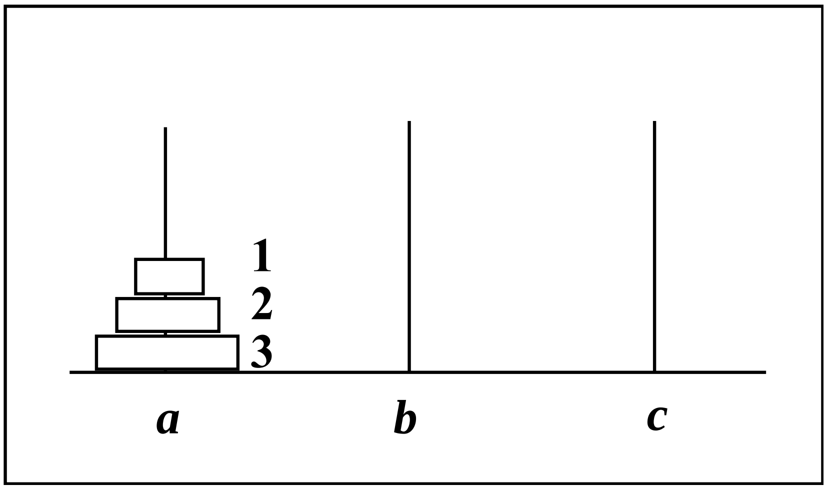
Tower of Hanoi Problem
(The general case)

The problem is as follows. There are three pivots ***a***, ***b***, and ***c***. On the first one there is a set of *n* disks, each of different radius. The task is to move all the disks to the pivot ***c*** moving only one disk at a time. In addition, at no time during the process may a disk be placed on top of a smaller disk. The pivot ***c*** can, of course, be used as a temporary resting place for the disks.



Let us designate an elementary step of moving disk number i from the pivot x to the pivot y as ***p***(i, x, y), a terminal symbol with parameters. Thus a solution of the Tower of Hanoi Problem might be represented as the following string of symbols with parameters:

$$\boldsymbol{p}(i_1, x_1, y_1)\boldsymbol{p}(i_2, x_2, y_2)\ldots\boldsymbol{p}(i_m, x_m, y_m).$$

This is the string of the language of all possible sequences of moves. Consider the controlled grammar shown in Figure 4. We will apply this grammar for derivation of a solution for the case of three disks: n =3, x=***a***, y=***c***. It means that the values of parameters for the starting symbol ***S*** are ***S***(3, ***a***, ***b***).

Controlled grammar generating solutions to the Tower of Hanoi Problem

L	Q	Kernel, π_k	π_n	F_T	F_F
1	Q_1	S(n, x, y) –> A(n, x, y)		2	ø
2	Q_2	A(n, x, y) –> A(f_1(n), x, f_2(x, y)) p(n, x, y) A(f_1(n), f_2(x, y), y)		2	3
3	Q_3	A(n, x, y) –> p(n, x, y)		2	ø

Here $V_T = \{p\}$

$V_N = \{S, A\}$

V_{PR}

Pred $= \{Q_1, Q_2, Q_3\}$,

$Q_1 = T$

$Q_2(n) = T$, if n>1; $Q_2(n) = F$, if n = 1.

$Q_3(n) = T$, if n=1; $Q_3(n) = F$, if n > 1.

Var = {n, x, y}

$F = Fcon \cup Fvar$,

$Fcon = \{f_1, f_2\}$

$f_1(n) = n-1$, n = 2, 3,...

$f_2(x, y)$ yields the value from $\{a, b, c\} \setminus \{x, y\}$, where values of x, y are from $\{a, b, c\}$

$Fvar = \{3, a, c\}$

$E = Z_+ \cup \{a, b, c\}$

Parm: $S \to Var$, $A \to Var$, $p \to Var$

$L = \{1, 2, 3\}$

At the beginning of derivation: x = a, y = c, n = 3.

Generation of a solution in case of n = 3:

$\boldsymbol{S}(3, \boldsymbol{a}, \boldsymbol{c})\ {}^{1}\Rightarrow \boldsymbol{A}(3, \boldsymbol{a}, \boldsymbol{c})\ {}^{2}\Rightarrow \boldsymbol{A}(2, \boldsymbol{a}, \boldsymbol{b})\boldsymbol{p}(3, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{A}(2, \boldsymbol{b}, \boldsymbol{c})$

${}^{2}\Rightarrow \boldsymbol{A}(1, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{p}(2, \boldsymbol{a}, \boldsymbol{b})\boldsymbol{A}(1, \boldsymbol{c}, \boldsymbol{b})\boldsymbol{p}(3, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{A}(2, \boldsymbol{b}, \boldsymbol{c})$

${}^{3}\Rightarrow \boldsymbol{p}(1, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{p}(2. \boldsymbol{a}, \boldsymbol{b})\boldsymbol{A}(1, \boldsymbol{c}, \boldsymbol{b})\boldsymbol{p}(3, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{A}(2, \boldsymbol{b}, \boldsymbol{c})$

${}^{3}\Rightarrow \boldsymbol{p}(1, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{p}(2, \boldsymbol{a}, \boldsymbol{b})\boldsymbol{p}(1, \boldsymbol{c}, \boldsymbol{b})\boldsymbol{p}(3, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{A}(2, \boldsymbol{b}, \boldsymbol{c})$

${}^{2}\Rightarrow \boldsymbol{p}(1, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{p}(2, \boldsymbol{a}, \boldsymbol{b})\boldsymbol{p}(1, \boldsymbol{c}, \boldsymbol{b})\boldsymbol{p}(3, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{A}(1, \boldsymbol{b}, \boldsymbol{a})\boldsymbol{p}(2, \boldsymbol{b}, \boldsymbol{c})\boldsymbol{A}(1, \boldsymbol{a}, \boldsymbol{c})$

${}^{3}\Rightarrow \boldsymbol{p}(1, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{p}(2, \boldsymbol{a}, \boldsymbol{b})\boldsymbol{p}(1, \boldsymbol{c}, \boldsymbol{b})\boldsymbol{p}(3, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{p}(1, \boldsymbol{b}, \boldsymbol{a})\boldsymbol{p}(2, \boldsymbol{b}, \boldsymbol{c})\boldsymbol{A}(1, \boldsymbol{a}, \boldsymbol{c})$

${}^{3}\Rightarrow \boldsymbol{p}(1,\boldsymbol{a}, \boldsymbol{c})\boldsymbol{p}(2,\boldsymbol{a}, \boldsymbol{b})\boldsymbol{p}(1, \boldsymbol{c}, \boldsymbol{b})\boldsymbol{p}(3, \boldsymbol{a}, \boldsymbol{c})\boldsymbol{p}(1, \boldsymbol{b}, \boldsymbol{a})\boldsymbol{p}(2, \boldsymbol{b}, \boldsymbol{c})\boldsymbol{p}(1, \boldsymbol{a}, \boldsymbol{c})$.

Generation of a solution in case of n = 4:

$S(4, a, c) \overset{1}{\Rightarrow} A(4, a, c) \overset{2}{\Rightarrow} \underline{A(3, a, b)}p(4, a, c)A(3, b, c)$

$\underline{A(3, a, b)} \overset{2}{\Rightarrow} A(2, a, c)p(3, a, b)A(2, c, b)$
$\overset{2}{\Rightarrow} A(1, a, b)p(2, a, c)A(1, b, c)p(3, a, b)A(2, c, b)$
$\overset{3}{\Rightarrow} p(1, a, b)p(2, a, c)A(1, b, c)p(3, a, b)A(2, c, b)$
$\overset{3}{\Rightarrow} p(1, a, b)p(2, a, c)p(1, b, c)p(3, a, b)A(2, c, b)$
$\overset{2}{\Rightarrow} p(1, a, b)p(2, a, c)p(1, b, c)p(3, a, b)A(1, c, a)p(2, c, b)A(1, a, b)$
$\overset{3}{\Rightarrow} p(1, a, b)p(2, a, c)p(1, b, c)p(3, a, b)p(1, c, a)p(2, c, b)A(1, a, b)$
$\overset{3}{\Rightarrow} p(1, a, b)p(2, a, c)p(1, b, c)p(3, a, b)p(1, c, a)p(2, c, b)p(1, a, b).$

$\overset{2,..,3}{\Rightarrow} p(1,a, b)p(2,a, c)p(1, b, c)p(3, a, b)p(1, c, a)p(2, c, b)p(1, a, b)$
$p(4, a, c)\underline{A(3, b, c)}$

$\underline{A(3, b, c)} \overset{2}{\Rightarrow} A(2, b, a)p(3, b, c)A(2, a, c)$
$\overset{2}{\Rightarrow} A(1, b, c)p(2, b, a)A(1, c, a)p(3, b, c)A(2, a, c)$
$\overset{3}{\Rightarrow} p(1, b, c)p(2. b, a)A(1, c, a)p(3, b, c)A(2, a, c)$
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$\overset{3}{\Rightarrow} p(1, b, c)p(2, b, a)p(1, c, a)p(3, b, c)p(1, a, b)p(2, a, c)A(1, b, c)$
$\overset{3}{\Rightarrow} p(1, b, c)p(2, b, a)p(1, c, a)p(3, b, c)p(1, a, b)p(2, a, c)p(1, b, c).$

$\overset{2,..,3}{\Rightarrow} p(1,a, b)p(2,a, c)p(1, b, c)p(3, a, b)p(1, c, a)p(2, c, b)p(1, a, b)$
$p(4, a, c)$
$p(1, b, c)p(2, b, a)p(1, c, a)p(3, b, c)p(1, a, b)p(2, a, c)p(1, b, c).$

Induction

Let's prove by induction that $1+2+3+...+n = n(n+1)/2$

1. For n=1 this is true: $1=1(1+1)/2$

2. Assume that the statement is true for n=m, i.e.,
$1+2+3+...+m = m(m+1)/2$

3. Let's prove that the statement is true for n=m+1.
Indeed, from the assumption 2 we have
$1+2+3+...+m + (m+1) = m(m+1)/2 + (m+1) =$
$(m/2+1)(m+1) = [(m+2)/2](m+1) = (m+1)(m+2)/2.$
The statement is proved for n = m+1.

4. By induction we conclude that this statement is true for all n = 1, 2, 3, ...

Consider the following problem:
$1+3+5+7+...+(2n-1) = ?$
Find the answer and prove by induction.

<u>Solution</u>

1. $1 = 1^2$

2. Assume that $1+3+5+...+(2m - 1) = m^2$

3. $1+2+3+...+(2(m+1) - 1) = 1+2+3+...+(2m -1) + (2m+1) = m^2 + (2m +1) =$
$m^2 + 2m +1 = (m +1)^2$

Examples of Induction

Prove that $1^2 + 2^2 + ... + n^2 = [n(n+1)(2n+1)] / 6$

Prove that $1^3 + 2^3 + ... + n^3 = [n^2(n+1)^2] / 4$

Prove that the number that consists of 3^n units (like $\underbrace{1111...1}_{3^n}$) is divisible by 3^n.

Solution

1. **111** is divisible by $\mathbf{3^1}$.

2. Assume that the statement is true for n = m, i.e., **A=** $\underbrace{\mathbf{111\ ...\ 1}}_{3^m}$.

3. Let's prove the statement for **B =** $\underbrace{\mathbf{111\ ...\ 1}}_{3^{m+1}}$.

Consider

$$\underbrace{111111111}_{3^{1+1}} = \underbrace{111}_{3^1}\underbrace{000000}_{2\cdot 3^1} + \underbrace{111}_{3^1}\underbrace{000}_{3^1} + \underbrace{111}_{3^1}$$

Let a = **111**, then

$$= a\cdot 10^{3^1\cdot 2} + a\cdot 10^{3^1} + a$$

$$= a \times (10^{3^1\cdot 2} + 10^{3^1} + 1)$$

$$B = \underbrace{111\ ...\ 1}_{3^m}\underbrace{000\ ...\ 0}_{2\cdot 3^m} + \underbrace{111\ ...\ 1}_{3^m}\underbrace{000\ ...\ 0}_{3^m} + \underbrace{111\ ...\ 1}_{3^m} = A\cdot 10^{3^m\cdot 2} + A\cdot 10^{3^m} + A$$

By assumption **A** is divisible by $\mathbf{3^m}$; hence, **B** = **A** × $(10^{3^m\cdot 2} + 10^{3^m} + 1)$ is divisible by $\mathbf{3^{m+1}}$, because the sum of digits of the expression in parenthesis is equal to 3 , i.e., it is divisible by 3.

Induction for the Tower of Hanoi problem

Let's prove that the grammar generates a solution of this problem in general case. We shall prove by induction.

1. We proved that the grammar generates a solution of this problem for n = 3.

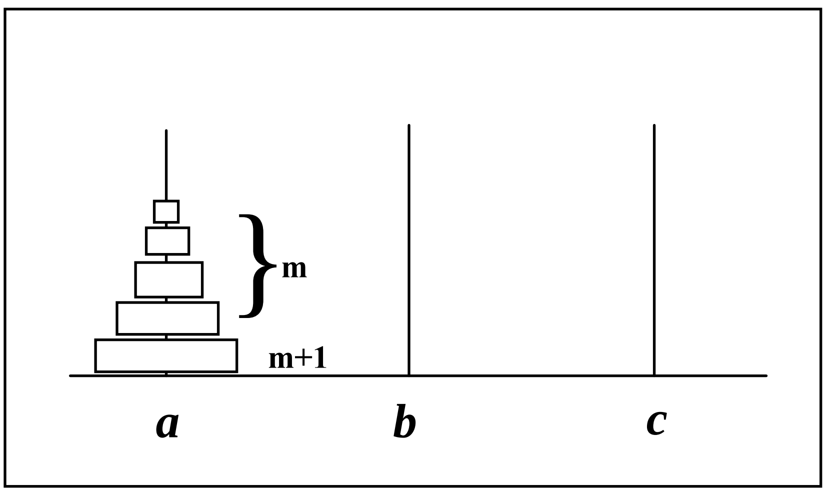
2. Assume that the grammar generates a solution for n = m.

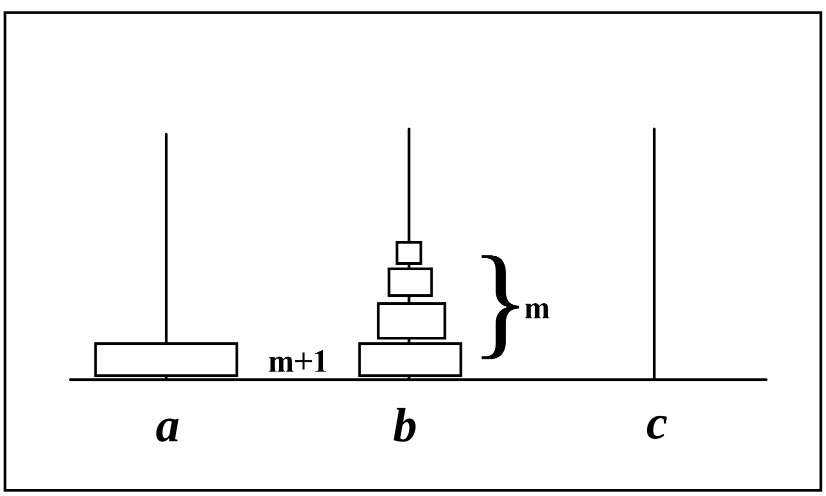
3. Let's prove that it generates a solution for n= m+1

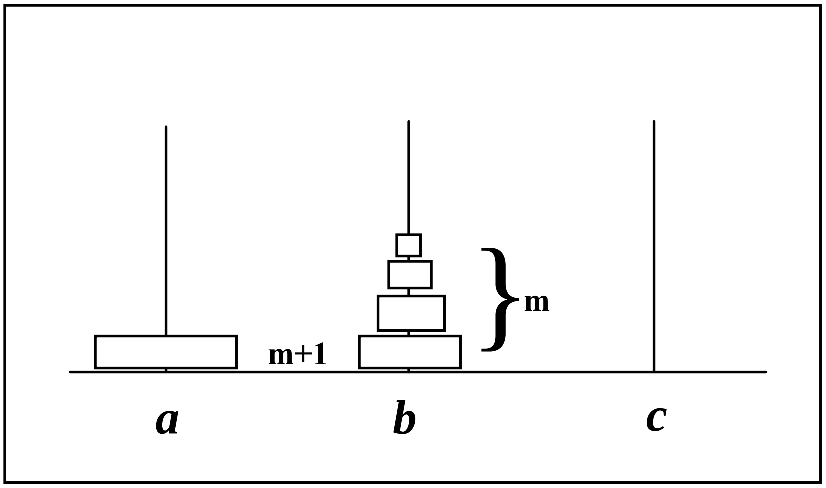
Consider the derivation in case of n = m+1:

$S(m+1, a, c) \overset{1}{\Rightarrow} A(m+1, a, c) \overset{2}{\Rightarrow} A(m, a, b)p(m+1, a, c)A(m, b, c)$.

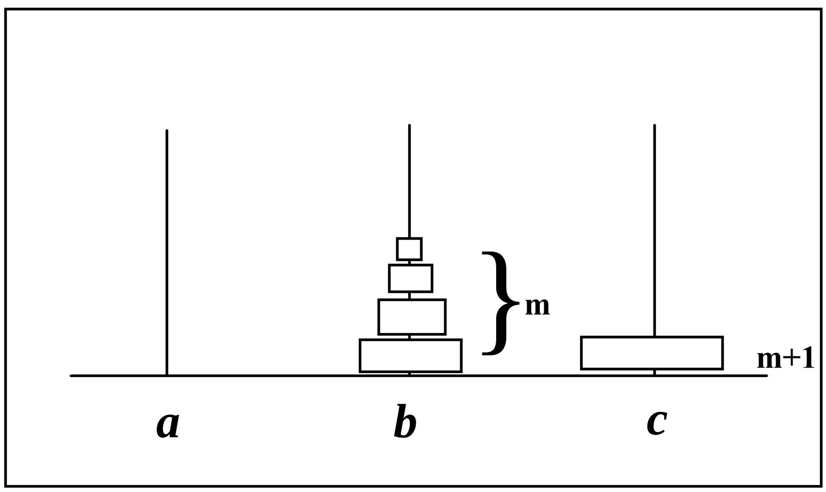
Obviously, the following application of the grammar to the symbol $A(m, a, b)$ **will generate the string of symbols. According to the assumption of induction 2. this string corresponds to the solution of the Tower of Hanoi problem with m disks on the pivot** ***a*****. These disks must be moved to the pivot** ***b*****.**





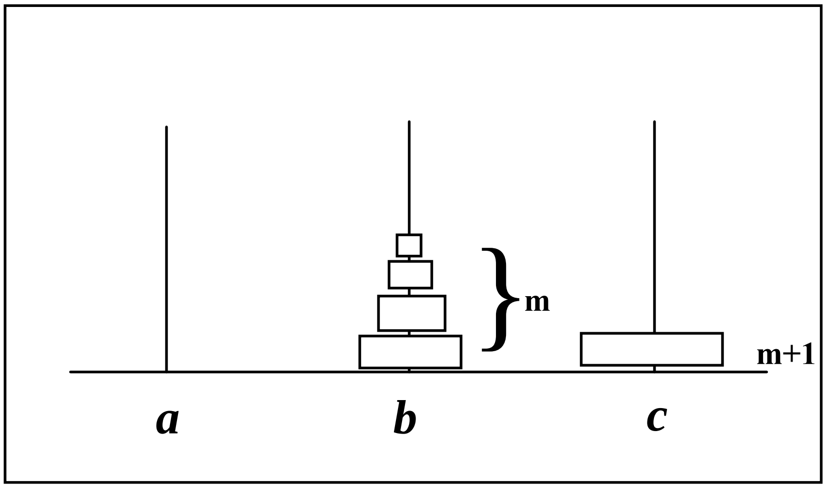


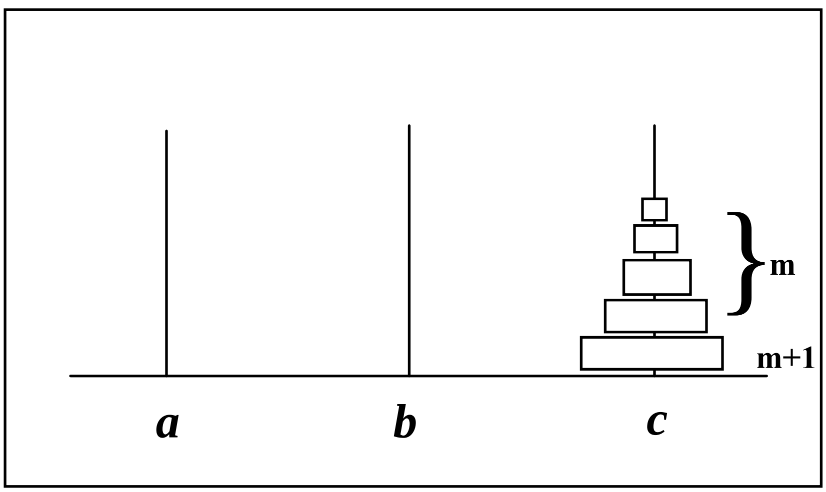
When these disks are moved to the pivot b we can apply *p*(m+1, *a*, *c*), i.e., we can move disk m+1 from pivot *a* to pivot *c*.



S(m+1, ***a***, ***c***) ¹=> ***A***(m+1, ***a***, ***c***) ²=> ***A***(m, ***a***, ***b***)***p***(m+1, ***a***, ***c***)***A***(m, ***b***, ***c***).

The following application of the grammar to the symbol *A*(m, *b*, *c*) will generate the string of symbols. According to the assumption of induction 2. this string corresponds to the solution of the Tower of Hanoi problem with m disks on the pivot *b*. These disks must be moved to the pivot *c*.





It means that the grammar generates a solution for n = m+1.

4. Conclusion: by induction the grammar generates a solution of the Tower of Hanoi problem for all n = 3, 4, 5, ...