Constructing Steady States Of Multispecies Size Spectrum Models

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We have a multispecies size spectrum model as described in the mizer vignette, except

$$\psi_i(w) = \begin{cases} 0 & \text{if } w < w_{*i} \\ \left(\frac{w}{W_{\infty.i}}\right)^{1-n} & \text{otherwise} \end{cases} \tag{1}$$

we also suppose our parameters are such that
$n = p$
$\lambda = q + 2 - n$
$\mu_f(w) = k = 0$
$\theta_{ij} = 1$
Given that, here is my attempted recipe for constructing a steady state for some size spectrum setups:
Choose any value of $\mu_0 > 0$.
We explicitly model s species, which may have any different species specific parameters. For each species i let

$$f_{0i} = \frac{\gamma_i}{\frac{h_i \beta_i^{2-\lambda} \exp(-(\lambda-2)^2 \sigma_i^2/2)}{\sqrt{2\pi}\kappa\sigma_i} + \gamma_i} \tag{2}$$

be the feeding level (as experienced by species i individuals feeding upon a community at abundance $\kappa w^{-\lambda}$) , and let $\hbar_i = \alpha_i h_i f_{0i} - k_{si}$.

Let $N_i(w)$ be a steady state of the MvF equation of species i with growth rate $\hbar_i w^n(1 - \psi_i(w))$ and death rate $\mu_0 w^{n-1}$. In particular

$$N_i(w) = \frac{1}{\hbar_i w^n} (w_{ei}/w)^{\mu_0/\hbar_i} \times \begin{cases} 1 & w < w_{*i} \\ \frac{\left(1-(w/W_{\infty_i})^{1-n}\right)^{(\mu_0/(\hbar_i(1-n)))-1}}{(1-(w_{*i}/W_{\infty i})^{1-n})^{\mu_0/(\hbar_i(1-n))}} & w > w_{*i} \end{cases} \tag{3}$$

Any multiple of $N_i(w)$ is also a steady state of the Mvf, and for our community construction, we can choose any abundance multipliers $A_i \geq 0$ and $A_i N_i(w)$ is also a steady state of the Mvf. We suppose that the abundance of background resource at weight w is $N_R(w) = \kappa w^{-\lambda} - \sum_{i=1}^{s} A_i N_i(w)$. In this resulting system, where species i has an abundance $A_i N_i(w)$ at weight w, and the background resource abundance has abundance $N_R(w)$, let $\mu_{pi}(w)$ denote the predation mortality level on species i from the explicitly modeled species $j \in 1, .., s$. Now if we choose a background mortality rate of $\mu_{bi}(w) = \mu_0 w^{n-1} - \mu_{p.i}(w)$ for the ith species, then its total mortality rate will be $\mu_i(w) = \mu_{p.i}(w) + \mu_{b.i}(w) = \mu_0 w^{n-1}$. Also, the growth rate will be $g_i(w) = \hbar_i w^n (1 - \psi_i(w))$. Since these are the same death and growth rates that we assumed in construction of our solutions $A_i N_i(w)$, we must have a steady state of the Mvf. We also have to choose parameters e.g., the reproductive efficiencies ϵ_i so that the reproduction boundary condition

$$g_i(w_{e.i}) N_i(w_{e.i}) = \frac{\epsilon_i}{2 w_{e.i}} \int_0^\infty N_i(w) E_{r.i}(w) \psi(w).dw \tag{4}$$

is met for each species. This should result in a steady state of the size spectrum dynamics.

The mizer code [stable_community_gurnard_really.R] we did before involves gurnard and background species.

$\mu_0 = (1 - f_{0B}) \sqrt{2\pi} \kappa \gamma_B \sigma_B \beta_B^{n-1} \exp(\sigma_B^2 (n-1)^2 / 2)$

describes the mortality rate induced by the background species.

In this case the the first $s - 1$ species are the background species so $\beta_1 = .. = \beta_{s-1} = \beta_B \neq \beta_s$ etc, and the sth and final species is the gurnard with its own different parameters. I guess this code works in a similar way to the above recipe, except this code has plankton dynamics, and we cut the plankton off at the first zero. We still need to test if things will work if we cut the background mortality term off.

Returning to the general steady state construction problem. If one is interested in modeling a multispecies ecosystem without so much artificial background death and mortality rates, one can include extra background species that are explicitly modeled, with parameters like $\beta_B$etc. above. Setting such modeled background species to the proper abundances to model induce the proper growth and death rates, then allows us to reduce the amount of artificial background death and background resource abundance we have to add to create a steady state according to the above recipe.

There are more general ways to make new steady states from old by making sub population transplants that preserve death and growth rates. If we have a pair of steady states of a size spectrum model (perhaps with different species extinct in either state), and then we do transplant surgery -cutting away some of the individuals from the second steady state, and replacing them with a subset of individuals from the first steady state, that are grafted on. If the growth and death rates are the same for the incumbent and grafted individuals before and after this type of population transplant then it will result in a new steady state.

More generally if we have two mizer steady states X and Y of different systems with different parameters if we can choose a subset x of species from X and a subset y of species from Y then we can sometimes transplant/swap populations x and y around in their ecosystems, (cutting their abundances out and swapping them round). The conditions required for the resulting two "post transplant" states to be steady states are:

(1) members of x experience the same growth and death rate as a result of their contact with $X \setminus x$ before the transplant, as they experience with the new surroundings (post-transplant) via their contact with $Y \setminus y$.

(2) members of $X \setminus x$ experience the same growth and death rate as a result of their contact with x before the transplant, as they experience with the new surroundings (post-transplant) via their contact with y.

(3) members of y experience the same growth and death rate as a result of their contact with $Y \setminus y$ before the transplant, as they experience with the new surroundings (post-transplant) via their contact with $X \setminus x$.

(4) members of $Y \setminus y$ experience the same growth and death rate as a result of their contact with y before the transplant, as they experience with the new surroundings (post-transplant) via their contact with x.