Non-Constructive Proof

$$T : \underbrace{\exists\, x, y \notin \mathbb{Q}}_{\text{Object}} : \underbrace{x^y \in \mathbb{Q}}_{\text{Property}}$$

Proof:

A well-known irrational number is $\sqrt{2}$

We now consider the number $\sqrt{2}^{\sqrt{2}}$ which might be rational (Case I) or irrational (Case II)

Case I: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$

then we can choose

$x = \sqrt{2}$

$y = \sqrt{2}$

and the theorem is proven for this case

Case II: $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$

(we know that $\sqrt{2} \notin \mathbb{Q}$ as well)

we now choose

$x = \sqrt{2}^{\sqrt{2}}$

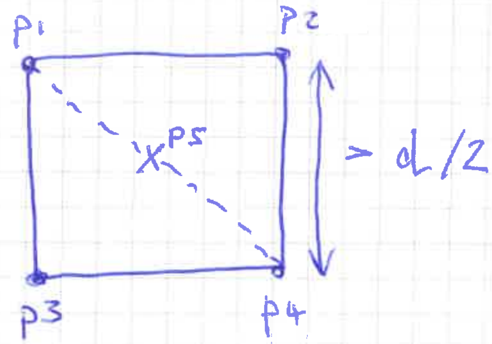
$y = \sqrt{2}$

now
$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2} \cdot \sqrt{2}}$$
$$= (\sqrt{2})^2$$
$$= 2$$

$\rightarrow x^y \in \mathbb{Q} \quad \square$

Pigeon Hole Principle

Proof <u>Idea</u> for T :



Wherever we place p5, the distance to some other point will be $d/2$ or less

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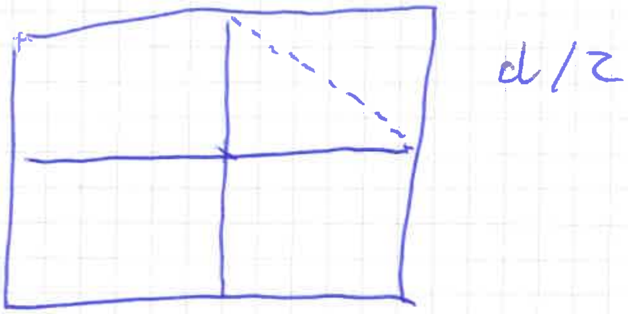
Proof of T based on Pigeon Hole Principle:

Partition S into 4 equal sub-squares



The max. distance in each sub-square is $d/2$

If we place 5 points in 4 squares there must be a square that contains at least two points

→ there will be at least two points with distance $d/2$ or less

T : $\forall n \geq 1$:

$$S(n): \sum_{i=1}^{n} i \;(= 1 + 2 + \dots + n) = \frac{n(n+1)}{2}$$

Base Case:

Show that $S(1): \sum_{i=1}^{1} i = \dfrac{1 \cdot (1+1)}{2}$ holds

Proof: $\sum_{i=1}^{1} i = 1 = \dfrac{2}{2} = \dfrac{1 \cdot 2}{2} = \dfrac{1 \cdot (1+1)}{2}$ $\square$

~~[illegible]~~ Inductive step: $S(k) \to S(k+1)$

Assume $S(k): \sum_{i=1}^{k} i = \dfrac{k(k+1)}{2}$ holds for some k

(Hypothesis)

Show that $S(k+1): \sum_{i=1}^{k+1} i = \dfrac{(k+1)(k+2)}{2}$ holds

Proof:
$$\sum_{i=1}^{k+1} i = 1 + 2 + \dots + k + (k+1)$$
$$= \left(\sum_{i=1}^{k} i\right) + (k+1)$$
$$= \frac{k \cdot (k+1)}{2} + (k+1) \qquad \text{(Hyp.)}$$
$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+2) \cdot (k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2} \qquad \square$$