

On Finding Steady States In Size Spectrum Models

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1 Getting a steady state by evolving with fixed egg production

This section is my interpretation of the steady state finding procedure conceived of by Gustav Delius.

We shall describe how to find a steady state of a given a multi-species size spectrum model, of the type described in Section 3 of [mizer vignette]. We assume all parameters are pre determined, although our procedure involves re-tuning the reproduction efficiency ϵ_i .

Step 1: Pick Initial Condition

Suppose we wish to construct the coexistent steady state (N_R, N) that maximizes some objective function $\Psi((N_R, N))$. For example Ψ could encode how well the biomass values for the different species in (N_R, N) agree with empirical data. Our initial condition (N'_R, N') may be chosen by any criterion. For example, one may wish to choose the initial condition (N'_R, N') that maximizes Ψ over all states.

Step 2: Set Initial Egg Production

We need to choose a value R_i at which to fix the egg production, of each species i in $\{1, 2, \dots, s\}$. For example, we could choose

$$R_i = \frac{R_{max,i} R_{p,i}}{R_{max,i} + R_{p,i}}$$

where

$$R_{p,i} = \frac{\epsilon}{2w_0} \int_0^\infty N'_i(w) E_{r,i}(w) \psi_i(w) dw$$

is the egg production that would be generated by state (N'_R, N') (before adjusting for the stock recruitment relationship).

Step 3: Evolve System With Egg Production Held Fixed

We evolve the system under the Mckendrick von Foerster equation (and using the semi-chemostat equation for the resource dynamics), as described in

Section 3 of [mizer vignette], except that we held the egg production fixed at $N_i(t, w_{0,i}) = R_i$ for each time t . Although it is unclear, experiments suggest that in these kinds of systems, where the egg production values are held fixed at R_i , have just one coexistent steady state (that is, a steady state where each species has none zero density). In most cases these dynamics will evolve to a steady state (N_R^*, N^*) . And although we have observed cases where such dynamics do not settle to a steady state (and instead end in an oscillation), hopefully a unique coexistent steady state (N_R^*, N^*) can still be found in such cases, using Netwon-Raphson. [Questions: does the final state/limit cycle depend on the initial condition ? In cases which cycles when egg reproduction is held fixed, what happens if egg production is then allowed to vary naturally, Do systems which are oscillatory when egg production is held fixed remain so, when egg production is properly generated in a density dependent manner.]

Step 4: Optimize (optional)

Compute $\Psi((N_R^*, N^*))$ for the resulting steady state. If desired one may keep re-selecting the egg production values (R_1, \dots, R_s) and re-running Step 3 (hopefully the dynamics are such that the initial condition is irrelevant, but we should check), in order to get a new steady state, .. etc. So essentially one could attempt to select (R_1, \dots, R_s) to optimize $\Psi((N_R^*, N^*))$, with the caveat that we may not be able to find a steady state for some values of (R_1, \dots, R_s) by simply evolving the system (although hopefully there is still a unique coexistent steady state to find using Netwon-Raphson).

Step 5: Re-select reproduction efficiency

We take the steady state (N_R^*, N^*) from the constant-egg-production system, and, for each species i in $\{1, \dots, s\}$, choose a new value ϵ'_i of the reproduction efficiency, so that the value R_i at which the egg production was previously held constant, is equal to the amount of eggs

$$\frac{R_{max,i} \times \frac{\epsilon'_i}{2w_{0,i}} \int_0^\infty N_i^*(w) E_{r,i}(w) \psi_i(w) . dw}{R_{max,i} + \frac{\epsilon'_i}{2w_{0,i}} \int_0^\infty N_i^*(w) E_{r,i}(w) \psi_i(w) . dw} \quad (1)$$

that would be generated by the resulting steady state, using the new reproduction efficiency value ϵ'_i (and accounting from possible egg loss due to R_{max}).

Result

(N_R^*, N^*) will be a steady state of the resulting multi-species size spectrum model, using new reproduction efficiency values ϵ'_i . The steady state construction procedure described above can be repeated for different parameter choices until one is satisfied with the resulting reproduction efficiency values. Alternatively, the objective function Ψ referred to in step 4 could be selected so as to reward the generation of the proper reproduction efficiency value as the output ϵ'_i .

Remaining Questions

- When does the algorithm fail ?
- Do basins of attraction matter ?
- Is there always one unique interior steady state ?
- What are the partial functions like: from (R_1, \dots, R_s) to (N_R^*, N^*) , and from (N_R^*, N^*) to $(\epsilon_1, \dots, \epsilon_s)$?

2 Newton-Raphson

Need to describe discretization scheme, and function to find zero of, for Newton-Raphson solving.

2.1 Grid point setup

In size spectrum models we wish to keep track of the abundance of fish for a continuum of different weights, however in computer simulations we only keep track of the abundance for weights belonging to a finite set of discrete grid points. The fish grid points w_k give the weights at which we track the abundance of fish. In mizer the fish grid points are specified by the following three numbers:

- The number of fish grid points $|w|$, the minimum fish egg size of a species W_e . The maximum fish size of a species W_+ .

The vector w has length $|w|$, and is such that $w_1 = W_e$ and $w_{|w|} = W_+$. Also the log of w is a vector with a constant difference between successive entries. In other words, $\forall k \in \{1, \dots, |w|\}$ we have $w_k = 10^{y_k}$ where

$$y_k = \log_{10}(W_e) + (k-1) \left(\frac{\log_{10}(W_+) - \log_{10}(W_e)}{|w| - 1} \right). \quad (2)$$

A vector Δw of differences is useful for Riemann sums and PDE schemes. For each $k \in \{1, \dots, |w| - 1\}$ this is defined as $(\Delta w)_k = w_{k+1} - w_k$. It is most natural to continue the pattern (that there is a constant ratio between successive entries of Δw) to specify $(\Delta w)_{|w|}$.

The plankton grid points w_k^{full} consist of the fish grid points together with extra points that refer to smaller weights of plankton, below the fish egg size. In mizer the plankton grid points are specified by two numbers: the number of plankton grid points $|w^{full}|$ and the smallest considered plankton size W_- . The vector w^{full} is defined such that $\forall k \in \{1, \dots, |w^{full}| - |w|\}$ we have $w_k = 10^{z_k}$ where

$$z_k = \log_{10}(W_-) + (k-1) \left(\frac{\log_{10}(W_e - (\Delta w)_1) - \log_{10}(W_-)}{|w^{full}| - |w| - 1} \right) \quad (3)$$

and the remaining $|w|$ terms are given by

$$\left(w_{|w^{full}|-|w|+1}^{full}, w_{|w^{full}|-|w|+2}^{full}, \dots, w_{|w^{full}|}^{full}\right) = (w_1, w_2, \dots, w_{|w|}). \quad (4)$$

This means, if we take the log of w^{full} we get a vector within which there is a constant spacing between points referring to the size range of fish, while there is a larger spacing between successive terms of the vector that refer to weights below that of a fish egg.

For each $k \in \{1, \dots, |w^{full}| - 1\}$ this is defined as $(\Delta w^{full})_k = w_{k+1}^{full} - w_k^{full}$.

3 Setting Plankton Separately

Sometimes we know the form that the plankton density $N_R(w)$ should take. And sometimes using the Newton-Raphson algorithm to find a steady state is faster if we only have to determine the density of fish species. We shall describe how to construct a steady state by varying the fish density $N_i(w)$, and the carrying capacity $c_p(w)$, given that the plankton density is $N_R(w)$.

The procedure is as follows:

First we solve for a steady state $N_i(w)$ for the fish. Under the assumption that we know $N_R(w)$, we can get the growth rate $g_i(w)$ for our fish, and solve for their steady state density. This can be done by using Newton Raphson, or by evolving the system with the plankton held fixed. Once we have determined the fish density functions $N_i(w)$ that correspond to steady states of the MVF, we determine the predation rate μ_p from $N_i(w)$, and then retune the plankton carrying capacity to be:

$$c_p(w) = N_R(w) \left(1 + \frac{\mu_p(w)}{r_0 w^{p-1}}\right)$$

This ensures that the resulting state (N_R, N) is a steady state when the semischemostat dynamics are considered, in addition to the MVF.

4 Ideas

Can we do larger scale experiments about finding steady states of systems with more species, and see how algos perform

5 Homogenous case

There is a faster way to find steady state in the case where all species have the same feeding kernel. In this case, the growth and death rate of each species just depends upon the total abundance, aggregated over all species. We shall suppose that $\theta_{ij} = 1, \forall i, j$ and $\phi_1 = \dots = \phi_s$ and $\gamma_1 = \dots = \gamma_s$. In this case, there is a way to decrease the dimension of the space that Newton-Raphson has to search through, to find a steady state. We shall consider a case where we are allowed to retune the carrying capacity of the background resource $c_p(w)$, and the reproduction efficiency ϵ_i at the end to complete our construction of

the steady state. We let $N_c(w)$ denote the aggregate abundance over all species of fish. We start by specifying a function F , that takes, as an input, N_c at all modeled discrete values of w , and outputs a vector of the same length. Also, suppose we know that the density of plankton at steady state should be $N_R(w)$. Our procedure will be to use Newton-Raphson to find a zero of F . Starting with N_c , we find $F[N_c]$ as follows:

- Determine growth $g_i(w)$ and $\mu_i(w)$ functions from N_c (this is possible since all species have similar feeding behaviors).
- Solve for the steady state density $N_i(w) = \frac{A_i}{g_i(w)} \exp\left(-\int_{w_{0,i}}^w \frac{\mu_i(w')}{g_i(w')} dw'\right)$ of each species i up to a multiplicative constant A_i
- Choose the A_i 's to minimize the integral $\int_0^\infty \left| N_c(w) - \sum_i \frac{A_i}{g_i(w)} \exp\left(-\int_{w_{0,i}}^w \frac{\mu_i(w')}{g_i(w')} dw'\right) \right|^2 .dw$ (or some similar, re-weighted integral)
- Return $F[N_c]$ such that $F[N_c](w) = N_c(w) - \sum_i \frac{A_i}{g_i(w)} \exp\left(-\int_{w_{0,i}}^w \frac{\mu_i(w')}{g_i(w')} dw'\right)$

We use Newton-Raphson to find an N_C that is a zero of F (all this has to be done after we have discretized the weight space). We extract the A_i 's that minimize the involved integral, and the resulting density functions $N_i(w) = \frac{A_i}{g_i(w)} \exp\left(-\int_{w_{0,i}}^w \frac{\mu_i(w')}{g_i(w')} dw'\right)$ for the different species i .

We can use this resulting N_c to determine the predation mortality rate on the plankton. Since we also assume that the plankton is at density $N_R(w)$ we can solve for the carrying capacity function $c_p(w)$ used to make sure that the steady state equation is satisfied by our semi-chemostat equation for the background resource. The values of the reproduction efficiency ϵ_i can then be re-selected so that the actual egg production is balanced by growing or dying fledglings, so that the system will be at steady state.