COS210 - Theoretical Computer Science
Decidable and Undecidable Languages: Part 2

The Halting Problem

Theorem

The language Halt is undecidable, where

$$Halt = \{\langle P, w\rangle : P \textit{ is a computer program that terminates on input } w\}$$

- P can be any program of Java, C, C++ etc.
- Intuitively, *Halt* is undecidable because we cannot know how long to run an arbitrary P until we can decide that it will never terminate

The Halting Problem

Proof by contradiction:

Assume that the language *Halt* is decidable

- Then there exists a program H such that for every input $\langle P, w\rangle$:
  - If $\langle P, w\rangle \in Halt$ (program P terminates on input w), then H terminates and returns **true**
  - if $\langle P, w\rangle \notin Halt$ (program P does not terminate on w), then H terminates and returns **false**
- In particular, H **always terminates**
- We can represent the program H with the shorthand

$$H(P, w) = \begin{cases} \textbf{true} \text{ if } P(w) \text{ terminates} \\ \textbf{false} \text{ if } P(w) \text{ does not terminate} \end{cases}$$

The Halting Problem

Proof cont:

We now define another program Q

- Q takes and arbitrary program $\langle P \rangle$ as and input, and:
  - If $H(P, \langle P \rangle) =$ **true** $(P(\langle P \rangle)$ terminates),
    then Q **loops forever**
  - if $H(P, \langle P \rangle) =$ **false** $(P(\langle P \rangle)$ does not terminate),
    then Q **terminates**
- In particular, $Q(\langle P \rangle)$ **only terminates** if $P(\langle P \rangle)$ **does not terminate**

What if we use $\langle Q \rangle$ itself as the input of Q?

- $Q(\langle Q \rangle)$ **only terminates** if $Q(\langle Q \rangle)$ **does not terminate**
- **Contradiction**. Hence, a program H that decides *Halt* cannot exist
  Consequently, the language *Halt* is **undecidable**

Countable Sets

Countability of sets (or languages) is a property that can be utilized in (un)decidability proofs

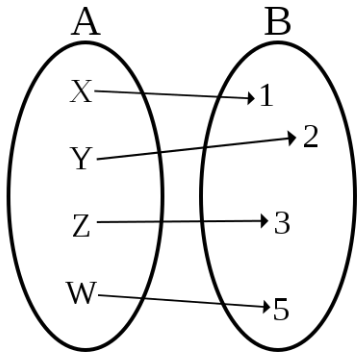
Definition

Let A be a set. Then A is **countable**, if A is finite, or A and $\mathbb{N}$ have the same size

Definition

Let A and B be two sets. We say that A and B have the **same size**, if there exists a **bijection** $f : A \rightarrow B$

A bijection pairs each element of A with exactly one element of B and vice versa



Countable sets

An infinite set A is **countable** , if we can map each $a \in A$ to a **unique natural number** $n \in \mathbb{N}$. For instance:

$$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\} \text{ is countable}$$

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The corresponding **bijection** is $f : \mathbb{N} \to \mathbb{Z}$ with

$$f(1) = 0, f(2) = 1, f(3) = -1, f(4) = 2, f(5) = -2, \ldots$$

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Definition of f:

$$f(n) = \begin{cases} \frac{n}{2} \text{ if } n \text{ is } \textbf{even} \\ \frac{-(n-1)}{2} \text{ if } n \text{ is } \textbf{odd} \end{cases}$$

Countable sets

The set of **rational numbers** $\mathbb{Q} = \{\frac{n}{m} : n, m \in \mathbb{Z}, m \neq 0\}$ is **countable**

- We illustrate this based on the positive rational numbers
- They can be arranged in a matrix as follows:

$$
\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Countable Sets

We can systematically count the elements of the matrix by following the arrows:

$$
\begin{array}{cccccccccc}
\frac{1}{1}\ (1) & \rightarrow & \frac{1}{2}\ (2) & & \frac{1}{3}\ (5) & \rightarrow & \frac{1}{4}\ (6) & & \frac{1}{5}\ (11) & \rightarrow \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \\
\frac{2}{1}\ (3) & & \frac{2}{2}\ (\cdot) & & \frac{2}{3}\ (7) & & \frac{2}{4}\ (\cdot) & & \frac{2}{5} & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & \\
\frac{3}{1}\ (4) & & \frac{3}{2}\ (8) & & \frac{3}{3}\ (\cdot) & & \frac{3}{4} & & \frac{3}{5} & \cdots \\
 & \swarrow & & \nearrow & & & & & & \\
\frac{4}{1}\ (9) & & \frac{4}{2}\ (\cdot) & & \frac{4}{3} & & \frac{4}{4} & & \frac{4}{5} & \cdots \\
\downarrow & \nearrow & & & & & & & & \\
\frac{5}{1}\ (10) & & \frac{5}{2} & & \frac{5}{3} & & \frac{5}{4} & & \frac{5}{5} & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Redundant elements such as $\frac{2}{2}$ or $\frac{2}{4}$ can be skipped

Countable Sets

$$
\begin{array}{ccccccccccc}
\frac{1}{1}\ (1) & \rightarrow & \frac{1}{2}\ (2) & & \frac{1}{3}\ (5) & \rightarrow & \frac{1}{4}\ (6) & & \frac{1}{5}\ (11) & \rightarrow & \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & & \\
\frac{2}{1}\ (3) & & \frac{2}{2}\ (\cdot) & & \frac{2}{3}\ (7) & & \frac{2}{4}\ (\cdot) & & \frac{2}{5} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & & \\
\frac{3}{1}\ (4) & & \frac{3}{2}\ (8) & & \frac{3}{3}\ (\cdot) & & \frac{3}{4} & & \frac{3}{5} & & \cdots \\
 & \swarrow & & \nearrow & & & & & & & \\
\frac{4}{1}\ (9) & & \frac{4}{2}\ (\cdot) & & \frac{4}{3} & & \frac{4}{4} & & \frac{4}{5} & & \cdots \\
\downarrow & \nearrow & & & & & & & & & \\
\frac{5}{1}\ (10) & & \frac{5}{2} & & \frac{5}{3} & & \frac{5}{4} & & \frac{5}{5} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

The corresponding bijection:

$$
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \frac{1}{2} & 2 & 3 & \frac{1}{3} & \frac{1}{4} & \frac{2}{3} & \frac{3}{2} & 4 & 5 & \frac{1}{5} & \cdots
\end{array}
$$

Countable Sets

Not all infinite sets are countable:

Theorem

The set $\mathbb{R}$ of real numbers is not countable

- We will prove that the subset $A \subset \mathbb{R}$ is not countable, where

$$A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$$

- If A is not countable, then $\mathbb{R}$ is not countable

Countable Sets

Proof by contradiction:

- Assume that A is countable
- Then there exists a bijective function $f : \mathbb{N} \to A$
- For each $n \in \mathbb{N}$, $f(n)$ is a real number between zero and one
- We can write A as

$$A = \{f(1), f(2), f(3), \ldots\}$$

- Consider the real number $f(1)$. We can write this number in decimal notation as

$$f(1) = 0.d_{11}d_{12}d_{13}\ldots$$

where each d_{1i} is a digit in the set $\{0, 1, 2, \ldots, 9\}$

Countable Sets

Proof cont:

- In general, for every $n \in \mathbb{N}$, we can write the real number $f(n)$ as

$$f(n) = 0.d_{n1}d_{n2}d_{n3}\ldots$$

  where each d_{ni} is a digit in the set $\{0, 1, 2, \ldots, 9\}$
- We now define the following real number

$$x = 0.d_1d_2d_3\ldots$$

  where, for each $n \geq 1$:

$$d_n = \begin{cases} 4 \text{ if } d_{nn} \neq 4 \\ 5 \text{ if } d_{nn} = 4 \end{cases}$$

  $x \in A$ and we have that $d_n \neq d_{nn}$ for each n

Countable Sets

Proof cont:

- In general, for every $n \in \mathbb{N}$, we can write the real number $f(n)$ as

$$f(n) = 0.d_{n1}d_{n2}d_{n3}\ldots$$

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where, for each $n \geq 1$:

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$x \in A$ and we have that $d_n \neq d_{nn}$ for each n

Example:

$f(1) = 0.\mathbf{0}12\ldots$
$f(2) = 0.3\mathbf{4}5\ldots$
$f(3) = 0.67\mathbf{8}\ldots$
...

$x = 0.454\ldots$

Countable Sets

Proof cont:

- $x = 0.d_1 d_2 d_3 \ldots$ where for each n, $d_n \neq d_{nn}$
- Since $x \in A$, there must be a natural number n with $f(n) = x$
- If we lay out the elements of A using the previous ordering

$$\begin{aligned} f(1) &= 0.d_{11} d_{13} d_{13} d_{14} d_{15} \ldots \\ f(2) &= 0.d_{21} d_{22} d_{23} d_{24} d_{25} \ldots \\ f(3) &= 0.d_{31} d_{33} d_{33} d_{34} d_{35} \ldots \\ &\vdots \qquad\qquad\qquad \vdots \end{aligned}$$

- We see that x is different from $f(1)$ in its 1st digit,
- x is different from $f(2)$ in its 2nd digit,
- for each n, x is different from $f(n)$ in its n-th digit
- There is no n with $f(n) = x$, which is a **contradiction** $\square$

Alternative Proof of the Halting Problem

Theorem

The language Halt is undecidable, where

$$Halt = \{\langle P, w\rangle : P \text{ is a computer program that terminates on input } w\}$$

Proof:

First we note that the set of all computer programs is **countable**

- The code of any program is of **finite length**
- We can describe any program P by a finite length string $\langle P\rangle$
- From here we can easily list all computer programs:
  - List all programs P, whose description $\langle P\rangle$ has length 1
  - List all programs P, whose description $\langle P\rangle$ has length 2
  - $\ldots$
- In each listing step there are finitely many programs to be considered
- So we can arrange all programs in the **countably infinite** list

$$P_1, P_2, P_3, P_4, \ldots$$

Alternative Proof of the Halting problem

Proof by contradiction:

- Assume that the language *Halt* is decidable
- Then there exists a computer program $H(\langle P, w\rangle)$ that decides *Halt*
  - H returns **true** if P **terminates** on input w
  - H returns **false** if P **does not terminate** on input w
- We construct a new program D that does the following:

On input $\langle P_n \rangle$, where $n \in \mathbb{N}$ do:

- **Step 1:** Run the program H on the input $\langle P_n, \langle P_n \rangle\rangle$
- **Step 2:**
  - ★ If H returns **true**, then **loop forever**
  - ★ If H returns **false**, then terminate and return **true**

Alternative Proof of the Halting problem

Proof cont:

- Computer programs are countable: $P_1, P_2, P_3, P_4, \ldots$
- Since D is a program, there must exists an $n \in \mathbb{N}$ such that $D = P_n$
- We know:
  - If D **terminates** on input $\langle P_n \rangle$, then H returns **false** on input$\langle P_n, \langle P_n \rangle \rangle$ i.e. P_n **does not terminate** on input $\langle P_n \rangle$.
  - If D **does not terminates** on $\langle P_n \rangle$, then H returns **true** on $\langle P_n, \langle P_n \rangle \rangle$ i.e. P_n **terminates** on input $\langle P_n \rangle$.
- To summarize
  - D **terminates** on input $\langle P_n \rangle$ **if and only if** P_n **does not terminate** on input $\langle P_n \rangle$.
- D **terminates** on input $\langle D \rangle$ **if and only if** D **does not terminate** on input $\langle D \rangle$
- **Contradiction** $\square$

Alternative Proof of the Halting problem: Remark

We defined the program D such that for each $n \in \mathbb{N}$, the computation of D on input $\langle P_n \rangle$ differs from the computation of P_n on input $\langle P_n \rangle$

- Hence, for each $n \in \mathbb{N}$ we have that $D \neq P_n$
- However, since D is a program, there must be an $n \in \mathbb{N}$ such that $D = P_n$.

This is the same argument that we used to prove that $\mathbb{R}$ was not countable