On Finding Steady States In Size Spectrum Models

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November 11, 2018

1 Getting a steady state by evolving with fixed egg production

This is my interpretation of the steady state finding procedure conceived of by Gustav Delius.

We shall describe how to find a steady state of a given a multi-species size spectrum model, of the type described in Section 3 of [mizer vignette]. We assume all parameters are pre-determined, although our procedure involves re-tuning the reproduction efficiency ϵ_i.

Step 1: Pick Initial Condition

Suppose we wish to construct the coexistent steady state (N_R, N) that maximizes some objective function $\Psi((N_R, N))$. For example Ψ could encode how well the biomass values for the different species in (N_R, N) agree with empirical data. Our initial condition (N'_R, N') may be chosen by any criterion. For example, one may wish to choose the initial condition (N'_R, N') that maximizes Ψ over all states.

Step 2: Set Initial Egg Production

We need to choose a value R_i at which to fix the egg production, of each species i in $\{1, 2, .., s\}$. For example, we could choose

$R_i = \frac{R_{max.i} R_{p.i}}{R_{max.i} + R_{p.i}}$

where

$R_{p.i} = \frac{\epsilon}{2w_0} \int_0^\infty N'_i(w) E_{r.i}(w) \psi_i(w).dw$

is the egg production that would be generated by state (N'_R, N') (before adjusting for the stock recruitment relationship).

Step 3: Evolve System With Egg Production Held Fixed

We evolve the system under the Mckendrick von Foerster equation (and using the semi-chemostat equation for the resource dynamics), as described in

Section 3 of [mizer vignette], except that we held the egg production fixed at $N_i(t, w_{0.i}) = R_i$ for each time t. Although it is unclear, experiments suggest that in these kinds of systems, where the egg production values are held fixed at R_i, have just one coexistent steady state (that is, a steady state where each species has none zero density). In most cases these dynamics will evolve to a steady state (N_R^*, N^*). And although we have observed cases where such dynamics do not settle to a steady state (and instead end in an oscillation), hopefully a unique coexistent steady state (N_R^*, N^*) can still be found in such cases, using Netwton-Raphson. [Questions: does the final state/limit cycle depend on the initial condition ? In cases which cycles when egg reproduction is held fixed, what happens if egg production is then allowed to vary naturally, Do systems which are oscillatory when egg production is held fixed remain so, when egg production is properly generated in a density dependent manner.]

Step 4: Optimize (optional)

Compute $\Psi((N_R^*, N^*))$ for the resulting steady state. If desired one may keep re-selecting the egg production values $(R_1, .., R_s)$ and re-running Step 3 (hopefully the dynamics are such that the initial condition is irrelevant, but we should check), in order to get a new steady state,.. etc. So essentially one could attempt to select $(R_1, .., R_s)$ to optimize $\Psi((N_R^*, N^*))$, with the caveat that we may not be able to find a steady state for some values of $(R_1, .., R_s)$ by simply evolving the system (although hopefully there is still a unique coexistant steady state to find using Netwon-Raphson).

Step 5: Re-select reproduction efficiency

We take the steady state (N_R^*, N^*) from the constant-egg-production system, and, for each species i in $\{1, .., s\}$, choose a new value ϵ_i' of the reproduction efficiency, so that the value R_i at which the egg production was previously held constant, is equal to the amount of eggs

$$\frac{R_{max.i} \times \frac{\epsilon_i'}{2w_{0.i}} \int_0^\infty N_i^*(w) E_{r.i}(w) \psi_i(w).dw}{R_{max.i} + \frac{\epsilon_i'}{2w_{0.i}} \int_0^\infty N_i^*(w) E_{r.i}(w) \psi_i(w).dw} \tag{1}$$

that would be generated by the resulting steady state, using the new reproduction efficiency value ϵ_i' (and accounting from possible egg loss due to R_{max}).

Result

(N_R^*, N^*) will be a steady state of the resulting multi-species size spectrum model, using new reproduction efficiency values ϵ_i'. The steady state construction procedure described above can be repeated for different parameter choices until one is satisfied with the resulting reproduction efficiency values. Alternatively, the objective function Ψ referred to in step 4 could be selected so as to reward the generation of the proper reproduction efficiency value as the output ϵ_i'.

Remaining Questions

- When does the algorithm fail ?
- Do basins of attraction matter ?
- Is there always one unique interior steady state ?
- What are the partial functions like: from $(R_1, .., R_s)$ to (N_R^*, N^*), and from (N_R^*, N^*) to $(\epsilon_1, .., \epsilon_s)$?

2 Newton-Raphson

Need to descibe discretization scheme, and function to find zero of, for Newton-Raphson solving.

3 Setting Plankton Seperately

Sometimes we know the form that the plankton density $N_R(w)$ should take. And sometimes using the Newton-Raphson algorithm to find a steady state is faster if we only have to determine the density of fish species. We shall describe how to construct a steady state by varing the fish density $N_i(w)$, and the carying capacity $c_p(w)$, given that the plankton density is $N_R(w)$.

The procedure is as follows:

First we solve for a steady state $N_i(w)$ for the fish. Under the assumption that we know $N_R(w)$, we can get the growth rate $g_i(w)$ for our fish, and solve for their steady state density. This can be done by using Netwton Raphson, or by evolving the system with the plankton held fixed. Once we have determined the fish densitiy functions $N_i(w)$ that correspond to steady states of the MVF, we determine the predation rate μ_p from $N_i(w)$, and then retune the plankton carrying capacity to be:

$c_p(w) = N_R(w) \left(1 + \frac{\mu_p(w)}{r_0 w^{p-1}}\right)$

This ensures that the resulting state (N_R, N) is a steady state when the semischemostat dynamics are considered, in addition to the MVF.

4 Ideas

Can we do larger scale experiments about finding steady states of systems with more species, and see how algos perform

5 Homogenous case

Faster way to find steady state in the case where all species have the same feeding kernel. In this case, the growth and death rate of each species just depeds upon the total abundance, aggregated over all species.