COS210 - Theoretical Computer Science
Finite Automata and Regular Languages (Part 5)

Closure of Operations on Regular Languages

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If we want to prove closure properties of the form

$$L, L' \text{ regular} \quad \Rightarrow \quad L \; op \; L' \text{ regular}$$

we can either construct a DFA **or** an NFA that accepts $L \; op \; L'$.

Closure of the Union Operation

Remember the following theorem we proved using a **DFA construction**:

Theorem (Closure of Union)

The set of regular languages R over Σ is closed under the Union *operation.*

$$L_1 \text{ and } L_2 \text{ regular} \Rightarrow L_1 \cup L_2 \text{ regular}$$

where

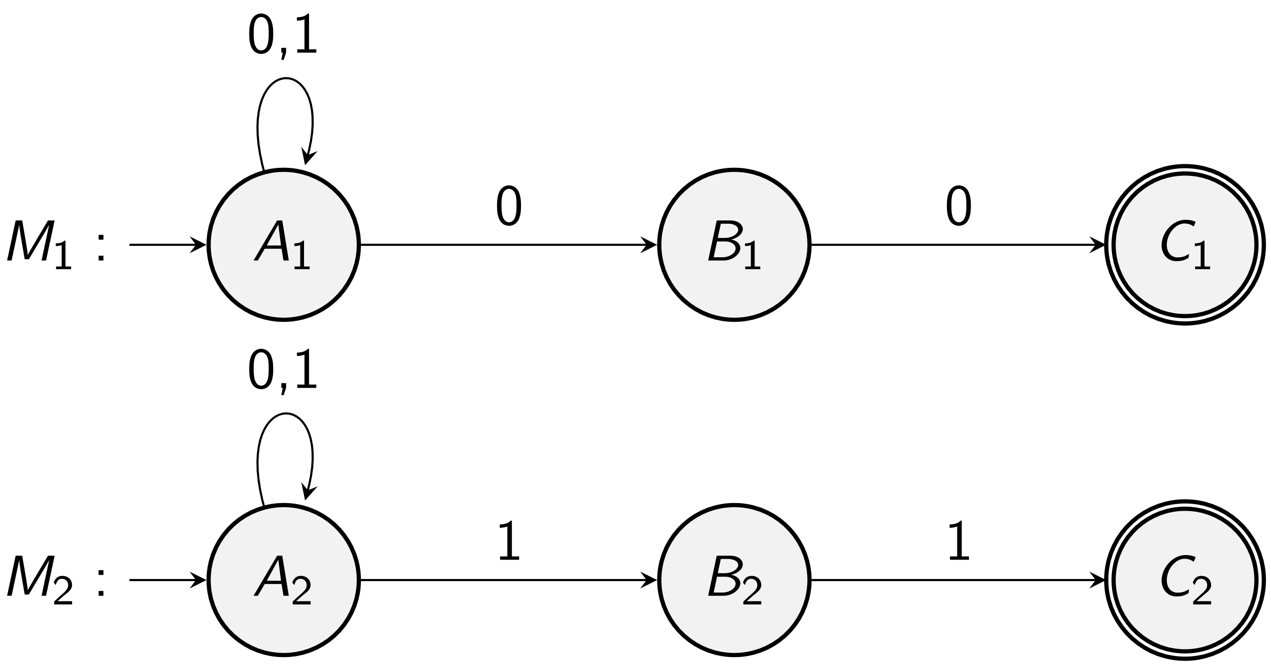
$$L_1 \cup L_2 \;=\; \{w : w \in L_1 \text{ or } w \in L_2\}$$

We will now prove it again using an **NFA construction** instead.

Closure of the Union Operation: Example

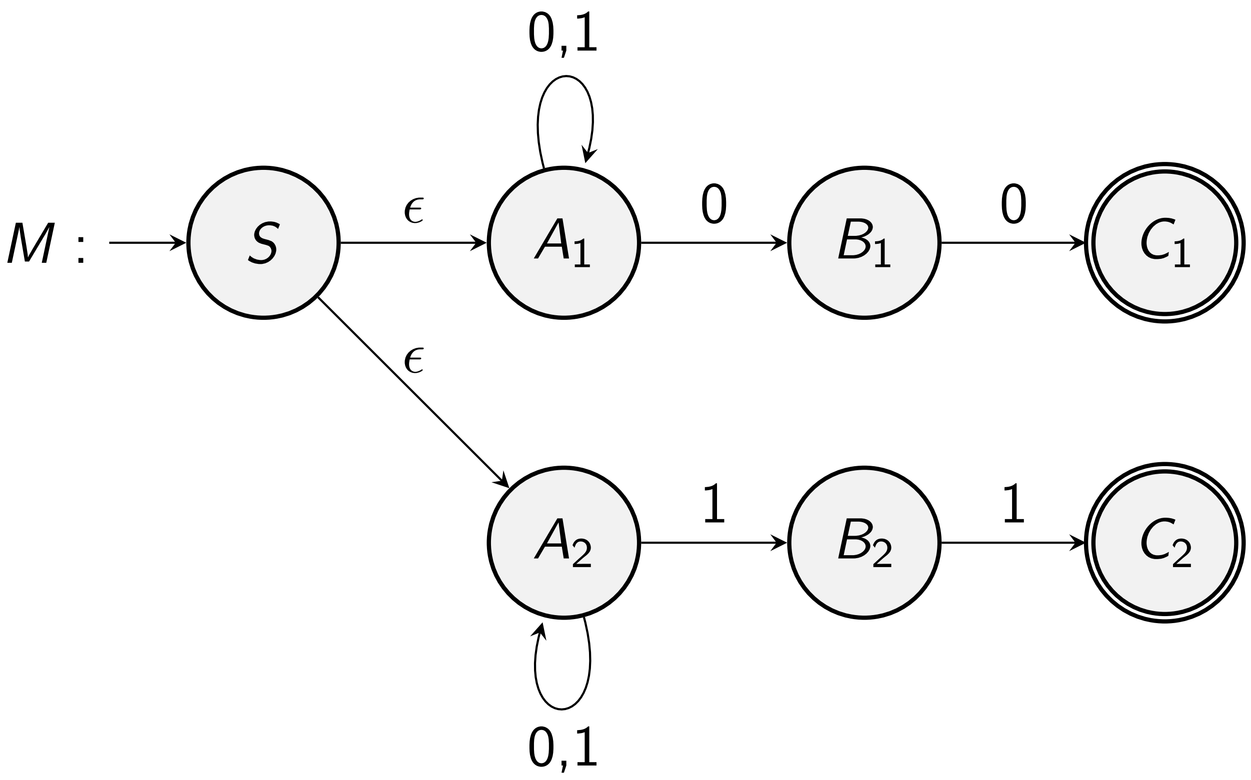
Consider the following regular languages of the automata below:

- $L_1(M_1) = \{w : \text{string } w \text{ ends with } 00\}$
- $L_2(M_2) = \{w : \text{string } w \text{ ends with } 11\}$



Closure of the Union Operation: Example

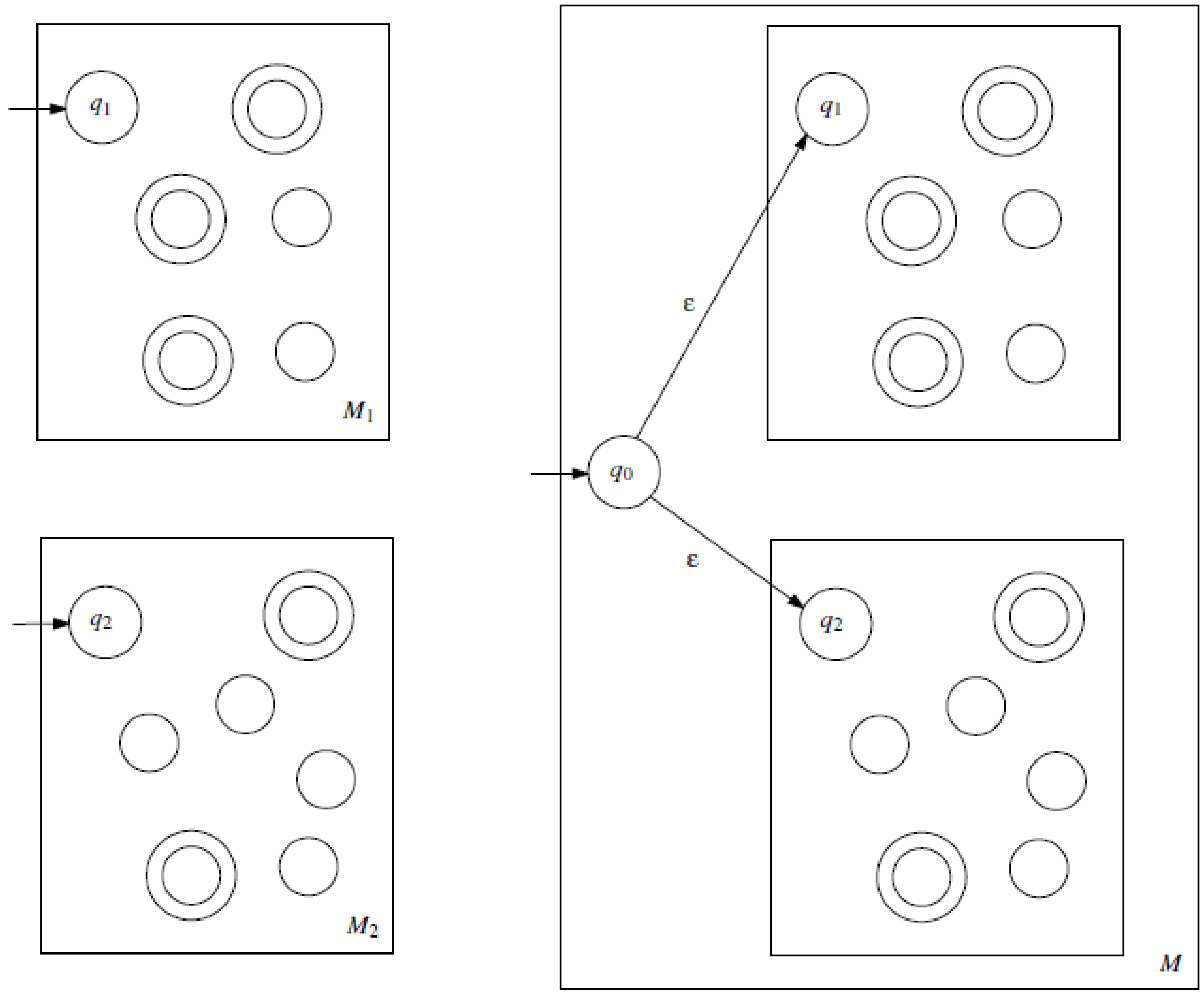
Proving that the union $L_1 \cup L_2$ is regular is a straightforward construction:



$$L(M) \;=\; L_1 \cup L_2 \;=\; \{w : \text{string } w \text{ ends with 00 or ends with 11}\}$$

Closure of the Union Operation

General construction for union:



Closure of the Union Operation

Theorem (Closure of Union)

The set of regular languages R over Σ is closed under the Union *operation.*

Proof:

- Show that L_1, L_2 regular $\Rightarrow L_1 \cup L_2$ regular
- L_1, L_2 regular $\Rightarrow$ NFAs M_1, M_2 exist (Theorem 2.5.2)
  - $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ with $L_1 = L(M_1)$
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- We can assume $Q_1 \cap Q_2 = \emptyset$

  (can be always established by renaming states)
- We now construct NFA M with $L(M) = L_1 \cup L_2$.

Closure of the Union Operation

Proof cont:

- $Q = \{q_0\} \cup Q_1 \cup Q_2$
- $q = q_0$
  (q_0 will be the new initial state)
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- It can be concluded that $L(M) = L_1 \cup L_2$. $\square$

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Theorem (Closure of Concatenation)

The set of regular languages R over Σ is closed under the Concatenation *operation.*

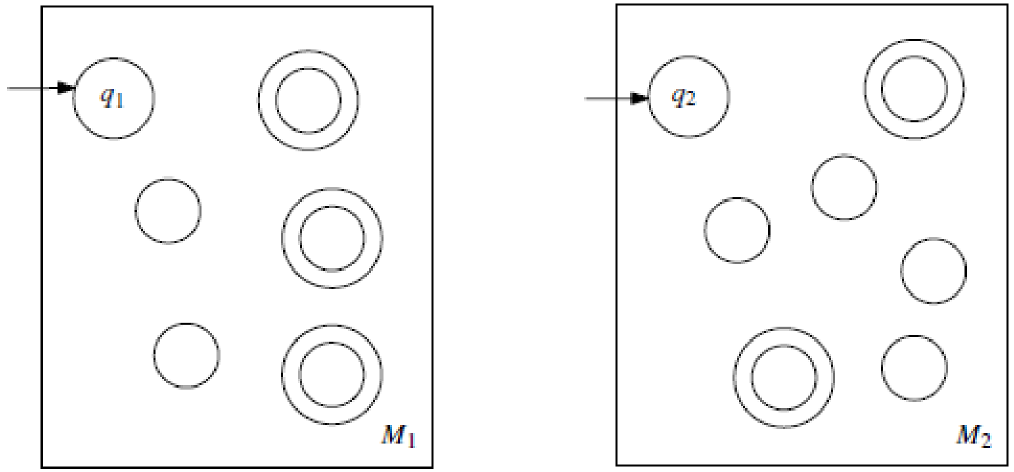
$$L_1 \text{ and } L_2 \text{ regular} \Rightarrow L_1L_2 \text{ regular}$$

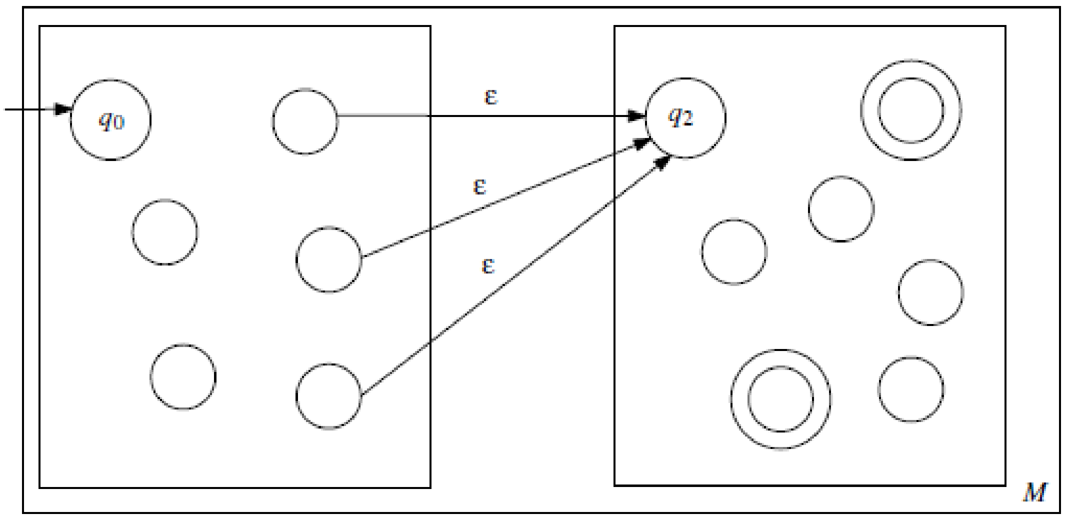
where

$$L_1L_2 \;=\; \{ww' : w \in L_1 \text{ and } w' \in L_2\}$$

Closure of the Concatenation Operation

Idea for proof by construction:





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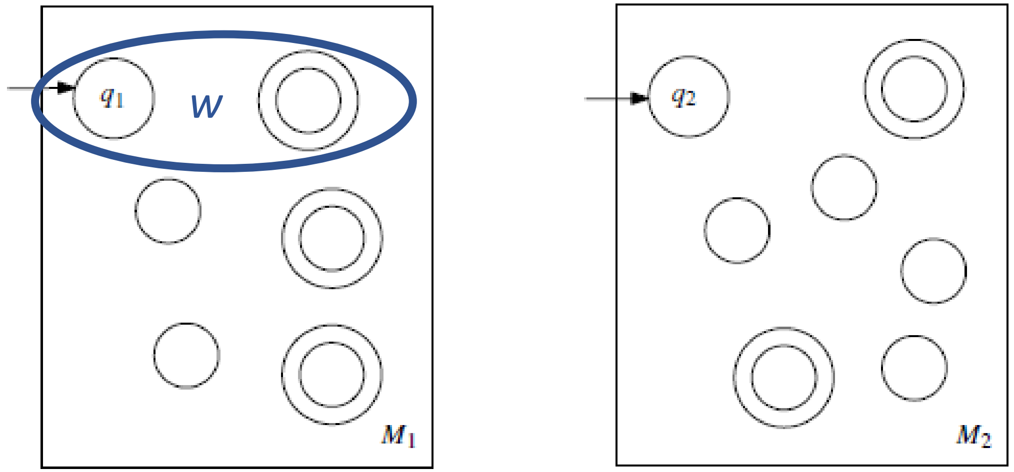

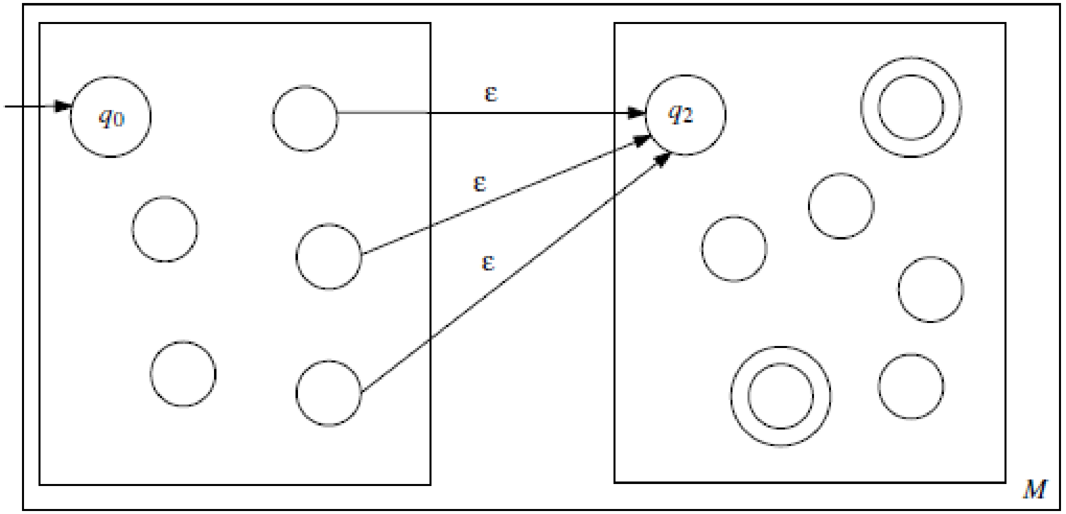

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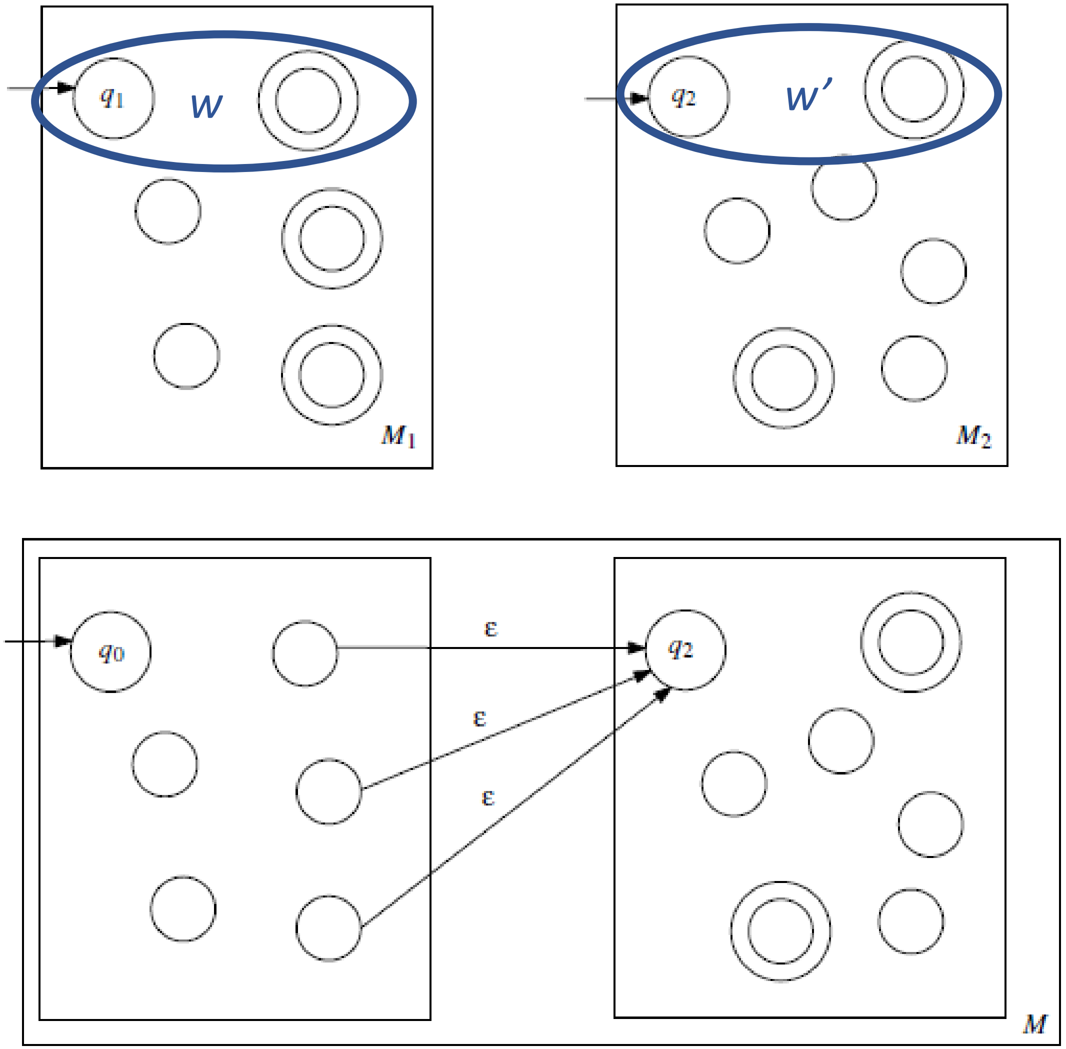

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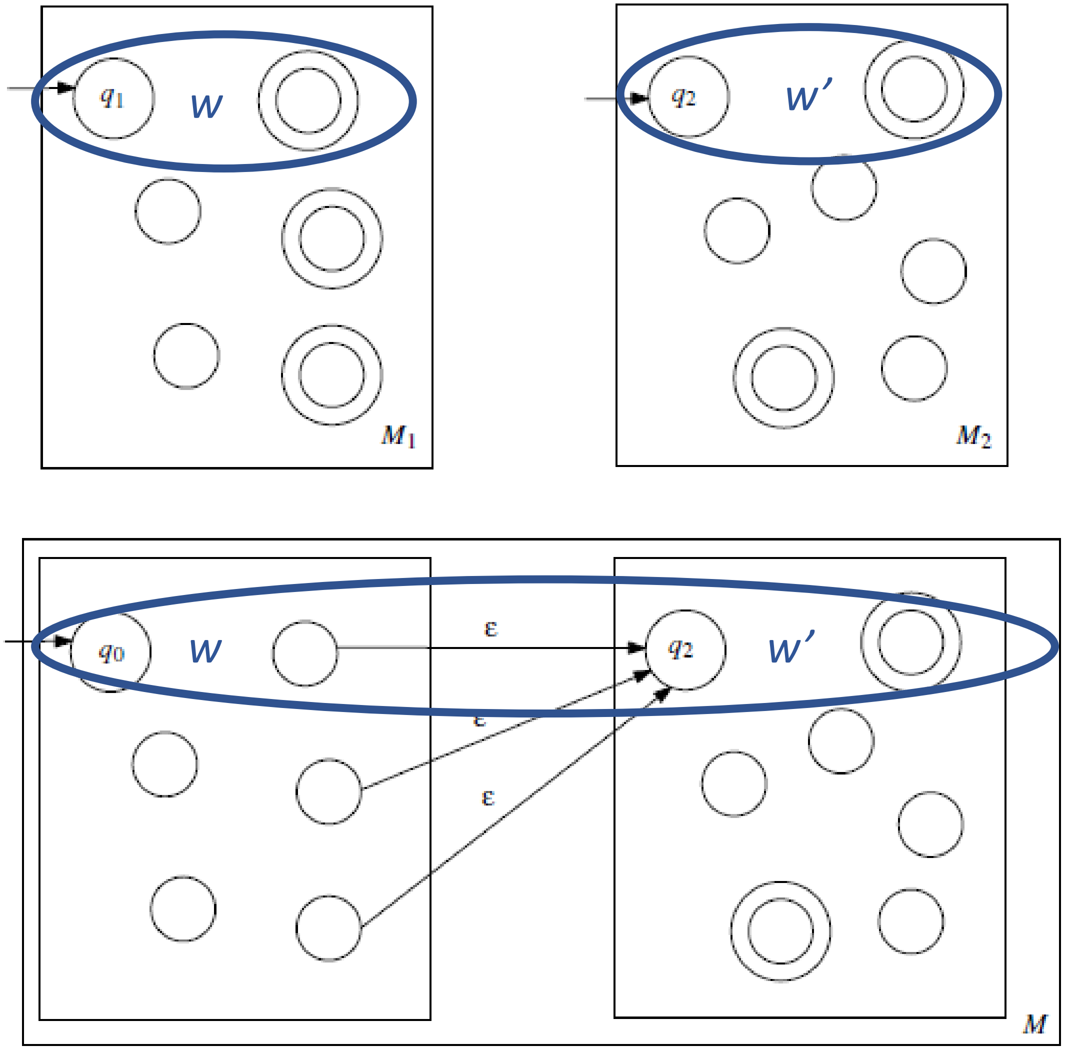

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Theorem (Closure of Star)

The set of regular languages R over Σ is closed under the Star *operation.*

$$A \text{ regular} \Rightarrow A^* \text{ regular}$$

where

$$A^* \;=\; \{w_1 w_2 \cdots w_k : k \geq 0 \text{ and } w_i \in A \text{ for all } i = 1, 2, \cdots, k\}$$

Set of all strings that are concatenations of zero or more elements of A

Closure of the Star Operation

Assume N accepts A. How to construct M accepting A^* based on N?

Cases to be considered:

- 0-concatenations: ϵ
- 1-concatenations: w
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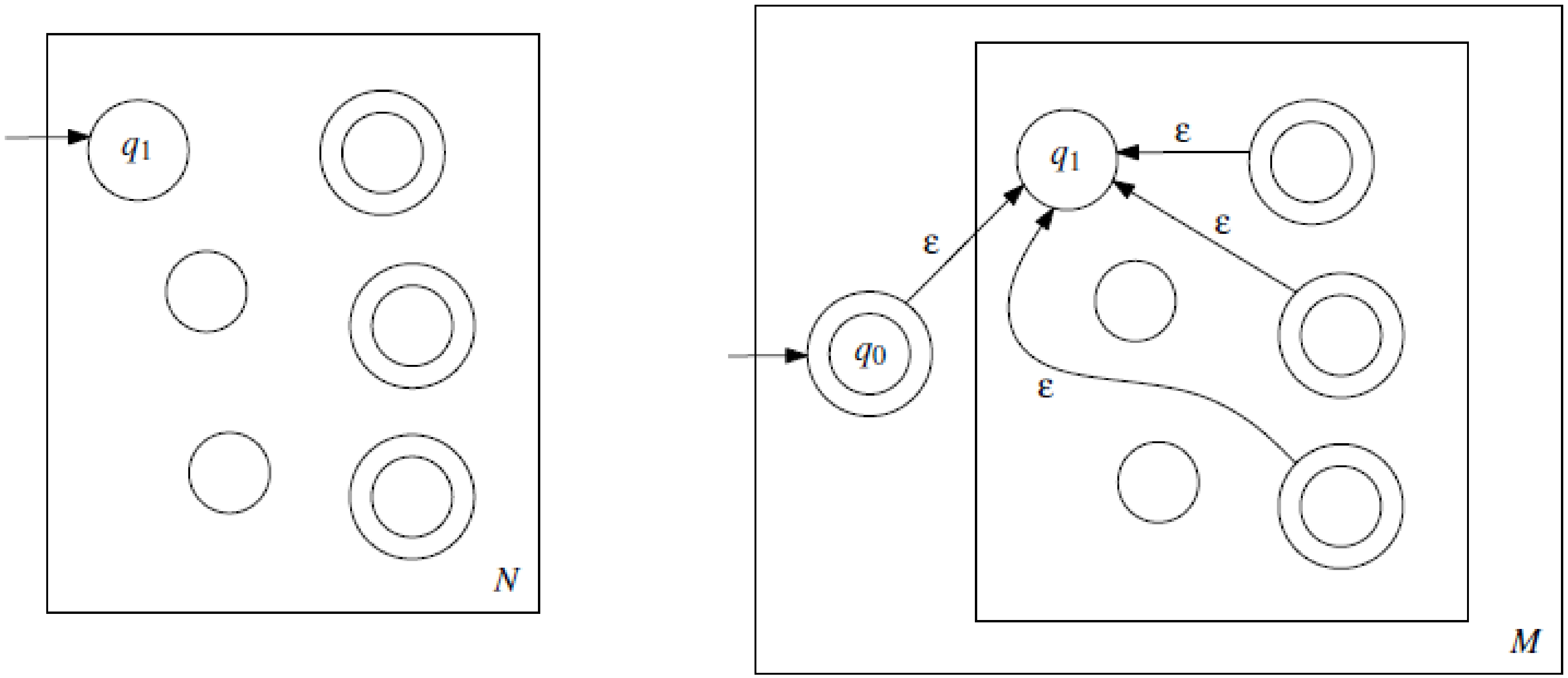
Idea for the construction of M:

- in order to accept ϵ, the initial state of M must be accepting state
- M accepts all 1-concatenations, thus, keep all behaviour of N
- after w_1 would have been accepted, M must reset in order to check if w_2 could be accepted next (and so on)

Closure of the Star Operation

Construction of M:



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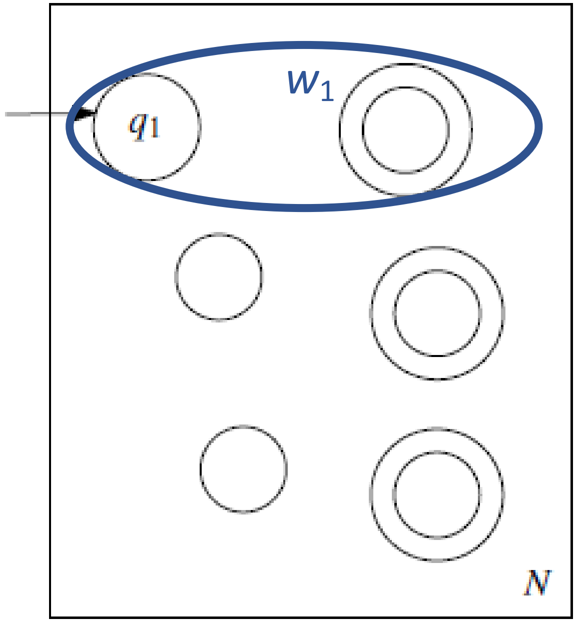

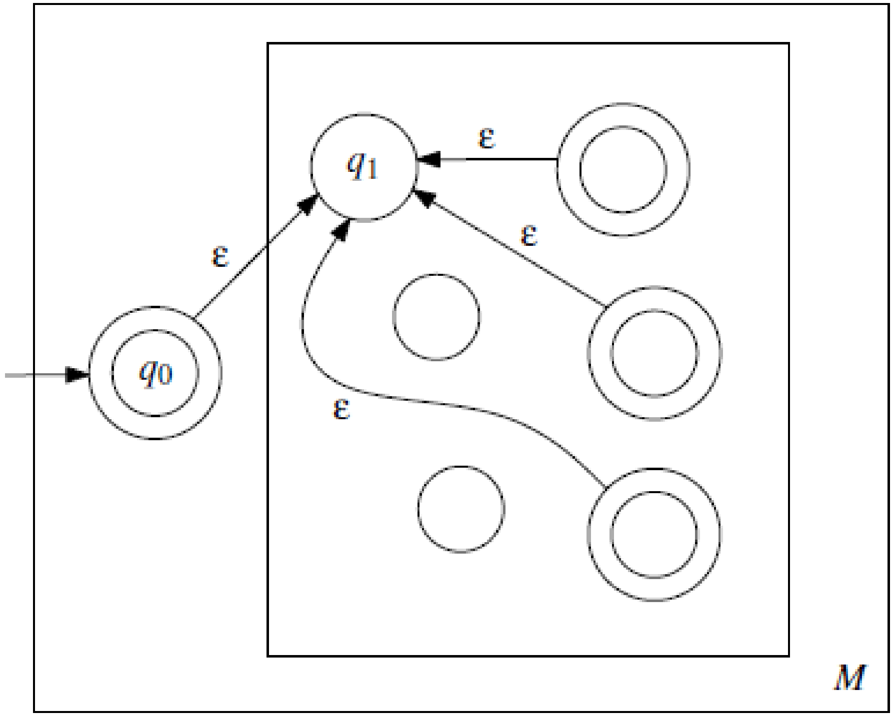

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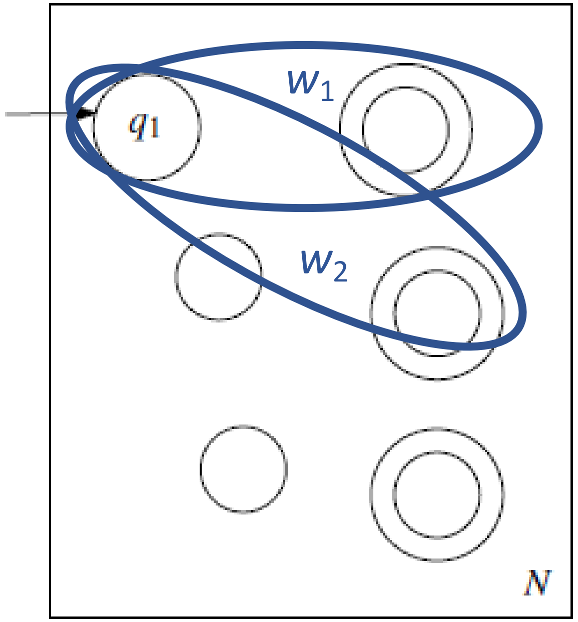

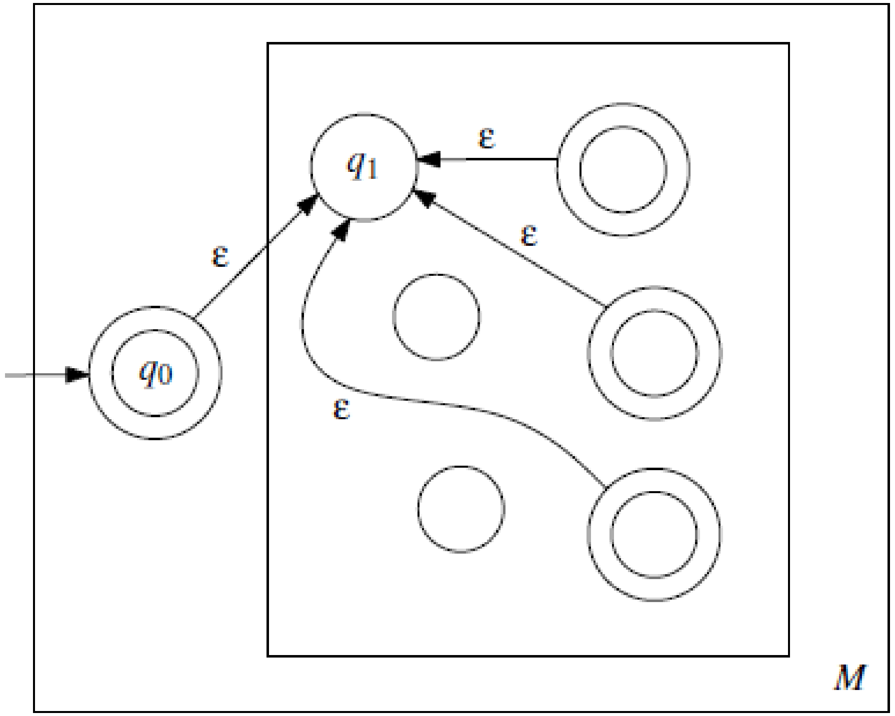

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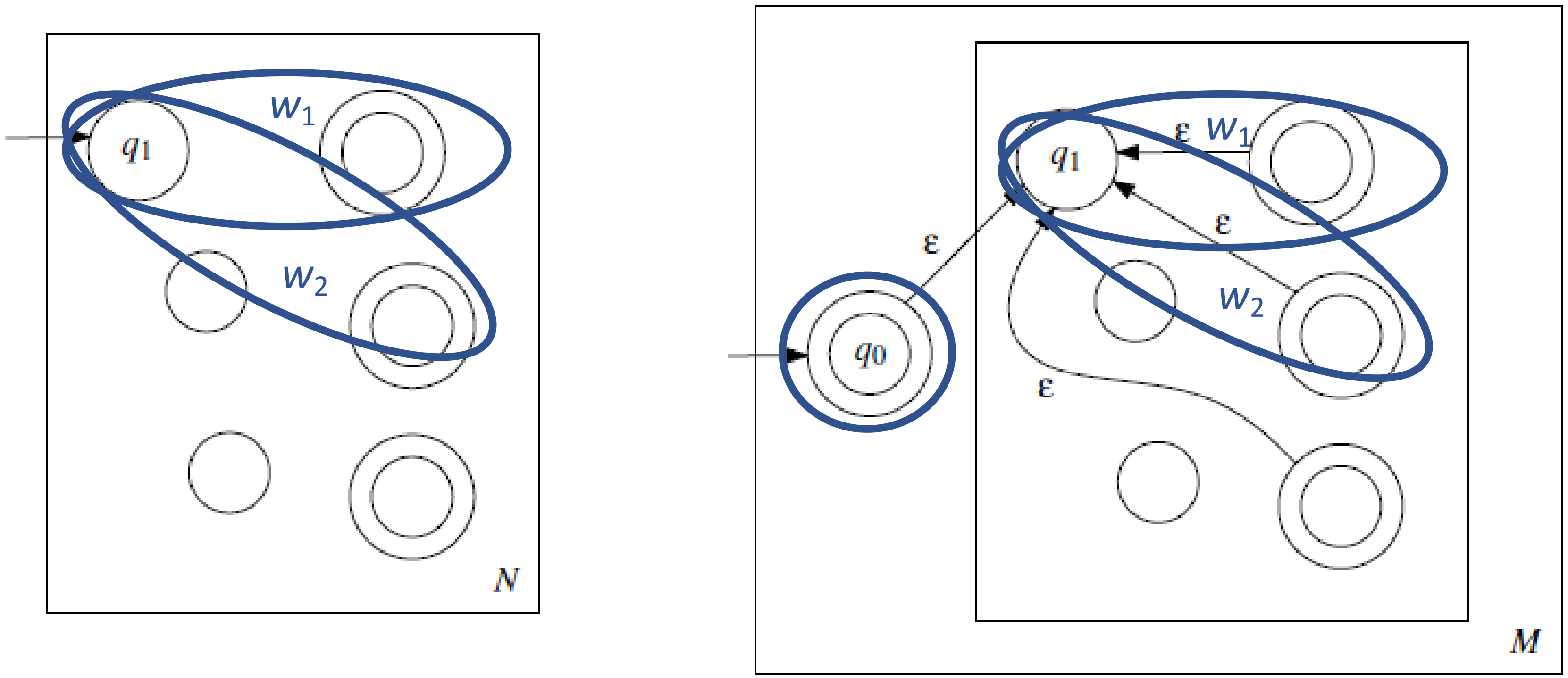

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- We now construct NFA $M = (Q, \Sigma, \delta, q, F)$ with $L(M) = A^*$.

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Proof cont:

- $Q = \{q_0\} \cup Q_1$, where q_0 is a new state.
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  keep all transitions of N

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  keep all transitions of N

  for each $r \in F_1$: extend the set of ϵ-successor states by $\{q_1\}$

  introduce ϵ-transition from q_0 to q_1

- It can be concluded that $L(M) = A^*$. $\square$

Closure of the Star Operation: Alternative Proof?

Remember:

$$\underbrace{A^*}_{\text{star}} = \underbrace{\{\epsilon\} \cup A \cup AA \cup AAA \cup \ldots \cup A^\infty}_{\text{union and concatenation}}$$

Question:
Can we prove the closure of star based on the following argumentation?

Closure of Union and **Closure of Concatenation** $\Rightarrow$ **Closure of Star**

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Question:
Can we prove the closure of star based on the following argumentation?

Closure of Union and Closure of Concatenation $\Rightarrow$ Closure of Star

Answer and justification:

- **No**
- infinite union means infinitely many states in proof by construction:
  $$Q' = \underbrace{Q \cup Q \cup \ldots \cup Q}_{\infty}$$
- result will be no longer a finite automaton

Closure of the Complement and Intersection Operation

Theorem (Closure of Complement)

The set of regular languages over Σ is closed under the Complement *operation.*

$$L \text{ regular} \Rightarrow \overline{L} \text{ regular} \quad \text{where} \quad \overline{L} = \{w \in \Sigma^* : w \notin L\}$$

Theorem (Closure of Intersection)

The set of regular languages over Σ is closed under the Intersection *operation.*

$$L_1 \text{ and } L_2 \text{ regular} \Rightarrow L_1 \cap L_2 \text{ regular}$$

where

$$L_1 \cap L_2 = \{w \in \Sigma^* : w \in L_1 \text{ and } w \in L_2\}$$

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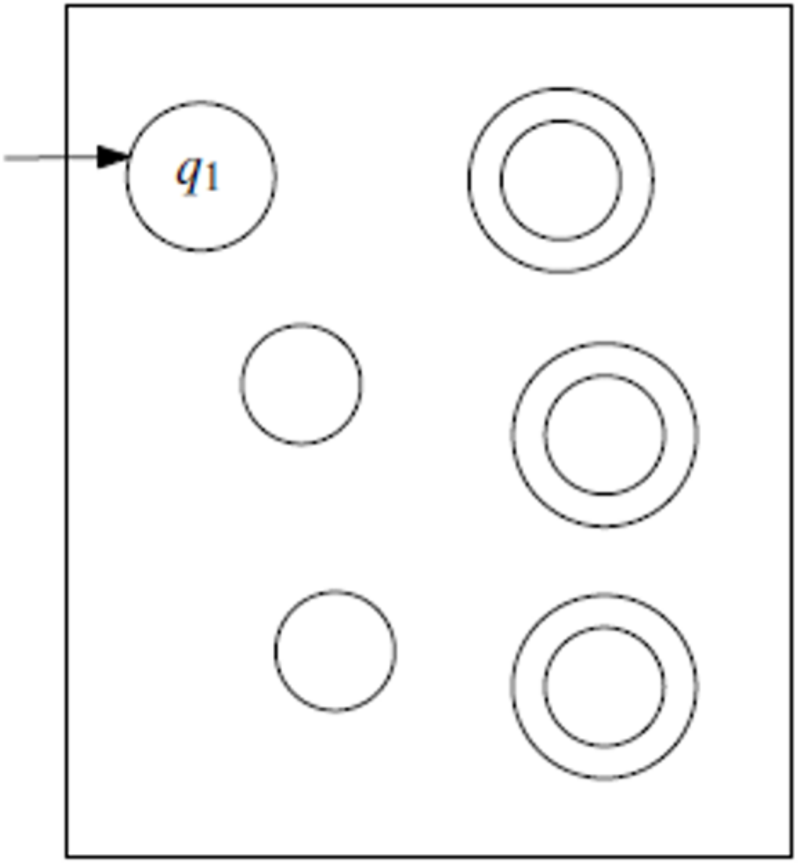
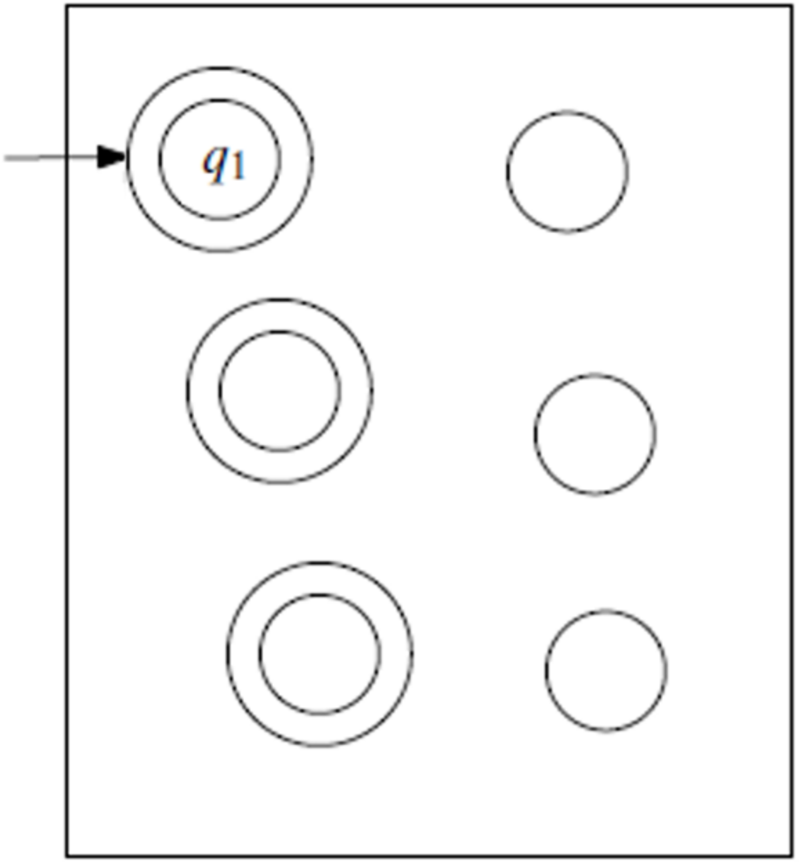
Closure of the Complement Operation

$D = (Q, \Sigma, \delta, q, \mathbf{F})$

$M = (Q, \Sigma, \delta, q, \mathbf{Q} \backslash \mathbf{F})$





If D accepts L

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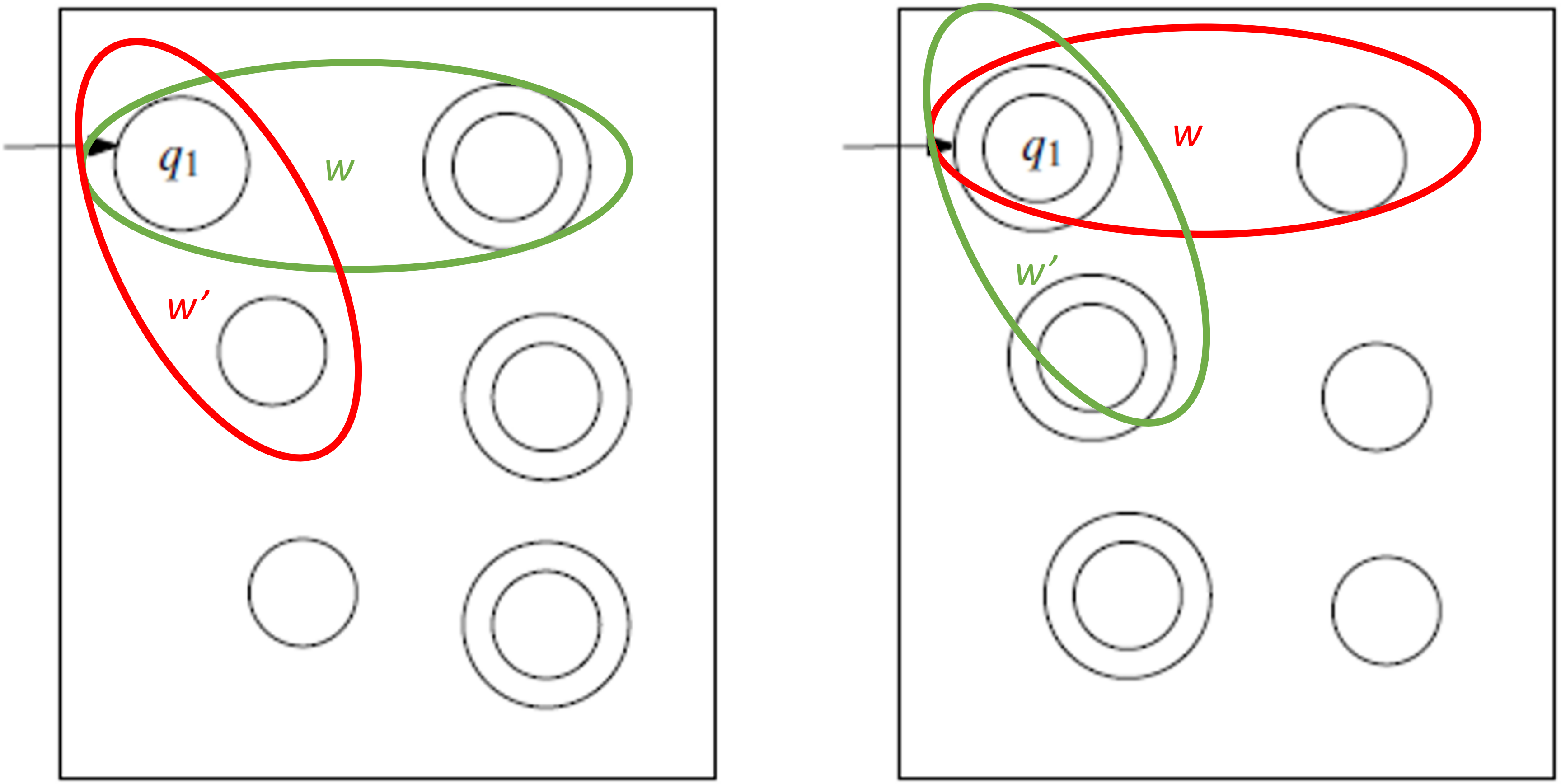

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Proof:

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\end{array}$$

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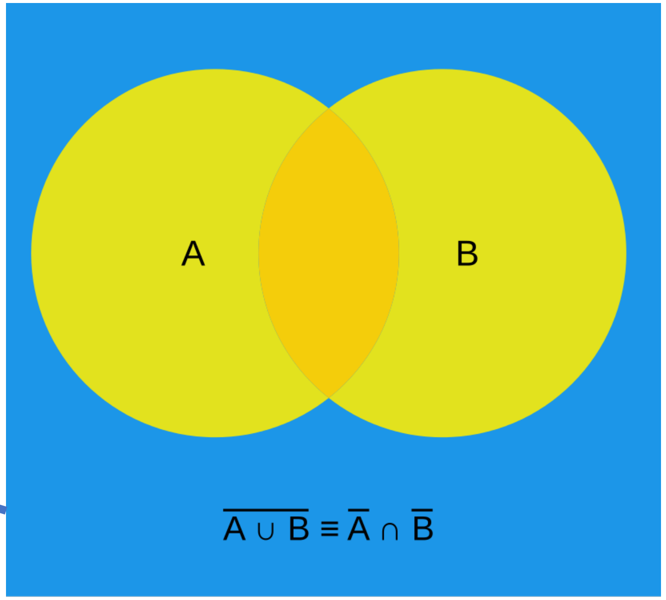
\Leftrightarrow\quad & \overline{\overline{L_1}} \cap \overline{\overline{L_2}} \quad \text{regular} & \text{(De Morgan)}
\end{aligned}$$



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\Rightarrow & \overline{\overline{L_1} \cup \overline{L_2}} \ \text{ regular} & \text{(closure of complement)} \\
\Leftrightarrow & \overline{\overline{L_1}} \cap \overline{\overline{L_2}} \ \text{ regular} & \text{(De Morgan)} \\
\Leftrightarrow & L_1 \cap L_2 \ \text{ regular} & \text{(double complement elemination)}
\end{array}$$