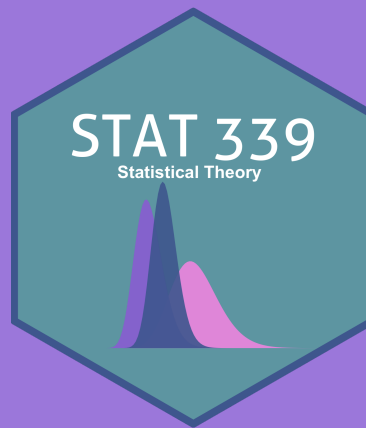


# STAT 339: Statistical Theory

## Frequentist Hypothesis Testing

Anthony Scotina



# Elements of a Statistical Test

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# Introduction

**Objective:** Make *inferences* about unknown **population parameters**

- Based on information contained in *sample data*

We can **estimate** the parameter using, for example, **point** or **interval** estimation.

- We can also **test specific claims about the parameter**.

A (Frequentist) **statistical hypothesis test** involves four components:

1. Null hypothesis,  $H_0$
2. Alternative hypothesis,  $H_A$
3. Test statistic
4. Rejection region

# My Awesome Function

I wrote an **R function** called `generateCat` that returns either "Moose" or "Cannoli". You don't know anything else about it.

I claim that this function returns "Moose" 50% of the time.

**Do you trust me?**<sup>1</sup>

```
generateCat(n = 20)
```

► Output

[1] No, you definitely don't trust me. You believe the "Moose" proportion,  $p$ , is *higher* than 50%.

# Null and Alternative Hypothesis

Because you **do not believe me** when I claim that the "Moose" proportion is 50%, you might seek to support the *research hypothesis* that the "Moose" proportion is *higher* than 50%.

- This research hypothesis is commonly known as the **alternative hypothesis**.

Support for the *alternative hypothesis* is obtained by showing (using **sample data** as evidence) that the **null hypothesis** is false.

- The **null hypothesis** is the converse of the **alternative hypothesis**:
  - $H_0 : p = 0.5$
  - $H_A : p > 0.5$

Our research objective is to show whether the **data** support rejecting the *null hypothesis* in favor of the *alternative*.

# Test Statistic

In order to decide between the null and alternative hypothesis, we need **data**.

- The **test statistic** (similar to an *estimator*) is a function of the **sample data**,  $Y_1, \dots, Y_n$ .

```
generateCat(n = 20)
```

```
##      n Moose_count Moose_prop
## 1 20           11      0.55
```

A **test statistic** for this test is  $Y = 11$ . This is the number of times `generateCat` returned "Moose".

- Our *statistical decision* will be based on this test statistic.

# Rejection Region

The **rejection region** (RR) specifies the values of the test statistic for which the null hypothesis is to be *rejected* in favor of the alternative hypothesis.

- We **reject the null hypothesis**, *in favor of the alternative*, if a particular sample yields a test statistic that falls in the RR.
- If the value of the test statistic does not fall into the RR, we **fail to reject the null hypothesis**.

In our example, larger values of  $Y$  would lead us to reject  $H_0$ .

- For example, one RR might be the set of all values of  $Y \geq 16$ :

$$RR = \{y : y \geq 16\}$$

But is this a *good* rejection region? It's clear that larger values of  $Y$ , say  $y \geq k$  are more contradictory to  $H_0 : p = 0.5$ , but how should we choose  $k$ ?

# Errors in Hypothesis Testing

For any rejection region, two types of errors can be made in reaching a decision for our test:

- **Type I Error:** Reject  $H_0$  when  $H_0$  is true.
  - The probability of a type I error is denoted by  $\alpha$ , which is also called the **level** of the test.
- **Type II Error:** Fail to reject  $H_0$  when  $H_A$  is true.
  - The probability of a type II error is denoted by  $\beta$ .

Clearly, we don't want to make incorrect decisions!

- Because  $\alpha$  and  $\beta$  measure the risks associated with the two possible *incorrect* decisions we might reach, they can help us find a "good" **rejection region**.



# Type I Error

In our example, I ran `generateCat`  $n = 20$  times.

We wish to test

$$H_0 : p = 0.5 \quad \text{versus} \quad H_A : p > 0.5$$

where  $p$  is the *true* proportion of times this function is supposed to return "Moose".

- The **test statistic** is  $Y$ , the number of sampled times that `generateCat` returned "Moose".
- Let's find  $\alpha$ , using  $RR = \{y \geq 16\}$ .

$$\begin{aligned}\alpha &= P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) \\ &= P(\text{value of test statistic is in RR when } H_0 \text{ is true}) \\ &= P(Y \geq 16 \mid p = 0.5)\end{aligned}$$

```
1 - pbinom(15, size = 20, prob = 0.5) # P(Y >= 16 | p = 0.5)
```

```
## [1] 0.005908966
```

# Type II Error

Using  $RR = \{y \geq 16\}$ , is our test equally good in protecting us from type II error?

- That is, do we have low risk of concluding that the "Moose" proportion is equal to 50% when in fact it is *greater*?

$$\begin{aligned}\beta &= P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_A \text{ is true}) \\ &= P(\text{value of test statistic is NOT in } RR \text{ when } H_A \text{ is true}) \\ &= P(Y < 16 \mid p = 0.6)\end{aligned}$$

- **Note:** We used  $p = 0.6$  as a particular value of  $p$  that satisfies  $H_A$ .

```
pbinom(15, size = 20, p = 0.6)
```

```
## [1] 0.949048
```

# Type I and II Error

When using  $RR = \{y \geq 16\}$ ...

- $\alpha = 0.006$  (low risk of type I error)
- $\beta = 0.949$  when  $p = 0.6$  (high risk of type II error)

We can balance  $\alpha$  and  $\beta$  by enlarging the RR. Suppose instead we used:

$$RR = \{y \geq 13\}$$

```
1 - pbinom(12, size = 20, p = 0.5)
```

```
## [1] 0.131588
```

- $\alpha = 0.132$

```
pbinom(12, size = 20, p = 0.6)
```

```
## [1] 0.5841071
```

- $\beta = 0.584$

💡 **Better solution:** Fix  $\alpha$ , increase the sample size!!!

# Common Large-Sample Tests

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# Introduction

**Goal:** Test a set of *hypotheses* concerning a parameter  $\theta$ , based on a *random sample*  $Y_1, Y_2, \dots, Y_n$ .

**Setting:** Hypothesis testing based on an estimator  $\hat{\theta}$  that has (approximately) a normal sampling distribution with mean  $\theta$  and standard error  $\sigma_{\hat{\theta}}$ .

- That is,

$$\hat{\theta} \sim N(\theta, \sigma_{\hat{\theta}})$$

Our **large-sample tests** will be based on the *large-sample* estimators from earlier this semester:

- $\bar{Y}$
- $\hat{p}$
- $\bar{Y}_1 - \bar{Y}_2$
- $\hat{p}_1 - \hat{p}_2$

# Large-Sample Estimators

(From WMS, page 397)

| Target<br>Parameter<br>$\theta$ | Sample<br>Size(s) | Point<br>Estimator<br>$\hat{\theta}$ | $E(\hat{\theta})$ | Standard<br>Error<br>$\sigma_{\hat{\theta}}$                        |
|---------------------------------|-------------------|--------------------------------------|-------------------|---|
| $\mu$                           | $n$               | $\bar{Y}$                            | $\mu$             | $\frac{\sigma}{\sqrt{n}}$   |
| $p$                             | $n$               | $\hat{p} = \frac{Y}{n}$              | $p$               | $\sqrt{\frac{pq}{n}}$   |
| $\mu_1 - \mu_2$                 | $n_1$ and $n_2$   | $\bar{Y}_1 - \bar{Y}_2$              | $\mu_1 - \mu_2$   | $\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}^{*\dagger}$ |
| $p_1 - p_2$                     | $n_1$ and $n_2$   | $\hat{p}_1 - \hat{p}_2$              | $p_1 - p_2$       | $\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}^{\dagger}$        |

# Setting Up The Test

Suppose we wish to test

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_A : \theta > \theta_0$$

where  $\theta_0$  is a specific value of  $\theta$ .

- If **in reality**  $\theta > \theta_0$ , then  $\hat{\theta}$  is *more likely* to be large.
  - In this case, larger values of  $\hat{\theta}$  favor rejection of  $H_0$ .

The **components of our statistical hypothesis test** are as follows:

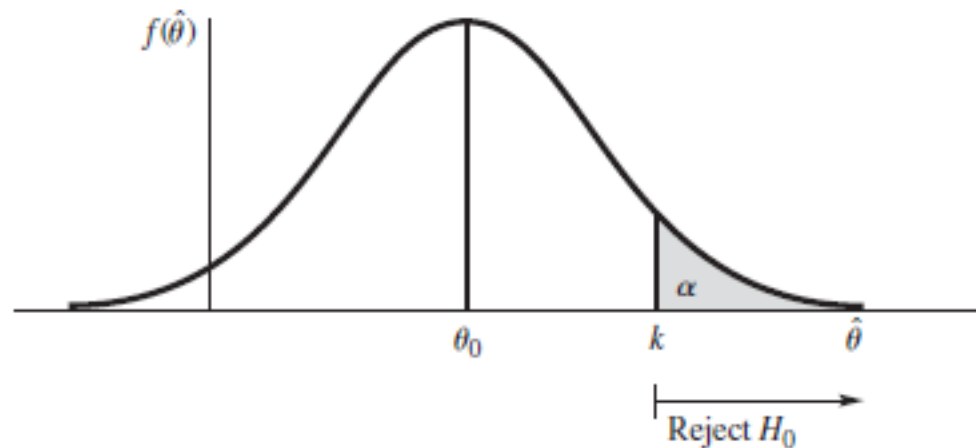
1. **Null Hypothesis:**  $H_0 : \theta = \theta_0$
2. **Alternative Hypothesis:**  $H_A : \theta > \theta_0$
3. **Test Statistic:**  $\hat{\theta}$
4. **Rejection Region:**  $RR = \{\hat{\theta} > k\}$  for some choice of  $k$

# The Rejection Region

To determine  $k$  in the **rejection region**, we'll fix the **type 1 error** probability  $\alpha$  (the **level** of the test) and choose  $k$  accordingly.

- $\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) = 0.05 \text{ (usually)}$

WMS Figure 10.3 (page 496)





# The Rejection Region

Remember,  $\theta$  is fixed but **unknown**.

- But when  $H_0$  is true,  $\theta = \theta_0$ , and by the **Central Limit Theorem**,

$$\hat{\theta} \sim N(\theta_0, \sigma_{\hat{\theta}}) \iff Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} \sim N(0, 1)$$

Therefore, an equivalent rejection region would be

$$RR = \left\{ \hat{\theta} : Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} > z_{\alpha} \right\}, \text{ where } P(Z > z_{\alpha}) = \alpha$$

# Large-Sample (right tailed) Test

A **large-sample test**, with level  $\alpha$ , is as follows:

1. **Null Hypothesis:**  $H_0 : \theta = \theta_0$
2. **Alternative Hypothesis:**  $H_A : \theta > \theta_0$
3. **Test Statistic:**  $Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$
4. **Rejection Region:**  $RR = \{z > z_{\alpha}\}$  (upper-tail rejection region)

## Example

The level of our test is  $\alpha = 0.05$ .

- This means  $RR = \{z > z_{0.05}\}$ :

```
qnorm(0.05, lower.tail = FALSE)
```

```
## [1] 1.644854
```

# My Awesome Function

I wrote an **R function** called `generateCat` that returns either "Moose" or "Cannoli". You don't know anything else about it.

I claim that this function returns "Moose" 50% of the time.

You don't trust me and *definitely* think it's higher. Test this hypothesis using an  $\alpha = 0.05$  level.

```
generateCat(n = 40)
```

```
##      n Moose_count Moose_prop  
## 1 40           24      0.6
```

- $H_0 : p = 0.5$
- $H_A : p > 0.5$

From our **sample**,  $n = 40$  and  $\hat{p} = 0.6$ .

# The Null Distribution

Under  $H_0$ ,  $p_0 = 0.5$ , and by the **Central Limit Theorem**,

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0, 1)$$

for **sufficiently large**  $n$ .

- **Note:** While we could've used  $\sqrt{\hat{p}(1-\hat{p})/n}$  to approximate the standard error of  $\hat{p}$ , we use  $p_0$  because we are considering the distribution of  $Z$  under  $H_0$ .
- $RR = \{z > 1.645\}$
- $Z = \frac{0.6-0.5}{\sqrt{\frac{0.5(1-0.5)}{40}}} = 1.265$

**Decision:** Fail to reject  $H_0$ ! At the  $\alpha = 0.05$  level, the evidence does not support your decision to not trust me...

- Does this mean I'm telling the truth?! 😊
- Only time will tell... We need to calculate the probability of **type II error**,  $\beta$ .

# Two-Tailed Tests

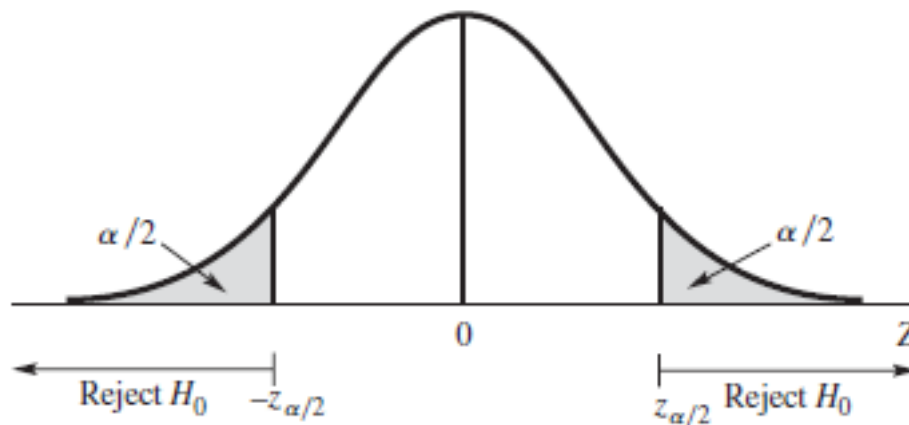
Suppose we wanted to test an alternative hypothesis of the form:  $\theta \neq \theta_0$ .

- That is, we're testing  $H_0 : \theta = \theta_0$  versus  $H_A : \theta \neq \theta_0$

Our test statistic would remain the same!

$$Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$$

- The one thing that would change is the **rejection region**. That's because, in this setting, we'll reject  $H_0$  if  $\hat{\theta}$  is (a) much smaller or (b) much larger than  $\theta_0$ .



# Common Grounds

I want to test whether the **mean** number of people in line at **Common Grounds** on a typical morning after STAT 339,  $\mu$ , is **different from 5** people.

- For all I know, the mean could be  $< 5$  or  $> 5$ . Considering I can't even get the *name* right, I don't know much else about Common Grounds...



I collect some **DATA** and obtain the following:

- $n = 30$  (yes, I checked the Common Grounds line for 30 mornings after STAT 339)
- $\bar{y} = 6$  people in line, with standard deviation  $s = 2$

Do the data present sufficient evidence to suggest that the mean number of people in line at Common Grounds after STAT 339 is *different from 5* people? Use  $\alpha = 0.01$ .

# Summary

## Large-Sample $\alpha$ -Level Hypothesis Tests

1. **Null Hypothesis:**  $H_0: \theta = \theta_0$

2. **Alternative Hypothesis:**

- $H_A: \theta > \theta_0$  (upper-tail alternative)
- $H_A: \theta < \theta_0$  (lower-tail alternative)
- $H_A: \theta \neq \theta_0$  (two-tailed alternative)

3. **Test Statistic:**  $Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$

4. **Rejection Region:**

- $\{z > z_\alpha\}$  (upper-tail RR)
- $\{z < z_\alpha\}$  (lower-tail RR)
- $\{|z| > z_{\alpha/2}\}$  (two-tailed RR)

# Type II Error Probabilities

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# Type II Error

**Recall:** **Type II Error** occurs when you *fail to reject the null*, when the alternative is true.

For a test with  $H_0 : \theta = \theta_0$  versus  $H_A : \theta > \theta_0$ , we can calculate type II error probabilities *only for specific values of  $\theta$  in  $H_A$* .

- For some specific alternative,  $\theta = \theta_A$ , the probability of type II error,  $\beta$ , is

$$\begin{aligned}\beta &= P(\hat{\theta} \text{ is not in RR when } H_A \text{ is true}) \\ &= P(\hat{\theta} \leq k \mid \theta = \theta_A) \\ &= P\left(\frac{\hat{\theta} - \theta_A}{\sigma_{\hat{\theta}}} \leq \frac{k - \theta_A}{\sigma_{\hat{\theta}}} \mid \theta = \theta_A\right),\end{aligned}$$

By the **Central Limit Theorem**,  $Z = (\hat{\theta} - \theta_A)/\sigma_{\hat{\theta}} \sim N(0, 1)$ , so we can calculate

$$\beta = P\left(Z \leq \frac{k - \theta_A}{\sigma_{\hat{\theta}}} \mid \theta = \theta_A\right)$$

by finding a corresponding area under the  $N(0, 1)$  curve, using `pnorm()`.

# Type II Error Probability

In **My Awesome Function** example, we had  $n = 40$  and  $\hat{p} = 0.6$ . Let's find  $\beta$  for a test of

$$H_0 : p = 0.5 \quad \text{versus} \quad H_A : p = 0.65.$$

- From the example, the **rejection region** for an  $\alpha = 0.05$  level test was

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} > 1.645 \iff \hat{p} > p_0 + 1.645 \sqrt{\frac{p_0(1-p_0)}{n}}.$$

- Plugging in  $p_0 = 0.5$  and  $n = 40$ , we have the rejection region  $\{\hat{p} > 0.63\}$ .

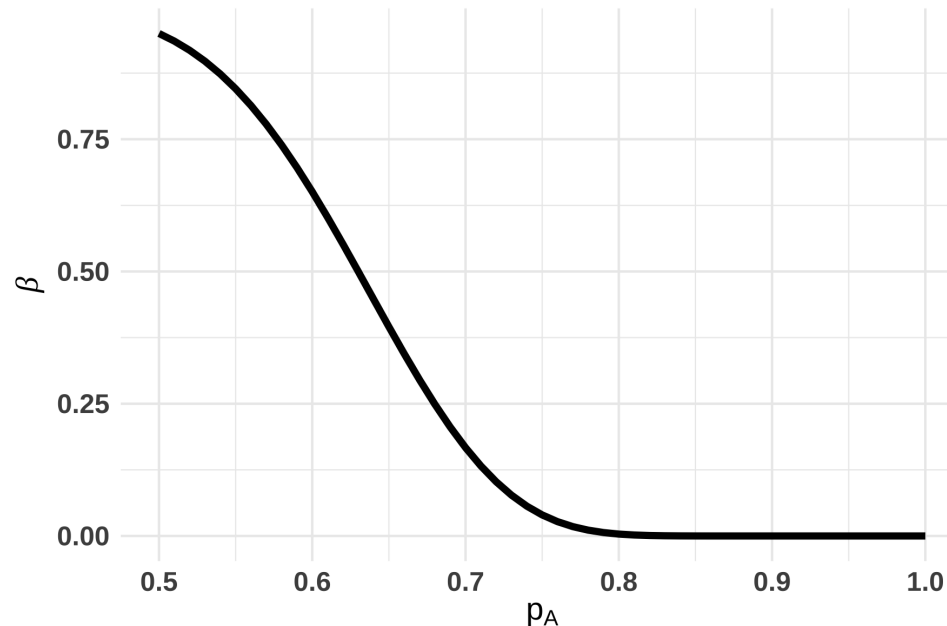
Therefore, by definition, the **type II error probability** is

$$\begin{aligned} \beta &= P(\hat{\theta} \text{ is not in RR when } H_A \text{ is true}) = P(\hat{p} \leq 0.63 \mid p = 0.65) \\ &= P\left(Z \leq \frac{0.63 - 0.65}{\sqrt{\frac{0.65(1-0.65)}{40}}}\right) \\ &= P(Z \leq -0.265) = \boxed{0.396} \end{aligned}$$

# Understanding $\beta$

While we can only calculate  $\beta$  for a fixed  $p_A$ , we can visualize how  $\beta$  depends on the distance between  $p_A$  and  $p_0$  (0.5).

- If  $p_A$  is far from  $p_0$ , then the true value of  $p$  is a bit easier to detect, hence  $\beta$  is much smaller.



# Back to Piéchart Emporium (yet again)

Piéchart Emporium claims that they pull in an average of 1000 dollars in *daily* revenue, but you *definitely* think it's lower. To check this, you could test

$$H_0 : \mu = 1000 \quad \text{versus} \quad H_A : \mu < 1000$$

at the  $\alpha = 0.05$  level, using the **rejection region**  $\{z < -1.645\}$  and the following observed data:

- $n = 30$  days
- $\bar{y} = 950$  dollars
- $s = 80$  dollars

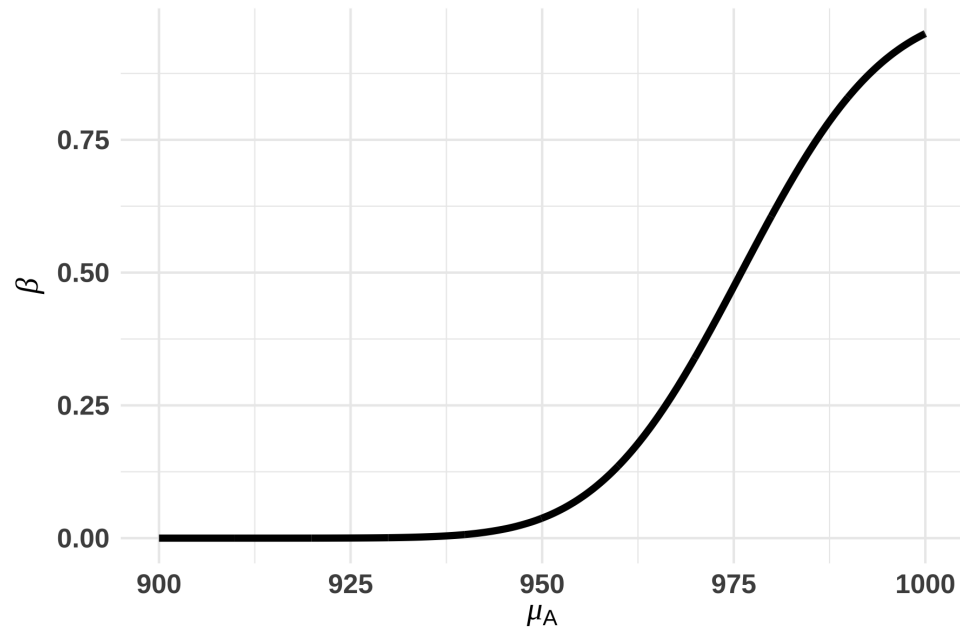
In *groups of 3*, calculate  $\beta$  for one of the following three specific values of  $\mu$  under  $H_A$ :

1.  $H_A : \mu = 980$

2.  $H_A : \mu = 960$

3.  $H_A : \mu = 940$

# Understanding $\beta$



# p-values

---



my p value:  $< 0.05$   
the null hypothesis:




# What is a p-value?

**p-value:** The probability of observing a test statistic *at least as extreme* as the one you observed, *if the null hypothesis is true*.

- In other words, this is the *smallest* level of significance,  $\alpha$ , for which the observed data indicate that the null hypothesis should be rejected.
- The *smaller* the p-value becomes, the **stronger** the evidence that the null hypothesis should be *rejected*.

Typically, we use p-values in the following way to reach decisions in a hypothesis test:

- $p \leq \alpha$ : **Reject the null hypothesis**
- $p > \alpha$ : **Fail to reject the null hypothesis**

 But be careful with this line of thinking! There is more to science than an *arbitrary, binary* decision making process!

# My Awesome Function

In the `generateCat` example, we tested

$$H_0 : p = 0.5 \quad \text{versus} \quad H_A : p > 0.5$$

using  $n = 40$  and  $\hat{p} = 0.6$ .

- Using  $\alpha = 0.05$ , our rejection region was  $RR = \{z > 1.645\}$ , and our observed test statistic was

$$Z = \frac{0.6 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{40}}} = 1.265.$$

Under  $H_0$ ,  $Z \sim N(0, 1)$ , so

$$\text{p-value} = P(Z \geq 1.265 \mid H_0 \text{ is true}) = 0.103$$

```
1-pnorm(1.265)
```

```
## [1] 0.1029357
```



# Common Grounds

I want to test whether the **mean** number of people in line at **Common Grounds** on a typical morning after STAT 339,  $\mu$ , is **different from 5** people.

- $H_0 : \mu = 5$
- $H_A : \mu \neq 5$

**Data:**  $n = 30, \bar{y} = 6, s = 2$

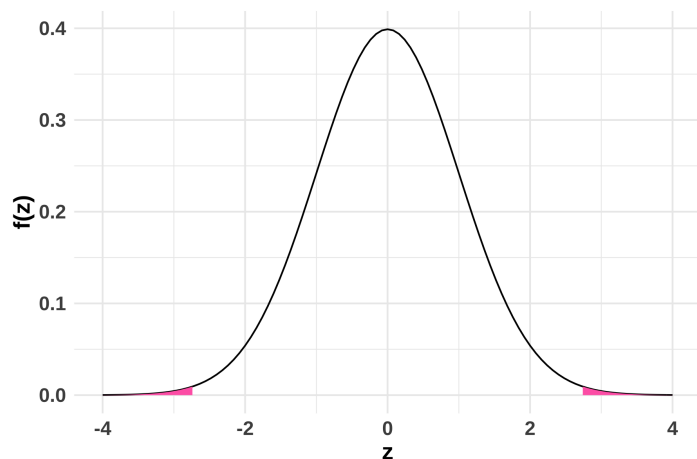
- Though rather than rejecting  $H_0$  for *only* large values of  $\bar{y}$ , we might also reject  $H_0$  for *small* values of  $\bar{y}$ . When calculating the p-value, we need to account for this!



# Common Grounds

Our **test statistic** for this test was

$$Z = \frac{6 - 5}{2/\sqrt{30}} = 2.74.$$



The pink regions in the plot represent the **p-value** for a two-tailed test.

In R:

```
2*pnorm(2.74, lower.tail = FALSE)
```

```
## [1] 0.006143918
```

**Decision:**  $p < 0.05$ , so reject the null hypothesis.

# Relationship with Confidence Intervals

(In my opinion) Confidence intervals are **more informative** and **less confusing** than p-values.

- *Not that p-values are useless; the definition just lends itself well to misinterpretation!*

There's also a *duality* that exists between large-sample CIs and two-tailed hypothesis tests:

- Do **not** reject  $H_0 : \theta = \theta_0$  at the  $\alpha$  level if  $\theta_0$  lies inside a  $100(1 - \alpha)\%$  confidence interval for  $\theta$ .
- **Reject**  $H_0$  if  $\theta_0$  lies outside the interval.

## Example

95% CI for the **mean** number of people in line at **Common Grounds** on a typical morning after STAT 339,  $\mu$ :

$$6 \pm 1.96 \times \frac{2}{\sqrt{30}} = (5.28, 6.72)$$

# Small-Sample Hypothesis Testing

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(for means)

# Introduction

The **large-sample** hypothesis testing procedures rely on the condition that the sample size is *large enough* that

$$Z = \frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} \sim (\textit{approx}) N(0, 1).$$

A common procedure that we can use with **small samples** relies on using the t-distribution as the sampling distribution of the **test statistic**.

- This procedure, the **t-test**, is *only appropriate for samples from Normal populations!*

# The t-test

## Setting

- $Y_1, Y_2, \dots, Y_n$  denote a **random sample** of size  $n$
- The sample comes from a **Normal** distribution with *unknown* mean  $\mu$  and *unknown* variance  $\sigma^2$ .

## Procedure

- Let  $\bar{Y}$  and  $S$  denote the sample mean and sample standard deviation, respectively.
- Under the **null hypothesis**,  $H_0 : \mu = \mu_0$ , the test statistic

$$T = \frac{\bar{Y} - \mu_0}{S/\sqrt{n}} \sim t(n-1).$$

- For  $H_A : \mu > \mu_0$ , the **rejection region** is  $RR = \{t > t_\alpha\}$ .
  - That is, any value of  $T$  such that  $P(T > t_\alpha) = \alpha$ , where  $t_\alpha$  is the **upper**  $\alpha$ -quantile of the  $t(n-1)$  distribution.

# Florida Lobsters



## WMS 10.75

A declaration by the Bahamian government that prohibits U.S. lobsterers from fishing on the Bahamian portion of the continental shelf was expected to dramatically reduce the landings (in pounds per lobster trap).

According to records, **the prior mean landings per trap was 30.31 pounds.**

A random sample of **20 lobster traps** since the Bahamian fishing restriction went into effect gave the following results (in pounds):

17.4, 18.9, 39.6, 34.4, 19.6, 33.7, 37.2, 43.4, 41.7, 27.5,  
24.1, 39.6, 12.2, 25.5, 22.1, 29.3, 21.1, 23.8, 43.2, 24.4

Do these landings provide evidence that the mean landings per trap has **decreased**? Test using  $\alpha = 0.05$

# Comparing Groups

We can also use the t-distribution to **compare the means** of two Normal populations that possess *equal variances*.

## Setting

- Independent samples from Normal distributions with  $\sigma_1^2 = \sigma_2^2$ .

## Hypotheses

- $H_0 : \mu_1 - \mu_2 = D_0$ , where  $D_0$  is usually 0.
- $H_A : \mu_1 - \mu_2 > D_0$  (or  $<$  or  $\neq$  depending on which alternative is appropriate)



# Comparing Groups

We can also use the  $t$ -distribution to **compare the means** of two Normal populations that possess *equal variances*.

## Test Statistic

- $T = \frac{(\bar{Y}_1 - \bar{Y}_2) - D_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ , where

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

## Rejection Region

- For an *upper-tail alternative*,  $RR = \{t > t_\alpha\}$ , where  $t_\alpha$  is the *upper  $\alpha$ -quantile* of the  $t(n_1 + n_2 - 2)$  distribution.

# Is the sequel always better?

I watched **Shrek** recently, because Netflix decided it was going to pull the movie at the end of April 2022.

- Not sure yet what this means for **Shrek 2**, and as of this writing I haven't watched it in a while. But let's compare them anyways!



Suppose random samples of  $n_1 = n_2 = 10$  critic reviews from *Metacritic* for each film yielded the ratings below. Check if the sequel is better at  $\alpha = 0.01$ .

- **Shrek**: 89.62, 82.03, 79.65, 88.20, 82.13, 86.06, 90.33, 89.44, 86.27, 79.82
- **Shrek 2**: 77.05, 73.93, 76.27, 70.89, 79.65, 64.49, 67.39, 74.15, 71.50, 86.16

# Connecting to Confidence Intervals

The connection between two-tailed **t-tests** and *small-sample confidence intervals* is similar to the duality between *large-sample* confidence intervals and tests!

- Do **not** reject  $H_0 : \mu = \mu_0$  if  $\mu_0$  lies *inside*  $\bar{y} \pm t_{\alpha/2} \times \frac{S}{\sqrt{n}}$
- **Reject**  $H_0 : \mu = \mu_0$  if  $\mu_0$  lies *outside*  $\bar{y} \pm t_{\alpha/2} \times \frac{S}{\sqrt{n}}$
- (Ditto for CIs for a **difference between means**)
  - Reject  $H_0 : \mu_1 - \mu_2 = D_0$  if  $D_0$  lies *outside* the CI.

## Example:

Calculate a 99% confidence interval for the *difference* in average Metacritic ratings between **Shrek** and **Shrek 2**.

# t-tests in R

Because the **t-test** is a common hypothesis testing procedure, it is supported by the **base R** `t.test()` function.

For example, we can repeat the *lobster* example in R using the following code:

```
lobster_data = c(17.4, 18.9, 39.6, 34.4, 19.6, 33.7, 37.2, 43.4, 41.7, 27.5,  
                 24.1, 39.6, 12.2, 25.5, 22.1, 29.3, 21.1, 23.8, 43.2, 24.4)
```

```
t.test(lobster_data, mu = 30.31, alternative = "less")
```

```
##  
##      One Sample t-test  
##  
## data:  lobster_data  
## t = -0.64678, df = 19, p-value = 0.2628  
## alternative hypothesis: true mean is less than 30.31  
## 95 percent confidence interval:  
##      -Inf 32.61098  
## sample estimates:  
## mean of x  
##      28.935
```

# Testing Hypotheses Concerning Variances

---

# Introduction

**Recall:** If  $Y_1, Y_2, \dots, Y_n \sim N(\mu, \sigma^2)$ , where both  $\mu$  and  $\sigma^2$  are *unknown*, then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

We can use this fact to test  $H_0 : \sigma^2 = \sigma_0^2$  for some fixed value  $\sigma_0^2$ .

If  $H_0 : \sigma^2 = \sigma_0^2$  is *true*, then

$$\chi^2 \equiv \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1),$$

and we can use  $\chi^2$  as our **test statistic**.

- For an  $\alpha$ -test of  $H_0 : \sigma^2 = \sigma_0^2$  versus  $H_A : \sigma^2 > \sigma_0^2$ ,

$$RR = \{\chi^2 > \chi_\alpha^2\}$$

# Example

(WMS 10.79)

The manufacturer of a machine to package soap powder claimed that their machine could load cartons at a given weight with a range of no more than 0.4 ounce. The mean and variance of a sample of  $n = 8$  three-pound boxes were found to equal 3.1 and 0.018, respectively.

Test the hypothesis that the variance of the population of weight measurements is  $\sigma^2 = 0.01$  against the alternative that  $\sigma^2 > 0.01$ . Use  $\alpha = 0.05$ .

# Comparing Groups

Sometimes rather than examining one group's variance, we might want to compare the variances of **two Normal distributions**.

- Usually to determine whether they are *equal*.

From **Exam 2**:

- If independent samples of size  $n_1$  and  $n_2$  are taken from two normally distributed populations with variances  $\sigma_1^2$  and  $\sigma_2^2$ , respectively, then

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(df_1 = n_1 - 1, df_2 = n_2 - 1)$$

Suppose we're testing  $H_0 : \sigma_1^2 = \sigma_2^2$  versus  $H_A : \sigma_1^2 > \sigma_2^2$ . Under  $H_0$ ,  $\sigma_1^2 = \sigma_2^2$  and

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2} \sim F(df_1 = n_1 - 1, df_2 = n_2 - 1)$$

is a **test statistic**!



# 90s Road Trip

Because the test statistic,  $F$ , follows an  $F(n_1 - 1, n_2 - 1)$  distribution under the null hypothesis, a **rejection region** for an  $\alpha$ -level test comparing variances is

$$RR = \{F > F_\alpha\},$$

where  $F_\alpha$  is the  $\alpha$ -quantile of the  $F(n_1 - 1, n_2 - 1)$  distribution.

## Example

|                         | 90s Road Trip | No Music |
|-------------------------|---------------|----------|
| Sample size, $n$        | 100           | 100      |
| Mean, $\bar{y}$         | 10.75         | 12.75    |
| Standard deviation, $s$ | 4             | 5        |

Using the data from **Exam 2**, is there sufficient evidence to conclude that the variance of *objects memorized* for the "No Music" group is *larger* than the variance for the "90s Road Trip" group?