STAT 339: Statistical Theory

Frequentist Hypothesis Testing

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Elements of a Statistical Test

Introduction

Objective: Make *inferences* about unknown **population parameters**

- Based on information contained in *sample data*

We can **estimate** the parameter using, for example, **point** or **interval** estimation.

- We can also **test specific claims about the parameter**.

A (Frequentist) **statistical hypothesis test** involves four components:

1. Null hypothesis, H_0
2. Alternative hypothesis, H_A
3. Test statistic
4. Rejection region

My Awesome Function

I wrote an **R function** called `generateCat` that returns either "*Moose*" or "*Cannoli*". You don't know anything else about it.

> I claim that this function returns "*Moose*" 50% of the time.
>
> **Do you trust me**?[1]

```
generateCat(n = 20)
```

▶ Output

[1] No, you definitely don't trust me. You believe the "Moose" proportion, p, is *higher* than 50%.

Null and Alternative Hypothesis

Because you **do not believe me** when I claim that the "Moose" proportion is 50%, you might seek to support the *research hypothesis* that the "Moose" proportion is *higher* than 50%.

- This research hypothesis is commonly known as the **alternative hypothesis**.

Support for the *alternative hypothesis* is obtained by showing (using **sample data** as *evidence*) that the **null hypothesis** is false.

- The **null hypothesis** is the *converse* of the **alternative hypothesis**:
  - $H_0 : p = 0.5$
  - $H_A : p > 0.5$

Our research objective is to show whether the **data** support rejecting the *null* hypothesis in favor of the *alternative*.

Test Statistic

In order to decide between the null and alternative hypothesis, we need **data**.

- The **test statistic** (similar to an *estimator*) is a function of the **sample data**, $Y_1, \ldots, Y_n$.

```
generateCat(n = 20)
```

```
##    n Moose_count Moose_prop
## 1 20          11       0.55
```

A **test statistic** for this test is $Y = 11$. This is the number of times `generateCat` returned "Moose".

- Our *statistical decision* will be based on this test statistic.

Rejection Region

The **rejection region** (*RR*) specifies the values of the *test statistic* for which the null hypothesis is to be *rejected* in favor of the alternative hypothesis.

- We **reject the null hypothesis**, *in favor of the alternative*, if a particular sample yields a test statistic that falls in the RR.
- If the value of the test statistic does *not* fall into the RR, we **fail to reject the null hypothesis**.

In our example, larger values of Y would lead us to reject H_0.

- For example, one RR might be the set of all values of $Y \geq 16$:

$$RR = \{y : y \geq 16\}$$

But is this a *good* rejection region? It's clear that larger values of Y, say $\mathbf{y} \geq \mathbf{k}$ are more contradictory to $H_0 : p = 0.5$, but how should we choose k?

Errors in Hypothesis Testing

For *any* rejection region, two types of errors can be made in reaching a decision for our test:

- **Type I Error**: Reject H_0 when H_0 is true.
  - The *probability of a type I error* is denoted by α, which is also called the **level** of the test.
- **Type II Error**: Fail to rejcet H_0 when H_A is true.
  - The *probability of a type II error* is denoted by β.

Clearly, we don't want to make incorrect decisions!

- Because α and β measure the risks associated with the two possible *incorrect* decisions we might reach, they can help us find a "good" **rejection region**.

Type I Error

In our example, I ran `generateCat` $n = 20$ times.

We wish to test

$$H_0 : p = 0.5 \quad \text{versus} \quad H_A : p > 0.5$$

where p is the *true* proportion of times this function is supposed to return "Moose".

- The **test statistic** is Y, the number of sampled times that `generateCat` returned "Moose".
- Let's find α, using $RR = \{y \geq 16\}$.

$$\begin{aligned} \alpha &= P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) \\ &= P(\text{value of test statistic is in RR when } H_0 \text{ is true}) \\ &= P(Y \geq 16 \mid p = 0.5) \end{aligned}$$

```
1 - pbinom(15, size = 20, prob = 0.5) # P(Y >= 16 | p = 0.5)
```

```
## [1] 0.005908966
```

Type II Error

Using $RR = \{y \geq 16\}$, is our test equally good in protecting us from type II error?

- That is, do we have low risk of concluding that the "Moose" proportion is equal to 50% when in fact it is *greater*?

$$\begin{aligned}
\beta &= P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_A \text{ is true}) \\
&= P(\text{value of test statistic is NOT in RR when } H_A \text{ is true}) \\
&= P(Y < 16 \mid p = 0.6)
\end{aligned}$$

- **Note**: We used $p = 0.6$ as a particular value of p that satisfies H_A.

```
pbinom(15,  size = 20,  p = 0.6)
```

```
## [1] 0.949048
```

Type I and II Error

When using $RR = \{y \geq 16\}$...

- $\alpha = 0.006$ (low risk of type I error)
- $\beta = 0.949$ when $p = 0.6$ (high risk of type II error)

We can balance α and β by enlarging the RR. Suppose instead we used:

$$RR = \{y \geq 13\}$$

```r
1 - pbinom(12, size = 20, p = 0.5)
```

```
## [1] 0.131588
```

- $\alpha = 0.132$

```r
pbinom(12, size = 20, p = 0.6)
```

```
## [1] 0.5841071
```

- $\beta = 0.584$

💡 **Better solution**: **Fix α, increase the sample size**!!!

Common Large-Sample Tests

Introduction

Goal: Test a set of *hypotheses* concerning a parameter θ, based on a *random sample* $Y_1, Y_2, \ldots, Y_n$.

Setting: Hypothesis testing based on an estimator $\hat{\theta}$ that has (approximately) a normal sampling distribution with mean θ and *standard error* $\sigma_{\hat{\theta}}$.

- That is,

$$\hat{\theta} \sim N(\theta, \sigma_{\hat{\theta}})$$

Our **large-sample tests** will be based on the *large-sample estimators* from earlier this semester:

- $\bar{Y}$
- $\hat{p}$
- $\bar{Y}_1 - \bar{Y}_2$
- $\hat{p}_1 - \hat{p}_2$

Large-Sample Estimators

(From WMS, page 397)

Target Parameter θ	Sample Size(s)	Point Estimator $\hat{\theta}$	$E(\hat{\theta})$	Standard Error $\sigma_{\hat{\theta}}$
μ	n	$\overline{Y}$	μ	$\dfrac{\sigma}{\sqrt{n}}$
p	n	$\hat{p} = \dfrac{Y}{n}$	p	$\sqrt{\dfrac{pq}{n}}$
$\mu_1 - \mu_2$	n_1 and n_2	$\overline{Y}_1 - \overline{Y}_2$	$\mu_1 - \mu_2$	$\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}^{*\dagger}$
$p_1 - p_2$	n_1 and n_2	$\hat{p}_1 - \hat{p}_2$	$p_1 - p_2$	$\sqrt{\dfrac{p_1 q_1}{n_1} + \dfrac{p_2 q_2}{n_2}}^{\dagger}$

Setting Up The Test

Suppose we wish to test

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_A : \theta > \theta_0$$

where θ_0 is a specific value of θ.

- If **in reality** $\theta > \theta_0$, then $\hat{\theta}$ is *more likely* to be large.
  - In this case, larger values of $\hat{\theta}$ favor rejection of H_0.

The **components of our statistical hypothesis test** are as follows:

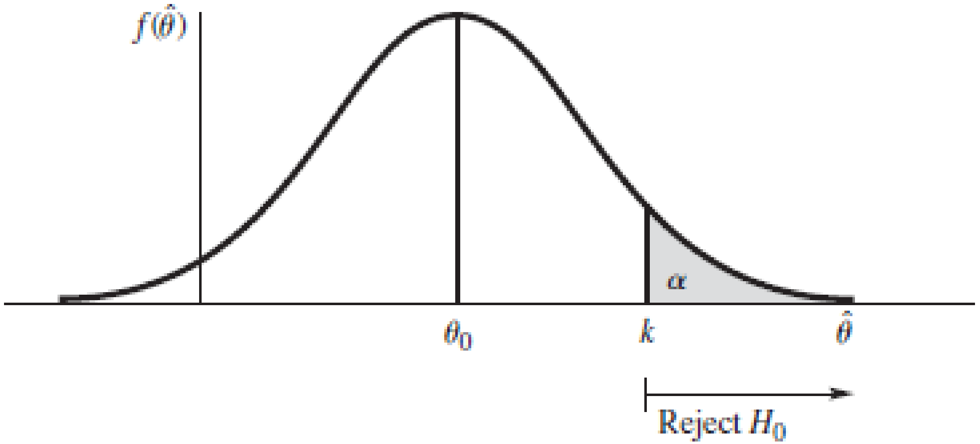
1. **Null Hypothesis**: $H_0 : \theta = \theta_0$
2. **Alternative Hypothesis**: $H_A : \theta > \theta_0$
3. **Test Statistic**: $\hat{\theta}$
4. **Rejection Region**: $RR = \{\hat{\theta} > k\}$ for some choice of k

The Rejection Region

To determine k in the rejection region, we'll fix the type 1 error probability α (the level of the test) and choose k accordingly.

- $\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) = 0.05$ (usually)

WMS Figure 10.3 (page 496)



The Rejection Region

Remember, θ is fixed but **unknown**.

- But *when* H_0 *is true*, $\theta = \theta_0$, and by the **Central Limit Theorem**,

$$\hat{\theta} \sim N(\theta_0, \sigma_{\hat{\theta}}) \iff Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} \sim N(0, 1)$$

Therefore, an equivalent rejection region would be

$$RR = \left\{ \hat{\theta} : Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} > z_\alpha \right\}$$

where $P(Z > z_\alpha) = \alpha$

Large-Sample (right tailed) Test

A **large-sample test**, with *level* α, is as follows:

1. **Null Hypothesis**: $H_0 : \theta = \theta_0$
2. **Alternative Hypothesis**: $H_A : \theta > \theta_0$
3. **Test Statistic**: $Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$
4. **Rejection Region**: $RR = \{z > z_\alpha\}$ (*upper-tail rejection region*)

Example

The *level* of our test is $\alpha = 0.05$.

- This means $RR = \{z > z_{0.05}\}$:

```
qnorm(0.05,  lower.tail = FALSE)
```

```
## [1] 1.644854
```

My Awesome Function

I wrote an **R function** called `generateCat` that returns either "*Moose*" or "*Cannoli*". You don't know anything else about it.

> I claim that this function returns "*Moose*" 50% of the time.
>
> You don't trust me and *definitely* think it's higher. Test this hypothesis using an $\alpha = 0.05$ level.

```
generateCat(n = 40)
```

```
##    n Moose_count Moose_prop
## 1 40          24        0.6
```

- $H_0 : p = 0.5$
- $H_A : p > 0.5$

From our **sample**, $n = 40$ and $\hat{p} = 0.6$.

The Null Distribution

Under $H_0, p_0 = 0.5$, and by the **Central Limit Theorem**,

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)$$

for **sufficiently large** n.

- **Note**: While we could've used $\sqrt{\hat{p}(1-\hat{p})/n}$ to *approximate* the standard error of $\hat{p}$, we use p_0 because we are considering the distribution of Z under H_0.
- $RR = \{z > 1.645\}$
- $Z = \frac{0.6-0.5}{\sqrt{\frac{0.5(1-0.5)}{40}}} = 1.265$

Decision: *Fail to reject* H_0! At the $\alpha = 0.05$ level, the evidence does not support your decision to not trust me...

- Does this mean I'm telling the truth?! 😇
- Only time will tell... We need to calculate the probability of **type II error**, β.

Two-Tailed Tests

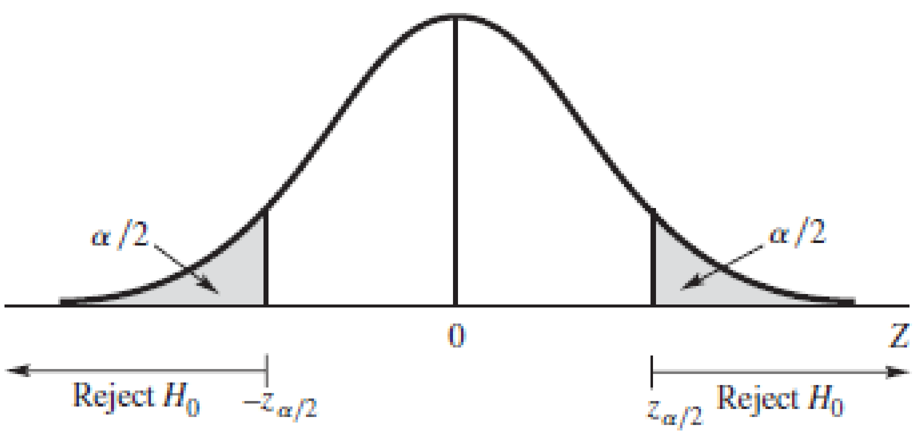
Suppose we wanted to test an alternative hypothesis of the form: $\theta \neq \theta_0$.

- That is, we're testing $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$

Our test statistic would remain the same!

$$Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$$

- The one thing that would change is the **rejection region**. That's because, in this setting, we'll reject H_0 if $\hat{\theta}$ is (a) much *smaller* or (b) much *larger* than θ_0.



Common Grounds

I want to test whether the **mean** number of people in line at **Common Grounds** on a typical morning after STAT 339, μ, is **different from** 5 people.

- For all I know, the mean could be < 5 or > 5. Considering I can't even get the *name* right, I don't know much else about Common Grounds...

I collect some **DATA** and obtain the following:

- $n = 30$ (yes, I checked the Common Grounds line for 30 mornings after STAT 339)
- $\bar{y} = 6$ people in line, with standard deviation $s = 2$

> Do the data present sufficient evidence to suggest that the mean number of people in line at Common Grounds after STAT 339 is *different from* 5 people? Use $\alpha = 0.01$.

Summary

Large-Sample $\boldsymbol{\alpha}$-Level Hypothesis Tests

1. **Null Hypothesis**: H_0: $\theta = \theta_0$

2. **Alternative Hypothesis**:

   - H_A: $\theta > \theta_0$ (upper-tail alternative)
   - H_A: $\theta < \theta_0$ (lower-tail alternative)
   - H_A: $\theta \neq \theta_0$ (two-tailed alternative)

3. **Test Statistic**: $Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$

4. **Rejection Region**:

   - $\{z > z_\alpha\}$ (upper-tail RR)
   - $\{z < z_\alpha\}$ (lower-tail RR)
   - $\{|z| > z_{\alpha/2}\}$ (two-tailed RR)

Type II Error Probabilities

Type II Error

Recall: **Type II Error** occurs when you *fail to reject the null*, when *the alternative is true*.

For a test with $H_0 : \theta = \theta_0$ versus $H_A : \theta > \theta_0$, we can calculate type II error probabilities *only for specific values of* θ in H_A.

- For some *specific alternative*, $\theta = \theta_A$, the *probability* of type II error, β, is

$$\begin{aligned}\beta &= P(\hat{\theta} \text{ is not in RR when } H_A \text{ is true}) \\ &= P(\hat{\theta} \le k \mid \theta = \theta_A) \\ &= P\left(\frac{\hat{\theta} - \theta_A}{\sigma_{\hat{\theta}}} \le \frac{k - \theta_A}{\sigma_{\hat{\theta}}} \mid \theta = \theta_A\right),\end{aligned}$$

By the **Central Limit Theorem**, $Z = (\hat{\theta} - \theta_A)/\sigma_{\hat{\theta}} \sim N(0,1)$, so we can calculate

$$\beta = P\left(Z \le \frac{k - \theta_A}{\sigma_{\hat{\theta}}} \mid \theta = \theta_A\right)$$

by finding a corresponding area under the $N(0,1)$ curve, using `pnorm()`.

Type II Error Probability

In **My Awesome Function** example, we had $n = 40$ and $\hat{p} = 0.6$. Let's find β for a test of

$$H_0 : p = 0.5 \quad \text{versus} \quad H_A : p = 0.65.$$

- From the example, the **rejection region** for an $\alpha = 0.05$ level test was

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} > 1.645 \iff \hat{p} > p_0 + 1.645\sqrt{\frac{p_0(1-p_0)}{n}}.$$

- Plugging in $p_0 = 0.5$ and $n = 40$, we have the rejection region $\{\hat{p} > 0.63\}$.

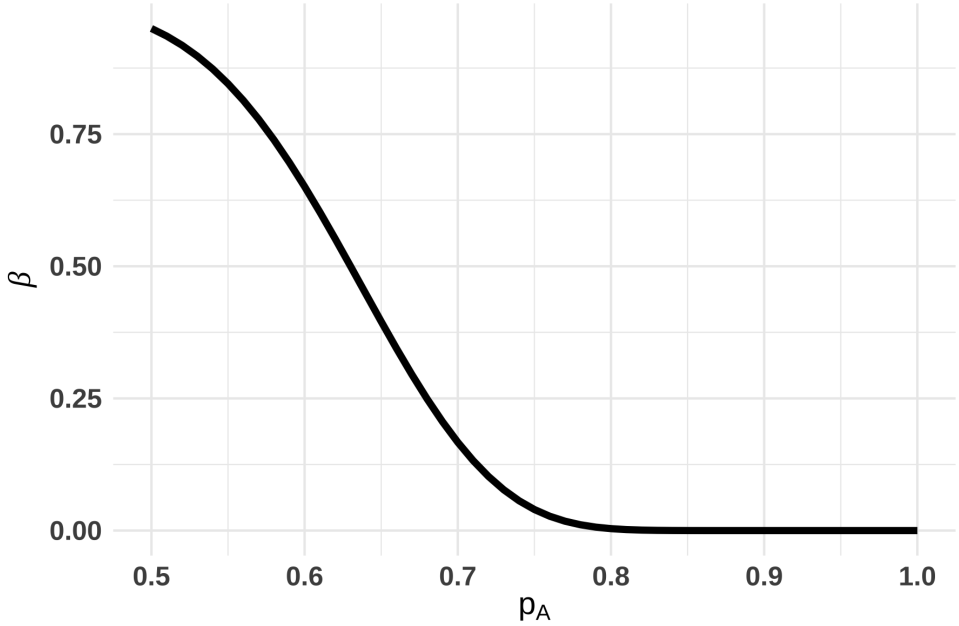
Therefore, *by definition*, the **type II error probability** is

$$\begin{aligned}
\beta &= P(\hat{\theta} \text{ is not in RR when } H_A \text{ is true}) = P(\hat{p} \leq 0.63 \mid p = 0.65) \\
&= P\left(Z \leq \frac{0.63 - 0.65}{\sqrt{\frac{0.65(1-0.65)}{40}}}\right) \\
&= P(Z \leq -0.265) = \boxed{0.396}
\end{aligned}$$

Understanding β

While we can only calculate β for a *fixed* p_A, we can visualize how β depends on the *distance* between p_A and p_0 (**0.5**).

- If p_A is far from p_0, then the true value of p is a bit easier to detect, hence β is much smaller.



Back to Piéchart Emporium (yet again)

Piéchart Emporium claims that they pull in an average of 1000 dollars in *daily revenue*, but you *definitely* think it's lower. To check this, you could test

$$H_0 : \mu = 1000 \quad \text{versus} \quad H_A : \mu < 1000$$

at the $\alpha = 0.05$ level, using the **rejection region** $\{z < -1.645\}$ and the following observed data:

- $n = 30$ days
- $\bar{y} = 950$ dollars
- $s = 80$ dollars

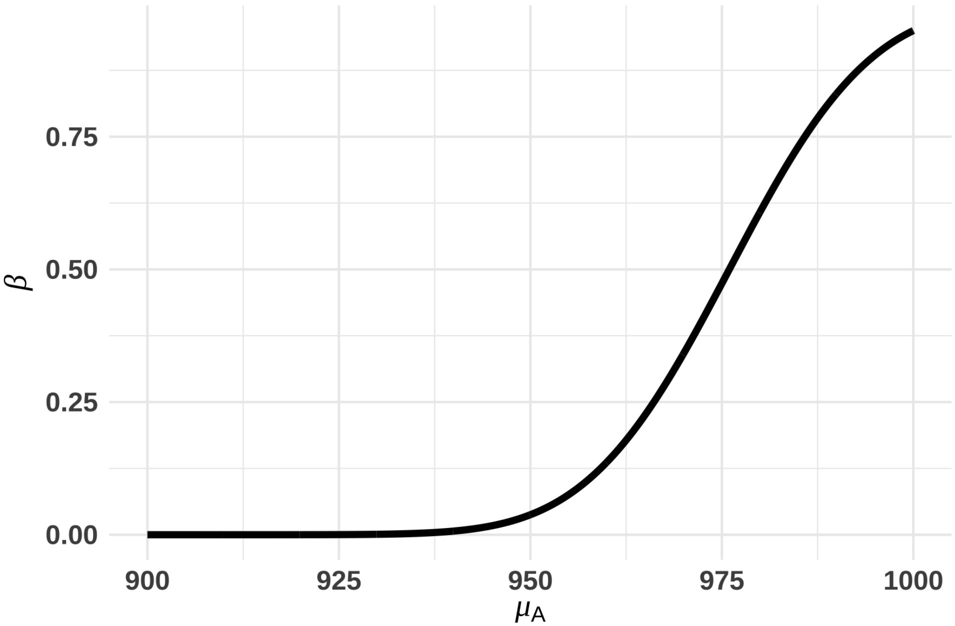
In *groups of 3*, calculate β for one of the following three specific values of μ under H_A:

1. $H_A : \mu = 980$

2. $H_A : \mu = 960$

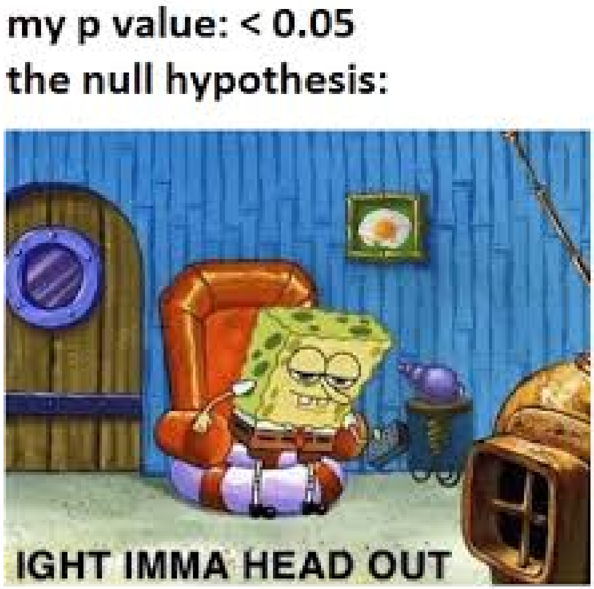
3. $H_A : \mu = 940$

Understanding β



p-values


P > 0.05
GAME OVER, TRY AGAIN


my p value: < 0.05
the null hypothesis:
IGHT IMMA HEAD OUT

What is a p-value?

> **p-value**: The probability of observing a test statistic *at least as extreme* as the one you observed, *if the null hypothesis is true.*

- In other words, this is the *smallest* level of significance, α, for which the observed data indicate that the null hypothesis should be rejected.
- The *smaller* the p-value becomes, the **stronger** the evidence that the null hypothesis should be *rejected.*

Typically, we use p-values in the following way to reach decisions in a hypothesis test:

- $p \leq \alpha$: **Reject the null hypothesis**
- $p > \alpha$: **Fail to reject the null hypothesis**

🚨 But be careful with this line of thinking! There is more to science than an *arbitrary*, *binary* decision making process!

My Awesome Function

In the `generateCat` example, we tested

$$H_0 : p = 0.5 \quad \text{versus} \quad H_A : p > 0.5$$

using $n = 40$ and $\hat{p} = 0.6$.

- Using $\alpha = 0.05$, our rejection region was $RR = \{z > 1.645\}$, and our *observed* test statistic was

$$Z = \frac{0.6 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{40}}} = 1.265.$$

Under H_0, $Z \sim N(0, 1)$, so

$$\text{p-value} = P(Z \geq 1.265 \mid H_0 \text{ is true}) = 0.103$$

```
1-pnorm(1.265)
```

```
## [1] 0.1029357
```

Common Grounds

I want to test whether the **mean** number of people in line at **Common Grounds** on a typical morning after STAT 339, μ, is **different from** 5 people.

- $H_0 : \mu = 5$
- $H_A : \mu \neq 5$

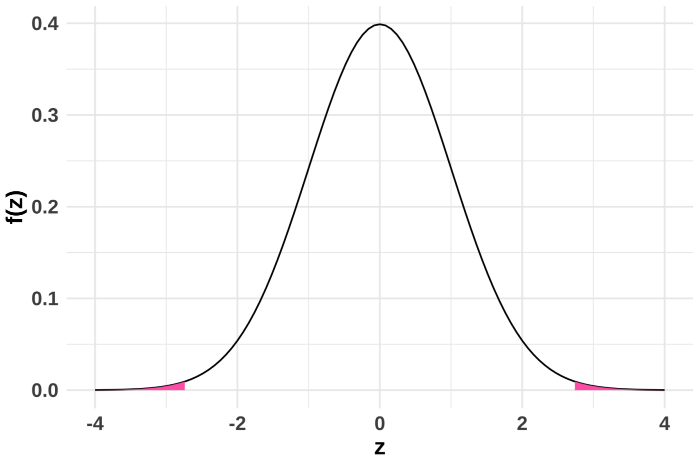
Data: $n = 30, \bar{y} = 6, s = 2$

- Though rather than rejecting H_0 for *only* large values of $\bar{y}$, we might also reject H_0 for *small* values of $\bar{y}$. When calculating the p-value, we need to account for this!

Common Grounds

Our **test statistic** for this test was

$$Z = \frac{6-5}{2/\sqrt{30}} = 2.74.$$



The pink regions in the plot represent the **p-value** for a *two-tailed* test.

In R:

```
2*pnorm(2.74, lower.tail = FALSE)
```

```
## [1] 0.006143918
```

Decision: $p < 0.05$, so *reject the null hypothesis.*

Relationship with Confidence Intervals

(In my opinion) Confidence intervals are **more informative** and **less confusing** than p-values.

- *Not that p-values are useless; the definition just lends itself well to misinterpretation!*

There's also a *duality* that exists between large-sample CIs and *two-tailed* hypothesis tests:

- Do **not** reject $H_0 : \theta = \theta_0$ at the α level if θ_0 lies *inside* a $100(1-\alpha)\%$ confidence interval for θ.

- **Reject** H_0 if θ_0 lies *outside* the interval.

Example

95% CI for the **mean** number of people in line at **Common Grounds** on a typical morning after STAT 339, μ:

$$6 \pm 1.96 \times \frac{2}{\sqrt{30}} = (5.28, 6.72)$$

Small-Sample Hypothesis Testing

(for means)

Introduction

The **large-sample** hypothesis testing procedures rely on the condition that the sample size is *large enough* that

$$Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} \sim (approx)\ N(0,1).$$

A common procedure that we can use with **small samples** relies on using the *t*-distribution as the sampling distribution of the **test statistic**.

- This procedure, the **t-test**, *is only appropriate for samples from Normal populations*!

The t-test

Setting

- $Y_1, Y_2, \ldots, Y_n$ denote a **random sample** of size n
- The sample comes from a **Normal** distribution with *unknown* mean μ and *unknown* variance σ^2.

Procedure

- Let $\bar{Y}$ and S denote the sample mean and sample standard deviation, respectively.
- Under the **null hypothesis**, $H_0 : \mu = \mu_0$, the *test statistic*

$$T = \frac{\bar{Y} - \mu_0}{S/\sqrt{n}} \sim t(n-1).$$

- For $H_A : \mu > \mu_0$, the **rejection region** is $RR = \{t > t_\alpha\}$.
  - That is, any value of T such that $P(T > t_\alpha) = \alpha$, where t_α is the **upper** α-quantile of the $t(n-1)$ distribution.

Florida Lobsters

WMS 10.75

A declaration by the Bahamian government that prohibits U.S. lobsterers from fishing on the Bahamian portion of the continental shelf was *expected* to dramatically reduce the landings (in pounds per lobster trap).

According to records, **the prior mean landings per trap was 30.31 pounds**.

A *random sample* of **20 lobster traps** since the Bahamian fishing restriction went into effect gave the following results (in pounds):

$$17.4,\ 18.9,\ 39.6,\ 34.4,\ 19.6,\ 33.7,\ 37.2,\ 43.4,\ 41.7,\ 27.5,$$
$$24.1,\ 39.6,\ 12.2,\ 25.5,\ 22.1,\ 29.3,\ 21.1,\ 23.8,\ 43.2,\ 24.4$$

Do these landings provide evidence that the mean landings per trap has **decreased**? Test using $\alpha = 0.05$

Comparing Groups

We can also use the t-distribution to **compare the means** of two Normal populations that possess *equal variances*.

Setting

- *Independent* samples from *Normal* distributions with $\sigma_1^2 = \sigma_2^2$.

Hypotheses

- $H_0 : \mu_1 - \mu_2 = D_0$, where D_0 is usually **0**.
- $H_A : \mu_1 - \mu_2 > D_0$ (or $<$ or $\neq$ depending on which alternative is appropriate)

Comparing Groups

We can also use the *t*-distribution to **compare the means** of two Normal populations that possess *equal variances*.

Test Statistic

- $T = \frac{(\bar{Y}_1 - \bar{Y}_2) - D_0}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$, where

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Rejection Region

- For an *upper-tail alternative*, $RR = \{t > t_\alpha\}$, where t_α is the *upper* α-quantile of the $t(n_1 + n_2 - 2)$ distribution.

Is the sequel always better?

I watched **Shrek** recently, because Netflix decided it was going to pull the movie at the end of April 2022.

- Not sure yet what this means for **Shrek 2**, and as of this writing I haven't watched it in a while. But let's compare them anyways!

Suppose random samples of $n_1 = n_2 = 10$ critic reviews from *Metacritic* for each film yielded the ratings below. Check if the sequel is better at $\alpha = 0.01$.

- **Shrek**: $89.62, 82.03, 79.65, 88.20, 82.13, 86.06, 90.33, 89.44, 86.27, 79.82$

- **Shrek 2**: $77.05, 73.93, 76.27, 70.89, 79.65, 64.49, 67.39, 74.15, 71.50, 86.16$

Connecting to Confidence Intervals

The connection between *two-tailed* **t-tests** and *small-sample confidence intervals* is similar to the duality between *large-sample* confidence intervals and tests!

- Do **not** reject $H_0 : \mu = \mu_0$ if μ_0 lies *inside* $\bar{y} \pm t_{\alpha/2} \times \frac{S}{\sqrt{n}}$
- **Reject** $H_0 : \mu = \mu_0$ if μ_0 lies *outside* $\bar{y} \pm t_{\alpha/2} \times \frac{S}{\sqrt{n}}$
- (Ditto for CIs for a **difference between means**)
  - *Reject* $H_0 : \mu_1 - \mu_2 = D_0$ if D_0 lies *outside* the CI.

Example:

Calculate a 99% confidence interval for the *difference* in average Metacritic ratings between **Shrek** and **Shrek 2**.

t-tests in R

Because the **t-test** is a common hypothesis testing procedure, it is supported by the **base R** `t.test()` function.

For example, we can repeat the *lobster* example in R using the following code:

```
lobster_data = c(17.4, 18.9, 39.6, 34.4, 19.6, 33.7, 37.2, 43.4, 41.7, 27.5,
                 24.1, 39.6, 12.2, 25.5, 22.1, 29.3, 21.1, 23.8, 43.2, 24.4)

t.test(lobster_data,  mu = 30.31, alternative = "less")
```

```
##
##  One Sample t-test
##
## data:  lobster_data
## t = -0.64678, df = 19, p-value = 0.2628
## alternative hypothesis: true mean is less than 30.31
## 95 percent confidence interval:
##      -Inf 32.61098
## sample estimates:
## mean of x
##    28.935
```

Testing Hypotheses Concerning Variances

Introduction

Recall: If $Y_1, Y_2, \ldots, Y_n \sim N(\mu, \sigma^2)$, where both μ and σ^2 are *unknown*, then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

We can use this fact to test $H_0 : \sigma^2 = \sigma_0^2$ for some fixed value σ_0^2.

If $H_0 : \sigma^2 = \sigma_0^2$ is *true*, then

$$\chi^2 \equiv \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1),$$

and we can use χ^2 as our **test statistic**.

- For an α-test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_A : \sigma^2 > \sigma_0^2$,

$$RR = \{\chi^2 > \chi^2_\alpha\}$$

Example

(WMS 10.79)

The manufacturer of a machine to package soap power claimed that their machine could load cartons at a given weight with a range of no more than 0.4 ounce. The mean and variance of a sample of $n = 8$ three-pound boxes were found to equal 3.1 and 0.018, respectively.

Test the hypothesis that the variance of the population of weight measurements is $\sigma^2 = 0.01$ against the alternative that $\sigma^2 > 0.01$. Use $\alpha = 0.05$.

Comparing Groups

Sometimes rather than examining one group's variance, we might want to *compare* the variances of **two Normal distributions**.

- Usually to determine whether they are *equal*.

From **Exam 2**:

- If independent samples of size n_1 and n_2 are taken from two normally distributed populations with variances σ_1^2 and σ_2^2, respectively, then

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(df_1 = n_1 - 1, df_2 = n_2 - 1)$$

Suppose we're testing $H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_A : \sigma_1^2 > \sigma_2^2$. Under H_0, $\sigma_1^2 = \sigma_2^2$ and

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2} \sim F(df_1 = n_1 - 1, df_2 = n_2 - 1)$$

is a **test statistic**!

90s Road Trip

Because the test statistic, F, follows an $F(n_1 - 1, n_2 - 1)$ distribution under the null hypothesis, a **rejection region** for an α-level test comparing variances is

$$RR = \{F > F_\alpha\},$$

where F_α is the α-quantile of the $F(n_1 - 1, n_2 - 1)$ distribution.

Example

	90s Road Trip	**No Music**
Sample size, n	100	100
Mean, $\bar{y}$	10.75	12.75
Standard deviation, s	4	5

Using the data from **Exam 2**, is there sufficient evidence to conclude that the variance of *objects memorized* for the "No Music" group is *larger* than the variance for the "90s Road Trip" group?