COS210 - Theoretical Computer Science
Finite Automata and Regular Languages (Part 4)

©Cleghorn, Marshall, Timm

DFAs vs. NFAs

Deterministic finite automaton $D = (Q, \Sigma, \delta, q, F)$:

- for each state $r \in Q$ and each symbol $a \in \Sigma$ there exists a **unique successor state** $s \in Q$
- for each input string w over Σ there exists **exactly one run** over w

DFAs vs. NFAs

Deterministic finite automaton $D = (Q, \Sigma, \delta, q, F)$:

- for each state $r \in Q$ and each symbol $a \in \Sigma$ there exists a **unique successor state** $s \in Q$
- for each input string w over Σ there exists **exactly one run** over w

Nondeterministic finite automaton $N = (Q, \Sigma, \delta, q, F)$:

- for each state $r \in Q$ and each symbol $a \in \Sigma$ there exists a **set of successor states** $R \subseteq Q$
- ϵ-**transitions** may exist that can be taken without reading a symbol
- for each input string w over Σ there may exist **multiple runs** over w

However, DFAs and NFAs are equally powerful.

©Cleghorn, Marshall, Timm

Equivalence of DFAs and NFAs

Each language that can be accepted by a DFA can be accepted by an NFA and vice versa:

Theorem (1)

Let D be a DFA with language $L(D)$.

Then there exists an NFA N with language $L(N) = L(D)$.

Theorem (2)

Let N be an NFA with language $L(N)$.

Then there exists a DFA D with language $L(D) = L(N)$.

Construction of NFA from DFA

Theorem (1)

Let D be a DFA with language $L(D)$.

Then there exists an NFA N with language $L(N) = L(D)$.

Proof by Construction:

- let $D = (Q, \Sigma, \delta, q, F)$
- then we construct $N = (\underline{Q, \Sigma}, \delta', \underline{q, F})$ where δ' is defined as follows:
  - for each $r \in Q$ and $a \in \Sigma$:

$$\text{if} \quad \delta(r, a) = s \quad \text{then} \quad \delta'(r, a) = \{s\}$$

  - for each $r \in Q$:

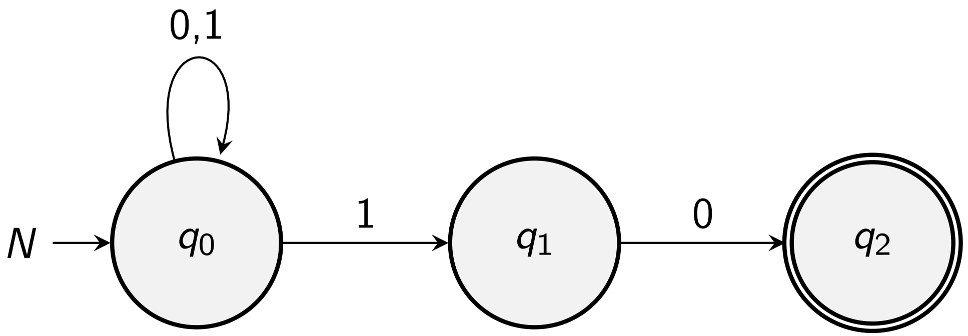
$$\delta'(r, \epsilon) = \emptyset$$

- state transition diagrams of D and N are equal and therefore accept the same language $\square$

Construction of DFA from NFA (without ϵ-Transitions)

Given NFA $N = (Q, \Sigma, \delta, q, F)$.

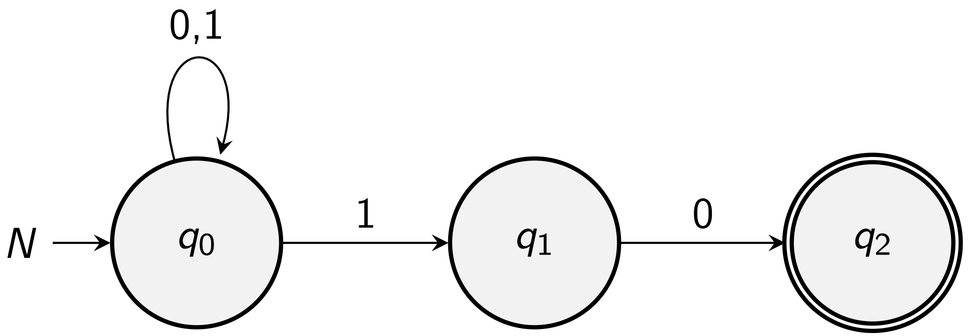
How to construct DFA $D = (Q', \Sigma, \delta', q', F')$ with $L(D) = L(N)$?



Construction of DFA from NFA (without ϵ-Transitions)

Given NFA $N = (Q, \Sigma, \delta, q, F)$.

How to construct DFA $D = (Q', \Sigma, \delta', q', F')$ with $L(D) = L(N)$?



Transitions of N map to **subsets of states** $R \subseteq Q$:

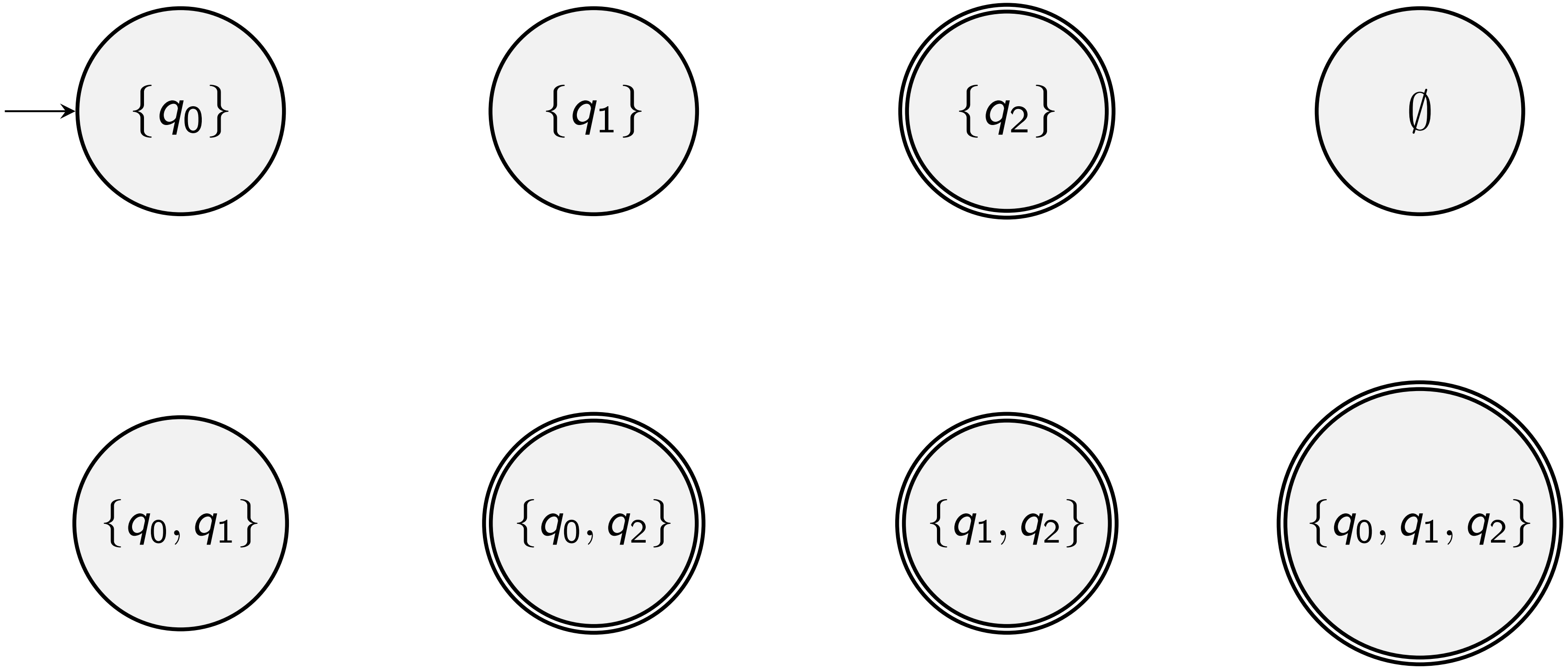
δ	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_2\}$	$\emptyset$
q_2	$\emptyset$	$\emptyset$

Transitions of D must map to **single states** $r \in Q'$.

$\rightarrow$ Each subset of states in N becomes a single state in D

Construction of DFA from NFA (without ϵ-Transitions)

DFA $D = (Q', \Sigma, \delta', q', F')$:



- $Q' = \{R : R \subseteq Q\}$ set of all subsets of Q
- $q' = \{q\}$ set that contains the initial state of N only
- $F' = \{R : \text{there exists } r \in R \text{ with } r \in F\}$
  set of states that contain at least one accepting state of N

Construction of DFA from NFA (without ϵ-Transitions)

Transition function δ' of D: for each $R \in Q'$, for each $a \in \Sigma$:

$$\underbrace{\delta'(R, a)}_{a\text{-successor of } R \text{ in } D} = \bigcup_{r \in R} \underbrace{\delta(r, a)}_{\text{set of } a\text{-successors of } r \text{ in } N}$$

Construction of DFA from NFA (without ϵ-Transitions)

Transition function δ' of D: for each $R \in Q'$, for each $a \in \Sigma$:

$$\underbrace{\delta'(R, a)}_{a\text{-successor of } R \text{ in } D} = \bigcup_{r \in R} \underbrace{\delta(r, a)}_{\text{set of } a\text{-successors of } r \text{ in } N}$$

δ'	0	1
$\{q_0\}$		
$\{q_1\}$		
$\{q_2\}$		
$\{q_0, q_1\}$	?	
$\{q_0, q_2\}$		
$\{q_1, q_2\}$		
$\{q_0, q_1, q_2\}$		
$\emptyset$		

Construction of DFA from NFA (without ϵ-Transitions)

Transition function δ' of D: for each $R \in Q'$, for each $a \in \Sigma$:

$$\underbrace{\delta'(R,a)}_{a\text{-successor of } R \text{ in } D} = \bigcup_{r \in R} \underbrace{\delta(r,a)}_{\text{set of } a\text{-successors of } r \text{ in } N}$$

δ	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_2\}$	$\emptyset$
q_2	$\emptyset$	$\emptyset$

δ'	0	1
$\{q_0\}$		
$\{q_1\}$		
$\{q_2\}$		
$\{q_0, q_1\}$	?	
$\{q_0, q_2\}$		
$\{q_1, q_2\}$		
$\{q_0, q_1, q_2\}$		
$\emptyset$		

Construction of DFA from NFA (without ϵ-Transitions)

Transition function δ' of D: for each $R \in Q'$, for each $a \in \Sigma$:

$$\underbrace{\delta'(R, a)}_{a\text{-successor of } R \text{ in } D} = \bigcup_{r \in R} \underbrace{\delta(r, a)}_{\text{set of } a\text{-successors of } r \text{ in } N}$$

δ	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_2\}$	$\emptyset$
q_2	$\emptyset$	$\emptyset$

δ'	0	1
$\{q_0\}$		
$\{q_1\}$		
$\{q_2\}$		
$\{q_0, q_1\}$	$\{q_0, q_2\}$	
$\{q_0, q_2\}$		
$\{q_1, q_2\}$		
$\{q_0, q_1, q_2\}$		
$\emptyset$		

$$\delta'(\{q_0, q_1\}, 0) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$$

Construction of DFA from NFA (without ϵ-Transitions)

Transition function δ' of D: for each $R \in Q'$, for each $a \in \Sigma$:

$$\underbrace{\delta'(R,a)}_{a\text{-successor of } R \text{ in } D} = \bigcup_{r \in R} \underbrace{\delta(r,a)}_{\text{set of } a\text{-successors of } r \text{ in } N}$$

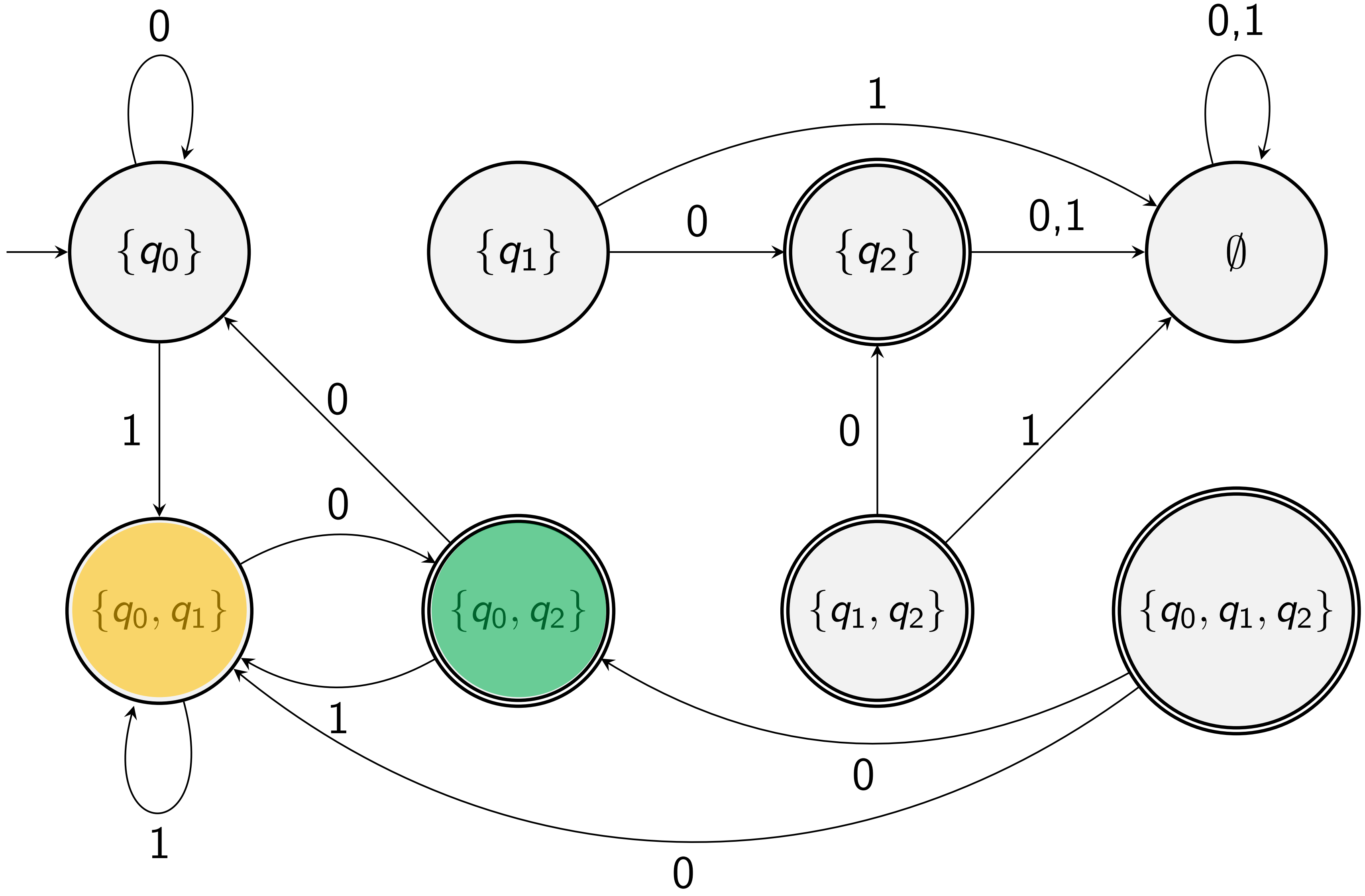
δ	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_2\}$	$\emptyset$
q_2	$\emptyset$	$\emptyset$

δ'	0	1
$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
$\{q_1\}$	$\{q_2\}$	$\emptyset$
$\{q_2\}$	$\emptyset$	$\emptyset$
$\{q_0, q_1\}$	$\{q_0, q_2\}$	$\{q_0, q_1\}$
$\{q_0, q_2\}$	$\{q_0\}$	$\{q_0, q_1\}$
$\{q_1, q_2\}$	$\{q_2\}$	$\emptyset$
$\{q_0, q_1, q_2\}$	$\{q_0, q_2\}$	$\{q_0, q_1\}$
$\emptyset$	$\emptyset$	$\emptyset$

$$\delta'(\{q_0, q_1\}, 0) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$$

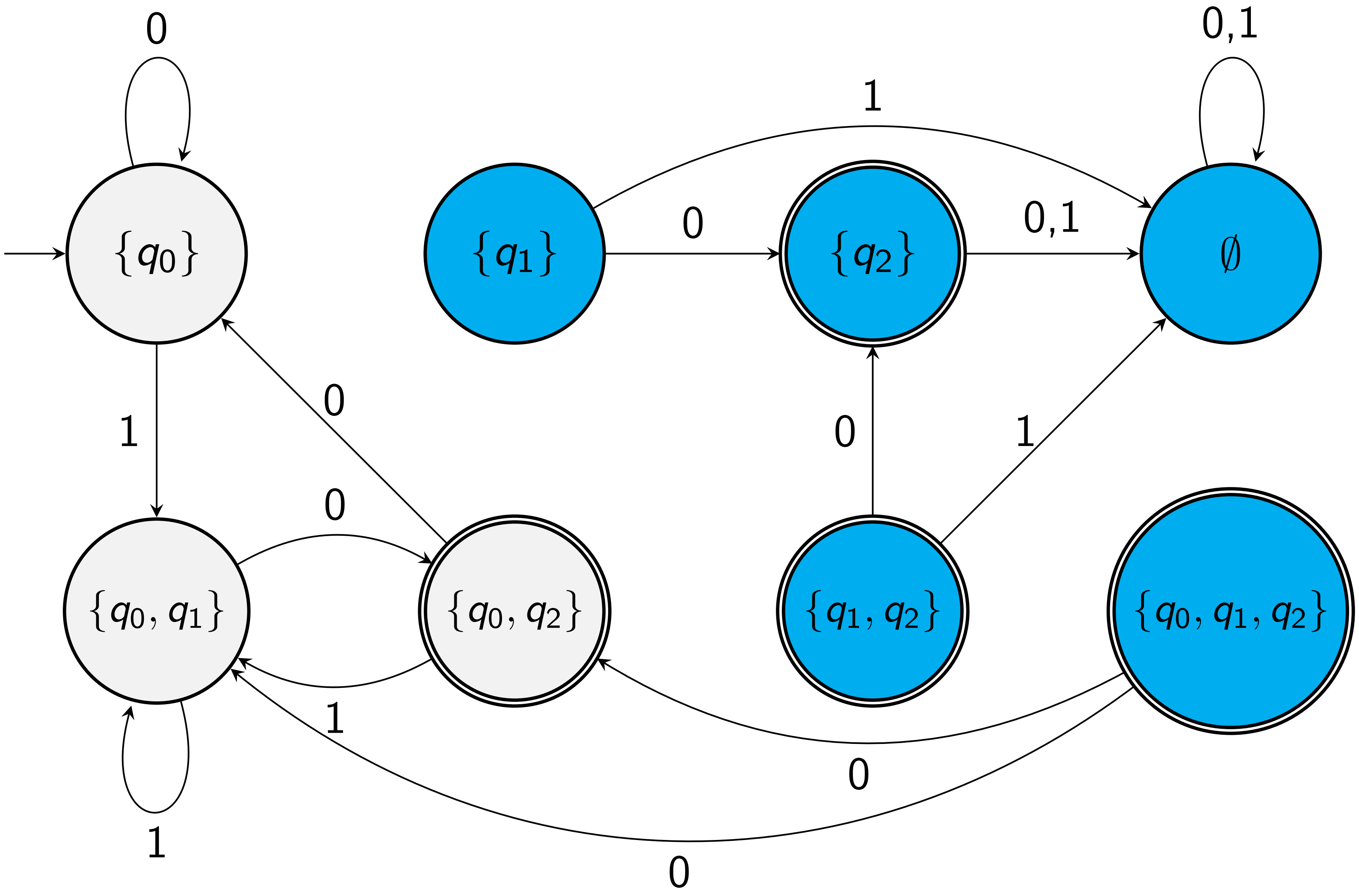
Construction of DFA from NFA (without ϵ-Transitions)

Complete DFA D:



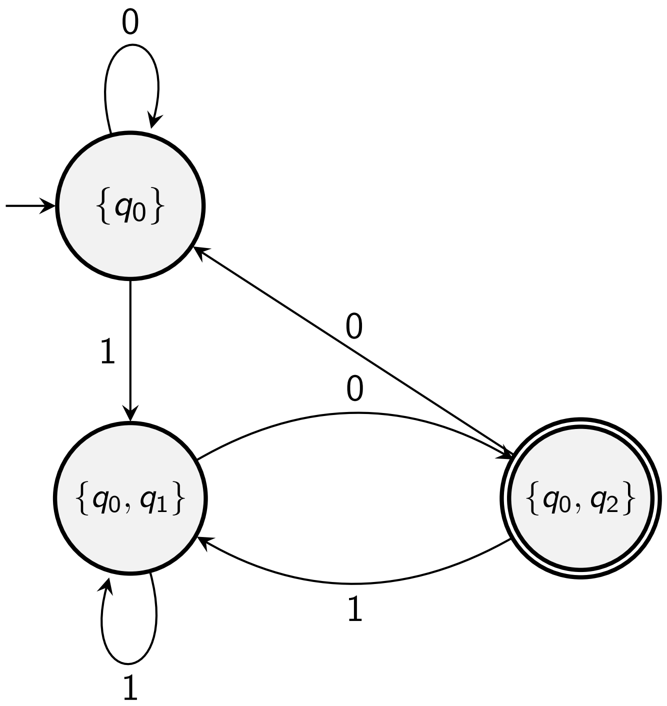
Construction of DFA from NFA (without ϵ-Transitions)

Several states of D are unreachable:

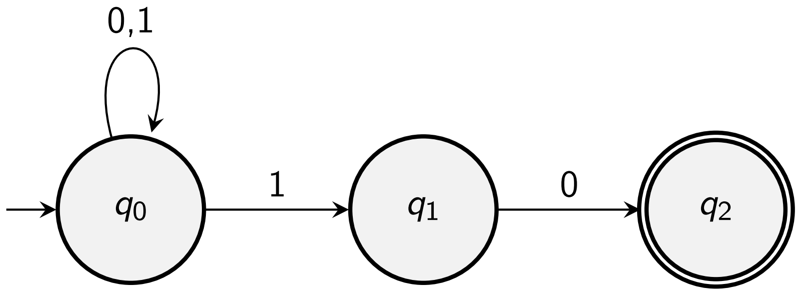


Construction of DFA from NFA (without ϵ-Transitions)

Reduced DFA D:



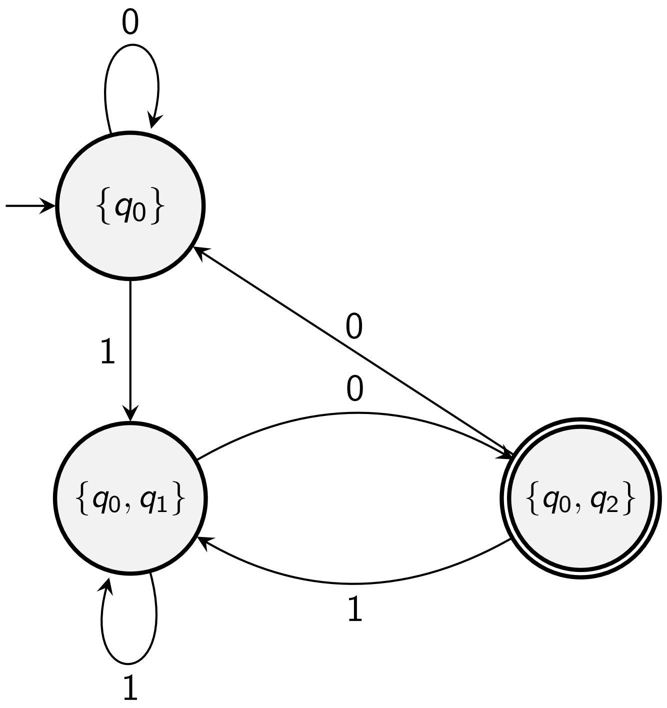
Original NFA N:



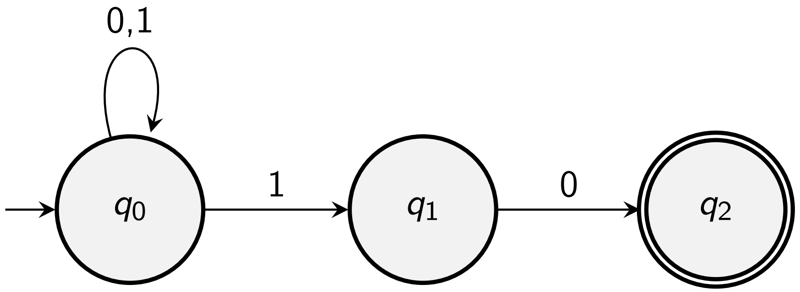
$$L(D) = L(N) = \{w : w \text{ ends with } 10\}$$

Construction of DFA from NFA (without ϵ-Transitions)

Reduced DFA D:



Original NFA N:



$w = 10$

$\{q_0\} \to \{q_0,q_1\} \to \{q_0,q_2\}$

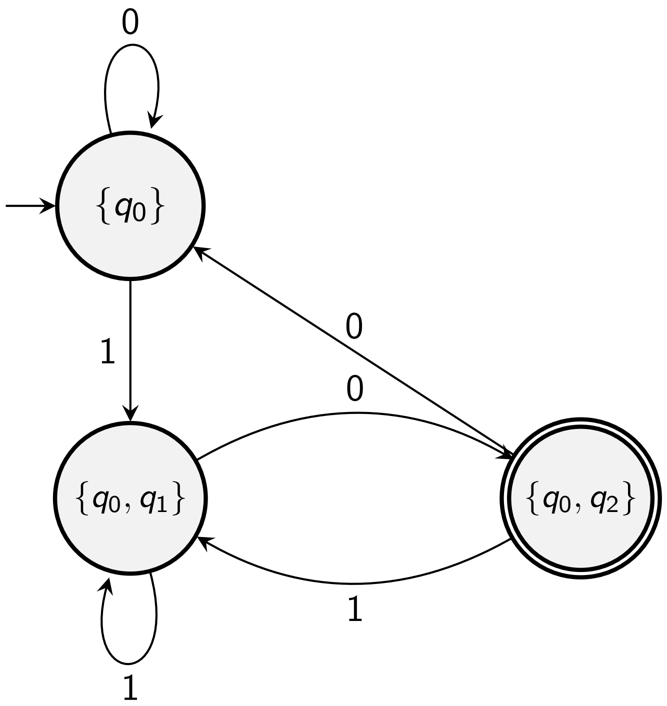
$q_0 \to q_0 \to q_0$

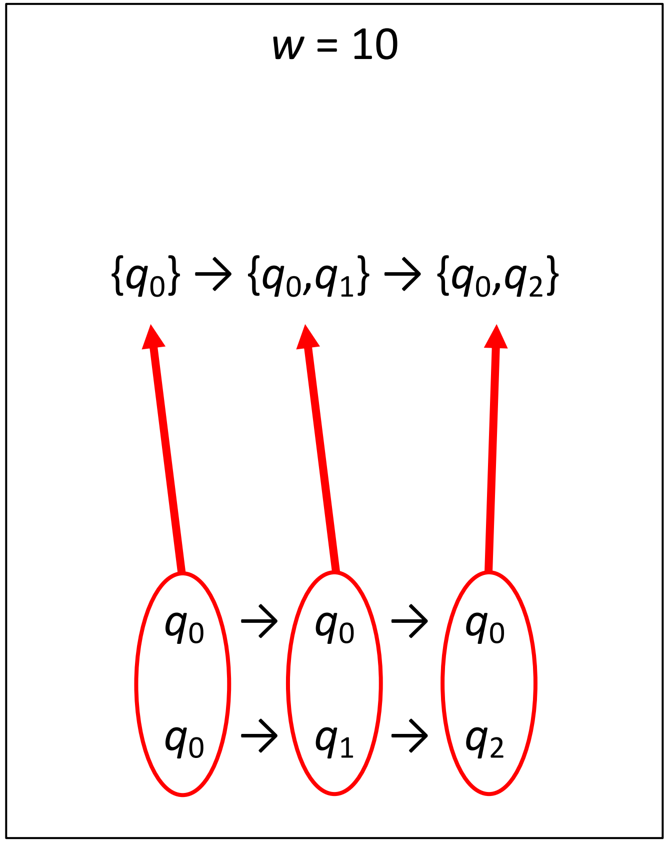
$q_0 \to q_1 \to q_2$

$$L(D) = L(N) = \{w : w \text{ ends with } 10\}$$

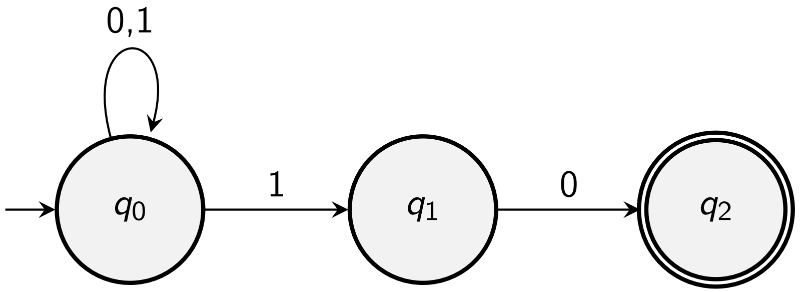
Construction of DFA from NFA (without ϵ-Transitions)

Reduced DFA D:





Original NFA N:



$$L(D) = L(N) = \{w : w \text{ ends with } 10\}$$

Construction of DFA from NFA (without ϵ-Transitions)

The following general theorem holds:

Theorem (3)

Let N be an NFA, without ϵ-transitions, *with language $L(N)$.*

Then there exists a DFA D with language $L(D) = L(N)$.

Proof by Construction:

Given $N = (Q, \Sigma, \delta, q, F)$, we construct $D = (Q^{'}, \Sigma, \delta^{'}, q^{'}, F^{'})$ as follows

- set of states $Q' = \{R : R \subseteq Q\}$
- initial state $q^{'} = \{q\}$
- set of accepting states $F' = \{R : \text{there exists } r \in R \text{ with } r \in F\}$

Construction of DFA from NFA (without ϵ-Transitions)

Proof Cont:

- transition function δ', for each $R \in Q'$ and each $a \in \Sigma$:

$$\underbrace{\delta'(R, a)}_{a\text{-successor of } R \text{ in } D} = \bigcup_{r \in R} \underbrace{\delta(r, a)}_{\text{set of } a\text{-successors of } r \text{ in } N}$$

Sketch of the remaining proof:

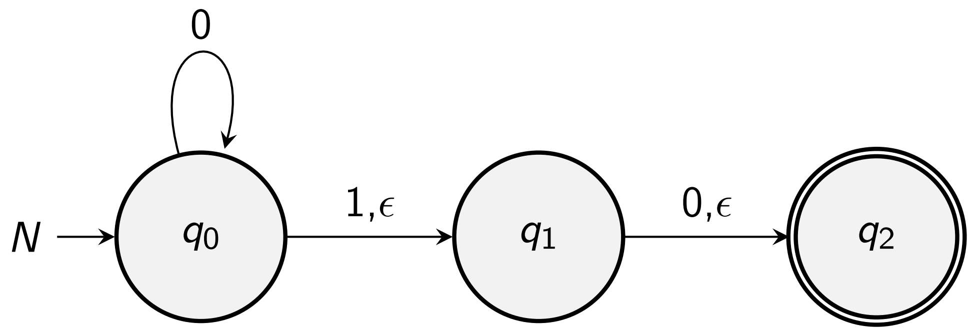
- in the constructed DFA D all possible runs of the NFA N over an input string w are considered **simultaneously**
- based on the definitions of q, q', δ, δ', F, and F' it can be shown that each string accepted by N is accepted by D and vice versa.
- It follows that $L(N) = L(D)$

$\square$

Construction of DFA from NFA (**with** ϵ-Transitions)

Given NFA $N = (Q, \Sigma, \delta, q, F)$.

How to construct DFA $D = (Q', \Sigma, \delta', q', F')$ with $L(D) = L(N)$?



- Q' and F' as before
- Presence of ϵ-transitions requires alternation of q' and δ'
- In which state can N start reading some input string w?
- Answer: q_0, q_1, or q_2. Hence, $\{q_0, q_1, q_2\}$ is initial state of D

ϵ-Closure of States

The idea used to determine the initial state of D is based on the ϵ-closure:

Definition

Let r be a state of an NFA N. Then the ϵ-closure of r, denoted by $C_\epsilon(r)$, is the set of all states that are reachable from r by zero or more ϵ-transitions.

This can be generalised to **sets of states**:

Definition

Let R be a subset of states of an NFA N. Then the ϵ-closure of R is

$$C_\epsilon(R) \;=\; \bigcup_{r \in R} C_\epsilon(r)$$

i.e. the union of all ϵ-closures of states $r \in R$.

Construction of DFA from NFA (**with** ϵ-Transitions)

Theorem (2)

Let N be an NFA, with ϵ-transitions, *with language $L(N)$.*

Then there exists a DFA D with language $L(D) = L(N)$.

Proof:

Given $N = (Q, \Sigma, \delta, q, F)$, we construct $D = (Q', \Sigma, \delta', q', F')$ as follows

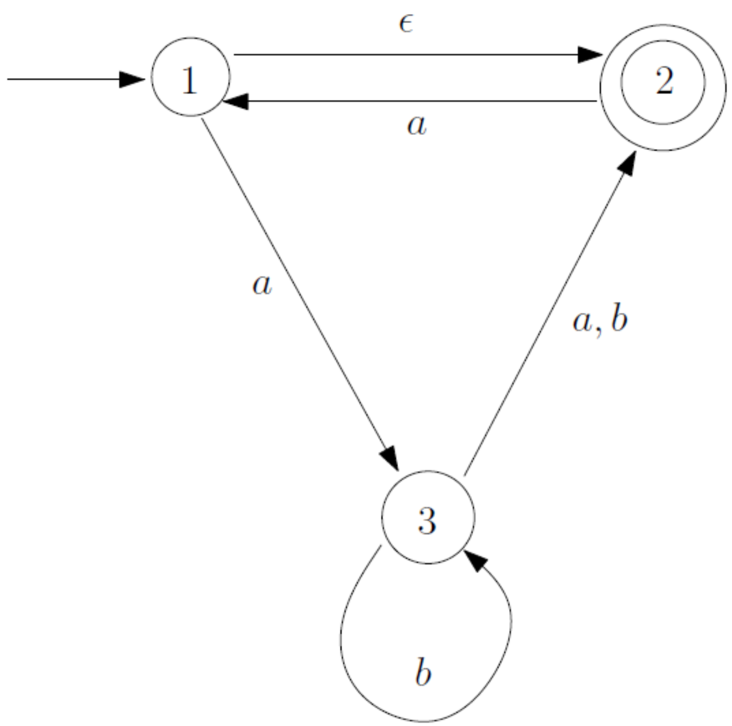
- set of states $Q' = \{R : R \subseteq Q\}$
- **initial state** $q' = C_\epsilon(q)$
- set of accepting states $F' = \{R : \text{there exists } r \in R \text{ with } r \in F\}$
- **transition function**, for each $R \in Q'$ and each $a \in \Sigma$:

$$\underbrace{\delta'(R, a)}_{a\text{-successor of } R \text{ in } D} = \bigcup_{r \in R} C_\epsilon(\underbrace{\delta(r, a)}_{\text{set of } a\text{-successors of } r \text{ in } N})$$

- it can be further shown that $L(D) = L(N)$

Construction of DFA from NFA: Example

Construct the DFA D corresponding to the following NFA N:

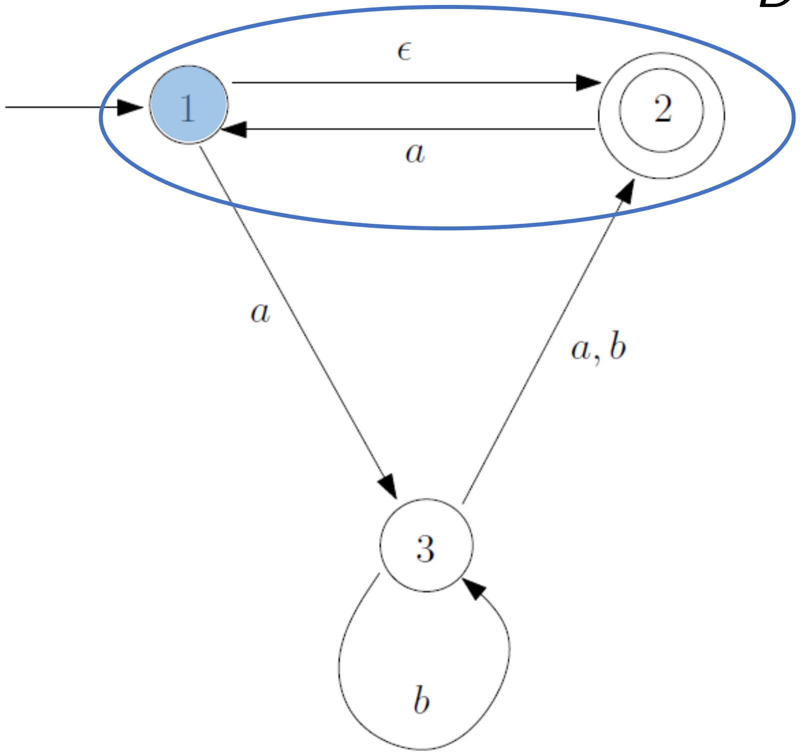


$D = (Q^{'}, \Sigma, \delta^{'}, q^{'}, F^{'})$:

- $\Sigma = \{a, b\}$
- $Q^{'} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- $q^{'} = C_{\epsilon}(\{1\}) = \{1, 2\}$
- $F' = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$

Construction of DFA from NFA: Example

Construct the DFA D corresponding to the following NFA N:

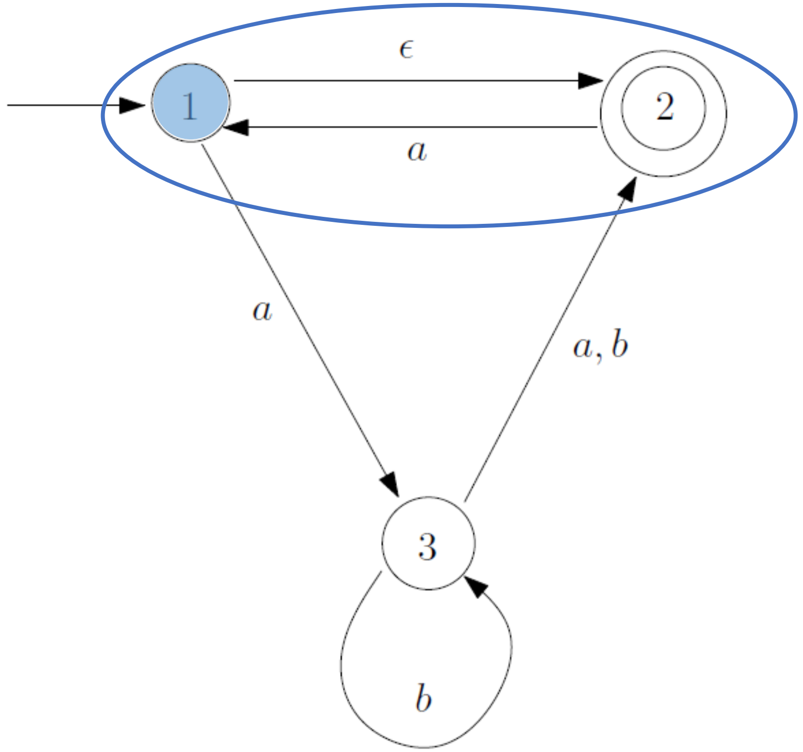


$D = (Q', \Sigma, \delta', q', F')$:

- $\Sigma = \{a, b\}$
- $Q' = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\},$
  $\{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $q' = C_\epsilon(\{1\}) = \{1,2\}$
- $F' = \{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

Construction of DFA from NFA: Example

Construct the DFA D corresponding to the following NFA N:

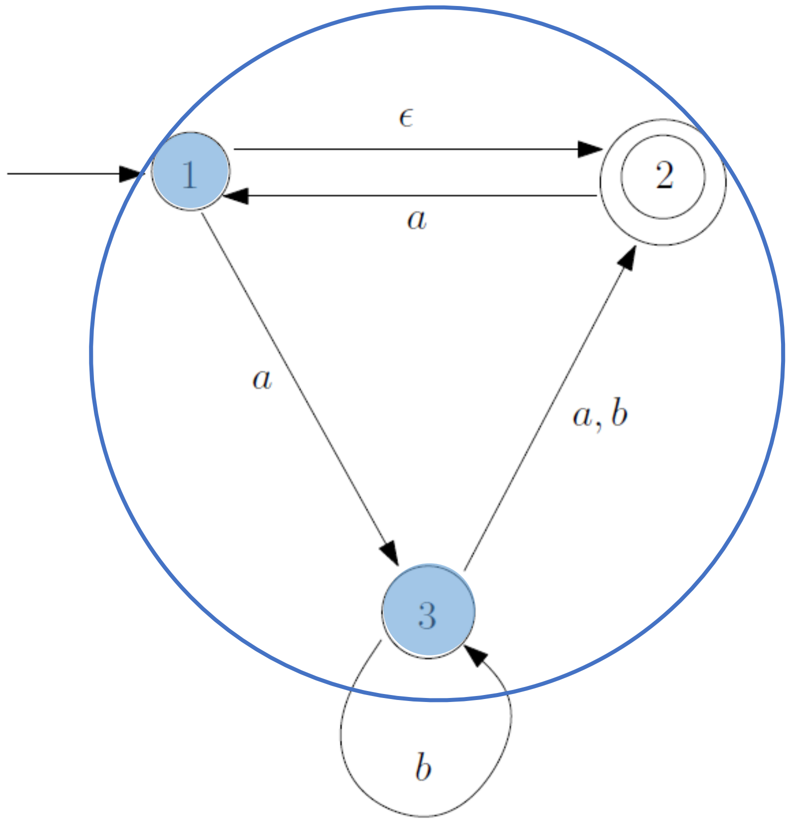


ϵ-closure:

R	$C_\epsilon(R)$
$\{1\}$	$\{1, 2\}$
$\{2\}$	
$\{3\}$	
$\{1, 2\}$	
$\{1, 3\}$	
$\{2, 3\}$	
$\{1, 2, 3\}$	
$\emptyset$	

Construction of DFA from NFA: Example

Construct the DFA D corresponding to the following NFA N:



ϵ-closure:

R	$C_\epsilon(R)$
$\{1\}$	$\{1,2\}$
$\{2\}$	$\{2\}$
$\{3\}$	$\{3\}$
$\{1,2\}$	$\{1,2\}$
$\{1,3\}$	$\{1,2,3\}$
$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$
$\emptyset$	$\emptyset$

Construction of DFA from NFA: Example

Transition function δ':

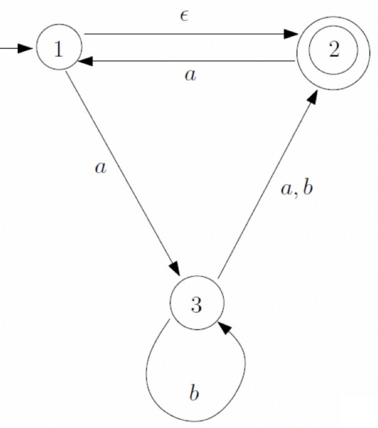
$$\delta'(R, a) = \bigcup_{r \in R} C_\epsilon(\delta(r, a))$$

δ'	a	b
$\{1\}$		
$\{2\}$		
$\{3\}$		
$\{1, 2\}$	?	
$\{1, 3\}$		
$\{2, 3\}$		
$\{1, 2, 3\}$		
$\emptyset$		

Construction of DFA from NFA: Example

Transition function δ':

$$\delta'(R, a) = \bigcup_{r \in R} C_\epsilon(\delta(r, a))$$



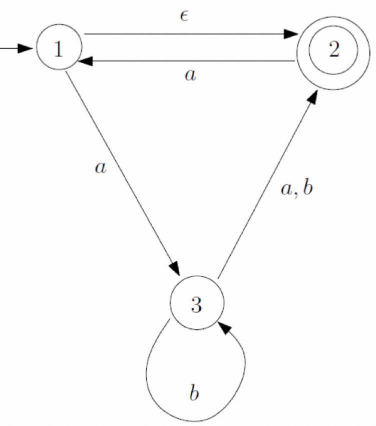
δ	a	b	ϵ
1	$\{3\}$	$\emptyset$	$\{2\}$
2	$\{1\}$	$\emptyset$	$\emptyset$
3	$\{2\}$	$\{2,3\}$	$\emptyset$

δ'	a	b
$\{1\}$		
$\{2\}$		
$\{3\}$		
$\{1,2\}$	?	
$\{1,3\}$		
$\{2,3\}$		
$\{1,2,3\}$		
$\emptyset$		

Construction of DFA from NFA: Example

Transition function δ':

$$\delta'(R, a) = \bigcup_{r \in R} C_\epsilon(\delta(r, a))$$



δ	a	b	ϵ
1	$\{3\}$	$\emptyset$	$\{2\}$
2	$\{1\}$	$\emptyset$	$\emptyset$
3	$\{2\}$	$\{2,3\}$	$\emptyset$

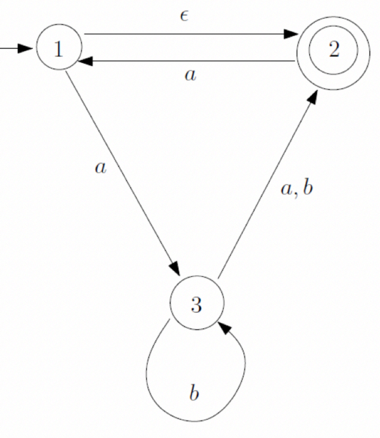
R	$C_\epsilon(R)$
$\{1\}$	$\{1,2\}$
$\{2\}$	$\{2\}$
$\{3\}$	$\{3\}$
$\{1,2\}$	$\{1,2\}$
$\{1,3\}$	$\{1,2,3\}$
$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$
$\emptyset$	$\emptyset$

δ'	a	b
$\{1\}$		
$\{2\}$		
$\{3\}$		
$\{1,2\}$	?	
$\{1,3\}$		
$\{2,3\}$		
$\{1,2,3\}$		
$\emptyset$		

Construction of DFA from NFA: Example

Transition function δ':

$$\delta'(R, a) = \bigcup_{r \in R} C_\epsilon(\delta(r, a))$$



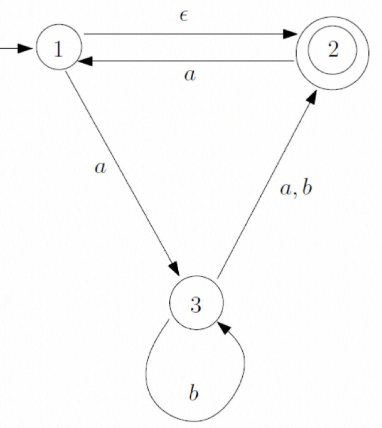
δ	a	b	ϵ
1	$\{3\}$	$\emptyset$	$\{2\}$
2	$\{1\}$	$\emptyset$	$\emptyset$
3	$\{2\}$	$\{2,3\}$	$\emptyset$

R	$C_\epsilon(R)$
$\{1\}$	$\{1,2\}$
$\{2\}$	$\{2\}$
$\{3\}$	$\{3\}$
$\{1,2\}$	$\{1,2\}$
$\{1,3\}$	$\{1,2,3\}$
$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$
$\emptyset$	$\emptyset$

δ'	a	b
$\{1\}$		
$\{2\}$		
$\{3\}$		
$\{1,2\}$	$\{1,2,3\}$	
$\{1,3\}$		
$\{2,3\}$		
$\{1,2,3\}$		
$\emptyset$		

Construction of DFA from NFA: Example

Transition function δ':



$$\delta'(R, a) = \bigcup_{r \in R} C_\epsilon(\delta(r, a))$$

δ	a	b	ϵ
1	$\{3\}$	$\emptyset$	$\{2\}$
2	$\{1\}$	$\emptyset$	$\emptyset$
3	$\{2\}$	$\{2,3\}$	$\emptyset$

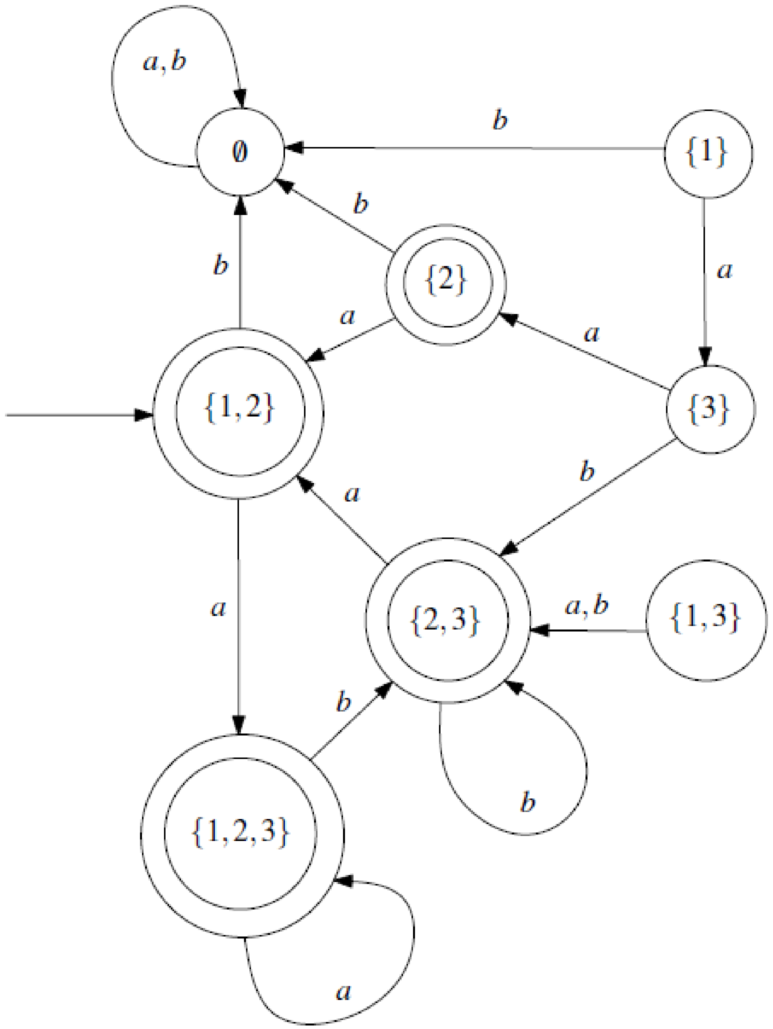
R	$C_\epsilon(R)$
$\{1\}$	$\{1,2\}$
$\{2\}$	$\{2\}$
$\{3\}$	$\{3\}$
$\{1,2\}$	$\{1,2\}$
$\{1,3\}$	$\{1,2,3\}$
$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$
$\emptyset$	$\emptyset$

δ'	a	b
$\{1\}$	$\{3\}$	$\emptyset$
$\{2\}$	$\{1,2\}$	$\emptyset$
$\{3\}$	$\{2\}$	$\{2,3\}$
$\{1,2\}$	$\{1,2,3\}$	$\emptyset$
$\{1,3\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2\}$	$\{2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$
$\emptyset$	$\emptyset$	$\emptyset$

$$\begin{aligned}\delta'(\{1,2\}, a) &= C_\epsilon(\delta(1, a)) \cup C_\epsilon(\delta(2, a)) \\ &= C_\epsilon(\{3\}) \cup C_\epsilon(\{1\}) \\ &= \{3\} \cup \{1,2\} = \{1,2,3\}\end{aligned}$$

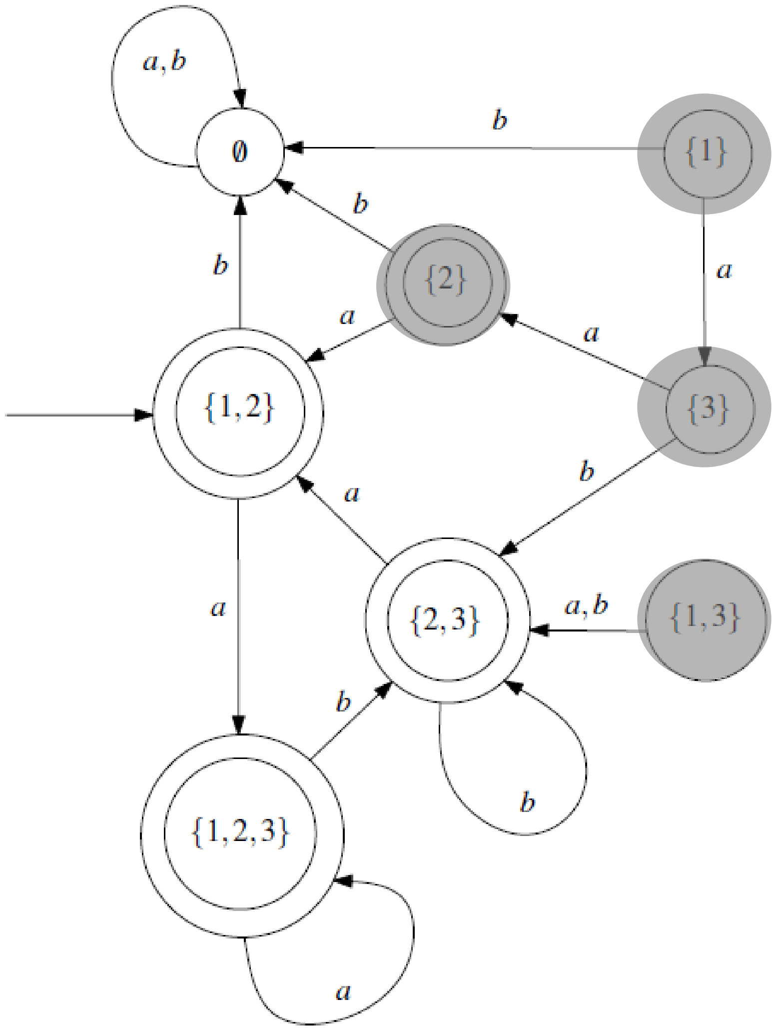
Construction of DFA from NFA: Example

Now we can draw the DFA:



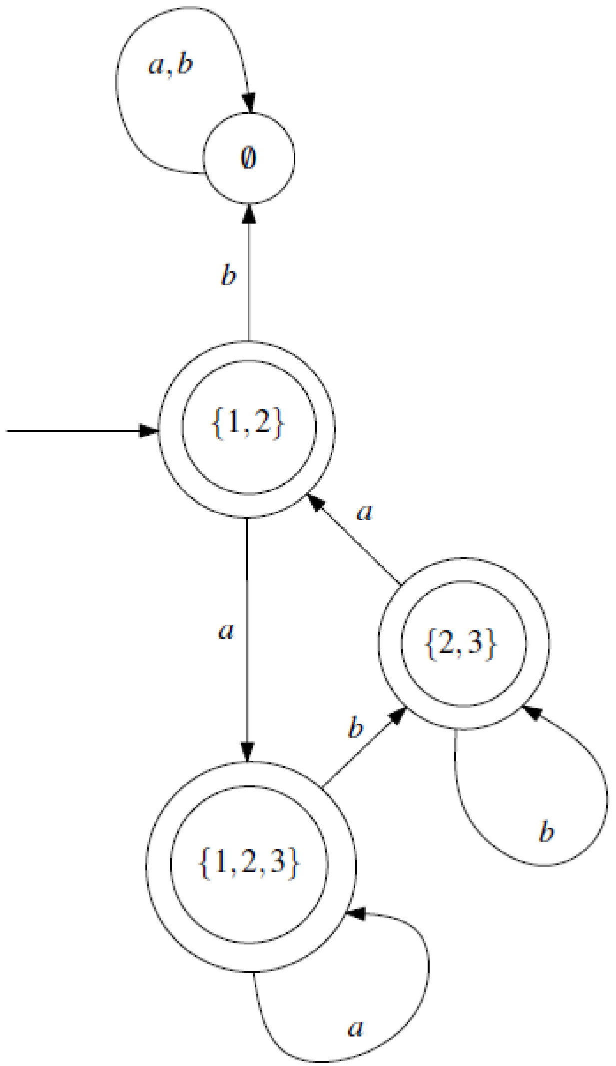
Construction of DFA from NFA: Example

Now we can draw the DFA:



Construction of DFA from NFA: Example

Reduced DFA:



Original NFA: