Direct Proof; Example

T: n odd $\longrightarrow$ n^2 odd
(premise) (conclusion)

Def. (Even Integer)

$n \in \mathbb{Z}$ even $\longleftrightarrow$ $n = 2 \cdot k$ for some $k \in \mathbb{Z}$

Def. (Odd Integer)

n odd $\longleftrightarrow$ $n = 2 \cdot k + 1$ for some $k \in \mathbb{Z}$

Proof of T:

n odd

$\longleftrightarrow$ $n = 2 \cdot k + 1$ for some $k \in \mathbb{Z}$ (Def. Odd)

$\longleftrightarrow$ $n^2 = (2k+1)^2$

$\longleftrightarrow$ $n^2 = 2 \cdot 2k^2 + 2 \cdot 2k + 1$

$\longleftrightarrow$ $n^2 = \underbrace{\underbrace{2 \cdot (2k^2 + 2k)}_{\text{even}} + 1}_{\text{odd}}$

$\longrightarrow$ n^2 odd (Def. Odd)

$\square$

QED

Constructive Proof: Example

T: $a, b \in \mathbb{R},\ a < b \longrightarrow \exists c \in \mathbb{R}:\ a < c < b$

Creating the object c:

$$\boxed{\begin{array}{ccc} 3 & 3.5 & 4 \\ \\ 3.5 & = & \dfrac{3+4}{2} \end{array}}$$

$$c = \frac{a+b}{2} \qquad \text{object}$$

---

to prove:
$$a < \frac{a+b}{2} < b$$

Part 1: $a < \dfrac{a+b}{2}$ (?)

$\Longleftrightarrow 2a < a + b$

$\Longleftrightarrow a + a < a + b \quad | -a$

$\Longleftrightarrow a < b$

$\Longleftrightarrow$ true (premise)

Part 2: $\dfrac{a+b}{2} < b$ (?)

$\Longleftrightarrow a + b < 2b$

$\Longleftrightarrow a + b < b + b \quad | -b$

$\Longleftrightarrow a < b$

$\Longleftrightarrow$ true (premise) $\square$

Proof by Contradiction

$T: n \in \mathbb{Z}^+ .\ n^2 \text{ even} \rightarrow n \text{ even}$

Proof:

Assume n^2 even $\wedge$ n odd

$$\rightarrow n = 2k + 1 \quad \text{for some } k$$

$$\leftrightarrow n^2 = (2k+1)^2$$

$$\leftrightarrow n^2 = \underbrace{\underbrace{2(2k^2+2k)}_{\text{even}} + 1}_{\text{odd}}$$

$\rightarrow n^2$ odd

(contradiction to assumption that n^2 is even)

$\rightarrow T$ is true $\square$

Proof by Contradiction

$T \quad : x \in \mathbb{Q}, y \notin \mathbb{Q} \rightarrow (x+y) \notin \mathbb{Q}$

Proof : $x \in \mathbb{Q}, y \notin \mathbb{Q}$

Assume $x + y = z, \quad z \in \mathbb{Q}$

$\Longleftrightarrow \frac{a}{b} + y = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$

$\Longleftrightarrow y = \frac{c}{d} - \frac{a}{b}$

$\Longleftrightarrow y = \frac{c \cdot b}{d \cdot b} - \frac{a \cdot d}{d \cdot b}$ common denominator

$\Longleftrightarrow y = \frac{c \cdot b - a \cdot d}{d \cdot b} \begin{matrix} \in \mathbb{Z} \\ \in \mathbb{Z} \end{matrix}$

$\longrightarrow$ y is rational ($y \in \mathbb{Q}$)

contradiction

$\longrightarrow$ T is true $\square$