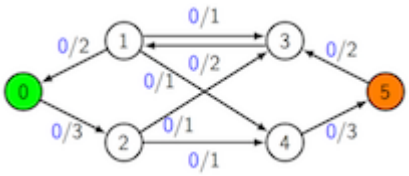
- same vertex set V
- for each edge $v \rightarrow^{c} w \in E$ …
  - $f(v,w) < c \Rightarrow$ add edge $(v \rightarrow^{c-f(v,w)} w)$ to E'
  - $f(v,w) > 0 \Rightarrow$ add edge $(v \leftarrow^{f(v,w)} w)$ to E'

Example:



Exercise #11: Augmenting Paths and Residual Networks

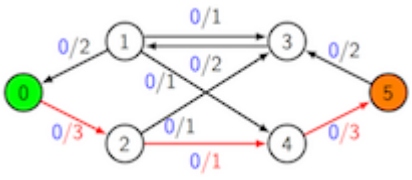
Find an augmenting path in:



and show the residual network after augmenting the flow

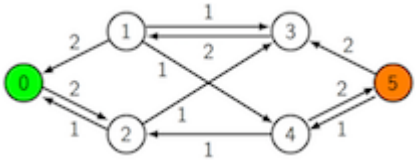
---

1. Augmenting path:



maximum additional flow = 1

2. Residual network:



Can you find a further augmenting path in the new residual network?

Edmonds-Karp Algorithm

One approach to solving maximum flow problem …

maxflow(G):

1. Find a shortest augmenting path
2. Update `flow[][]` so as to represent residual graph
3. Repeat until no augmenting path can be found

... Edmonds-Karp Algorithm

Algorithm:

```
flow[][]  // V×V array of current flow
visited[] /* array of predecessor nodes on shortest path
             from source to sink in residual network */

maxflow(G):
|  Input  flow network G with source s and sink t
|  Output maximum flow value
|
|  initialise flow[v][w]=0 for all vertices v, w
|  maxflow=0
|  while ∃shortest augmenting path visited[] from s to t do
|  |  df = maximum additional flow via visited[]
|  |  // adjust flow so as to represent residual graph
|  |  v=t
|  |  while v≠s do
|  |  |  flow[visited[v]][v] = flow[visited[v]][v] + df;
|  |  |  flow[v][visited[v]] = flow[v][visited[v]] - df;
|  |  |  v=visited[v]
|  |  end while
|  |  maxflow=maxflow+df
|  end while
|  return maxflow
```

Shortest augmenting path can be found by standard BFS

... Edmonds-Karp Algorithm

Time complexity analysis …

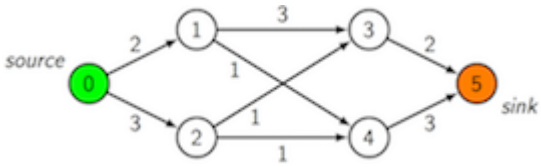
- *Theorem*. The number of augmenting paths needed is at most $V \cdot E/2$.
  $\Rightarrow$ Outer loop has $O(V \cdot E)$ iterations.
- Finding augmenting path $\Rightarrow O(E)$ (consider only vertices connected to source and sink $\Rightarrow O(V+E)=O(E)$)

Overall cost of Edmonds-Karp algorithm: $O(V \cdot E^2)$

Note: Edmonds-Karp algorithm is an implementation of general *Ford-Fulkerson method*

Exercise #12: Edmonds-Karp Algorithm

Show how Edmonds-Karp algorithm runs on:



flow	[0]	[1]	[2]	[3]	[4]	[5]
[0]						
[1]						
[2]						
[3]						
[4]						
[5]						

c−f	[0]	[1]	[2]	[3]	[4]	[5]
[0]		**2**	**3**			
[1]				**3**	**1**	
[2]				**1**	**1**	
[3]						**2**
[4]						**3**
[5]						

---

flow	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**0**	**0**	**0**	**0**	**0**	**0**
[1]	**0**	**0**	**0**	**0**	**0**	**0**
[2]	**0**	**0**	**0**	**0**	**0**	**0**
[3]	**0**	**0**	**0**	**0**	**0**	**0**
[4]	**0**	**0**	**0**	**0**	**0**	**0**
[5]	**0**	**0**	**0**	**0**	**0**	**0**

c−f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**–**	**2**	**3**	**–**	**–**	**–**
[1]	**–**	**–**	**–**	**3**	**1**	**–**
[2]	**–**	**–**	**–**	**1**	**1**	**–**
[3]	**–**	**–**	**–**	**–**	**–**	**2**
[4]	**–**	**–**	**–**	**–**	**–**	**3**
[5]	**–**	**–**	**–**	**–**	**–**	**–**

augmenting path: 0-1-3-5, `df`: 2

flow	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**0**	**2**	**0**	**0**	**0**	**0**
[1]	**-2**	**0**	**0**	**2**	**0**	**0**
[2]	**0**	**0**	**0**	**0**	**0**	**0**
[3]	**0**	**-2**	**0**	**0**	**0**	**2**
[4]	**0**	**0**	**0**	**0**	**0**	**0**
[5]	**0**	**0**	**0**	**-2**	**0**	**0**

c−f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**–**	**0**	**3**	**–**	**–**	**–**
[1]	**2**	**–**	**–**	**1**	**1**	**–**
[2]	**–**	**–**	**–**	**1**	**1**	**–**
[3]	**–**	**2**	**–**	**–**	**–**	**0**
[4]	**–**	**–**	**–**	**–**	**–**	**3**
[5]	**–**	**–**	**–**	**2**	**–**	**–**

augmenting path: 0-2-4-5, `df`: 1

flow	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**0**	**2**	**1**	**0**	**0**	**0**
[1]	**-2**	**0**	**0**	**2**	**0**	**0**
[2]	**-1**	**0**	**0**	**0**	**1**	**0**

c−f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**–**	**0**	**2**	**–**	**–**	**–**
[1]	**2**	**–**	**–**	**1**	**1**	**–**
[2]	**1**	**–**	**–**	**1**	**0**	**–**

[3]	**0**	**-2**	**0**	**0**	**0**	**2**	[3]	**–**	**2**	**–**	**–**	**–**	**0**
[4]	**0**	**0**	**-1**	**0**	**0**	**1**	[4]	**–**	**–**	**1**	**–**	**–**	**2**
[5]	**0**	**0**	**0**	**-2**	**-1**	**0**	[5]	**–**	**–**	**–**	**2**	**1**	**–**

augmenting path: 0-2-3-1-4-5, `df`: 1

`flow`	[0]	[1]	[2]	[3]	[4]	[5]	`c-f`	[0]	[1]	[2]	[3]	[4]	[5]
[0]	**0**	**2**	**2**	**0**	**0**	**0**	[0]	**–**	**0**	**1**	**–**	**–**	**–**
[1]	**-2**	**0**	**0**	**1**	**1**	**0**	[1]	**2**	**–**	**–**	**2**	**0**	**–**
[2]	**-2**	**0**	**0**	**1**	**1**	**0**	[2]	**2**	**–**	**–**	**0**	**0**	**–**
[3]	**0**	**-1**	**-1**	**0**	**0**	**2**	[3]	**–**	**1**	**1**	**–**	**–**	**0**
[4]	**0**	**-1**	**-1**	**0**	**0**	**2**	[4]	**–**	**1**	**1**	**–**	**–**	**1**
[5]	**0**	**0**	**0**	**-2**	**-2**	**0**	[5]	**–**	**–**	**–**	**2**	**2**	**–**

Summary

88/88

- Digraphs, weighted graphs: representations, applications
- Reachability
  - Warshall
- Minimum Spanning Tree (MST)
  - Kruskal, Prim
- Shortest path problems
  - Dijkstra (single source SPP)
  - Floyd (all-pair SSP)

- Suggested reading (Sedgewick):
  - digraphs … Ch. 19.1-19.3
  - weighted graphs … Ch. 20-20.1
  - MST … Ch. 20.2-20.4
  - SSP … Ch. 21-21.3

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