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Monte Carlo Simulation for Option Pricing and Improving Accuracy

Goal: Explore methods to improve Monte Carlo accuracy

Main Topics:

- Review of Monte Carlo simulation for European Options
- Variance reduction techniques (antithetic and control variates)
- Strengths and limitations of Monte Carlo vs. Black–Scholes

Review: Monte Carlo Pricing of a European Option

Intuition

- A European option payoff depends only on the stock price at maturity T.
- Under the risk–neutral measure, the *fair price today* is the discounted expected payoff.
- Monte Carlo approximates this expectation by:
  - simulating many possible future stock prices $S_T^{(i)}$
  - computing the payoff in each scenario
  - averaging and discounting back to today.

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Review: Monte Carlo Pricing of a European Option -2

Risk–neutral pricing formula

$$V_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\big[\mathrm{payoff}(S_T)\big]$$

For a European call with strike K:

$$V_0^{\mathrm{call}} = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\big[(S_T - K)^+\big]$$

Review: Monte Carlo Pricing of a European Option -3

Monte Carlo

1. Simulate $S_T^{(1)}, \dots, S_T^{(N)}$ under the risk–neutral model.
2. Compute payoffs $P^{(i)} = \text{payoff}\big(S_T^{(i)}\big)$.
3. Take the average payoff and discount:

$$\hat{V}_0 = e^{-rT}\,\frac{1}{N}\sum_{i=1}^{N} P^{(i)}.$$

Variance Reduction Techniques

Why do we need variance reduction?

- Monte Carlo estimates improve slowly: error $\sim \frac{1}{\sqrt{N}}$.
- We want more accuracy *without* increasing the number of simulations.
- Variance reduction techniques use clever tricks to make simulations “cancel noise”.

Monte Carlo Error and Convergence Rate

Monte Carlo estimator for a European option

$$\hat{V}_0 = e^{-rT}\,\frac{1}{N}\sum_{i=1}^{N} P^{(i)}, \qquad P^{(i)} = \text{payoff in scenario } i.$$

Assume $P^{(1)}, \ldots, P^{(N)}$ are i.i.d. with $\mathbb{E}[P] = \mu$ and $\mathrm{Var}(P) = \sigma^2$.

Mean (signal):

$$\mathbb{E}[\hat{V}_0] = e^{-rT}\,\mu = V_0$$

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Monte Carlo Error and Convergence Rate -2

Variance (noise):

$$\mathrm{Var}(\hat{V}_0) = e^{-2rT}\,\frac{\sigma^2}{N} \quad \Rightarrow \quad \text{Std. error} \;\approx\; \frac{\text{const}}{\sqrt{N}}.$$

- Error decreases like $1/\sqrt{N}$: to cut error in half, need $4\times$ more paths.
- **Noise** $=$ random fluctuation $\hat{V}_0 - V_0$ due to finite sampling.
- Variance reduction methods aim to reduce this noise without changing the true mean.

Variance Reduction Techniques -2

1. **Antithetic Variates (simple and intuitive)**
   - If $Z \sim N(0,1)$ drives a simulated stock path, also use $-Z$.
   - These two paths tend to move in opposite directions.
   - Averaging the two payoffs cancels randomness and reduces noise.

$$\hat{V}_{\text{anti}} = \frac{1}{2}\Big(P(Z) + P(-Z)\Big)$$

Benefit: Works very well for monotonic payoffs like European calls.

Why Antithetic Variates Work Well

1. Stock price randomness comes from a single normal variable

Under Black–Scholes (risk–neutral GBM):

$$S_T = S_0 \exp\Big((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,Z\Big), \qquad Z \sim N(0,1).$$

- The only randomness is the shock Z.
- Using Z and $-Z$ creates two *symmetric* shocks.

Why Antithetic Variates Work Well -2

2. Antithetic pair: opposite shocks

$$S_T^{(+)} = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}, \qquad S_T^{(-)} = S_0 e^{(r-\frac{1}{2}\sigma^2)T-\sigma\sqrt{T}Z}.$$

- If Z pushes the stock up, $-Z$ pushes it down (symmetrically).
- For monotonic payoffs (calls/puts), this creates opposite payoffs:

$$P(Z) \text{ large} \quad \Rightarrow \quad P(-Z) \text{ small.}$$

Why Antithetic Variates Work Well -3

3. Variance reduction comes from negative correlation

Define the antithetic estimator:

$$\hat{V}_{\text{anti}} = \frac{1}{2}\left(P(Z) + P(-Z)\right).$$

Its variance is:

$$Var(\hat{V}_{\text{anti}}) = \frac{1}{4}\Big(Var(P(Z)) + Var(P(-Z)) + 2\,Cov(P(Z), P(-Z))\Big).$$

- Since $P(-Z)$ decreases when $P(Z)$ increases, the covariance is **negative**.

Thus:

$$Cov(P(Z), P(-Z)) < 0 \quad \Longrightarrow \quad Var(\hat{V}_{\text{anti}}) < Var(P(Z)).$$

Variation Reduction Technique -2

2. Control Variates (use something we know exactly)

- Choose a related quantity whose true value is known (e.g., Black–Scholes price).
- Simulate both the target payoff and the control at the same time.
- Adjust the Monte Carlo estimate to "pull" it toward the exact known value.

$$\hat{V}_{\mathrm{CV}} = \bar{P} - \beta(\bar{C} - C_{\mathrm{exact}})$$

- C_{exact} = known analytical price (e.g., Black–Scholes).
- $\bar{C}$ = simulated control variate value.
- β chosen to minimise variance (controls how strongly we correct the Monte Carlo estimate.)

Why the Control Variate Adjustment "Pulls" Toward the True Value

- When we simulate both P (target payoff) and C (control), they tend to move together. If the simulated control value $\bar{C}$ is too high, this usually means $\bar{P}$ is also too high.
- Since the true value $\mathbb{E}[C] = C_{\text{exact}}$ is known, the quantity $(\bar{C} - C_{\text{exact}})$ tells us how much the simulation has over- or under-shot.
- The control variate estimator

$$\hat{V}_{\mathrm{CV}} = \bar{P} - \beta(\bar{C} - C_{\text{exact}})$$

  subtracts this error. Therefore:
  - If $\bar{C} > C_{\text{exact}}$, the estimate is corrected downwards.
  - If $\bar{C} < C_{\text{exact}}$, the estimate is corrected upwards.

Deriving the Optimal Control Variate Coefficient β

Goal: Choose β to minimise the variance of the estimator

$$\hat{V}_{\mathrm{CV}} = \bar{P} - \beta(\bar{C} - C_{\mathsf{exact}}).$$

Since C_{exact} is constant,

$$Var(\hat{V}_{\mathrm{CV}}) = Var(\bar{P} - \beta\bar{C}).$$

Using the variance identity:

$$Var(X - \beta Y) = Var(X) + \beta^2 Var(Y) - 2\beta\, Cov(X, Y),$$

we get:

$$Var(\hat{V}_{\mathrm{CV}}) = Var(\bar{P}) + \beta^2 Var(\bar{C}) - 2\beta\, Cov(\bar{P}, \bar{C}).$$

Deriving the Optimal Control Variate Coefficient β -2

Minimise variance: take derivative w.r.t. β and set to zero

$$\frac{d}{d\beta} Var(\hat{V}_{\mathrm{CV}}) = 2\beta Var(\bar{C}) - 2Cov(\bar{P}, \bar{C}) = 0.$$

$$\Rightarrow \quad \beta^* = \frac{Cov(\bar{P}, \bar{C})}{Var(\bar{C})}.$$

Deriving the Optimal Control Variate Coefficient β -3

Interpretation:

- Higher correlation $\Rightarrow$ bigger $\beta \Rightarrow$ stronger correction.
- If correlation is zero, $\beta = 0$ and control variates give no benefit.
- Variance reduces to:

$$Var(\hat{V}_{\mathrm{CV}}) = Var(\bar{P})(1 - \rho_{PC}^2),$$

where ρ_{PC} is the correlation.

Strengths and Limitations: Monte Carlo vs. Black–Scholes

Monte Carlo Simulation

- **Strengths:**
  - Works for almost *any* payoff (exotic, path-dependent, multi-asset).
  - Easy to implement and flexible: just simulate more paths.
  - Accuracy improves systematically as $1/\sqrt{N}$.
  - Variance reduction methods can improve efficiency.
- **Limitations:**
  - Slow for very high accuracy (need many simulations).
  - Not ideal for early exercise options (American).
  - Requires careful variance reduction to be practical.

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Strengths and Limitations: Monte Carlo vs. Black–Scholes -2

Black–Scholes Formula

- **Strengths:**
  - Closed-form, extremely fast, zero simulation error.
  - Provides intuition (Greeks, volatility, hedging).
- **Limitations:**
  - Only works for simple European options.
  - Based on strong assumptions (constant volatility, GBM model).
  - Cannot easily handle path-dependence or complex payoffs.