We have a set of $s$ species, for each species $i \in \{1, 2, .., s\}$ , and each $w > w_{e.i}$, the abundance $N_i(w)$ satisfies the McKendrik-von Foerster equation

$$\frac{\partial N_i(w)}{\partial t} = -\frac{\partial [N_i(w) g_i(w)]}{\partial w} - \mu_i(w) N_i(w) \tag{1}$$

(steady state solution is $N_i(w) \propto \frac{1}{g_{i(w)}} \exp\left(-\int_{w_{e.i}}^{w} \frac{\mu(w')}{g(w')}.dw'\right)$ ),
also, reproduction is modeled by the boundary condition

$$g_i(w_{e.i}) N_i(w_{e.i}) = \frac{\epsilon_i}{2 w_{e.i}} \int_0^\infty N_i(w) E_{r.i}(w) \psi(w).dw. \tag{2}$$

Here

$$g_i(w) = g_i[N](w) = \begin{cases} (1 - \psi_i(w))\, E_{g.i}[N](w) & \text{if } E_{g.i}[N](w) > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

denotes the growth rate of a weight $w$ member of species $i$ when the system is in state $N$. Here

$$E_g[N](w) = \alpha_i h_i w^n \left( \frac{E_{e.i}[N](w)}{E_{e.i}[N](w) + h_i w^n} \right) - k_{s.i} w^p - k_i w. \tag{4}$$

denotes the energy available for growth and reproduction, and $hw^n$ is the maximum food intake rate of a weight $w$ individual. The total energy intake $f(w)hw^n$, however we suppose energy is only assimilated with an efficiency $\alpha$ and a weight $w$ individual must spend energy $kw^p$ in order to satisfy its metabolic needs or else it will be subject to potential starvation. Often we assume $n = p$ and $k = 0$. Here

$$E_{e.i}[N](w) = \gamma_i w^q \int_0^\infty w' \phi_i\left(\frac{w'}{w}\right) \left( N_R(w') + \sum_{j=1}^{s} \theta_{ij} N_j(w') \right) dw' \tag{5}$$

denotes the energy encountered and

$$\phi_i\left(\frac{w'}{w}\right) = \exp\left[ -\left( \ln\left( \frac{\beta_i w'}{w} \right) \right)^2 / \left(2\sigma_i^2\right) \right] \tag{6}$$

represents the preference that a predator of weight $w$ has for a prey of weight $w'$.

The mortality rate is given by

$$\mu_i(w) = \mu_{b.i}(w) + \mu_{F.i}(w) + \mu_{p.i}(w) \tag{7}$$

where the predation mortality rate is

$$\mu_p(w) = \int_0^\infty \sum_j \left(\frac{hw'^n}{E_e(w')}\right)\left(\frac{E_{e.i}[N](w)}{E_{e.i}[N](w)+h_i w^n}\right) N_j(w',t)\phi_i\left(\frac{w}{w'}\right)\gamma w'^q dw' \tag{8}$$

A state $N$ is a specification of $N_i(w) \in \mathbb{R}$ for each $i \in \{1,..,s\}$ , and each $w > 0$. This is said to be a steady state if $\frac{\partial N_i(w)}{\partial t} = 0$, for each $i \in \{1,..,s\}$ , and each $w > 0$.

## Transplant Principle

We can construct an equilibria where $g_i(w) = P_i(w)(1-\psi_i(w))$, and $P_i(w) = \hbar_i w^n$, and $\mu_i(w) = \mu_0 w^{n-1}$. Any abundance multipliers, dodgy background and resource setup

We should also consider the case where $P_i(w) = \sum_k a_{i,k} w^{\alpha_k}$ and $\mu_i(w) = \sum_k b_{i,k} w^{\beta_k}$

because we can solve the MvF exactly for such growth and death rates, I presume, and it should be possible to reconstruct size spectra so that (for certain parameter choices at least), we can consider

*I'm wondering if it might be possible to get the analytic form of a steady state N such that such that aggregate abundance equals the sum of two powers (so $\sum_i N_i(w) = \kappa_1 w^{-\lambda_1} + \kappa_2 w^{-\lambda_2}$, which consists of a combination of two stable_community.R style assemblages of background species which use different parameters to each other. If we have the case where growth and death are given by sums of powers, we could solve the Mvf in this case, and if we have two background assemblages which vary over continuua of characteristic size then perhaps we can combine them to produce the above size spectrum, the associated death and growth rates would also be described by sums of powers, and we can solve the Mvf in such a case, and we will have two different solution shapes, but since each is associated with a continuua of charactertic sizes, and can set the abundances so that the integral over the abundances of the two assemblages is the sum of the two power laws. This seems to be possible if we make the exponents, $n, p, q$ species specific. Indeed it would be good to explore generalizations of the single homogeneous steady state construction procedure in this regard, of adding extra collections of background species, associated with different exponents, so that the aggregation of abundance over all species equals a sum of power laws.

*We assume $h_i = \infty, \forall i$ so that growth and death are linear, so $g_i[N^1 + N^2](w) = g_i[N^1] + g_i[N^2](w)$ and $\mu_i[N^1 + N^2](w) = \mu_i[N^1] + \mu_i[N^2](w)$ (do we also have to assume $\phi_1 = .. = \phi_s$ in order to get the death rates to act linearly.

*It is important to distiguish two different type of steady states in multi-species hetrogenous size spectrum models. A uniform steady state is a steady state $N$ such that for any pair of species $i, j \in \{1,..,s\}$ there exists a rescaling constant $Q_{ij}$ such that $N_j(w) = Q_{ij} N_i(w)$. In this case, we can find all the solutions are the same, if we then set things up so there are similar species accross many characteristic sizes, then we may be able to demonstrate that we can setup equilibria which have something

* simplified form of transplant principal in this linear case: If $N^1 + N^2$ and $M^1 + M^2$ are steady states, each composed into two parts (with disjoint species sets $sp$), and for each species $i \in sp(N^1) = sp(M^1)$ we have $g_i[N^2](w) = g_i[M^2](w)$, $\mu_i[N^2](w) = \mu_i[M^2](w)$ and for each species $j \in sp(N^2) = sp(M^2)$ we have $g_j[N^1](w) = g_j[M^1](w)$, $\mu_j[N^1](w) = \mu_j[M^1](w)$ then $N^1 + M^2$ and $N^2 + M^1$ are steady states

*If there are all these extra steady states/neutrality that could be created by including extra species with different parameters, then it could be quite important to properly model the presence of rare species, which could increase their abundance in various combinations to replace a common species that goes extinct. I guess there there will be cases where common species goes extinct and is replaced with a rather non-trivial combination of many other species.

* if you have a steady state, and then you replace one species with another new species, at half the abundance which has twice the gamma, and pre-existing species have twice the theta preference, for the new species, then that will lead to another steady state.

* under the further assumptions that phi_1=phi_2=..=phi_s, and [N_1,N_2,..,N_s are all proportional to one another], we can actively describe the affine set of steady states, by considering the linear algebra problem involving theta.

* homogeneous case as a special case of the above

* special system where all species are identical, apart from that we can have arbitrary theta values

* Transplants using tricks with alpha, gamma, theta and A_i

* demonstrate why we saw a bunch of steady states in our work on steady states by converting the code

* We say a size spectrum model has unlimited intake if $h_i = \infty, \forall i$

* We say an unlimited intake size spectrum model in state $N$ has 'power law sum aggregate abundance' if $\exists \lambda_1, ..\lambda_k$, $\exists \kappa_1, .., \kappa_k$,$\exists B_1, .., B_s$ such that $N_R(w) + \sum_j \theta_{ij} N_j(w) = B_i \sum_{e=1}^k \kappa_e w^{-\lambda_e}$, and $\exists D_1, .., D_s, \exists \mu_{0.1}, .., \mu_{0.k}$, $\exists n_1, .., n_k$ such that$\mu_i(w) = \mu_{p.i}[N](w) + \mu_{b.i}(w) = D_i \sum_{e=1}^k \mu_{0.e} w^{n_e - 1}$

* An unlimited intake size spectrum model in state $N$ has 'species-scaled power law aggregate abundance' if it has power law sum aggregate abundance with $k = 1$. In this case $\exists \kappa^{(1)}, .., \kappa^{(s)}, \exists \mu_0^{(1)}, .., \mu_0^{(s)}$ : such that $N_R(w) + \sum_j \theta_{ij} N_j(w) = \kappa^{(i)} w^{-\lambda}$ and $\mu_{p.i}[N](w) + \mu_{b.i}(w) = \mu_0^{(i)} w^{n-1}$.

* An unlimited intake size spectrum model in state $N$ has ' power law aggregate abundance' if it has 'species-scaled power law aggregate abundance' with $\kappa^{(1)} = .. = \kappa^{(s)} = \kappa$, $\mu_0^{(1)} = .. = \mu_0^{(s)} = \mu_0$.

* A spectrum model is [gamma,theta,alpha,feeding-kernel,..etc.] homogeneous, if every species has the same parameter/function [gamma,theta,alpha,feeding-kernel,..etc.]

* A state $N$ is 'single shaped ' if $\forall i, j\ \exists Q_{i,j} : Q_{i,j} N_i(w) = N_j(w), \forall j$

*If the intake is unlimited, and the feeding-kernels are homogeneous, then the set of steady states that can be converted between by altering the steady state in such a way that the death and growth rates of individuals can be preserved, corresponds to an affine set (the same I observed with R.L.).

* The above can be generalized to allow different species to be associated with different death rates, and therefore to have different solutions, but one can still describe the affine structure of the steady states, but now it corresponds to a ‘species-scaled power law aggregate abundance‘ with $\kappa^{(1)} = .. = \kappa^{(s)} = \kappa$, and it no longer has single shaped solutions, but by tuning the species specific background mortality, we can set things up.

*For a feeding kernel homogeneous, unlimited intake, size spectrum model, we know how to construct a power law aggregate abundance steady state that is single shaped (stable_community.R does this type of thing). Using the linearity of the growth and death rates in this case

* An unlimited intake size spectrum model in state $N$ has ‘power law aggregate abundance‘ when it has power law sum aggregate abundance and when $k = 1$,$B_1 = .. = B_s = 1, \theta_{ij} = 1, D_1 = .. = D_s = 1$. This is the standard power law case for which we can obtain a complete analytic solution. In other words, an unlimited intake size spectrum model in state $N$ has ‘power law aggregate abundance‘ if $\theta_{ij} = 1$, and

* A generalization of the above case is to allow $B_1, .., B_s, D_1, .., D_s$ to take general values.