PDE scheme Using formula to solve MVF over small time blocks where growth and death rates are assumed fixed.

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Idea: try and describe a formula based PDE solver to get abundance value $N_i(t,w) = \frac{X_i(t,w)}{g_i(w)}$ over end to end time blocks $[L,H]$ that are sufficiently small. Solutions are done repeatedly, for end to end time blocks, to provide a, hopefully, more accurate method of PDE evolution than the upwind difference scheme currently implemented in mizer (which relieves on truncating the Taylor expansion for the abundance, and so loses more info than just assuming constant growth and death over its time steps). Perhaps the scheme we propose here is inferior to the upwind difference scheme for some reason we have not yet considered, or perhaps there are further problems which could crop up with trying to implement the PDE evolver we describe here. We shall have to wait and see as we have not yet tried to implement this PDE evolver idea with code. But, it still looks worth investigating, irrespective of its practical benefits, because considering how this new algorithm works seems (at least to me) quite informative about how information flows across characteristic curves, and is influenced by reproduction, and by the initial condition, in size spectrum models.

This scheme we propose, "of repeatedly using formula to solve the PDE exactly over small time blocks, under the assumption that the growth and death rates are the same across the whole block, as they were at the start of the time block", repeatedly over many small end to end time blocks could provide improvements

Let $A_i(w) = \int_{w_{0.i}}^{w} \frac{1}{g_i(w')} dw'$ denote the age of a weight w member of species i with egg size $w_{0.i}$.

$X_i(t,w) = F_i(t - A_i(w)) \exp\left(-\int_{w_{0.i}}^{w} \frac{\mu_i(w')}{g_i(w')}.dw'\right) = N_i(t,w) g_i$.

If $\tau \geq 0$ [case 1] then $F_i(\tau) = R_i(\tau) := \frac{\epsilon_i}{2w_{0.i}} \int_0^\infty N_i(\tau,w') E_{r.i}(w') \psi_i(w').dw' = X_i(t, w_{0.i})$

or if $\tau < 0$ [case 2] then $F_i(\tau)$ is defined, so that $\forall w \in [w_{0.i}, W_{\infty.i}]$ we have

$X_{0.i}(w) = X_i(0,w) = F_i(-A_i(w)) \exp(-B_i(w))$ where $B_i(w) = \int_{w_{0.i}}^{w} \frac{\mu_i(w')}{g_i(w')}.dw'$.
In particular, let $C_i(w) = -A_i(w)$, and since

$C_i(w)$ is continuous and monotonic, we can define $C_i^{-1}(u)$ as the unique w such that $C_i(w) = u$.

Now $X_i(0,w) = F_i(C_i(w)) \exp(-B_i(w))$ so $X_i(0, C_i^{-1}(u)) = F_i(u) \exp(-B_i(C_i^{-1}(u)))$ so $F_i(u) = X_i(0, C_i^{-1}(u)) \exp(B_i(C_i^{-1}(u)))$ in this case.

$$F_i(\tau) = \begin{cases} RDD_i(\tau) & \text{if } \tau \geq 0 \\ X_i(0, C_i^{-1}(\tau)) \exp(B_i(C_i^{-1}(\tau))) & \text{if } \tau < 0 \end{cases}$$

where $RDD_i(\tau)=$

$X_{0.i}(-A_i^{-1}(u)) = F_i(-A_i(-A_i^{-1}(u))) \exp(-B_i(-A_i^{-1}(u)))$

there exists a function $V_{0.i}()$ such that we can write the initial condition as

We can divide time into blocks, over which we assume g and mu have constant values, and we run over a block,

(what is the physical interpretation of $B_i(w)$?)

B_{i}(w)

weight like or time like trajectories of growth

Description

Under the assumption of time invariant growth rate $g_i(w)$ and time invariant death rate $\mu_i(w)$ the McKendrick–von Foerster equation $\frac{\partial N_i(t,w)}{\partial t} = -\frac{\partial[N_i(t,w)g_i(w)]}{\partial t} - \mu_i(w)N_i(t,w)$. This equation can be rewritten as $\frac{1}{g_i(w)}\frac{\partial X_i(t,w)}{\partial t} + \frac{\partial X_i(t,w)}{\partial w} = -\frac{\mu_i(w)}{g_i(w)}X_i(t,w)$ where $X_i(t,w) := N_i(t,w)g_i(w)$. According to the MathItUp-Canada YouTube video on solving partial differential equations,

PDE YouTube Video

we can find a solution to this type of partial differential equation by equating the following three infinitesimal quantities:

$g_i(w).dt = dw = -\frac{g_i(w)}{\mu_i(w)}\frac{1}{X_i}dX_i$.

Equating the first and second quantities yields the equation

$t = \int_0^t dt = a + \int_{w_{0.i}}^{w} \frac{1}{g_i(w')}dw'$, which can be rearranged to describe the constant of integration as $a = t - \int_{w_{0.i}}^{w} \frac{1}{g_i(w')}dw'$ which corresponds to the birth time of fish which is of weight w at time t. Equating the first and third of our infinitesimal quantities yields the equation

$b + \int_{w_{0.i}}^{w} -\frac{\mu_i(w')}{g_i(w')}dw' = \int_0^{X_i} \frac{1}{X_i'}dX_i' = \ln X_i$.

Now

$X_i(t,w) = B\exp\left(-\int_{w_{0.i}}^{w} \frac{\mu_i(w')}{g_i(w')}\right)$, where $B = \exp(b)$. Now, to complete the solution, we write the constant from integration B, as an arbitrary function F_i of the previous constant of integration a, so we write $B = F_i(a) = F_i(t - A_i(w))$ where $A_i(w) = \int_{w_{0.i}}^{w} \frac{1}{g_i(w')}dw'$ is the age of a weight w member of species i, which began life at egg size $w_{0.i}$. So, provided we can find the form of our function F_i (which we can), it is possible to write an algebraic description of the solution of this system. In particular we have $N_i(t,w) = \frac{X_i(t,w)}{g_i(w)}$ where $X_i(t,w) = F_i(t - A_i(w))\exp(-B_i(w))$ where $B_i(w) = \int_{w_{0.i}}^{w} \frac{\mu_i(w')}{g_i(w')}.dw'$. Soon we shall describe how to determine the value of $F_i(\tau)$ where $\tau = t - A_i(w)$.

However, first, let us define $C_i(w) = -A_i(w)$. Since $w \in [w_{0,i}, W_{\infty.i}) \Rightarrow g_i(w) > 0$, we have that $A_i(w)$ is monotonically increasing with w, and $C_i(w)$ is monotonically decreasing. I guess that if $C_i(w)$ is strictly decreasing and continuous then it will be invertible, but anyhow, suppose $C_i(w)$ is invertible, and let $C_i^{-1}(u)$ denote the w such that $C_i(w) = -A_i(w) = u$.

Now we can continue our description of how to find $F_i(\tau)$ for $\tau = t - A_i(w)$. (Note that τ corresponds to the birth time of the fish which is of size w at time t)

If $\tau \geq 0$ then we let $F_i(\tau) = RDD_i(\tau) = \frac{R_{max.i}R_i(\tau)}{R_{max.i}+R_i(\tau)}$ where $R_i(\tau) = \frac{\epsilon_i}{2w_{0.i}}\int_0^\infty N_i(\tau,w')E_{r.i}(w')\psi_i(w').dw'$ is the egg production at the previous time

τ. This corresponds to the case where we have to look at a the amount of reproduction on the previous year τ in order to determine the size of the current cohort.

If $\tau < 0$ then we let $F_i(\tau) = X_i(0, C_i^{-1}(\tau)) \exp(B_i(C_i^{-1}(\tau)))$ where $X_i(0,w) = N_i(0,w)g_i(w)$. The reasoning for this is as follows, we have $X_i(0,w) = F_i(-A_i(w)) \exp(-B_i(w)) = F_i(C_i(w)) \exp(-B_i(w))$ and , substituting $w = C_i^{-1}(\tau)$ into this expression for $X_i(0,w)$ gives us $X_i(0, C_i^{-1}(\tau)) = F_i(\tau) \exp(-B_i(C_i^{-1}(\tau)))$ and we rearrange this expression to get our recipe $F_i(\tau) = X_i(0, C_i^{-1}(\tau)) \exp(B_i(C_i^{-1}(\tau)))$ for choosing $F_i(\tau)$ when $\tau < 0$. This corresponds to the case where the 'time of birth' τ is negative, and therefore the fish was alive (and in post birth state) at the beginning of the [partial differential equation trajectory], and it turns out that, in a similar way to how to 'time like' $RDD_i(\tau)$ information, informs us about the initial condition, by shaping F when $\tau \geq 0$, we similarly have that when $\tau < 0$ we have that $F_i(w)$ is determined by the 'weight like' initial condition $X_i(0,w)$ we wish to calculate. One way to state this relationship is to say that the values of $F_i(\tau)$ for $\tau < 0$ are chosen so that, $\forall w \in [w_{0.i}, W_{\infty.i}]$ we have $X_i(0,w) = F_i(-A_i(w)) \exp(-B_i(w)) = F_i(C_i(w)) \exp(-B_i(w))$ and we have that [*need to check this] this turns out to be equivalent to saying, for some generic $\tau < 0$ that $F_i(\tau) = X_i(0, C_i^{-1}(\tau)) \exp(B_i(C_i^{-1}(\tau)))$ as argued above.

So the above describes how to solve the partial differential equation, now we will describe how this idea could be used to provide a, hopefully, more accurate time evolution scheme then the PDE solver currently employed. The above is an accurate way of determining $X_i(t,w)$, and hence $N_i(t,w)$ for all $t \in [L,H]$ and all $w \in [w_{0.i}, W_{\infty.i}]$, provided [either [the time interval $[L,H]$ is sufficiently short] or [the populations involved in the $N_i(t,w)$ fluctuations are insufficiently large to have a significant effect on changing the growth rates and death rates], we will have an instance where our assumption $g_i(t,w) \approx g_i(L,w)$, $\mu_i(t,w) \approx \mu_i(L,w)$ that the growth rate and the death rate can be approximated by the value at the left hand side of the '(time) block' $[L,H]$, for each time point $t \in [L,H]$ and for each weight $w \in$ As I said above (perhaps in somewhat confusing language), ensuring that $[L,H]$ is sufficiently small, is one way to make sure that this "growth and death rates do not change" hypothesis is sufficiently valid to justify this approach.

So now we have described how we used a formula based PDE solution to get abundance value $N_i(t,w) = \frac{X_i(t,w)}{g_i(w)}$ over a time block $[L,H]$ that is sufficiently small, and this could be used repeatedly, for end to end time blocks, to provide a, hopefully, more accurate method of PDE evolution than the upwind difference scheme currently implemented in mizer (which relieves on truncating the Taylor expansion for the abundance, and so loses more info than just assuming constant growth and death over its time steps. This scheme we propose, "of repeatedly using formula to solve the PDE exactly over small time blocks, under the assumption that the growth and death rates are the same across the whole block, as they were at the start of the time block", repeatedly over many small end to end time blocks could provide improvements (maybe not, we have to wait and try it first, but it seems interesting to think of mizer from this perspective anyhow, since these ideas can be studied with analytically solveable cases, and we could use such cases as guinie pigs for linear stability analysis, to determine when this 'g and mu dont change' criterion may occur in more practical situations, but I digress.,).

So a basic idea in this "repeated formula based PDE solution thing over end to end time blocks" thing to work, is that in order to determine $N_i(t,w) = \frac{X_i(t,w)}{g_i(w)}$ we need to determine $X_i(t,w) = F_i(t - A_i(w))\exp(-B_i(w))$ (recall $A_i(w) = \int_{w_{0.i}}^{w} \frac{1}{g_i(w')}dw'$ and $B_i(w) = \int_{w_{0.i}}^{w} \frac{\mu_i(w')}{g_i(w')}.dw'$.) so we need to determine $F_i(\tau)$ for each $\tau = t - A_i(w)$ such that $t \in [L,H]$ and $w \in [w_{0.i}, W_{\infty.i}]$. We need to determine the values of $F_i(\tau)$ that appear for the different values of $\tau = t - A_i(w)$ for points $(t,w) \in [L,H] \times [w_{0.i}, W_{\infty.i}]$ that appear during the evolution of the time block. When $\tau < 0$ we can determine the value of $F_i(\tau)$ by considering the initial condition $X_i(0,w) = N_i(0,w)g_i(w)$, using the 'c-inversion' based formula given above. We have already described how to determine $F_i(\tau)$ when $\tau < 0$ by consulting the initial condition $X_i(0,w)$. The other case, which we must describe how to deal with, in order to complete the description of the this new PDE evolving scheme, is how to determine $F_i(\tau)$ when $\tau \geq 0$. In this case, since $0 \leq \tau \leq t$ we have $F_i(\tau) = RDD_i(\tau) = \frac{R_{max.i}R_i(\tau)}{R_{max.i}+R_i(\tau)}$ where $R_i(\tau) = \frac{\epsilon_i}{2w_{0.i}}\int_0^{\infty} N_i(\tau,w')E_{r.i}(w')\psi_i(w').dw'$ is the egg production at the previous time τ. . One approach (?) could be using the exact formula for the $N_i(\tau,w')$ values, for the chosen τ, to calculate $R_i(\tau)$ exactly, 'on demand'. (is there some recursive process that works with exact formula, and looks back over generations, to get explicit formula for current condition in terms of it ?) process recursively, if one wishes to exactly solve the system this, one could try this way.

In practice, filling out $X_i(t,w)$ can be done by working over blocks, and saving all evolution values down to some fine grain time values. To save memory, one could just store the initial condition, and keep record of the high resolution reproduction RDD(t) data, over previous slots, although, actually, come to think of it, I wonder if there is any need to carry such discrete data around, I mean, under the supposed assumptions of this PDE evolver, we get algebraic formula for how the N's change over out successive time blocks, and so we have this information directly at our disposal, to calculate at any resolution we wish. So rather than a time series, I am hoping this approach could yield a formula for the final state, in terms of the initial condition (somehow successive generations get re-threaded into the reproduction integral, in order to give a description of $X_i(t,w)$ directly in terms of the initial condition).

Anyhow, so summarize, we break the time up into blocks, and these are sufficiently small that the assumption that growth and death are held fixed across a block is valid, then we can use the PDE solution formula to describe how abundances change across any given time block. Our scheme may hence be used, much like the current mizer scheme, to iterate forwards, although I hope that the PDE evolver I propose here will be more accurate because it tracks the way that the abundances change in a higher order way. However, as I have mentioned, there may be ways to use this scheme to provide information for arbitrary (t, w) by appealing directly to formula, I am a little confused about this (think it has something to do with iterating over the reproduction equation, will check later), but that is something else to look into later.

Notes:

(It is also worth bearing in mind that the interaction matrix theta having a peculiar structure (so the population fluctuations from the PDE evolution over the block do not have a significant effect on the growth and mortality rates, and , so the assumption of static growth and mortality rates remains valid).

(It is interesting, if somewhat tangential, to note that, for a PDE solver this scheme can be useful, because it allows one to simulate the simulation down to as fine a time scale at high resolution quickly, because it gets the information about all the brief time evolution quickly from a formula, rather than as one wishes (i.e., as one knows about , or cares to interpolate, the values of tangential case, is what the population fluctuations are among a subset of the population which, for whatever reason (perhaps they are infuffiectly numerous etc.) they display this type .