Imperial College London

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Why Randomness in Finance?

Intuition: Prices move because new information arrives randomly. To model uncertainty, we treat stock prices as **random processes**.

Key Ideas for Today:

- Financial markets are unpredictable $\rightarrow$ randomness is essential.
- Simple random walk models price movements.
- Moving from discrete steps to continuous models.
- Monte Carlo simulation: estimating expected values.

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Price Movements as Random Processes

Real-world intuition:

- Each day brings news: earnings, macro data, rumours.
- Prices react unpredictably $\rightarrow$ randomness captures this.

Mathematical idea:

$$\text{Price change} = \text{trend} + \text{random shock}$$

Examples from data:

- Stock returns are noisy.
- Size of jumps varies every day.

The Discrete Random Walk

Toy model: At each step the price goes:

$$S_{n+1} = S_n + \epsilon_n, \qquad \epsilon_n \in \{-1, +1\}$$

Intuition:

- Price moves up/down randomly.
- Each step independent (news arrives unpredictably).

Properties:

- Easy to simulate.
- Captures unpredictability.
- But unrealistic: price can be negative.

Example: Coin Toss Random Walk

Example setup:

- Start with price $S_0 = 100$.
- Toss a fair coin each step.
- Heads $\Rightarrow \epsilon_n = +1$ (price goes up by 1).
- Tails $\Rightarrow \epsilon_n = -1$ (price goes down by 1).

One possible path:

$$S_0 = 100$$

$$\text{T, H, H, T, H} \quad \Rightarrow \quad -1, +1, +1, -1, +1$$

$$\Rightarrow \begin{aligned} S_1 &= 100 - 1 = 99 \\ S_2 &= 99 + 1 = 100 \\ S_3 &= 100 + 1 = 101 \\ S_4 &= 101 - 1 = 100 \\ S_5 &= 100 + 1 = 101 \end{aligned}$$

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Example: Coin Toss Random Walk - 2

Key takeaway: Even with simple coin flips, the path becomes unpredictable.

From Random Walk to Continuous Models

Motivation: Financial markets move in tiny increments $\rightarrow$ better to model continuous time.

Brownian Motion:

- Continuous version of a random walk.
- Path wiggles randomly at every instant.

$$W_t \sim \mathcal{N}(0, t)$$

Key property: future changes are independent of the past.

Use in finance (preview): Model stock prices as a random process driven by Brownian motion.

Price Movements as Random Processes (GBM)

Mathematical model:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Interpretation:

- S_t: stock price at time t.
- $\mu S_t \, dt$ = **drift term** (average trend or expected return).
- $\sigma S_t \, dW_t$ = **random shock**, scaled by volatility σ.
- dW_t = increment of **Brownian motion** (pure randomness).

Why this model?

- Ensures prices stay positive, captures noisy, unpredictable returns.
- Matches empirical fact: returns (not prices) are approximately normal.

Returns remark

Remark (returns vs. log-returns):

- The simple return is: $R_t = \frac{S_t - S_{t-1}}{S_{t-1}}$
- For very small time steps, price changes are tiny.
- Therefore we use **log-returns**:

$$r_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$$

- Log returns are additive

Why log-returns are similar to simple returns

Recall:

$$R_t = \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad r_t = \ln\left(\frac{S_t}{S_{t-1}}\right) = \ln(1 + R_t)$$

Use the Taylor expansion of $\ln(1 + x)$ for small x:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Set $x = R_t$:

$$r_t = \ln(1 + R_t) = R_t - \frac{R_t^2}{2} + \frac{R_t^3}{3} - \cdots$$

Why log-returns are similar to simple returns -2

Conclusion:

- For small returns $|R_t| \ll 1$, the higher-order terms are tiny.
- So $r_t \approx R_t$ (log-return $\approx$ simple return).
- But log-returns add nicely over time and are normal under GBM.

Solving using Ito's Formula

$$S_t = S_0 \exp\big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\big)$$

- S_0 — initial price
- $(\mu - \frac{1}{2}\sigma^2)t$ — deterministic trend (Ito correction)
- σW_t — random shock
- Solution ensures $S_t > 0$ and log-returns are normal

Distribution of Stock Prices under GBM

From the GBM solution:

$$S_t = S_0 \exp\big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\big)$$

Taking logs:

$$\ln S_t = \ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t$$

Since $W_t \sim \mathcal{N}(0, t)$:

$$\boxed{\ln S_t \sim \mathcal{N}\left(\ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right) t,\ \sigma^2 t\right)}$$

Ref: Brigo (SDE.pdf): GBM solution and log-transform.

Monte Carlo: Core Idea

Problem: We want to compute expectations (e.g., future price, option payoff).

Monte Carlo method:

- Simulate many random price paths.
- Compute payoff for each.
- Take the average.

$$\mathbb{E}[f(S_T)] \approx \frac{1}{N}\sum_{i=1}^{N} f(S_T^{(i)})$$

Intuition: Like polling many random samples give the true average. (By Law of Large Numbers)

Law of Large Numbers (LLN)

Intuition:

- If you repeat a random experiment many times,
- the average outcome gets closer and closer to the true expected value.

Example: Toss a fair coin.

$$\mathbb{E}[\text{Heads}] = 0.5$$

If you toss the coin:

- 10 times $\rightarrow$ the fraction of heads may be very noisy,
- 1,000 times $\rightarrow$ the fraction will be closer to 0.5,
- 100,000 times $\rightarrow$ extremely close to 0.5.

Law of Large Numbers (LLN) -2

Formal statement (simplified):

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{n\to\infty} \mathbb{E}[X]$$

for independent, identically distributed (i.i.d.) random variables X_i.

Why we care in finance:

- Monte Carlo simulation relies on LLN.
- More simulated price paths $\Rightarrow$ estimate becomes more accurate.

Simulating a Stock Price Path (Simple Version)

Discretize the GBM model:

$$S_{t+\Delta t} = S_t \exp\left[(\mu - \tfrac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\, Z\right], \quad Z \sim \mathcal{N}(0,1)$$

Interpretation:

- Drift term pushes price upward/downward.
- Random term adds uncertainty.

Use cases:

- Pricing options.
- Stress testing.
- Risk management (VaR).

Summary of Week 2

We learned:

- Price changes can be modelled as random processes.
- Discrete random walk $\rightarrow$ intuitive starting point.
- Brownian motion = continuous limit.
- GBM = standard model for stock prices.
- Monte Carlo simulation = estimate expectations by averaging.

Next Week: Pricing options using Monte Carlo and Black–Scholes.