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Week 3: European Option Pricing and Black–Scholes Intuition

Goal: Understand how options are priced under the no-arbitrage principle and risk-neutral valuation.

Main Topics:

- Concept of no-arbitrage and replication
- Payoff of a European call and put option
- Intuitive explanation of the Black–Scholes model
- Expected value under the risk-neutral measure

Concept of No-Arbitrage and Replication

What is Arbitrage?

- A trading strategy that requires *zero initial cost*, *no risk*, and yields a *guaranteed profit*.
- **Mathematical definition:** A strategy with initial wealth

$$V_0 = 0,$$

final wealth

$$V_T \geq 0 \quad \text{almost surely,}$$

and

$$\mathbb{P}(V_T > 0) > 0.$$

Concept of No-Arbitrage and Replication - 3

In other terms, you have an arbitrage if, with zero initial money, you can only draw or win, and there is a strictly positive probability that you win: *"money from nothing"*.

We say that the market is **arbitrage–free**, or simply that we have **no arbitrage**, if there are no arbitrage opportunities.

We will work under no-arbitrage conditions, because no one wants to be in an arbitrageable market: arbitrageurs could make money from nothing, putting everyone else at a disadvantage and causing them to lose money to the arbitrageurs.

Concept of No-Arbitrage and Replication - 4

Simple example (starting with zero wealth).

Suppose a stock costs £10 on Exchange A and £11 on Exchange B.

- Start with **initial wealth** $V_0 = 0$.
- Borrow £10 from the bank at the risk–free rate.
- Use the £10 to **buy** the stock on Exchange A.
- Immediately **sell** the same stock on Exchange B for £11.
- Repay the £10 loan.

Your final wealth is:

$$V_T = 11 - 10 = 1 > 0,$$

which is a guaranteed profit with zero initial investment and no risk.

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Concept of No-Arbitrage and Replication - 5

Replication Strategy

- An option payoff can be reproduced by a portfolio consisting of:
  - Δ: units of the underlying stock,
  - B: amount invested (or borrowed if negative) in the risk–free asset.
- The value of the replicating portfolio at time t is:

$$V_t = \Delta S_t + Be^{rt}.$$

- If (Δ, B) is chosen such that:

$$V_T = \text{Option Payoff at } T,$$

then the portfolio *perfectly replicates* the option.

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Concept of No-Arbitrage and Replication - 6

Why must the prices be the same? (Law of One Price)

- Two portfolios that deliver the *same payoff in every possible state* must have the *same price today*.
- If the option price were different from $\Delta S_0 + B$:
  - Buy the cheaper one,
  - Sell the more expensive one,

  generating a *risk-free arbitrage profit*.
- Therefore:

$$\text{Option Price} = \Delta S_0 + B.$$

- An option's payoff can be reproduced using a portfolio of:
  - Δ: units of the underlying stock
  - B: amount invested/borrowed in the risk-free asset
- If a portfolio (Δ, B) *perfectly replicates* the option payoff, then the option must have the same price.

Concept of No-Arbitrage and Replication - 7

No-Arbitrage Pricing Principle

- If two portfolios have the same payoff in every state, they must have the same price:

$$\text{Option Price} = \Delta S_0 + B$$

- Otherwise, you could buy the cheaper one and sell the more expensive one $\rightarrow$ arbitrage.

Payoff Diagram: European Call Option

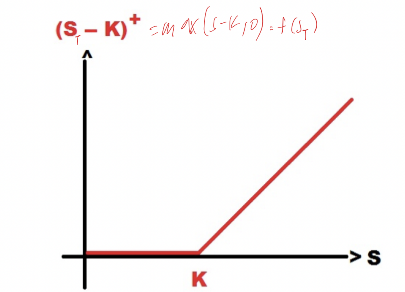

Figure 1: Call Option Pay off

Payoff Diagram: European Put Option

Figure 2: Put Option Pay off

Intuitive Derivation of Black–Scholes (1/5)

Step 1: Model the underlying and the option.

- Stock price follows a geometric Brownian motion:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

  where μ is the drift, σ the volatility.

- Risk–free asset (bank account):

$$dB_t = rB_t \, dt, \qquad B_t = e^{rt}.$$

- The option price at time t is a function of time and stock price:

$$V_t = V(t, S_t).$$

Idea: Use Itô's formula to find dV_t, then use replication and no–arbitrage to eliminate risk and obtain the Black–Scholes equation.

Intuitive Derivation of Black–Scholes (2/5)

Step 2: Apply Itô's formula to $V(t, S_t)$.

Assume $V(t, S)$ is smooth enough: $V \in C^{1,2}([0, T] \times \mathbb{R}_+)$.

- Itô's formula gives

$$dV(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)\, dt + \frac{\partial V}{\partial S}(t, S_t)\, dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\, dS_t^2.$$

- Using $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $dW_t^2 = dt$:

$$dS_t^2 = (\mu S_t dt + \sigma S_t dW_t)^2 = \sigma^2 S_t^2 dt.$$

- Substitute back:

$$dV(t, S_t) = \left(\frac{\partial V}{\partial t}(t, S_t) + \mu S_t \frac{\partial V}{\partial S}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t)\right) dt + \sigma S_t \frac{\partial V}{\partial S}(t, S_t)\, dW_t.$$

Intuitive Derivation of Black–Scholes (3/5)

Step 3: Build a replicating, self–financing strategy.

- Consider a trading strategy (ϕ_t^S, ϕ_t^B):

$$V_t = \phi_t^S S_t + \phi_t^B B_t$$

(same idea as our replication slide: stock + bank account).

- **Self–financing condition:** no extra money in or out, changes in V_t come only from changes in S_t and B_t:

$$dV_t = \phi_t^S \, dS_t + \phi_t^B \, dB_t.$$

- Substitute dS_t and dB_t:

$$\begin{aligned} dV_t &= \phi_t^S(\mu S_t dt + \sigma S_t dW_t) + \phi_t^B(rB_t dt) \\ &= \left(\phi_t^S \mu S_t + \phi_t^B rB_t\right) dt + \phi_t^S \sigma S_t dW_t. \end{aligned}$$

Key idea: This portfolio is intended to *replicate* the option, so its value and dynamics must match V_t and dV_t from the Itô formula.

Intuitive Derivation of Black–Scholes (4/5)

Step 4: Match the two expressions for dV_t.
From Itô (previous slide):

$$\boxed{dV_t = \left(V_t^{(t)} + \mu S_t V_t^{(S)} + \tfrac{1}{2}\sigma^2 S_t^2 V_t^{(SS)}\right) dt + \sigma S_t V_t^{(S)} dW_t}$$

where we use shorthand $V_t^{(t)} = \frac{\partial V}{\partial t}$, $V_t^{(S)} = \frac{\partial V}{\partial S}$, etc.
From the replicating strategy:

$$\boxed{dV_t = \left(\phi_t^S \mu S_t + \phi_t^B r B_t\right) dt + \phi_t^S \sigma S_t dW_t.}$$

Intuitive Derivation of Black–Scholes (4/5)

1. **Match the dW_t terms (eliminate risk):**

$$\sigma S_t V_t^{(S)} = \phi_t^S \sigma S_t \quad \Rightarrow \quad \phi_t^S = V_t^{(S)}.$$

2. **Use $V_t = \phi_t^S S_t + \phi_t^B B_t$ to find ϕ_t^B:**

$$\phi_t^B = \frac{V_t - \phi_t^S S_t}{B_t} = \frac{V_t - S_t V_t^{(S)}}{B_t}.$$

Intuitive Derivation of Black–Scholes (5/5)

Step 5: Plug back and use no–arbitrage.

Substitute ϕ_t^S and ϕ_t^B into the dt part of dV_t from the strategy:

$$\begin{aligned} dV_t &= \left[\phi_t^S \mu S_t + \phi_t^B r B_t\right] dt \\ &= \left[V_t^{(S)} \mu S_t + r\left(V_t - S_t V_t^{(S)}\right)\right] dt \\ &= \left[r V_t + (\mu - r) S_t V_t^{(S)}\right] dt. \end{aligned}$$

But from Itô we also have:

$$dV_t = \left(V_t^{(t)} + \mu S_t V_t^{(S)} + \tfrac{1}{2}\sigma^2 S_t^2 V_t^{(SS)}\right) dt.$$

By no–arbitrage, these dt terms must be equal:

$$V_t^{(t)} + \mu S_t V_t^{(S)} + \tfrac{1}{2}\sigma^2 S_t^2 V_t^{(SS)} = r V_t + (\mu - r) S_t V_t^{(S)}.$$

Intuitive Derivation of Black–Scholes (5/5)

Simplify (the μ terms cancel) to get the **Black–Scholes PDE**:

$$\boxed{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.}$$

For a European call with payoff $V(T,S) = (S-K)^+$, the solution is the Black–Scholes formula:

$$V(t,S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

with

$$d_{1,2} = \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Put–Call Parity and Black–Scholes Put Formula

Put–Call Parity

For European options (no early exercise), the following no–arbitrage relationship holds:

$$C_t - P_t = S_t - Ke^{-r(T-t)}.$$

- Intuition: A call + discounted strike is equivalent to a put + stock.
- This comes directly from comparing payoffs at maturity for both portfolios.

Put–Call Parity and Black–Scholes Put Formula -2

Deriving the European Put Price

Given the Black–Scholes call price

$$C_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

substitute into put–call parity:

$$P_t = C_t - S_t + Ke^{-r(T-t)}.$$

Simplifying:

$$\boxed{P_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1).}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Risk–Neutral Interpretation of Black–Scholes

Why another view of Black–Scholes?

- The PDE / replication derivation is mathematically heavy (Itô calculus, hedging, self-financing condition).
- The risk–neutral approach gives the *same price* but with a much simpler formula:

$$V_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\big[\text{Payoff at } T\big].$$

- This is extremely convenient for numerical methods.

Risk–Neutral Interpretation of Black–Scholes -2

Risk–neutral idea (informal)

- Under the risk–neutral measure $\mathbb{Q}$, all assets grow at the risk–free rate r.
- The stock follows the **risk–neutral SDE**:

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}.$$

- Pricing becomes taking a discounted expectation:

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\text{Payoff}(S_T)].$$

Risk–Neutral Interpretation of Black–Scholes -3

Why is this convenient for Monte Carlo?

- We no longer need to hedge or solve a PDE.
- We only need to simulate S_T under the simple drift r:

$$S_T = S_0 \exp\Big((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,Z\Big), \quad Z \sim N(0,1).$$

- Monte Carlo recipe becomes:
  1. simulate many paths of S_T under $\mathbb{Q}$,
  2. compute the payoff on each path,
  3. average the payoffs,
  4. discount by e^{-rT}.
- Very flexible: works for exotic payoffs where no closed form exists.

Assumptions of the Black–Scholes Model

Market Assumptions

- No arbitrage opportunities exist.
- Markets are frictionless:
  - no transaction costs,
  - no bid–ask spreads,
  - assets are perfectly divisible.
- Continuous trading is possible.
- Investors can borrow and lend at the same constant risk–free rate r.

Assumptions of the Black–Scholes Model -2

Underlying Asset Assumptions

- Stock price follows a geometric Brownian motion:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

- Volatility σ is constant.
- The stock pays no dividends.

Assumptions of the Black–Scholes Model -3

Option/Model Assumptions

- The option is European (exercise only at maturity).
- Trading strategies are self–financing.
- The probability distribution of returns is lognormal.

Summary: What We Learned Today

Key Takeaways

- **No-arbitrage and replication** If an option payoff can be perfectly replicated using stock + bank account, then by the Law of One Price it must have the same price.
- **Black–Scholes via hedging** Model S_t as GBM, apply Itô's formula to $V(t, S_t)$, build a self-financing replicating portfolio, and use no-arbitrage to obtain the Black–Scholes PDE and closed-form call/put prices.

Summary: What We Learned Today

- **Put–call parity** Relates European calls and puts:

$$C_t - P_t = S_t - Ke^{-r(T-t)},$$

  and lets us derive the put price from the call.

- **Risk-neutral pricing** Under the risk-neutral measure $\mathbb{Q}$,

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\text{Payoff}(S_T)],$$

  giving an alternative, expectation-based view of pricing.

- **Monte Carlo link** Simulate S_T under the risk-neutral dynamics, compute payoffs, average, and discount: a flexible numerical method for more complex options (beyond Black–Scholes).