

COS210 - Theoretical Computer Science  
Turing Machines and the Church-Turing Thesis (Part 2)

# Turing Machine Example: Palindromes

We will construct a Turing machine that accepts the language

$$L = \{w \in \{a, b\}^* : w \text{ is a } \mathbf{palindrome}\}$$

A palindrome is a string that reads the **same backward as forward**, e.g.

*aba, ababa, bbaabb*

Different Turing machines can solve the problem. We will consider two:

- A **one tape** Turing machine (less efficient)
- A **two tape** Turing machine (more efficient)

# One Tape Solution

The transition function  $\delta$  is defined by the following instructions:

$$q_0a \rightarrow q_a \square R$$

$$q_0b \rightarrow q_b \square R$$

$$q_0\square \rightarrow q_{accept}$$

$$q_aa \rightarrow q_aaR$$

$$q_ab \rightarrow q_abR$$

$$q_a\square \rightarrow q'_a\square L$$

$$q_ba \rightarrow q_baR$$

$$q_bb \rightarrow q_bbR$$

$$q_b\square \rightarrow q'_b\square L$$

$$q'_aa \rightarrow q_L\square L$$

$$q'_ab \rightarrow q_{reject}$$

$$q'_a\square \rightarrow q_{accept}$$

$$q'_ba \rightarrow q_{reject}$$

$$q'_bb \rightarrow q_L\square L$$

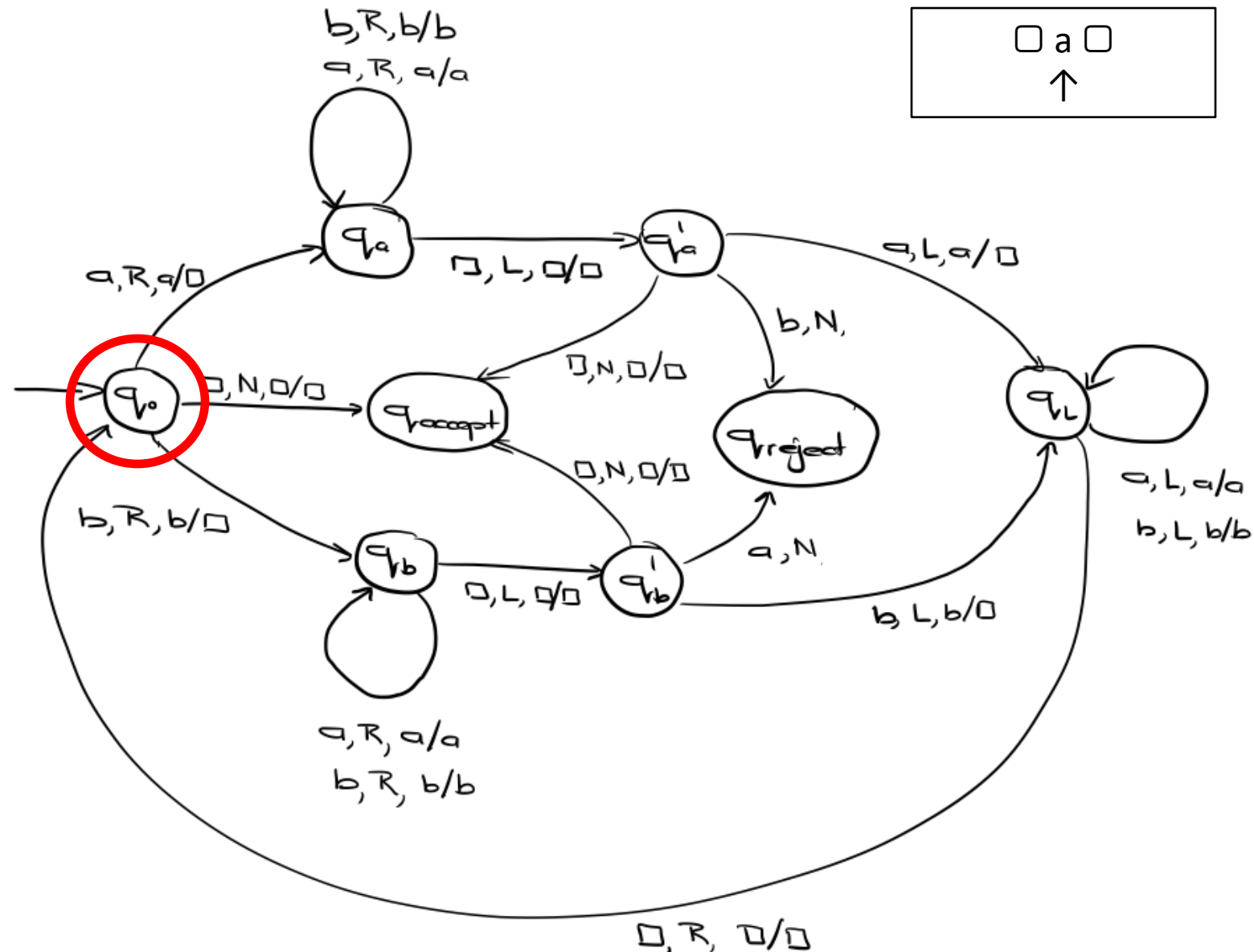
$$q'_b\square \rightarrow q_{accept}$$

$$q_La \rightarrow q_LaL$$

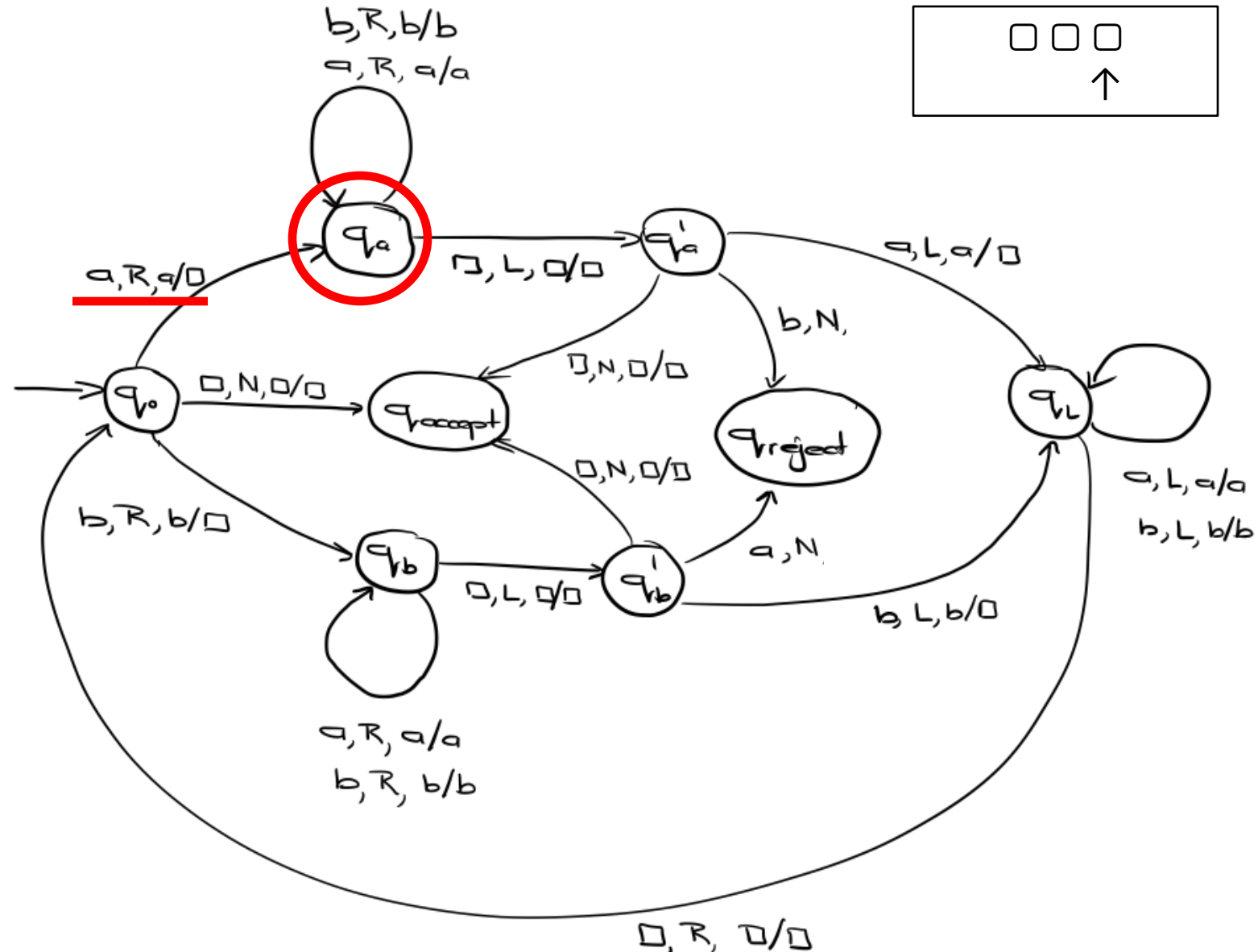
$$q_Lb \rightarrow q_LbL$$

$$q_L\square \rightarrow q_0\square R$$

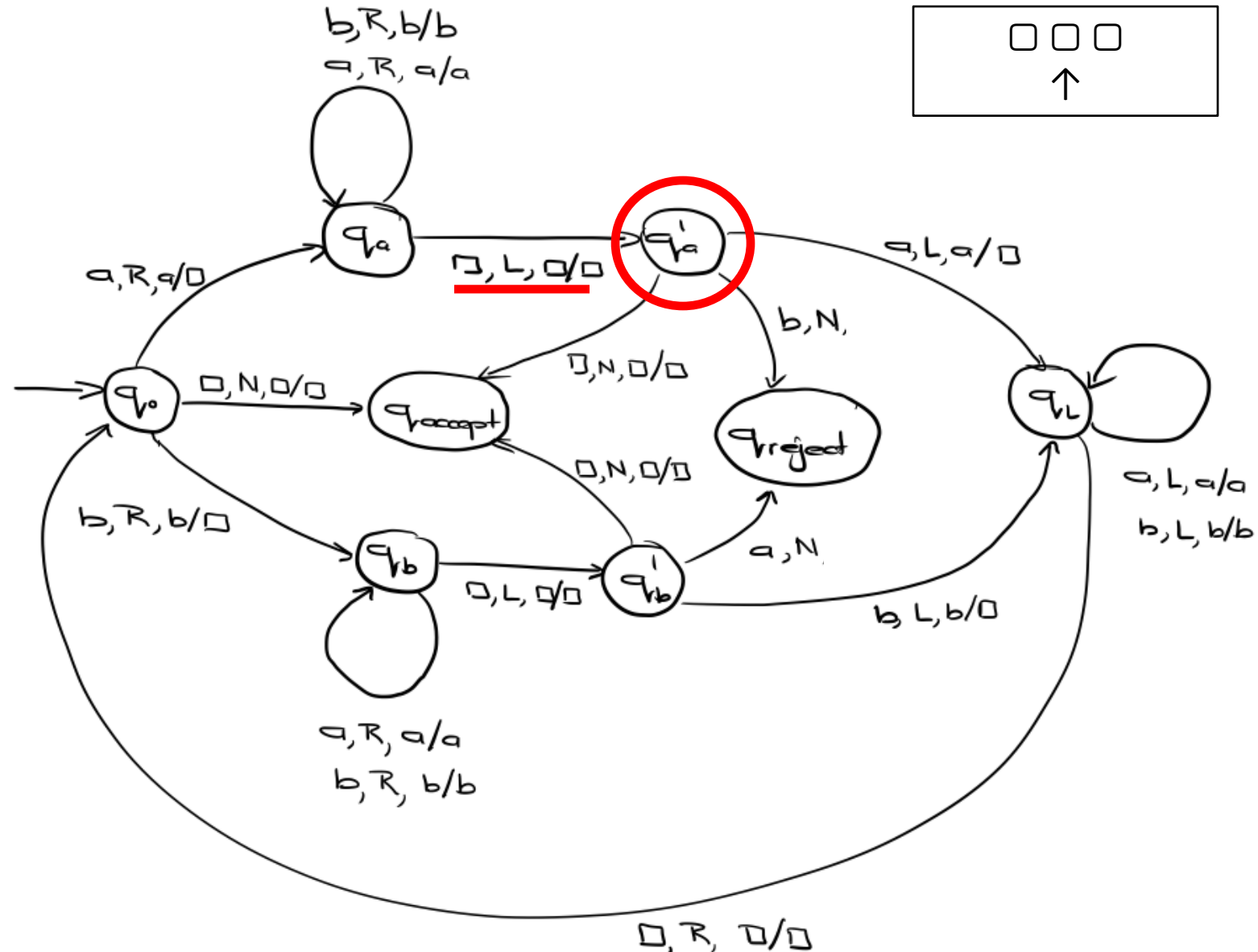
# One Tape Solution



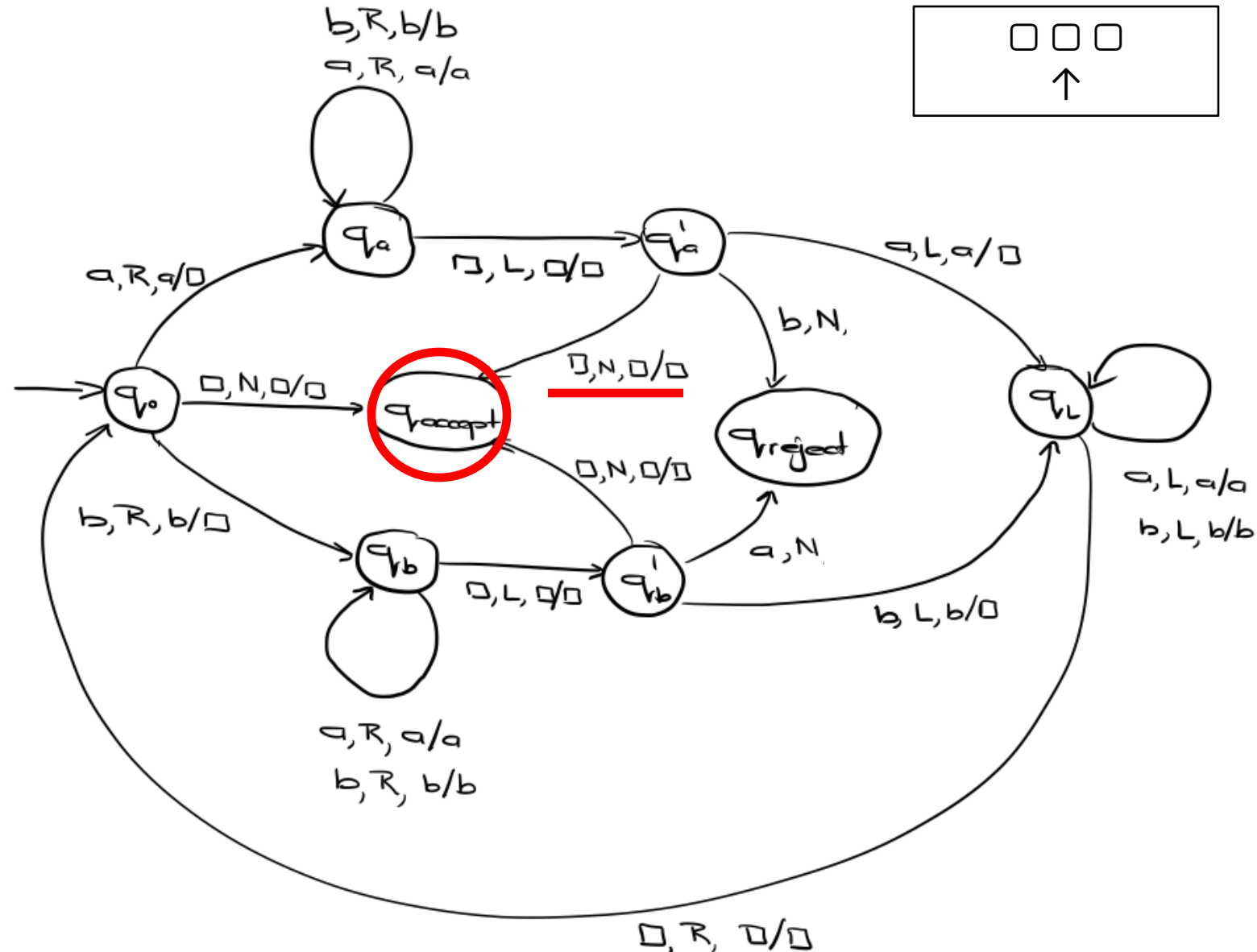
# One Tape Solution



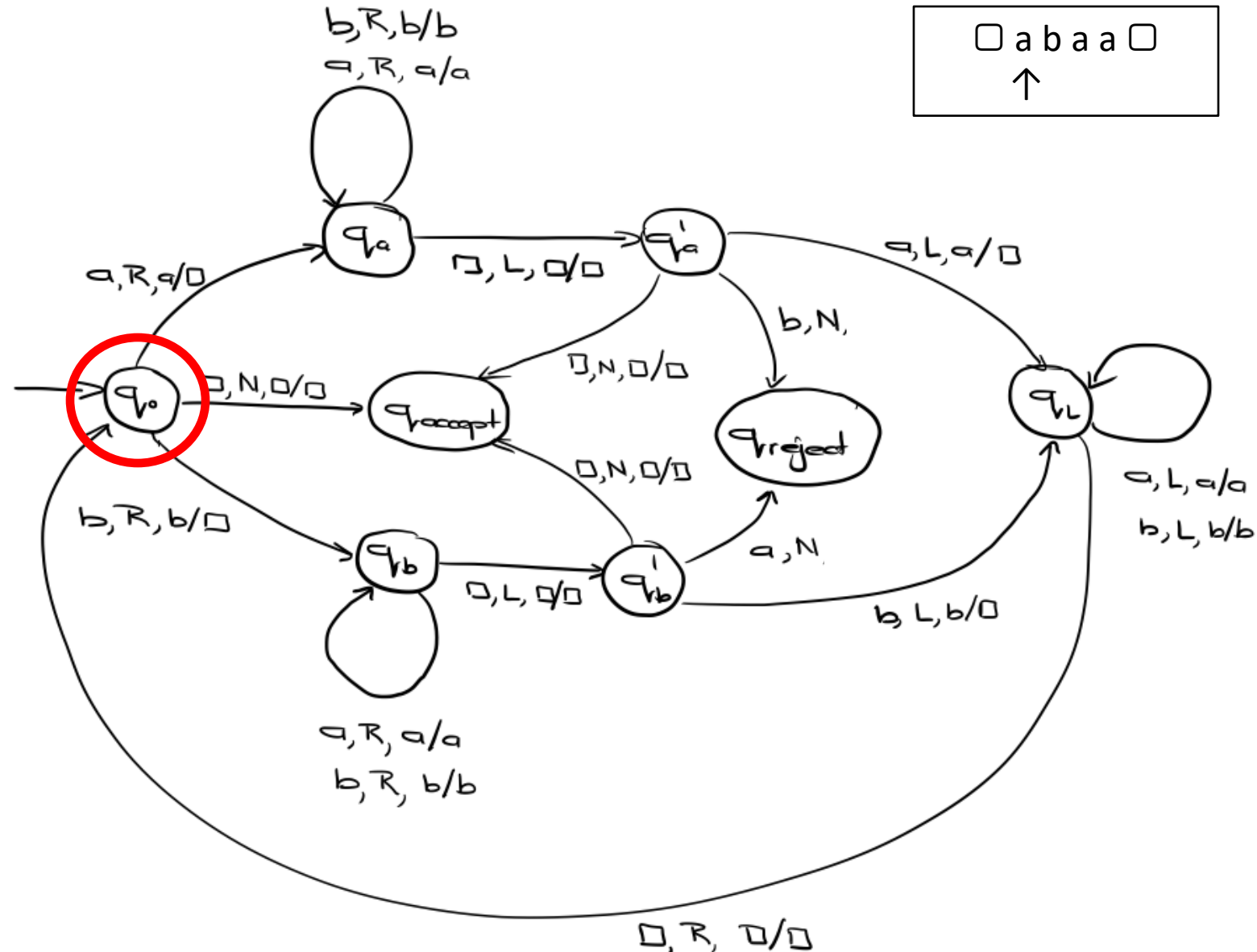
# One Tape Solution



# One Tape Solution

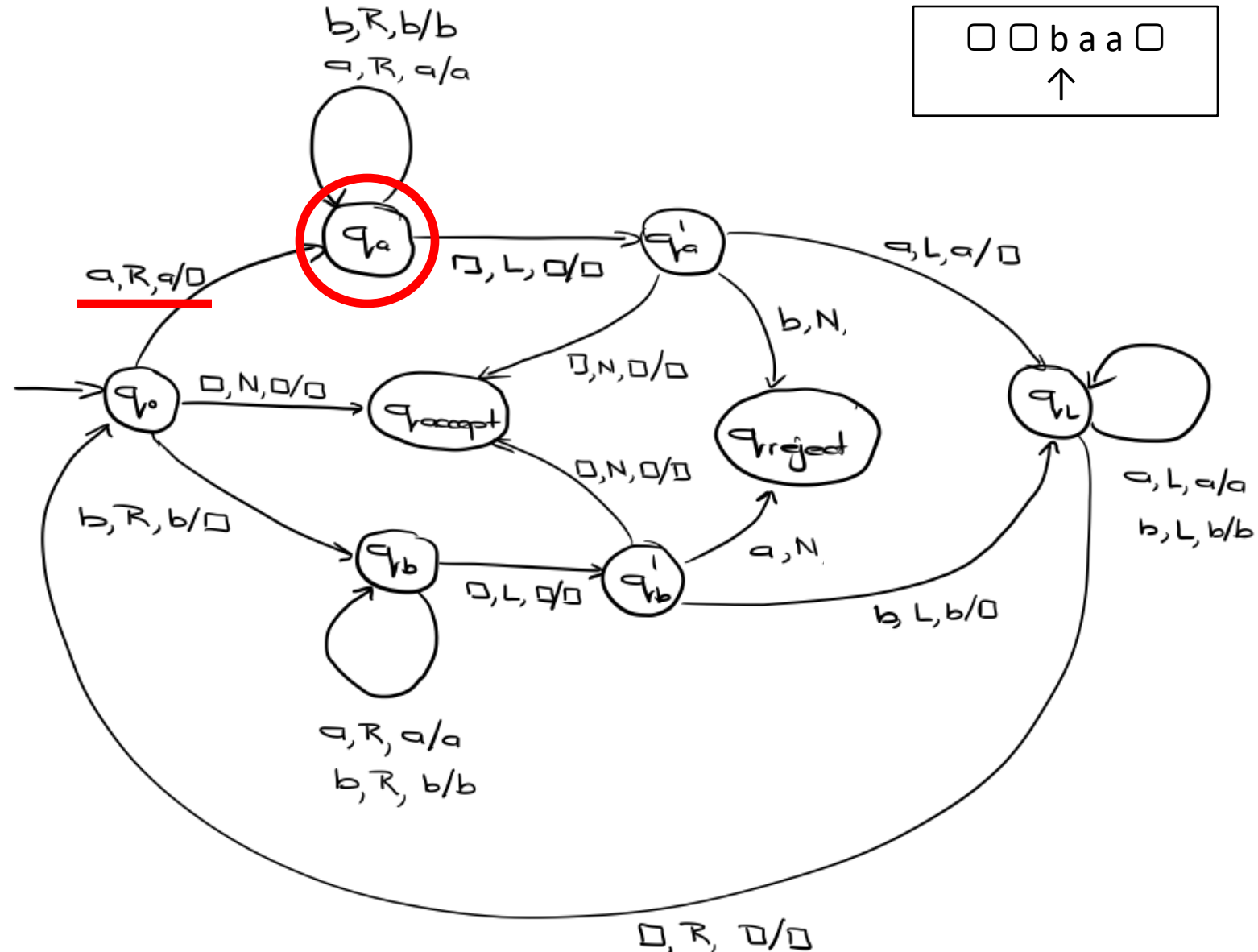


# One Tape Solution

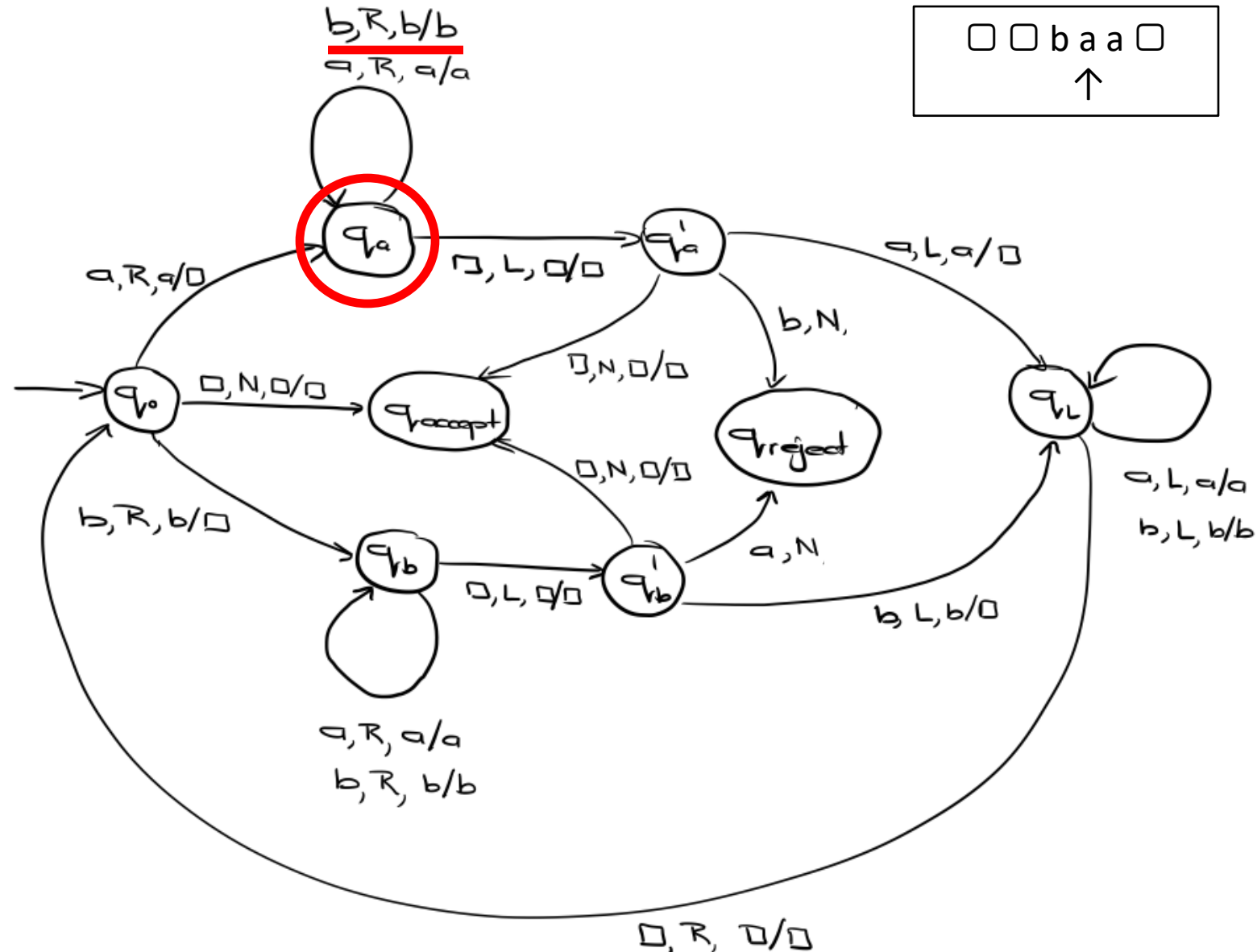




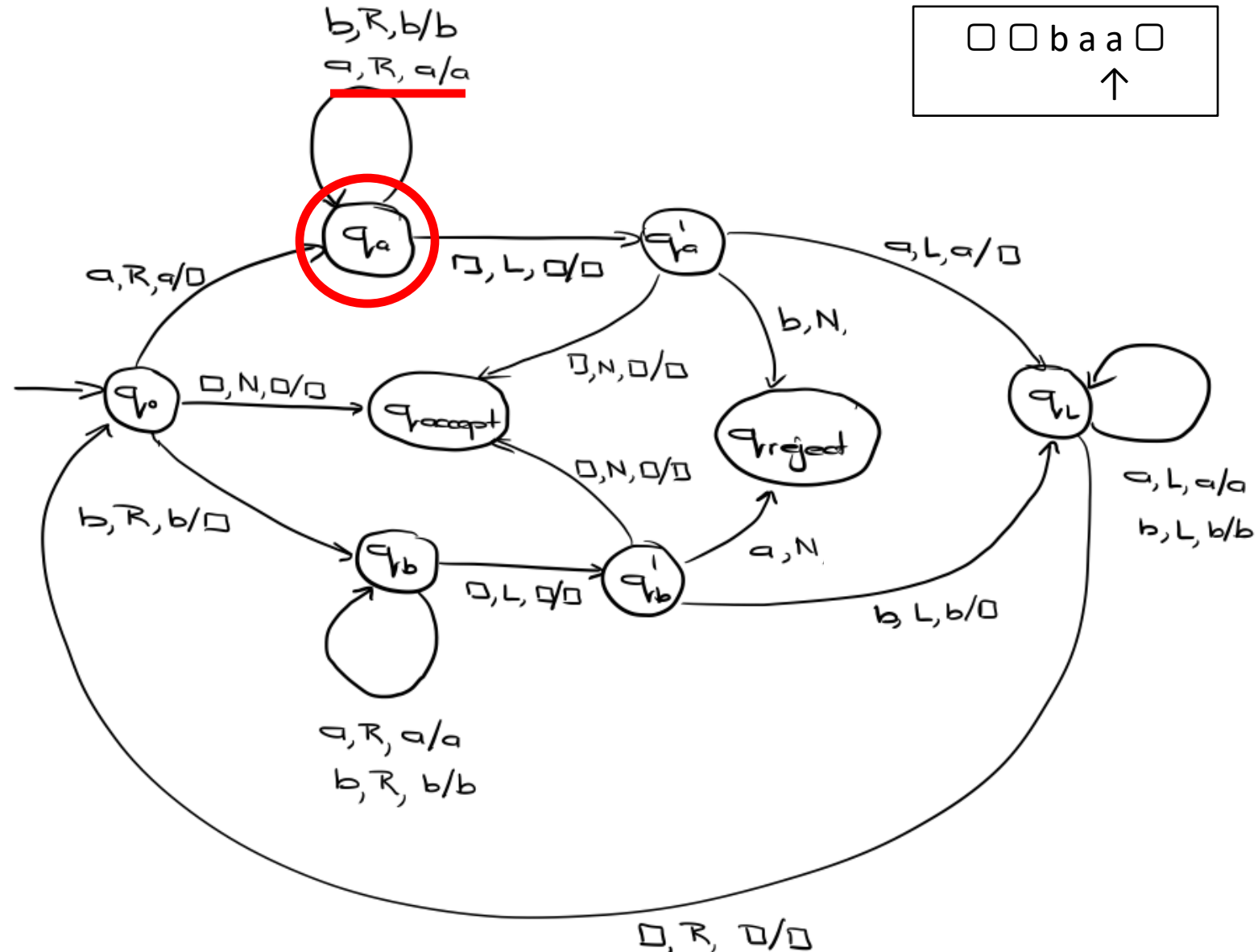
# One Tape Solution



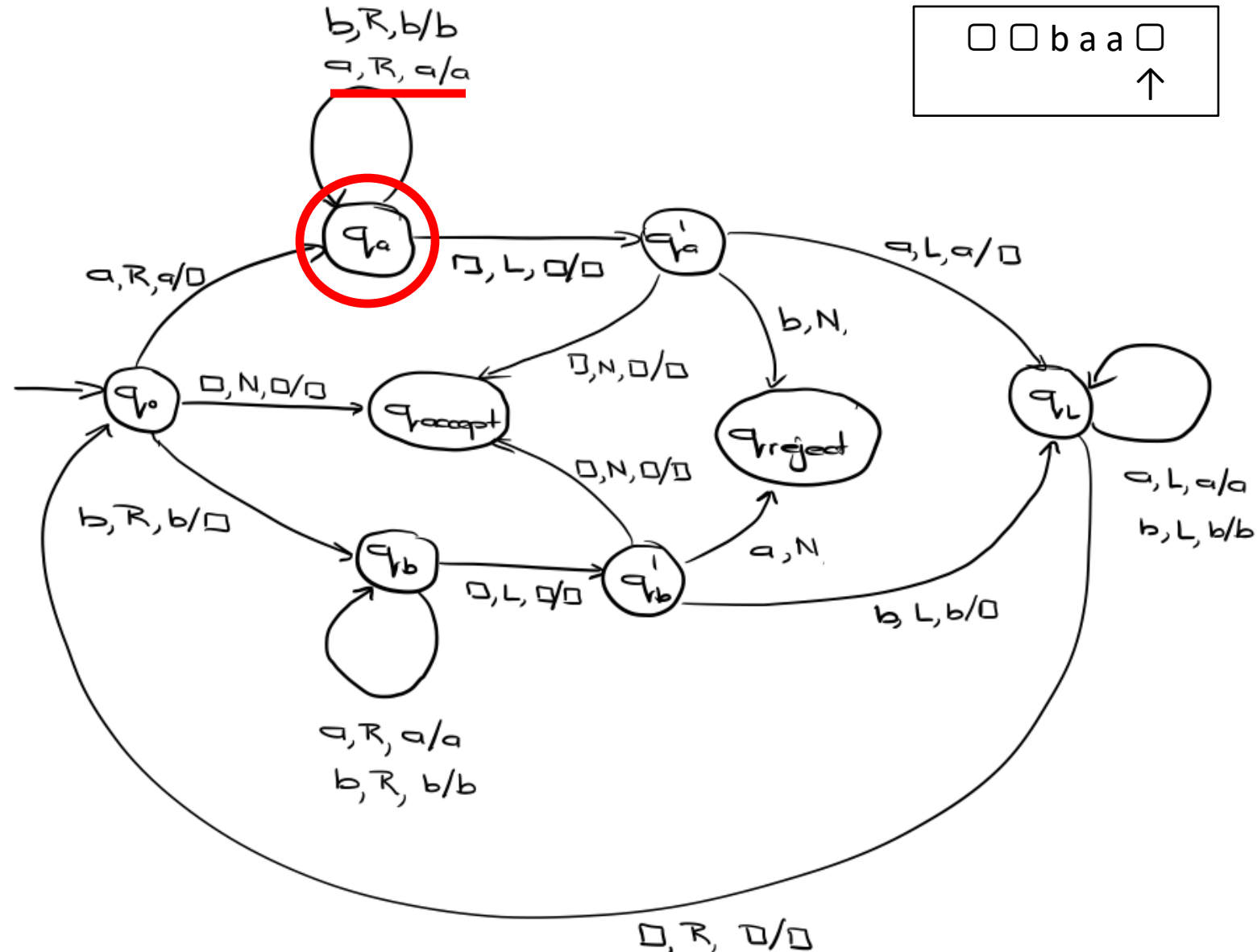
# One Tape Solution



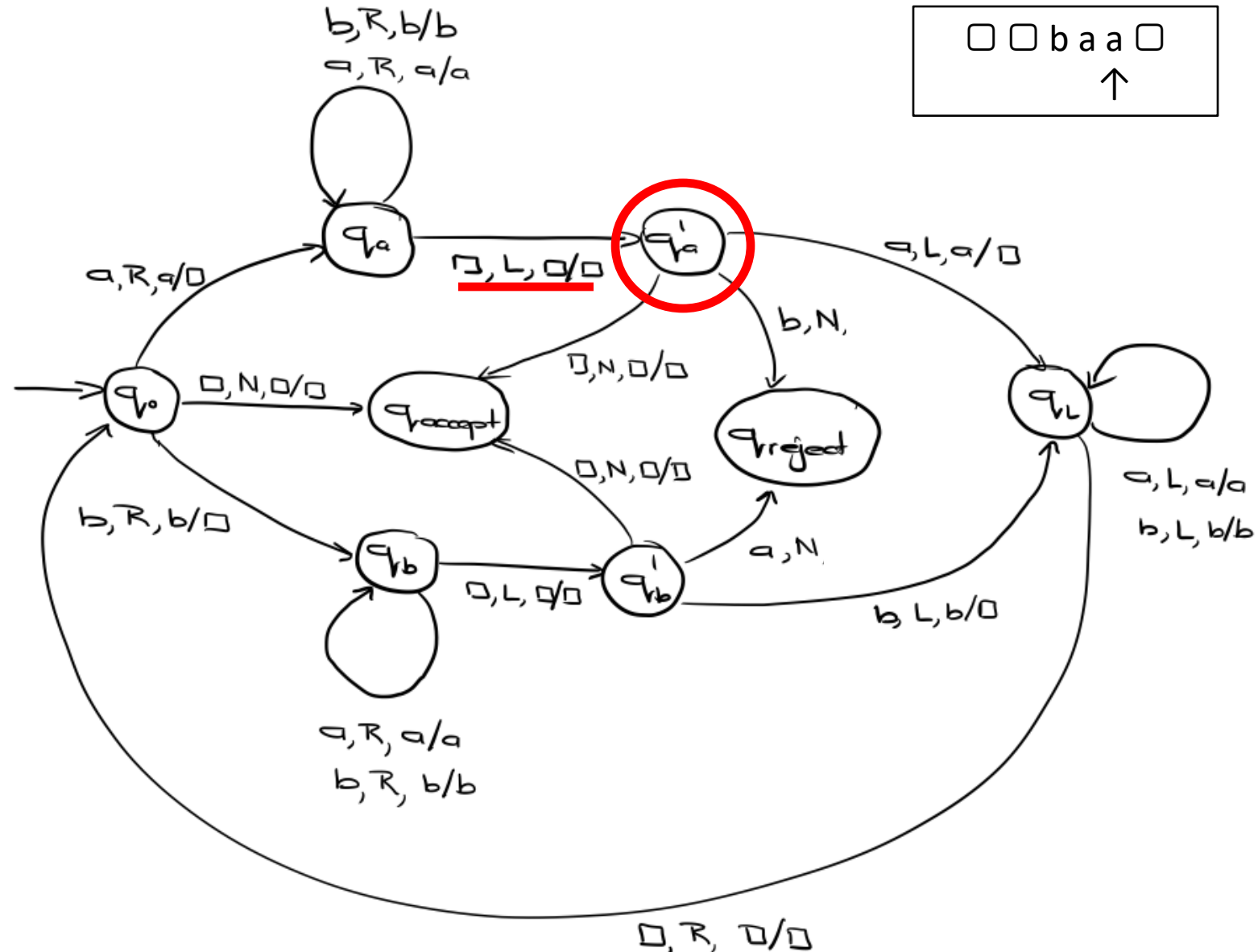
# One Tape Solution



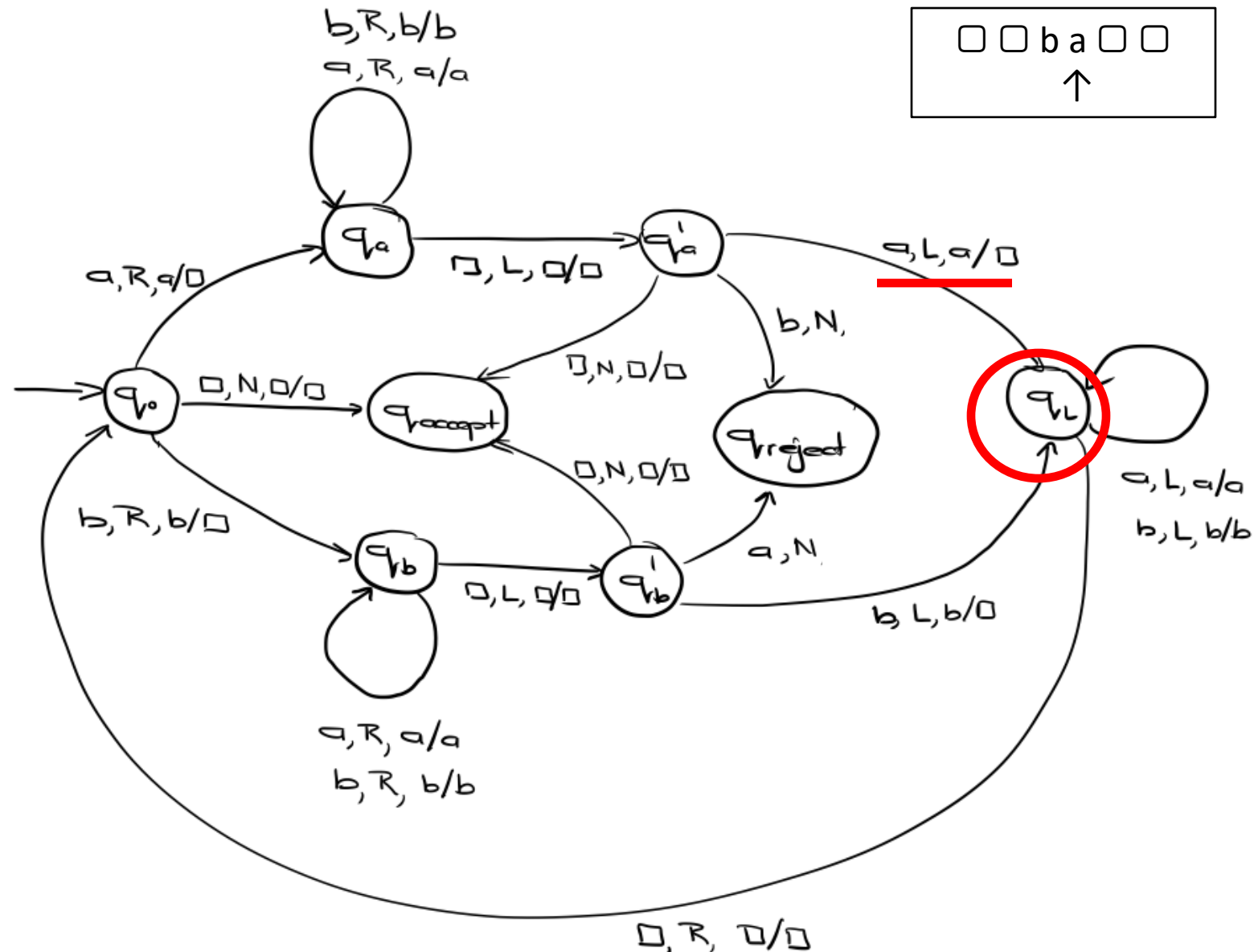
# One Tape Solution



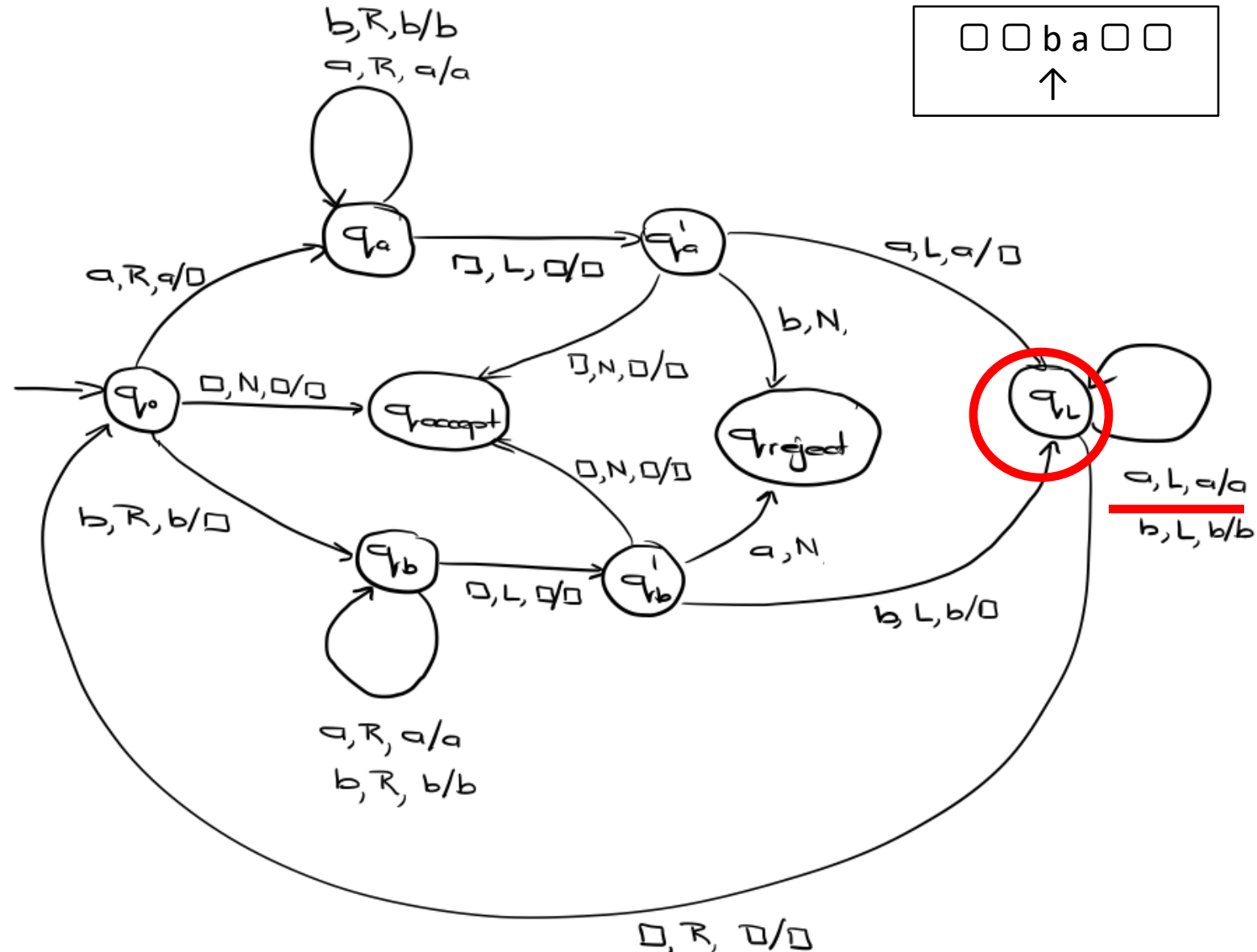
# One Tape Solution



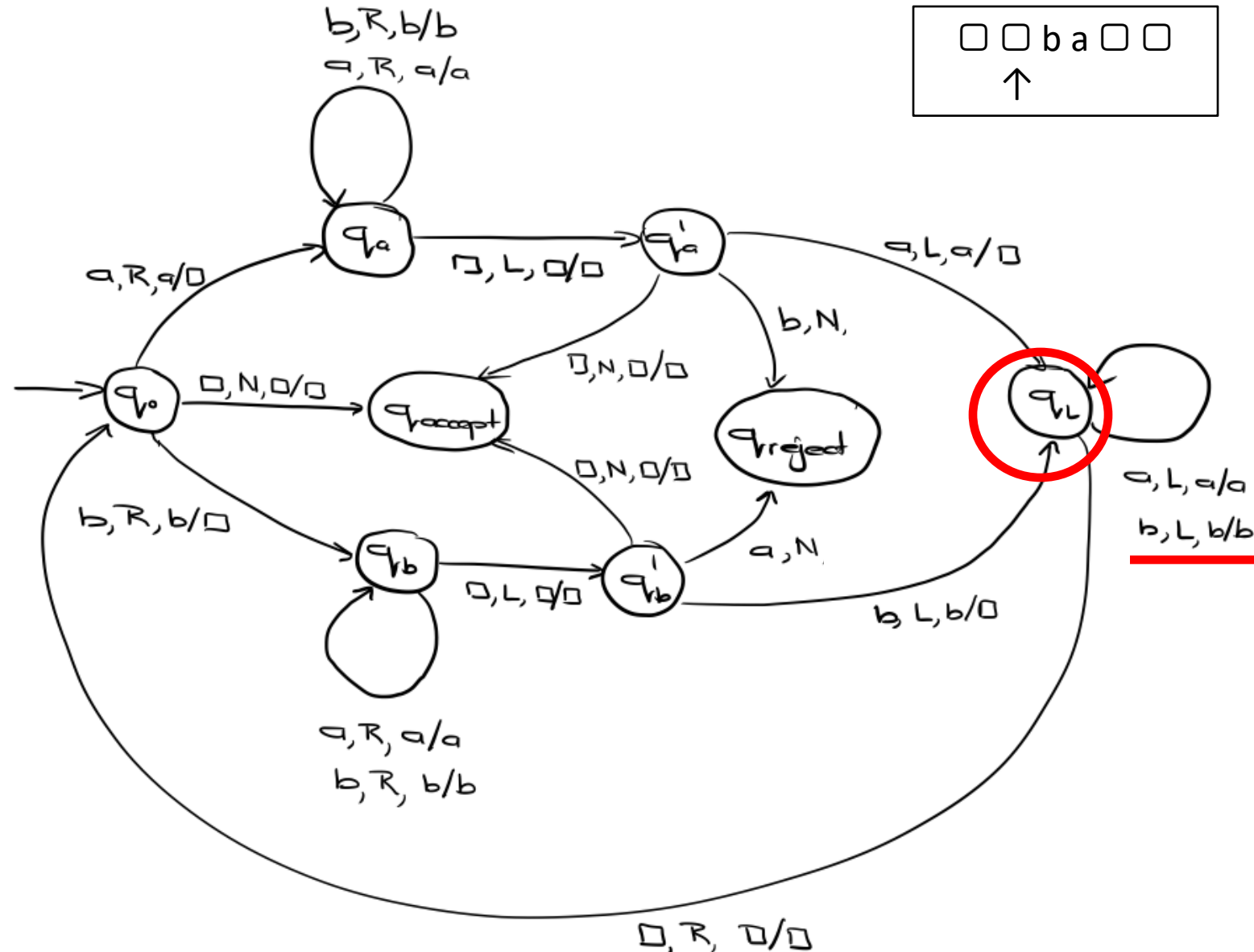
# One Tape Solution



# One Tape Solution

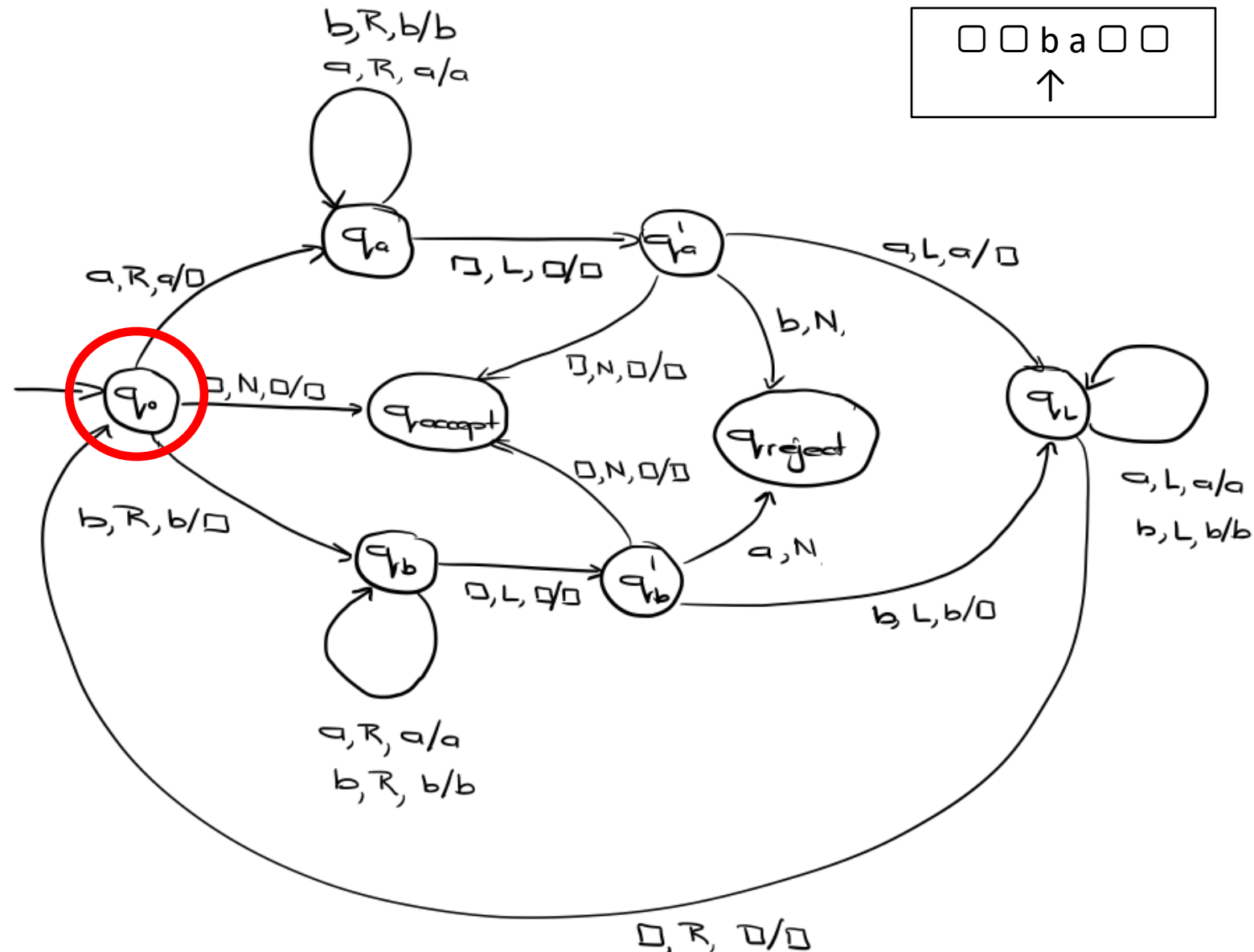


# One Tape Solution

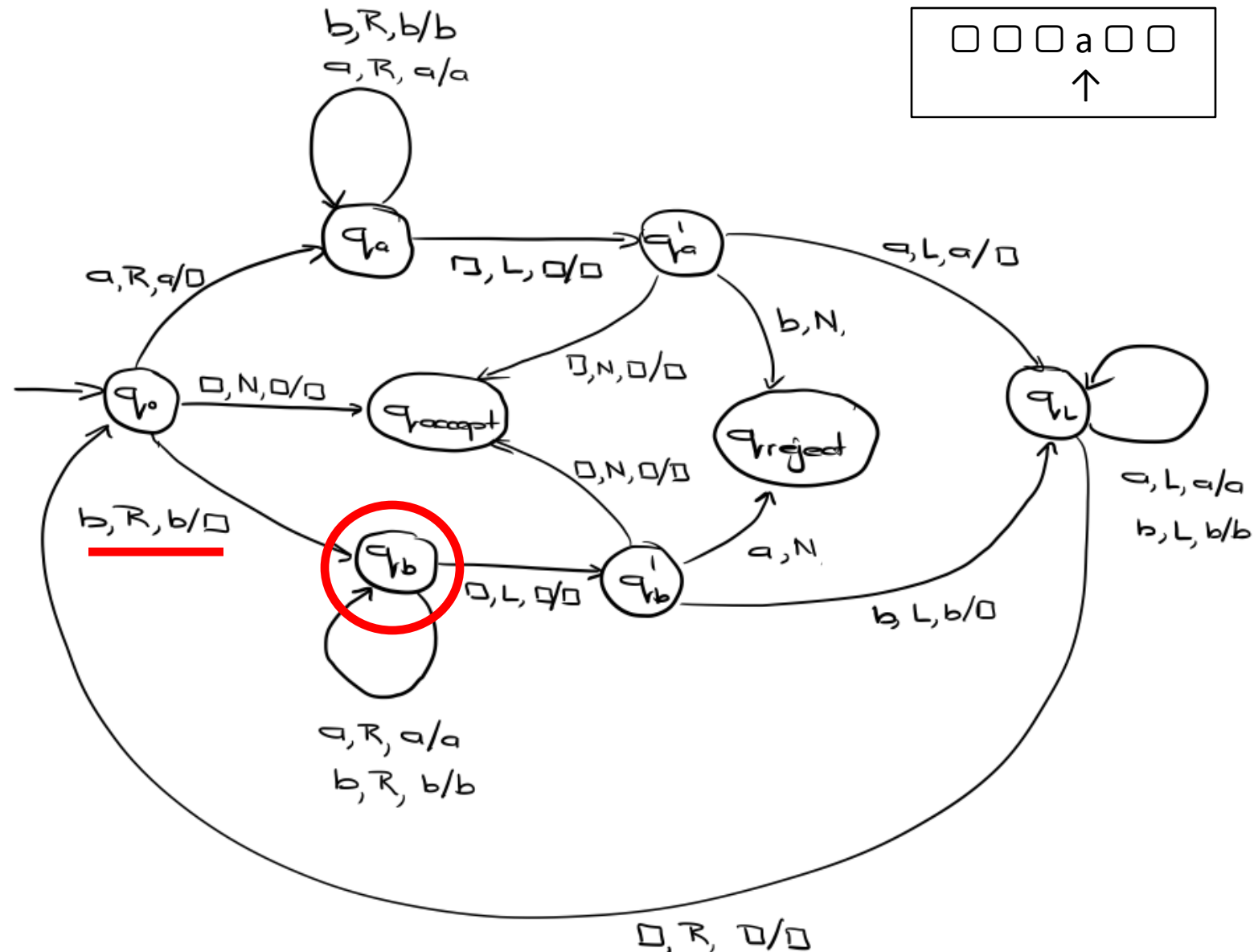




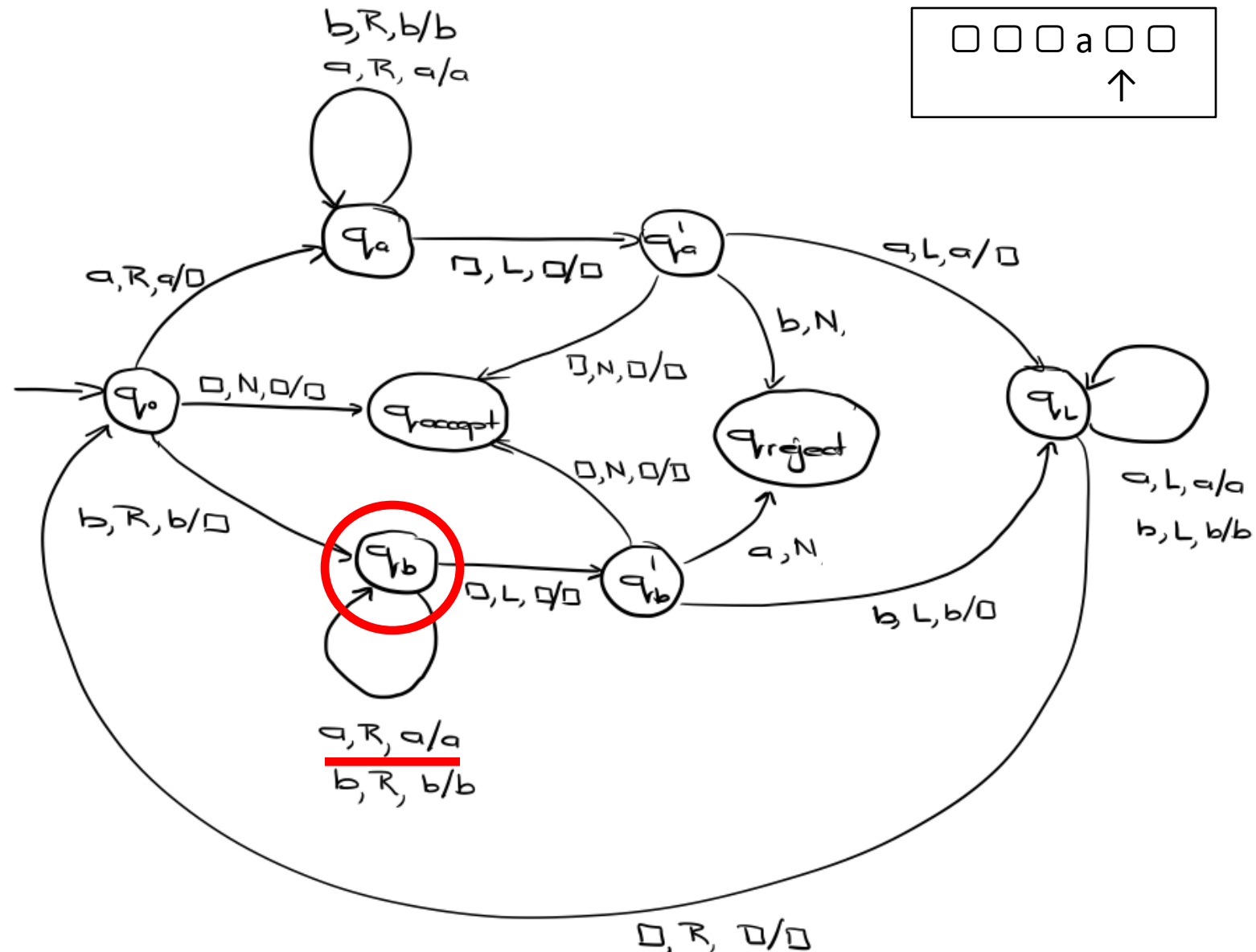
# One Tape Solution



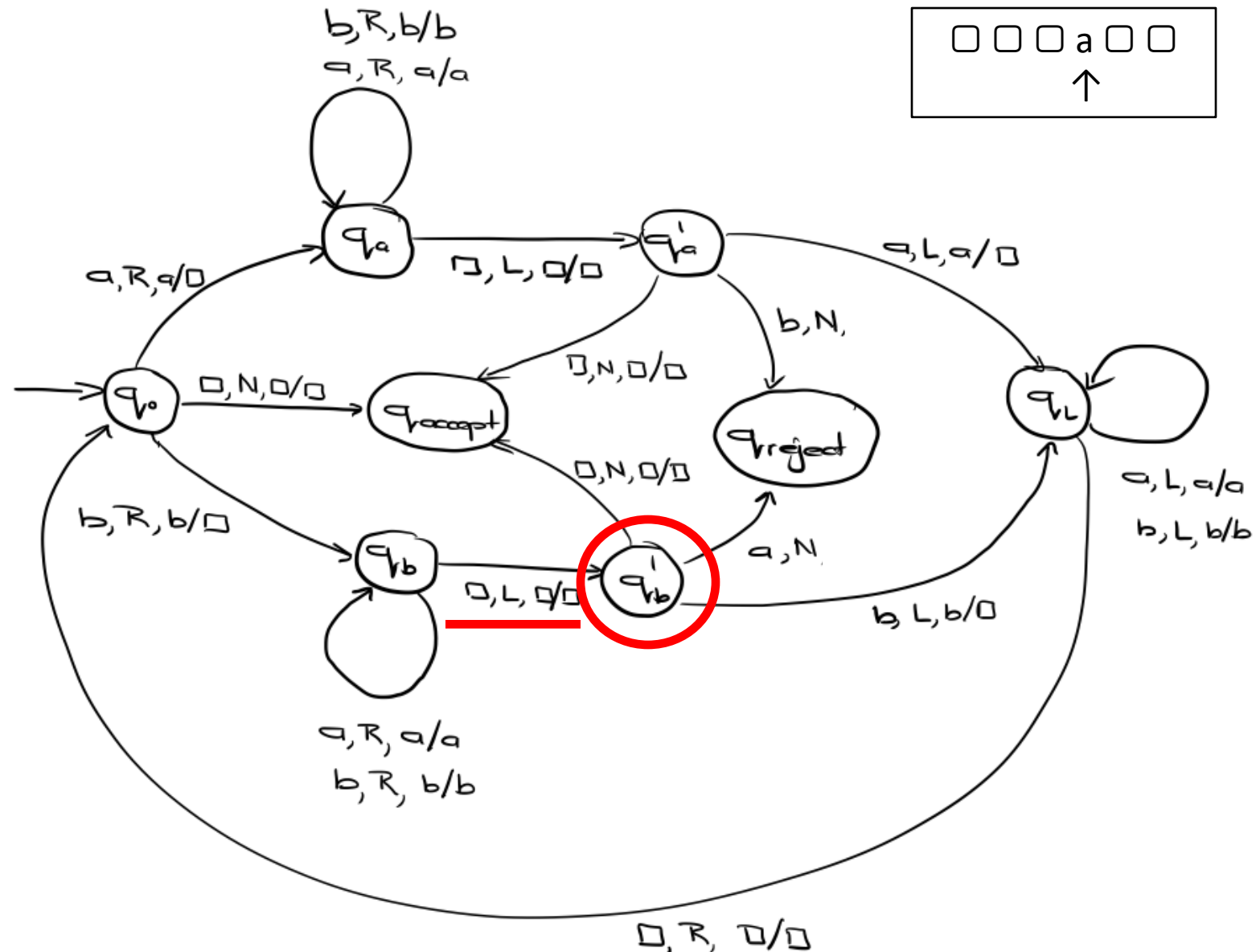
# One Tape Solution



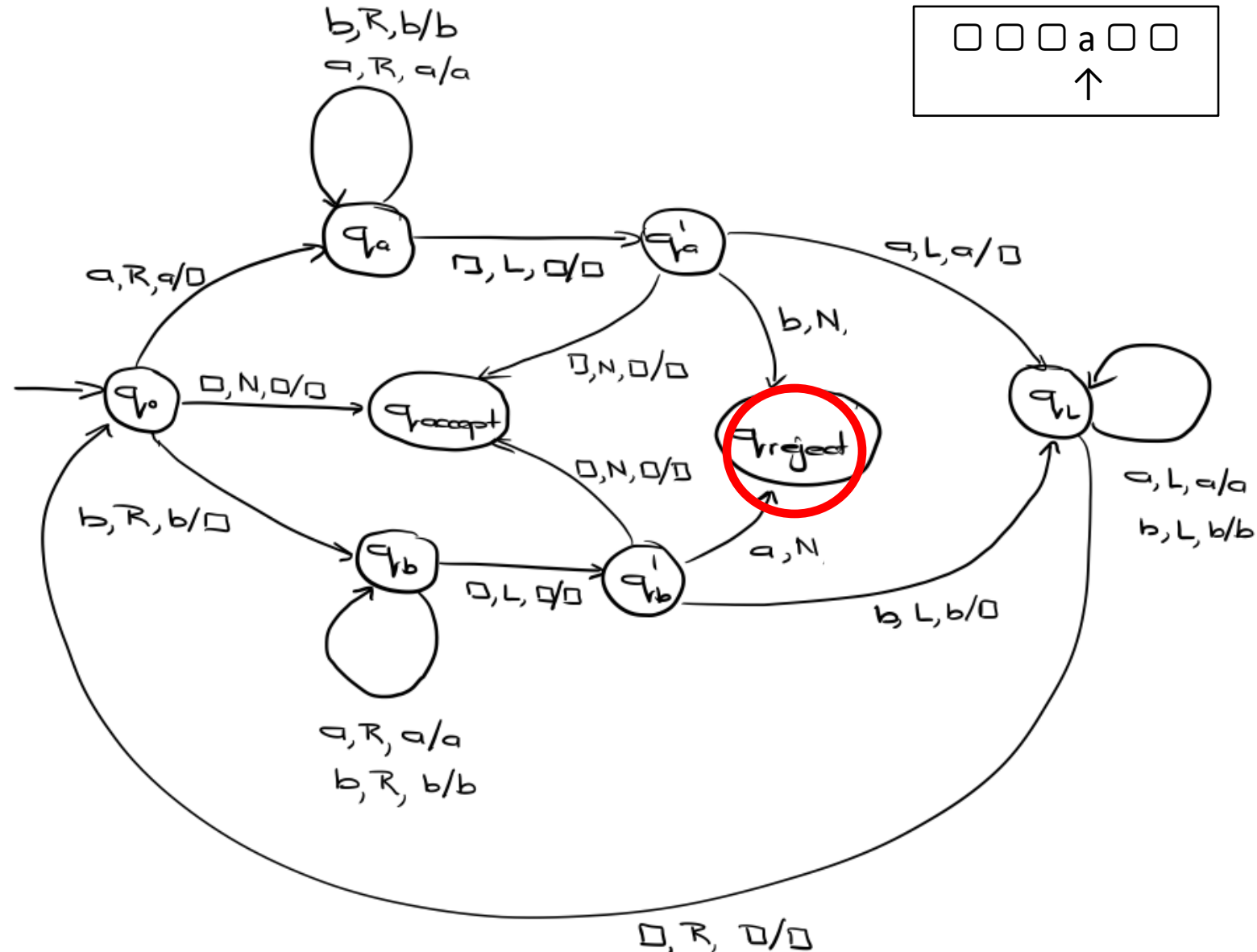
# One Tape Solution



# One Tape Solution



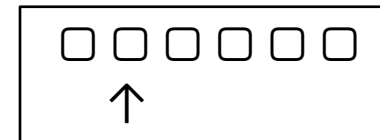
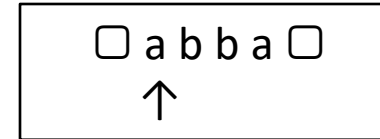
# One Tape Solution



# Two Tape Solution – Idea

## Initial configuration:

- The first tape contains the input string,  $w$
- The second tape is empty



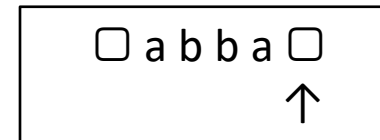
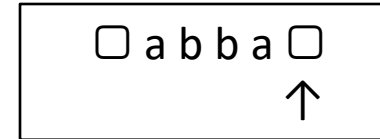
## Approach:

- Copy  $w$  to the second tape
- Move head of first tape back to leftmost symbol of  $w$
- Head of second tape remains at the rightmost symbol of  $w$
- Head of the first tape moves to the right at the same time as head of the second tape moves to the left
- In each step, check if the symbols at the heads of first and second tape are equal

# Two Tape Solution – Idea

## Initial configuration:

- The first tape contains the input string,  $w$
- The second tape is empty



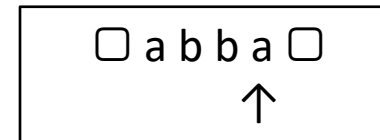
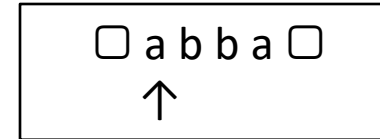
## Approach:

- Copy  $w$  to the second tape
- Move head of first tape back to leftmost symbol of  $w$
- Head of second tape remains at the rightmost symbol of  $w$
- Head of the first tape moves to the right at the same time as head of the second tape moves to the left
- In each step, check if the symbols at the heads of first and second tape are equal

# Two Tape Solution – Idea

## Initial configuration:

- The first tape contains the input string,  $w$
- The second tape is empty



## Approach:

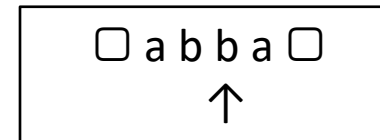
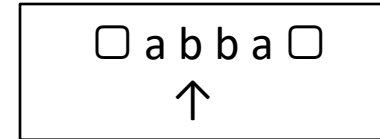
- Copy  $w$  to the second tape
- Move head of first tape back to leftmost symbol of  $w$
- Head of second tape remains at the rightmost symbol of  $w$
- Head of the first tape moves to the right at the same time as head of the second tape moves to the left
- In each step, check if the symbols at the heads of first and second tape are equal



# Two Tape Solution – Idea

## Initial configuration:

- The first tape contains the input string,  $w$
- The second tape is empty



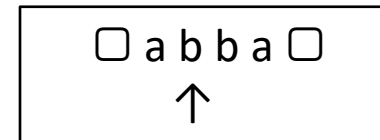
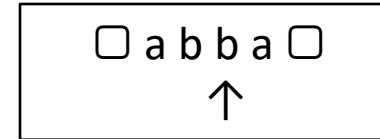
## Approach:

- Copy  $w$  to the second tape
- Move head of first tape back to leftmost symbol of  $w$
- Head of second tape remains at the rightmost symbol of  $w$
- Head of the first tape moves to the right at the same time as head of the second tape moves to the left
- In each step, check if the symbols at the heads of first and second tape are equal

# Two Tape Solution – Idea

## Initial configuration:

- The first tape contains the input string,  $w$
- The second tape is empty



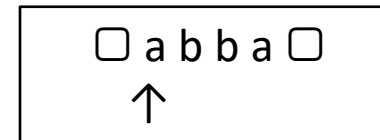
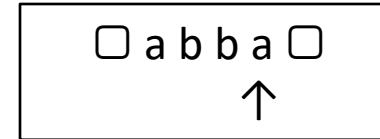
## Approach:

- Copy  $w$  to the second tape
- Move head of first tape back to leftmost symbol of  $w$
- Head of second tape remains at the rightmost symbol of  $w$
- Head of the first tape moves to the right at the same time as head of the second tape moves to the left
- In each step, check if the symbols at the heads of first and second tape are equal

# Two Tape Solution – Idea

## Initial configuration:

- The first tape contains the input string,  $w$
- The second tape is empty



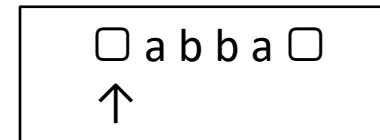
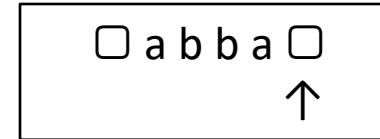
## Approach:

- Copy  $w$  to the second tape
- Move head of first tape back to leftmost symbol of  $w$
- Head of second tape remains at the rightmost symbol of  $w$
- Head of the first tape moves to the right at the same time as head of the second tape moves to the left
- In each step, check if the symbols at the heads of first and second tape are equal

# Two Tape Solution – Idea

## Initial configuration:

- The first tape contains the input string,  $w$
- The second tape is empty



## Approach:

- Copy  $w$  to the second tape
- Move head of first tape back to leftmost symbol of  $w$
- Head of second tape remains at the rightmost symbol of  $w$
- Head of the first tape moves to the right at the same time as head of the second tape moves to the left
- In each step, check if the symbols at the heads of first and second tape are equal

# Two Tape Solution – Details

- We use the input alphabet  $\Sigma = \{a, b\}$  and the tape alphabet  $\Gamma = \{a, b, \square\}$
- The set  $Q$  consists of the following states:

$q_0$  : initial state, copy  $w$  to the second tape

$q_1$  :  $w$  has been copied; head of first tape moves to left

$q_2$  : head of first tape moves to the right

at the same time as head of second tape moves to the left

remain in  $q_2$  as long as equality tests are positive

$q_{accept}$  : accept state

$q_{reject}$  : reject state

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 aaRR$$

$$q_0 b \square \rightarrow q_0 bbRR$$

$$q_0 \square \square \rightarrow q_1 \square \square LL$$

$$q_1 aa \rightarrow q_1 aaLN$$

$$q_1 ab \rightarrow q_1 abLN$$

$$q_1 ba \rightarrow q_1 baLN$$

$$q_1 bb \rightarrow q_1 bbLN$$

$$q_1 \square a \rightarrow q_2 \square aRN$$

$$q_1 \square b \rightarrow q_2 \square bRN$$

$$q_1 \square \square \rightarrow q_{accept}$$

$$q_2 aa \rightarrow q_2 aaRL$$

$$q_2 ab \rightarrow q_{reject}$$

$$q_2 ba \rightarrow q_{reject}$$

$$q_2 bb \rightarrow q_2 bbRL$$

$$q_2 \square \square \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 aaRR$$

$$q_0 b \square \rightarrow q_0 bbRR$$

$$q_0 \square \square \rightarrow q_1 \square \square LL$$

$$q_1 aa \rightarrow q_1 aaLN$$

$$q_1 ab \rightarrow q_1 abLN$$

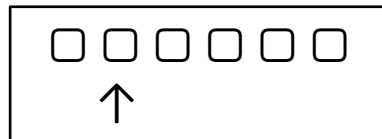
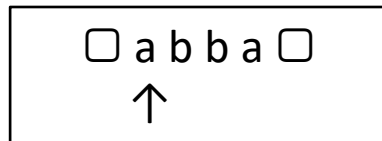
$$q_1 ba \rightarrow q_1 baLN$$

$$q_1 bb \rightarrow q_1 bbLN$$

$$q_1 \square a \rightarrow q_2 \square aRN$$

$$q_1 \square b \rightarrow q_2 \square bRN$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 aa \rightarrow q_2 aaRL$$

$$q_2 ab \rightarrow q_{reject}$$

$$q_2 ba \rightarrow q_{reject}$$

$$q_2 bb \rightarrow q_2 bbRL$$

$$q_2 \square \square \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 aaRR$$

$$q_0 b \square \rightarrow q_0 bbRR$$

$$q_0 \square \square \rightarrow q_1 \square \square LL$$

$$q_1 aa \rightarrow q_1 aaLN$$

$$q_1 ab \rightarrow q_1 abLN$$

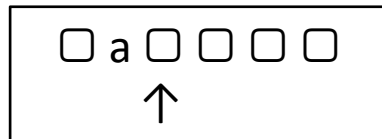
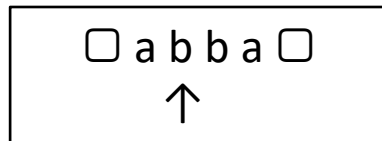
$$q_1 ba \rightarrow q_1 baLN$$

$$q_1 bb \rightarrow q_1 bbLN$$

$$q_1 \square a \rightarrow q_2 \square aRN$$

$$q_1 \square b \rightarrow q_2 \square bRN$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 aa \rightarrow q_2 aaRL$$

$$q_2 ab \rightarrow q_{reject}$$

$$q_2 ba \rightarrow q_{reject}$$

$$q_2 bb \rightarrow q_2 bbRL$$

$$q_2 \square \square \rightarrow q_{accept}$$



# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 aaRR$$

$$q_0 b \square \rightarrow q_0 bbRR$$

$$q_0 \square \square \rightarrow q_1 \square \square LL$$

$$q_1 aa \rightarrow q_1 aaLN$$

$$q_1 ab \rightarrow q_1 abLN$$

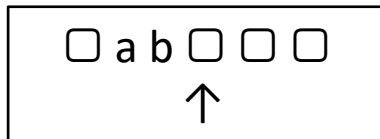
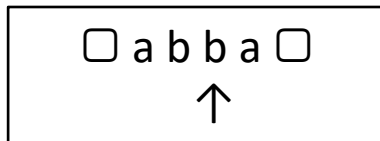
$$q_1 ba \rightarrow q_1 baLN$$

$$q_1 bb \rightarrow q_1 bbLN$$

$$q_1 \square a \rightarrow q_2 \square aRN$$

$$q_1 \square b \rightarrow q_2 \square bRN$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 aa \rightarrow q_2 aaRL$$

$$q_2 ab \rightarrow q_{reject}$$

$$q_2 ba \rightarrow q_{reject}$$

$$q_2 bb \rightarrow q_2 bbRL$$

$$q_2 \square \square \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 aaRR$$

$$q_0 b \square \rightarrow q_0 bbRR$$

$$q_0 \square \square \rightarrow q_1 \square \square LL$$

$$q_1 aa \rightarrow q_1 aaLN$$

$$q_1 ab \rightarrow q_1 abLN$$

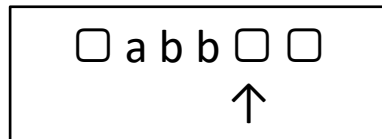
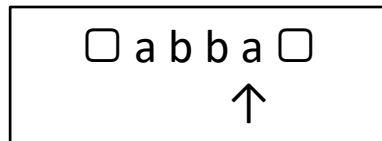
$$q_1 ba \rightarrow q_1 baLN$$

$$q_1 bb \rightarrow q_1 bbLN$$

$$q_1 \square a \rightarrow q_2 \square aRN$$

$$q_1 \square b \rightarrow q_2 \square bRN$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 aa \rightarrow q_2 aaRL$$

$$q_2 ab \rightarrow q_{reject}$$

$$q_2 ba \rightarrow q_{reject}$$

$$q_2 bb \rightarrow q_2 bbRL$$

$$q_2 \square \square \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 aaRR$$

$$q_0 b \square \rightarrow q_0 bbRR$$

$$q_0 \square \square \rightarrow q_1 \square \square LL$$

$$q_1 aa \rightarrow q_1 aaLN$$

$$q_1 ab \rightarrow q_1 abLN$$

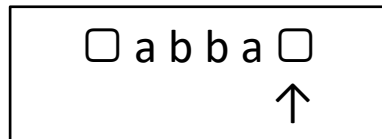
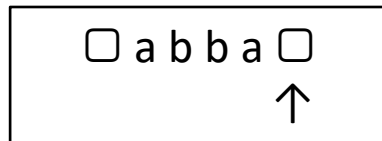
$$q_1 ba \rightarrow q_1 baLN$$

$$q_1 bb \rightarrow q_1 bbLN$$

$$q_1 \square a \rightarrow q_2 \square aRN$$

$$q_1 \square b \rightarrow q_2 \square bRN$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 aa \rightarrow q_2 aaRL$$

$$q_2 ab \rightarrow q_{reject}$$

$$q_2 ba \rightarrow q_{reject}$$

$$q_2 bb \rightarrow q_2 bbRL$$

$$q_2 \square \square \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 aaRR$$

$$q_0 b \square \rightarrow q_0 bbRR$$

$$q_0 \square \square \rightarrow q_1 \square \square LL$$

$$q_1 aa \rightarrow q_1 aaLN$$

$$q_1 ab \rightarrow q_1 abLN$$

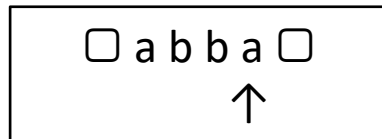
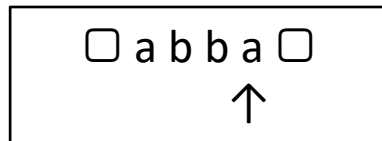
$$q_1 ba \rightarrow q_1 baLN$$

$$q_1 bb \rightarrow q_1 bbLN$$

$$q_1 \square a \rightarrow q_2 \square aRN$$

$$q_1 \square b \rightarrow q_2 \square bRN$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 aa \rightarrow q_2 aaRL$$

$$q_2 ab \rightarrow q_{reject}$$

$$q_2 ba \rightarrow q_{reject}$$

$$q_2 bb \rightarrow q_2 bbRL$$

$$q_2 \square \square \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 a a R R$$

$$q_0 b \square \rightarrow q_0 b b R R$$

$$q_0 \square \square \rightarrow q_1 \square \square L L$$

$$q_1 a a \rightarrow q_1 a a L N$$

$$q_1 a b \rightarrow q_1 a b L N$$

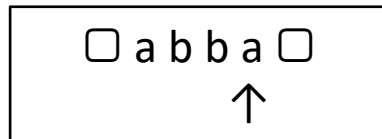
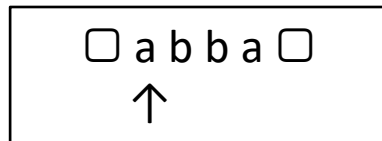
$$q_1 b a \rightarrow q_1 b a L N$$

$$q_1 b b \rightarrow q_1 b b L N$$

$$q_1 \square a \rightarrow q_2 \square a R N$$

$$q_1 \square b \rightarrow q_2 \square b R N$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 a a \rightarrow q_2 a a R L$$

$$q_2 a b \rightarrow q_{reject}$$

$$q_2 b a \rightarrow q_{reject}$$

$$q_2 b b \rightarrow q_2 b b R L$$

$$q_2 \square \square \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0a\Box \rightarrow q_0aaRR$$

$$q_0b\Box \rightarrow q_0bbRR$$

$$q_0\Box\Box \rightarrow q_1\Box\Box LL$$

$$q_1aa \rightarrow q_1aaLN$$

$$q_1ab \rightarrow q_1abLN$$

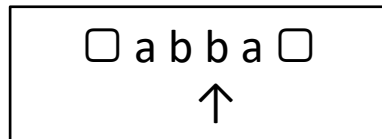
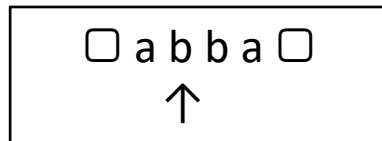
$$q_1ba \rightarrow q_1baLN$$

$$q_1bb \rightarrow q_1bbLN$$

$$q_1\Box a \rightarrow q_2\Box aRN$$

$$q_1\Box b \rightarrow q_2\Box bRN$$

$$q_1\Box\Box \rightarrow q_{accept}$$



$$q_2aa \rightarrow q_2aaRL$$

$$q_2ab \rightarrow q_{reject}$$

$$q_2ba \rightarrow q_{reject}$$

$$q_2bb \rightarrow q_2bbRL$$

$$q_2\Box\Box \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 a a R R$$

$$q_0 b \square \rightarrow q_0 b b R R$$

$$q_0 \square \square \rightarrow q_1 \square \square L L$$

$$q_1 a a \rightarrow q_1 a a L N$$

$$q_1 a b \rightarrow q_1 a b L N$$

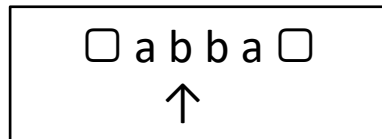
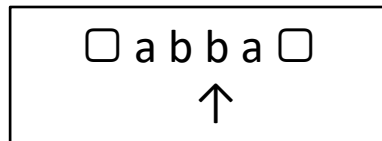
$$q_1 b a \rightarrow q_1 b a L N$$

$$q_1 b b \rightarrow q_1 b b L N$$

$$q_1 \square a \rightarrow q_2 \square a R N$$

$$q_1 \square b \rightarrow q_2 \square b R N$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 a a \rightarrow q_2 a a R L$$

$$q_2 a b \rightarrow q_{reject}$$

$$q_2 b a \rightarrow q_{reject}$$

$$q_2 b b \rightarrow q_2 b b R L$$

$$q_2 \square \square \rightarrow q_{accept}$$

# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 a a R R$$

$$q_0 b \square \rightarrow q_0 b b R R$$

$$q_0 \square \square \rightarrow q_1 \square \square L L$$

$$q_1 a a \rightarrow q_1 a a L N$$

$$q_1 a b \rightarrow q_1 a b L N$$

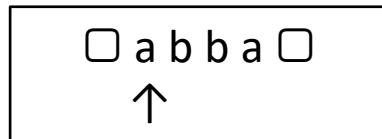
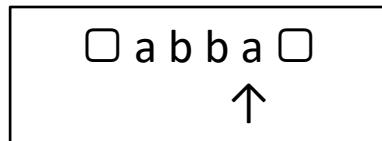
$$q_1 b a \rightarrow q_1 b a L N$$

$$q_1 b b \rightarrow q_1 b b L N$$

$$q_1 \square a \rightarrow q_2 \square a R N$$

$$q_1 \square b \rightarrow q_2 \square b R N$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 a a \rightarrow q_2 a a R L$$

$$q_2 a b \rightarrow q_{reject}$$

$$q_2 b a \rightarrow q_{reject}$$

$$q_2 b b \rightarrow q_2 b b R L$$

$$q_2 \square \square \rightarrow q_{accept}$$



# Two Tape Solution – Details

The transition function  $\delta$  is defined by the following instructions:

$$q_0 a \square \rightarrow q_0 a a R R$$

$$q_0 b \square \rightarrow q_0 b b R R$$

$$q_0 \square \square \rightarrow q_1 \square \square L L$$

$$q_1 a a \rightarrow q_1 a a L N$$

$$q_1 a b \rightarrow q_1 a b L N$$

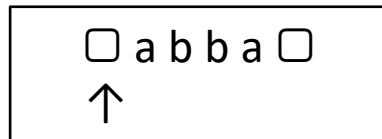
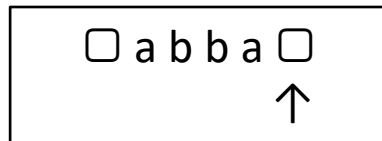
$$q_1 b a \rightarrow q_1 b a L N$$

$$q_1 b b \rightarrow q_1 b b L N$$

$$q_1 \square a \rightarrow q_2 \square a R N$$

$$q_1 \square b \rightarrow q_2 \square b R N$$

$$q_1 \square \square \rightarrow q_{accept}$$



$$q_2 a a \rightarrow q_2 a a R L$$

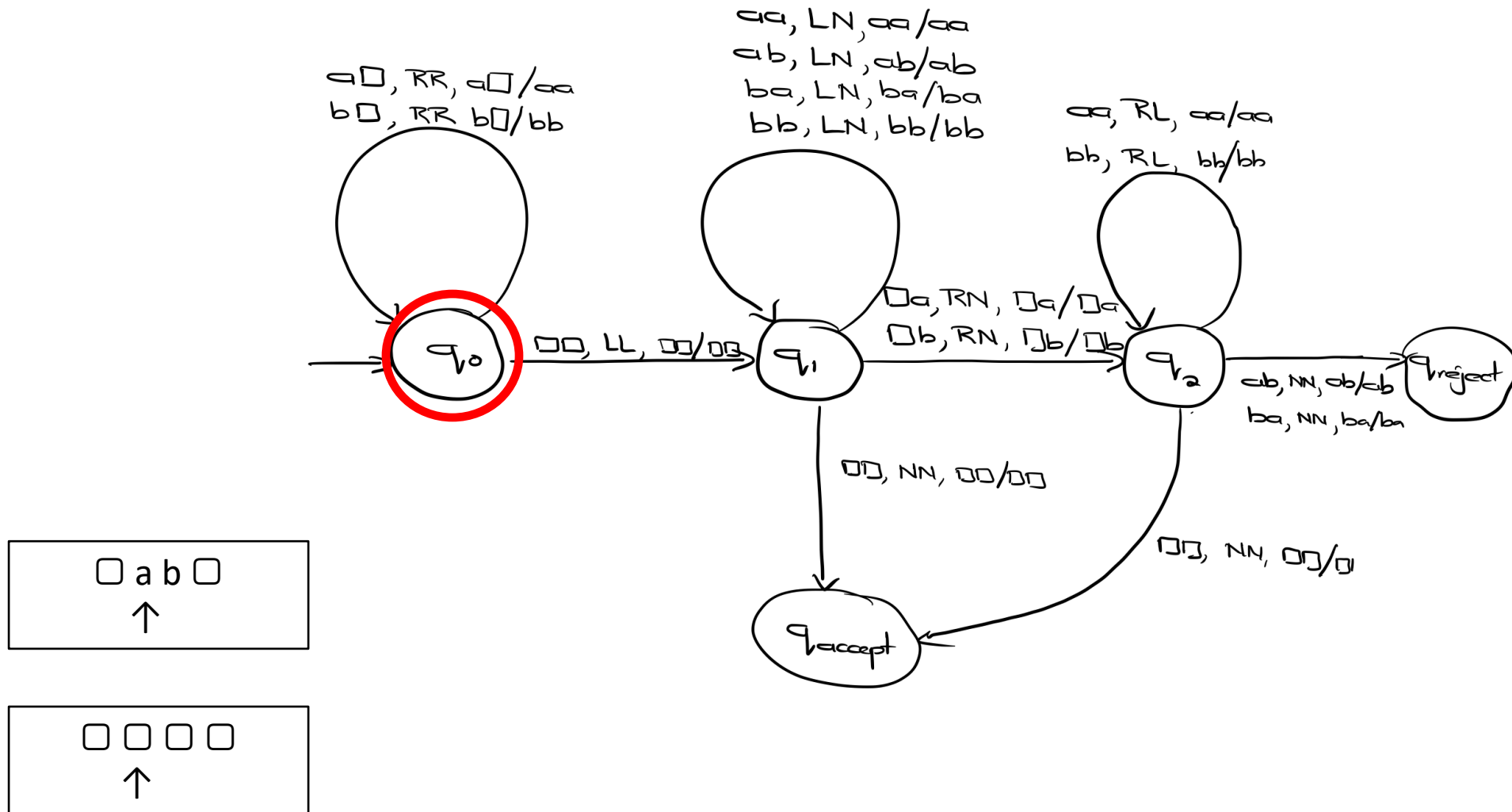
$$q_2 a b \rightarrow q_{reject}$$

$$q_2 b a \rightarrow q_{reject}$$

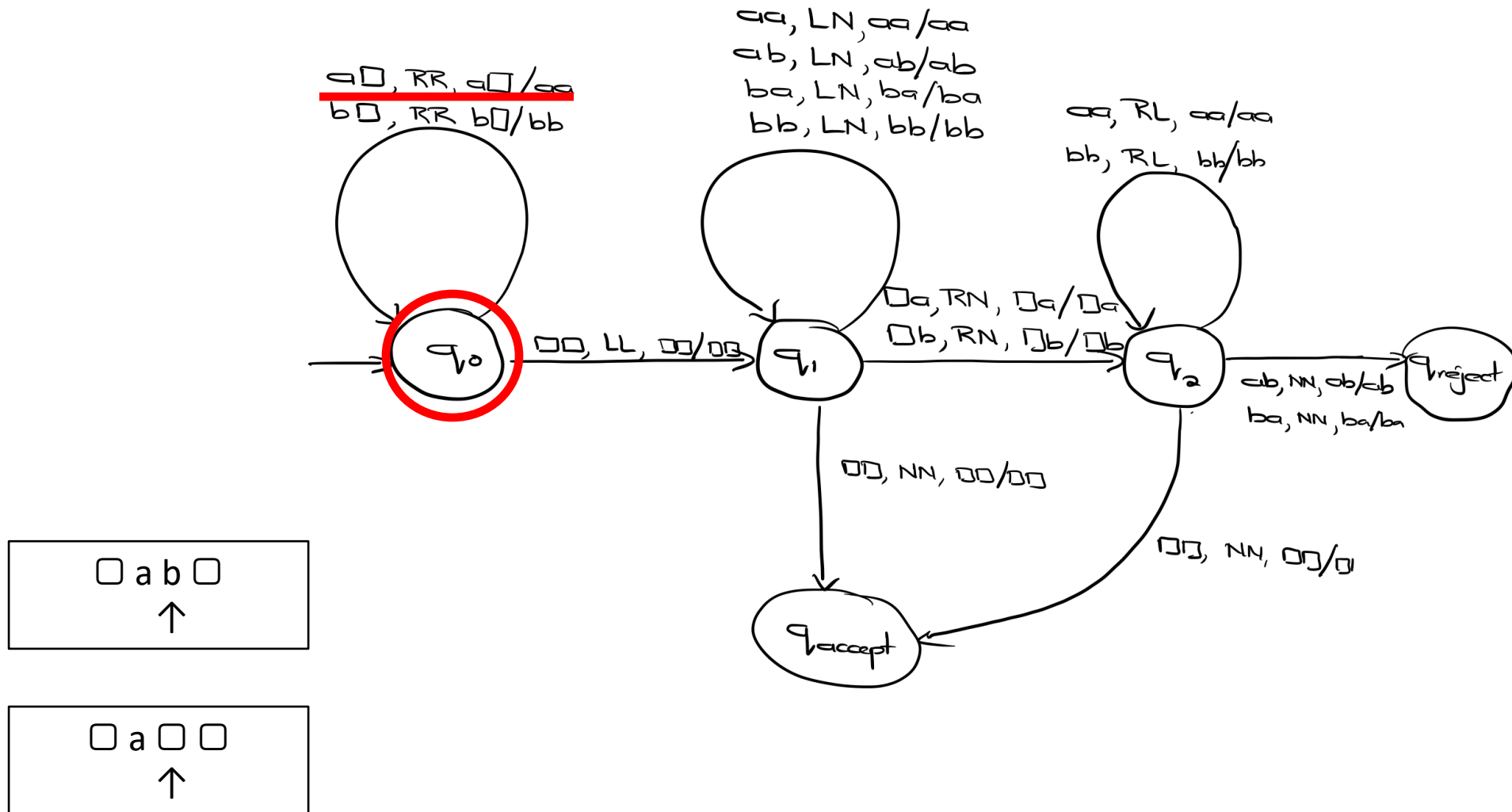
$$q_2 b b \rightarrow q_2 b b R L$$

$$q_2 \square \square \rightarrow q_{accept}$$

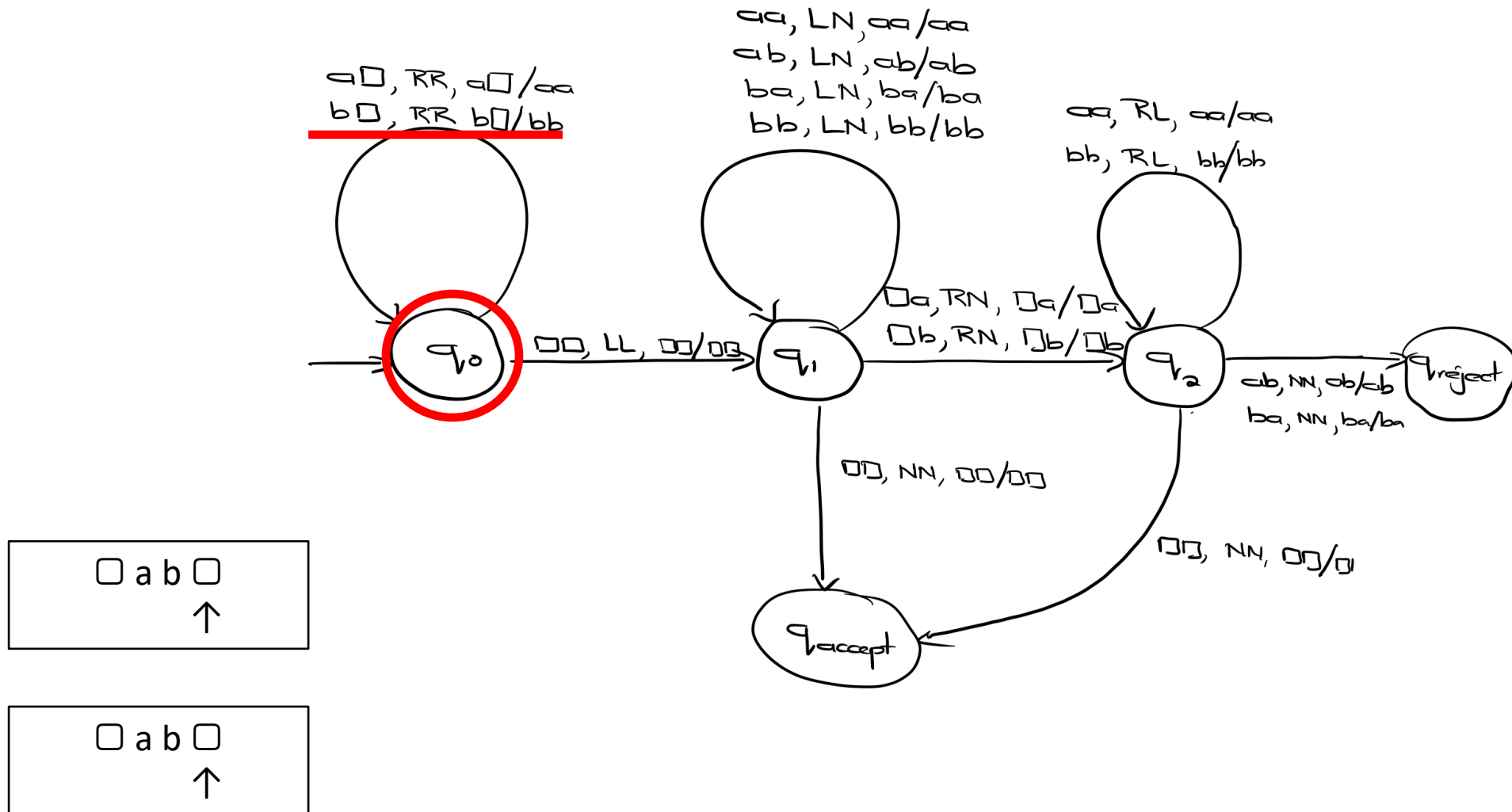
# Two Tape Solution – Illustration



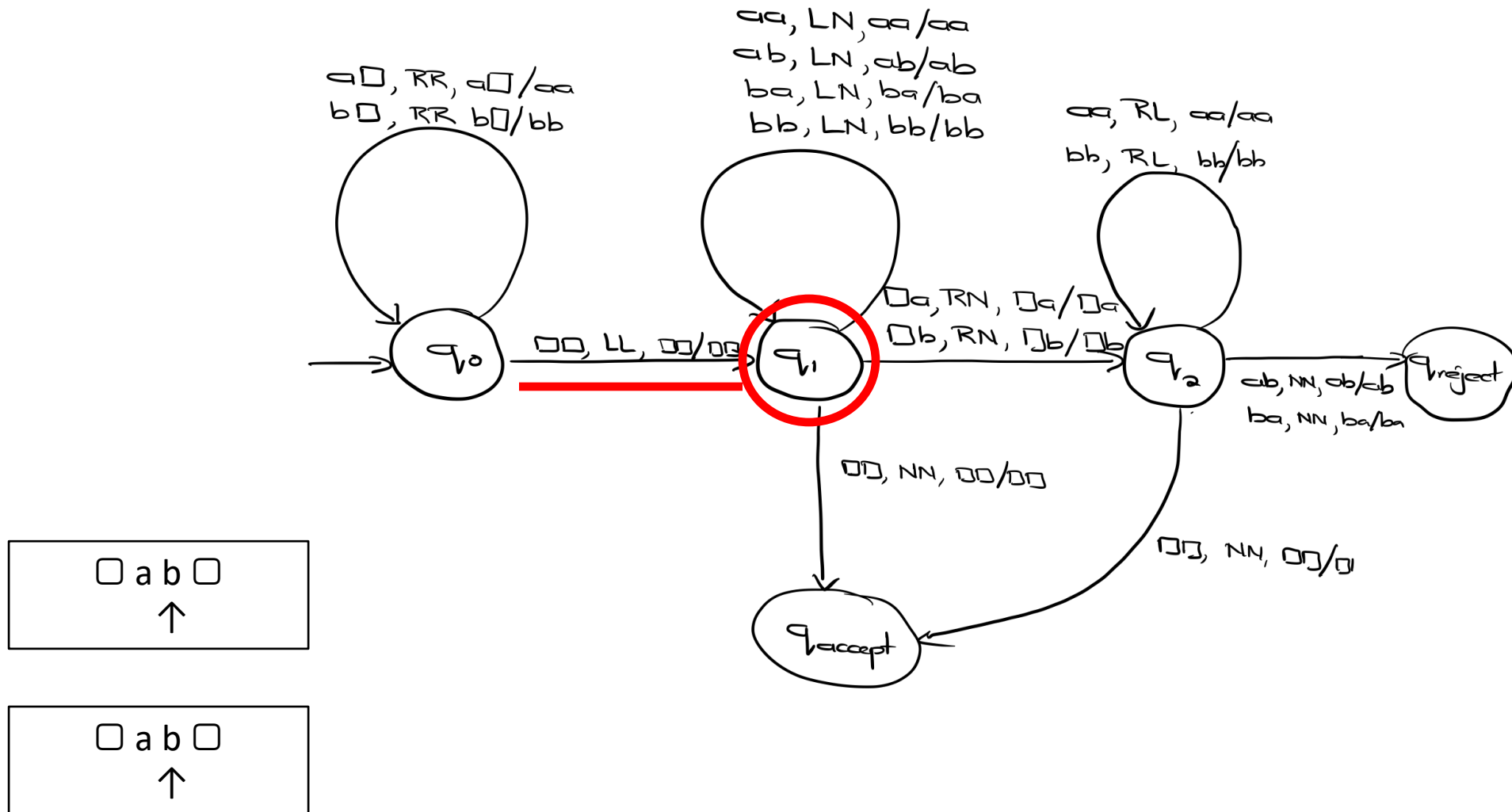
# Two Tape Solution – Illustration



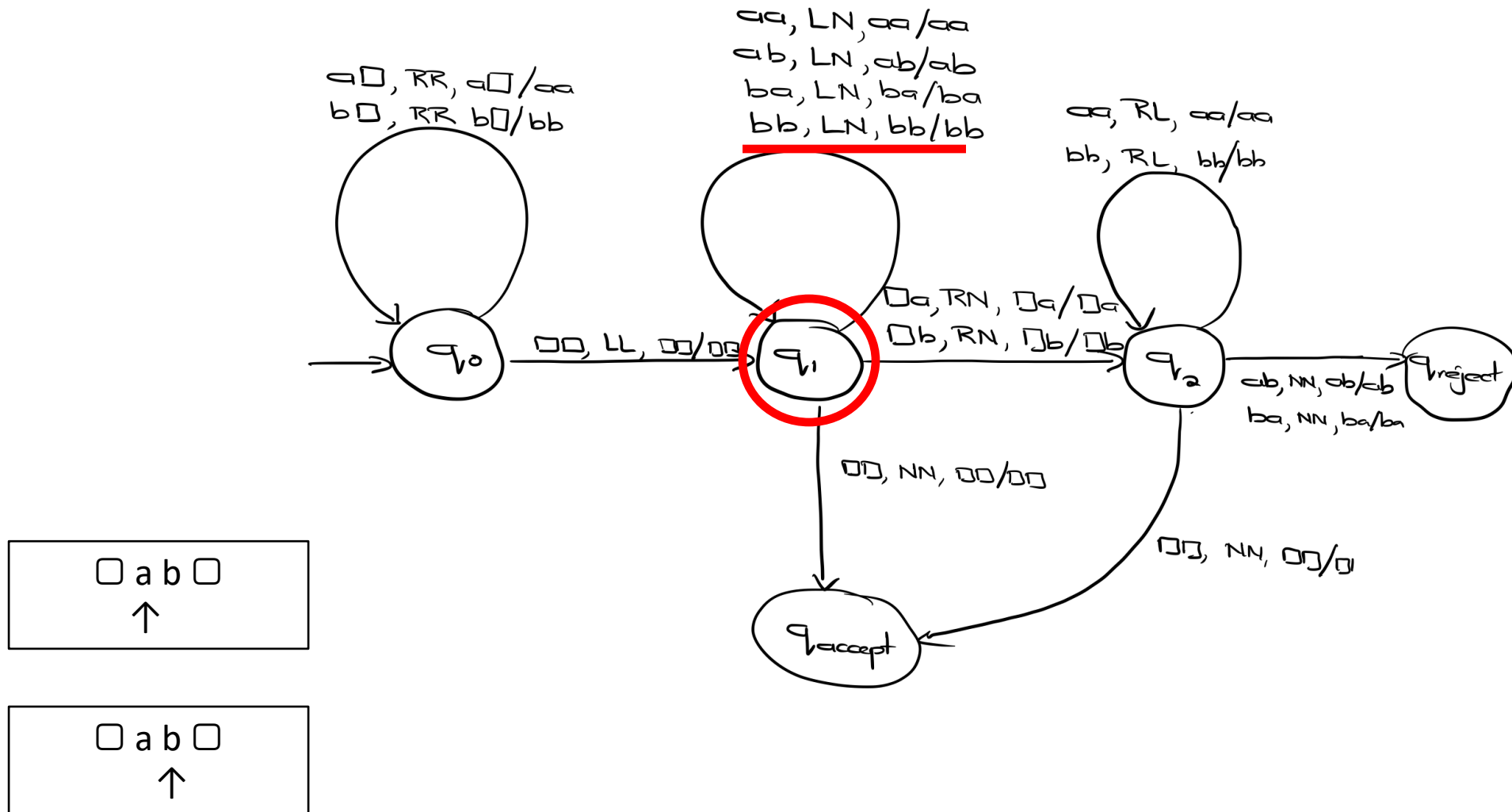
# Two Tape Solution – Illustration



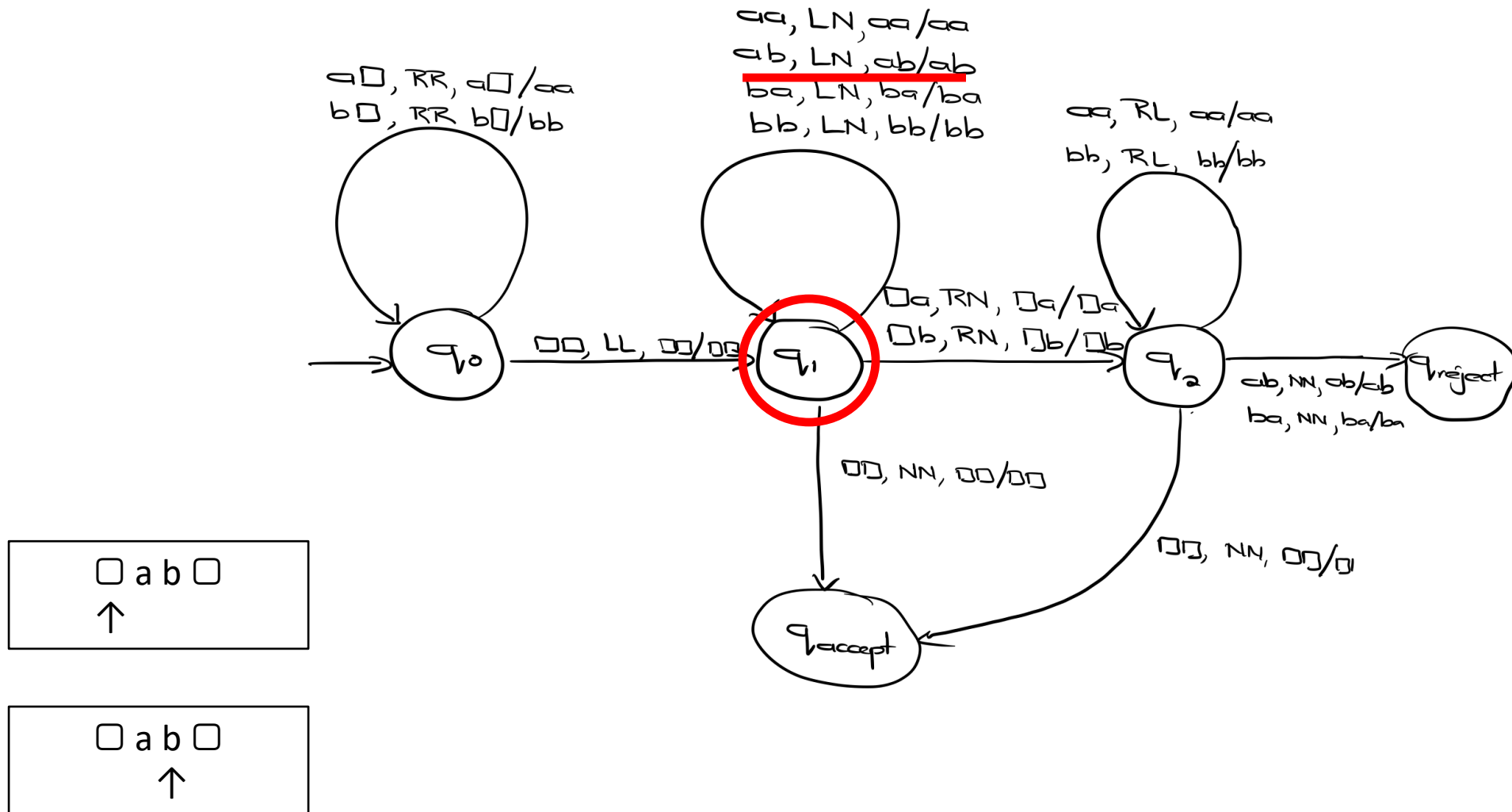
# Two Tape Solution – Illustration



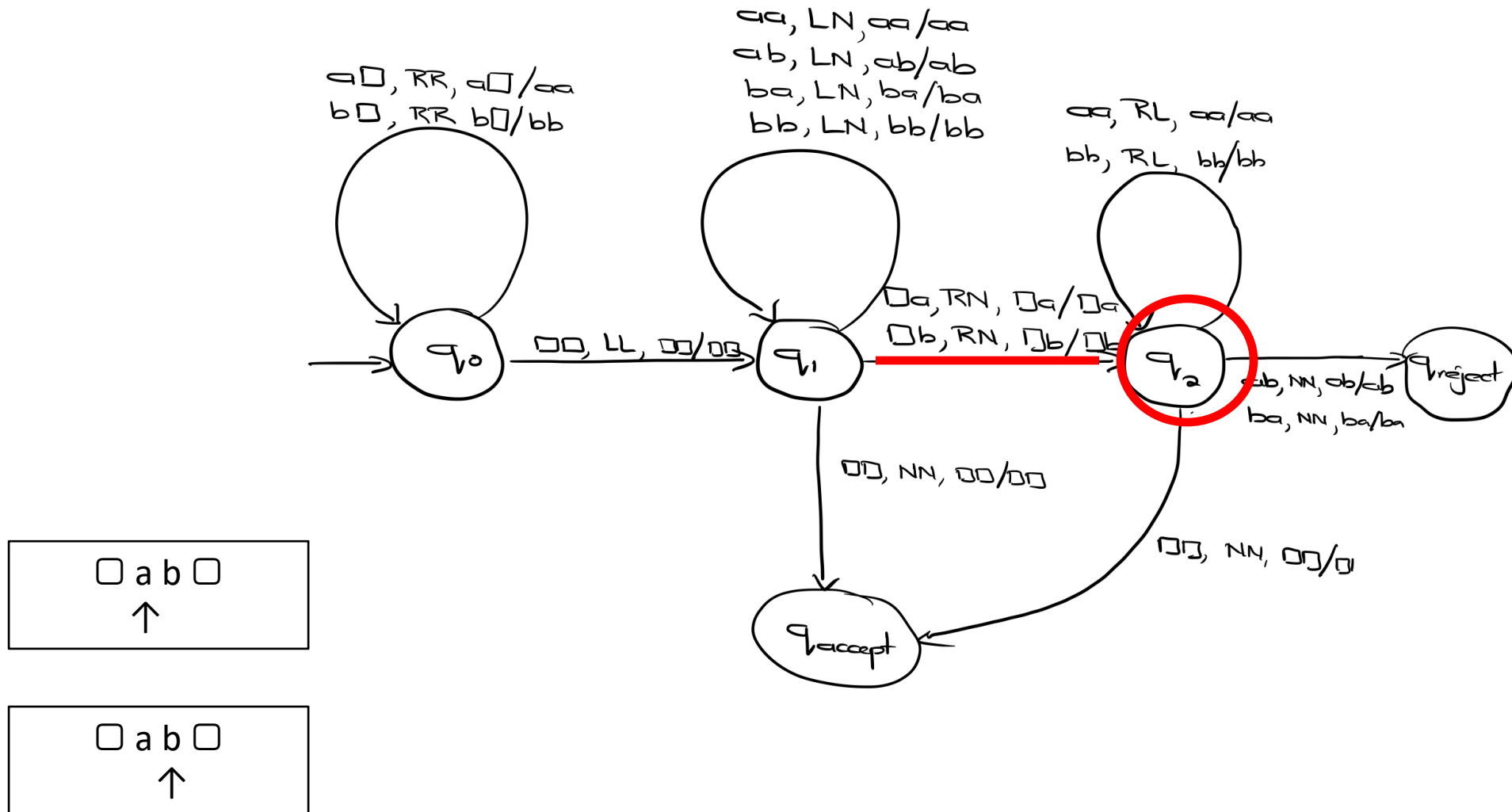
# Two Tape Solution – Illustration



# Two Tape Solution – Illustration

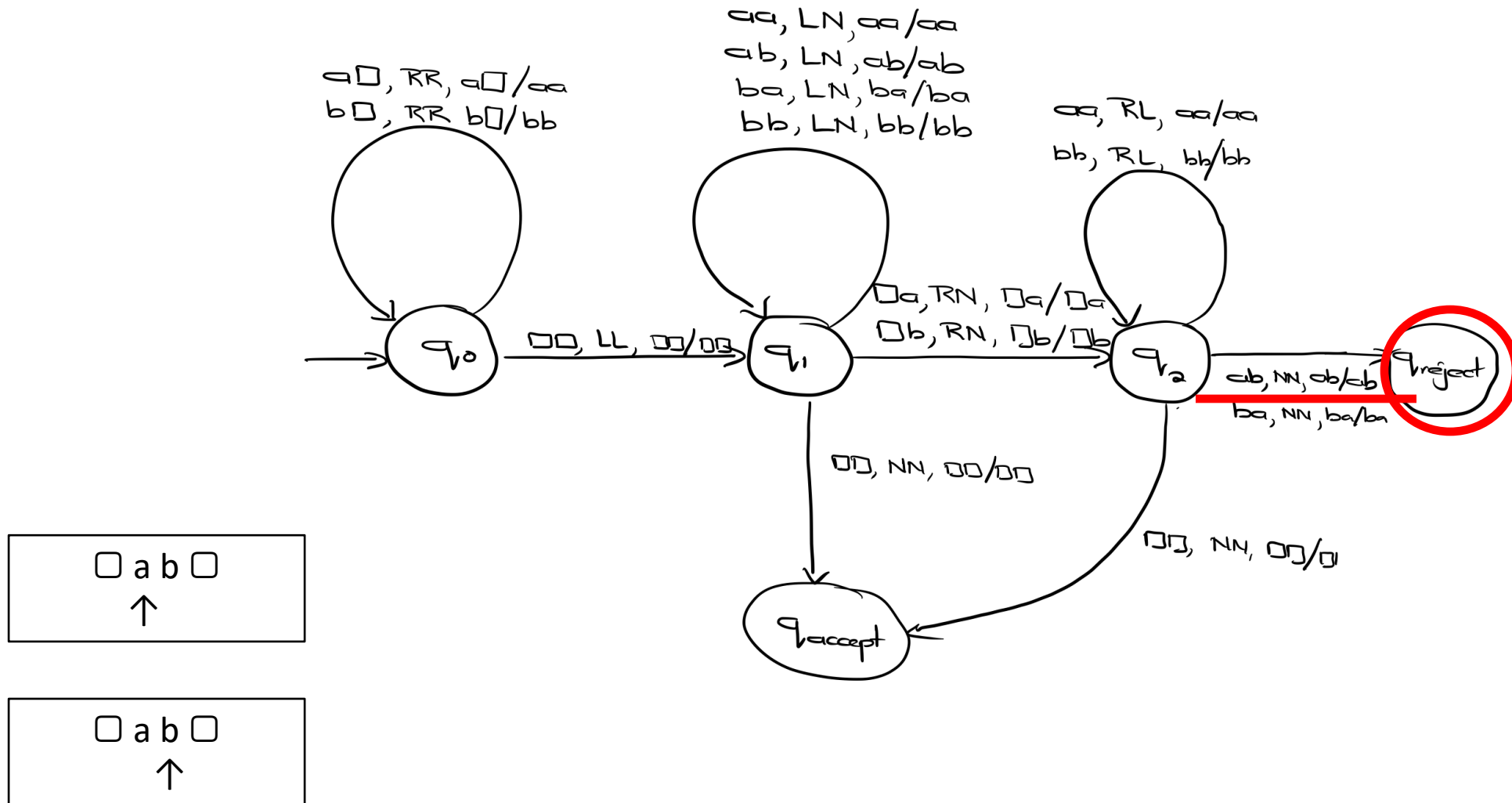


# Two Tape Solution – Illustration





# Two Tape Solution – Illustration



# Turing Machine Accepting $a^n b^n c^n$

We know that that following language is neither regular nor context-free

$$L = \{a^n b^n c^n : n \geq 0\}$$

but we can build a Turing machine that accepts  $L$ .

Design choices to be made:

- How many tapes?
- How many tape symbols?
- How many states?

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

□ a b b c c c □
↑

We will use the following two phased approach:

- **Phase 1:** Verify if input string  $w$  is of the form  $a^* b^* c^*$
- **Phase 2:** If it is of the correct form, then confirm if the number of **a**'s is the same as the number of **b**'s and **c**'s

The Turing machine

- will use a single tape and the tape symbols  $\Gamma = \{a, b, c, d, \square\}$
- the symbol  $d$  will be used in Phase 2 only

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

**Phase 1:** Verifying if input string  $w$  is of the form  $a^* b^* c^*$

- For this phase, we use the following states,

$q_a$  : start state; we are reading the block of  $a$ 's

$q_b$  : we are reading the block of  $b$ 's

$q_c$  : we are reading the block of  $c$ 's

$q_L$  : walk to the leftmost symbol

- Once we leave the state  $q_L$  we move to Phase 2

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$$q_a a \rightarrow q_a a R$$

$$q_a b \rightarrow q_b b R$$

$$q_a c \rightarrow q_c c R$$

$$q_a d \rightarrow \text{cannot happen}$$

$$q_a \square \rightarrow q_L \square L$$

$$q_c a \rightarrow q_{\text{reject}}$$

$$q_c b \rightarrow q_{\text{reject}}$$

$$q_c c \rightarrow q_c c R$$

$$q_c d \rightarrow \text{cannot happen}$$

$$q_c \square \rightarrow q_L \square L$$

$$q_b a \rightarrow q_{\text{reject}}$$

$$q_b b \rightarrow q_b b R$$

$$q_b c \rightarrow q_c c R$$

$$q_b d \rightarrow \text{cannot happen}$$

$$q_b \square \rightarrow q_L \square L$$

$$q_L a \rightarrow q_L a L$$

$$q_L b \rightarrow q_L b L$$

$$q_L c \rightarrow q_L c L$$

$$q_L d \rightarrow \text{cannot happen}$$

$$q_L \square \rightarrow q'_a \square R$$

Phase 2 transition

$\square$	a	b	b	c	c	c	$\square$
		↑					

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$$q_a a \rightarrow q_a a R$$

$$q_a b \rightarrow q_b b R$$

$$q_a c \rightarrow q_c c R$$

$$q_a d \rightarrow \text{cannot happen}$$

$$q_a \square \rightarrow q_L \square L$$

$$q_c a \rightarrow q_{\text{reject}}$$

$$q_c b \rightarrow q_{\text{reject}}$$

$$q_c c \rightarrow q_c c R$$

$$q_c d \rightarrow \text{cannot happen}$$

$$q_c \square \rightarrow q_L \square L$$

$$q_b a \rightarrow q_{\text{reject}}$$

$$q_b b \rightarrow q_b b R$$

$$q_b c \rightarrow q_c c R$$

$$q_b d \rightarrow \text{cannot happen}$$

$$q_b \square \rightarrow q_L \square L$$

$$q_L a \rightarrow q_L a L$$

$$q_L b \rightarrow q_L b L$$

$$q_L c \rightarrow q_L c L$$

$$q_L d \rightarrow \text{cannot happen}$$

$$q_L \square \rightarrow q'_a \square R$$

Phase 2 transition

$\square$	a	b	b	c	c	c	$\square$
		↑					

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$q_a a \rightarrow q_a a R$   
 $q_a b \rightarrow q_b b R$   
 $q_a c \rightarrow q_c c R$   
 $q_a d \rightarrow \text{cannot happen}$   
 $q_a \square \rightarrow q_L \square L$

$q_c a \rightarrow q_{\text{reject}}$   
 $q_c b \rightarrow q_{\text{reject}}$   
 $q_c c \rightarrow q_c c R$   
 $q_c d \rightarrow \text{cannot happen}$   
 $q_c \square \rightarrow q_L \square L$

$q_b a \rightarrow q_{\text{reject}}$   
 $q_b b \rightarrow q_b b R$   
 $q_b c \rightarrow q_c c R$   
 $q_b d \rightarrow \text{cannot happen}$   
 $q_b \square \rightarrow q_L \square L$

$q_L a \rightarrow q_L a L$   
 $q_L b \rightarrow q_L b L$   
 $q_L c \rightarrow q_L c L$   
 $q_L d \rightarrow \text{cannot happen}$

$q_L \square \rightarrow q'_a \square R$

Phase 2 transition

$\square a b b c c c \square$   
 $\uparrow$

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$q_a a \rightarrow q_a a R$   
 $q_a b \rightarrow q_b b R$   
 $q_a c \rightarrow q_c c R$   
 $q_a d \rightarrow \text{cannot happen}$   
 $q_a \square \rightarrow q_L \square L$

$q_c a \rightarrow q_{\text{reject}}$   
 $q_c b \rightarrow q_{\text{reject}}$   
 $q_c c \rightarrow q_c c R$   
 $q_c d \rightarrow \text{cannot happen}$   
 $q_c \square \rightarrow q_L \square L$

$q_b a \rightarrow q_{\text{reject}}$   
 $q_b b \rightarrow q_b b R$   
 $q_b c \rightarrow q_c c R$   
 $q_b d \rightarrow \text{cannot happen}$   
 $q_b \square \rightarrow q_L \square L$

$q_L a \rightarrow q_L a L$   
 $q_L b \rightarrow q_L b L$   
 $q_L c \rightarrow q_L c L$   
 $q_L d \rightarrow \text{cannot happen}$

$q_L \square \rightarrow q'_a \square R$

Phase 2 transition

$\square a b b c c c \square$   
 $\uparrow$



# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$q_a a \rightarrow q_a a R$   
 $q_a b \rightarrow q_b b R$   
 $q_a c \rightarrow q_c c R$   
 $q_a d \rightarrow \text{cannot happen}$   
 $q_a \square \rightarrow q_L \square L$

$q_c a \rightarrow q_{\text{reject}}$   
 $q_c b \rightarrow q_{\text{reject}}$   
 $q_c c \rightarrow q_c c R$   
 $q_c d \rightarrow \text{cannot happen}$   
 $q_c \square \rightarrow q_L \square L$

$q_b a \rightarrow q_{\text{reject}}$   
 $q_b b \rightarrow q_b b R$   
 $q_b c \rightarrow q_c c R$   
 $q_b d \rightarrow \text{cannot happen}$   
 $q_b \square \rightarrow q_L \square L$

$q_L a \rightarrow q_L a L$   
 $q_L b \rightarrow q_L b L$   
 $q_L c \rightarrow q_L c L$   
 $q_L d \rightarrow \text{cannot happen}$

$q_L \square \rightarrow q'_a \square R$

Phase 2 transition

$\square a b b c c c \square$   
 $\uparrow$

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$q_a a \rightarrow q_a a R$   
 $q_a b \rightarrow q_b b R$   
 $q_a c \rightarrow q_c c R$   
 $q_a d \rightarrow \text{cannot happen}$   
 $q_a \square \rightarrow q_L \square L$

$q_c a \rightarrow q_{\text{reject}}$   
 $q_c b \rightarrow q_{\text{reject}}$   
 $q_c c \rightarrow q_c c R$   
 $q_c d \rightarrow \text{cannot happen}$   
 $q_c \square \rightarrow q_L \square L$

$q_b a \rightarrow q_{\text{reject}}$   
 $q_b b \rightarrow q_b b R$   
 $q_b c \rightarrow q_c c R$   
 $q_b d \rightarrow \text{cannot happen}$   
 $q_b \square \rightarrow q_L \square L$

$q_L a \rightarrow q_L a L$   
 $q_L b \rightarrow q_L b L$   
 $q_L c \rightarrow q_L c L$   
 $q_L d \rightarrow \text{cannot happen}$

$q_L \square \rightarrow q'_a \square R$

Phase 2 transition

$\square a b b c c c \square$   
 $\uparrow$

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$q_a a \rightarrow q_a a R$   
 $q_a b \rightarrow q_b b R$   
 $q_a c \rightarrow q_c c R$   
 $q_a d \rightarrow \text{cannot happen}$   
 $q_a \square \rightarrow q_L \square L$

$q_c a \rightarrow q_{\text{reject}}$   
 $q_c b \rightarrow q_{\text{reject}}$   
 $q_c c \rightarrow q_c c R$   
 $q_c d \rightarrow \text{cannot happen}$   
 $q_c \square \rightarrow q_L \square L$

$q_b a \rightarrow q_{\text{reject}}$   
 $q_b b \rightarrow q_b b R$   
 $q_b c \rightarrow q_c c R$   
 $q_b d \rightarrow \text{cannot happen}$   
 $q_b \square \rightarrow q_L \square L$

$q_L a \rightarrow q_L a L$   
 $q_L b \rightarrow q_L b L$   
 $q_L c \rightarrow q_L c L$   
 $q_L d \rightarrow \text{cannot happen}$

$q_L \square \rightarrow q'_a \square R$

Phase 2 transition

$\square a b b c c c \square$   
 $\uparrow$

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$q_a a \rightarrow q_a a R$   
 $q_a b \rightarrow q_b b R$   
 $q_a c \rightarrow q_c c R$   
 $q_a d \rightarrow \text{cannot happen}$   
 $q_a \square \rightarrow q_L \square L$

$q_c a \rightarrow q_{\text{reject}}$   
 $q_c b \rightarrow q_{\text{reject}}$   
 $q_c c \rightarrow q_c c R$   
 $q_c d \rightarrow \text{cannot happen}$   
 $q_c \square \rightarrow q_L \square L$

$q_b a \rightarrow q_{\text{reject}}$   
 $q_b b \rightarrow q_b b R$   
 $q_b c \rightarrow q_c c R$   
 $q_b d \rightarrow \text{cannot happen}$   
 $q_b \square \rightarrow q_L \square L$

$\square$	a	b	b	c	c	c	$\square$
						$\uparrow$	

$q_L a \rightarrow q_L a L$   
 $q_L b \rightarrow q_L b L$   
 $q_L c \rightarrow q_L c L$   
 $q_L d \rightarrow \text{cannot happen}$

$q_L \square \rightarrow q'_a \square R$

Phase 2 transition

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 1** we need the following transitions:

$$q_a a \rightarrow q_a a R$$

$$q_a b \rightarrow q_b b R$$

$$q_a c \rightarrow q_c c R$$

$$q_a d \rightarrow \text{cannot happen}$$

$$q_a \square \rightarrow q_L \square L$$

$$q_b a \rightarrow q_{\text{reject}}$$

$$q_b b \rightarrow q_b b R$$

$$q_b c \rightarrow q_c c R$$

$$q_b d \rightarrow \text{cannot happen}$$

$$q_b \square \rightarrow q_L \square L$$

$$q_c a \rightarrow q_{\text{reject}}$$

$$q_c b \rightarrow q_{\text{reject}}$$

$$q_c c \rightarrow q_c c R$$

$$q_c d \rightarrow \text{cannot happen}$$

$$q_c \square \rightarrow q_L \square L$$

$$q_L a \rightarrow q_L a L$$

$$q_L b \rightarrow q_L b L$$

$$q_L c \rightarrow q_L c L$$

$$q_L d \rightarrow \text{cannot happen}$$

$$q_L \square \rightarrow q'_a \square R$$

Phase 2 transition

$\square$ a b b c c c $\square$
↑

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

Approach to **Phase 2**:

- We start from the leftmost symbol of the input string
- We replace the leftmost **a** with **d**
- Then we search for the leftmost **b** and the leftmost **c** and replace them by **d**
- We repeat these steps until there are no more **a**'s
- If the remaining string consists of **d**'s only, then we accept

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

For **Phase 2** we use the following states:

- $q'_a$  : start state of Stage 2; search for the leftmost  $a$
- $q'_b$  : leftmost  $a$  has been replaced by  $d$ ;  
search for the leftmost  $b$
- $q'_c$  : leftmost  $a$  has been replaced by  $d$ ;  
leftmost  $b$  has been replaced by  $d$ ;  
search for the leftmost  $c$
- $q'_L$  : leftmost  $a$  has been replaced by  $d$ ;  
leftmost  $b$  has been replaced by  $d$ ;  
leftmost  $c$  has been replaced by  $d$ ;  
walk to the leftmost symbol

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

Transitions of **Phase 2:**

$$q'_a a \rightarrow q'_b d R$$

$$q'_a b \rightarrow q_{reject}$$

$$q'_a c \rightarrow q_{reject}$$

$$q'_a d \rightarrow q'_a d R$$

$$q'_a \square \rightarrow q_{accept}$$

$$q'_b a \rightarrow q'_b a R$$

$$q'_b b \rightarrow q'_c d R$$

$$q'_b c \rightarrow q_{reject}$$

$$q'_b d \rightarrow q'_b d R$$

$$q'_b \square \rightarrow q_{reject}$$

$$q'_c a \rightarrow q_{reject}$$

$$q'_c b \rightarrow q'_c b R$$

$$q'_c c \rightarrow q'_L d L$$

$$q'_c d \rightarrow q'_c d R$$

$$q'_c \square \rightarrow q_{reject}$$

$$q'_L a \rightarrow q'_L a L$$

$$q'_L b \rightarrow q'_L b L$$

$$q'_L c \rightarrow q'_L c L$$

$$q'_L d \rightarrow q'_L d L$$

$$q'_L \square \rightarrow q'_a \square R$$

$\square$	a	b	b	c	c	c	$\square$
	↑						



# Single-Tape Turing Machine Accepting $a^n b^n c^n$

Transitions of **Phase 2:**

$$\begin{aligned} q'_a a &\rightarrow q'_b d R \\ q'_a b &\rightarrow q_{reject} \\ q'_a c &\rightarrow q_{reject} \\ q'_a d &\rightarrow q'_a d R \\ q'_a \square &\rightarrow q_{accept} \end{aligned}$$
$$\begin{aligned} q'_c a &\rightarrow q_{reject} \\ q'_c b &\rightarrow q'_c b R \\ q'_c c &\rightarrow q'_L d L \\ q'_c d &\rightarrow q'_c d R \\ q'_c \square &\rightarrow q_{reject} \end{aligned}$$
$$\begin{aligned} q'_b a &\rightarrow q'_b a R \\ q'_b b &\rightarrow q'_c d R \\ q'_b c &\rightarrow q_{reject} \\ q'_b d &\rightarrow q'_b d R \\ q'_b \square &\rightarrow q_{reject} \end{aligned}$$
$$\begin{aligned} q'_L a &\rightarrow q'_L a L \\ q'_L b &\rightarrow q'_L b L \\ q'_L c &\rightarrow q'_L c L \\ q'_L d &\rightarrow q'_L d L \\ q'_L \square &\rightarrow q'_a \square R \end{aligned}$$

$\square$	d	b	b	c	c	c	$\square$
		↑					

# Single-Tape Turing Machine Accepting $a^n b^n c^n$

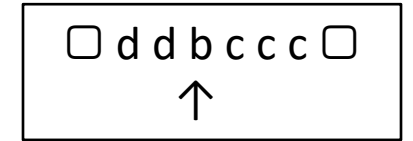
Transitions of **Phase 2**:

$$\begin{aligned}q'_a a &\rightarrow q'_b d R \\ q'_a b &\rightarrow q_{reject} \\ q'_a c &\rightarrow q_{reject} \\ q'_a d &\rightarrow q'_a d R \\ q'_a \square &\rightarrow q_{accept}\end{aligned}$$

$$\begin{aligned}q'_c a &\rightarrow q_{reject} \\ \boxed{q'_c b &\rightarrow q'_c b R} \\ q'_c c &\rightarrow q'_L d L \\ q'_c d &\rightarrow q'_c d R \\ q'_c \square &\rightarrow q_{reject}\end{aligned}$$

$$\begin{aligned}q'_b a &\rightarrow q'_b a R \\ q'_b b &\rightarrow q'_c d R \\ q'_b c &\rightarrow q_{reject} \\ q'_b d &\rightarrow q'_b d R \\ q'_b \square &\rightarrow q_{reject}\end{aligned}$$

$$\begin{aligned}q'_L a &\rightarrow q'_L a L \\ q'_L b &\rightarrow q'_L b L \\ q'_L c &\rightarrow q'_L c L \\ q'_L d &\rightarrow q'_L d L \\ q'_L \square &\rightarrow q'_a \square R\end{aligned}$$



# Single-Tape Turing Machine Accepting $a^n b^n c^n$

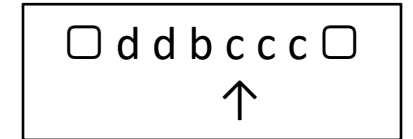
Transitions of **Phase 2**:

$$\begin{aligned}q'_a a &\rightarrow q'_b d R \\ q'_a b &\rightarrow q_{reject} \\ q'_a c &\rightarrow q_{reject} \\ q'_a d &\rightarrow q'_a d R \\ q'_a \square &\rightarrow q_{accept}\end{aligned}$$

$$\begin{aligned}q'_c a &\rightarrow q_{reject} \\ q'_c b &\rightarrow q'_c b R \\ q'_c c &\rightarrow q'_L d L \\ q'_c d &\rightarrow q'_c d R \\ q'_c \square &\rightarrow q_{reject}\end{aligned}$$

$$\begin{aligned}q'_b a &\rightarrow q'_b a R \\ q'_b b &\rightarrow q'_c d R \\ q'_b c &\rightarrow q_{reject} \\ q'_b d &\rightarrow q'_b d R \\ q'_b \square &\rightarrow q_{reject}\end{aligned}$$

$$\begin{aligned}q'_L a &\rightarrow q'_L a L \\ q'_L b &\rightarrow q'_L b L \\ q'_L c &\rightarrow q'_L c L \\ q'_L d &\rightarrow q'_L d L \\ q'_L \square &\rightarrow q'_a \square R\end{aligned}$$



# Single-Tape Turing Machine Accepting $a^n b^n c^n$

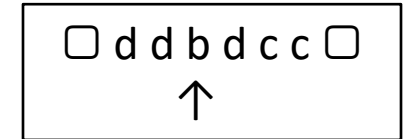
Transitions of **Phase 2**:

$$\begin{aligned}q'_a a &\rightarrow q'_b d R \\ q'_a b &\rightarrow q_{reject} \\ q'_a c &\rightarrow q_{reject} \\ q'_a d &\rightarrow q'_a d R \\ q'_a \square &\rightarrow q_{accept}\end{aligned}$$

$$\begin{aligned}q'_c a &\rightarrow q_{reject} \\ q'_c b &\rightarrow q'_c b R \\ q'_c c &\rightarrow q'_L d L \\ q'_c d &\rightarrow q'_c d R \\ q'_c \square &\rightarrow q_{reject}\end{aligned}$$

$$\begin{aligned}q'_b a &\rightarrow q'_b a R \\ q'_b b &\rightarrow q'_c d R \\ q'_b c &\rightarrow q_{reject} \\ q'_b d &\rightarrow q'_b d R \\ q'_b \square &\rightarrow q_{reject}\end{aligned}$$

$$\begin{aligned}q'_L a &\rightarrow q'_L a L \\ q'_L b &\rightarrow q'_L b L \\ q'_L c &\rightarrow q'_L c L \\ q'_L d &\rightarrow q'_L d L \\ q'_L \square &\rightarrow q'_a \square R\end{aligned}$$



# Single-Tape Turing Machine Accepting $a^n b^n c^n$

Transitions of **Phase 2**:

$$q'_a a \rightarrow q'_b d R$$

$$q'_a b \rightarrow q_{reject}$$

$$q'_a c \rightarrow q_{reject}$$

$$q'_a d \rightarrow q'_a d R$$

$$q'_a \square \rightarrow q_{accept}$$

$$q'_b a \rightarrow q'_b a R$$

$$q'_b b \rightarrow q'_c d R$$

$$q'_b c \rightarrow q_{reject}$$

$$q'_b d \rightarrow q'_b d R$$

$$q'_b \square \rightarrow q_{reject}$$

$$q'_c a \rightarrow q_{reject}$$

$$q'_c b \rightarrow q'_c b R$$

$$q'_c c \rightarrow q'_L d L$$

$$q'_c d \rightarrow q'_c d R$$

$$q'_c \square \rightarrow q_{reject}$$

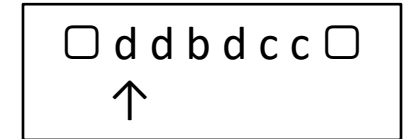
$$q'_L a \rightarrow q'_L a L$$

$$q'_L b \rightarrow q'_L b L$$

$$q'_L c \rightarrow q'_L c L$$

$$q'_L d \rightarrow q'_L d L$$

$$q'_L \square \rightarrow q'_a \square R$$



# Single-Tape Turing Machine Accepting $a^n b^n c^n$

Transitions of **Phase 2**:

$$q'_a a \rightarrow q'_b d R$$

$$q'_a b \rightarrow q_{reject}$$

$$q'_a c \rightarrow q_{reject}$$

$$q'_a d \rightarrow q'_a d R$$

$$q'_a \square \rightarrow q_{accept}$$

$$q'_b a \rightarrow q'_b a R$$

$$q'_b b \rightarrow q'_c d R$$

$$q'_b c \rightarrow q_{reject}$$

$$q'_b d \rightarrow q'_b d R$$

$$q'_b \square \rightarrow q_{reject}$$

$$q'_c a \rightarrow q_{reject}$$

$$q'_c b \rightarrow q'_c b R$$

$$q'_c c \rightarrow q'_L d L$$

$$q'_c d \rightarrow q'_c d R$$

$$q'_c \square \rightarrow q_{reject}$$

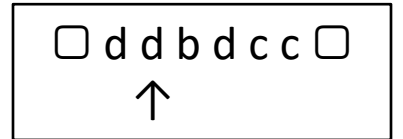
$$q'_L a \rightarrow q'_L a L$$

$$q'_L b \rightarrow q'_L b L$$

$$q'_L c \rightarrow q'_L c L$$

$$q'_L d \rightarrow q'_L d L$$

$$q'_L \square \rightarrow q'_a \square R$$



# Single-Tape Turing Machine Accepting $a^n b^n c^n$

Transitions of **Phase 2**:

$$q'_a a \rightarrow q'_b d R$$

$$q'_a b \rightarrow q_{reject}$$

$$q'_a c \rightarrow q_{reject}$$

$$q'_a d \rightarrow q'_a d R$$

$$q'_a \square \rightarrow q_{accept}$$

$$q'_b a \rightarrow q'_b a R$$

$$q'_b b \rightarrow q'_c d R$$

$$q'_b c \rightarrow q_{reject}$$

$$q'_b d \rightarrow q'_b d R$$

$$q'_b \square \rightarrow q_{reject}$$

$$q'_c a \rightarrow q_{reject}$$

$$q'_c b \rightarrow q'_c b R$$

$$q'_c c \rightarrow q'_L d L$$

$$q'_c d \rightarrow q'_c d R$$

$$q'_c \square \rightarrow q_{reject}$$

$$q'_L a \rightarrow q'_L a L$$

$$q'_L b \rightarrow q'_L b L$$

$$q'_L c \rightarrow q'_L c L$$

$$q'_L d \rightarrow q'_L d L$$

$$q'_L \square \rightarrow q'_a \square R$$

$\square$	d	d	b	d	c	c	$\square$
			↑				

## Exercise: Multi-Tape Turing Machine Accepting $a^n b^n c^n$

Try and replace **Phase 2** with a four-tape solution:

- The input is stored on tape 1, the remaining tapes are initially empty
- For every **a** of the input string, write an **a** onto tape 2
- For every **b** of the input string, write a **b** onto tape 3
- For every **c** of the input string, write a **c** onto tape 4
- Now check if each of the tapes 2, 3, and 4 has the same number of symbols