PDE solution for non-interacting fish

December 19, 2017

Our system is

$$\frac{\partial N(w,t)}{\partial t} = -\frac{\partial\left[g(w)N(w,t)\right]}{\partial t} - \mu(w)N(w,t), \forall w \in (w_e, w_*) \tag{1}$$

where

$$X(w_e,t) = KN_m \tag{2}$$

$$\frac{dN_m}{dt} = X(w_*,t) - \bar{\mu}N_m \tag{3}$$

Here $X(w,t) = N(w,t)g(w)$ and we consider the case where $\mu(w) = \mu_0 w^{n-1}$ and $g(w) = \hbar w^n$.

The system can be rewritten as the PDE

$$g(w)\frac{\partial X(w,t)}{\partial w} + \frac{\partial X(w,t)}{\partial t} = -\mu(w)X(w,t), \forall w \in (w_e, w_*) \tag{4}$$

with boundary conditions given by the delay ODE

$$\frac{\partial X(w_e,t)}{\partial t} = KX(w_*,t) - \bar{\mu}X(w_e,t). \tag{5}$$

The general solution to the PDE is

$$X(w,t) = F(L(w) - t)w^{-\frac{\mu_0}{\hbar}} \tag{6}$$

where F is an arbitrary function, and

$$L(w) = \frac{w^{1-n}}{\hbar(1-n)} \tag{7}$$

If $t < \frac{w^{1-n} - w_e^{1-n}}{\hbar(1-n)}$ then

$$X(w,t) = I\left[s(w,t)\right]s(w,t)^{\frac{\mu_0}{\hbar}}w^{-\frac{\mu_0}{\hbar}} \tag{8}$$

where

$$s(w,t) = \left[\hbar(1-n)(L(w)-t)\right]^{\frac{1}{1-n}} \tag{9}$$

is such that $L(w) - t = L(s(w,t)) - 0$

so that characteristic curve through w, t crosses the intial condittion $I(v) = X(v, 0)$ at $v = s(w,t)$.

Next we suppose that

$$\bar{t} < \frac{w_*^{1-n} - w_e^{1-n}}{\hbar(1-n)} \tag{10}$$

So

$$X(w*, \bar{t}) = I\left[s(w_*, \bar{t})\right] s(w_*, \bar{t})^{\frac{\mu_0}{\hbar}} w_*^{-\frac{\mu_0}{\hbar}} \tag{11}$$

is known, we can subsitute this into our delay ODE, changing it into a normal ODE with solution

$$X(w_e, \bar{t}) = e^{-\bar{\mu}\bar{t}} \left(c_1 + \Delta(\bar{t})\right) \tag{12}$$

where $c_1 = X(w_e, 0) = I(w_e)$ and

$$\Delta(\bar{t}) = \int_0^{\bar{t}} e^{\bar{\mu}t'} K X(w*, t') dt' \tag{13}$$

Now, in addition to our inequality for $\bar{t}$, if w, t are such that

$$L(w) - t = L(w_*) - \bar{t} \tag{14}$$

Then since $F(L(w) - t) = F(L(w_*) - \bar{t})$ we have

$$X(w,t) = \left(\frac{w}{w_e}\right)^{-\frac{\mu_0}{\hbar}} X(w, \bar{t}) \tag{15}$$

Hence

$$X(w,t) = \left(\frac{w}{w_e}\right)^{-\frac{\mu_0}{\hbar}} e^{-\bar{\mu}\bar{t}} \left(c_1 + \Delta(\bar{t})\right) \tag{16}$$

where $\bar{t} = t + L(w_*) - L(w)$.

Try initial condition $I(w) = w^{1-n-\alpha}$