Week 2: Analysis of Algorithms

Week 2 1/85

Things to Note ...

- Weekly Assessments (2 marks, every week), starting from this week. Keep up with the weekly schedule.

In This Lecture ...

- Principles of algorithm analysis (Slides, [S] 2.1-2.4,2.6)

Coming Up ...

- Dynamic data structures ([M] Ch.6, Ch.10, [S] Ch.3, Ch.4)

Nerds You Should Know 2/85

First in a series on famous computer scientists ...

Who's the guy standing in the bus door?

... Nerds You Should Know 3/85

Alan Turing

- Founder of Computer Science
  ("Nobel prize of computing" named after him)
- 1930's Maths/Physics at Cambridge
- 1936 The Turing Machine
  (framework for computability and complexity theory)

- 1940-45 Code breaker
  (cracked Enigma code at Bletchley Park)
- 1946-50 Designed early computer
- Papers on neural nets, programming, chess computers
- 1950 Posed the "Turing Test" for AI
- 1954 Suicide by poisoned apple

Biography: "Alan Turing: The Enigma" by Andrew Hodges

Analysis of Algorithms

Running Time

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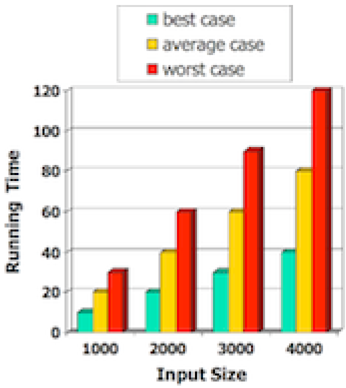
An algorithm is a step-by-step procedure

- for solving a problem
- in a finite amount of time

Most algorithms map input to output

- running time typically grows with input size
- *average time* often difficult to determine
- Focus on *worst case* running time
  - easier to analyse
  - crucial to many applications: finance, robotics, games, …



Empirical Analysis

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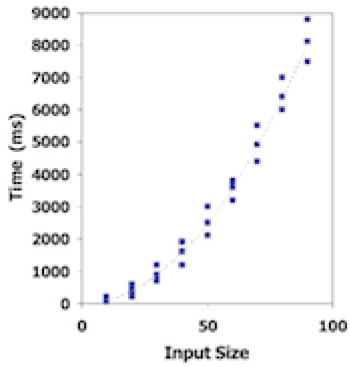
1. Write program that implements an algorithm
2. Run program with inputs of varying size and composition
3. Measure the actual running time
4. Plot the results



Limitations:

- requires to implement the algorithm, which may be difficult
- results may not be indicative of running time on other inputs
- same hardware and operating system must be used in order to compare two algorithms

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Theoretical Analysis 7/85

- Uses high-level description of the algorithm instead of implementation ("pseudocode")
- Characterises running time as a function of the input size, *n*
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

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Pseudocode 8/85

Example: Find maximal element in an array

```
arrayMax(A):
|  Input  array A of n integers
|  Output maximum element of A
|
|  currentMax=A[0]
|  for all i=1..n-1 do
|  |  if A[i]>currentMax then
|  |     currentMax=A[i]
|  |  end if
|  end for
|  return currentMax
```

... Pseudocode 9/85

Control flow

- **if** … **then** … [else] … **end if**
- **while .. do** … **end while**
  repeat … **until**
  for [all][each] .. do … **end for**

Function declaration

- f(arguments):
  Input …
  Output …
  …

Expressions

- = assignment
- = equality testing
- n^2 superscripts and other mathematical formatting allowed
- swap A[i] and A[j] verbal descriptions of *simple* operations allowed

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... Pseudocode

10/85

- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Exercise #1: Pseudocode

11/85

Formulate the following verbal description in pseudocode:

To reverse the order of the elements on a stack S with the help of a queue:

1. *In the first phase, pop one element after the other from S and enqueue it in queue Q until the stack is empty.*
2. *In the second phase, iteratively dequeue all the elements from Q and push them onto the stack.*

As a result, all the elements are now in reversed order on S.

Sample solution:

```
while S is not empty do
   pop e from S, enqueue e into Q
end while
while Q is not empty do
   dequeue e from Q, push e onto S
end while
```

Exercise #2: Pseudocode

13/85

Implement the following pseudocode instructions in C

1. `A` is an array of `int`s

```
...
swap A[i] and A[j]
...
```

2. `S` is a stack

```
...
swap the top two elements on S
...
```

1.
```
int temp = A[i];
A[i] = A[j];
A[j] = temp;
```

2.
```
x = StackPop(S);
y = StackPop(S);
StackPush(S, x);
StackPush(S, y);
```

The following pseudocode instruction is problematic. Why?

```
...
swap the two elements at the front of queue Q
...
```

The Abstract RAM Model

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RAM = Random Access Machine

- A CPU (central processing unit)
- A potentially unbounded bank of memory cells
  - each of which can hold an arbitrary number, or character
- Memory cells are numbered, and accessing any one of them takes CPU time

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Primitive Operations

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- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent of the programming language
- Exact definition not important (we will shortly see why)
- Assumed to take a constant amount of time in the RAM model

Examples:

- evaluating an expression
- indexing into an array
- calling/returning from a function

Counting Primitive Operations

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By inspecting the pseudocode …

- we can determine the maximum number of primitive operations executed by an algorithm
- as a function of the input size

Example:

```
arrayMax(A):
|  Input  array A of n integers
|  Output maximum element of A
|
|  currentMax=A[0]                  1
|  for all i=1..n-1 do              n+(n-1)
|  |  if A[i]>currentMax then       2(n-1)
|  |     currentMax=A[i]            n-1
|  |  end if
|  end for
|  return currentMax                1
                                    -------
                           Total    5n-2
```

Estimating Running Times

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Algorithm `arrayMax` requires $5n - 2$ primitive operations in the *worst* case

- *best* case requires $4n - 1$ operations (why?)

Define:

- a … time taken by the fastest primitive operation
- b … time taken by the slowest primitive operation

Let $T(n)$ be worst-case time of `arrayMax`. Then

$$a \cdot (5n - 2) \leq T(n) \leq b \cdot (5n - 2)$$

Hence, the running time $T(n)$ is bound by two linear functions

---

... Estimating Running Times

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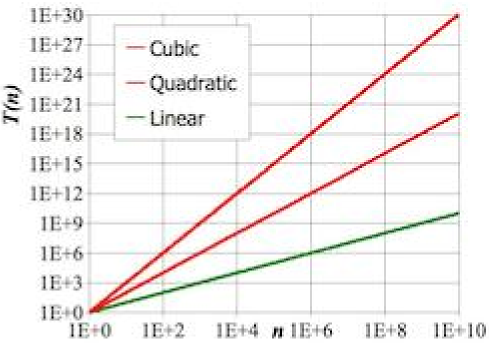
Seven commonly encountered functions for algorithm analysis

- Constant ≅ 1
- Logarithmic ≅ $\log n$
- Linear ≅ n
- N-Log-N ≅ $n \log n$
- Quadratic ≅ n^2
- Cubic ≅ n^3
- Exponential ≅ 2^n

... Estimating Running Times

20/85

In a log-log chart, the slope of the line corresponds to the growth rate of the function



... Estimating Running Times

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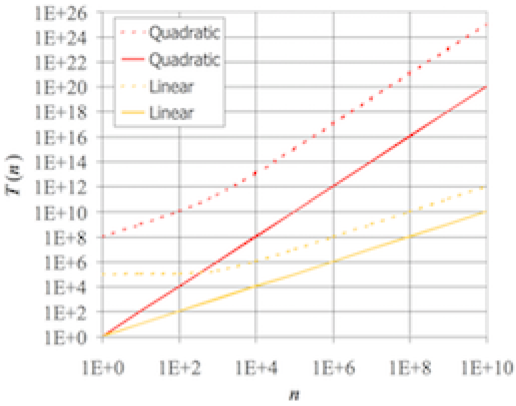
The growth rate is not affected by constant factors or lower-order terms

- Examples:
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



... Estimating Running Times

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Changing the hardware/software environment

- affects *T(n)* by a constant factor
- but does not alter the growth rate of *T(n)*

⇒ *Linear* growth rate of the running time *T(n)* is an intrinsic property of algorithm `arrayMax`

Exercise #3: Estimating running times

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Determine the number of primitive operations

```
matrixProduct(A,B):
|  Input  n×n matrices A, B
|  Output n×n matrix A·B
|
|  for all i=1..n do
|  |  for all j=1..n do
|  |  |  C[i,j]=0
|  |  |  for all k=1..n do
|  |  |     C[i,j]=C[i,j]+A[i,k]·B[k,j]
|  |  |  end for
|  |  end for
|  end for
|  return C
```

```
matrixProduct(A,B):
|  Input  n×n matrices A, B
|  Output n×n matrix A·B
|
|  for all i=1..n do                      2n+1
|  |  for all j=1..n do                   n(2n+1)
|  |  |  C[i,j]=0                         n^2
|  |  |  for all k=1..n do                n^2(2n+1)
|  |  |     C[i,j]=C[i,j]+A[i,k]·B[k,j]   n^3·4
|  |  |  end for
|  |  end for
|  end for
|  return C                               1
                                          ------------
                                  Total   6n^3+4n^2+3n+2
```

Big-Oh

Big-Oh Notation

Given functions f(n) and g(n), we say that

$$f(n) \in O(g(n))$$

if there are positive constants c and n_0 such that

$$f(n) \le c \cdot g(n) \quad \forall n \ge n_0$$

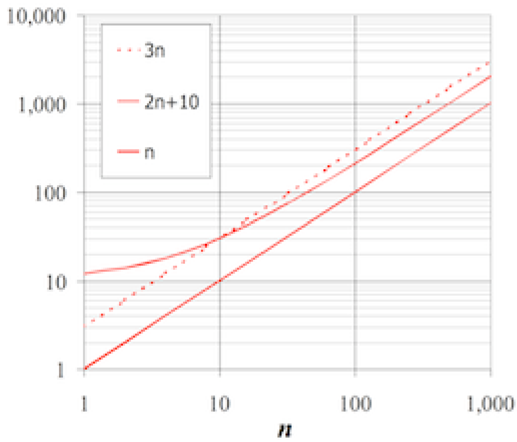
Hence: *O(g(n)) is the set of all functions that do not grow faster than g(n)*

Big-Oh notation gives an upper bound on the growth rate of a function

- "f(n) ∈ O(g(n))" means growth rate of f(n) no more than growth rate of g(n) (i.e., f(n) grows no faster than g(n)

... Big-Oh Notation

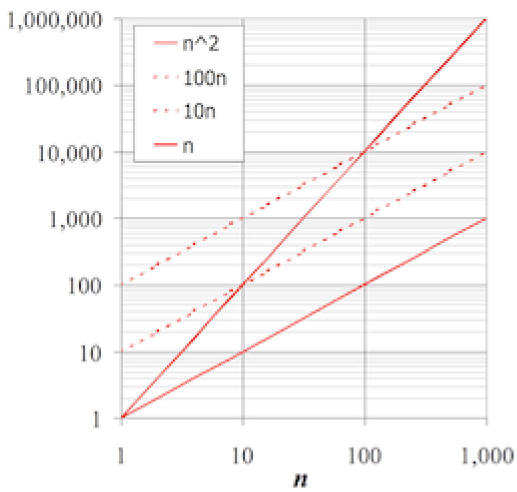
Example: function $2n + 10$ is in O(n)



- $2n+10 \le c \cdot n$
  $\Rightarrow$ $(c-2)n \ge 10$
  $\Rightarrow$ $n \ge 10/(c-2)$
- pick c=3 and n_0=10

... Big-Oh Notation

Example: function n^2 is not in O(n)



- $n^2 \le c \cdot n$
  $\Rightarrow \quad n \le c$
- inequality cannot be satisfied since c must be a constant

Big-Oh Rules

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- If f(n) is a polynomial of degree d $\Rightarrow$ f(n) is O(n^d)
  - lower-order terms are ignored
  - constant factors are ignored
- Use the smallest possible class of functions
  - say "2n is O(n)" instead of "2n is O(n^2)"
    - but keep in mind that, 2n is in O(n^2), O(n^3), …
- Use the simplest expression of the class
  - say "3n + 5 is O(n)" instead of "3n + 5 is O(3n)"

Exercise #4: Big-Oh

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Show that $\sum_{i=1}^{n} i$ is O(n^2)

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

which is O(n^2)

Asymptotic Analysis of Algorithms

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Asymptotic analysis of algorithms determines running time in big-Oh notation:

- find worst-case number of primitive operations as a function of input size
- express this function using big-Oh notation

Example:

- algorithm `arrayMax` executes at most 5n – 2 primitive operations
  $\Rightarrow$ algorithm `arrayMax` "*runs in O(n) time*"

Constant factors and lower-order terms eventually dropped
$\Rightarrow$ can disregard them when counting primitive operations

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Example: Computing Prefix Averages

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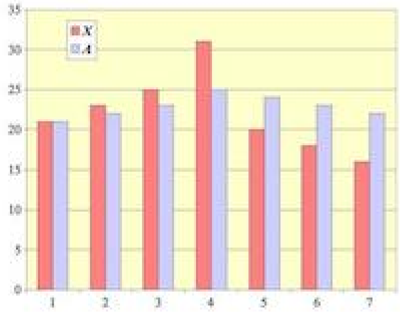
- The *i-th prefix average* of an array X is the average of the first i elements:

A[i] = (X[0] + X[1] + … + X[i]) / (i+1)



NB. computing the array A of prefix averages of another array X has applications in financial analysis

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... Example: Computing Prefix Averages

34/85

A *quadratic* algorithm to compute prefix averages:

```
prefixAverages1(X):
|  Input  array X of n integers
|  Output array A of prefix averages of X
|
|  for all i=0..n-1 do          O(n)
|  |  s=X[0]                    O(n)
|  |  for all j=1..i do         O(n^2)
|  |     s=s+X[j]               O(n^2)
|  |  end for
|  |  A[i]=s/(i+1)              O(n)
|  end for
|  return A                     O(1)
```

$$2 \cdot O(n^2) + 3 \cdot O(n) + O(1) = O(n^2)$$

⇒ *Time complexity* of algorithm `prefixAverages1` is $O(n^2)$

... Example: Computing Prefix Averages

35/85

The following algorithm computes prefix averages by keeping a running sum:

```
prefixAverages2(X):
|  Input  array X of n integers
|  Output array A of prefix averages of X
|
|  s=0
|  for all i=0..n-1 do          O(n)
|     s=s+X[i]                  O(n)
|     A[i]=s/(i+1)              O(n)
|  end for
|  return A                     O(1)
```

Thus, `prefixAverages2` is $O(n)$

Example: Binary Search

36/85

The following recursive algorithm searches for a value in a *sorted* array:

```
search(v,a,lo,hi):
|  Input  value v
|         array a[lo..hi] of values
|  Output true if v in a[lo..hi]
|         false otherwise
```

```
|
|  mid=(lo+hi)/2
|  if lo>hi then return false
|  if a[mid]=v then
|     return true
|  else if a[mid]<v then
|     return search(v,a,mid+1,hi)
|  else
|     return search(v,a,lo,mid-1)
|  end if
```

... Example: Binary Search

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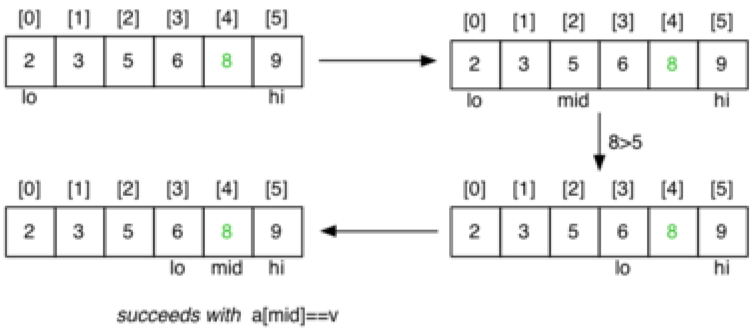
Successful search for a value of 8:



... Example: Binary Search

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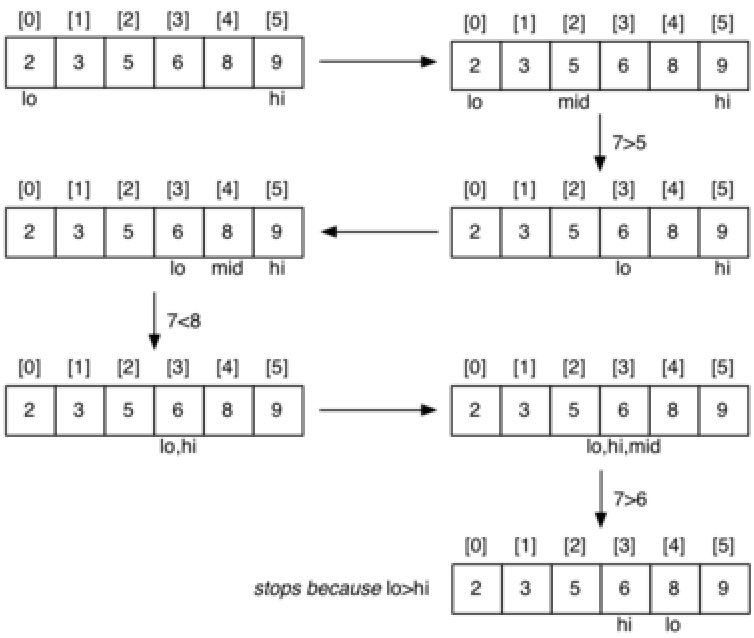
Unsuccessful search for a value of 7:



... Example: Binary Search

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Cost analysis:

- C_i = #calls to `search()` for array of length i
- for best case, C_n = 1
- for `a[i..j]`, `j<i` (length=0)

  - $C_0 = 0$
- for `a[i..j]`, `i≤j` (length=n)
  - $C_n = 1 + C_{n/2} \;\Rightarrow C_n = \log_2 n$

Thus, binary search is $O(\log_2 n)$ or simply $O(\log n)$ (why?)

... Example: Binary Search

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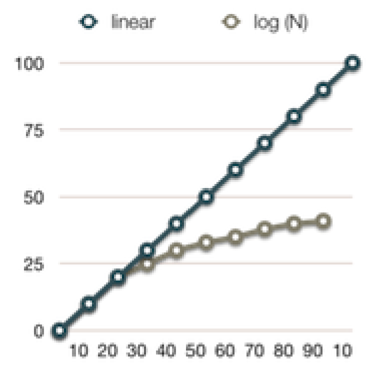
Why logarithmic complexity is good:



Math Needed for Complexity Analysis

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- Logarithms
  - $\log_b (xy) = \log_b x + \log_b y$
  - $\log_b (x/y) = \log_b x - \log_b y$
  - $\log_b x^a = a \log_b x$
  - $\log_a x = \log_b x \cdot (\log_c b / \log_c a)$
- Exponentials
  - $a^{(b+c)} = a^b a^c$
  - $a^{bc} = (a^b)^c$
  - $a^b / a^c = a^{(b-c)}$
  - $b = a^{\log_a b}$
  - $b^c = a^{c \cdot \log_a b}$
- Proof techniques
- Summation (addition of sequences of numbers)
- Basic probability (for average case analysis, randomised algorithms)

Exercise #5: Analysis of Algorithms

42/85

What is the complexity of the following algorithm?

```
enqueue(Q,Elem):
|  Input  queue Q, element Elem
|  Output Q with Elem added at the end
|
|  Q.top=Q.top+1
|  for all i=Q.top down to 1 do
|     Q[i]=Q[i-1]
|  end for
|  Q[0]=Elem
|  return Q
```

Answer: O(|Q|)

Exercise #6: Analysis of Algorithms

44/85

What is the complexity of the following algorithm?

```
binaryConversion(n):
|  Input  positive integer n
|  Output binary representation of n on a stack
|
|  create empty stack S
|  while n>0 do
|  |  push (n mod 2) onto S
|  |  n=⌊n/2⌋;
|  end while
|  return S
```

Assume that creating a stack and pushing an element both are O(1) operations ("constant")

Answer: O(log n)

Complexity Analysis: Arrays vs. Linked Lists

Static/Dynamic Sequences

47/85

Previously we have used an *array* to implement a stack

- fixed size collection of homogeneous elements
- can be accessed via index or via "moving" pointer

The "fixed size" aspect is a potential problem:

- how big to make the (dynamic) array? (big … just in case)
- what to do if it fills up?

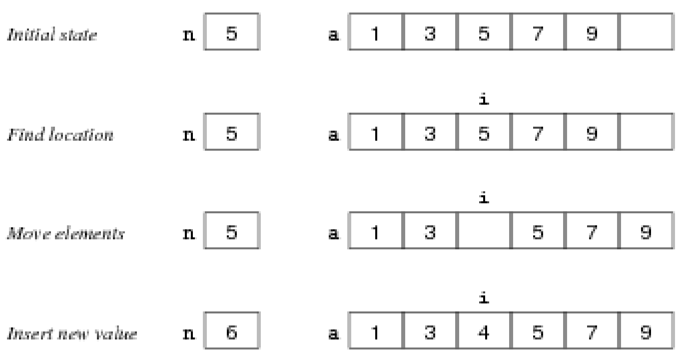
The rigid sequence is another problems:

- inserting/deleting an item in middle of array

... Static/Dynamic Sequences

48/85

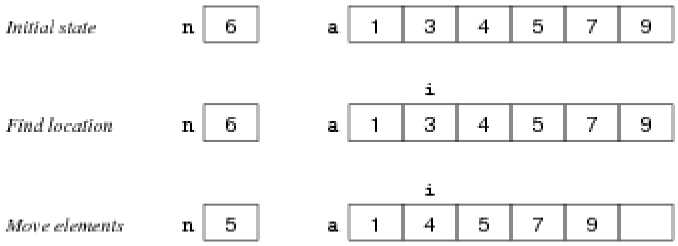
Inserting a value (*4*) into a sorted array `a` with `n` elements:



... Static/Dynamic Sequences

49/85

Deleting a value (*3*) from a sorted array `a` with `n` elements:

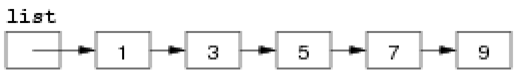


... Static/Dynamic Sequences

50/85

The problems with using arrays can be solved by

- allocating elements individually
- linking them together as a "chain"



Benefits:

- insertion/deletion have minimal effect on list overall
- only use as much space as needed for values

Self-referential Structures

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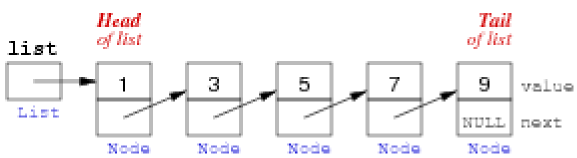
To realise a "chain of elements", need a *node* containing

- a value
- a link to the next node

To represent a chained (linked) *list* of nodes:

- we need a *pointer* to the first node
- each node contains a pointer to the next node
- the `next` pointer in the last node is `NULL`



... Self-referential Structures

52/85

Linked lists are more flexible than arrays:

- values do not have to be adjacent in memory
- values can be rearranged simply by altering pointers
- the number of values can change dynamically
- values can be added or removed in any order

Disadvantages:

- it is not difficult to get pointer manipulations wrong

- each value also requires storage for `next` pointer

... Self-referential Structures

53/85

Create a new list node:

```
makeNode(v)
|  Input  value v
|  Output new linked list node with value v
|
|  new.value=v      // initialise data
|  new.next=NULL    // initialise link to next node
|  return new       // return pointer to new node
```

Exercise #7: Creating a Linked List

54/85

Write pseudocode to create a linked list of three nodes with values 1, 42 and 9024.

```
mylist=makeNode(1)
mylist.next=makeNode(42)
(mylist.next).next=makeNode(9024)
```

Iteration over Linked Lists

56/85

When manipulating list elements

- typically have pointer `p` to current node
- to access the data in current node: `p.value`
- to get pointer to next node: `p.next`

To iterate over a linked list:

- set `p` to point at first node (head)
- examine node pointed to by `p`
- change `p` to point to next node
- stop when `p` reaches end of list (`NULL`)

... Iteration over Linked Lists

57/85

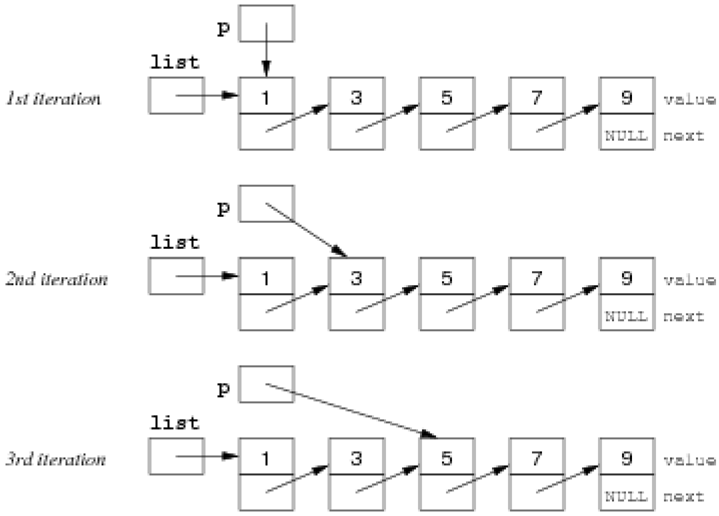
Standard method for scanning all elements in a linked list:

```
list  // pointer to first Node in list
p     // pointer to "current" Node in list

p=list
while p≠NULL do
|  … do something with p.value …
|  p=p.next
end while
```

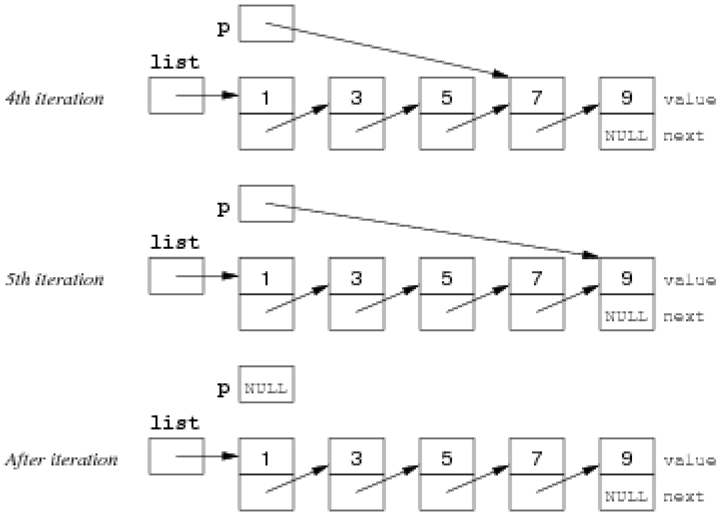
... Iteration over Linked Lists

58/85



... Iteration over Linked Lists

59/85



... Iteration over Linked Lists

60/85

Check if list contains an element:

```
inLL(L,d):
|  Input  linked list L, value d
|  Output true if d in list, false otherwise
|
|  p=L
|  while p≠NULL do
|  |  if p.value=d then   // element found
|  |     return true
|  |  end if
|  |  p=p.next
|  end while
|  return false          // element not in list
```

Time complexity: O(|L|)

... Iteration over Linked Lists

61/85

Print all elements:

```
showLL(L):
|  Input linked list L
```

```
|
|   p=L
|   while p≠NULL do
|      print p.value
|      p=p.next
|   end while
```

Time complexity: O(|L|)

Exercise #8: Traversing a linked list

62/85

What does this code do?

```
1  p=list
2  while p≠NULL do
3  |  print p.value
4  |  if p.next≠NULL then
5  |     p=p.next.next
6  |  else
7  |     p=NULL
8  |  end if
9  end while
```

What is the purpose of the conditional statement in line 4?

Every second list element is printed.

If `p` happens to be the last element in the list, then `p.next.next` does not exist.
The if-statement ensures that we do not attempt to assign an undefined value to pointer `p` in line 5.

Exercise #9: Traversing a linked list

64/85

Rewrite **`showLL()`** as a recursive function.

```
showLL(L):
|  Input linked list L
|
|  if L≠NULL do
|     print L.value
|     showLL(L.next)
|  end if
```

Modifying a Linked List

66/85

Insert a new element at the beginning:

```
insertLL(L,d):
|  Input  linked list L, value d
|  Output L with d prepended to the list
|
|  new=makeNode(d)  // create new list element
|  new.next=L       // link to beginning of list
|  return new       // new element is new head
```

Time complexity: O(1)

... Modifying a Linked List

67/85

Delete the *first* element:

```
deleteHead(L):
|  Input  non-empty linked list L, value d
|  Output L with head deleted
|
|  return L.next    // move to second element
```

Time complexity: O(1)

Delete a *specific* element (recursive version):

```
deleteLL(L,d):
|  Input  linked list L
|  Output L with element d deleted
|
|  if L=NULL then            // element not in list
|     return L
|  else if L.value=d then    // d found at front
|     return deleteHead(L)   // delete first element
|  else                      // delete element in tail list
|     L.next=deleteLL(L.next,d) // the rest of the list (i.e., L.next)
|     return L               // is what you get by deleting 'd' from the rest of the list
|  end if
```

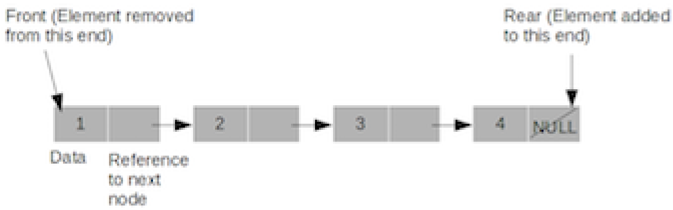
Time complexity: O(|L|)

Exercise #10: Implementing a Queue as a Linked List 68/85

Develop a datastructure for a queue based on linked lists such that …

- enqueuing an element takes constant time
- dequeuing an element takes constant time

Use pointers to both ends



Dequeue from the front …

```
dequeue(Q):
|  Input  non-empty queue Q
|  Output front element d, dequeued from Q
|
|  d=Q.front.value       // first element in the list
|  Q.front=Q.front.next  // move to second element
|  return d
```

Enqueue at the rear …

```
enqueue(Q,d):
|  Input queue Q
|
|  new=makeNode(d)      // create new list element
```

```
|   Q.rear.next=new     // add to end of list
|   Q.rear=new          // link to new end of list
```

Comparison Array vs. Linked List 70/85

Complexity of operations, n elements

	array	linked list
insert/delete at beginning	$O(n)$	$O(1)$
insert/delete at end	$O(1)$	$O(1)$ ("doubly-linked" list, with pointer to rear)
insert/delete at middle	$O(n)$	$O(n)$
find an element	$O(n)$ ($O(\log n)$, if array is sorted)	$O(n)$
index a specific element	$O(1)$	$O(n)$

Complexity Classes

Complexity Classes 72/85

Problems in Computer Science …

- some have *polynomial* worst-case performance (e.g. n^2)
- some have *exponential* worst-case performance (e.g. 2^n)

Classes of problems:

- P = problems for which an algorithm can compute answer in polynomial time
- NP = includes problems for which no P algorithm is known

Beware: NP stands for "nondeterministic, polynomial time (on a theoretical *Turing Machine*)"

... Complexity Classes 73/85

Computer Science jargon for difficulty:

- tractable … have a polynomial-time algorithm (useful in practice)
- intractable … no tractable algorithm is known (feasible only for small n)
- non-computable … no algorithm can exist

Computational complexity theory deals with different degrees of intractability

---

Generate and Test 74/85

The next topic on algorithm analysis is called 'Generate and Test'.

For this week, we will skip this one, but we are leaving the slides and relevant code examples for you. We may come back to this topic later in the term, if we can.

This topic (slide 79-90) is not covered in the mid-term exam. But maybe part of the final exam if we end up covering the topic.

Generate and Test 75/85

In scenarios where

- it is simple to test whether a given state is a solution
- it is easy to generate new states (preferably likely solutions)

then a *generate and test* strategy can be used.

It is necessary that states are generated systematically

- so that we are guaranteed to find a solution, or know that none exists
  - some randomised algorithms do not require this, however
    (more on this later in this course)

... Generate and Test 76/85

Simple example: checking whether an integer n is prime

- generate/test all possible factors of n
- if none of them pass the test $\Rightarrow$ n is prime

Generation is straightforward:

- produce a sequence of all numbers from 2 to n-1

Testing is also straightforward:

- check whether next number divides n exactly

... Generate and Test 77/85

Function for primality checking:

```
isPrime(n):
|  Input  natural number n
|  Output true if n prime, false otherwise
|
|  for all i=2..n-1 do      // generate
|  |  if n mod i = 0 then   // test
|  |     return false       // i is a divisor => n is not prime
|  |  end if
|  end for
|  return true              // no divisor => n is prime
```

Complexity of `isPrime` is O(n)

Can be optimised: check only numbers between 2 and $\lfloor\sqrt{n}\rfloor$ $\Rightarrow$ O($\sqrt{n}$)

Example: Subset Sum 78/85

Problem to solve ...

Is there a subset S of these numbers with $\Sigma_{x\in S}\, x = 1000$?

```
 34,  38,  39,  43,  55,  66,  67,  84,  85,  91,
101, 117, 128, 138, 165, 168, 169, 182, 184, 186,
234, 238, 241, 276, 279, 288, 386, 387, 388, 389
```

General problem:

- given *n* arbitrary integers and a target sum *k*
- is there a subset that adds up to exactly *k*?

... Example: Subset Sum

79/85

Generate and test approach:

```
subsetsum(A,k):
|  Input  set A of n integers, target sum k
|  Output true if Σ_{x∈S}x=k for some S⊆A
|         false otherwise
|
|  for each subset B⊆A do
|  |  if Σ_{b∈B}b=k then
|  |     return true
|  |  end if
|  end for
|  return false
```

- How many subsets are there of *n* elements?
- How could we generate them?

... Example: Subset Sum

80/85

Given: a set of `n` distinct integers in an array `A` …

- produce all subsets of these integers

A method to generate subsets:

- represent sets as *n* bits (e.g. *n=4*, `0000`, `0011`, `1111` etc.)
- bit *i* represents the i^{th} input number
- if bit *i* is set to 1, then `A[i]` is in the subset
- if bit *i* is set to 0, then `A[i]` is not in the subset
- e.g. if `A[]=={1,2,3,5}` then `0011` represents `{1,2}`

... Example: Subset Sum

81/85

Algorithm:

```
subsetsum1(A,k):
|  Input  set A of n integers, target sum k
|  Output true if Σ_{x∈S}x=k for some S⊆A
|         false otherwise
|
|  for s=0..2^n−1 do
|  |  if k = Σ_{(i^th bit of s is 1)} A[i] then
|  |     return true
|  |  end if
|  end for
|  return false
```

Obviously, `subsetsum1` is $O(2^n)$

... Example: Subset Sum

82/85

Alternative approach …

`subsetsum2(A,n,k)`
(returns true if any subset of A[0..n-1] sums to k; returns false otherwise)

- if the n^{th} value A[n-1] is part of a solution …
  - then the first n-1 values must sum to k – A[n-1]
- if the n^{th} value is not part of a solution …
  - then the first n-1 values must sum to k
- base cases: k=0 (solved by {}); n=0 (unsolvable if k>0)

```
subsetsum2(A,n,k):
|  Input  array A, index n, target sum k
|  Output true if some subset of A[0..n-1] sums up to k
|         false otherwise
|
|  if k=0 then
|     return true   // empty set solves this
|  else if n=0 then
|     return false  // no elements => no sums
|  else
|     return subsetsum(A,n-1,k-A[n-1]) or subsetsum(A,n-1,k)
|  end if
```

... Example: Subset Sum

83/85

Cost analysis:

- C_i = #calls to `subsetsum2()` for array of length i
- for worst case,
  - $C_1 = 2$
  - $C_n = 2 + 2 \cdot C_{n-1} \Rightarrow C_n \cong 2^n$

Thus, `subsetsum2` also is $O(2^n)$

... Example: Subset Sum

84/85

Subset Sum is typical member of the class of *NP-complete problems*

- intractable … only algorithms with exponential performance are known
  - increase input size by 1, double the execution time
  - increase input size by 100, it takes 2^{100} = 1,267,650,600,228,229,401,496,703,205,376 times as long to execute
- but if you can find a polynomial algorithm for Subset Sum, then any other *NP*-complete problem becomes *P* …

Summary

85/85

- Big-Oh notation
- Asymptotic analysis of algorithms
- Examples of algorithms with logarithmic, linear, polynomial, exponential complexity
- Linked lists vs. arrays

- Suggested reading:
  - Sedgewick, Ch. 2.1-2.4, 2.6

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