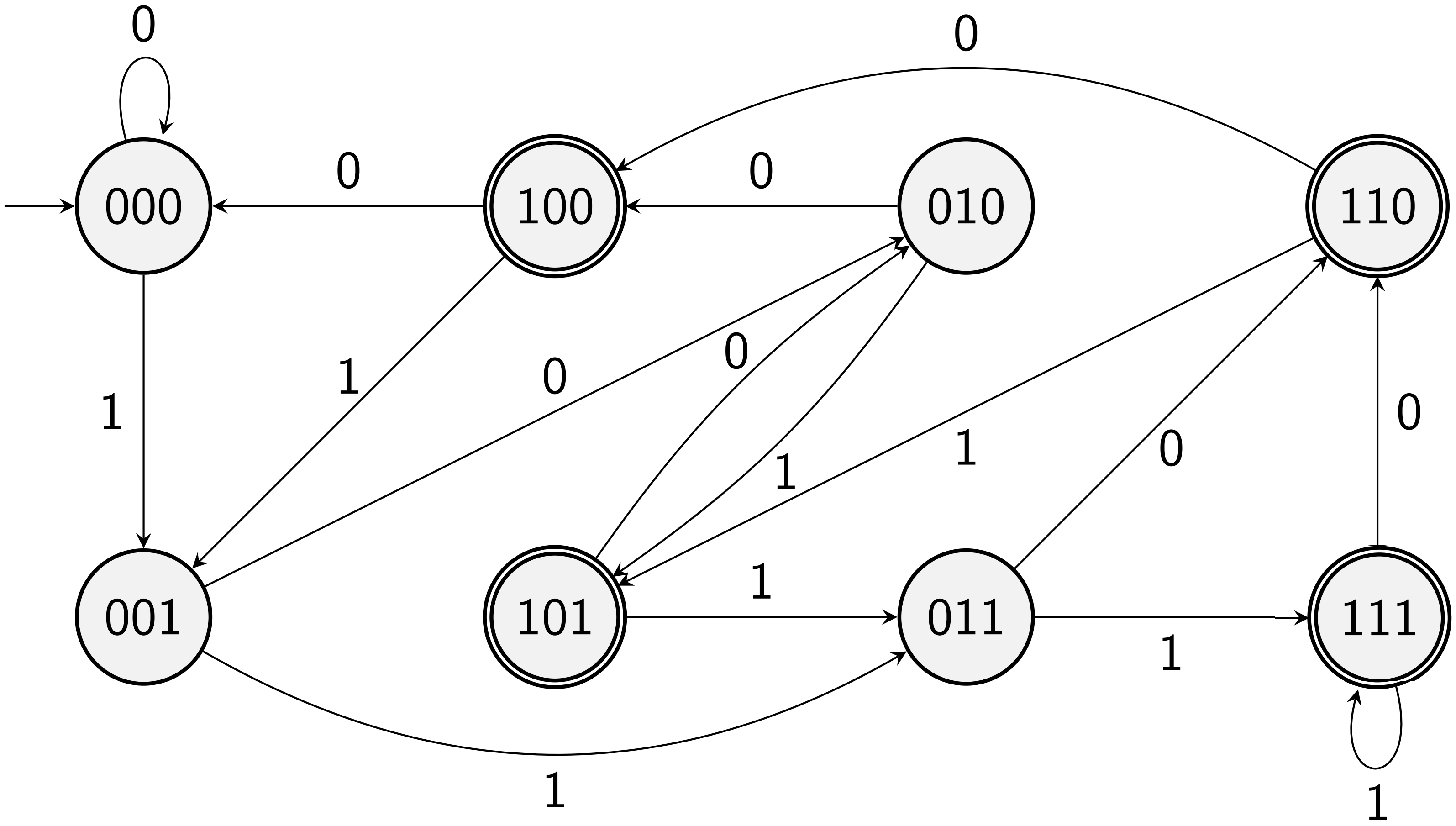
COS210 - Theoretical Computer Science
Finite Automata and Regular Languages (Part 3)

Nondeterministic Finite Automata (NFA) - Motivation

$$A = \{w \in \{0,1\}^* : w \text{ has a 1 in the third postion from the right}\}$$

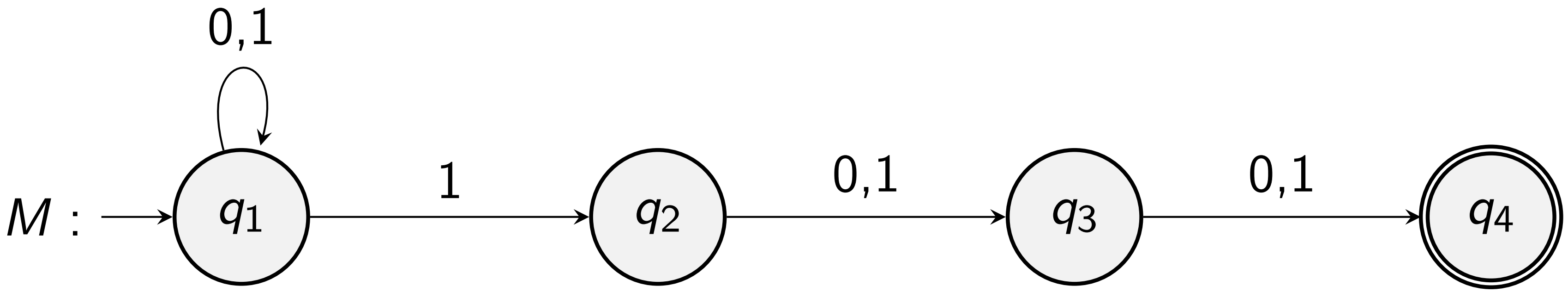
Deterministic solution:



Nondeterministic Finite Automata (NFA) - Motivation

$$A = \{w \in \{0,1\}^* : w \text{ has a 1 in the third postion from the right}\}$$

Alternative solution?



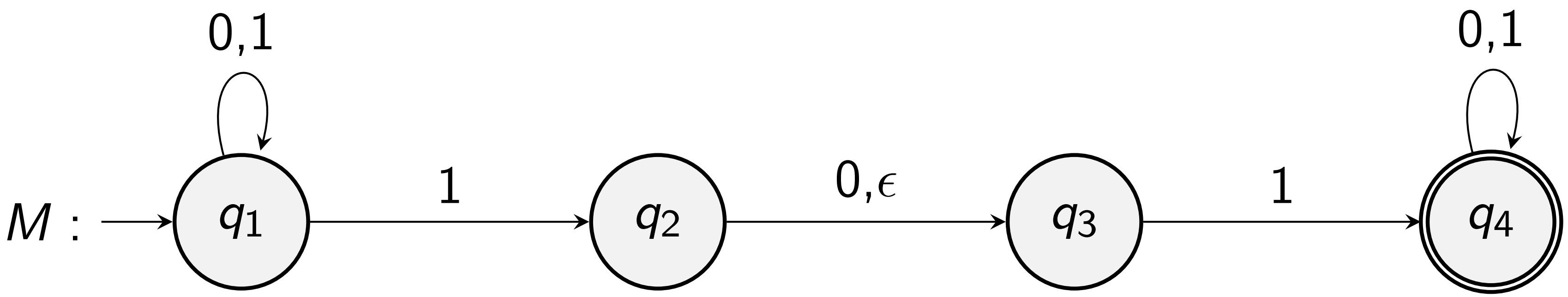
Questions:

- What happens for input 1 in state q_1?
- What happens for any input in state q_4?
- Does M accept 1111 and reject 1000?

Alternative definitions of the **transition function** and of **acceptance** needed

Nondeterministic Finite Automata - Example

$$L(M) = \{w \in \{0,1\}^* : w \text{ contains the substring 101 or 11}\}$$



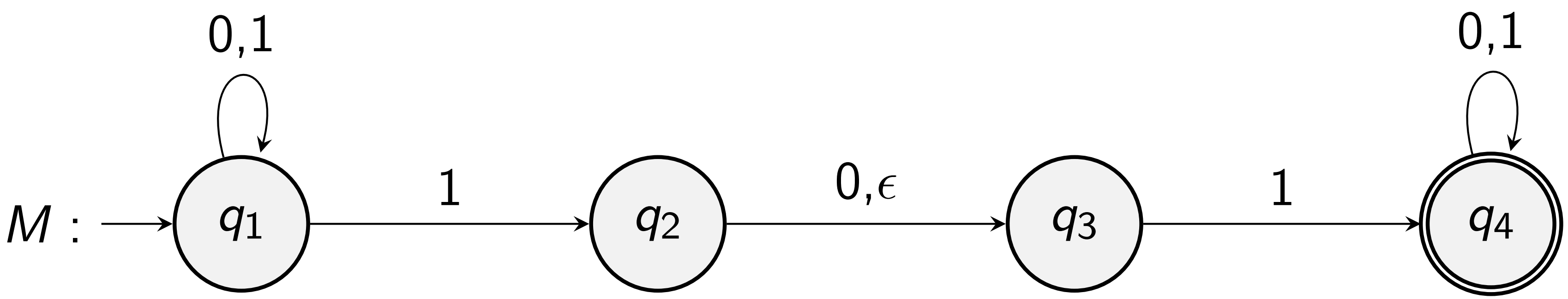
Visual differences to a DFA:

- In state q_1 there are two **choices** for an input of 1
- Transition **without an input symbol** is possible (q_2 to q_3)
- The automata can **hang** in state q_3 if input of 0 is read

Nondeterministic Finite Automata - Example

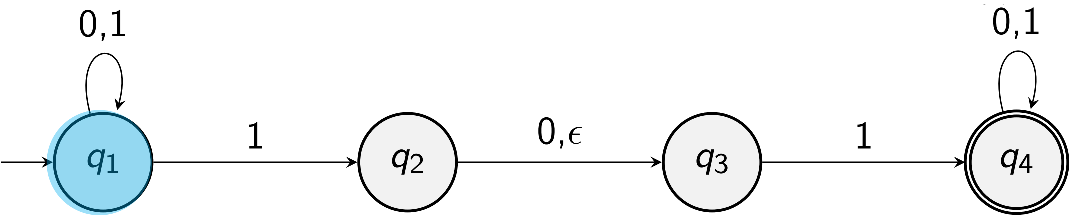
Runs in an automaton:

Let $w = 010110$

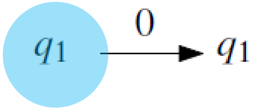


- In a DFA there is **exactly one run** over an input string w
- In an NFA there may be **multiple runs** over a string w

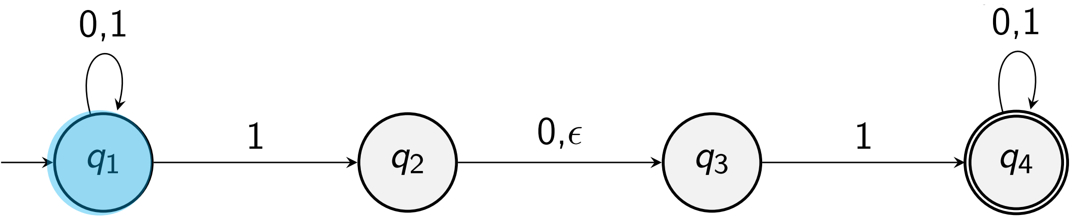
Nondeterministic Finite Automata - Example



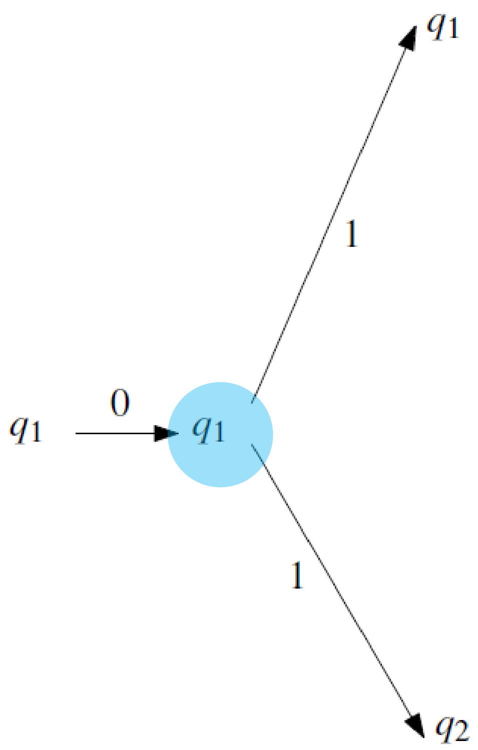
Possible runs over $w = \boxed{0}10110$:



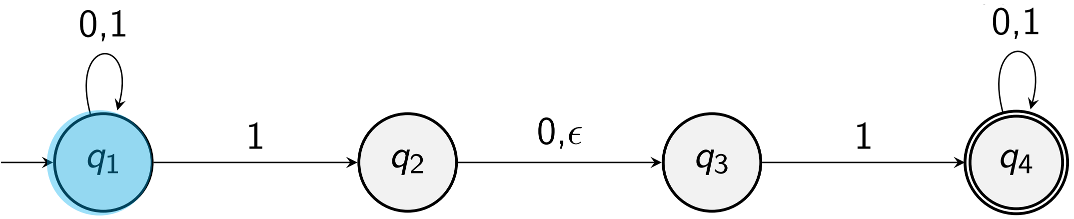
Nondeterministic Finite Automata - Example



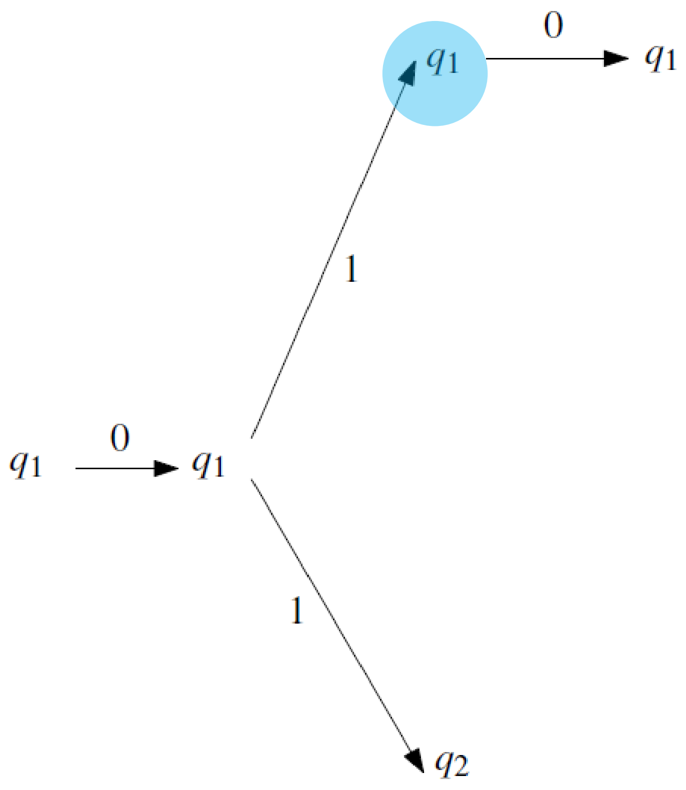
Possible runs over $w = 0\boxed{1}0110$:



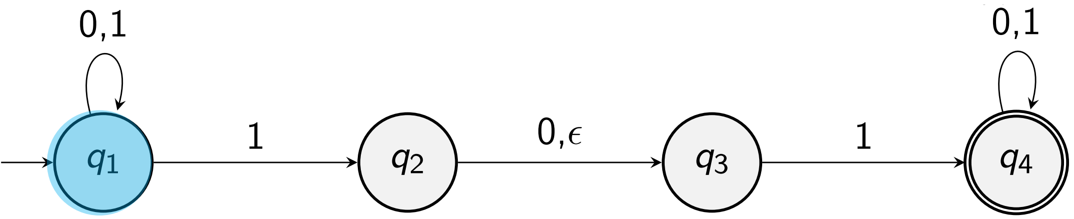
Nondeterministic Finite Automata - Example



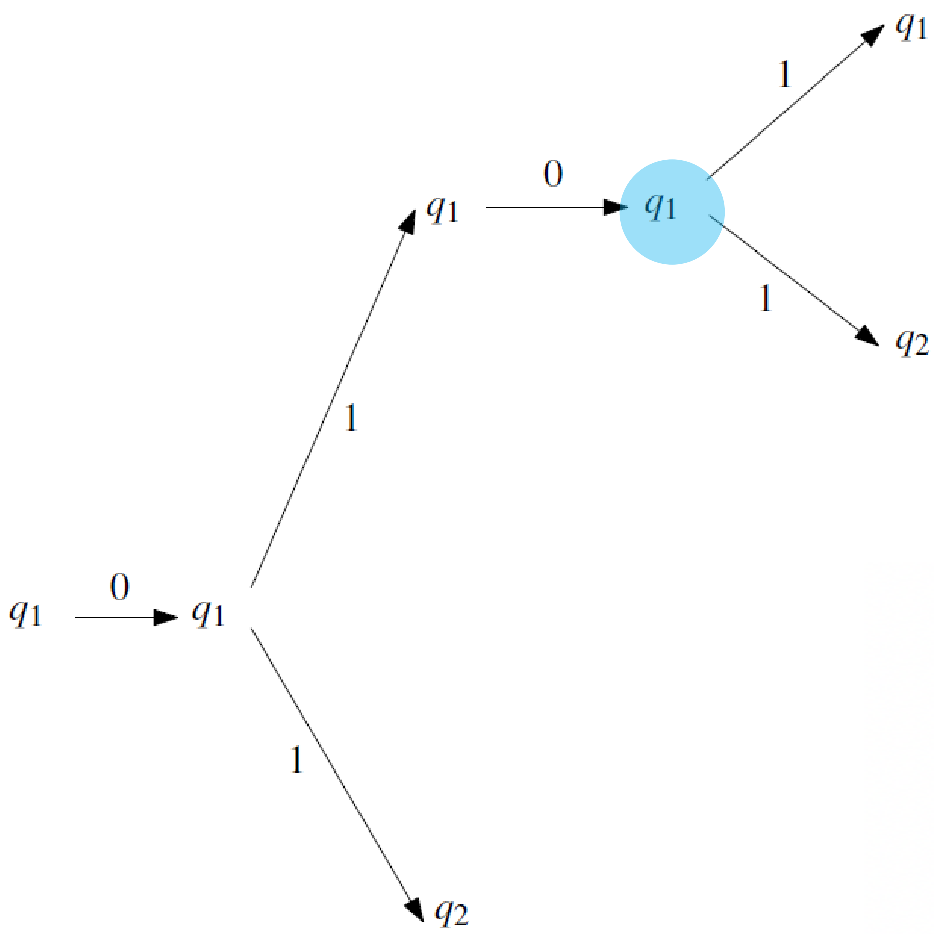
Possible runs over $w = 01\boxed{0}110$:



Nondeterministic Finite Automata - Example



Possible runs over $w = 010\boxed{1}10$:



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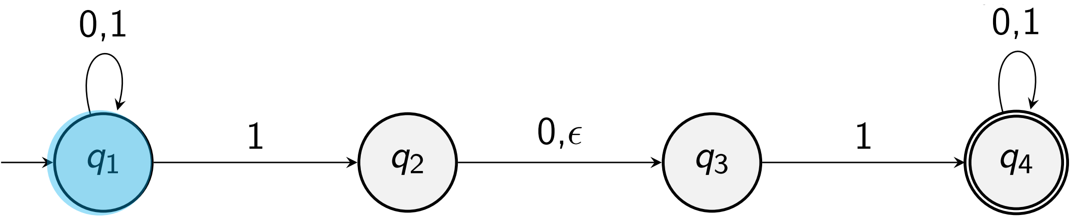
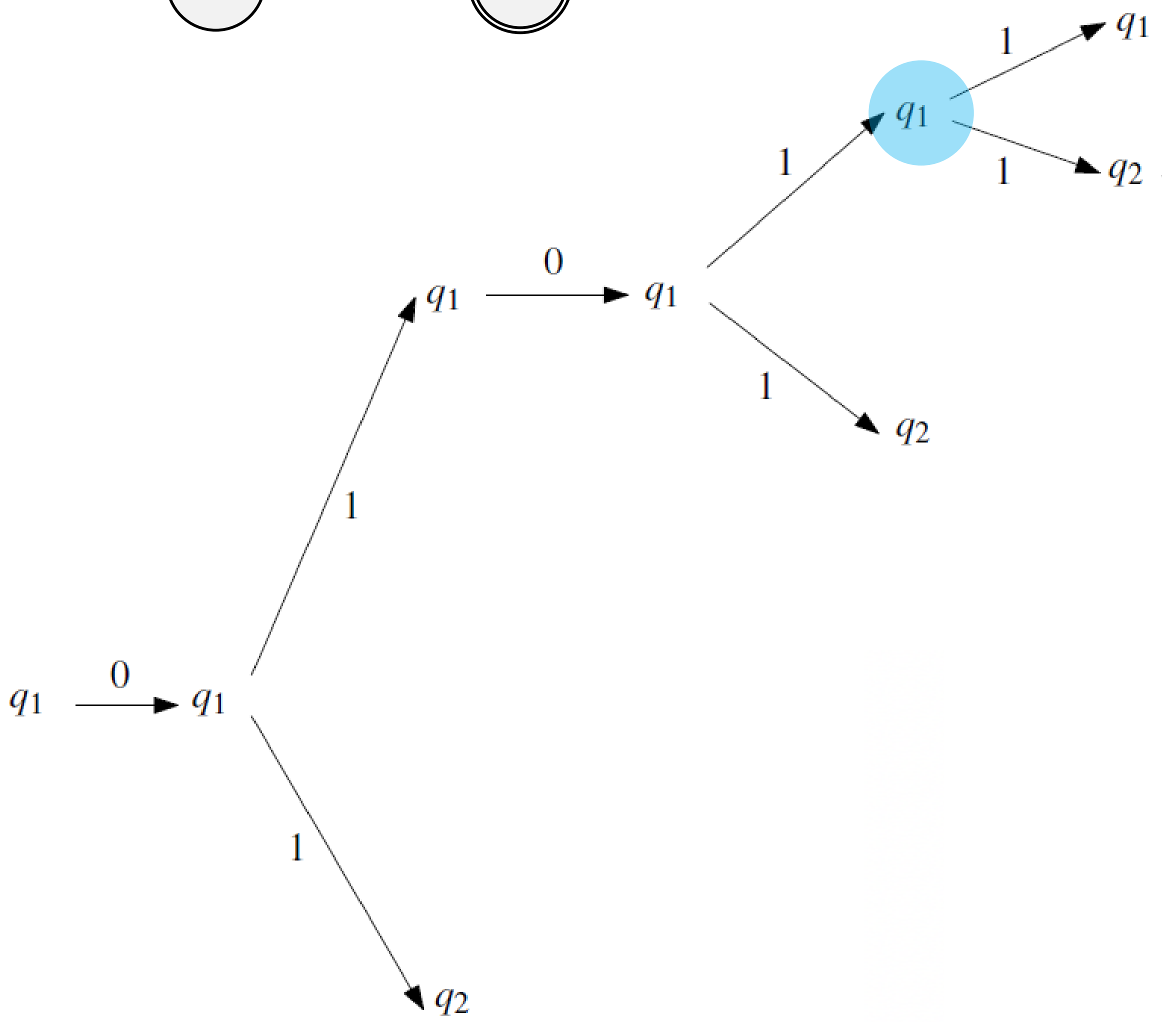


Possible runs over $w = 0101\boxed{1}0$:



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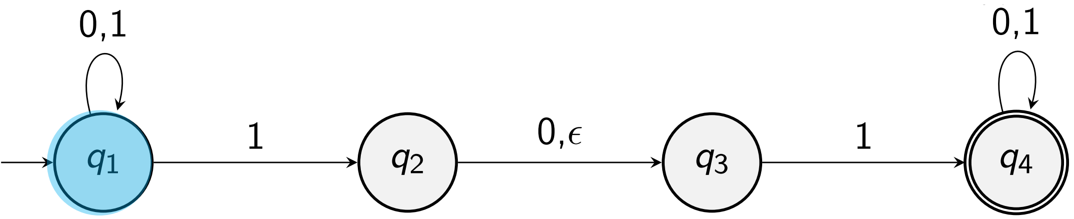
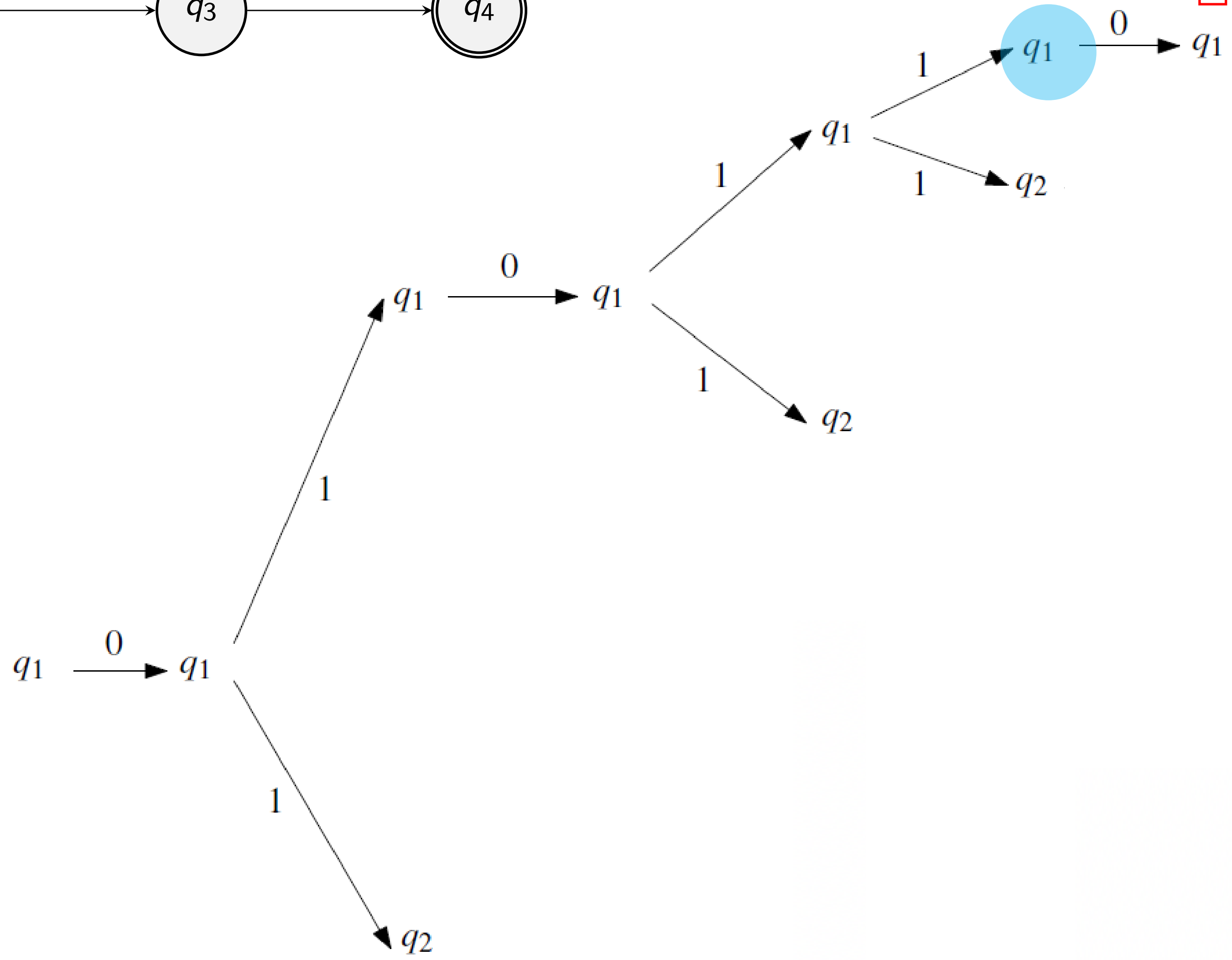
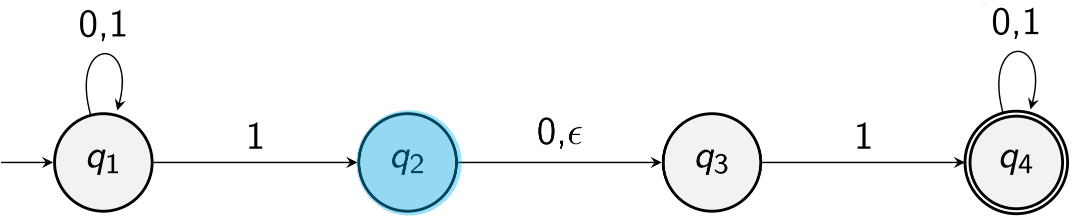


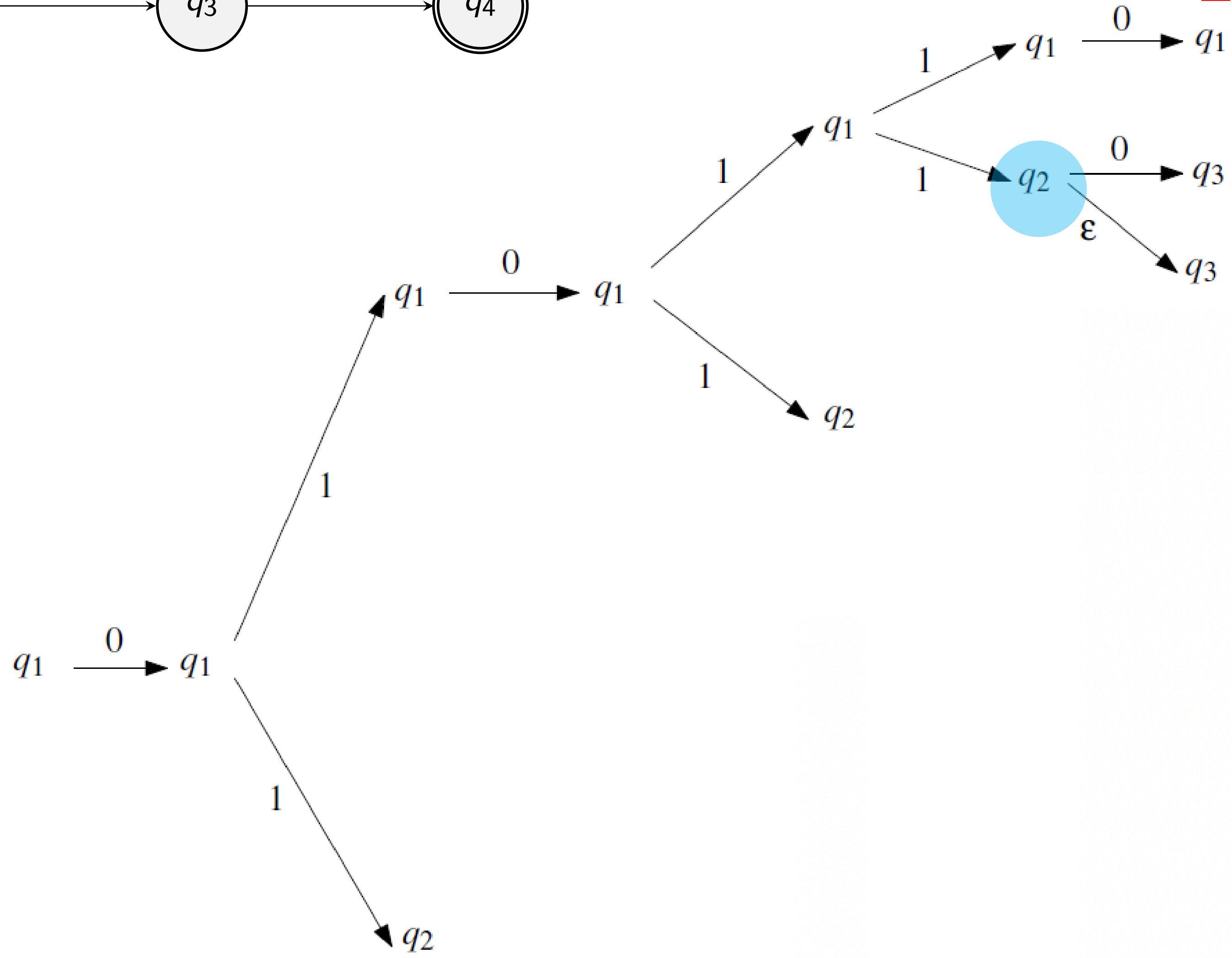
Possible runs over $w = 010110$:



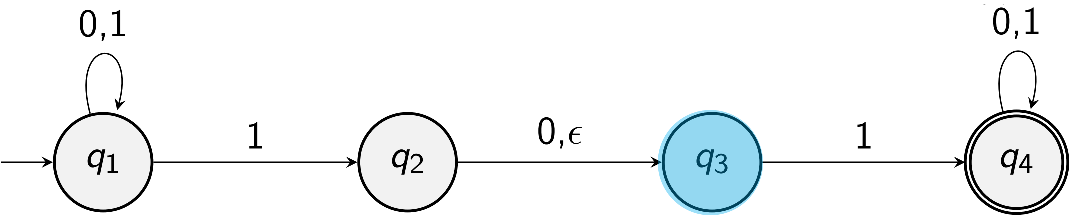
Nondeterministic Finite Automata - Example



Possible runs over $w = 01011\boxed{0}$:



Nondeterministic Finite Automata - Example



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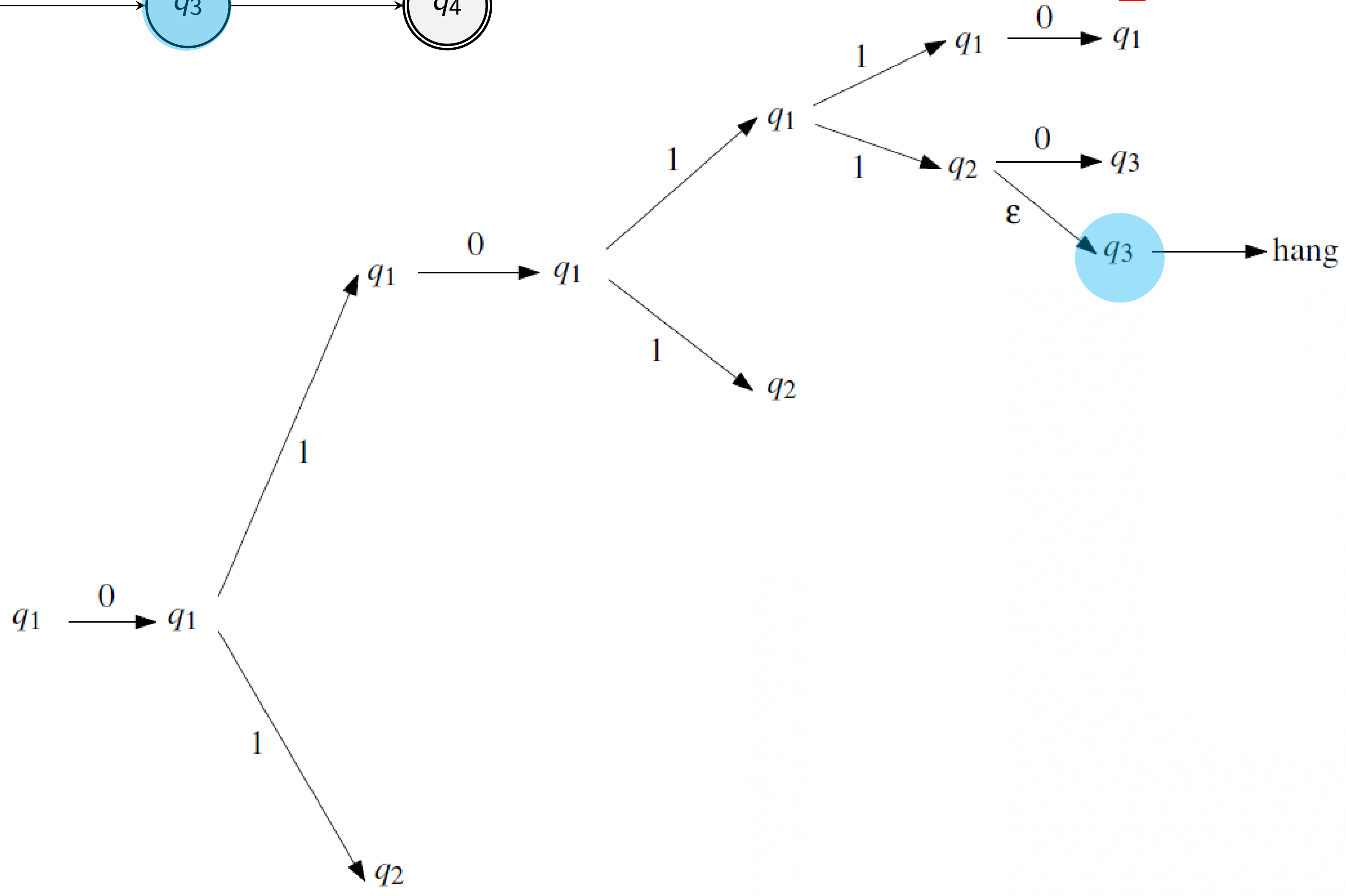
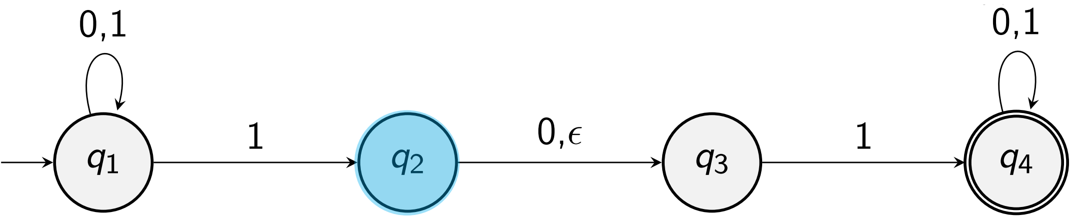
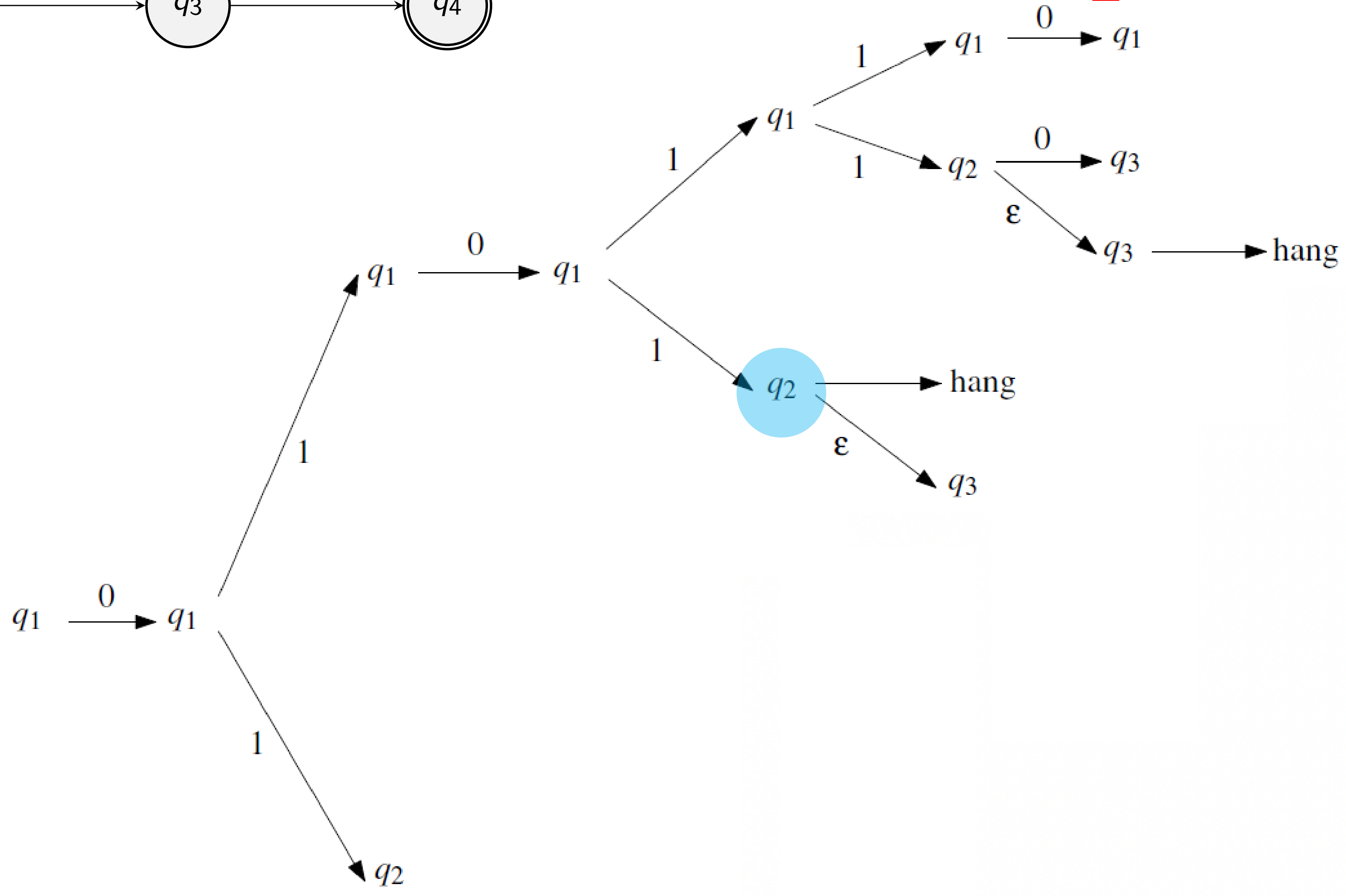


Nondeterministic Finite Automata - Example



Possible runs over $w = 010\boxed{1}10$:



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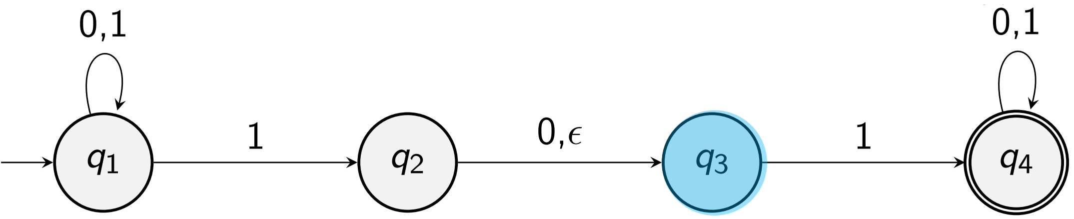
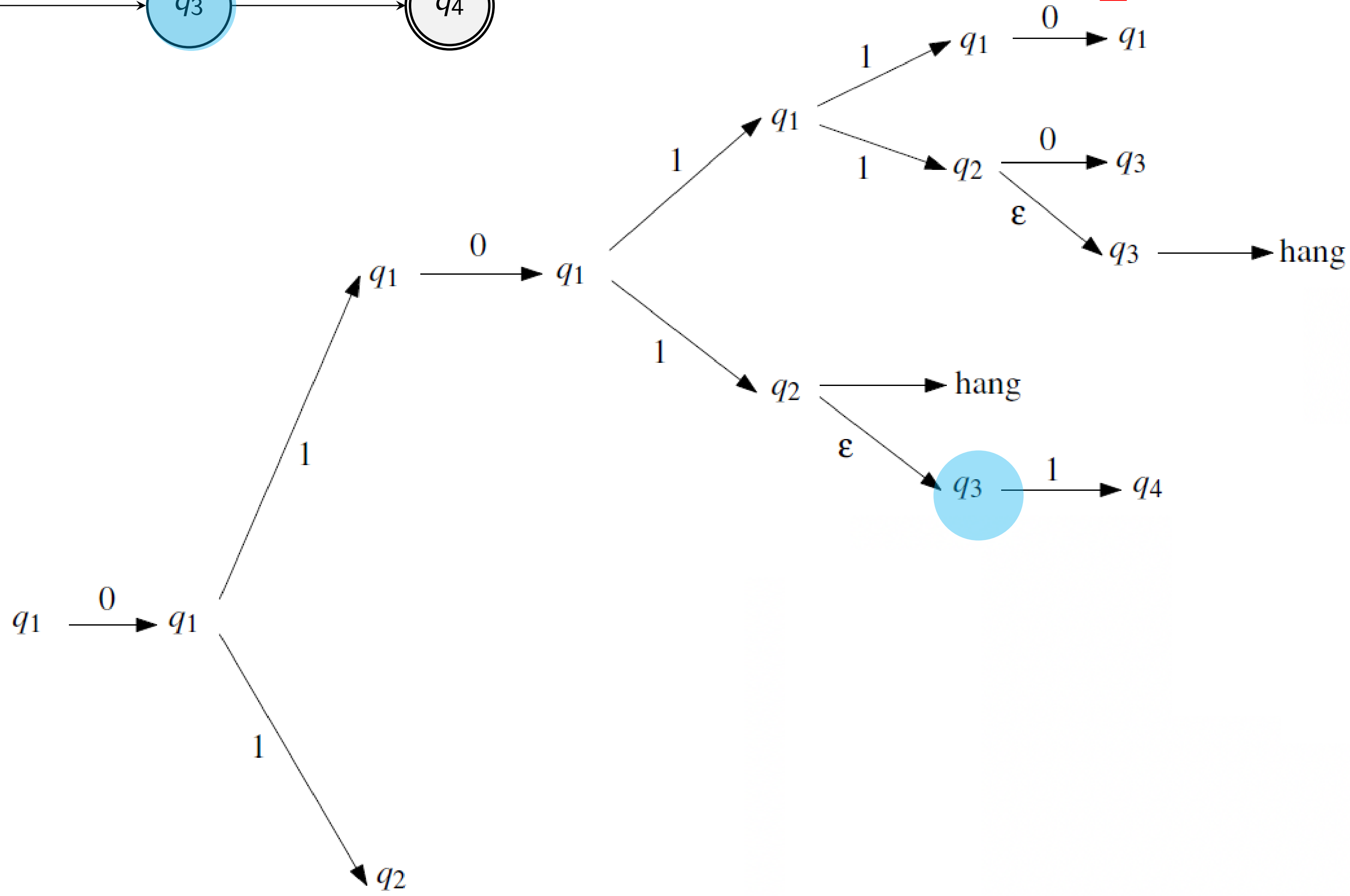


Possible runs over $w = 010\boxed{1}10$:



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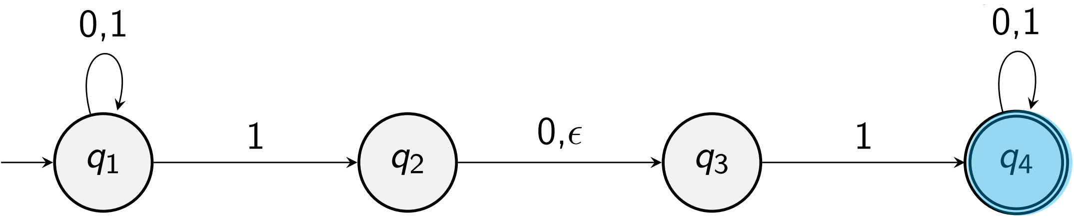
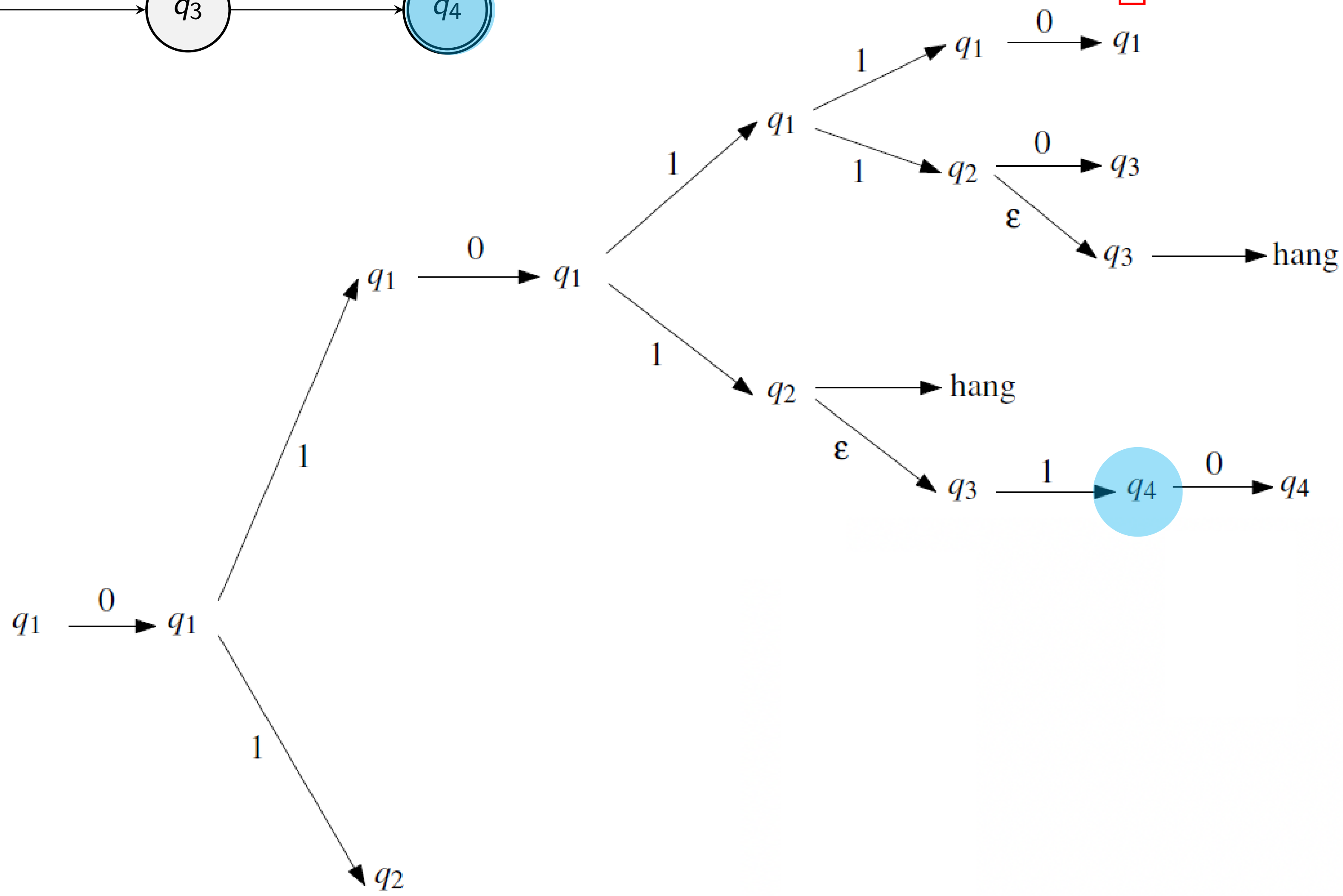


Possible runs over $w = 010110$:



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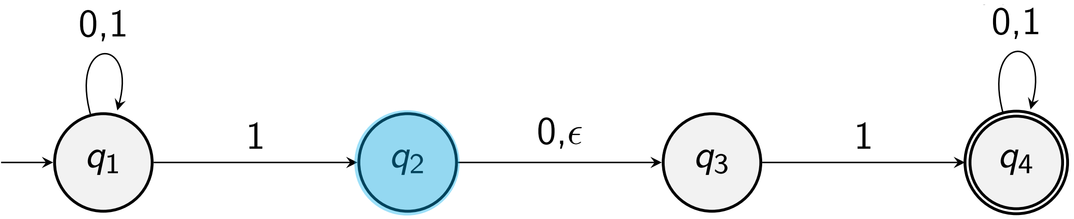
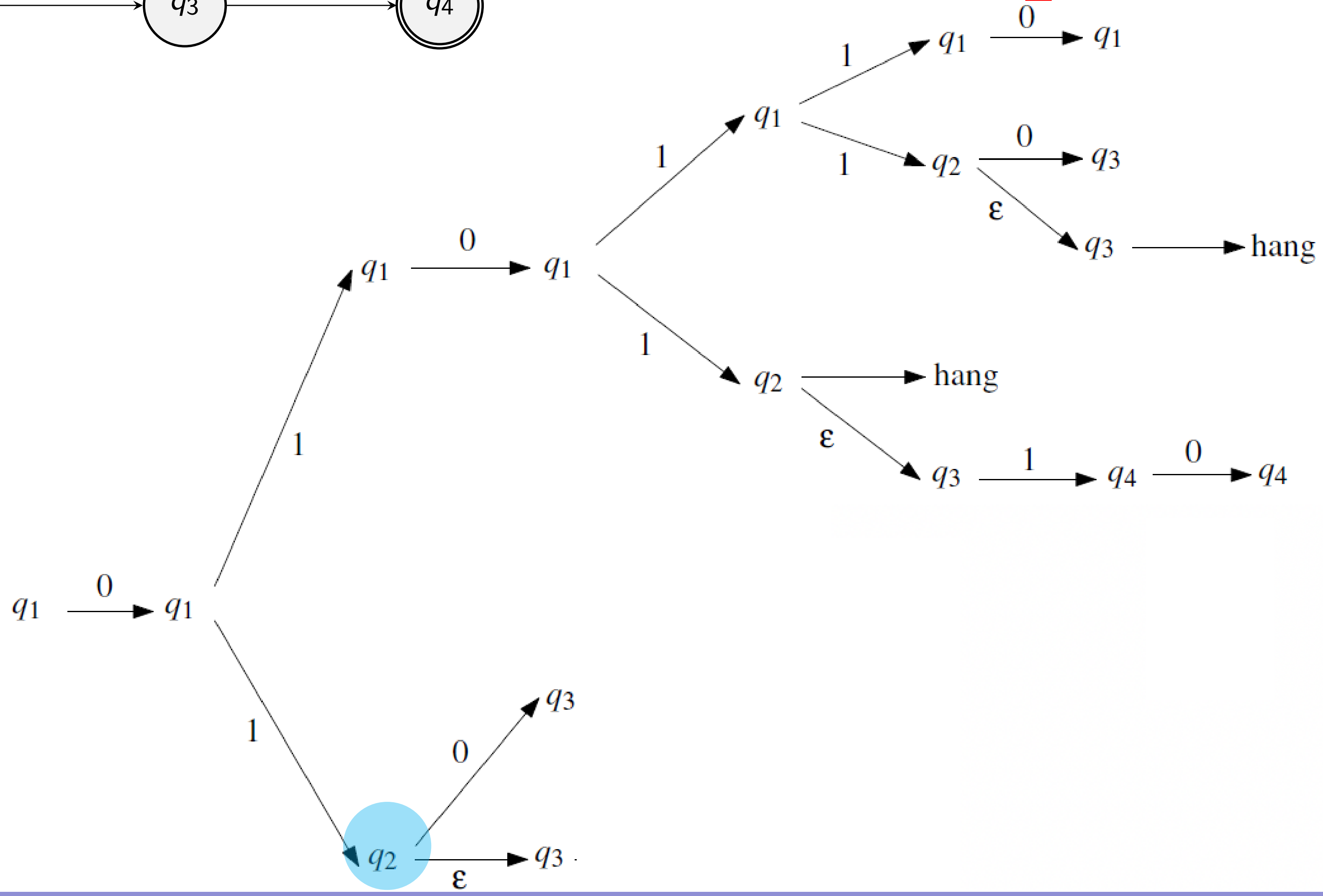
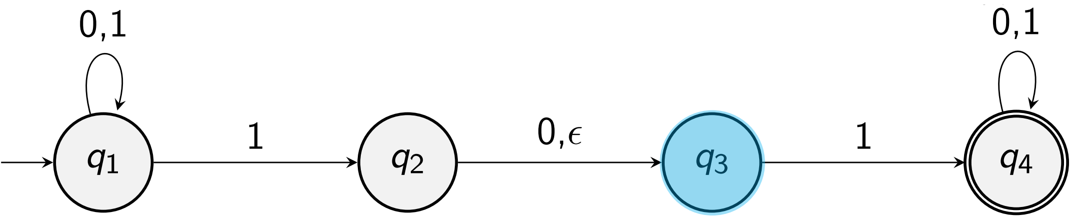


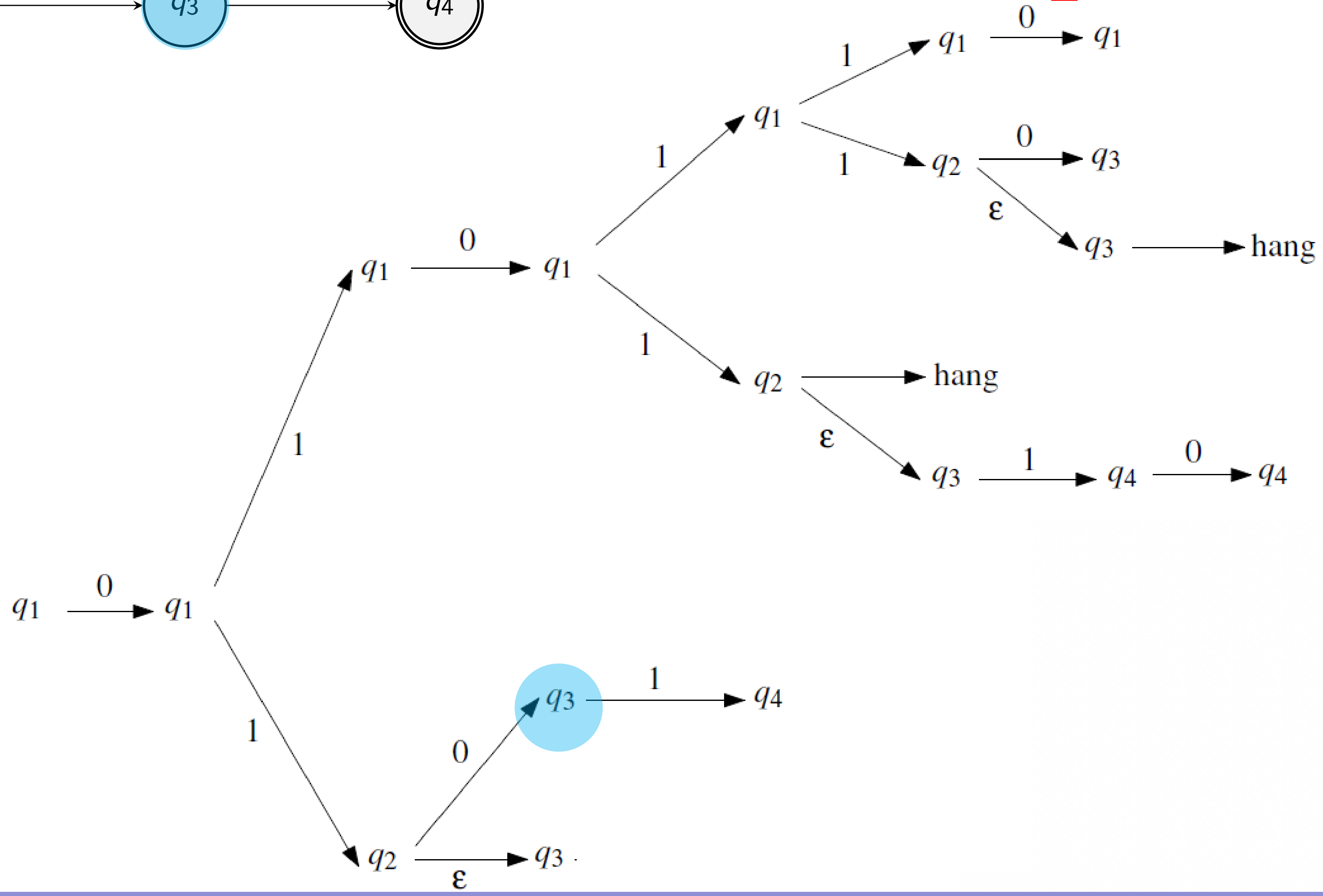
Possible runs over $w = 01\boxed{0}110$:



Nondeterministic Finite Automata - Example



Possible runs over $w = 010\boxed{1}10$:



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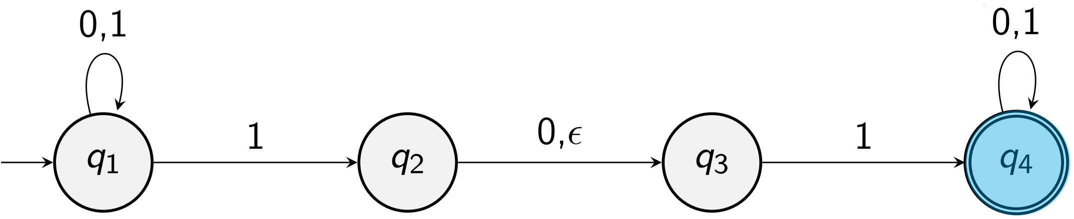

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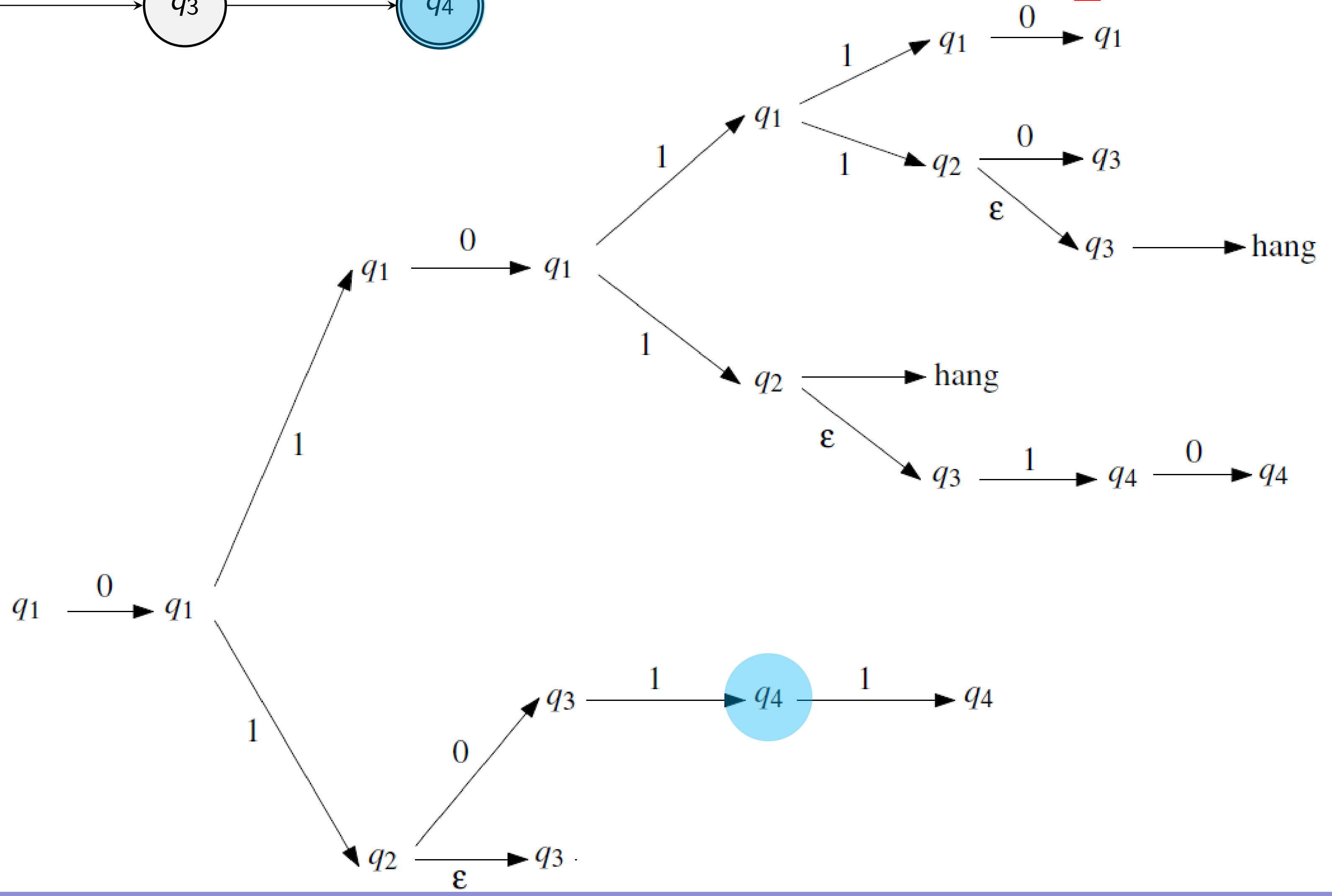

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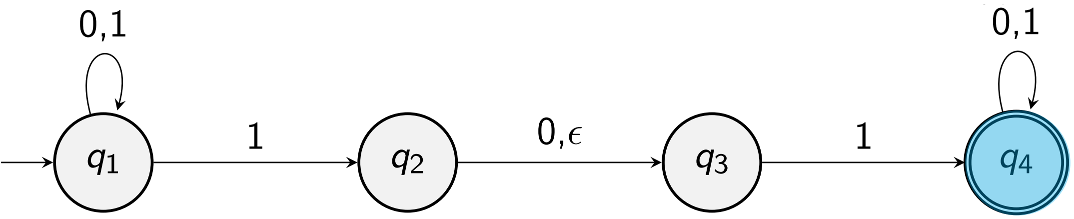

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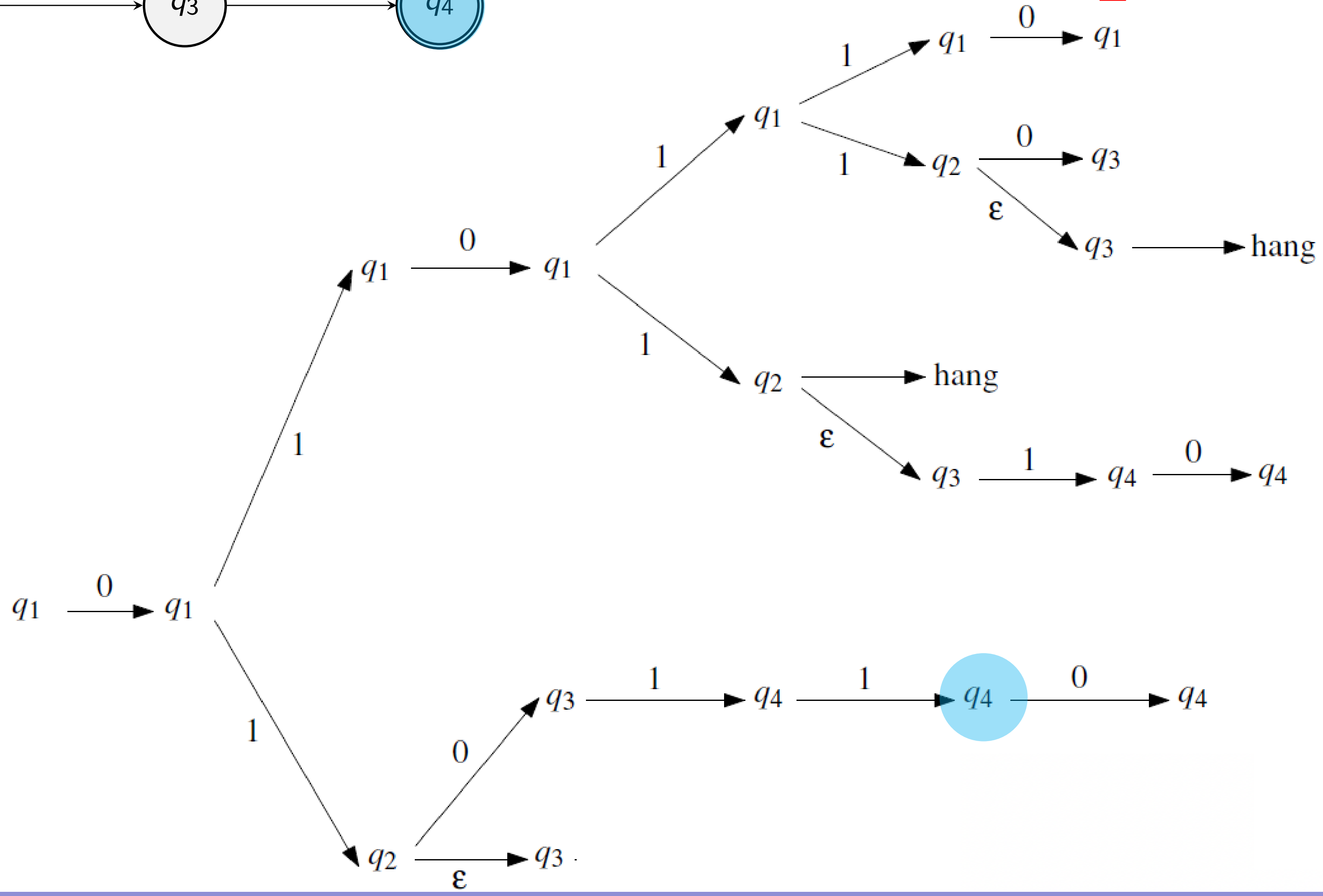
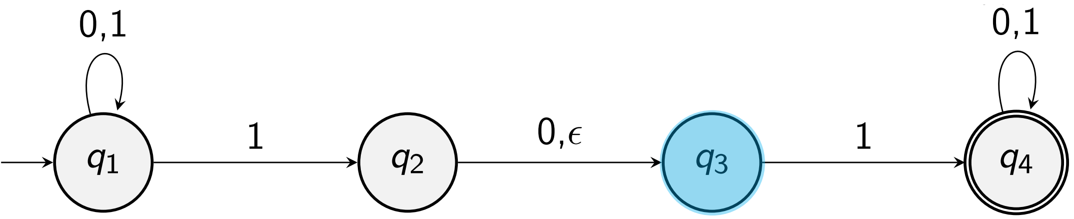


Nondeterministic Finite Automata - Example



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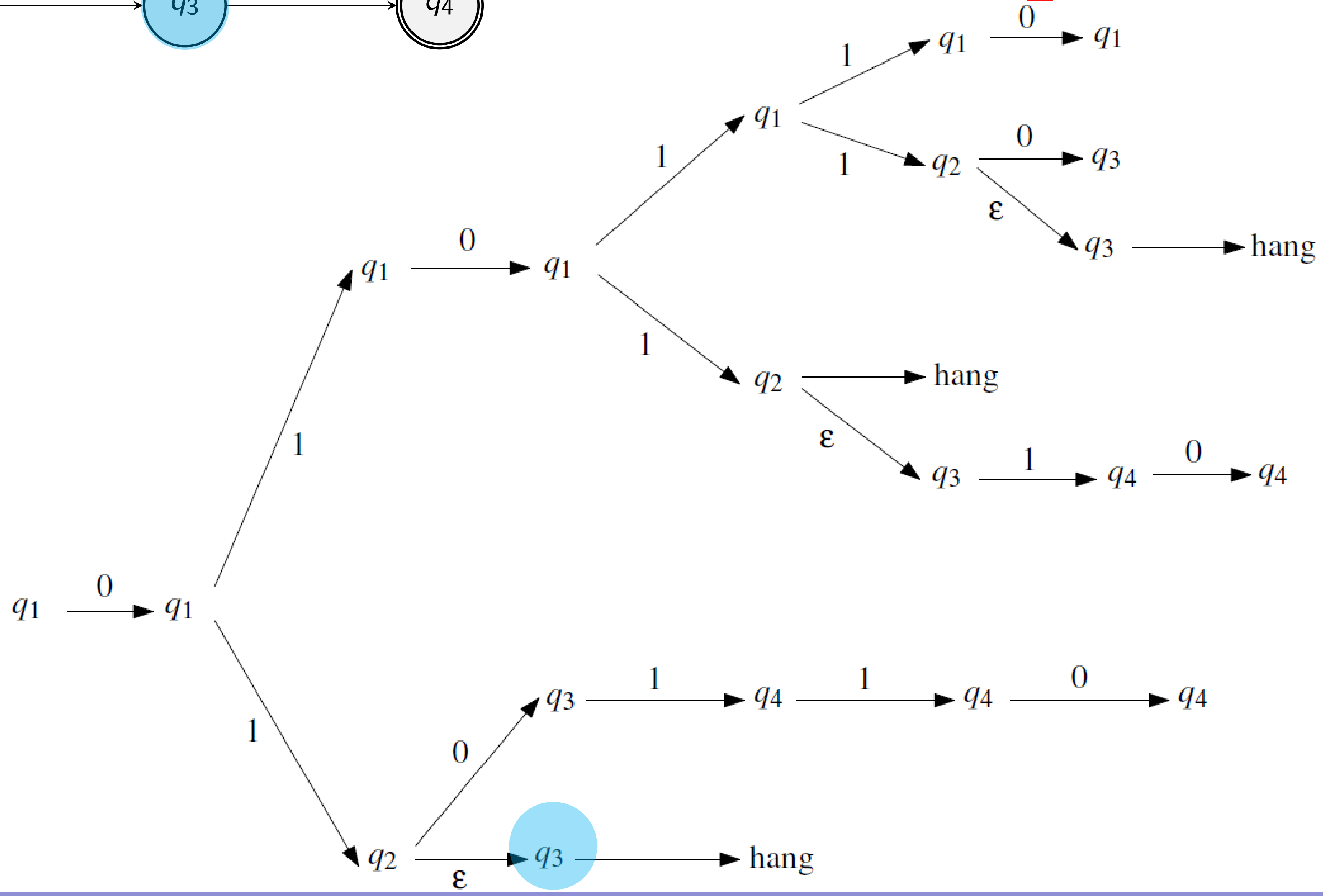

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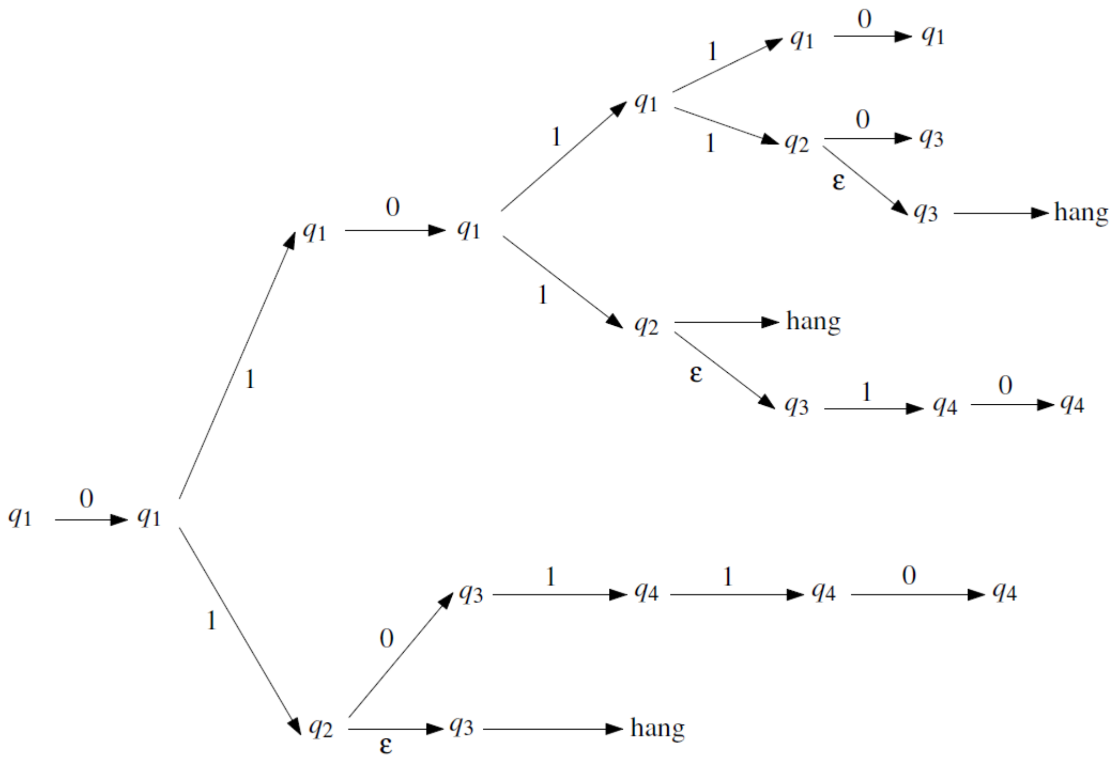


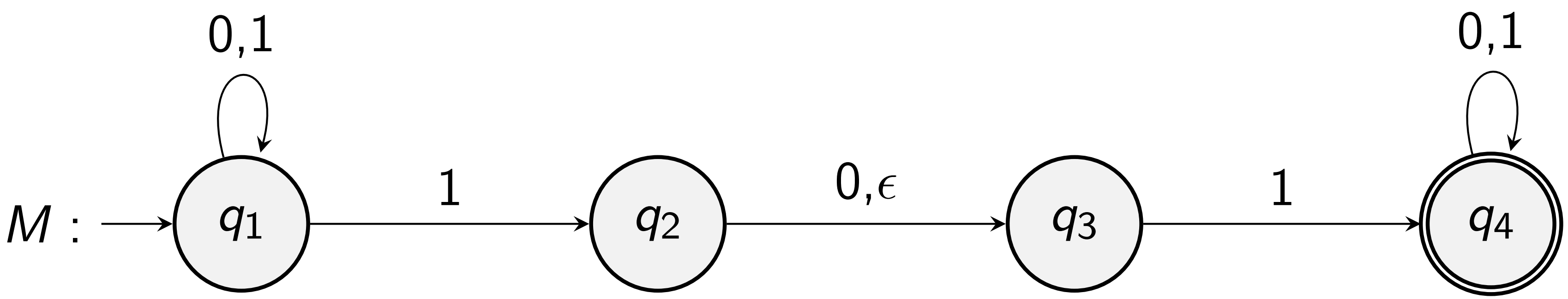
- 2 runs end in the **accept** state q_4
- 2 runs end in a **non-accept** state
- 3 runs cannot complete and **hang**

An NFA accepts a string w if there exists at least one run over w that ends in an accept state

Nondeterministic Finite Automata - Example

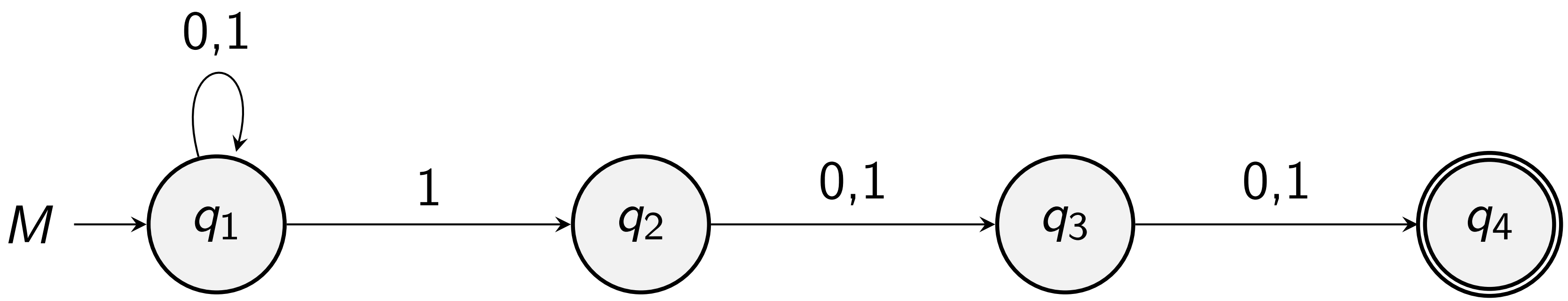
$$L(M) = \{w \in \{0,1\}^* : w \text{ contains the substring 101 or 11}\}$$

Let $w = 010110$



Nondeterministic Finite Automata - Exercise

$$A = \{w \in \{0,1\}^* : w \text{ has a 1 in the third postion from the right}\}.$$



Draw the possible runs of M over the following input strings:

- $w = 01100$
- $w' = 1000$

Nondeterministic Finite Automata - Exercise

$w = 011\,00$

$$\begin{tikzcd}
q_1 \arrow[r, "0"] & q_1 \arrow[r, "1"] \arrow[ddr, "1"] & q_1 \arrow[r, "1"] \arrow[dr, "1"] & q_1 \arrow[r, "0"] & q_1 \arrow[r, "0"] & q_1 \\
& & & q_2 \arrow[r, "0"] & q_3 \arrow[r, "0"] & q_4 \\
& & q_2 \arrow[r, "1"] & q_3 \arrow[r, "0"] & q_4 \arrow[r] & \text{hang}
\end{tikzcd}$$

Nondeterministic Finite Automata - Exercise

$w' = 1000$

$$\begin{tikzcd}
q_1 \arrow[r, "1"] \arrow[dr, "1"] & q_1 \arrow[r, "0"] & q_1 \arrow[r, "0"] & q_1 \arrow[r, "0"] & q_1 \\
& q_2 \arrow[r, "0"] & q_3 \arrow[r, "0"] & q_4 \arrow[r] & \text{hang}
\end{tikzcd}$$

Nondeterministic Finite Automata: Example 2

$$\text{Let } \Sigma = \{0\} \text{ and } 0^k = \underbrace{0\ldots0}_{k \text{ times}}$$

$$A = \{0^k : k \text{ is divisible by 2 or divisible by 3}\}$$

How to construct an NFA that accepts A:

Nondeterministic Finite Automata: Example 2

$$\text{Let } \Sigma = \{0\} \text{ and } 0^k = \underbrace{0\ldots0}_{k \text{ times}}$$

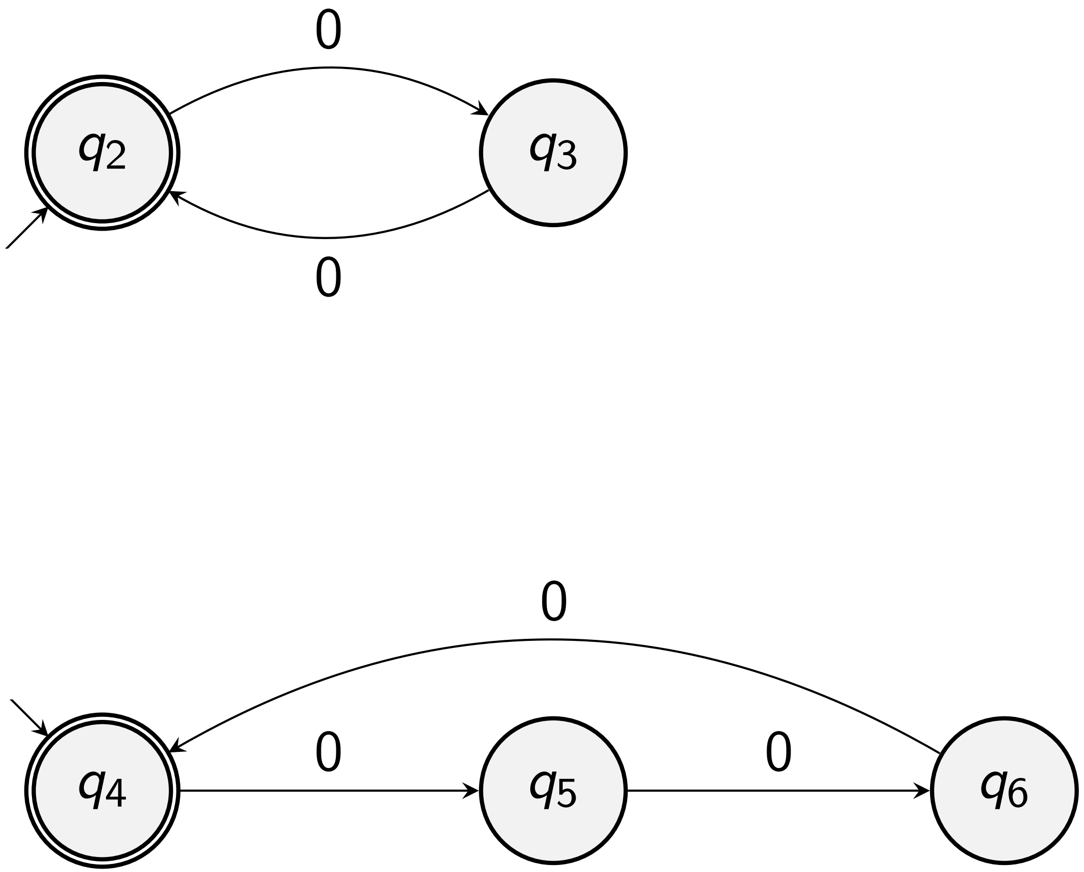
$$A = \{0^k : k \text{ is divisible by 2 or divisible by 3}\}$$

How to construct an NFA that accepts A:

- A can be written as the **union** of the two languages
  - $A_1 = \{0^k : k \text{ is divisible by 2}\}$
  - $A_2 = \{0^k : k \text{ is divisible by 3}\}$
- NFA can be composed of two DFA's
  - One that accepts A_1
  - One that accepts A_2

Nondeterministic Finite Automata: Example 2

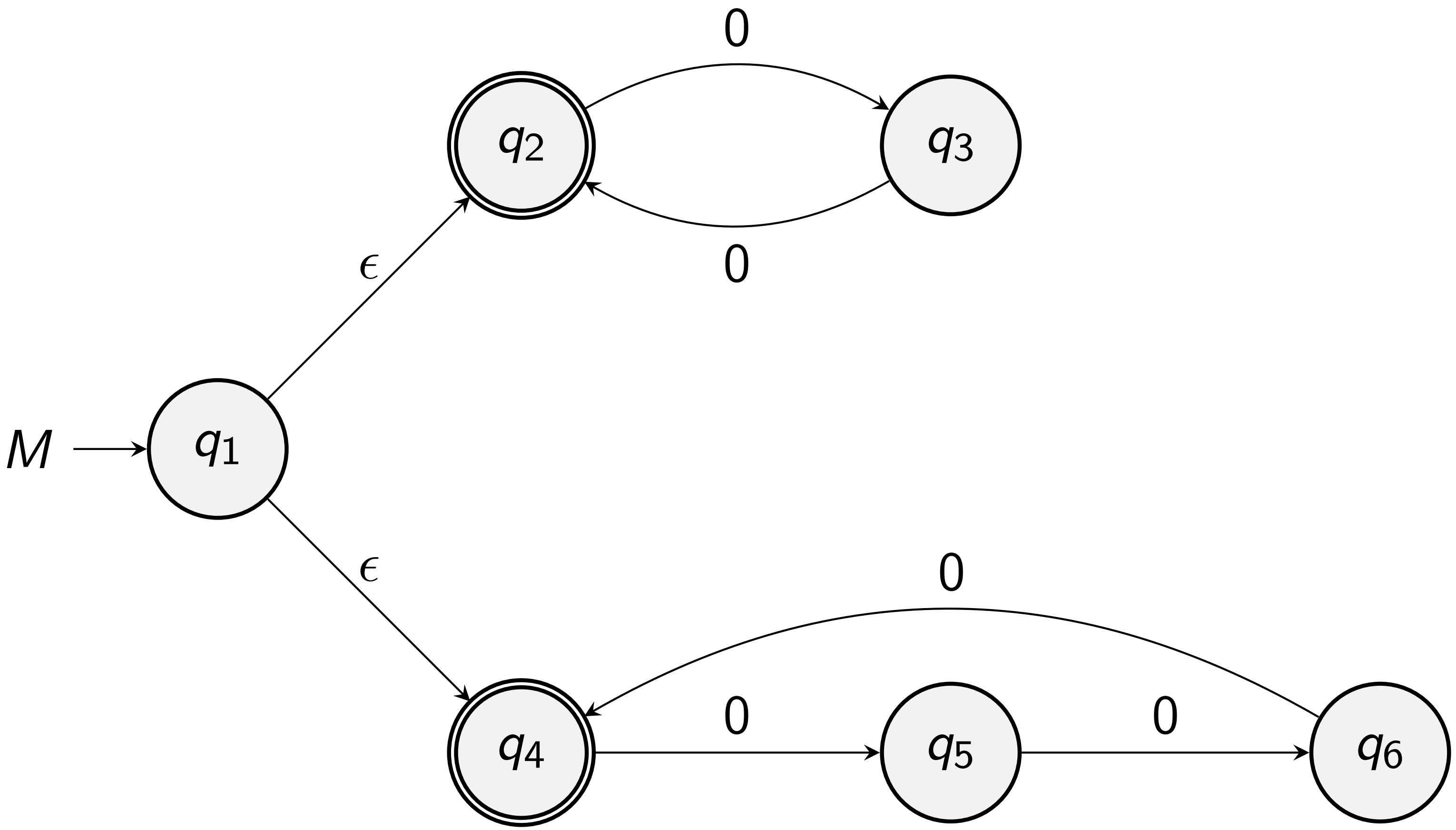
$$A = A_1 \cup A_2 = \{0^k : k \text{ is divisible by } 2\} \cup \{0^k : k \text{ is divisible by } 3\}$$



For every w of length divisible by 2 there exists an accepting run,
for every w of length divisible by 3 there exists an accepting run as well

Nondeterministic Finite Automata: Example 2

$$A = A_1 \cup A_2 = \{0^k : k \text{ is divisible by } 2\} \cup \{0^k : k \text{ is divisible by } 3\}$$



For every w of length divisible by 2 there exists an accepting run, for every w of length divisible by 3 there exists an accepting run as well

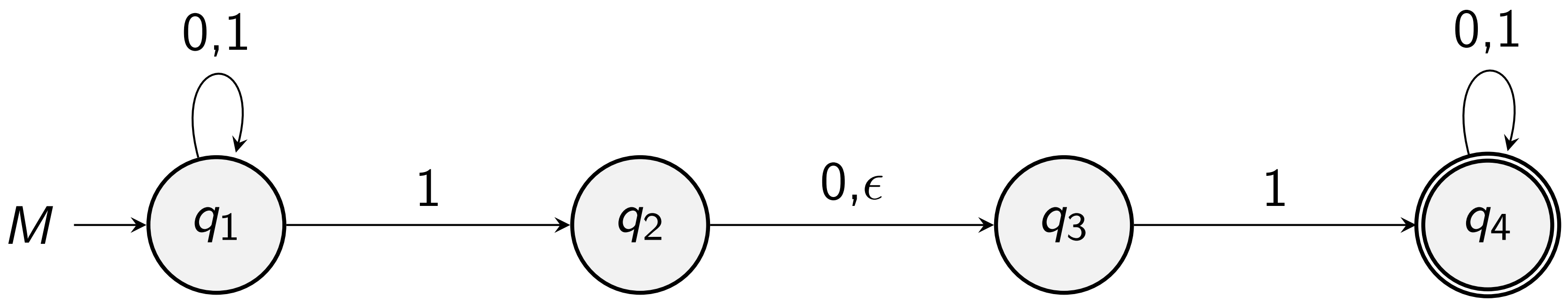
Nondeterministic Finite Automata: Formal Definition

Definition

A nondeterministic finite automaton is a 5-tuple $M = (Q, \Sigma, \delta, q, F)$, where

- Q is a finite set of states
- Σ is an alphabet of symbols
- $\delta : Q \times \Sigma_\epsilon \to \mathcal{P}(Q)$ is a **nondeterministic transition function**
  - $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$
    ϵ-transitions possible
  - $\mathcal{P}(Q) = \{R : R \subseteq Q\}$
    a state may have multiple successor states for the same symbol
- $q \in Q$ is the initial state
- $F \subseteq Q$ is the set of accept states

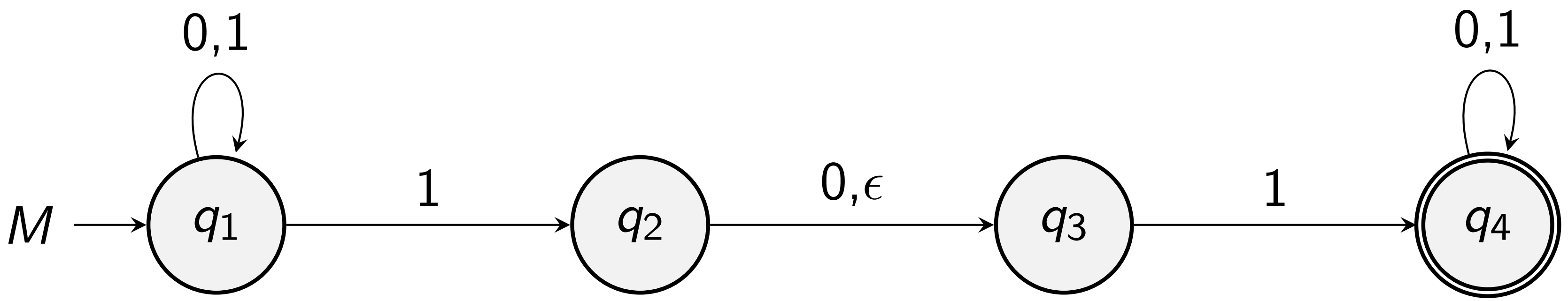
Nondeterministic Finite Automata: Transition Function



Nondeterministic transition function δ of M:

	0	1	ϵ
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

Nondeterministic Finite Automata: Transition Function



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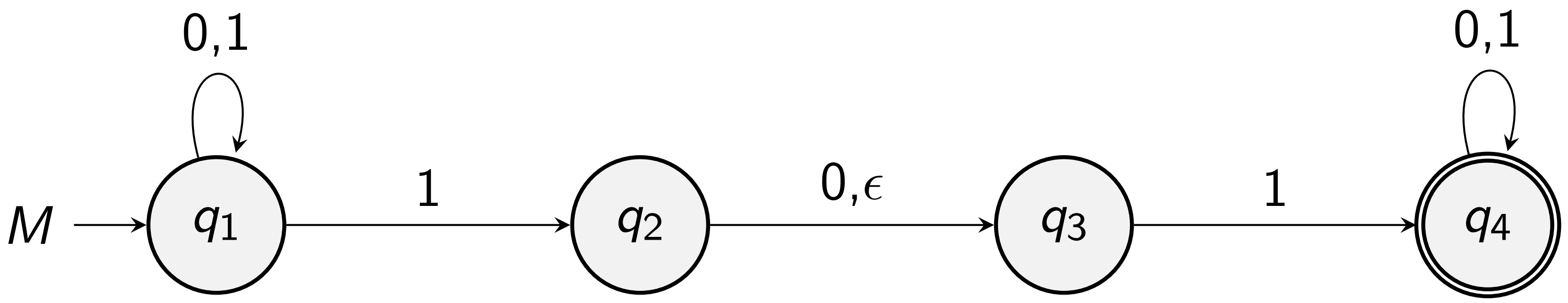
Nondeterministic Finite Automata: Transition Function



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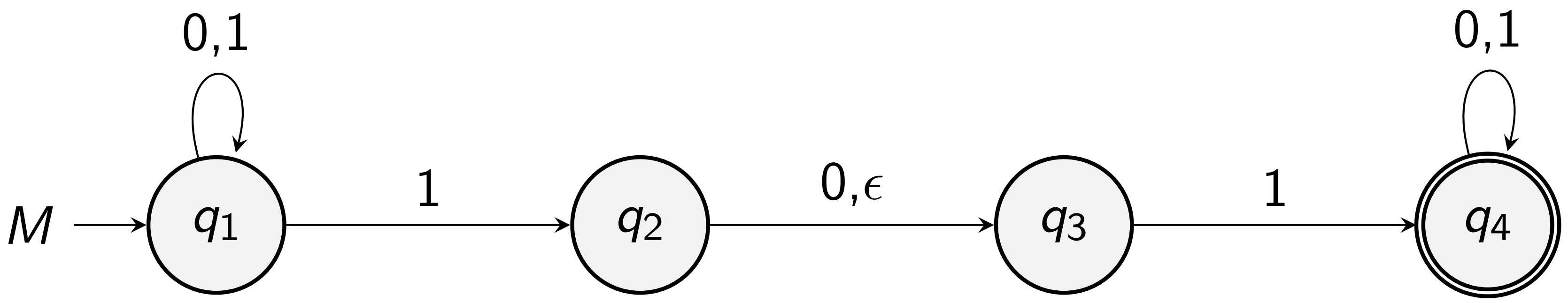
Nondeterministic Finite Automata: Transition Function



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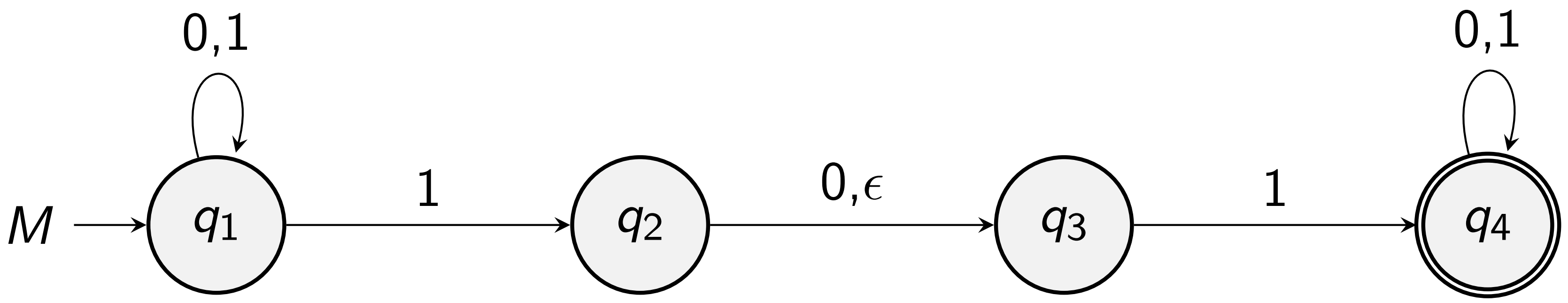
Nondeterministic Finite Automata: Transition Function



Nondeterministic transition function δ of M:

	0	1	ϵ
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

Nondeterministic Finite Automata: Acceptance

Definition

Let $M = (Q, \Sigma, \delta, q, F)$ be an NFA and let w be a string over Σ.
M **accepts** w, if w can be written as $w = y_1 y_2 \cdots y_m$, where $y_i \in \Sigma \cup \{\epsilon\}$ for all i with $1 \leq i \leq m$, and there exists a run $q_0, \ldots, q_m$ such that

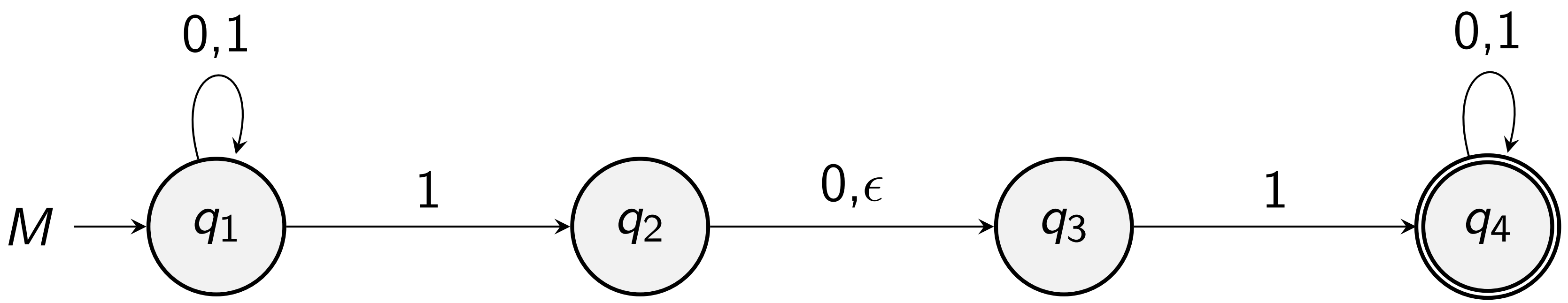
- $q_0 = q$,
- $q_{i+1} \in \delta(q_i, y_{i+1})$, for $i < m$
- $q_m \in F$

A run over $w = 10011$ may look as follows:

$$q_0 \xrightarrow{1} q_1 \xrightarrow{0} q_2 \xrightarrow{\epsilon} q_3 \xrightarrow{0} q_4 \xrightarrow{1} q_5 \xrightarrow{\epsilon} q_6 \xrightarrow{0} q_7$$

Nondeterministic Finite Automata: Acceptance Example

Is $w = 01100$ accepted by M?



Yes:

- rewrite w as $01\epsilon100$
- the following run over $01\epsilon100$ exists:

$$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{\epsilon} q_3 \xrightarrow{1} q_4 \xrightarrow{0} q_4 \xrightarrow{0} q_4$$

- run starts in initial state $q_1 = q$
- and ends in accept state $q_4 \in F$

Nondeterministic Finite Automata: Language

Definition

Let $M = (Q, \Sigma, \delta, q, F)$ be a nondeterministic finite automaton. The **language** $L(M)$ of M is the set of all strings that are accepted by M:

$$L(M) = \{w : w \text{ is a string over } \Sigma \text{ and } M \text{ accepts } w\}$$