DRP summary

December 2025

1 Big Picture: What Are We Doing?

- Understand **financial assets** and **derivatives**.
- Focus on **options** (calls & puts), their **payoffs** and **profit**.
- Model stock prices as **random processes** (GBM) and use **Monte Carlo** to estimate expectations.
- Use **no-arbitrage + replication** to get **Black–Scholes prices** and connect them to **risk-neutral Monte Carlo**.
- Improve Monte Carlo **accuracy** via **variance reduction** (antithetic variates & control variates).
- Implement everything in **Python** (basic syntax + NumPy).

2 Assets, Securities, Derivatives & Options

2.1 Assets & Securities

Asset: anything of value that provides **future economic benefit** (e.g. house, stock, bond).

Security: a **tradable claim** with monetary value (e.g. share, bond, option).

Types of assets:

- **Real assets**: physical items – real estate, gold, machinery.
- **Financial assets**: claims on cash flows – **equities (stocks)**, **bonds**, **cash**.

Equity (share/stock):

- Represents **ownership** in a company.
- Traded on stock exchanges (e.g. LSE, NYSE).
- **Market value** = price investors are currently willing to pay.

Fixed income (bonds):

- Investor **lends money** to borrower (company/government).
- Receives fixed **coupons** (interest) + **principal** at maturity.
- Example: Lend £20k at 1% for 5 years $\Rightarrow$ £200 per year + £20k at the end.

2.2 What Are Derivatives?

Derivative: instrument whose value **depends on** an underlying asset (stock, index, rate, commodity).

Examples:

- **Forwards**: private agreement to buy/sell at fixed price in future.
- **Futures**: standardised forwards traded on an exchange.
- **Options**: right (not obligation) to buy/sell at fixed price.
- **Swaps**: agreements to exchange cash flows.

2.3 Call and Put Options

Key terms:

- **Underlying**: asset the option is written on (e.g. Apple stock).
- **Strike** K: fixed price at which you can buy/sell.
- **Maturity** T: expiry date.
- **European**: can exercise only at **maturity** T.
- **American**: can exercise any time **up to** T.

Rights:

- **Call option** (long call): right to **buy** the underlying at K.
- **Put option** (long put): right to **sell** the underlying at K.

Payoffs at maturity (European):
Let S_T be the stock price at time T.

$$\text{Call payoff} = (S_T - K)^+ = \max(S_T - K, 0),$$

$$\text{Put payoff} = (K - S_T)^+ = \max(K - S_T, 0).$$

Profit = payoff − premium (initial option price).

Typical shapes:

- Long call: zero until $S_T > K$, then increases linearly.
- Long put: falls linearly when $S_T < K$, zero above K.

3 Randomness in Finance & Stock Price Models

3.1 Why Randomness?

- Prices react to **random news**: earnings, macro data, rumours, ...
- Price change ≈ **trend + random shock**.
- We model prices as **random processes**.

3.2 Discrete Random Walk

Simple toy model:

$$S_{n+1} = S_n + \varepsilon_n, \quad \varepsilon_n \in \{-1, +1\}.$$

- Each step is an **up or down move** chosen at random (like coin tosses).
- Captures unpredictability but can give **negative prices**, so not realistic long term.

3.3 Brownian Motion & Geometric Brownian Motion (GBM)

Brownian motion (W_t):

- Continuous-time random process.
- $W_0 = 0$, increments $W_t - W_s \sim N(0, t-s)$, independent of the past.

Geometric Brownian Motion (GBM) for stock price:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,$$

where

- S_t: stock price at time t,
- $\mu S_t dt$: **drift** (average trend/expected return),
- $\sigma S_t dW_t$: **random shock**, scaled by **volatility** σ.

GBM solution (Itô calculus):

$$S_t = S_0 \exp\Big[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\Big].$$

Taking logs:

$$\ln S_t = \ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t,$$

so

$$\ln S_t \sim N\Big(\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)t,\ \sigma^2 t\Big).$$

Hence stock prices are **log-normal** and log-returns are **normal**.

3.4 Simple Returns vs Log-Returns

Simple return:

$$R_t = \frac{S_t - S_{t-1}}{S_{t-1}}.$$

Log-return:

$$r_t = \ln \frac{S_t}{S_{t-1}} = \ln(1 + R_t).$$

Using Taylor expansion of ln:

$$\ln(1 + x) = x - \frac{x^2}{2} + \cdots$$

For small $|R_t| \ll 1$, we have $r_t \approx R_t$.

Why we like log-returns:

- They **add over time**.
- Under GBM, sums of normals $\Rightarrow$ normal.

4 No-Arbitrage, Replication & Black–Scholes

4.1 Arbitrage & Law of One Price

Arbitrage: risk-free profit with **zero initial investment**.

Example: same stock £10 on Exchange A, £11 on Exchange B:

1. Borrow £10 at risk-free rate.
2. Buy stock at £10 on A.
3. Immediately sell at £11 on B.
4. Repay £10 loan $\Rightarrow$ guaranteed £1 profit.

Law of One Price:

- Two portfolios with **identical payoffs in every state** must have **same price today**.
- Otherwise, buy cheap one and sell expensive one $\Rightarrow$ arbitrage.

4.2 Replicating an Option

Build a **portfolio** of:

- Δ units of stock,
- B amount in risk-free asset (bank account).

Portfolio value:

$$V_t = \Delta S_t + Be^{rt}.$$

If we can choose (Δ, B) such that at maturity

$$V_T = \text{Option payoff at } T,$$

then this portfolio **perfectly replicates** the option.

By **no-arbitrage**:

$$\text{Option price at } t = V_t = \Delta S_t + Be^{rt}.$$

4.3 Black–Scholes via PDE (Idea Only)

1. Model stock: $dS_t = \mu S_t dt + \sigma S_t dW_t$ (GBM).
2. Option price $V(t, S)$. Apply **Itô's formula** to $V(t, S_t)$.
3. Construct a **self-financing** replicating portfolio using ϕ_t^S (stock), ϕ_t^B (bond).
4. Choose portfolio to **eliminate randomness**, then use no-arbitrage to get **Black–Scholes PDE**.
5. Solve PDE for European call/put $\Rightarrow$ **closed-form prices**.

4.4 Black–Scholes Call & Put Formulas

European call (no dividends):

$$C_t = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2).$$

European put:

$$P_t = Ke^{-r(T-t)} \Phi(-d_2) - S_t \Phi(-d_1).$$

Where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t},$$

and $\Phi(\cdot)$ is the standard normal CDF.

4.5 Put–Call Parity

For European options:

$$C_t - P_t = S_t - Ke^{-r(T-t)}.$$

Rearrange to get put from call:

$$P_t = C_t - S_t + Ke^{-r(T-t)}.$$

Intuition: **Call + discounted strike = Put + stock** (same payoff at T).

5 Risk-Neutral Pricing & Monte Carlo

5.1 Risk-Neutral Measure $\mathbb{Q}$

Under the **risk-neutral measure**:

- All assets grow on average at the **risk-free rate** r.
- Stock dynamics:
$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}.$$
- Risk-neutral pricing formula (any European payoff):
$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\text{Payoff}(S_T)].$$

Under risk-neutral GBM:
$$S_T = S_0 \exp\left[(r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right], \quad Z \sim N(0,1).$$

5.2 Monte Carlo Recipe for a European Option

For a European call payoff $(S_T - K)^+$:

1. **Simulate** N i.i.d. risk-neutral paths:
   - Draw $Z_i \sim N(0,1)$.
   - Compute
   $$S_T^{(i)} = S_0 \exp\left[(r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z_i\right].$$
2. **Compute payoff** each time:
$$P^{(i)} = \max\left(S_T^{(i)} - K, 0\right).$$
3. **Average & discount**:
$$\hat{V}_0 = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} P^{(i)}.$$

As $N \to \infty$, by the **Law of Large Numbers**, $\hat{V}_0 \to V_0$.

5.3 Monte Carlo Error & Convergence

If payoffs $P^{(i)}$ are i.i.d. with mean $\mu = \mathbb{E}[P]$ and variance $\sigma^2 = \text{Var}(P)$:
Estimator:
$$\hat{V}_0 = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} P^{(i)}.$$

Mean:

$$\mathbb{E}[\hat{V}_0] = e^{-rT}\mu = V_0.$$

Variance:

$$\mathrm{Var}(\hat{V}_0) = \frac{e^{-2rT}\sigma^2}{N}.$$

Standard error scales like const$/\sqrt{N}$.

Implications:

- To **halve** error $\Rightarrow$ need about $\mathbf{4}\times$ more simulations.
- To gain **one decimal place** of accuracy $\Rightarrow$ roughly $\mathbf{100}\times$ more simulations.

6 Variance Reduction Techniques

Goal: **Reduce variance** (noise) of Monte Carlo estimator **without increasing** N.

6.1 Antithetic Variates

Idea:

- Stock price randomness under BS (risk-neutral GBM) comes from a **single normal variable** Z:

$$S_T = S_0 \exp\left[(r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right], \quad Z \sim N(0,1).$$

- For each Z, also use $-Z$ (the **antithetic** pair).

Paths:

$$S_T^{(+)} = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}, \quad S_T^{(-)} = S_0 e^{(r-\frac{1}{2}\sigma^2)T-\sigma\sqrt{T}Z}.$$

Payoffs:

$$P(Z) = \text{payoff}(S_T^{(+)}), \qquad P(-Z) = \text{payoff}(S_T^{(-)}).$$

Antithetic estimator:

$$\hat{V}_{\text{anti}} = e^{-rT} \cdot \frac{1}{2}\big(P(Z) + P(-Z)\big).$$

Why it works:

- For **monotonic payoffs** (calls/puts), if $P(Z)$ is large, $P(-Z)$ tends to be small $\Rightarrow$ **negative correlation**.
- Variance:

$$\mathrm{Var}(\hat{V}_{\text{anti}}) = \frac{1}{4}\Big(\mathrm{Var}(P(Z)) + \mathrm{Var}(P(-Z)) + 2\mathrm{Cov}(P(Z), P(-Z))\Big),$$

and since $\mathrm{Cov}(P(Z), P(-Z)) < 0$, total variance is **smaller**.

6.2 Control Variates

Idea:

- Use a related quantity C whose **true value** $C_{\text{exact}} = \mathbb{E}[C]$ is known (e.g. Black–Scholes price).
- Simulate both **target payoff** P and **control** C on the *same* paths.
- Adjust estimate to **pull it towards** the known value.

Control variate estimator:

$$\hat{V}_{\text{CV}} = \bar{P} - \beta\big(\bar{C} - C_{\text{exact}}\big),$$

where

$$\bar{P} = \frac{1}{N}\sum_{i=1}^{N} P^{(i)}, \qquad \bar{C} = \frac{1}{N}\sum_{i=1}^{N} C^{(i)}.$$

Variance:

$$\operatorname{Var}(\hat{V}_{\text{CV}}) = \operatorname{Var}(\bar{P}) + \beta^2 \operatorname{Var}(\bar{C}) - 2\beta \operatorname{Cov}(\bar{P}, \bar{C}).$$

Optimal β^* (minimises variance):

$$\beta^* = \frac{\operatorname{Cov}(\bar{P}, \bar{C})}{\operatorname{Var}(\bar{C})}.$$

Resulting variance:

$$\operatorname{Var}(\hat{V}_{\text{CV}}) = \operatorname{Var}(\bar{P})(1 - \rho_{PC}^2),$$

where ρ_{PC} is the **correlation** between P and C.

Interpretation:

- If P and C are **highly correlated**, $|\rho_{PC}| \approx 1 \Rightarrow$ **huge variance reduction**.
- If $\rho_{PC} = 0 \Rightarrow$ **no benefit**.

7 Monte Carlo vs Black–Scholes – Strengths & Weaknesses

Monte Carlo:

- Works for almost **any payoff** (Asian, barrier, multi-asset, path-dependent).
- Very flexible; just change the payoff function.
- Accuracy improves systematically as $1/\sqrt{N}$.

- Can use variance reduction (antithetic, control variates).
- *Cons:* Slow for very high accuracy (need many simulations); not ideal for **early exercise** (American options) without extra tricks.

Black–Scholes:

- Closed-form formula $\Rightarrow$ **extremely fast**, zero simulation noise.
- Provides intuition about **Greeks**, volatility, hedging.
- *Cons:* Only for simple **European options** under GBM, constant σ, frictionless markets.
- Hard to extend to exotic/path-dependent payoffs.