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Mathematical Modeling of Hexacopter

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Abstract

The purpose of this paper is to present the basic mathematical modeling of microcopters, which could be used to develop proper methods for stabilization and trajectory control. The microcopter taken into account consists of six rotors, with three pairs of counter-rotating fixed-pitch blades. The microcopter is controlled by adjusting the angular velocities of the rotors which are spun by electric motors. It is assumed as a rigid body, so the differential equations of the microcopter dynamics can be derived from both the Newton-Euler and Euler-Lagrange equations. Euler-angle parametrization of three-dimensional rotations contains singular points in the coordinate space that can cause failure of both dynamical model and control. In order to avoid singularities, the rotations of the microcopter are parametrized in terms of quaternions. This choice has been made taking into consideration the linearity of quaternion formulation, their stability and efficiency.

Keywords Hexacopter; Newton-Euler equations; Quaternions.

1 Introduction

The interest for Unmanned Aerial Vehicle (UAV) in military and civil applications is fast growing up with the aim of developing cheaper and more capable machines. Among the multicopter layout typology, the four rotors, also called quadrotor, have been widely chosen by many researchers as a very promising vehicle for indoor/outdoor navigation. Multidisciplinary concepts are necessary because this type of rotorcraft attempts to achieve stable hovering and precise flight by balancing the forces produced by the four rotors. Nowadays, the design of multicopter with more than four rotors, i.e. hexacopter and octocopter, is developing thanks to the possibility of managing one or more engine failures and to increase the total payload. In this paper a hexacopter is considered whose six-rotors are located on vertices of a hexagon and are equidistant from the center of gravity; moreover, the propulsion system consists of three pairs of counter-rotating fixed-pitch blades. The aircraft dynamic behavior is here presented by the mathematical model, by considering all its external and internal influences. Assuming the hexacopter as a rigid body, the differential equations describing its dynamic behaviour can be derived from the Newton-Euler equations, leading to equivalent mathematical models. Euler-angle parameterization of three-dimensional rotations contains singular points in the coordinate space that can cause failure of both dynamical models and control. These singularities are not present if the three-dimensional rotations are parametrized in terms of quaternions. As shown in [1], the strength of quaternions depends on the linearity of their formulation, on the easiness of their algebraic structure and, overall, on their stability and efficiency. In the following the mathematical model of the hexacopter is presented; then, the quaternions parametrization is introduced; finally, the Newton-Euler equations, describing the dynamics of the hexacopter, are obtained.

2 Reference systems for the hexacopter

This section deals with the coordinate systems and the reference frames chosen to describe the hexacopter dynamics. First of all, the classical Euler parametrization is treated; in other words, the angular orientation of the aircraft's body is described by three Euler angels, that represent an ordered set of sequential rotations from a reference frame to a frame fixed in the body. Although this formulation is easy to develop and visualize, it fails in specific configurations and therefore it is not an effective method for aircraft dynamics [4]. In the following the quaternions are introduced and then used to formulate the dynamical equations [2, 5, 7].

2.1 Euler angles

The schematic structure of the hexacopter is illustrated in Figure 1. In order to describe the hexacopter motion only two reference systems are necessary: earth fixed frame and body frame. The motion of an aircraft is always planned by

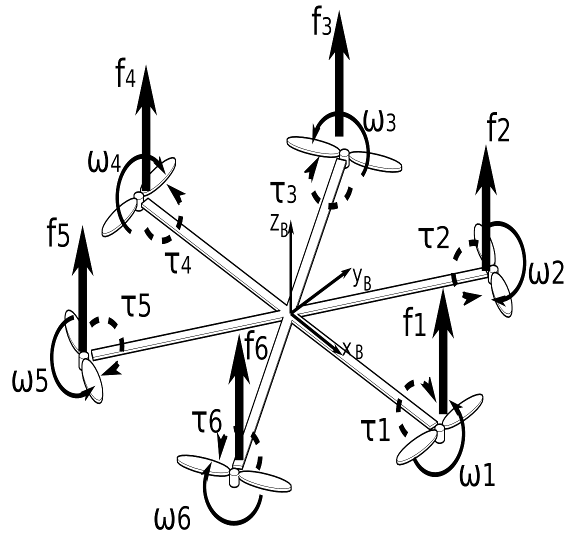

Figure 1: The body frame of an hexacopter

using geographical maps, so it is useful to define an earth fixed frame tangent to the earth surface. One of such a frame is the system that uses the North, East and Down (NED) coordinates. The origin of this reference system is fixed in one point located on the earth surface and the X, Y, and Z axes are directed to the North, East and down, respectively. This earth fixed frame is seen as an inertial frame in which the absolute linear position (x, y, z) of the hexacopter is defined. The mobile frame (X_B, Y_B, Z_B) is the body fixed frame, that is centered in the hexacopter center of gravity and oriented as shown in Figure 1. The angular position of the body frame with respect to the inertial one, is usually defined by means of the Euler angles: roll ϕ, pitch θ and yaw ψ. For sake of simplicity, let us denote the inertial position vector and the Euler angle vector by means of $\xi = [\,\mathbf{x\ y\ z}]^{\mathbf{T}}$ and $\eta = [\,\phi\ \theta\ \psi]^{\mathbf{T}}$, respectively. The transformation from the body frame to the inertial frame is realized by using the well known rotation matrix $\mathbf{R}$

$$\begin{bmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\phi\cos\psi\sin\theta + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

which is orthogonal. As a consequence, the transformation matrix from the inertial frame to the body frame is $\mathbf{R^{-1}} = \mathbf{R^T}$. As shown in [1], the transformation matrix for angular velocities from the body frame to the inertial one is

$$\mathbf{W}_\eta^{\mathbf{-1}} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sec\theta\sin\phi & \cos\phi\sec\theta \end{bmatrix}.$$

Then the transformation laws are $\nu = \mathbf{W}_\eta \dot{\eta}$ and $\dot{\eta} = \mathbf{W}_\eta^{\mathbf{-1}} \nu$. in which the angular velocity ν is defined by the vector $\nu = [\mathbf{p}\ \mathbf{q}\ \mathbf{r}]^{\mathbf{T}}$. It is important to observe that $\mathbf{W}_\eta^{\mathbf{-1}}$ can be defined if and only if $\theta \neq \pi/2 + k\pi$, $(k \in Z)$. This is the main effect of Euler formulation that leads to the *gimbal lock*, typical situation in which a degree of freedom is lost. To overcame this problem, it is possible to consider a different representation for the hexacopter orientation in space. The aircraft rotation from one frame of reference to another will be identified by four parameters, known as quaternions, whose general structure is briefly summarized afterwards. The advantages of an approach based on quaternions consist not only in the absence of singularities but also in the simplicity of computation, as will be shown later on.

2.2 Quaternions

In the above section, we noted that the rotation of a rigid body in space could be represented by Euler angles; however, the singularity of the transformation laws leads to adopt a new parametrization, the quaternions, with the aim of describing the orientation of the aircraft respect to the earth fixed frame [3, 6, 8]. Quaternions are commonly used in several application fields, such as computer game development and 3D virtual worlds, but also as a method for rigid body rotation in three-dimensional space. The quaternion representation is based on the Euler's rotation theorem which states that any rigid body displacement where a point is fixed is equivalent to a rotation. Therefore, if α is the rotation angle about the unit vector $\mathbf{u} = (u_1, u_2, u_3)$, it is possible to define a quaternion as $\mathbf{q} = [q_0\ q_1\ q_2\ q_3]^T$, with $q_0 = \cos(\alpha/2)$, $q_1 = \sin(\alpha/2)u_1$, $q_2 = \sin(\alpha/2)u_2$, $q_3 = \sin(\alpha/2)u_3$. Unlike Euler angles, quaternion rotations do not require a set of pre-defined rotation axes because they can change its single axis continuously. Due to the fact that the method of rotating around an arbitrary direction has only one axis of rotation, degrees of freedom cannot be lost; therefore gimbal lock cannot occur. The transformation of the translational velocities representation from the body frame to the inertial one can be expressed by $\xi = \mathbf{Q}\xi_{\mathbf{B}}$, where $\mathbf{Q}$ is the following matrix

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}.$$

As the matrix $\mathbf{R}$, $\mathbf{Q}$ is orthogonal; therefore, it is $\mathbf{Q}^{\mathbf{-1}} = \mathbf{Q}^{\mathbf{T}}$. As the angular velocities concerns, the involved transformation can be written as $\dot{\mathbf{q}} = \mathbf{S}\,\nu$, where the matrix $\mathbf{S}$ depends on quaternion components as follows

$$\mathbf{S} = \frac{1}{2} \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}.$$

In conclusion, we remark that the advantage in considering the quaternion reference is twofold because it avoids critical positions and, thanks to the linearity of the coefficients of the transformation matrix, it is also numerically more efficient and stable compared to traditional rotation formulation.

3 Mathematical model of the hexacopter

In this section the mathematical model for the hexacopter is illustrated. This model is basically obtained by representing the mini-rotorcraft as a solid body evolving in three dimensional space. The six electric motor dynamics is relatively fast and therefore it will be neglected as well as the flexibility of the blades. Our aim is to provide the mathematical equations driving the dynamical behavior of the hexacopter by means of a generalization of the quadrotor model presented in [1]. The motion of a rigid body can be decomposed into the translational and rotational components. Therefore, in order to describe the dynamics of the hexacopter, assumed to be a rigid body, the Newton-Euler equations, that govern linear and angular motion, are taken into account. First of all, the force acting on the hexacopter is provided by

$$\mathbf{F} = \mathbf{d}(\mathbf{m}\,\mathbf{v_B})/\mathbf{dt} + \nu \times (\mathbf{m}\,\mathbf{v_B})$$

where the mass m is assumed to be constant. Every rotor i has an angular velocity $\omega_\mathbf{i}$, which generates a force $\mathbf{f}_i = [0\ 0\ \omega_i^2]$ being k the lift constant, thus the total thrust $\mathbf{T_B}$ is given by $\mathbf{T_B} = [\mathbf{0}\ \mathbf{0}\ \mathbf{T}]^\mathbf{T}$ with

$$T = \sum_{i=1}^{6} f_i = k \sum_{i=1}^{6} \omega_i.$$

Total thrust together with gravitational force represents the total force acting on the hexacopter,

$$\mathbf{F} = \mathbf{Q}^T \mathbf{F}_g + \mathbf{T}_B.$$

As a consequence, the translation component of the motion referred to the body frame is

$$m\,\dot{\mathbf{v}}_B + \nu \times (\mathbf{m}\,\mathbf{v_B}) = \mathbf{Q^T F_g} + \mathbf{T_B}$$

In order to get the same equation with respect to the inertial frame, let notice that centrifugal force is nullified because the inertial frame does not rotate and thus

$$m\,\ddot{\xi} = \mathbf{F}_g + \mathbf{Q}\,\mathbf{T}_B.$$

By now, let $\mathbf{I}$ be the inertia matrix. The hexacopter has a symmetric structure with respect to the X_B-axis, Y_B-axis and Z_B-axis, thus the inertia matrix is the diagonal one $I = diag(I_{xx}, I_{yy}, I_{zz})$. As the total external moment $\mathbf{M}$ concerns, the rate of change of the angular momentum $\mathbf{H} = \mathbf{I}\nu$ is considered and the moment acting on the hexacopter is provided by

$$\mathbf{M} = d(\mathbf{I}\,\nu)/\mathbf{dt} + \nu \times (\mathbf{I}\,\nu).$$

Moreover, angular velocity and acceleration of the rotor create a torque

$$\tau_{M_i} = b\,\omega_i^2 + I_{M_i}\,\dot{\omega}_i$$

around the rotor axis, where b is the drag constant and $\mathbf{I}_M$ is the inertia moment of the rotor i. From the geometrical structure of the hexacopter and from the components of $\mathbf{f}_i$ and $\tau_{\mathbf{M_i}}$ over the body frame, it is possible to get the information on roll, pitch and yaw moment, namely

$$\begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} \frac{3}{4}kl\,(\omega_2^2 + \omega_3^2 - \omega_5^2 - \omega_6^2) \\ kl\left(-\omega_1^2 - \frac{\omega_2^2}{4} + \frac{\omega_3^2}{4} + \omega_4^2 + \frac{\omega_5^2}{4} - \frac{\omega_6^2}{4}\right) \\ b\,(-\omega_1^2 + \omega_2^2 - \omega_3^2 + \omega_4^2 - \omega_5^2 + \omega_6^2) + \\ +I_M\,(\dot{\omega}_1 + \dot{\omega}_2 + \dot{\omega}_3 + \dot{\omega}_4 + \dot{\omega}_5 + \dot{\omega}_6) \end{bmatrix}$$

Here l is the distance between the rotor and the center of gravity of the hexacopter and $\dot{x}\omega_i$ denotes the derivative of $\omega_i(t)$ with respect to time, i.e. $d\omega_i(t)/dt$. Afterwards, the equation that governs the rotational dynamic can be summarized as

$$\mathbf{I}\,\dot{\nu} + \nu \times (\mathbf{I}\,\nu) + \mathbf{\Gamma} = \tau_{\mathbf{B}},$$

in which $\mathbf{\Gamma}$ represents the gyroscopic forces and $\tau_{\mathbf{B}}$ the external torque. After some algebra, it results

$$\dot{\nu} = \mathbf{I^{-1}}\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx}\,p \\ I_{yy}\,q \\ I_{zz}\,r \end{bmatrix} - \mathbf{I_r}\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\omega_{\mathbf{\Gamma}} + \tau\right), \tag{1}$$

in which $\omega_\Gamma = \omega_1 - \omega_2 + \omega_3 - \omega_4 + \omega_5 - \omega_6$. Otherwise, the equation (1) can be written as

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} (I_{yy} - I_{zz})\,q\,r/I_{xx} \\ (I_{zz} - I_{xx})\,p\,r/I_{yy} \\ (I_{xx} - I_{yy})\,p\,q/I_{zz} \end{bmatrix} - I_r\begin{bmatrix} q/I_{xx} \\ -p/I_{yy} \\ 0 \end{bmatrix}\omega_\Gamma + \begin{bmatrix} \tau_\phi/I_{xx} \\ \tau_\theta/I_{yy} \\ \tau_\psi/I_{zz} \end{bmatrix} \tag{2}$$

Finally, once the angular velocity has been evaluated, the angular acceleration in the inertial frame can be easily deduced as $\ddot{\mathbf{q}} = d(\mathbf{S}\,\nu)/\mathbf{dt}$.

4 Conclusion

In this paper we presented the mathematical model of a mini rotorcraft with six rotors. We defined the equations by means of quaternion because, unlike Euler angles, they do not suffer from the gimbal lock; they are also more efficient in terms of numerical computation and, in addition, any operations involving them is trivial. We suppose that in real applications our aircraft will assume configurations far form the gimbal lock. However, quaternion parametrization is taken into account because of its simplicity for computation and its numerical stability, that allow more efficient and fast algorithm implementation with higher control system. Future works extend in several directions, such improving our model with more realistic features (aerodynamic effects) and supporting it with a wide numerical experiments.

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References

[1] T.S. Alderete Simulator aero model implementation, *NASA Ames Research Center, Moffett Field, California*, available at `http://www.aviationsystemsdivision.arc.nasa.gov/publications/hitl/rtsim/Toms.pdf`

[2] E. Bekir (2007) *Introduction to Modern Navigation System*, (World Scientific Publishing Co. Pte. Ltd.).

[3] M. Cefalo, J.M. Mirats-Tur (2011) A comprehensive dynamic model for class-1 tensegrity systems based on quaternions, *International Journal of Solids and Structures* **48** 785–802.

[4] K. Großekatthöfer, Z. Yoon (2012) *Introduction into quaternions for spacecraft attitude representation*, (TU Berlin).

[5] W. F. Phillips (2004) *Mechanics of Flight*, (John Wilet & Sons).

[6] J.M. Rico-Martinez, J. Gallardo-Alvarado (2000) A simple method for the determination of angular velocity and acceleration of a spherical motion through quaternions, *Meccanica* **35** 111–118.

[7] R. F. Stengel (2004) *Flight Dynaimcs*, (Princeton University Press).

[8] A. P. Yefremov (2004) Quaternions: Algebra, Geometry and Physical Theories, *Hypercomplex Numbers in Geometry and Physics* **1**.

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