COS210 - Theoretical Computer Science
Proofs (Part 2)

Non-Constructive Proofs

Remember:
In a *constructive proof*, we construct an object O and show that O satisfies a property P

In contrast,

- in a *non-constructive proof* we **only** show that an object with property P exists,
- but we do not **construct** the specific object O.

Non-Constructive Proofs: Example

Theorem

There exist irrational numbers x and y such that x^y is rational.

Proof:

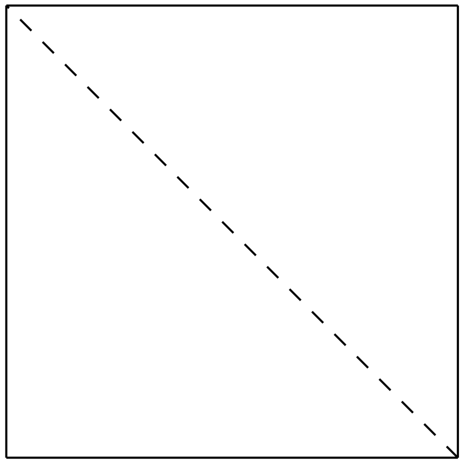
Pigeon Hole Principle

Definition (Pigeon Hole Principle)

If $n+1$ or more objects are placed into n boxes, then there is at least one box that contains two or more objects.

The pigeon hole principle can used as an argument in proofs

Pigeon Hole Principle: Application



Theorem

Let S be a square with diameter d. For any five points $p_1, \ldots, p_5$ within S, there will be at least two points with a distance of at most $d/2$.

Proof:

Proof by Contrapositive

The following conditional theorems are equivalent

Theorem (X)

$$A \rightarrow B$$

Theorem (Y)

$$\neg B \rightarrow \neg A$$

- Argumentation:
  $(A \rightarrow B) = \neg A \vee B = B \vee \neg A = \neg(\neg B) \vee \neg A = (\neg B \rightarrow \neg A)$
- Theorem Y is the *contrapositive* of Theorem X
- If we directly prove Theorem Y, then we prove Theorem X by contrapositive

Proof by Contrapositive: Example

In some cases a proof by contrapositive can be easier than a direct proof

Theorem (X)

$$(x^4 - x^3 + x^2 \neq 1) \rightarrow (x \neq 1)$$

Theorem (Y)

$$(x = 1) \rightarrow (x^4 - x^3 + x^2 = 1)$$

Proof of Y:

$$x = 1$$

$$\rightarrow x^4 - x^3 + x^2 = (1)^4 - (1)^3 + (1)^2 = 1 \quad \square$$

Proof by Induction

Proof technique for mathematical statements that hold for all natural numbers $n = (0), 1, 2, 3, \ldots$

Theorem

For all natural numbers $n \geq 1$:

The statement $S(n)$ holds.

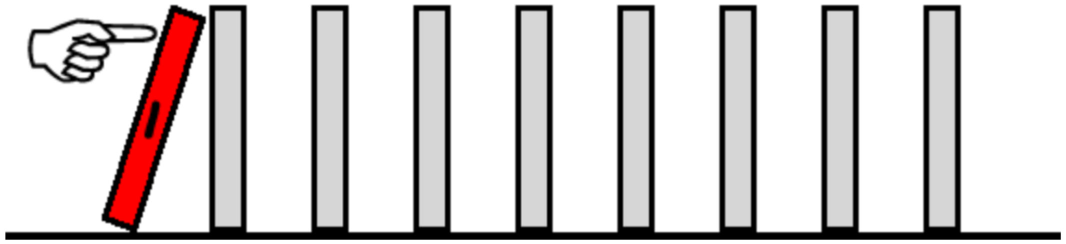
Inductive Proof:

- Base case: Prove $S(1)$
- Inductive step: Prove $S(k) \rightarrow S(k+1)$ for an arbitrary k
  If $S(k)$ holds (Hypothesis), then $S(k+1)$ holds as well
- Theorem then follows from proven base case and step

Note that the range of n may not always start with the number 1

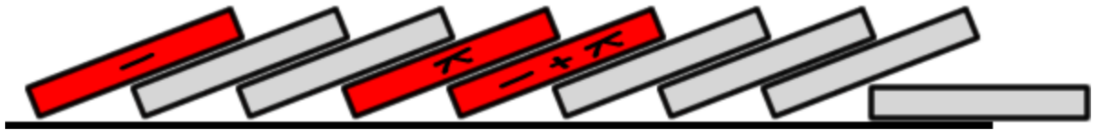
Proof by Induction: Domino Illustration

Base case:



Inductive step:

Conclusion:



Proof by Induction: Example

Theorem

For all $n \geq 1$:

$$S(n) \;:\; \sum_{i=1}^{n} i \;(= 1 + 2 + 3 + \cdots + n) \;=\; \frac{n(n+1)}{2}$$