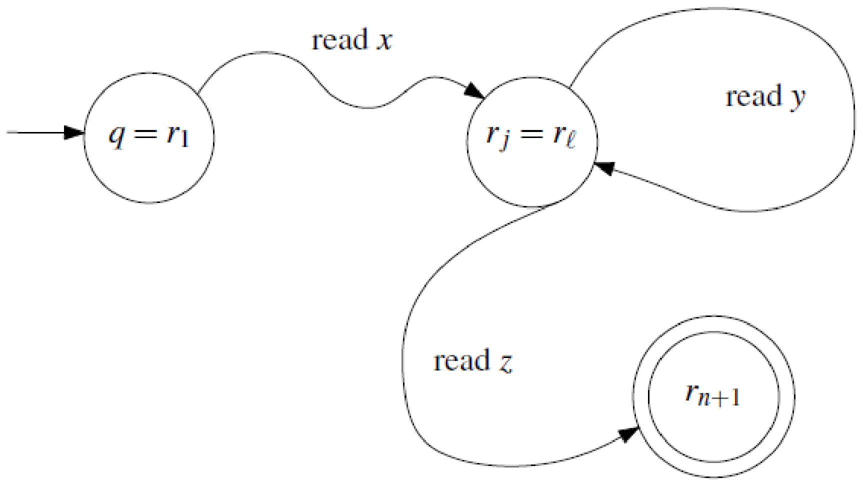
COS210 - Theoretical Computer Science
Finite Automata and Regular Languages (Part 9)

Pumping Lemma

Theorem (Pumping Lemma for Regular Languages)

Let A be a regular language. Then there exists an integer $p \geq 1$, called the pumping length, such that the following holds: Every string w in A, with $|w| \geq p$, can be written as $w = xyz$, such that

1. $y \neq \epsilon$ *(non-empty middle part y)*
2. $|xy| \leq p$ *(finite prefix xy)*
3. $xy^k z \in A$ *for all* $k \geq 0$ *(repeatable middle part)*



Pumping Lemma

Proof:

- Let A be a regular language over Σ

$\Rightarrow$ There exists a DFA $M = (Q, \Sigma, \delta, q, F)$ that accepts A

- Choose the number of states of M as the pumping length: $p = |Q|$

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- Choose the number of states of M as the pumping length: $p = |Q|$
- Let $w = w_1 \ldots w_n \in A$ a string of length $n \geq p$

$\Rightarrow$ There exists a run over $n + 1$ states

$$r_1 \xrightarrow{w_1} r_2 \xrightarrow{w_2} \ldots \xrightarrow{w_{n-1}} r_n \xrightarrow{w_n} r_{n+1}$$

in M where $r_1 = q$ and $r_{n+1} \in F$

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$\Rightarrow$ There exists a DFA $M = (Q, \Sigma, \delta, q, F)$ that accepts A

- Choose the number of states of M as the pumping length: $\boxed{p = |Q|}$
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- Since $\boxed{n+1 > |Q|}$ there must be a state r_i that occurs twice along the first $|Q|+1$ states of the run (pigeon hole principle)

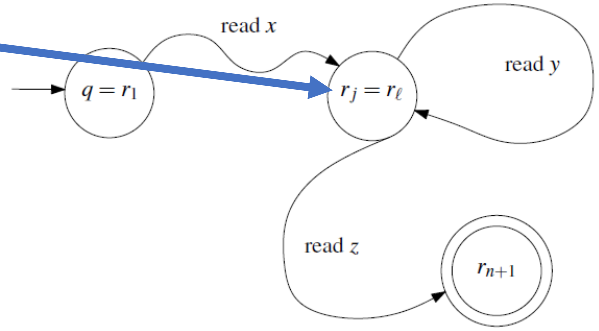
$$r_1 \xrightarrow{w_1} \dots \xrightarrow{w_{i-1}} \underbrace{r_i \xrightarrow{w_i} \dots \xrightarrow{w_{j-1}} r_j}_{r_i \, = \, r_j} \xrightarrow{w_j} \dots \xrightarrow{w_n} r_{n+1}$$

where $i < j$ and $j \leq |Q| + 1$

Pumping Lemma

- Some state r_i occurs twice along the run

$$r_1 \xrightarrow{w_1} \ldots \xrightarrow{w_{i-1}} \underbrace{r_i \xrightarrow{w_i} \ldots \xrightarrow{w_{j-1}} r_j}_{r_i \;=\; r_j} \xrightarrow{w_j} \ldots \xrightarrow{w_n} r_{n+1}$$



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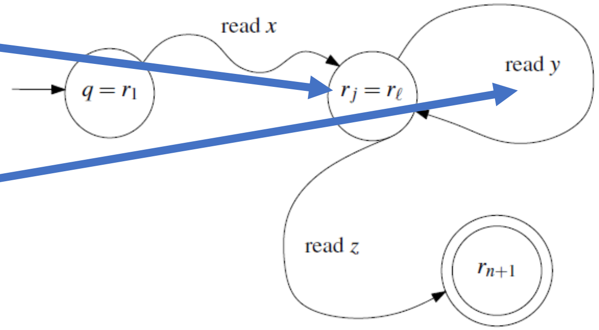

- String w can be written as $w = xyz$ where

$$\overbrace{r_1 \xrightarrow{w_1} \ \dots \ \xrightarrow{w_{i-1}}}^{x} r_i \overbrace{\xrightarrow{w_i} \ \dots \ \xrightarrow{w_{j-1}}}^{y} r_j \overbrace{\xrightarrow{w_j} \ \dots \ \xrightarrow{w_n}}^{z} r_{n+1}$$

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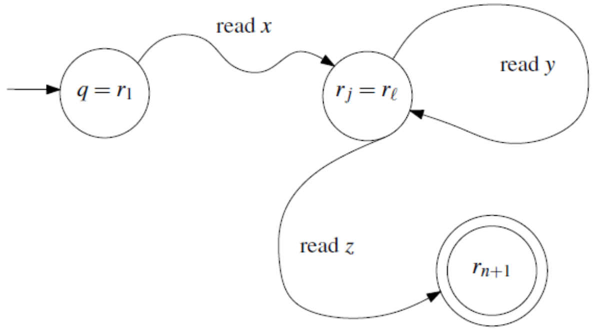

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We have that:

1. $y \neq \epsilon$, since $i < j$

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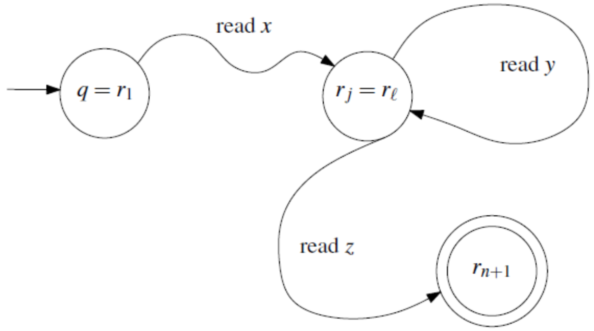

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We have that:

1. $y \neq \epsilon$, since $i < j$
2. $|xy| \leq p$, since $|xy| = j - 1 \leq |Q| = p$

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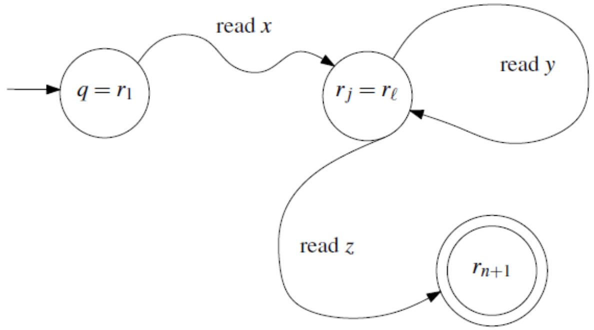

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We still need to show that:

3. $xy^k z \in A$ for all $k \geq 0$

Premises:

- $xyz \in A$
- $r_1 = q$
- $\bar{\delta}(r_1, x) = r_i$
- $\bar{\delta}(r_i, y) = r_j$
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Conclusions:

$\Rightarrow \bar{\delta}(r_i, y) = r_i$

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Conclusions:

$\Rightarrow$ $\bar{\delta}(r_i, y) = r_i$

$\Rightarrow$ $\bar{\delta}(r_i, y^k) = r_i$ for all $k \geq 0$

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$\Rightarrow \boxed{x}y^k z \in A$ for all $k \geq 0$

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$\Rightarrow xy^k\boxed{z} \in A$ for all $k \geq 0$ $\square$

Pumping Lemma: Application 1

The language A is not regular

$$A = \{0^n 1^n : n \geq 0\}$$

Proof:

- Assume that A is regular

$\Rightarrow$ There exists a pumping length $p \geq 1$

Theorem (Pumping Lemma for Regular Languages)

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$\boxed{xyz = 0^p 1^p \Rightarrow xz = 0^q 1^p \text{ with } q < p}$

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- $|xy| \leq p \;\Rightarrow\; xy = 0 \ldots 0$ (zeros only)
- $y \neq \epsilon \;\Rightarrow\; y$ consists of a non-zero number of 0s

$\Rightarrow$ xz contains less 0s than 1s

$\Rightarrow$ $xz \notin A$ (contradiction)

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$\boxed{xyz = 0^p 1^p \;\Rightarrow\; xz = 0^q 1^p \text{ with } q < p}$

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Pumping Lemma: Application 2

The language A is not regular

$$A = \{ss : s \text{ is a string over } \{0,1\}\} \text{ (concatenations of strings with itself)}$$

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Pumping Lemma: Application 2

The language A is not regular

$A = \{ss : s \text{ is a string over } \{0,1\}\}$ (concatenations of strings with itself)

Proof

- Assume that A is regular

$\Rightarrow$ There exists a pumping length $p \geq 1$

- Consider $w = 0^p 1 0^p 1 \in A$
- We have that $|w| = p + 1 + p + 1 \geq p$

$\Rightarrow$ w can be written as $w = xyz$, where

  - $y \neq \epsilon$, $|xy| \leq p$, and $xy^kz \in A$ for all $k \geq 0$

- $|xy| \leq p \;\Rightarrow\; xy = 0\ldots0$ (zeros only)
- $y \neq \epsilon \;\Rightarrow\; y$ consists of a non-zero number of 0s

$\Rightarrow$ $xyyz = \underbrace{0\ldots0}_{p+|y|}1\underbrace{0\ldots0}_{p}1$

Theorem (Pumping Lemma for Regular Languages)

Let A be a regular language. Then there exists an integer $p \geq 1$, called the pumping length, such that the following holds: Every string w in A, with $|w| \geq p$, can be written as $w = xyz$, such that

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$$A = \{1^{(n^2)} : n \geq 0\} \text{ (strings that are sequences of } n^2 \text{ 1s)}$$

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$\boxed{|xyyz| = p^2 + |y| \geq p^2 + 1 > p^2}$

$$p^2 < |xyyz| \qquad \textbf{(1,2)}$$

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$$p^2 < |xyyz| \quad \textbf{(1,2)}$$

$$|xyyz| \leq p^2 + p < (p+1)^2 \quad \textbf{(1,3)}$$

$|xyyz| = p^2 + |y| \leq p^2 + p < p^2 + 2p + 1 = (p + 1)^2$

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$$\boxed{\begin{aligned} p^2 &< |xyyz| & \\ &\phantom{<}\ |xyyz| < (p+1)^2 \end{aligned}} \quad \begin{matrix}\textbf{(1,2)} \\ \textbf{(1,3)}\end{matrix}$$

$\Rightarrow$ Length of $xyyz$ is strictly between the squares p^2 and $(p+1)^2$

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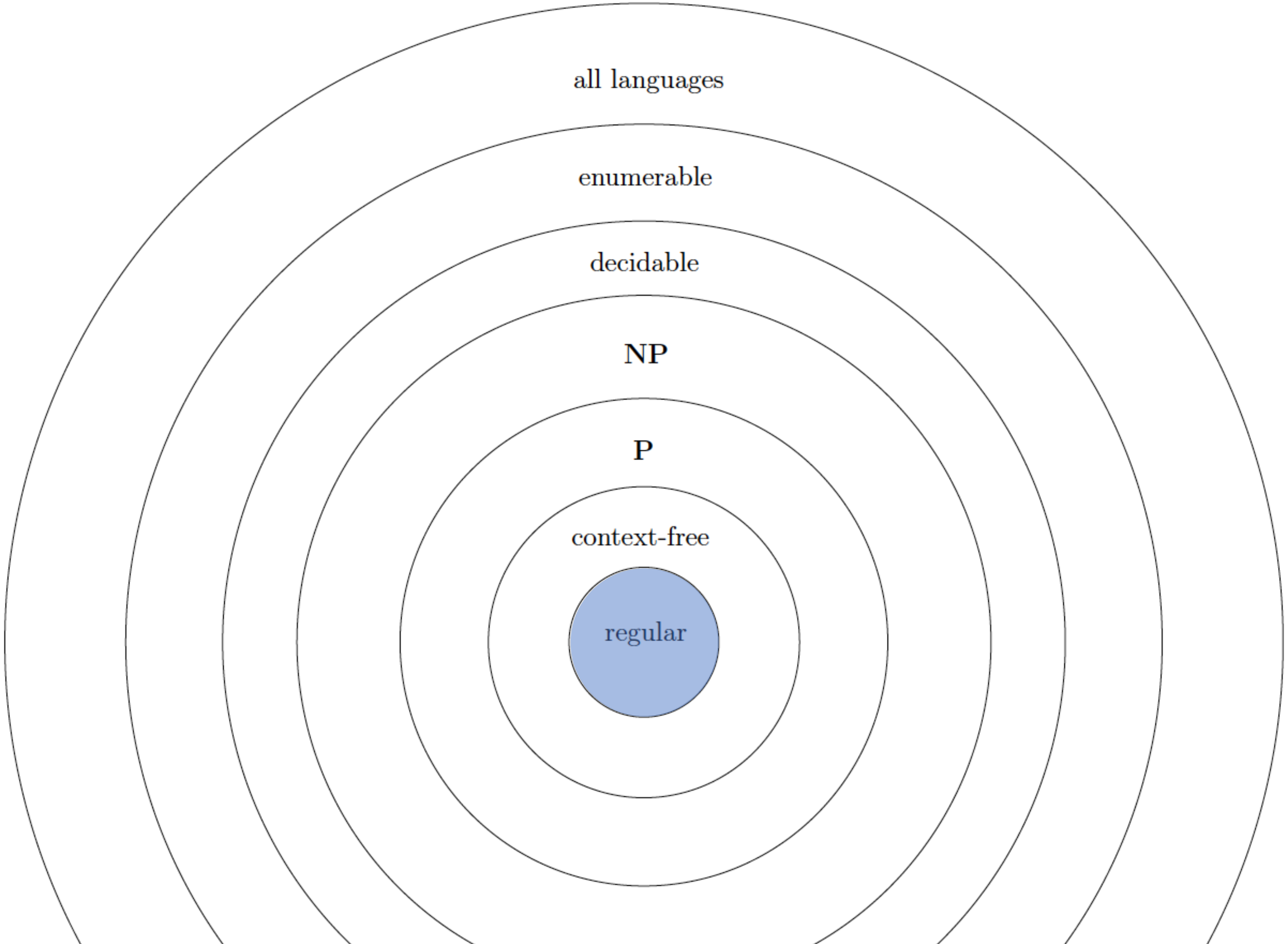
Pumping Lemma: Exercise

The language A is not regular

$$A = \{w : w \text{ contains twice as many 0s as 1s}\}$$

Proof:

End of Chapter 2



End of Chapter 2