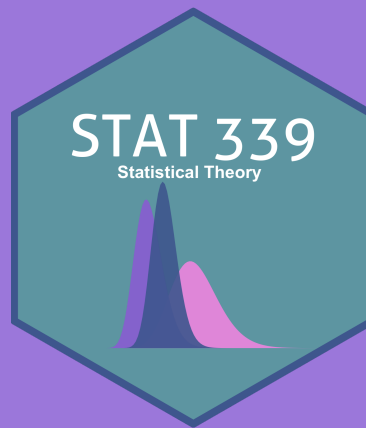


# STAT 339: Statistical Theory

## Frequentist Parameter Estimation

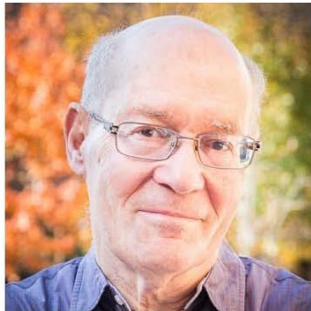
Anthony Scotina



# Reminder!

## Personal and General Reflections on 50 years of Teaching Statistics

- Event for *Undergraduate Teaching Award*, Boston Chapter of the ASA



📅 Tuesday, January 25

🕒 6-7pm ET

📍 Register [here](#)

💰 **FREE** to attend

🙏 Please go, if you can!

# Examples

1. **Clinical Trial**: What is the risk of major adverse cardiovascular events (MACE) for T2D patients while taking certain treatment regimens?
  - Estimating  $p$ , the *unknown* proportion of MACE for a large group of T2D patients taking a specific treatment
2. **Piéchart Emporium**: What is the average wait time at the checkout counter for PE customers?
  - Estimating  $\mu$ , the *unknown* average wait time for PE customers
3. **Cell Phone Batteries**: How can we best quantify battery life in a certain type of smart phone?
  - Estimating  $\mu$ , the *unknown* average battery life

**Considerations**: What is the *best* estimator? How do we determine what makes an estimator *best*?

# Estimators and Estimates

In general, we will refer to  $\theta$  as the **target parameter** of interest.

- Can be equal to  $\mu, p, \sigma^2$ , etc., but we'll use  $\theta$  as a "catch-all".

To estimate one (or more) parameters, we need **data**!

- For example, suppose the average wait time of a *random sample* of 20 PE customers was **five minutes**.
  - This is a **point estimate** - it is an estimate of  $\theta$  in the form of a *single value*.

A **point estimator** (or *statistic*),  $\hat{\theta}$ , is the rule/formula used to calculate the value of an estimate based on *sample data*.

**Examples:**

- $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$
- $s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2}$

# Estimators and Estimates

# Unbiased Estimation

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# Bias of Point Estimators

Estimators are not perfect! Some are **good**, some are **bad**, and others are 🤡🤡🤡

Let  $\hat{\theta}$  be a point estimator for the parameter  $\theta$ . Then  $\hat{\theta}$  is an **unbiased estimator** if  $E(\hat{\theta}) = \theta$ . If  $E(\hat{\theta}) \neq \theta$ , then  $\hat{\theta}$  is *biased*.

- The **bias** of a point estimator  $\hat{\theta}$  is given by  $Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$ .

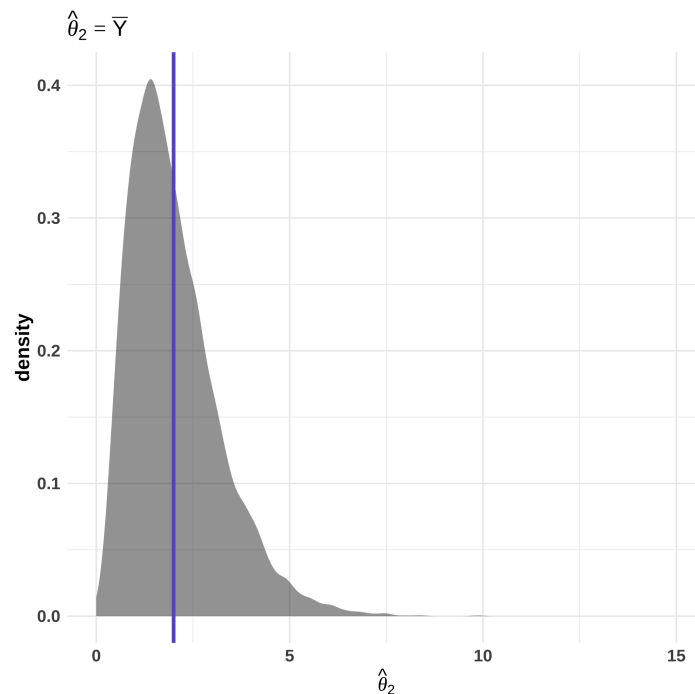
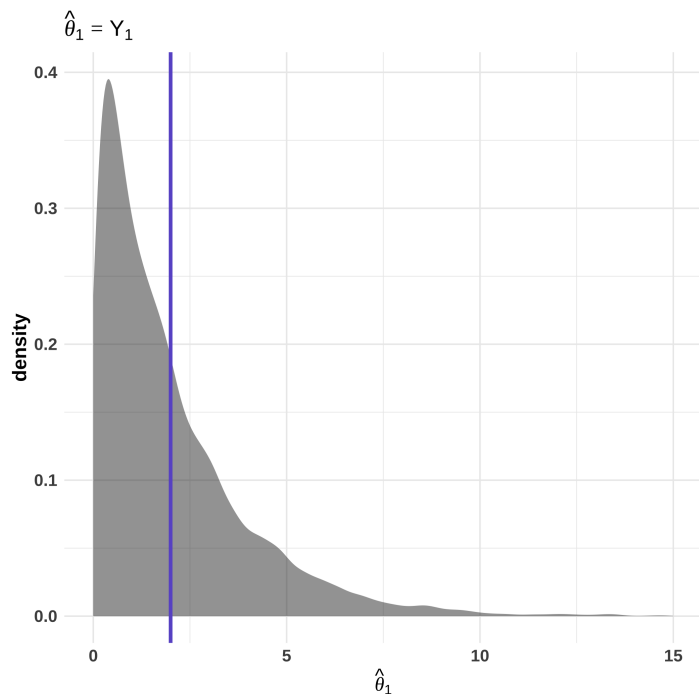
Ideally, the expected value of our estimator  $\hat{\theta}$  will equal the parameter ( $\theta$ ) that we're trying to estimate.

- But we also want  $\hat{\theta}$  to have a **small variance** - this means a higher fraction of  $\hat{\theta}$  values (in *repeated sampling*) will be close to  $\theta$ .

# Two Unbiased Estimators

$Y_1, Y_2, Y_3 \sim \text{Exponential}(2)$

- Suppose  $\theta = E(Y_i) = 2$ . let's try to estimate  $\theta$  using different  $\hat{\theta}$ .





# Mean Square Error (MSE)

The **mean square error (MSE)** of a point estimator is the *average of the square of the distance between the estimator and target parameter*:

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

It can be shown that

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + [Bias(\hat{\theta})]^2.$$

- In other words,  $MSE(\hat{\theta})$  is a function of both the **variance** and **bias** of  $\hat{\theta}$ .

**Note:** For unbiased estimators,  $MSE(\hat{\theta}) = Var(\hat{\theta})$ .

# Biased Estimators

If an estimator  $\hat{\theta}$  is **biased** we can usually correct it to make it *unbiased*.

## Example

Suppose that  $\hat{\theta}$  is an estimator for a parameter  $\theta$  and  $E(\hat{\theta}) = a\theta + b$  for some nonzero constants  $a$  and  $b$ .

1. In terms of  $a$ ,  $b$ , and  $\theta$ , what is  $Bias(\hat{\theta})$ ?
2. find a function of  $\hat{\theta}$  - say,  $\hat{\theta}^*$  - that is an unbiased estimator for  $\theta$ .
3. Express  $MSE(\hat{\theta}^*)$  as a function of  $Var(\hat{\theta})$ .

# Order Statistics as Estimators

Let  $Y_1, Y_2, \dots, Y_n \sim \text{Uniform}(0, \theta)$ , where the target parameter is  $\theta$ .

- Because  $\theta$  is the upper bound of the support for the  $Y_i$ , let's try to use

$$Y_{(n)} = \max(Y_1, Y_2, \dots, Y_n)$$

as an estimator for  $\theta$ .

## Questions

1. Is  $\hat{\theta} = Y_{(n)}$  unbiased for  $\theta$ ?
2. What is the variance of  $\hat{\theta}$ ?

**From STAT 338:** The PDF for  $Y_{(n)}$  is

$$g_{(n)}(y) = n[F(y)]^{n-1}f(y),$$

where  $f(y)$  is the PDF for  $Y$ , and  $F(y) = P(Y \leq y)$ .

# Order Statistics as Estimators

Let  $Y_1, Y_2, \dots, Y_n$  denote a random sample of size  $n$  from a population whose density is given by

$$f(y \mid \alpha) = 3\alpha^3 y^{-4}, \quad \alpha \leq y,$$

where  $\alpha > 0$  is unknown. That is,  $Y_i \sim \text{Pareto}(\alpha, \beta = 3)$ , where in general

$$E(Y_i) = \alpha\beta/(\beta - 1).$$

Show that  $\hat{\alpha} = Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n)$  is a *biased* estimator for  $\alpha$ .

# Common Unbiased Point Estimators

## Sample Mean

Suppose  $Y_1, \dots, Y_n$  are a *random sample* from some population with mean  $\mu$  and variance  $\sigma^2$ .

- Our target parameter is  $\theta = \mu$ . Let's show that  $\hat{\theta} = \bar{Y}$  is **unbiased**.

## Sample Variance

It turns out that

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2$$

is **biased** for  $\sigma^2$ .

- How can we find an **unbiased** estimator for  $\sigma^2$ ? 🤔

# Estimator for Binomial Variance

If  $Y$  has a binomial distribution with parameters  $n$  and  $p$ , then we have seen that  $\hat{p} = Y/n$  is an unbiased estimator for  $p$ .

To estimate the variance of  $Y$ , where  $\text{Var}(Y) = np(1 - p)$ , we generally use

$$\widehat{\text{Var}}(Y) = n\hat{p}(1 - \hat{p}).$$

1. Show that the suggested estimator is a *biased* estimator of  $\text{Var}(Y)$ .
2. Modify  $n\hat{p}(1 - \hat{p})$  slightly to form an *unbiased* estimator of  $\text{Var}(Y)$ .

# Bias/Variance Trade-off

While **unbiased** estimators sound *desirable*, they are not always the best estimators.

In general, we'd like for  $Bias(\hat{\theta})$  to be close to zero. But we also want  $Var(\hat{\theta})$  to be close to zero!

- Higher variance means that estimates might be very *different* across **repeated samples**.
- Ideally,  $MSE(\hat{\theta})$  will be as small as possible.

# Bias/Variance Trade-off

$Y_1, Y_2, \dots, Y_n \sim iid \text{Uniform}(0, \theta).$

Consider three estimators for  $\theta$ :

1.  $\hat{\theta}_1 = 2\bar{Y}$

2.  $\hat{\theta}_2 = Y_{(n)}.$

3.  $\hat{\theta}_3 = 2Y_1.$

Let's find the bias and variance for each.



# Methods of Estimation

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## The Method of Moments

# Finding Estimators

Up to this point, we've mostly used *intuition* to find estimators  $\hat{\theta}$  of  $\theta$ .

- The **sample mean**,  $\bar{Y}$ , seems like it would be a good estimator for the **population mean**,  $\mu$ .
- The **sample variance**,  $s^2$ , seems like it would be a good estimator for the **population variance**,  $\sigma^2$ .

But what if we wanted to find estimators for the  $\alpha$  and  $\beta$  parameters, using a sample of observations from the  $Gamma(\alpha, \beta)$  distribution?

- $E(Y) = \alpha\beta$ , but we want to find estimators for each of  $\alpha$  and  $\beta$ !

## Two estimation techniques

1. Method of Moments
2. Method of Maximum Likelihood

# Method of Moments

**Recall:** The  $k$ th moment of a random variable  $Y$  is

$$\mu'_k = E(Y^k)$$

- Therefore,  $\mu'_1 = E(Y)$ ,  $\mu'_2 = E(Y^2)$ , etc.

We define the  $k$ th **sample moment** as the average,

$$m'_k = \frac{1}{n} \sum_{i=1}^n Y_i^k.$$

**Method of Moments (MOM):** Set  $\mu'_k = m'_k$ , for  $k = 1, 2, \dots, t$  ( $t$  = number of parameters to be estimated) and solve for the parameter(s) of interest.

# Uniform MOM Estimator

Let  $Y_1, Y_2, \dots, Y_n \sim iid \text{Uniform}(0, \theta)$ .

- $\mu'_1 = E(Y) = \theta/2$
- $m'_1 = \bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$

# MOM Estimators for Gamma parameters

Let  $Y_1, Y_2, \dots, Y_n \sim iid \text{Gamma}(\alpha, \beta)$ , where  $\alpha$  and  $\beta$  are unknown.

- Find the MOM estimators for  $\alpha$  and  $\beta$ .
- $\mu'_1 = E(Y) = \alpha\beta$ .
  - Set this equal to  $\frac{1}{n} \sum_{i=1}^n Y_i = \bar{Y}$ .
- $\mu'_2 = E(Y^2) = Var(Y) + [E(Y)]^2 = \alpha\beta^2 + \alpha^2\beta^2$ .
  - Set this equal to  $\frac{1}{n} \sum_{i=1}^n Y_i^2$ .

We need to solve the system of equations for  $\alpha$  and  $\beta$ .

- $\tilde{\alpha} = \frac{n\bar{Y}^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2}$
- $\tilde{\beta} = \frac{\bar{Y}}{\tilde{\alpha}} = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n\bar{Y}}$

# MOM Estimators for Normal parameters

Suppose we have a random sample  $Y_1, Y_2, \dots, Y_n \sim iid \text{Normal}(\mu, \sigma^2)$ .

- Find the MOM estimators for  $\mu$  and  $\sigma^2$ .
- $\tilde{\mu} = \bar{X} \implies \text{unbiased}$  for  $\mu$ !
- $\tilde{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2 \implies \text{biased}$  for  $\sigma^2$

# Pros and Cons of MOM

## Benefits

- Simple to use (just equate sample and population moments)
- Can be used to estimate multiple parameter families

## Limitations

- Generate *biased* estimators in many cases
- Need the moments to exist! (Sorry, [Cauchy distribution...](#))
- MLEs are typically *closer* to the target quantity...

# Methods of Estimation

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## The Method of Maximum Likelihood



# Likelihood Function

**Setting:**  $Y_1, Y_2, \dots, Y_n$  are iid from a distribution with parameter  $\theta$  (which might be a single value or a vector of multiple parameters).

- The **likelihood function**,  $f(\mathbf{y} \mid \theta)$ , gives the *likelihood* of observing our sample

$$(Y_1 = y_1, Y_2 = y_2, \dots, Y_n = y_n)$$

when the parameter is  $\theta$ .

- For simplicity, we define  $\mathbf{y} = (y_1, y_2, \dots, y_n)$ .

**Example** (Continuous random sample)

$$\begin{aligned} f(\mathbf{y} \mid \theta) &= f(y_1, y_2, \dots, y_n \mid \theta) \\ &= f(y_1 \mid \theta) \times f(y_2 \mid \theta) \times \dots \times f(y_n \mid \theta) \end{aligned}$$

**Note:** The likelihood function is sometimes written as  $L(\mathbf{y} \mid \theta)$  or  $L(\theta)$ .

# Maximum Likelihood Estimation

**Intuition:** Choose  $\hat{\theta}$  as the estimate of  $\theta$  that **maximizes** the likelihood function!

- In this context,  $\hat{\theta}$  is called the **maximum likelihood estimator (MLE)**.

**Example:** Moose's favorite toys

This box came with 45 balls. Sadly, Moose lost most of them under furniture, and there are **four left**.

- Some are *red*, and some are *yellow*, but we don't know *exactly* how many of each.
- Moose really only cares about the **red** balls, so let's try to estimate how many are red!



# Moose's Favorite Toys

We have four balls - some are **red**, and some are yellow. Let's try to estimate *how many red balls there are among the four remaining*.

- I allow Moose to choose three of these balls at *random*. Suppose all three are red; yay!
- If our sample yields *three red balls*, what would be a good estimate of the total number of red balls remaining,  $n_r$ ?

The parameter,  $n_r$  can be either **3** or **4**. We know that Moose choose  $Y = 3$  red balls, so  $n_r$  cannot equal 0, 1, or 2.

- Let's find the *likelihood* of obtaining our sample, in two separate worlds: one with  $n_r = 3$ , and one with  $n_r = 4$

$$P(Y = 3 \mid n_r = 3) = \frac{\binom{3}{3} \binom{1}{0}}{\binom{4}{3}} = 0.25$$

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- Let's find the *likelihood* of obtaining our sample, in two separate worlds: one with  $n_r = 3$ , and one with  $n_r = 4$

$$P(Y = 3 \mid n_r = 4) = \frac{\binom{4}{3}}{\binom{4}{3}} = 1$$

Because  $n_r = 4$  maximizes the likelihood of the observed sample, our **MLE** of  $n_r$  is  $\hat{n}_r = 4$ .

# Lifetimes of Electrical Components

Suppose the lifetimes of electrical components (in years),  $Y$ , are modeled from an exponential distribution. That is,  $Y_1, Y_2, \dots, Y_n \sim \text{Exponential}(\theta)$ .

- We observe a sample of  $n = 5$  component lifetimes:  $\mathbf{y} = (3, 1.5, 2, 1.7, 2.1)$ .  
Let's find the MLE  $\hat{\theta}_{MLE}$  for  $\theta$  that maximizes the likelihood of this sample.

1. Write likelihood:

$$\begin{aligned} L(\theta) &= f(\mathbf{y} \mid \theta) = f(y_1 \mid \theta) \times \dots \times f(y_5 \mid \theta) \\ &= \left(\frac{1}{\theta}\right) e^{-y_1/\theta} \times \dots \times \left(\frac{1}{\theta}\right) e^{-y_5/\theta} \\ &= \frac{1}{\theta^5} \exp\left(\frac{-\sum_{i=1}^5 y_i}{\theta}\right) \\ &= \frac{1}{\theta^5} \exp\left(\frac{-10.3}{\theta}\right) \end{aligned}$$

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Let's find the MLE  $\hat{\theta}_{MLE}$  for  $\theta$  that maximizes the likelihood of this sample.

2. Take derivative of **log-likelihood** with respect to  $\theta$ :

- $\log L(\theta) = -5 \log \theta - (10.3/\theta)$
- $\frac{d \log L(\theta)}{d\theta} = (-5/\theta) + (10.3/\theta^2)$

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Let's find the MLE  $\hat{\theta}_{MLE}$  for  $\theta$  that maximizes the likelihood of this sample.

3. Solve for  $\theta$ :

- $(-5/\theta) + (10.3/\theta^2) = 0 \implies \theta = 10.3/5 = 2.06$

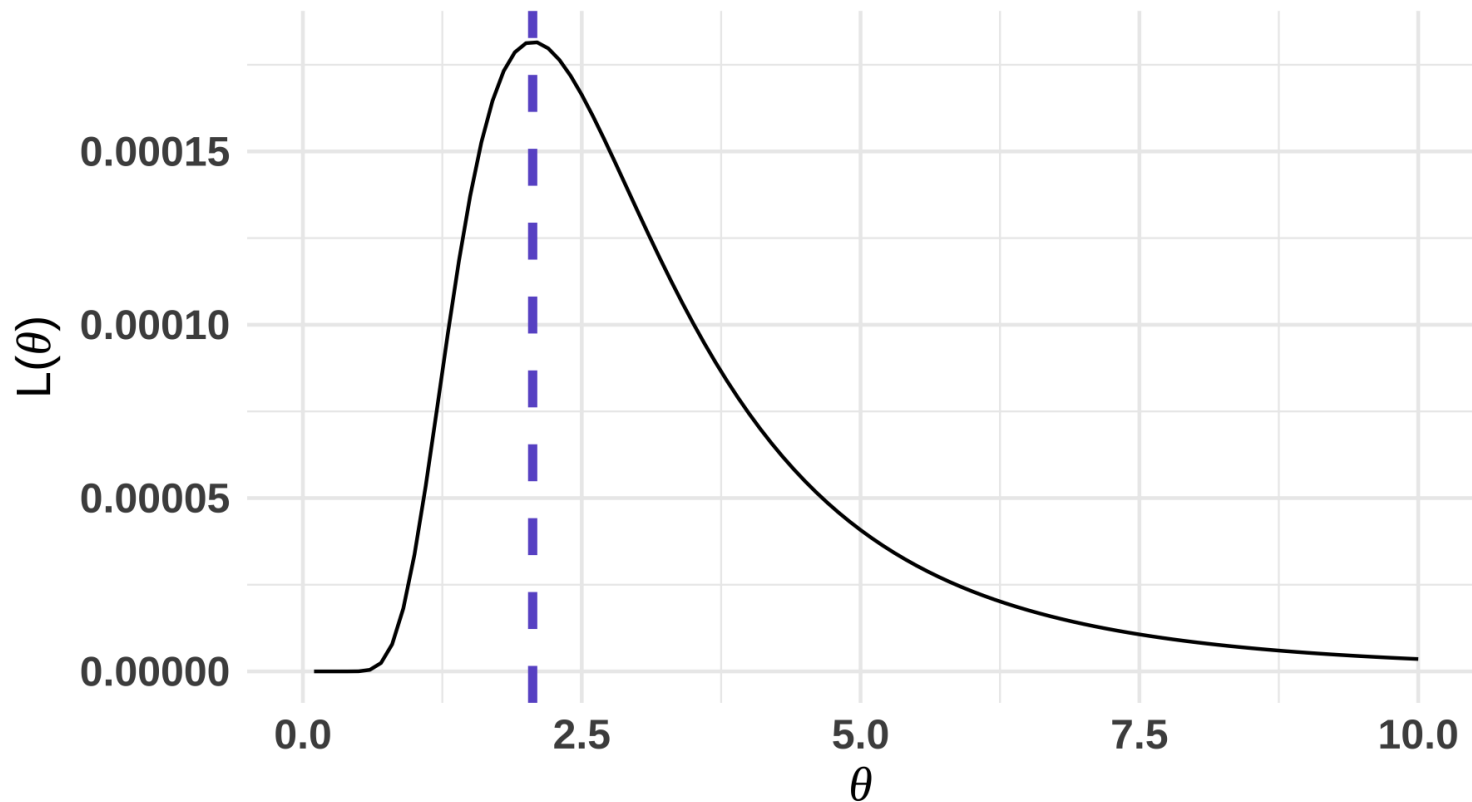
Therefore,  $\hat{\theta}_{MLE} = 2.06$ . Because the data  $\mathbf{y}$  are observed here, 2.06 is a maximum likelihood **estimate** of  $\theta$ .

4. (Bonus) Take second derivative of log-likelihood, make sure it is negative at  $\theta = 2.06$ .

# Exponential Likelihood

Likelihood function for  $\text{Exp}(\theta)$

$n = 5, \hat{\theta}_{\text{MLE}} = 2.06$





# Normal Distribution MLEs

Suppose that  $Y_1, Y_2, \dots, Y_n$  form a *random sample* from a  $Normal(\mu, \sigma^2)$  distribution.

- Find the MLEs of  $\mu$  and  $\sigma^2$ .

**Note:**  $\theta = (\mu, \sigma^2)$ , so we need to take two different derivatives of  $\log L(\theta)$ .

## Solution

- $\hat{\mu}_{MLE} = \bar{Y}$
- $\hat{\sigma}_{MLE}^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2$  (a **biased** estimator!)

# Uniform MLE

Suppose that  $Y_1, Y_2, \dots, Y_n$  form a *random sample* from a  $Uniform(0, \theta)$  distribution.

- Find the MLE of  $\theta$ .

1. Write likelihood:

$$\begin{aligned} L(\theta) &= f(y_1 | \theta) \times \cdots \times f(y_n | \theta) \\ &= \frac{1}{\theta^n}, \quad \text{if } 0 \leq y_i \leq \theta \end{aligned}$$

- The first derivative of  $L(\theta)$  does not equal zero for any  $\theta > 0$ .
- However,  $1/\theta^n$  **increases** as  $\theta$  decreases, so we want to select  $\theta$  to be as small as possible in order to maximize the likelihood.
  - One constraint: All of the  $y_i$  values are between 0 and  $\theta$ .
  - The *smallest* value of  $\theta$  that satisfies this constraint is  $Y_{(n)} = \max(Y_1, \dots, Y_n)$

Therefore,  $\hat{\theta}_{MLE} = Y_{(n)}$ .

# Pros and Cons of MLE

## Benefits

- MLEs are *invariant*! This means that, if  $\hat{\theta}$  is an MLE for  $\theta$ , then  $g(\hat{\theta})$  is an MLE for  $g(\theta)$ .
- MLEs are *consistent*.

## Limitations

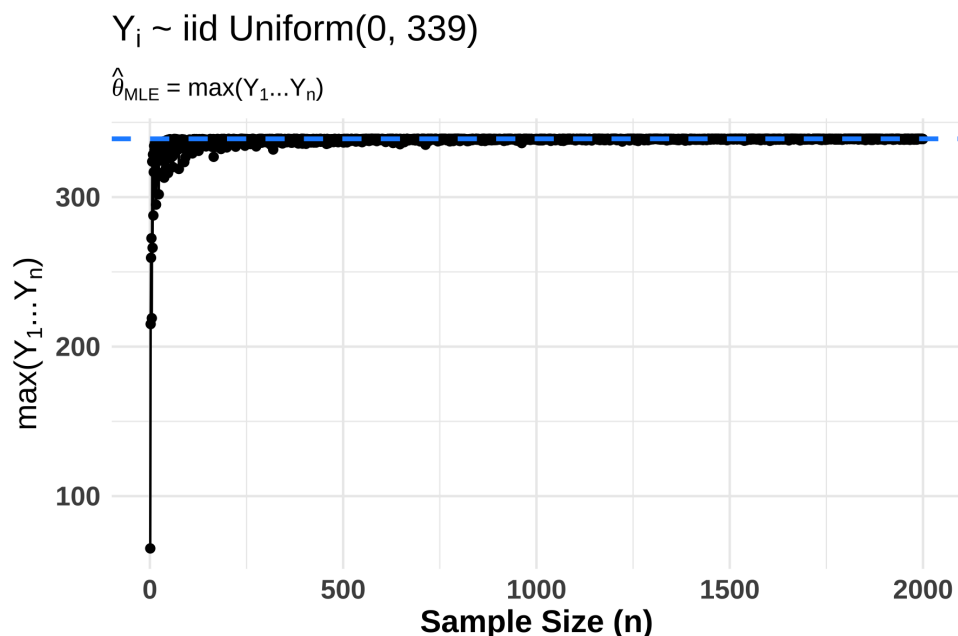
- MLEs do not always exist.
- The MLE is **NOT** the most likely parameter, given the data ( $E(\theta \mid Y)$ ). It estimates the parameter  $\theta$  that maximizes the distribution of  $Y \mid \theta$ 
  - In other words, the MLE gives the parameter estimate most likely to have produced the observed data.

# Consistency of the MLE

The estimator  $\hat{\theta}_n$  is said to be **consistent** for  $\theta$  if, for any  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} P(|\hat{\theta}_n - \theta| \leq \epsilon) = 1.$$

- Basically, this means that if  $n$  is **large enough**, there is a probability of 1 that  $\hat{\theta}_n$  will be very close to  $\theta$ .



# Data Reduction: Sufficiency

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# Sufficient Statistics

Most of the estimators we've chosen have seemed like they would be good estimators.

- The sample mean,  $\bar{Y}$ , is *probably* a solid estimator for the population mean  $\mu$ .

Once we calculate  $\bar{Y}$ , the actual sample values  $Y_1, \dots, Y_n$  are *no longer important*; the information in the sample is *summarized* by  $\bar{Y}$ .

- But does this summary retain all of the information about  $\mu$  contained in the original  $n$  observations?

A statistic that summarizes *all* information in a sample about a target parameter is said to be **sufficient**.

- We'll use sufficient statistics to help determine *best* (unbiased) estimators.

# Factorization Criterion

## Theorem

Let  $U$  be a statistic based on the random sample  $Y_1, \dots, Y_n$ . Then  $U$  is a **sufficient statistic** for the estimation of a parameter  $\theta$  if and only if the likelihood  $L(\theta) = f(y_1, \dots, y_n \mid \theta)$  can be factored into two nonnegative functions:

$$L(\theta) = g(u, \theta) \times h(y_1, \dots, y_n)$$

where:

- $g(u, \theta)$  is a function only of  $u$  and  $\theta$ , and
- $h(y_1, \dots, y_n)$  is not a function of  $\theta$ .

## Process for Finding a Sufficient Statistic

1. Write out the **likelihood**,  $L(\theta) = f(y_1 \mid \theta) \times \dots \times f(y_n \mid \theta)$ .
2. Given some statistic  $U$ , check if  $L(\theta)$  can be broken down into  $g(u, \theta)$  and  $h(y_1, \dots, y_n)$ .
  - **Note:** There are often *more than one* sufficient statistic for any parameter.

# Sufficient Statistic Examples

1. Let  $Y_1, Y_2, \dots, Y_n$  be a random sample such that  $Y_i \sim \text{Exponential}(\theta)$  with PDF

$$f(y_i | \theta) = \frac{1}{\theta} e^{-y_i/\theta}, \quad y_i > 0.$$

Show that  $U = \bar{Y}$  is a sufficient statistic for  $\theta$ .

2. Let  $Y_1, Y_2, \dots, Y_n$  be a random sample such that  $Y_i \sim \text{Beta}(\theta, 1)$  with PDF

$$f(y_i | \theta) = \theta y^{\theta-1}, \quad 0 < y < 1.$$

Show that  $U = \prod_{i=1}^n Y_i = Y_1 \times \dots \times Y_n$  is a sufficient statistic for  $\theta$ .



# Rao-Blackwell Theorem

Let  $\hat{\theta}$  be an unbiased estimator for  $\theta$  If  $\hat{\theta}$  has a smaller variance than *all other unbiased estimators* for  $\theta$ , then  $\hat{\theta}$  is the **best unbiased estimator (BUE)** (the "boo").



## Rao-Blackwell Theorem

Let  $h(U)$  be some function of a statistic,  $U$ . If:

- $U$  is a sufficient statistic for  $\theta$
- $E[h(U)] = \theta$

then it follows that  $\hat{\theta} = h(U)$  is the best unbiased estimator for  $\theta$ .



# Sampling Distributions of Estimators

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# Recap

What have we done so far? ⌚ ⌚ ⌚

We've used **point estimators** (or statistics),  $\hat{\theta}$ , to estimate unknown target **parameters**,  $\theta$ .

- These estimators are functions of:
  - observable random variables in a sample
  - known constants (usually the sample size,  $n$ )
- While *unknown*,  $\theta$  is assumed to be **fixed** at some value.

Because statistics are *functions* of random variables...

**All statistics are random variables!**

Because **all statistics are random variables**, all statistics have *probability distributions* that illustrate (among other things) how much they vary from sample to sample.

- These "special" probability distributions are called **sampling distributions**.

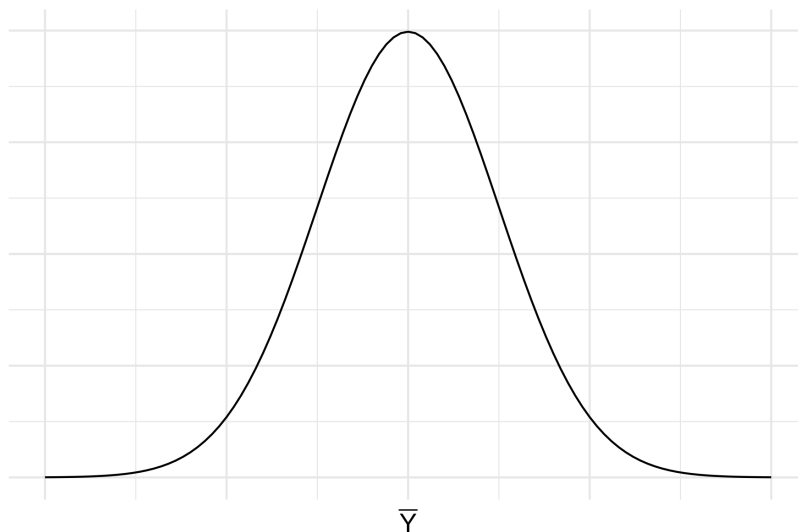
# Why sampling distributions?

Before the sample has been taken, we can use the sampling distribution of  $\hat{\theta}$  to calculate the probability that  $\hat{\theta}$  will be close to  $\theta$ .

## Example

Let  $Y_1, Y_2, \dots, Y_n \sim iid \text{Normal}(\mu, \sigma^2)$ . Then:

$$\bar{Y} \sim \text{Normal}\left(\mu, \frac{\sigma^2}{n}\right).$$



# Chi-Squared Distribution

Again, suppose that  $Y_1, Y_2, \dots, Y_n \sim iid \text{Normal}(\mu, \sigma^2)$ .

- Though now we want to work with the sample variance,  $S^2$ .

## 1. Unbiased Estimator:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2$$

## 2. MLE (and MOM Estimator):

$$\hat{\sigma}_{MLE}^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2$$

It turns out that, except for a scale factor, the sample variance follows a  $\chi^2$  (chi-squared) distribution with  $n - 1$  **degrees of freedom**.

# Chi-Squared Distribution

## Theorem

Let  $Y_1, Y_2, \dots, Y_n$  be a random sample from a  $Normal(\mu, \sigma^2)$  distribution. Then

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2(n-1).$$

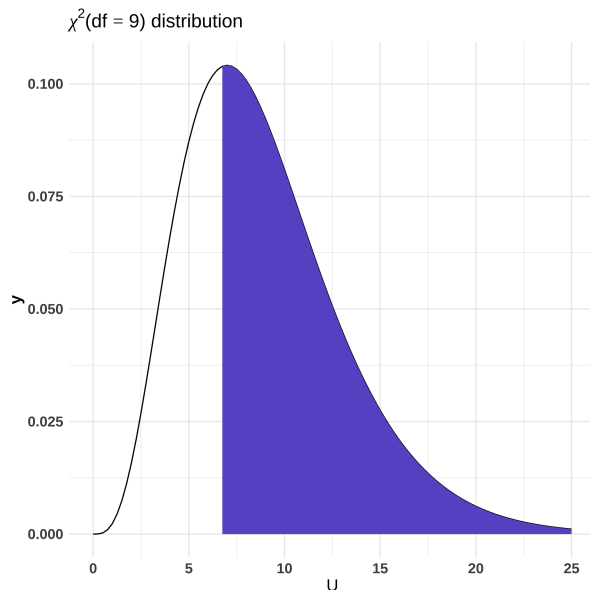
# Example

Suppose  $Y_1, Y_2, \dots, Y_{10} \sim iid \text{Normal}(\mu, \sigma^2 = 4)$ .

- $\mu$  is unknown, and  $\sigma^2$  is known.

Because  $n = 10$  and  $\sigma^2 = 4$ , the sampling distribution  $U = 9S^2/4 \sim \chi^2(df = 9)$ .

- Let's use this to find  $P(S^2 > 3)$ .



$$P(S^2 > 3) = P\left(\frac{9S^2}{4} > \frac{9 \times 3}{4}\right) = P(U > 6.75)$$

```
1 - pchisq(6.75, df = 9)
```

```
## [1] 0.6631296
```



# Student's t Distribution

When the population standard deviation,  $\sigma$ , is *unknown*, it can be estimated by  $S = \sqrt{S^2}$ , and the quantity

$$T = \frac{\bar{Y} - \mu}{s/\sqrt{n}}$$

is used in certain procedures for inference about  $\mu$ .

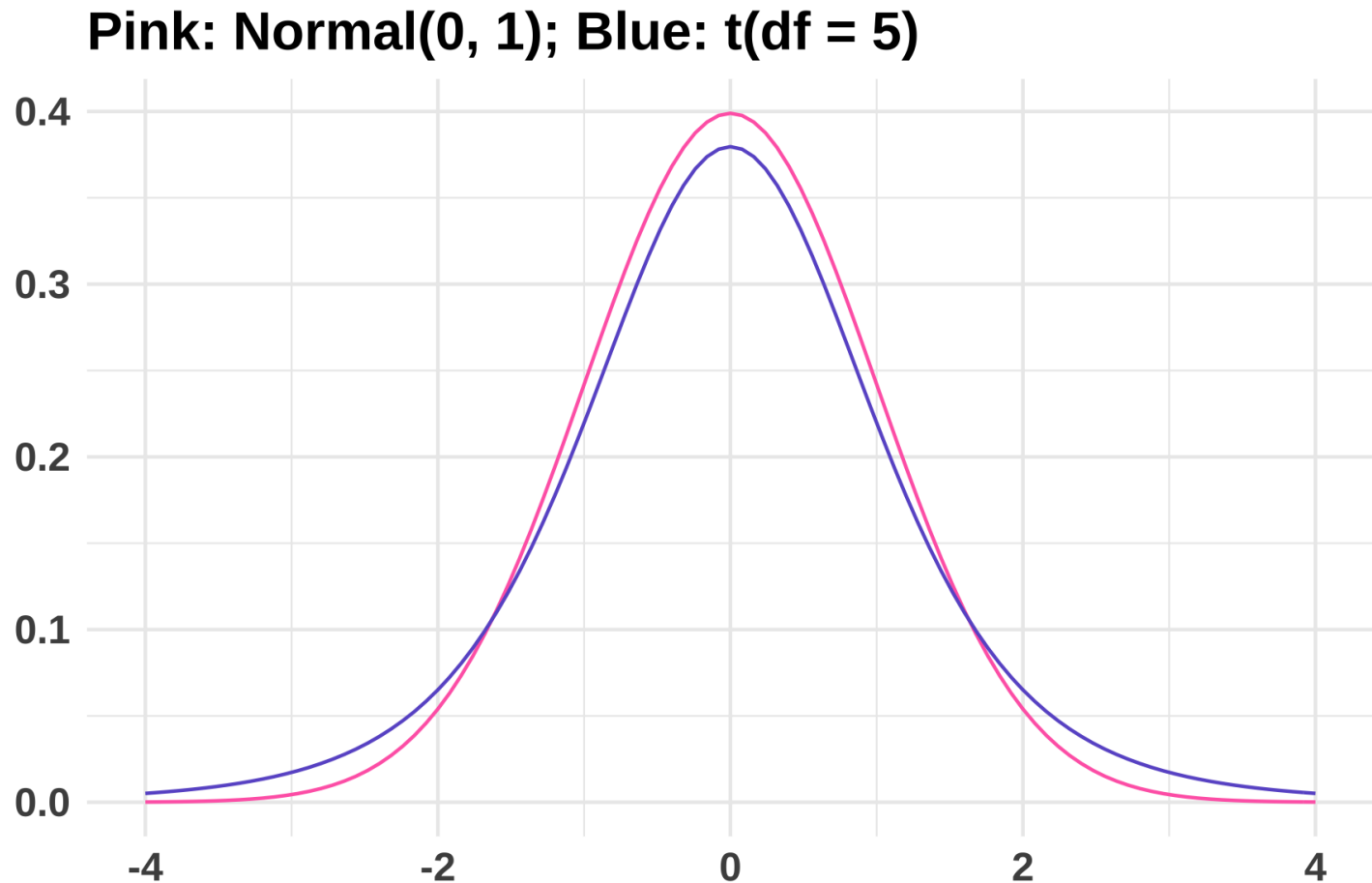
- This quantity,  $T$ , has a **t distribution** with  $n - 1$  degrees of freedom!

**Definition:** Let  $Z$  be a **standard Normal** random variable, and let  $W$  be a  $\chi^2$ -distributed random variable with  $\nu$  degrees of freedom. Then, if  $Z$  and  $W$  are independent,

$$T = \frac{Z}{\sqrt{W/\nu}}$$

is said to have a **t distribution** with  $\nu$  degrees of freedom.

# Student's t vs. Normal



# F Distribution

Suppose now that we are comparing the variances from **two normal samples**:

- $X_1, X_2, \dots, X_n \sim N(\mu_X, \sigma_X^2)$
- $Y_1, Y_2, \dots, Y_n \sim N(\mu_Y, \sigma_Y^2)$

## Question

Are the sample data consistent with the assumption that  $\sigma_X^2 = \sigma_Y^2$ ?

- We know that  $S_X^2$  and  $S_Y^2$  are unbiased estimators of  $\sigma_X^2$  and  $\sigma_Y^2$ , respectively.
  - Let's look at the *ratio*,  $S_X^2/S_Y^2$ .
- It turns out that, if we divide each  $S^2$  by its respective  $\sigma^2$ , then the ratio

$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F(df_1 = n_X - 1, df_2 = n_Y - 1)$$

# F Distribution

## General Definition

Let  $W_1$  and  $W_2$  be independent  $\chi^2$ -distributed random variables with  $\nu_1$  and  $\nu_2$  df, respectively. Then

$$F = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F(\nu_1, \nu_2).$$

**Example:** Suppose...

- $X_1, \dots, X_{21} \sim N(\mu_X, \sigma_X^2), S_X^2 = 994.7$
- $Y_1, \dots, Y_{15} \sim N(\mu_Y, \sigma_Y^2), S_Y^2 = 250.3$

Is it reasonable to assume that  $\sigma_X^2 = \sigma_Y^2$ ?

**IF**  $\sigma_X^2 = \sigma_Y^2$ , then  $\frac{S_X^2}{S_Y^2} \sim F(20, 14)$ :

$$P\left(\frac{S_X^2}{S_Y^2} > \frac{994.7}{250.3}\right) = P(F > 3.97) = 0.006$$

# Recap

We have developed **sampling distributions** of statistics calculated by using observations in random samples from **Normal** populations.

If  $Y_1, \dots, Y_n \sim iid N(\mu, \sigma^2)$ , then...

1.  $\sqrt{n}(\bar{Y} - \mu)/\sigma \sim N(0, 1)$

2.  $(n - 1)S^2/\sigma^2 \sim \chi^2(n - 1)$

3.  $\sqrt{n}(\bar{Y} - \mu)/S \sim t(n - 1)$

4.  $F = (S_1^2/\sigma_1^2)/(S_2^2/\sigma_2^2) \sim F(n_1 - 1, n_2 - 1)$ , provided that the samples are independent.

These sampling distributions will help us quite a bit later on with **confidence intervals** and **hypothesis tests**!

# Frequentist Estimation

This first unit of STAT 339 has been devoted to estimation from a **frequentist perspective**.

- Frequentists view probability as a representation of a *long-run frequency* over a *large* (sometimes infinite) number of repetitions of an experiment.
- The true value of a parameter,  $\theta$ , is **fixed** and **unknown**.

In the next unit, we will focus on estimation from a **Bayesian perspective**.

- Bayesians view probability as a representation of a *relative plausibility* of an event.
- Parameters,  $\theta$ , are themselves treated as *random variables*, assigned some **prior distribution**.
  - Gives weight to prior knowledge.

While we will study various procedures through both *frequentist* and *Bayesian* lenses, these are **not** competing!

- Both perspectives aim to learn from data, both use data to fit models, evaluate hypotheses, etc.