Math finance: an intro to Option Pricing

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The Forward Contract

What are derivatives?

Key questions and answers of this course

1. *What are derivatives (a.k.a. options)?*

   A type of contract, which allows to transfer risk.

2. *How to price derivatives?*

   By replication, using the NA (no-arbitrage) principle.

Example: Forward Contract

If easyJet 'buys' (at cost 0) from a bank a forward contract for A litres of jet fuel at *expiry* T and T-*forward price* K, then they agree that easyJet will buy from the bank A litres of jet fuel at price K at time T.

The Forward Contract

Then at maturity, what happens depends on the fuel price P_T:

- if $P_T \geq K$, easyJet will have the right to buy the fuel from the bank at price K.
- if $P_T < K$, easyJet will also have the obligation to buy at the pre-agreed price K.
- So easyJet, buying a forward contract, transfers the 'risk' of rising fuel prices to the bank.

Payoff of Forward Contract

- The value of the forward contract at time T (a.k.a. its *payoff*) is

$$F_T = P_T - K = f(P_T), \qquad \text{with } f(x) := x - K, \quad x \in \mathbb{R}.$$

- The forward contract is then called a *derivative*.
- A derivative is a financial contract that derives its value from the value of other quantities, called the *underlying*.

The simplest hedge

- To reduce the risk, the bank will seek to *hedge*, i.e. to trade in a way that the total risks cancel each other out.

Example: the simplest hedge

Koch industries produces jet fuel and wants to cover itself against the risk of dropping fuel prices. Thus, the bank buys the forward contract from Koch industries and sells 'it' to easyJet.

- This trade reduces the risk for easyJet and Koch industries. In exchange, the bank will make some small profits, because the forward price at which it will buy and sell the forward are slightly different.

- In reality hedging is much harder than this, so easyJet and Koch industries will choose to use the bank and pay its fees.

The Binary Option

The binary option

- At 10am gold is trading at \$58 (per gram), and Bob believes its price will close above \$59.
- He ask his trader to buy from a bank 1500 *binary options* on gold with strike price \$59; they cost \$$B_0$ each.
- So Bob enters a contract with the bank, which states that:
  1. if Bob is right (i.e. $P_T \geq 59$), the bank owes him \$100 for each option, i.e. \$150000 in total. The option is *in the money*.
  2. if Bob is wrong (i.e. $P_T < 59$), the bank owes him nothing. The option is *out of the money* (i.e. worth nothing).
- Options allow to place bets of market predictions, so clearly their costs should be tied to the probability that their prediction comes true.

Payoff and P&L of binary option

The payoff B_T of the binary option at time T is $B_T = f(P_T)$, with

$$f(x) := 100 \cdot 1_{\{x \geq 59\}}, \quad \text{where} \qquad 1_{\{x \geq b\}} := \begin{cases} 1 & \text{if } x \geq b, \\ 0 & \text{otherwise.} \end{cases}$$

Bob's P&L (*Profit and Loss*) in \$ is

$$\text{P\&L options} = \begin{cases} 150000 - 1500 B_0 & \text{if } P_T \geq 59, \\ -1500 B_0 & \text{otherwise.} \end{cases}$$

Speculation

Trading gold instead of options

What if Bob bought gold instead? Assume $B_0 = 37$. If Bob invests $1500B_0 = \$55500$ in gold instead of options, he buys $55500/58 \sim 957$g of gold, and his final wealth is $957 \cdot P_T$, i.e.

$$\text{P\&L gold } = 957 \cdot (P_T - 58).$$

If $P_T = 59.3$ then

$$\text{P\&L options} = \$94500, \quad \text{P\&L gold} = \$957 \cdot 1.3 = \$1244.1,$$

If $P_T = 57.7$ then

$$\text{P\&L options} = -\$55500, \quad \text{P\&L gold} = \$957 \cdot (-0.3) = -\$287.1.$$

Speculation

1. Both P&L's are strictly positive if $P_T \geq 59$, but trading options can lead to much bigger gains and losses.
2. Derivatives can be used to create investments that are *riskier* than the underlying, making it a more effective way of betting on its future prices.
3. Speculating with derivatives can be extremely risky (e.g. Barings bank folded after losing \$1.4 billion in futures).

Derivatives: good or bad?

Derivatives can be used both for hedging and for speculating. so, even experts can have very different views on them, e.g.:

1. the legendary investor Warren Buffett wrote

   [derivatives are] financial weapons of mass destruction [...] time bombs, both for the parties that deal in them and the economic system

2. the Nobel laureate Merton Miller wrote

   Contrary to the widely held perception, derivatives have made the world a safer place, not a more dangerous one. They have made it possible for firms and institutions to deal efficiently and cost effectively with risks and hazards that have plagued them for decades, if not for centuries.

Pricing by replication

Law of one price

If a derivative gives a payoff $f((P_t)_{t \le T})$ at expiry $T > 0$ (one time, not random), at what price 'should' a bank sell it?

Law of one price

If there are two possible investments which have, under all *possible* market outcomes, the same value at time T, then they must have the same value also at all previous times.

Example: finding the forward price K

Suppose easyJet, decided to take out a loan from a bank and use it to buy the jet fuel today at price P_0. Then at maturity easyJet owns the jet fuel, valued at P_T, and owes the bank $L := P_0(1 + r)$. This investment has payoff $P_T - L$ and cost 0.
The Forward has payoff $P_T - K$ and cost $c(K)$, for K s.t. $c(K) = 0$. So the *T-forward price* K of jet fuel must equal L.

Pricing via replication

If we can find a portfolio (a.k.a. investment) which replicates (has same payoff as) a derivative, then at any time $t \leq T$ by the law of one price:

The derivative's price must equal the value of the replicating portfolio.

Replicating portfolios all have the same value, so:

there is only **one price** which at which to trade a **replicable** derivative.

Hedging = - Replicating

We say that the trader has a *short position* in some derivative if she has *sold* it, and has a long position in the derivative if she has bought it.

Assume a trader sold a derivative with expiry T to easyJet, and used the proceeds to replicate the derivative. At maturity the trader owes the final value of the derivative X_T to easyJet, and her replicating portfolio is worth X_T.

Thus, replicating a derivative means hedging a short position in the derivative.

Pricing a binary option

Pricing a binary option

Market:

1. trade gold
2. borrow/deposit money from/into a bank.

Assume: now gold is trading at \$58 (per gram), and interest rate $r = 0$.

We model the price P_T of gold at maturity as

$$P_T(\omega) = \begin{cases} 61 & \text{if } \omega = g \\ 56 & \text{if } \omega = b. \end{cases}, \qquad \text{and so } B_T(\omega) := \begin{cases} 100 & \text{if } \omega = g \\ 0 & \text{if } \omega = b \end{cases}$$

equals the payoff $100 \cdot 1_{\{P_T \geq 59\}}$ of Bob's binary option.

Replication in a given model

If initial wealth $V_0 = x$, buy h grams of gold at \$58 each, then

$$V_T = V_T^{x,h} := \text{final wealth} = x - h \cdot 58 + h \cdot P_T.$$

(x, h) is a replicating portfolio if $V_T^{x,h} = B_T$, i.e.

$$\begin{cases} x + h \cdot (61 - 58) = 100 \\ x + h \cdot (56 - 58) = 0, \end{cases}$$

whose unique solution is $h = 20, x = 40$.

Replicating strategy: initial capital $V_0 = \$40$, buy 20 grams of gold, wait until maturity.

The price of the binary option is $V_0 = 40$.

Replication depends on the model

If we assumed instead

$$P_T(\omega) = \begin{cases} 60 & \text{if } \omega = g \\ 56 & \text{if } \omega = b, \end{cases}$$

then the replicating portfolio would have changed to $h = 25, x = 50$, and the price of the binary to \$50.

If we had modelled P_T as taking the three values $61, 57, 56$, we would have found that the binary option is not even replicable!

By 'P_T take values $61, 57, 56$' we mean that:

$$\mathbb{P}(\{P_T = x\}) > 0 \text{ for } x \in \{61, 57, 56\}, \quad \mathbb{P}(\{P_T \notin \{61, 57, 56\}\}) = 0.$$

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Dependence of prices on probabilities

Pricing in the binomial market

Consider a one-period market model made of:

- a bank account with the interest rate $r = 0$,
- gold, trading at S_0 \$58 per gram.

Suppose Bob believes that

$$S_T(\omega) = \begin{cases} 61 & \text{if } \omega = \omega_1 \\ 56 & \text{if } \omega \neq \omega_1, \end{cases} \qquad \text{with } \mathbb{P}(\{\omega_1\}) = 2/3.$$

Consider the binary option C with payoff $100 \cdot 1_{\{S_T \geq 59\}}$

At what prices C_0 should Bob be willing to sell this option?

Replicating strategy: initial capital $V_0 = 40$, buy 20 shares.
Thus, he is willing to sell it at any price $\geq$ than 40.

Price does not depend on probabilities

1. Suppose that I believe that $\mathbb{P}(\{\omega_1\}) = 999\%$. At what price should I be willing to sell it?
2. Answer: the same as Bob. Indeed, the probabilities do not appear in the replication equation, so nothing changes.
3. Suppose Alice believes that

$$S_T(\omega) = \begin{cases} 61 & \text{if } \omega = \omega_1 \\ 56 & \text{if } \omega = \omega_2 \\ 55 & \text{if } \omega = \omega_3 \end{cases} \quad \text{with } \mathbb{P}(\{\omega\}) = \begin{cases} 2/3 - \epsilon \\ 1/3 - \epsilon \\ 2\epsilon, \end{cases}$$

with $\epsilon = 10^{-9}$. At what price is she willing to sell it?

Selling without risk

1. She cannot replicate C, how should she price it?
2. Suppose that she is infinitely risk-averse.
3. Then Alice is willing to sell at any price $\geq u^{\epsilon}(C)$, where $u^{\epsilon}(C)$ is the *smallest* x for which: $\exists h$ s.t. $V_1^{x,h} \geq C_T$
4. We need to find the smallest x such that
$$\begin{cases} x + h \cdot (61 - 58) \geq 100 \\ x + h \cdot (56 - 58) \geq 0 \\ x + h \cdot (55 - 58) \geq 0. \end{cases}$$
5. Draw the three half planes described by
$$\begin{cases} x + 3h \geq 100 \\ x - 2h \geq 0 \\ x - 3h \geq 0. \end{cases}$$

Prices depend on the probabilities

1. Their intersection is the wedge between the two lines

$$x + 3h = 100, \quad x - 3h = 0,$$

   to the right of their intersection at $x = 50, h = 50/3$.
2. Thus $u^{\epsilon}(C) = 50$, for any $\epsilon > 0$, whereas $u^0(C) = 40$.
3. This is **not** reasonable! In reality it is impossible to replicate anything with absolute certainty.
4. The outputs of a good model should depend continuously on the inputs, since one can never know anything with absolute precision.
5. Even if not reasonable, this is how no-arbitrage pricing works.
6. Never blindly trust the output of any model.

Model uncertainty

Model uncertainty

1. To price options we work with a model, e.g. we assumed P_T takes values $61, 56$, each with proba $\frac{1}{2}$.
2. The realised outcomes are unknown, but the possible outcomes have a known distribution. Frank Knight called this the 'known unknown', i.e. an unknown which is 'a quantity susceptible of measurement'.
3. Given one such market model, we can find prices of options. However, in reality, we do not even know the distribution of the possible outcomes.
4. Choosing a model introduces *model risk*, i.e. the **unquantifiable** risk that we have chosen an inappropriate model.

Storage Costs

Let us see why one might want a more complicated model, in the setting of easyJet's example.

To replicate the forward, easyJet would have to pay to buy fuel now...and store it for 6 months!

We have ignored storage costs. Doing so can make sense, depending on the market. E.g., if the underlying is:

1. Shares of a corporation (or gold), can ignore these costs
2. Crude oil, it depends
3. Wheat or milk, can **not** ignore these costs

Some goods cannot be stored (e.g. flowers).

Counter-party risk

Suppose easyJet buys the forward contract from Koch industries, jet fuel prices rise, and at maturity Koch industries is unable to sell jet fuel at the pre-agreed prices because it has gone bankrupt.

Then easyJet's forward contract is worthless! We have ignored *counter-party risk*. If we admit the possibility of default, then the trading strategy we described is not a replicating strategy.

Choosing the right model

One has to choose a model which is appropriate to the specific market one is considering. This is hard...but often a model predictions work well even if its assumptions are violated:

> *Essentially all models are wrong, but some are useful. (George Box, 1987)*

E.g. the famed economist Stephen Ross said in 1987:

> *[option pricing theory is] the most successful theory not only in finance, but in all of economics.*

This is one of the main reasons why the derivatives' market has expanded massively, and is now huge.

What can be used as underlying

What can be used as underlying

Which assets can be used as an underlying? Historically, on

1. commodities (e.g. precious metals, agricultural products, crude oil);

nowadays instead most derivatives are based on *financial assets* (or related quantities):

2. bonds (e.g. US Treasury bonds, UK gilts), interest rate,
3. stocks (e.g. Amazon (AMZN), HSBC, Alibaba (BABA)),
4. stock market indices (e.g. SP500, NASDAQ)
5. FX rates (e.g. GBP/EUR, USD/RMB), . . .
6. other derivatives (e.g. a call option based on a call option).

Some derivatives are based on

7. a weather-related quantity (e.g. temperature, wind precipitation)

but there is no accepted framework for pricing them.

The bank account and the bond

Many debt instruments

1. We said an investor can 'put money in the bank' and receive interest. This is a very idealised representation of the money-market.

2. In reality, there are many bonds and other many debt instruments, which describe how the seller (i.e. the borrower) will repay at an interest, and they all have different characteristics and 'interest rates'

3. Money markets are an important but complicated subject. We do not study it.

The bank account and 'the bond'

To simplify, we will consider 'the bond' (or 'the bank account'), i.e. a contract with value B_t s.t.

1. $B_0 = 1$, by normalisation
2. in discrete time: it satisfies $B_{t+1} = B_t(1 + R_t)$
3. in continuous time: it satisfies $dB_t = B_t R_t dt$
4. R_t is the interest rate at time t (often $R_t(\omega) = r$)
5. We will *not* assume $R_t > 0$, and so wealth invested in bonds might decrease
6. We do assume $R_t > -1$: bonds always maintain some value.

If $R = r$ is constant then $B_t = B_0(1 + r)^t$ or $B_t = B_0 e^{rt}$.

Unspoken modelling assumptions

Unspoken modelling assumptions

In binomial model we made very common modelling assumptions:

1. all market participants have the same information
2. portfolio of K bonds and H shares has value

$$V_t^{(K,H)} := K_t B_t + H_t \cdot S_t$$

In particular:

Linearity assumption

$$(K, H) \mapsto V^{(K,H)} \text{ is linear.}$$

How is linearity violated?

In reality:

1. one can only buy or sell an integer number of shares.
2. bid price $\neq$ ask price, the interest rates for lending $\neq$ for borrowing
3. if buying one share of a stock costs S_t, buying a big number H of shares costs *more* than $H_t S_t$, by the law of demand and supply
4. a trader cannot choose to hold whatever amount H of shares he wants, cannot borrow how much money he wants
5. market frictions: transaction costs, taxes, running costs
6. short-selling is not as innocent as - buying: let's see why

Short-selling

Short-selling := selling shares which you do not own

Like with cash, you have to borrow the share to sell it; then later buy it from the market and return it.

Complications:

1. one can lose arbitrary amount
2. have to pay a fee, so even more risky
3. companies and countries hate short-sellers, and fight them
4. short-selling is sometimes out-lawed

If we assume small trader, only significant over-simplifications are: market-frictions and short-selling

Justification of the law of one price

Law of one price fails $\implies \exists$ arbitrage

Example

Suppose that the \$/£ exchange rate in NYC is A, and in London is B. Then $A = B$, otherwise:

1. Consider the case $A > B$ (the other being analogous)
2. The trader Bob could buy \$1 in London using £B, and sell the \$1 in NYC for £A, making the profit $A - B > 0$ (in £)
3. Bob just made an *arbitrage*, i.e., without investing any money, and without any risk of losses, he made some money.
4. Bob repeats the above trade $n \gg 1$ times!

Arbitrage $\implies$ not at equilibrium

1. Now in London there is a (strongly) increased demand for \$ so, by the law of demand and supply, the price B of \$ goes up
2. Analogously the price A of \$ in NYC goes down
3. This keeps happening until $A = B$
4. Bob now has no reason to keep investing as above, so A, B stop changing, the market reached a (stable) equilibrium.
5. So: law of one price fails $\implies \exists$ arbitrage $\implies$ no equilibrium
6. Models with arbitrage represent markets which are not at equilibrium, and in which things just don't behave sensibly
7. We should only work with models in which there exists no arbitrage.

The no-arbitrage principle

Arbitrage

Definition: Arbitrage

An arbitrage is a portfolio with 0 initial capital if its final wealth V_T of such portfolio satisfies $V_T > 0$ with proba > 0, and $V_T < 0$ with proba $= 0$.

Definition: a.s.

We say that a property holds a.s.(=almost surely) if the event where it holds has probability 1 (equivalently, its complement has probability 0).

For example, a portfolio is without any risk if $V_T \geq 0$ a.s..

NA holds iff, given any portfolio with 0 initial capital, if its wealth V satisfies $V_T \geq 0$ a.s. then necessarily $V_T = 0$ a.s..

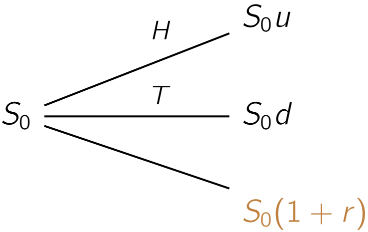
Arbitrage in models with one stock

- Suppose from now on my initial capital is 0
- Consider a model *with only one stock* S, plus the bank
- To buy 1 share, I have to borrow £S_0 from the bank
- If I do, my final wealth is $W := S_1 - S_0(1 + r)$
- If instead I buy $h \in \mathbb{R}$ shares, my wealth is hW
- $h > 0$ is an arbitrage $\iff$ $h = 1$ is an arbitrage
- $h < 0$ is an arbitrage $\iff$ $h = -1$ is an arbitrage
- $h = 0$ is never an arbitrage

Binomial model and arbitrage, buy stock

If $d \geq 1 + r$

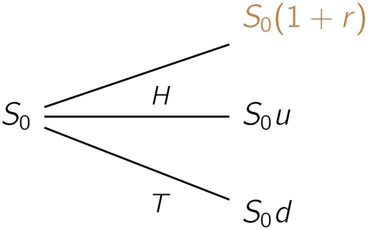
- Buy 1 share (and borrow £S_0 from the bank)
- Final wealth is $W := S_1 - S_0(1 + r)$
- $W(T) \geq 0$, $W(H) > 0 \implies$ arbitrage



Binomial model and arbitrage, buy bond

If $u \leq 1 + r$

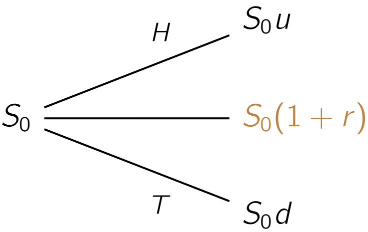
- Short-sell 1 share (and deposit £S_0 in the bank)
- Final wealth is $W(-1) = S_0(1 + r) - S_1 = -W$
- $-W(H) \geq 0, -W(T) > 0 \implies$ arbitrage



Binomial model and arbitrage, NA

If $d < 1 + r < u$

- If I buy 1 share, wealth is W, $W(T) < 0$, not an arbitrage
- If I sell 1 share, wealth is $-W$, $-W(H) < 0$, not an arbitrage
- So no $h > 0, h = 0, h < 0$ is an arbitrage $\implies \nexists$ arbitrage



Theorem

In binomial model: $\quad d < 1 + r < u \quad \iff \quad \nexists$ Arbitrage

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Prices of liquid and illiquid goods

What about the underlying?

1. Why did it make sense to model the price of the underlying?
2. What determines the price of the underlying?
3. The law of demand and supply determines prices of liquidly traded goods, i.e. goods traded a lot, by many parties
4. We cannot apply the law of demand and supply to price illiquid (i.e. non-traded) goods
5. Most goods are neither perfectly liquid nor illiquid. Most derivatives are fairly illiquid.
6. Finance deals with how to price *illiquid* derivatives based on a *liquid* underlying, using the no-arbitrage principle.

Examples of (il)liquid goods

Examples of liquid goods:

- very liquid are financial assets: bonds, stocks of major companies, stock indexes, and FX
- quite liquid: small stocks, commodities, some derivatives

Examples of illiquid goods:

- real estate
- luxury items: antiques, art pieces, Ferraris, ...
- heavy machinery: industrial equipment, battle tanks, ...
- most derivatives

Discounting and Numeraire

Discounting

- Since interest rates are normally > 0, when comparing values across times, it make little sense to use currency (say e.g. \$) as a unit of measure
- Instead of looking at the value W_t in \$, we should consider

$$\overline{W}_t := W_t/B_t,$$

  the value in units of the bond B; this is called *discounting*.
- W is the price in *nominal terms*, and $\overline{W}$ can be called
  - the price in *real terms*
  - the value of W *adjusted for inflation*
  - the *present value* of W

Formula for the value in real terms

Describe portfolio using (x, h) (where $x =$ initial capital, $h = \#$ of shares of S) instead of (k, h) (where $k = \#$ of bonds).

The value of the portfolio $(x, h) \in \mathbb{R} \times \mathbb{R}^m$ in nominal terms is

$$V_0^{x,h} := x, \quad V_1^{x,h} := x(1+r) + h \cdot (S_1 - S_0(1+r)).$$

Whereas in real terms the value is $\overline{V}_0^{x,h} = V_0^{x,h} = x$ and

$$\overline{V}_1^{x,h} = x + h \cdot \left(\frac{S_1}{1+r} - S_0 \right).$$

and so in summary

$$\overline{V}_t^{x,h} = x + h \cdot (\overline{S}_t - \overline{S}_0), \quad t = 0, 1.$$

Advantages of working in discounted terms

1. Intuitive way to describe values
2. Since $\overline{B}_t = 1$, when working in discounted terms the interest rate r is always 0...and vice-versa. This is handy.
3. In the multi-period setting it will allow us to automatically take care of the self-financing condition.
4. The *gains from trade* between times 0 and t are

$$\overline{V}_t^{x,h} - \overline{V}_0^{x,h} = h \cdot (\overline{S}_t - \overline{S}_0).$$

This involves *the increment* of $\overline{S}$, so in continuous-time

$$d\overline{V}_t^{x,H} = H_t \cdot d\overline{S}_t, \qquad \text{or equiv. } \overline{V}_s^{x,H} - \overline{V}_0^{x,H} = \int_0^s H_t \cdot d\overline{S}_t,$$

We can obtain formula for $V^H = B\overline{V}^H$

Numeraire

1. More generally, if V^L if the value in £ of some portfolio L s.t. $V^L > 0$, if something has value W_t in £, we can express its value as W_t / V_t^L at time t in units of V^L.
2. Such L is then called a *numeraire*.
3. Which numeraire we use to perform the calculations does not change the qualitative properties of our models.
4. Normally numeraire:='the bond', but it can be useful to consider other numeraires (e.g. foreign bonds).

Finite Probability Spaces

Finite Probability Spaces

Given $\Omega = \{\omega_i : i = 1, \ldots, n\}$, take $\mathcal{F} := \{A | A \subseteq \Omega\}$, then:

1. Represent rv X on Ω with $x = (x_i)_i \in \mathbb{R}^n$: $x_i = X(\omega_i)$.
2. Represent random vector $(X^j)_{j=1}^m$ with the matrix $(X^j(\omega_i))_{i,j}$.
3. Represent proba $\mathbb{P}$ with $p = (p_i)_i \in \mathbb{R}^n_+$ s.t. $\sum_i p_1 = 1$:

$$p_i = \mathbb{P}(\omega_i) := \mathbb{P}(\{\omega_i\}) \qquad \mathbb{P}(A) = \sum_{i:\omega_i \in A} p_i, \quad \text{for all } A \in \mathcal{F}.$$

4. W.l.o.g. assume $\mathbb{P}(\{\omega_i\}) > 0$ for all i. In this case we call $(\Omega, \mathcal{A}, \mathbb{P})$ a *finite proba space*
5. Write X to mean x and $\mathbb{P}$ to mean p
6. Linearity + finite proba space $\Longrightarrow$ all problems become questions about systems of *linear* (in)equalities, i.e. about vector spaces (resp. polyhedra).

How to find arbitrage

How to find arbitrage

In our linear market model (B, S) on finite Ω, how can we discover if $\exists$ arbitrage $h \in \mathbb{R}^m$, and find one?

1. Work in discounted terms. Represent rv and proba as vectors.
2. h arb $\iff \overline{V}_1^{0,h} = h \cdot (\overline{S}_1 - \overline{S}_0) \geq 0$ a.s., not $(\overline{V}_1^{0,h} = 0$ a.s.$)$
3. I.e. if $w_i^h := \sum_j h^j (\overline{S}_1 - \overline{S}_0)^j (\omega_i) \geq 0$ for all i, $w^h := (w_i^h)_i \neq 0$
4. Space W of discounted payoffs replicable at cost 0 is

$$W := \{h \cdot (\overline{S}_1 - \overline{S}_0) : h \in \mathbb{R}^m\} = \text{ span } \left\{\overline{S}_1^j - \overline{S}_0^j : j = 1, \ldots, m\right\}$$

5. The set of all arbitrage payoffs is $W \cap (\mathbb{R}_+^n \setminus \{0\})$
6. $\nexists$ arbitrage $\iff W \cap \mathbb{R}_+^n = \{0\}$

The Fourier-Motzkin algorithm

The Fourier-Motzkin algorithm

1. Given a system of linear *equalities*, we can find its solutions eliminating the variables one by one
2. The FM (Fourier-Motzkin) elimination algorithm generalises this to solve systems of linear *inequalities*
3. Eliminating variables corresponds to taking projections
4. The FM algo explicitly computes which points belong to a given polyhedron
5. The FM can be used to determine if $\exists$ arbitrage
6. The FM algo is computationally inefficient, but very simple to explain

Elimination algo with system of *equalities*

For example, to solve

$$\begin{cases} 2x + 3y + 0z = 6 \\ x + 3y + 0z = 1 \end{cases} \tag{1}$$

we isolate z and rewrite it as

$$\begin{cases} 2x + 3y = 6 \\ x + 3y = 1 \end{cases} \tag{2}$$

which has one fewer variable. We then isolate y and write

$$\begin{cases} y = & 2 - \frac{2}{3}x \\ y = & \frac{1}{3} - \frac{1}{3}x \end{cases} \tag{3}$$

1. This leads to $2 - \frac{2}{3}x = \frac{1}{3} - \frac{1}{3}x$, with solution $x = 5$.
2. Taking $y = 2 - \frac{2}{3}x$ gives $y = -\frac{4}{3}$: we solved eq. (2).
3. Set of solutions of eq. (1) is $\{(5, -\frac{4}{3})\} \times \mathbb{R}$.

Example of FM algo

To solve the system

$$\begin{cases} x_1 + x_2 + 2x_3 \geq 2 \\ x_1 + x_2 \geq 1 \\ x_1 - 4x_3 \geq 4 \\ 2x_1 + 3x_3 \geq 3 \\ 2x_1 - x_2 + x_3 \leq -5 \end{cases}$$

we isolate x_3 and get

$$\begin{cases} x_3 \geq 1 - \frac{1}{2}x_1 - \frac{1}{2}x_2 \\ x_3 \geq 1 - \frac{2}{3}x_1 \\ x_3 \leq -1 + \frac{1}{4}x_1 \\ x_3 \leq -5 - 2x_1 + x_2 \\ 0 \geq 1 - x_1 - x_2 \end{cases} \tag{4}$$

Since (y, z) solves

$$\begin{cases} z \geq e^i \cdot y + f^i & \text{for } i \in I_< \\ z \leq e^i \cdot y + f^i & \text{for } i \in I_> \end{cases} \tag{5}$$

iff y solves

$$\left\{ e^i \cdot y + f^i \geq e^j \cdot y + f^j \quad \text{for } i \in I_>, j \in I_< \right. \tag{6}$$

and then

$$z \in [\max_{j \in I_<} e^j \cdot y + f^j, \min_{i \in I_>} e^i \cdot y + f^i], \tag{7}$$

writing eq. (6) taking eq. (5) given by eq. (4) we get

$$\begin{cases} -1 + \frac{1}{4}x_1 & \geq 1 - \frac{1}{2}x_1 - \frac{1}{2}x_2 \\ -1 + \frac{1}{4}x_1 & \geq 1 - \frac{2}{3}x_1 \\ -5 - 2x_1 + x_2 & \geq 1 - \frac{1}{2}x_1 - \frac{1}{2}x_2 \\ -5 - 2x_1 + x_2 & \geq 1 - \frac{2}{3}x_1 \\ 0 & \geq 1 - x_1 - x_2 \end{cases}$$

isolating x_2 we get

$$\begin{cases} x_2 & \geq 4 - \frac{3}{2}x_1 \\ x_2 & \geq 4 + x_1 \\ x_2 & \geq 6 + \frac{4}{3}x_1 \\ x_2 & \geq 1 - x_1 \\ 0 & \geq 2 - \frac{11}{12}x_1 \end{cases} \tag{8}$$

This leads to $0 \geq 2 - \frac{11}{12}x_1$, whose solution is any $x_1 \geq \frac{24}{11}$. For any such x_1, and taking any

$$x_2 \geq \max\left(4 - \frac{3}{2}x_1, 4 + x_1, 6 + \frac{4}{3}x_1, 1 - x_1\right), \tag{9}$$

gives all sols of eq. (8). For each such (x_1, x_2), taking any x_3 in

$$[\max\left(1 - \frac{1}{2}x_1 - \frac{1}{2}x_2, 1 - \frac{2}{3}x_1\right), \min\left(-1 + \frac{1}{4}x_1, -5 - 2x_1 + x_2\right)]$$

gives all solutions of eq. (4), because of eq. (7).
E.g. You can take $x_1 = 3$, and then $x_2 = 10$, and then $x_3 = -1$.

How to find arbitrage with the FM algo

How to find arbitrage with the FM algo

1. The FM algorithm can be use to find out if $\exists$ arbitrage, and to explicitly compute an arbitrage

2. Take $r = 1/9$, $S_0^1 = 5, S_0^2 = 10$,

$$S_1^1 = \frac{10}{9}\begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix}, \quad S_1^2 = \frac{10}{9}\begin{pmatrix} 12 \\ 8 \\ 8 \end{pmatrix}, \tag{10}$$

3. We first compute their discounted values as

$$\overline{S}_1^1 - \overline{S}_0^1 = \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$\overline{S}_1^2 - \overline{S}_0^2 = \begin{pmatrix} 12 \\ 8 \\ 8 \end{pmatrix} - \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}.$$

1. Plugging values into $\overline{V}_1^{x,h} = x + h \cdot (\overline{S}_1 - \overline{S}_0)$ gives

$$\overline{V}_1^{0,h} = h^1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + h^2 \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}.$$

2. $h = (h^1, h^2)$ is an arbitrage iff $\overline{V}_1^{0,h} \geq 0$, and $\overline{V}_1^{0,h} \neq 0$.

3. Since the vectors multiplying h^1 and h^2 are independent, the *only* solution of $\overline{V}_1^{0,h} = 0$ is $h = 0$, so any $h \neq 0$ s.t. $\overline{V}_1^{0,h} \geq 0$ is an arbitrage.

4. Let us solve $\overline{V}_1^{0,h} \geq 0$ using FM algo

1. We isolate h^1 and get

$$\begin{cases} h^1 \geq -2h^2 \\ h^1 \ \ \geq \ \ 2h^2 \\ h^1 \leq -2h^2 \end{cases}$$

2. eliminating h^1 leads to the following system

$$\begin{cases} -2h^2 \geq -2h^2 \\ -2h^2 \ \ \geq \ \ 2h^2 \end{cases}$$

3. its solutions are all $h^2 \leq 0$, and then taking

$$h^1 \in [\max\Big(-2h^2, 2h^2\Big), -2h^2] = \{-2h^2\}$$

   we find the solutions $h = (h^1, h^2)$ of $\overline{V}_1^{0,h} \geq 0$.

4. Thus, h is an arbitrage iff $h^2 < 0$, $h^1 = -2h^2$; so $\exists$ arbitrage.

Math finance: an intro to Option Pricing

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The no-arbitrage and the domination principles

The Weak Domination Principle

So far, to price a replicable derivative we used the LOP

Law of One Price

If two portfolios L, M satisfy $V_T^L = V_T^M$ a.s. (i.e. with proba 1) then they satisfy $V_t^L = V_t^M$ a.s. for all $t \in [0, T)$.

To price an arbitrary derivative, we should then use the following general assumption

Weak Domination Principle

If two portfolios L, M satisfy $V_T^L \leq V_T^M$ a.s. then they satisfy $V_t^L \leq V_t^M$ a.s. for all $t \in [0, T)$.

Pricing via super- and sub-replication

Definition

A portfolio U is *super-replicating* the derivative X if its value V^U satisfies $X_T \leq V_T^U$ a.s.; D is *sub-replicating* if $X_T \geq V_T^D$ a.s..

Corollary

At any time $t \in [0, T]$, the derivative's price must be $\leq$ (*resp.* $\geq$) than the value of *any* super- (*resp. sub-*) replicating portfolio.

Define the price bounds for X as

$$u(X) := \inf\{V_0^U : V_T^U \geq X_T\}, \quad d(X) := \sup\{V_0^D : V_T^D \leq X_T\}.$$

If the WDP holds then $d(X) \leq u(X)$.

The Domination Principle

Domination Principle

The SPD (*Strict Domination Principle*) holds if, given portfolios L, M s.t.

1. $V_t^L \leq V_t^M$ a.s.
2. V_t^L is not a.s.$= V_t^M$

for $t = T$, then necessarily items 1 and 2 hold for all $t \in [0, T)$.
We will say that the DP (*Domination Principle*) holds if both the LOP and the SDP hold.

The one-period linear model

Consider the one-period linear model:

1. constants $B_0 = 1, B_1 = B_0(1+r) > 0, r > -1, S_0 \in \mathbb{R}^m$
2. random vector $S_1 \in \mathbb{R}^m$
3. portfolio (x, h): $x =$ initial capital, and h^j the number of shares of stock S^j, for $j = 1, \ldots, m$.
4. The value of (x, h) is $V^{x,h}$, where $V_0^{x,h} := x$ and

$$V_1^{x,h} := x(1+r) + h \cdot (S_1 - S_0(1+r)), \quad (x, h) \in \mathbb{R} \times \mathbb{R}^m.$$

When working in one-period models, items 1 and 2 in the DP for $t < T$ become simply $V_0^L < V_0^M$.

Domination Principle $\iff$ No-Arbitrage

Theorem

In the linear one-period market model the DP holds $\iff \exists$ NA (No-Arbitrage).

Proof ($\Longrightarrow$:) By def. M is an arbitrage if $V_0^M = 0$, $V_T^M \geq 0$ and $=$ does not hold a.s.; this violates the SDP (for $t = 0$) with $L = 0$.
($\Longleftarrow$:) If LOP fails, $\exists$ portfolios $L = (x, g), M = (y, h)$ s.t. $V_T^L = V_T^M$ and $x = V_0^L \neq V_0^M = y$, so w.l.o.g $x > y$.
Since $M - L = (y - x, h - g)$ has $M_0 - L_0 = y - x < 0$ and $V_T^M - V_T^L = 0$, the portfolio

$$N := (0, h - g) = (M - L) + (x - y, 0)$$

is an *arbitrage*, since it has final value

$$V_T^N = \underbrace{(V_T^M - V_T^L)}_{=0} + \underbrace{(x - y)(1 + r)}_{>0}.$$

Proof of DP $\Longleftarrow$ NA

If SDP fails, $\exists$ portfolio $L = (x, g), M = (y, h)$ s.t.

1. $V_T^L \leq V_T^M$ a.s.
2. $V_T^L = V_T^M$ does *not* hold a.s.
3. $x \geq y$

The same N is again an arbitrage, since

$$V_T^N = \underbrace{(V_T^M - V_T^L)}_{\geq 0, \text{ and not a.s. } = 0} + \underbrace{(x - y)(1 + r)}_{\geq 0}$$

$\square$

No-arbitrage prices

Definition of AFP

Consider the linear one-period market model (B, S) with NA, let X be an illiquid derivative X with payoff X_T at maturity T.

Definition of AFP

$p \in \mathbb{R}$ is a *fair* price (a.k.a. *Arbitrage-Free Price*) of X in the market (B, S) if the *enlarged* market (B, S, X), in which X is traded has price $X_0 = p$ at time 0, is also arbitrage-free.

To characterise the set of AFP of X, we will need the following

Lemma The following infimum and supremum are attained

$$u(X) := \inf\{V_0^U : V_T^U \geq X_T\}, \quad d(X) := \sup\{V_0^D : V_T^D \leq X_T\}.$$

Characterisation of $\mathcal{AFP}(X)$

We can now give an intuitive characterisation of the set $\mathcal{AFP}(X)$.

Proposition

In the arbitrage-free one-period market (B, S), if a derivative X is

1. *replicable*, then its fair price X_0 is unique, it equals the initial value x of any replicating portfolio, and $u(X) = X_0 = d(X)$.
2. *not replicable*, then the set of its fair prices is the *open* interval $(d(X), u(X))$.

Proof: We will use that, if p is a fair price for X, then the DM and LOP hold in the market (B, S, X) where X is sold at price $X_0 = p$.

Proof of $\mathcal{AFP}(X) = \ldots$ for X replicable

If X can be replicated by the portfolio $M = (x, g)$, then the market (B, S) has the same set of portfolios as the market (B, S, X) where X is sold at price $X_0 = x$. Thus x is a fair price for X: by assumption (B, S) has no arbitrage.

By the LOP, the *only* fair price for X is $V_0^M = x$.

Since M replicates X we get $u(X) \leq V_0^M$ and $V_0^M \leq d(X)$. Using $d(X) \leq u(X)$ we conclude $d(X) = V_0^M = u(X)$.

Proof of $\mathcal{AFP}(X) = \ldots$ for X non-replicable

If X is not replicable, let U and D be super- and sub-replicating portfolios with extremal initial values $V_0^U = u(X)$ and $V_0^D = d(X)$. Then the inequalities $V_T^D \leq X_T \leq V_T^U$ hold a.s., and do not hold a.s. with equality.

If X sold at a fair price X_0, by the DP $V_0^D < X_0 < V_0^U$, i.e. any fair price belongs to $(d(X), u(X))$.

Let us prove that conversely any $X_0 \in (d(X), u(X))$ is a fair price. The portfolio $N = (0, g, h)$ in the (B, S, X) market has wealth

$$V_T^N = g \cdot (S_1 - S_0(1 + r)) + h(X_1 - X_0(1 + r)).$$

If N is an arbitrage then V_T^N is ≥ 0 a.s., and is not a.s. $= 0$. This implies $h \neq 0$: otherwise $(0, g)$ would be an arbitrage in the (B, S) market $\lightning$

If $h > 0$ we get that a.s.

$$X_1 \geq X_0(1+r) - \frac{g}{h} \cdot (S_1 - S_0(1+r))$$

If $h > 0$ we get that a.s.

$$X_1 \geq X_0(1+r) - \frac{g}{h} \cdot (S_1 - S_0(1+r)) = V_1^L$$

where L is the portfolio $(X_0, -\frac{g}{h})$ in the (B, S) market. Thus L sub-replicates X, so it satisfies $d(X) \geq V_0^L$ by the DP.

If $h < 0$ analogously we get that $V_0^L \geq u(X)$.
Since $V_0^L = X_0$, we proved that if $\exists$ an arbitrage N in the (B, S, X) market then $X_0 \notin (d(X), u(X))$, i.e. any $X_0 \in (d(X), u(X))$ is a fair price for X.

Other notions of price in incomplete models

Traders need to come up with *one* price at which they should trade a derivative, not a whole interval of them.

Definition

A market model (B, S) is called *complete* if any derivative X can be replicated (in such market); otherwise it is called *incomplete*.

The way traders deal with incomplete models in the real world, is to use statistical considerations to pick *one* price inside the interval of arbitrage-free prices; we will not discuss how.

Alternatively, one could consider other notions of price, which lead to a smaller interval of prices. This topic is of little relevance to how option pricing is done in the real world

Linear Programming, option pricing and arbitrage

Polyhedra, Linear Programs

Definition

$P \subseteq \mathbb{R}^k$ is a *polyhedron* if

$$P := \{z \in \mathbb{R}^k : Az \geq b, \quad Cz \leq d, \quad Ez = f\},$$

for matrices A, C, E and vectors b, d, f.

Definition

If $a \in \mathbb{R}$, c, z are vectors and P is a polyhedron, the problems

$$\begin{aligned} \text{minimise / maximise} \quad & a + c \cdot z \\ \text{subject to} \quad & z \in P, \end{aligned}$$

are called a LPs (*Linear Programs*).

LP terminology

1. Solving the LP

$$\begin{aligned} \inf \quad & c \cdot z \\ \text{subject to} \quad & z \in P \end{aligned} \tag{11}$$

   means computing such inf y^*, and all $z^* \in P$ s.t. $c \cdot z^* = y^*$.

2. y^* is called the *optimal value*, and z^* an *optimiser*, of the LP. If such z^* exists, the LP is called *solvable*.

3. The LP (11) is called *feasible* if $P \neq \emptyset$. If (11) is feasible, we say that it is *bounded* if $y^* > -\infty$.

4. Conventions: $\sup \emptyset := -\infty$, $\inf \emptyset := \infty$. So, the LP (11) has $y^* = \infty$ iff it is unfeasible, and $y^* = -\infty$ iff it is unbounded.

Theorem 1

A LP is solvable $\iff$ it is feasible and bounded (i.e. $y^* \in \mathbb{R}$).

$u(X), d(X)$ as solutions to LPs

- rv $h \cdot (\overline{S}_1 - \overline{S}_0)$ 'is' the vector $(h \cdot (\overline{S}_1 - \overline{S}_0)(\omega_i))_i$, so using

$$M_{i,j} := (\overline{S}_1^j - \overline{S}_0^j)(\omega_i) \text{ we can write } h \cdot (\overline{S}_1 - \overline{S}_0) = Mh.$$

- If $z := (x, h) \in \mathbb{R} \times \mathbb{R}^m$, $c = (1, 0 \ldots, 0)$, then $c \cdot z = x$
- Problems defining $u(X), d(X)$ are LPs, since

$$u(X) = \inf\{c \cdot z : z \in P_{\geq}\}, \quad d(X) = \sup\{c \cdot z : z \in P_{\leq}\},$$

where, $\overline{V}_1^{x,h} = x + Mh$, $b = \overline{X}_1$, and so

$$P_{\geq} := \{(x, h) : x + Mh \geq b\}, \quad P_{\leq} := \{(x, h) : x + Mh \leq b\}$$

represent the super/sub replication constraints

LP have rich theory, awesome algorithms

- LPs arise whenever model is linear (= all the time), so are very well studied
- LPs have a rich and beautiful theory (*Linear Programming*)
- LP be solved by hand using some simple algorithms
- LP with $\sim 10^5$ variables can be solved on a laptop by complex algorithms, can be proved to always converge fast.

Solving LPs with the FM algo

Computations involving FM algo

1. To solve the LP

$$\begin{aligned} \inf \quad & c \cdot z \\ \text{subject to} \quad & z \in P \end{aligned} \tag{12}$$

   define the polyhedron $R := \{(y, z) : y = c \cdot z, z \in P\}$ and use the FM algorithm to compute its projection $\pi_1^{n+1}(R)$, where $\pi_1^{n+1} = \pi_y$ is defined on $\mathbb{R}_y^1 \times \mathbb{R}_z^n$.

2. $\pi_1^{n+1}(R) = \emptyset \iff R = \emptyset \iff P = \emptyset$

3. $\pi_1^{n+1}(R) \subseteq \mathbb{R}$ is a polyhedron in one variable. If $\pi_1^{n+1}(R) \neq \emptyset$ then it is a closed interval ($(-\infty, b]$, or $[a, b]$, or $[a, \infty)$), easily computed

4. inf of $\pi_1^{n+1}(R) = \emptyset / (-\infty, b] / [a, b] / [a, \infty)$ is $\infty / -\infty / a / a$.

Link between LP (12) and polyhedron R

Theorem 2

$y^* = \inf \pi_1^{n+1}(R)$, and z^* solves LP (12) iff $(y^*, z^*) \in R$.

Proof: By definition of projection we get

$$\pi_1^{n+1}(R) = \{y : \exists z \text{ s.t. } (y, z) \in R\} =$$

and, by def. of R, this equals

$$= \{y : \exists z \in P \text{ s.t. } y = c \cdot z\},$$

whose $\inf$ is y^*. By definition

$$z^* \text{ solves LP (12)} \iff z^* \in P \text{ and } c \cdot z^* = y^* \iff (y^*, z^*) \in R$$

LPs are solvable

Corollary 3

A LP is solvable $\iff$ its optimal value is finite

Proof:

1. W.l.o.g. consider a *minimisation* LP, as in (12).
2. If $y^* \in \{\infty, -\infty\}$, then trivially (12) is not solvable.
3. If $y^* = \inf \pi_1^{n+1}(R) \in \mathbb{R}$, $\pi_1^{n+1}(R)$ is a closed interval, of the form $[a, b], [a, \infty)$, and so $y^* = a \in \pi_1^{n+1}(R)$.
4. We can then use the FM algorithm to compute a z^* s.t. $(y^*, z^*) \in R$ by proceeding backwards, one variable at the time
5. Theorem 2 then shows that z^* solves (12).

How to price a derivative using the FM algorithm

Pricing a derivative with the FM algo

1. Model with $r = 1$, and two stocks with values $S_0^1 = 5 = S_0^2$,
$$S_1^1 = \left(12, 12, 8, 6\right)^T \quad , \quad S_1^2 = \left(16, 8, 6, 4\right)^T \quad .$$
2. We want to use the FM algo to compute
$$u(X) = \inf\left\{x : (x, h) \in \mathbb{R} \times \mathbb{R}^m \text{ s.t. } \overline{V}_1^{x,h} \geq \overline{X}_1\right\}. \qquad (13)$$
where $X_1 = (S_1^2 - 14)^+$.
3. We should check that the model is free of arbitrage (it is).
4. To solve (13), apply the FM algo to the polyhedron
$$R' = \{(x, h) \in \mathbb{R} \times \mathbb{R}^m : \overline{V}_1^{x,h} \geq \overline{X}_1\},$$
eliminate $h \in \mathbb{R}^m$ to compute the interval $\pi_x(R') = \pi_1^{m+1}(R')$, then $u(X) = \inf \pi_x(R)$
5. Go backwards in the algo to compute h^* s.t. $(u(X), h^*) \in R'$, then $(u(X), h^*)$ are the optimisers of (13).

FM algo: eliminate h^1

R' given by

$$\begin{cases} x + h^1 + 3h^2 \geq 1 \\ x + h^1 - h^2 \geq 0 \\ x - h^1 - 2h^2 \geq 0 \\ x - 2h^1 - 3h^2 \geq 0 \end{cases}, \tag{14}$$

To eliminate h^1, we write

$$\begin{cases} -x & -3h^2 & +1 & \leq & h^1 \\ -x & +h^2 & & \leq & h^1 \\ x & -2h^2 & & \geq & h^1 \\ \frac{1}{2}x & -\frac{3}{2}h^2 & & \geq & h^1 \end{cases} \tag{15}$$

we can now eliminate h^1:

FM algo: eliminate h^2

$$\begin{cases} -x & -3h^2 & +1 & \leq & x & -2h^2 \\ -x & -3h^2 & +1 & \leq & \frac{1}{2}x & -\frac{3}{2}h^2 \\ -x & +h^2 & & \leq & x & -2h^2 \\ -x & +h^2 & & \leq & \frac{1}{2}x & -\frac{3}{2}h^2 \end{cases}$$

To eliminate h^2, we rewrite this as

$$\begin{cases} h^2 & \geq & -2x & +1 \\ h^2 & \geq & -x & +\frac{2}{3} \\ h^2 & \leq & \frac{2}{3}x & \\ h^2 & \leq & \frac{3}{5}x & \end{cases} \tag{16}$$

we can now eliminate h^2, to get the system describing $\pi_1^3(R')$:

FM algo: last step

$$\begin{cases} \frac{2}{3}x & \geq & -2x & +1 \\ \frac{3}{5}x & \geq & -2x & +1 \\ \frac{2}{3}x & \geq & -x & +\frac{2}{3} \\ \frac{3}{5}x & \geq & -x & +\frac{2}{3} \end{cases}$$

which is equivalent to

$$\begin{cases} x & \geq & \frac{3}{8} \\ x & \geq & \frac{5}{13} \\ x & \geq & \frac{2}{5} \\ x & \geq & \frac{5}{12} \end{cases} \tag{17}$$

i.e. $x \geq \max(\frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{5}{12}) = \frac{5}{12}$, whose inf is $u(X) = \frac{5}{12}$.

How to find the optimisers of a LP using the FM algo

Finding the optimisers

1. Let's find the solutions (x, h) of the LP (13) defining $u(X)$,
2. Using (16), and then substituting $x = \frac{5}{12}$, we find

$$h^2 \in \left[\max(-2x+1, -x+\frac{2}{3}), \min(\frac{2}{3}x, \frac{3}{5}x)\right] = \left[\frac{1}{4}, \frac{1}{4}\right] = \{\frac{1}{4}\}$$

3. Using (15) we find

$$h^1 \in \left[\max(-x-3h^2+1, -x+h^2), \min(x-2h^2, \frac{1}{2}x-\frac{3}{2}h^2)\right]$$

4. Substituting $x = \frac{5}{12}$ and $h^2 = \frac{1}{4}$ we get $h^1 \in \{-\frac{1}{6}\}$
5. The LP (13) has optimal value $x = \frac{5}{12}$ and unique solution

$$x = \frac{5}{12}, \quad h^1 = -\frac{1}{6}, \quad h^2 = \frac{1}{4}.$$

Solution of pricing problem via FM algo

1. To find $u(X)$ we computed $x \geq \max(\frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{5}{12}) = \frac{5}{12}$.
2. To solve the LP

$$d(X) = \sup\Big\{x : (x, h) \in \mathbb{R} \times \mathbb{R}^m \text{ s.t. } \overline{V}_1^{x,h} \leq \overline{X}_1\Big\}. \tag{18}$$

   results in replacing $\geq$ with $\leq$, and vice versa, in all the inequalities above

3. this leads to $x \leq \min(\frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{5}{12}) = \frac{3}{8}$, and so $d(X) = \frac{3}{8}$.
4. In summary the set of arbitrage-free prices of X is the interval

$$(d(X), u(X)) = \left(\frac{3}{8}, \frac{5}{12}\right)$$

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Formulas for the binomial model

Delta-hedging formula in binomial model

To price a derivative X_1 in one-period binomial model (B, S) we replicate it, i.e. find x, h:

$$\overline{V}_1^{x,h}(H) = \overline{X}_1(H), \quad \overline{V}_1^{x,h}(T) = \overline{X}_1(T). \tag{19}$$

To find h (*resp.* x) take linear combos the two eq. (19) and get eq. only in h (*resp.* x). For h we get

$$h(\overline{S}_1(H) - \overline{S}_1(T)) = \overline{V}_1^{x,h}(H) - \overline{V}_1^{x,h}(T) = \overline{X}_1(H) - \overline{X}_1(T)$$

Delta-hedging formula

$$h = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)}. \tag{20}$$

Finding AFP in binomial model

To find x we instead fix $\tilde{p}$ and consider

$$\tilde{p}\overline{V}_1^{x,h}(H) + (1-\tilde{p})\overline{V}_1^{x,h}(T) = \tilde{p}\overline{X}_1(H) + (1-\tilde{p})\overline{X}_1(T); \tag{21}$$

rewrite its LHS as

$$x + hC_{\tilde{p}} := x + h\Big((\tilde{p}\overline{S}_1(H) + (1-\tilde{p})\overline{S}_1(T)) - \overline{S}_0\Big),$$

then choose $\tilde{p}$ s.t. $C_{\tilde{p}} = 0$, so that eq. (21) becomes

$$x = \tilde{p}\overline{X}_1(H) + (1-\tilde{p})\overline{X}_1(T). \tag{22}$$

Finding $\tilde{p}$

Now find $\tilde{p}$ solving

$$0 = C_{\tilde{p}} := (\tilde{p}\overline{S}_1(H) + (1-\tilde{p})\overline{S}_1(T)) - \overline{S}_0, \tag{23}$$

which gives

$$\tilde{p} := \frac{(1+r)-d}{u-d}, \quad \tilde{q} := 1-\tilde{p} = \frac{u-(1+r)}{u-d}. \tag{24}$$

Notice that NA $\iff d < 1+r < u \iff \tilde{p} \in (0,1)$.

The RNPF in the binomial model

If NA then $\mathbb{Q}(H) := \tilde{p}, \mathbb{Q}(T) := \tilde{q}$ define a proba $\mathbb{Q}$ on $\{H, T\}$ s.t. $\mathbb{Q}(\{\omega\}) > 0$ for all $\omega \in \{H, T\}$ and $\overline{S}_0 = \mathbb{E}^{\mathbb{Q}} \overline{S}_1$ (eq. (23)).

Conversely, if $\exists$ a proba $\mathbb{Q}$ on $\{H, T\}$ s.t. $\mathbb{Q}(\{\omega\}) > 0$ for all $\omega \in \{H, T\}$, and $\overline{S}_0 = \mathbb{E}^{\mathbb{Q}} \overline{S}_1$, then $\tilde{p} = \mathbb{Q}(H) \in (0, 1)$, and so NA.

Moreover, if such $\mathbb{Q}$ exists, then for $\forall X_1$ eq. (22) holds, i.e.

Risk-Neutral Pricing Formula

$$\overline{X}_0 = \mathbb{E}^{\mathbb{Q}} \overline{X}_1, \tag{25}$$

The Fundamental Theorem of Asset Pricing

Equivalent Martingale Measures

In binomial model, NA $\iff \exists$ proba $\mathbb{Q}$ s.t. $\overline{S}_0 = \mathbb{E}^{\mathbb{Q}}\overline{S}_1$ and $\mathbb{Q}(\{\omega\}) > 0$ for all ω. This suggests the following defs and thm.

We assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a *finite* probability space (as usual), and $\mathbb{Q}$ a probability on $\mathcal{A}$.

1. $\mathbb{Q}$ is a $\overline{S}$-*Martingale Measure* if

$$\overline{S}_0^j = \mathbb{E}^{\mathbb{Q}}(\overline{S}_1^j), \quad \forall j = 1, \ldots, m. \tag{26}$$

2. We say that $\mathbb{Q} \sim \mathbb{P}$ if $\mathbb{Q}(\{\omega\}) > 0$ for all ω.
3. A MM $\mathbb{Q} \sim \mathbb{P}$ is called an *Equivalent MM*.
4. We denote with $\mathbb{M}(\overline{S}), \mathcal{M}(\overline{S})$ the set of all MM/EMM.
5. The *physical measure* $\mathbb{P}$ satisfies $\mathbb{E}^{\mathbb{P}}(\overline{S}_1) > S_0$, because investors are *risk-averse*

The Fundamental Theorem of Asset Pricing

Theorem 4 (1st FTAP)

In the one-period market model (B, S) *it holds*

$$NA \iff \mathcal{M}(\overline{S}) \neq \emptyset.$$

Proof: $\Longrightarrow$ already proved for binomial model; proof for finite Ω sketched in lecture notes.
$(\Longleftarrow)$ If by contradiction $\exists$ arb. h then $\overline{V}_1^{0,h}(\omega) \geq 0$ $\mathbb{P}$ for all ω, and $\overline{V}_1^{0,h}(\omega) > 0$ for some $\omega = \omega'$. Since $\mathbb{Q}(\{\omega'\}) > 0$, it follows that $\mathbb{E}^{\mathbb{Q}}(\overline{V}_1^{0,h}) > 0$. This contradicts

$$\mathbb{E}^{\mathbb{Q}}(\overline{V}_1^{0,h}) = h \cdot (\mathbb{E}^{\mathbb{Q}}(\overline{S}_1) - \overline{S}_0) = 0. \qquad \square$$

$\exists$ similar link between no uniform arbitrage and $\exists \mathbb{Q} \in \mathbb{M}(\overline{S})$.

The Risk Neutral Pricing Formula

AFPs and EMMs

Corollary 5 (Risk-Neutral Pricing Formula)

The set of arbitrage-free prices for an illiquid derivative with payoff X_1 in a one-period arbitrage-free market model (B, S) is

$$\mathcal{AFP}(X_1) = \{\mathbb{E}^{\mathbb{Q}}[\overline{X}_1] : \mathbb{Q} \in \mathcal{M}(\overline{S})\}.$$

Proof: By theorem 4 $\mathcal{M}(\overline{S}) \neq \emptyset$, and $X_0 = \overline{X}_0$ is an AFP for X iff $\mathcal{M}(\overline{S}, \overline{X}) \neq \emptyset$. Since $\mathbb{Q} \in \mathcal{M}(\overline{S})$ belongs to $\mathcal{M}(\overline{S}, \overline{X})$ iff $\overline{X}_0 = \mathbb{E}^{\mathbb{Q}}[\overline{X}_1]$, the thesis follows. □

Corollary 6 (Replicability criterion)

X_1 *is replicable* $\iff \mathbb{E}^{\mathbb{Q}}[\overline{X}_1]$ *is const. over* $\mathbb{Q} \in \mathcal{M}(\overline{S})$

Proof: Follows from corollary 5 and the fact that X_1 is replicable $\iff$ its price is unique. □

Pricing using martingale measures

Assume finite Ω.

1. Since $\mathcal{AFP}(X_1) = (d(X), u(X))$, it follows

$$u(X) = \mathsf{sup}\{\mathbb{E}^{\mathbb{Q}}(\overline{X}_1) : \mathbb{Q} \in \mathcal{M}(\overline{S})\} \tag{27}$$

2. One can prove that $\mathbb{M}$ is the closure of $\mathcal{M}$
3. So we can replace sup over $\mathcal{M}$ with max over $\mathbb{M}$ in eq. (27).
4. Since $\mathbb{M}$ 'is' the polyhedron

$$\{q \in \mathbb{R}^n : q \geq 0, \textstyle\sum_i q_i = 1, \sum_i q_i(\overline{S}_1^j - \overline{S}_0^j)(\omega_i) = 0 \,\forall j\},$$

   we get the LP $u(X) = \mathsf{max}\{b \cdot q : q \in \mathbb{M}\}$, where b 'is' $\overline{X}_1$.

5. Analogously $d(X) = \mathsf{inf} \ldots$, so we can replace inf over $\mathcal{M}$ with min over $\mathbb{M}$ and get the LP $d(X) = \mathsf{min}\{b \cdot q : q \in \mathbb{M}\}$.

2nd FTAP

Corollary 7 (Characterisation of complete models)

Let (B, S) be a market model free of arbitrage. Then (B, S) is complete $\iff$ the EMM is unique.

Proof: $(\Longleftarrow)$ If the EMM $\mathbb{Q}$ is unique then by corollary 6 any derivative is replicable.
$(\Longrightarrow)$ Given arbitrary $A \in \mathcal{F}$, since $X_1 := 1_A$ is replicable, by theorem 5 we have that, for any $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{M}(\overline{S})$,

$$\mathbb{Q}^1(A)/(1+r) = \mathbb{E}^{\mathbb{Q}^1}[\overline{X}_1] = \mathbb{E}^{\mathbb{Q}^2}[\overline{X}_1] = \mathbb{Q}^2(A)/(1+r).$$

This shows $\mathbb{Q}^1(A) = \mathbb{Q}^2(A)$ for all $A \in \mathcal{F}$, i.e. $\mathbb{Q}^1 = \mathbb{Q}^2$. $\square$

Dividends

What are dividends

1. When a company makes profits, it can invest them, or give *handouts* in cash to its share-holders, called *dividends*
2. The handouts given to bond holders are called *coupons*.
3. The stock-holder can reinvest such cash in the same stock, or deposit it in the bank account, or use it in other ways.
4. The dividend amounts, and times of payment (*ex-dividend dates*), *could* depend on the time, the stock price, etc.
5. In general, stock-holders get paid on a quarterly basis, always the same amount.
6. The price S of one share just *after* the dividend D is paid is called *ex-dividend* (or *post-dividend*)
7. The price V of one share just *before* the dividend D is paid is called *cum-dividend*
8. Clearly $V = S + D$, $S \geq 0$, $D \geq 0$.

Arbitrage and and short-selling with dividends

1. Dividends create some complications when pricing options; let's illustrate some.
2. Consider a one-period binomial model with $r = 3$, one asset with $S_0 = 2, S_1$ can take values $4, 8$, dividend $D = D_1 = 2$. Is the model is arbitrage free?
3. Yes, because $d < 1 + r < u$ is satisfied if d, u are the values taken by V_1/V_0 (*not by* S_1/S_0), where $V_0 = S_0$, $V_1 = S_1 + D$
4. It is (B, V) which must have no arbitrage, not (B, S).
5. When short-selling a share, you borrow it. When you return it to its owner, you owe him also compensation for the dividends issued in the meantime.

Pricing a forward contract with dividends

1. Let us compute the forward price F of the stock.
2. The forward contract pays $S_1 - F$ (not $V_1 - F$), so to replicate it we have to:
   - Buy one share, at cost S_0
   - Borrow $\frac{F}{1+r}$ from the bank
   - Borrow $\frac{D}{1+r}$ from the bank
3. Indeed, your final wealth will then be

$$(S_1 + D) - (1+r)\left(\frac{F}{1+r}\right) - (1+r)\left(\frac{D}{1+r}\right) = S_1 - F.$$

4. The initial cost of this strategy is $S_0 - \frac{F+D}{1+r}$, and setting this $= 0$ yields $F = S_0(1+r) - D = 2 \cdot (1+3) - 2 = 6$.

Foreign currency

The value of a foreign investment

1. FX market is huge and very liquid
2. To avoid *FX risk*, can trade options based on exchange rates
3. The exchange rate $E := E_{\pounds}^{€}$ between £ and € , measured in £/€ (defined as the cost of one € in £)
4. It should be modelled as some stochastic process $E > 0$
5. We chose to describe the (domestic) money market (in £) with 'the (British) bond' B^d, with interest rate r^d
6. Analogously, if we can trade also in € we should also consider a € bond B^f, with interest rate r^f
7. If $V =$ value in € of an investment, its value in £ is VE, since
$$€ \cdot \frac{\pounds}{€} = \pounds$$
8. If $W =$ value in £ of an investment, its value in € is V/E

A market with multiple currencies

For example:

1. If at time 0 I deposit €1 in the foreign bank, at time 1 I will have €$(1 + r^f)$, which I can convert to £$(1 + r^f)E_1$.
2. If instead at time 0 I convert the €1 in £ I get £E_0, and if I deposit that in the domestic bank I get £$(1 + r^d)E_0$ at time 1

If I consider £, €, a British stock S^d and a European stock S^f

1. S^d is the price in £ of the domestic stock
2. S^f is the price in € of the foreign stock. Its price in £ is $S^f E$
3. So, I should model the market as being $(B^d, S^d, B^f E, S^f E)$
4. A European investor would measure values in € instead of £, so should model the same market as $(B^d/E, S^d/E, B^f, S^f)$, and use B^f instead of B^d to discount

An example of a market with £ and €

Determine if the following one-period market admits arbitrage:

$$r^d = \frac{1}{8}, r^f = \frac{1}{3}, E_0 := 2, S_0^f := 5,$$

ω	ω_1	ω_2	ω_3
$E_1(\omega)$	3	1	2
$S_1^f(\omega)$	6	12	4

where E is measured in £/€, S in €.

It does iff the market $(B^d, B^f E, S^f E)$ does, where

B_0^d	1
$B_0^f E_0$	2
$S_0 E_0$	10

ω	ω_1	ω_2	ω_3
$B_1^d(\omega)$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{9}{8}$
$B_1^f E_1(\omega)$	$\frac{8}{3}$	$\frac{4}{3}$	4
$S_1^f E_1(\omega)$	18	12	8

One can now do all the calculations as usual for the market $(B, S^1, S^2) := (B^d, B^f E, S^f E)$, in which all values are in £.

Math finance: an intro to Option Pricing

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Table of Contents of Week 6

Measurability

Multi-period models: measurability

- A stochastic processes is $X = (X_t)_{t \in I}$
- Normally $I \subseteq [0, \infty]$, now $I = \mathbb{T} := \{0, 1, \ldots, T\}$.
- A multi-period market model is $(B_t, S_t)_{t \in \mathbb{T}}$ with $B > 0$
- We say that processes $X \leq Y$ if $X_t \leq Y_t$ holds $\forall t \in I$ a.s., i.e.

$$\mathbb{P}(\{\omega : X_t(\omega) \leq Y_t(\omega) \, \forall t \in I\}) = 1$$

How can we express that $H = (H_t)_{t \in \mathbb{T}}$ is *non-anticipative*, i.e. H_t depends *only* on info known at time t?

1. Declare set $\mathcal{S}_t$ of rv known at t, e.g. $\mathcal{S}_t = \{B_u, S_u\}_{u \in \mathbb{T}, u \leq t}$
2. Ask that $\exists f$ s.t. $H_t = f(t, B_0, S_0, \ldots, B_t, S_t)$ for all t
3. Must restrict to *Borel* f (so $f(t, B_0, S_0, \ldots, B_t, S_t)$ is rv)...
4. ...unless $\mathcal{S}$ countable, each $X \in \mathcal{S}$ has countably-many values.

Doob-Dynkin Lemma

1. For $X : \Omega \to \mathbb{R}^n$, define $\sigma(X) := \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^n)\}$.
2. If X has countably-many values $\sigma(X) = \{X^{-1}(B) : B \subseteq \mathbb{R}^n\}$.

Lemma 8 (Doob-Dynkin)

Suppose X and Y are random vectors with n and k components, defined on the measurable space $(\Omega, \mathcal{A})$, then t.f.a.e.:

1. *$\exists$ Borel $f : \mathbb{R}^n \to \mathbb{R}^k$ s.t. $X = f(Y)$*
2. *X is $\sigma(Y)$-measurable, i.e. $\sigma(X) \subseteq \sigma(Y)$*

Lemma 9

X, Y rv on $(\Omega, \mathcal{A})$, if Y only takes values $\{y_n\}_{n\in\mathbb{N}}$ then t.f.a.e.:

1. *$\exists$ f s.t. $X = f(Y)$*
2. *X takes a constant value x_n on $\{Y = y_n\}$ for all $n \in \mathbb{N}$*

Then $f(y_n) = x_n$, and one can choose $f = 0$ on $\mathbb{R}^k \setminus \{y_n\}_{n\in\mathbb{N}}$.

Adapted processess and filtrations

Definitions

- $(\mathcal{F}_t)_{t\in\mathbb{T}}$ is a *filtration* if $\mathcal{F}_t$ σ-algebra and $\mathcal{F}_s \subseteq \mathcal{F}_t\, \forall s \leq t$.
- $(X_t)_{t\in\mathbb{T}}$ is *adapted* to filtration $(\mathcal{F}_t)_t$ if X_t is $\mathcal{F}_t$-meas. $\forall t$.
- $\mathcal{F}_t^X := \sigma((X_u)_{u\leq t, u\in\mathbb{T}}), t \in \mathbb{T}$, is the *natural filtration* of X.

- Given a market (B, S) on $(\Omega, \mathcal{A}, \mathbb{P})$, consider a filtration $\mathcal{F} \supseteq \mathcal{F}^{(B,S)}$, and the *filtered proba. space* $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$
- Trading strategy: $\mathcal{F}$-adapted processes K, H
- Stream of payoffs of derivative: $\mathcal{F}^{(B,S)}$-adapted processes X
- The interest rate R must be adapted, so B_t is $\mathcal{F}_{t-1}$-meas. for all t, since $B_t = B_0(1 + R_0) \cdot \ldots \cdot (1 + R_{t-1})$

Self-financing portfolios

Self-financing portfolios

1. At t own K_t bonds with price B_t and H_t^j shares with price S_t^j, so wealth is $K_t B_t + H_t \cdot S_t$

2. At $t+1$ wealth becomes $K_t B_{t+1} + H_t \cdot S_{t+1}$, then re-adjust our portfolio and wealth is $K_{t+1} B_{t+1} + H_{t+1} \cdot S_{t+1}$

3. (K, H) must be *self-financing*, i.e.

$$K_t B_{t+1} + H_t \cdot S_{t+1} = K_{t+1} B_{t+1} + H_{t+1} \cdot S_{t+1}$$

4. K_0 and $H = (H_t)_{t<T}$ are free, other K_t's determined by

$$K_{t+1} = \frac{K_t B_{t+1} + (H_t - H_{t+1}) \cdot S_{t+1}}{B_{t+1}}$$

5. Once determined K from K_0 and H, one can compute the wealth $K_t B_t + H_t \cdot S_t$

Describing a portfolio with K_0, H

1. $$V_{t_{i+1}} - V_{t_i} = H_{t_i}(S_{t_{i+1}} - S_{t_i}) + K_{t_i}(B_{t_{i+1}} - B_{t_i})$$

2. Since $B_{t_{i+1}} - B_{t_i} = B_{t_i} r$, and $K_{t_i} B_{t_i} = V_{t_i} - H_{t_i} S_{t_i}$ we get
$$K_{t_i}(B_{t_{i+1}} - B_{t_i}) = K_{t_i} B_{t_i} r = (V_{t_i} - H_{t_i} S_{t_i}) r$$

3. Combining the two gives
$$V_{t_{i+1}} - V_{t_i} = H_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V_{t_i} - H_{t_i} S_{t_i}) r \tag{28}$$

4. $$V_{t_k} - V_0 = \sum_{i=0}^{k-1} H_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V_{t_i} - H_{t_i} S_{t_i}) r,$$

5. $V_0 = K_0 B_0 + H_0 \cdot S_0$ gives expression of V using only K_0, H

Describing a portfolio with x, H

1. Let us use instead the variables $x = V_0, H$
2. Instead of eq. (28) we write $\overline{V}_{t+1}^{x,H} - \overline{V}_t^{x,H} = H_t \cdot (\overline{S}_{t+1} - \overline{S}_t)$
3. Define $(H \cdot Y)_t := \sum_{s=0}^{t-1} H_s \cdot (Y_{s+1} - Y_s)$
4. Convenient formula:

$$\overline{V}_t^{x,H} = x + (H \cdot \overline{S})_t$$

5. If you want K anyway, use $V_t^{x,H} = B_t(K_t + H_t \cdot \overline{S}_t)$ to get

$$K_t = \overline{V}_t^{x,H} - H_t \cdot \overline{S}_t = x + (H \cdot \overline{S})_t - H_t \cdot \overline{S}_t$$

6. The value V_t of a derivative with expiry N is defined for $t \leq N$; its replicating strategy H_t only for $t \leq N - 1$.

Arbitrage and arbitrage-free prices

Arbitrage-free prices

Consider a multi-period model $(B_t, S_t)_{t\in\mathbb{T}}$, with $S_t = (S_t^1, \dots, S_t^m)$

1. a trading strategy is an adapted process H with values in $\mathbb{R}^m$
2. H is an *arbitrage* if $\overline{V}_T^{0,H} \geq 0$ a.s., and $\overline{V}_T^{0,H}$ is not a.s. $= 0$.
3. Assume $(B_t, S_t)_{t\in\mathbb{T}}$ is arbitrage-free

Theorem 10

There exists an arbitrage in the multi-period model $(B_t, S_t)_{t\in\mathbb{T}}$ if and only if there exists a $s \in \mathbb{T}, s < T$ such that there exists an arbitrage in the one-period sub-model $(B_t, S_t)_{t=s,s+1}$.

Definition 11

An adapted process $(P_t)_{t\in\mathbb{T}}$ is an *Arbitrage Free Price* for the derivative with payoff X_T at maturity T in $(B_t, S_t)_{t\in\mathbb{T}}$ if $P_T = X_T$ and the enlarged market $(B_t, S_t, P_t)_{t\in\mathbb{T}}$ is arbitrage-free.

The multi-period binomial model

The multi-period binomial model

Bond with interest R, and one underlying with price $(S_n)_{n\leq N}$ s.t.

$$S_{n+1}(\omega) = \begin{cases} (S_n U_n)(\omega) & \text{if } X_{n+1}(\omega) = H, \\ (S_n D_n)(\omega) & \text{if } X_{n+1}(\omega) = T, \end{cases} \tag{29}$$

Build the N-period binomial model on this $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P}_{\mathbf{p}})$:

1. $\Omega = \{H, T\}^N$, i.e. $\omega = (\omega_1, \cdots, \omega_N) \in \Omega$ with $\omega_n \in \{H, T\}$
2. $\mathcal{A} := \mathcal{P}(\Omega) :=$ all subsets of Ω.
3. $\mathcal{F} = (\mathcal{F}_n)_{n=0}^N =$ natural filtration of X, where $X_n(\omega) := \omega_n$
4. For $p \in (0,1)$ let $\mathbb{P}_p$ be the proba on $\{H, T\}$: $\mathbb{P}_p(\{H\}) = p$.
   For $\mathbf{p} = (p_n)_{n=1}^N \in (0,1)^N$ take $\mathbb{P}_{\mathbf{p}}(\{\omega\}) = \Pi_{n=1}^N \mathbb{P}_{p_n}(\{\omega_n\})$

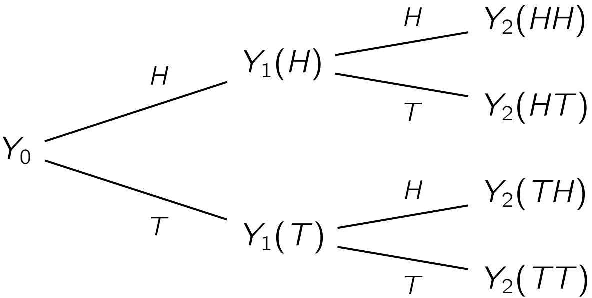
Adapted processes in binomial model

Y is $\mathcal{F}$-adapted $\iff$ $Y_n = f(X(n))$ for $X(n) := (X_k)_{k \leq n}$

We improperly write e.g. $Y_2(H, T, \omega_3) = Y_2(H, T)$ (if $N = 3$), and

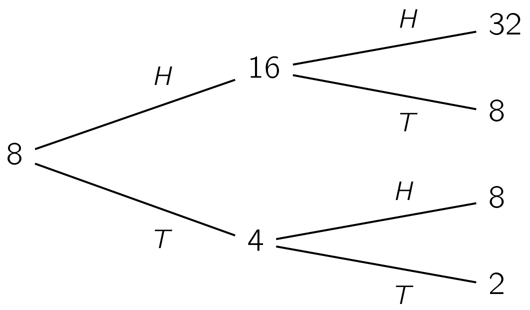
$$\{HT\} = \{\omega \in \{H, T\}^3 : X_1(\omega) = H, X_2(\omega) = T\}.$$

We identify adapted processes with binary trees: if Y is an adapted



Adapted processes and binary trees

Conversely, writing on a binary tree some values, e.g.



identifies a unique adapted Y which takes those values:

$$Y_0 = 8, Y_1(H) = 16, Y_1(T) = 4$$

$$Y_2(HH) = 32, Y_2(HT) = 8 = Y_2(TH), Y_2(TT) = 2.$$

Arbitrage in multi-period binomial model

Inputs R_n, U_n, D_n are $\mathcal{F}$-adapted and s.t. $R > -1$, $0 < D < U$.

Theorem 12

Multi-period binomial model is arb.-free $\iff D < 1 + R < U$.

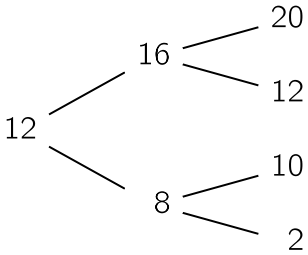
Proof: It follows from NA $\iff$ NA in $(B_t, S_t)_{t=n,n+1}$ for each n, and from $(B_t, S_t)_{t=n,n+1} \iff D_n < 1 + R_n < U_n$

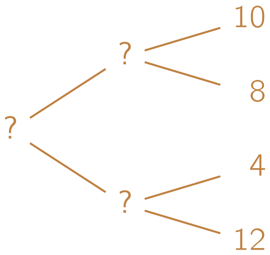
Example of pricing in the multi-period binomial model

Example of pricing in multi-period binomial model

To price derivative Y with expiry T, try to solve $V_T^{x,G} = Y_T$.

Example: 2-period binomial model with $R = 0$, and stock price S and derivative's payoff Y given by





Computational considerations

- The system $V_2^{x,G}(\omega) = Y_2(\omega)$, $\omega \in \{H, T\}^2$ has 4 eqs., in the 4 variables: x, G_0, $G_1(H)$, $G_1(T)$.
- The eqs. are linearly independent $(D < U)$, so $\exists!$ sol. (x^*, G^*).
- The derivative Y has price $Y_0 = x^*$ and $Y_1 = V_1^{x^*,G^*}$.
- If $T = N$ the system $V_T^{x,G} = Y_T$ has 2^N eqs. and 2^N variables
- Computationally unfeasible to solve by hand if $N \geq 3$, and even on a computer for big N.
- In practice $N \geq 100$; $2^{100} \sim 10^{30}$, so we cannot price this way.
- The way out will be:
  1. Break the big system into many small sub-systems.
  2. Identify which of these many sub-systems are identical, so we need to solve only one of these!

Pricing by backward induction

To find $Y_1(H)$, look at one-period sub-model $(S_t, Y_t)(H\,\cdot)$, $t = 1, 2$:

$$\begin{tikzcd}
 & (20, 10) \\
(16, 9) \arrow[ur, dash] \arrow[dr, dash] & \\
 & (12, 8)
\end{tikzcd}$$

We can now find $Y_1(H)$ as usual, e.g. compute

$$U_1(H) = \frac{S_2(HH)}{S_1(H)} = \frac{20}{16} = \frac{5}{4}, \qquad D_1(H) = \frac{S_2(HT)}{S_1(H)} = \frac{12}{16} = \frac{3}{4},$$

$$\tilde{P}_1(H) = \frac{1 + R_1 - D_1}{U_1 - D_1}(H) = \frac{1 + 0 - 3/4}{5/4 - 3/4} = \frac{1}{2}.$$

and then

$$Y_1(H) = \frac{\tilde{P}_1(H)Y_2(HH) + (1 - \tilde{P}_1(H))Y_2(HT)}{1 + R_1(H)} = \frac{10 + 8}{2} = 9$$

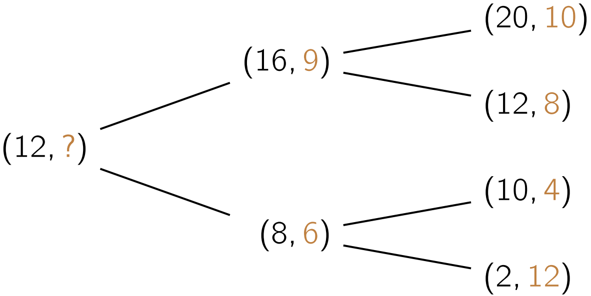
Replicating strategy G^* is given by delta-hedging formula

$$G_1^*(H) = \frac{Y_2(HH) - Y_2(HT)}{S_2(HH) - S_2(HT)} = \frac{10-8}{20-12} = \frac{2}{8} = \frac{1}{4}$$

Analogously we can take $\omega_1 = T$ and compute

$$Y_1(T) = V_1^{x^*,G^*}(T) = 6, \quad G_1^*(T) = -1.$$

So (S, Y) is given by



Thus (S_t, Y_t) for $t = 0, 1$ is given by

$$\begin{tikzcd}[row sep=small] & (16, 9) \\ (12, ?) \arrow[ur, no head] \arrow[dr, no head] & \\ & (8, 6) \end{tikzcd}$$

so we can use the RNPF to compute the initial price

$$Y_0 = x^* = V_0^{x^*, G^*} = \frac{15}{2}$$

and the initial value of the replicating strategy

$$G_0 = \frac{V_1^{x^*, G^*}(H) - V_1^{x^*, G^*}(T)}{S_1(H) - S_1(T)} = \frac{3}{8}.$$

Math finance: an intro to Option Pricing

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Table of Contents of Week 7

Derivatives paying a cashflow

The cashflow of a derivative

1. So far we considered only derivatives which provide a payoff P_T at only one, deterministic time $T = N$, called *expiry*.
2. Some derivatives provide a payoff P_t at any time $t = 0, 1, \ldots, T$, where $T \in \mathbb{N} \cup \{\infty\}$ is called *expiry*.
3. $P = (P_t)_{t=0,\ldots,T}$ is called the *cashflow* of the derivative.
4. P must be an *adapted* process.

How to price derivatives with a cashflow

1. To price derivative with a cashflow P we work by replication.
2. Denote with P_k^n the value at time $k \leq n$ of the derivative which only has payoff P_n at time n, and with H_k^n the number of shares one should hold at time $k \leq n-1$ to replicate it.
3. To replicate the derivative we only need to replicate the future cash flows, and the value V_k of the derivative at time k equals the present value of its *future* cash flows, and so

$$H_k = \sum_{n=k+1}^{N-1} H_k^n, \qquad V_k = \sum_{n=k}^{N} P_k^n. \tag{30}$$

Derivatives with random maturity

Random expiry

1. A derivative can have a payoff D at one *random* time τ.
2. τ has values in $\mathbb{N} \cup \{\infty\}$, and $\{\tau = \infty\} = \{D = 0\}$.
3. Derivative with payoff D at random time τ = derivative with cashflow P s.t. $P_k(\omega) \neq 0$ iff $k \neq \tau(\omega) < \infty$, where $P_k = D$ on $\{\tau = k\}$, $P_k = 0$ on $\{\tau \neq k\}$, for $k \in \mathbb{N}$.
4. As P is adapted, the time of payment $\tau := \inf\{k : P_k \neq 0\}$ is a *stopping* time, i.e., $\{\tau \leq k\}$ is $\mathcal{F}_k$-meas. for all k; equivalently, $A_k := 1_{\{\tau \leq k\}}, k \in \mathbb{N}$ is adapted.
5. Often τ is a *hitting time*, i.e. the first time an adapted process X hits a set C, i.e. $\tau := \inf\{k \geq 0 : X_k \in C\}$
6. Though one could price and replicate derivatives with random expiry with the same method as for those paying a cashflow, it is simpler and quicker to instead set up and *solve the replication equation only up to time τ*.

Example of pricing an option with random expiry

In the binomial model $N = 2, r = 0, S_0 = 4, u = 2, d = \frac{1}{2}$, find the price X the option which pays the rebate 2 at the (stopping) time

$$\sigma := \inf\{k = 0, 1, 2 : S_k \geq U\}$$

the price S of the stock crosses the barrier $U = 6$, and pays $(S_N - 3)^+$ at time N if the barrier U is never crossed.

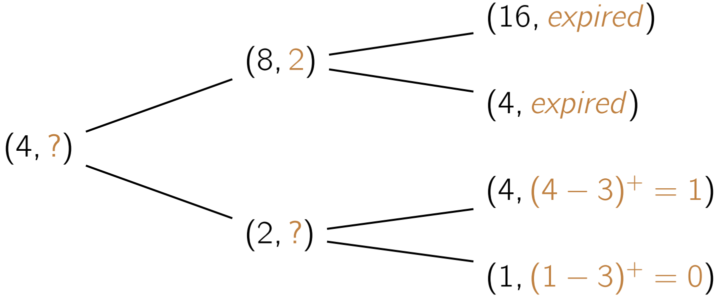


Figure 1: Tree of (S, X).

Computing the price X_1 at time 1

1. The first (and only) time of payment is τ

ω	HH	HT	TH	TT
$\sigma(\omega)$	1	1	∞	∞
$\tau(\omega)$	1	1	2	∞

2. When $\omega_1 = T$ we get

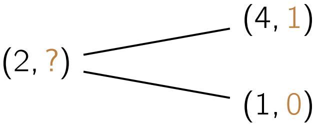


Figure 2: Branch of tree of (S, X) from $\omega_1 = T$.

3. Since $r = 0$, $\tilde{p} := \frac{1-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1}{3}$, the RNPF gives

$$X_1(T) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 0 = \frac{1}{3}.$$

Computing X_0, and the replicating strategy

1. We can now compute X_0:

$$\begin{tikzcd}[row sep=small] & (8, 2) \\ (4, ?) \arrow[ur, no head] \arrow[dr, no head] & \\ & (2, \frac{1}{3}) \end{tikzcd}$$

Figure 3: Root of tree of (S, X).

2. The RNPF gives $X_0 = \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot \frac{1}{3} = \frac{8}{9}$.
3. The delta-hedging formula gives the replicating strategy

$$G_1(T) = \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2(TT)} = \frac{1-0}{4-1} = \frac{1}{3},$$

$$G_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = \frac{2 - \frac{1}{3}}{8-2} = \frac{5}{18}.$$

4. If $\omega_1 = H$ then expiry $\tau(H\omega_2) = 1$, so $G_1(H) = 0$.

Chooser Options

Chooser, American and Bermudan Options

Many derivatives offer the holder (buyer) choices, for example:

1. If I buy a *chooser option* (at time 0), I get to choose, at time s, whether I will receive the payoff A at time $u \geq s$, or the payoff B at time $v \geq s$.
2. If I buy an *American* call option, I get to choose the (stopping) time τ at which I will receive the payoff $(S_\tau - K)^+$.
3. If I buy an *Bermudan* call option, I get to choose the (stopping) time τ at which I will receive the payoff $(S_\tau - K)^+$, among those τ with values in a set D of possible dates.
4. To compute the price C of a chooser, work by backward induction to compute, for $t \in [s, u]$, the value A_t at time t of getting A at time u. Analog. compute B_t, for $t \in [s, v]$. Then set $C_s := \max(A_s, B_s)$. Then, for $r \leq s$, compute C_r by backward induction from C_s.

Example of pricing a chooser option

In the binomial model $N = 2, r = 0, S_0 = 4, u = 2, d = \frac{1}{2}$, find the price C_0 at which you are willing to sell the option which gives the buyer the right to choose at time 1 whether to receive $A_2 := (S_2 - 5)^+$ at time 2, or $B_1 := (S_1 - 5)^+$ at time 1.

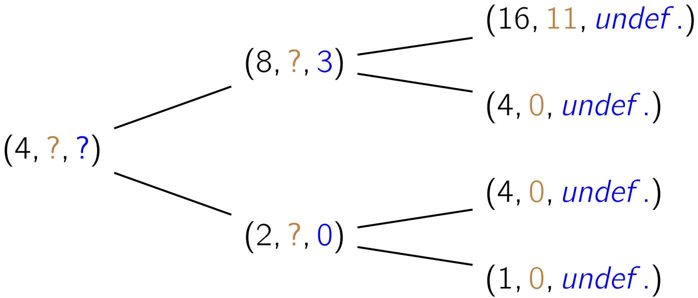


Figure 4: Tree of (S, A, B).

Computing the values at time 1

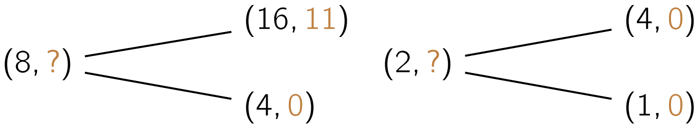


Figure 5: Branches of tree of (S, A) from $\omega_1 = H$, and from $\omega_1 = T$.

Since $r = 0$, $\tilde{p} = \frac{1}{3}$, the RNPF $A_1 = \mathbb{E}_1^{\mathbb{Q}}\left[\frac{A_2}{1+r}\right]$ gives

$$A_1(H) = \frac{1}{3} \cdot 11 + \frac{2}{3} \cdot 0 = \frac{11}{3}, \quad A_1(T) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 0 = 0.$$

Comparing the values at the time of choice

$$
(4, \textit{irrelevant}, \textit{irrelevant}, ?) \quad
\begin{array}{l}
\diagup \; (8, \tfrac{11}{3}, 3, \max(\tfrac{11}{3}, 3) = \tfrac{11}{3}) \\[2ex]
\diagdown \; (2, 0, 0, \max(0, 0) = 0)
\end{array}
$$

Figure 6: Root of tree of (S, A, B, C).

So, the option holder should choose A_2 if $\omega_1 = H$, and is indifferent between A_2 and B_1 if $\omega_1 = T$.

Since $r = 0, \tilde{p} = \frac{1}{3}$, the RNPF gives

$$C_0 = \frac{1}{3} \cdot \frac{11}{3} + \frac{2}{3} \cdot 0 = \frac{11}{9}.$$

Hedging a chooser option

1. The hedging strategy G is given by the delta-hedging formula

$$G_0 = \frac{C_1(H) - C_1(T)}{S_1(H) - S_1(T)} = \frac{\frac{11}{3} - 0}{8 - 2} = \frac{11}{18}$$

2. If at time 1 the option holder chooses A_2, you should hold

$$G_1(\omega_1) = \frac{A_2(\omega_1 H) - A_2(\omega_1 T)}{S_2(\omega_1 H) - S_2(\omega_1 T)}, \text{ so } G_1(H) = \frac{11}{12}, G_1(T) = 0.$$

3. If instead at time 1 the holder chooses B_1, then the chooser expires, so no more hedging is needed, i.e. $G_1 = 0$. In this case, if $\omega_1 = H$ the option seller makes a profit without risk!

American Options

American options

1. A standard (European) option has a payoff I_T for some adapted functional I of S (i.e. $I_t = f(t, S_0, \dots, S_t)$), e.g.

$$I_t = (S_t - K)^+, \quad I_t = S_t - \min_{u \le t} S_u, \quad I_t = (\tfrac{1}{t}\textstyle\sum_{u \le t} S_u - K)^+$$

2. The corresponding American option gives its buyer the right to choose at which (stopping) time τ to get paid I_τ.
3. I is the *intrinsic value* of the derivative, τ the *exercise* date.
4. The buyer can choose $\tau \le N$, or $\tau = \infty$ (=no payment), so (s)he would only choose $\tau(\omega) = t \le N$ if $I_t(\omega) \ge 0$.
5. The value at time n of receiving Y_{n+1} at time $n+1$ is given by the RNPF. We denote it with $\mathbb{E}_n^{\mathbb{Q}}\left[\frac{Y_{n+1}}{1+r}\right]$.
6. If V_n is the value at time n of the American option which hasn't yet been exercised, then for all $n = 0, \dots, N-1$,

$$V_N = I_N^+, \quad V_n = \max\left(I_n, C_n\right), \quad C_n := \mathbb{E}_n^{\mathbb{Q}}\left[\frac{V_{n+1}}{1+r}\right],$$

where C_n is the called *continuation value*.

Example of pricing an American put option

1. Model the underlying S with a 2-period binomial model with

$$S_0 = 4, \quad u = 2, \quad d = \frac{1}{2}, \quad r = \frac{1}{4}.$$

2. Compute the price at time 0 of the American put option on S with strike price $K = 5$. Thus $I_n := (5 - S_n)^+$.

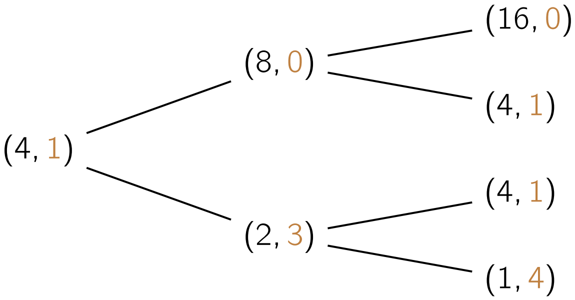


Figure 7: Tree of $(S, (5 - S)^+)$.

Calculation of price

1. $V_2 = I_2^+ := (5 - S_2)^+$.
2. Since $r = \frac{1}{4}$, $u = 2$, $d = 1/2$ we get
$$\tilde{p} = \frac{1}{2} = 1 - \tilde{p}, \quad \frac{1}{1+r} = \frac{4}{5}.$$
3. Compute the continuation value $C_n := \mathbb{E}_n^{\mathbb{Q}}\left[\frac{V_{n+1}}{1+r}\right]$
$$C_1(H) = \frac{1}{2} \cdot \frac{4}{5}(0+1) = \frac{2}{5}, \quad C_1(T) = \frac{1}{2} \cdot \frac{4}{5}(1+4) = 2.$$
4. Since $V_1 = I_1 \vee C_1 := \max(I_1, C_1)$, we get
$$V_1(H) = \frac{2}{5}, \quad V_1(T) = 3.$$
5. Analogously
$$C_0 = \frac{2}{5}\left(\frac{2}{5} + 3\right) = \frac{34}{25}, \quad V_0 = I_0 \vee C_0 = 1 \vee \frac{34}{25} = \frac{34}{25}.$$

Optimal exercise time(s)

What is the optimal exercise time τ^* of the American option?

1. At time 0: wait (since $C_0 > I_0$). So, $\{\tau^* = 0\} = \emptyset$.
2. At time 1: if $\omega_1 = H$ then wait (since $C_1(H) > I_1(H)$); if $\omega_1 = T$ exercise (since $C_1(T) < I_1(T)$). So, $\{\tau^* = 1\} = \{T\}$.
3. At time 2: if HT then exercise (since $I_2(HT) > 0$)
4. At time 2: if HH then indifferent between exercising vs. letting the option expire (since $I_2(HH) = 0$). So

$$\tau^*(T) := 1, \quad \tau^*(H) := 2, \quad \sigma(\omega) := \begin{cases} \infty & \text{if } \omega = HH \\ \tau(\omega) & \text{otherwise} \end{cases}$$

are both optimal. Notice that $\sigma \geq \tau^*$.

So $V_n = I_n \vee C_n \geq I_n$ and the *smallest* optimal exercise time is

$$\tau^* := \inf\{n \leq N : V_n = I_n\}.$$

Conditional probability and conditional expectation

Conditional Probability given a set B

Given $(\Omega, \mathcal{A}, \mathbb{P})$ and $B \in \mathcal{A}$ s.t. $\mathbb{P}(B) > 0$, we can define the *conditional probability* $\mathbb{P}$ given B:

$$\mathbb{P}(\cdot|B) : \mathcal{A} \to [0,1], \quad \mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{for } A \in \mathcal{A}.$$

Theorem 13

$\mathbb{P}(\cdot|B)$ *is a probability on* $\mathcal{A}$*, and for any random variable* X

$$\mathbb{E}^{\mathbb{P}(\cdot|B)}[X] = \frac{\mathbb{E}^{\mathbb{P}}[X 1_B]}{\mathbb{P}(B)}. \tag{31}$$

Conditional Probability given a partition Π

1. $\{B_k\}_{k=1}^n =: \Pi$ is a (finite) *partition* of Ω, i.e. a family of disjoint, non-empty sets whose union is Ω.

2. The *conditional probability of A given* Π is defined as

$$\mathbb{P}(A|\Pi) := \sum_{k=1}^{n} 1_{B_k}\mathbb{P}(A|B_k) = \begin{cases} \mathbb{P}(A|B_1) & \text{if } \omega \in B_1 \\ \dots & \text{if } \dots \\ \mathbb{P}(A|B_n) & \text{if } \omega \in B_n \ . \end{cases} \tag{32}$$

3. If X only takes values $x_1, \dots, x_n$ and $\Pi_X := \{\{X = x_k\}\}_{k=1}^n$ then $\mathbb{P}(A|X) := \mathbb{P}(A|\Pi_X)$ is the *conditional probability of A given X*.

The σ-algebra generated by a partition

1. If Π finite partition of Ω then $\sigma(\Pi)$ is finite, since

$$\sigma(\Pi) = \{\cup_{P \in I} P | I \subseteq \Pi\},$$

2. $B \in \Pi$ are the smallest (non-empty) sets in $\sigma(\Pi)$.

Example 14

If

$$Y(HH) = 9, Y(HT) = 6 = Y(TH), Y(TT) = 3$$

then the atoms of $\sigma(Y)$ are

$$\{Y = 9\} = \{HH\}, \{Y = 6\} = \{HT, TH\}, \{Y = 3\} = \{TT\}$$

and the elements of $\sigma(Y)$ are

$$\emptyset, \{Y = 9\}, \{Y = 6\}, \{Y = 3\}, \{Y = 9\} \cup \{Y = 6\},$$
$$\{Y = 9\} \cup \{Y = 3\}, \{Y = 6\} \cup \{Y = 3\}, \Omega.$$

Characterisation of finite σ-algebras

Definition 15

An *atom* of a σ-algebra $\mathcal{F}$ is a non-empty $A \in \mathcal{F}$ such that $B \subseteq A, B \in \mathcal{F}$ imply that either $B = A$ or $B = \emptyset$. The family of atoms of $\mathcal{F}$ is denoted by $\mathcal{A}(\mathcal{F})$.

Theorem 16

If $\mathcal{F}$ is a finite σ-algebra on Ω, then:

1. $A_{\mathcal{F}}(\omega) := \cap\{A \in \mathcal{F} : \omega \in A\}$ *is the smallest* $A \in \mathcal{F}$ *s.t.* $A \ni \omega$,
2. $\mathcal{A}(\mathcal{F}) = \{A_{\mathcal{F}}(\omega) : \omega \in \Omega\}$,
3. $\mathcal{A}(\mathcal{F})$ *is a finite partition of* Ω.

The maps $\Pi \mapsto \sigma(\Pi)$ *and* $\mathcal{F} \mapsto \mathcal{A}(\mathcal{F})$ *are inverse of one another, when applied to* finite *partitions/σ-algebras on* Ω.

If $\mathcal{F}$ is finite, we can define $\mathbb{P}(A|\mathcal{F}) := \mathbb{P}(A|\mathcal{A}(\mathcal{F}))$.

Conditional expectation

1. The *conditional* $\mathbb{P}$*-expectation of* W *given* $\mathcal{F}$ ($\mathcal{F}$ finite) is

$$\mathbb{E}[W|\mathcal{F}] = \mathbb{E}^{\mathbb{P}}[W|\mathcal{F}](\omega) := \mathbb{E}^{\mathbb{P}(\cdot|\mathcal{F})(\omega)}[W],$$

2. i.e., if $\omega \in B \in \mathcal{A}(\mathcal{F})$ then $\mathbb{E}^{\mathbb{P}}[W|\mathcal{F}](\omega)$ equals

$$\mathbb{E}^{\mathbb{P}}(W|B) := \mathbb{E}^{\mathbb{P}(\cdot|B)}(W) = \mathbb{E}^{\mathbb{P}}[W1_B]/\mathbb{P}(B).$$

3. So, if $\mathcal{F} = \sigma(X)$ then, on $\{X = x_k\}$, $\mathbb{E}^{\mathbb{P}}[W|X]$ equals

$$\mathbb{E}^{\mathbb{P}}(W|X = x_k) := \mathbb{E}^{\mathbb{P}(\cdot|X=x_k)}(W) = \mathbb{E}^{\mathbb{P}}[W1_{\{X=x_k\}}]/\mathbb{P}(X = x_k).$$

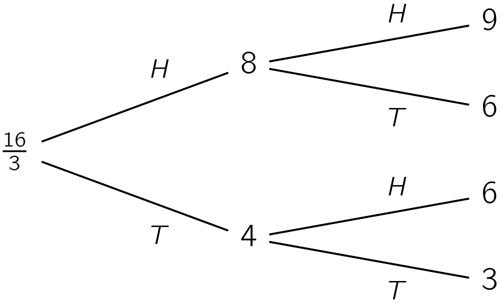
Example with binomial model

Example 17

In binomial model with maturity $N = 2$, assume

$$\mathbb{P}(HH) = \frac{1}{9}, \quad \mathbb{P}(HT) = \frac{2}{9}, \quad \mathbb{P}(TH) = \frac{1}{3}, \quad \mathbb{P}(TT) = \frac{1}{3}.$$

Assume the stock price S is given by



Compute conditional probability and expectation

Let us compute $\mathbb{P}(S_2 = 6|S_1), \mathbb{E}[S_2|S_1]$; first, we compute

$$\mathbb{P}(S_2 = 6|S_1 = 8) = \frac{\mathbb{P}(S_1=8, S_2=6)}{\mathbb{P}(S_1=8)} = \frac{\mathbb{P}(HT)}{\mathbb{P}(HH, HT)} = \frac{\frac{2}{9}}{\frac{1}{9}+\frac{2}{9}} = \frac{2}{3}$$

Since $\mathbb{P}(\cdot|S_1 = 8)$ is a probability, we get that

$$\mathbb{E}(S_2|S_1 = 8) = 6 \cdot \frac{2}{3} + 9 \cdot \frac{1}{3} = 7.$$

Analogously

$$\mathbb{P}(S_2 = 6|S_1 = 4) = \ldots = \tfrac{1}{2}, \quad \mathbb{E}(S_2|S_1 = 4) = 6 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{2} = \tfrac{9}{2},$$

and so

$$\mathbb{P}(S_2 = 6|S_1) = \frac{2}{3} 1_{\{S_1=8\}} + \frac{1}{2} 1_{\{S_1=4\}},$$

$$\mathbb{E}(S_2|S_1) = 7 1_{\{S_1=8\}} + \frac{9}{2} 1_{\{S_1=4\}}.$$

Properties of the conditional expectation

Definition and characterisation of $\mathbb{E}[X|\mathcal{G}]$

1. We have defined $\mathbb{E}[X|\mathcal{H}]$ for any *finite* σ-algebra $\mathcal{H} \subseteq \mathcal{F}$
2. For arbitrary $\mathcal{G} \subseteq \mathcal{F}$, define $\mathbb{E}[X|\mathcal{G}]$, as the limit in L^1 of $\mathbb{E}[X|\mathcal{H}]$ as the *finite* $\mathcal{H} \subseteq \mathcal{G}$ become bigger and bigger.
3. Such limit exists for all $X \in L^1$ and σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, so $\mathbb{E}[X|\mathcal{G}]$ is always defined.
4. $Z := \mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable and satisfies

$$\mathbb{E}[ZW] = \mathbb{E}[XW] \text{ for all } \mathcal{G}\text{-measurable bounded } W, \tag{33}$$

   and it is the *unique* $\mathcal{G}$-measurable rv $Z \in L^1$ with this property.
5. So, $\mathbb{E}[X|\mathcal{G}]$ can alternatively be defined as the $\mathcal{G}$-measurable $Z \in L^1$ s.t. (33) holds (Kolmogorov's definition).

Some properties of the conditional expectation

1. *Linearity*: $\mathbb{E}(X+Z|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) + \mathbb{E}(Z|\mathcal{G})$
2. *Independence*: if X is independent of $\mathcal{G}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$. In particular if c is constant then $\mathbb{E}(c|\mathcal{G}) = c$, and if $\mathcal{G} = \{\emptyset, \Omega\}$ then *any* X satisfies $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}[X]$.
3. *Taking out what is know*: if X is $\mathcal{G}$-measurable then $\mathbb{E}(XZ|\mathcal{G}) = X\mathbb{E}(Z|\mathcal{G})$, and in particular $\mathbb{E}(X|\mathcal{G}) = X$.
4. *Iterated conditioning:* If $\mathcal{H} \subseteq \mathcal{A}$ is a σ-algebra and $\mathcal{G} \subseteq \mathcal{H}$ then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})$; in partic. $\mathbb{E}[\mathbb{E}(X|\mathcal{H})] = \mathbb{E}[X]$.
5. *Jensen inequality*: if ϕ is convex then $\mathbb{E}[\phi(X)|\mathcal{G}] \geq \phi(\mathbb{E}[X|\mathcal{G}])$

The best $\mathcal{G}$-measurable approximation

The best approximation of X with a constant is $\mathbb{E}[X]$, in that $\mathbb{E}[(X-c)^2] \geq \mathbb{E}[(X-\mathbb{E}[X])^2]$ for all $c \in \mathbb{R}$. Analogously:

Theorem 18

If $\mathbb{E}[X^2] < \infty$ then, for all $\mathcal{G}$-measurable C,

$$\mathbb{E}[(X-C)^2] \geq \mathbb{E}[(X-\mathbb{E}[X|\mathcal{G}])^2].$$

Sketch of Proof: Adding and subtracting W we get

$$\mathbb{E}(X-C)^2 = \mathbb{E}(X-W)^2 + \mathbb{E}(W-C)^2 + 2\mathbb{E}(X-W)(W-C),$$

which is $\geq \mathbb{E}[(X-W)^2]$ because $(W-C)^2 \geq 0$, and

$$\mathbb{E}[(X-W)(W-C)|\mathcal{G}] = (W-C)\mathbb{E}[X-W|\mathcal{G}] = 0,$$

which implies $\mathbb{E}[(X-W)(W-C)] = 0$.

Math finance: an intro to Option Pricing

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Table of Contents of Week 8

The RNPF in the multi-period binomial model

The Risk-Neutral Pricing Formula for $(B_t, S_t)_{t=n}^{n+1}$

As in the example, to price in multi-period binomial model: for $\omega \in \Omega_N := \{H, T\}^N$, set $\omega(n) := (\omega_1, \dots, \omega_n)$ and compute

$$U_n(\omega) := \frac{S_{n+1}((\omega(n), H))}{S_n(\omega(n))}, \quad D_n(\omega) := \frac{S_{n+1}((\omega(n), T))}{S_n(\omega(n))},$$

define the risk-neutral *transition-probabilities* $\tilde{P}_n$, $\tilde{Q}_n := 1 - \tilde{P}_n$ by

$$\overline{S}_n(\omega(n)) = \tilde{P}_n(\omega(n))\overline{S}_{n+1}((\omega(n), H)) + \tilde{Q}_n(\omega(n))\overline{S}_{n+1}((\omega(n), T))$$

solving for $\tilde{P}_n$ gives

$$\tilde{P}_n(\omega) = \tilde{P}_n(\omega(n)) := \frac{(1 + R_n) - D_n}{U_n - D_n}(\omega(n))$$

The RNPF for multi-period binomial model

Then, compute $V_n = V_n^{x,G}$ by backward induction

$$\overline{V}_n(\omega(n)) = \tilde{P}_n(\omega(n))\overline{V}_{n+1}((\omega(n), H)) + \tilde{Q}_n(\omega(n))\overline{V}_{n+1}((\omega(n), T)).$$

Let us use the convenient notation, e.g. write $\mathbb{Q}(T|HT)$ for

$$\mathbb{Q}(\{\omega' \in \Omega_N : X_3(\omega') = T\}|\{\omega' \in \Omega_N : X_1(\omega') = H, X_2(\omega') = T\}).$$

Using the probability $\mathbb{Q}$ on $\Omega = \{H, T\}^N$ s.t.

$$\mathbb{Q}(H|\{X(n) = \omega(n)\}) = \tilde{P}_n(\omega(n)) \tag{34}$$

the above formula becomes $\overline{V}_n(\omega(n)) = \mathbb{E}^{\mathbb{Q}|\{X(n)=\omega(n)\}}[\overline{V}_{n+1}]$, i.e.

$$\overline{V}_n = \mathbb{E}^{\mathbb{Q}}[\overline{V}_{n+1}|\mathcal{F}_n]. \tag{RNPF}$$

Proba vs Transition Proba

Here is why in the binomial model $\exists ! \mathbb{Q}$ s.t. eq. (34) holds

Lemma 19

The map $\mathbb{Q} \mapsto \tilde{P}$ *given by eq.* (34) *is a bijection between*

1. *probabilities* $\mathbb{Q}$ *on* $(\Omega, \mathcal{A})$
2. $\mathcal{F}$*-adapted processes* $\tilde{P}$ *on* $(\Omega, \mathcal{A})$ *with values in* $[0, 1]$

Moreover $\mathbb{Q} \sim \mathbb{P} \iff 0 < \tilde{P} < 1$.

Proof: Only for simpler notation consider $N = 2$. Then

$$\mathbb{Q}(HT) = \mathbb{Q}(H)\mathbb{Q}(T|H) = \mathbb{Q}(X_1 = H)\mathbb{Q}(X_2 = T|X_1 = H),$$

so $\mathbb{Q}(HT) = \tilde{P}_0(1 - \tilde{P}_1(H)) = \tilde{P}_0\tilde{Q}_1(H)$. Analogously

$$\mathbb{Q}(HH) = \tilde{P}_0\tilde{P}_1(H), \quad \mathbb{Q}(TT) = \tilde{Q}_0\tilde{Q}_1(T), \quad \mathbb{Q}(TH) = \tilde{Q}_0\tilde{P}_1(T).$$

The FTAP in the multi-period setting

Martingales and Martingale Measures

By RNFP and the tower property of conditional expectation

$$\overline{V}_k = \mathbb{E}^{\mathbb{Q}}[\overline{V}_n|\mathcal{F}_k] \quad \text{for any } 0 \leq k \leq n \leq N.$$

So $\overline{V}_0 = \mathbb{E}^{\mathbb{Q}}[\overline{V}_N]$: we can compute $\overline{V}_0$ *directly* (without backward induction), but we need to first compute $\mathbb{Q}$ from $\mathbb{Q}(\cdot|\mathcal{F}_n)$.

Since $\mathbb{Q}$ is determined by $\overline{S}_n = \mathbb{E}^{\mathbb{Q}}[\overline{S}_{n+1}|\mathcal{F}_n]$, we define:

Definition

1. $Y = (Y_t)_{t\in\mathbb{T}}$ $\mathcal{F}$-adapted is a *martingale* if $Y_t \in L^1(\mathbb{P})$ and $\mathbb{E}[Y_t|\mathcal{F}_s] = Y_s$ for each $s \leq t$.
2. A proba $\mathbb{Q}$ on $\mathcal{F}$ is a *Martingale Measure* (for $\overline{S} := S/B$) if $\overline{S}$ is a $\mathbb{Q}$-martingale; $\mathcal{M}(\overline{S})$ is the set of EMM (Equivalent MM).

Theorem (1st FTAP)

A multi-period market $(B_t, S_t)_{t\in\mathbb{T}}$ is arb. free $\iff \mathcal{M}(\overline{S}) \neq \emptyset$.

Proof: ($\Longrightarrow$ for binomial model) NA $\iff D < 1 + R < U$ $\iff 0 < \tilde{P} < 1 \iff \mathbb{Q} \in \mathcal{M}(\overline{S})$.

($\Longleftarrow$) By contradiction. If G arbitrage: $\overline{V}_N^{0,G} \geq 0$ $\mathbb{P}$ a.s., so $\mathbb{Q}$ a.s.. $\mathbb{R}(\{\overline{V}_N^{0,G} > 0\}) > 0$ if $\mathbb{R} = \mathbb{P}$, so if $\mathbb{R} = \mathbb{Q}$. So $\mathbb{E}^{\mathbb{Q}}(\overline{V}_N^{0,G}) > 0$. However, since $\mathbb{Q}$ is a $\overline{S}$-MM

$$\mathbb{E}^{\mathbb{Q}}(G_n \cdot (\overline{S}_{n+1} - \overline{S}_n)|\mathcal{F}_n) = G_n \cdot \mathbb{E}^{\mathbb{Q}}(\overline{S}_{n+1} - \overline{S}_n|\mathcal{F}_n) = G_n \cdot 0 = 0,$$

and so

$$\mathbb{E}^{\mathbb{Q}}(\overline{V}_N^{0,G}) = \sum_{n=0}^{N-1} \mathbb{E}^{\mathbb{Q}}(G_n \cdot (\overline{S}_{n+1} - \overline{S}_n)) = 0 \quad \lightning$$

Permutation-invariant processes and recombinant trees

Permutation-invariant processes

1. In binomial model AFP Y of Y_N calculated by backward induction
2. The computation time to get $\overline{Y}_n = \mathbb{E}^{\mathbb{Q}}(\overline{Y}_{n+1}|\mathcal{F}_n)$ is proportional to $\#\{H, T\}^n = 2^n$
3. In practice $N \geq 100$; since $2^{100} \sim 10^{30} \sim \infty$, computing Y like this is impossible
4. Solution: consider only *permutation-invariant* (B, S) and work with Markov processes
5. An adapted W is *permutation-invariant* if, for each n,
$$W_n(\omega_1, \dots, \omega_n) = W_n(\sigma(\omega_1), \dots, \sigma(\omega_n))$$
for any permutation σ of $\Omega_n := \{H, T\}^n$
6. If W is permutation-invariant, W_n takes at most $n + 1$ values

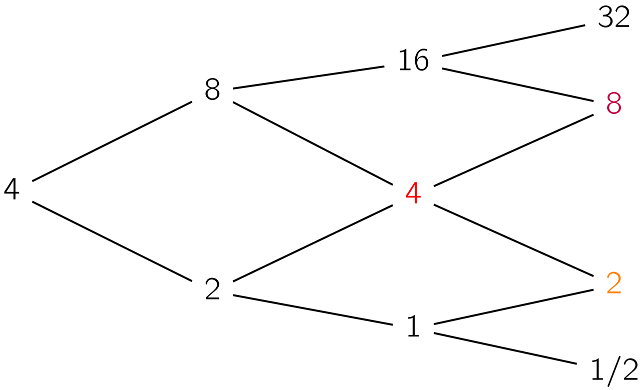
Example of binary tree of permutation-invariant S

One underlying S with $S_0 = 4$, $u = 2 = 1/d$, $r = 0$; then $B = 1$ and S is permutation-invariant and its tree is

$$
\begin{tikzcd}[arrows=dash]
 & & S_2(HH) = 16 \arrow[r] \arrow[dr] & S_3(HHH) = 32 \\
 & S_1(H) = 8 \arrow[ur] \arrow[dr] & & S_3(HHT) = 8 \\
 & & S_2(HT) = 4 \arrow[r] \arrow[dr] & S_3(HTH) = 8 \\
S_0 = 4 \arrow[uur] \arrow[ddr] & & & S_3(HTT) = 2 \\
 & & S_2(TH) = 4 \arrow[r] \arrow[dr] & S_3(THH) = 8 \\
 & S_1(T) = 2 \arrow[ur] \arrow[dr] & & S_3(THT) = 2 \\
 & & S_2(TT) = 1 \arrow[r] \arrow[dr] & S_3(TTH) = 2 \\
 & & & S_3(TTT) = 1/2
\end{tikzcd}
$$

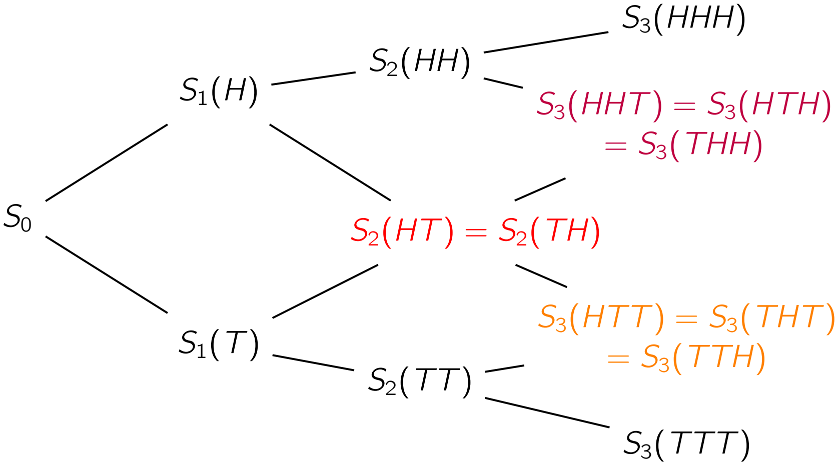
Example of recombinant tree of same S

Represent the same S more simply as



General recombinant tree

If S is permutation-invariant and $N = 3$ its recombinant tree is



Pricing with permutation-invariant (B, S)

1. (Perm.-inv.) adapted proc. $\iff$ (recombinant) bin. trees
2. Assume below that (B, S) is permutation-invariant (e.g. binomial model with constant r, d, u).
3. If $\overline{Y}_N = f_N(S_N)$ then $\overline{Y}_N$ takes at most $N+1$ values!
4. For many derivatives $\overline{Y}_N = f_N(S_N, M_N, L_N, A_N)$ where

   $$M_n := \max_{k \le n} S_k, \quad L_n := \min_{k \le n} S_k, \quad A_n := \tfrac{1}{n+1} \textstyle\sum_{k \le n} S_k.$$

   Such $\overline{Y}_N$ also take few values.
5. For $n < N$: in general $\overline{Y}_n = \mathbb{E}^{\mathbb{Q}}(\overline{Y}_{n+1} | \mathcal{F}_n) = f_n(X(n))$, and $X(n)$ takes $\#\{H, T\}^n = 2^n$ possible values...too many.
6. Examples with $\overline{Y}_N = f_N(S_N)$ show that somehow $\overline{Y}_k = f_k(S_k)$ for some f_k. Why? For which processes W does $\overline{Y}_N = f_N(W_N)$ imply $\overline{Y}_k = f_k(W_k)$ for some f_k?
7. To price $\overline{Y}_N$: write $\overline{Y}_N = f_N(W_N)$ for one such process. Choose W which takes few values.

Independence

Independent sets

Definition 20

Given $(\Omega, \mathcal{A}, \mathbb{P})$, the events $\{F_{i_j}\}_{j=1}^{n} \subseteq \mathcal{A}$ are $\mathbb{P}$-*independent* if

$$\mathbb{P}\left(G_{i_1} \cap \ldots \cap G_{i_n}\right) = \prod_{k=1}^{n} \mathbb{P}\left(G_{i_k}\right). \tag{35}$$

for any $G_{i_j} \in \{F_{i_j}, F_{i_j}^c\}, j = 1, \ldots, n$ The events $\{G_i\}_{i \in I} \subseteq \mathcal{A}$ are independent if $\{G_i\}_{i \in J}$ are independent for every finite $J \subseteq I$.

Intuition? If $\mathbb{P}(B) > 0$, $A, B \in \mathcal{A}$ are indep. $\iff \mathbb{P}(A|B) = \mathbb{P}(A)$.

It is easy to show that $A, B \in \mathcal{A}$ are indep $\iff$ C, D are indep. for every $C \in \sigma(A), D \in \sigma(B)$, which suggests the following def:

Independent random variables

Definition 21

Given $(\Omega, \mathcal{F}, \mathbb{P})$ and sub-σ-algebras $\{\mathcal{G}_i\}_{i\in I}$, we say that $\{\mathcal{G}_i\}_{i\in I}$ are *independent* if, for every choice of $G_i \in \mathcal{G}_i$, $i \in I$, $\{G_i\}_{i\in I}$ are independent.We then say that random vectors $\{X_i\}_{i\in I}$ are *independent* if $\{\sigma(X_i)\}_{i\in I}$ are independent.

Theorem 22

Given rv $X_j : \Omega \to \mathbb{R}^{k_j}$, $j = 1, \dots, n$, the following are equivalent:

1. $(X_j)_j$ *are independent*
2. *for any bounded and Borel functions* $f_j : \mathbb{R}^{k_j} \to \mathbb{R}, j = 1, \ldots, n$

$$\mathbb{E}\left[\prod_{j=1}^{n} f_j(X_j)\right] = \prod_{j=1}^{n} \mathbb{E}\left[f_j(X_j)\right] \tag{36}$$

3. *eq.* (36) *holds for any functions* $f_j : \mathbb{R}^{k_j} \to \mathbb{C}, j = 1, \ldots, n$ *of the form* $f_j(x) = \exp(i\ t_j \cdot x)$, $t_j \in \mathbb{R}^{k_j}$.

Independence and conditional probability

Theorem 23

σ-algebras $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ are $\mathbb{P}$-independent $\iff$ for all $B \in \mathcal{B}$, the rv $\mathbb{P}(B|\mathcal{C})$ is constant. In this case $\mathbb{P}(B|\mathcal{C}) = \mathbb{P}(B)$.

Proof: ($\Longrightarrow$) trivial. ($\Longleftarrow$) Assume $\mathcal{C}$ finite and whose atoms have proba > 0. Then $\mathbb{P}(B|\mathcal{C}) = c$ means $\mathbb{P}(B|C) = c$ for every atom C of $\mathcal{C}$. Thus $\mathbb{P}(B \cap C) = c\mathbb{P}(C)$ for every atom C of $\mathcal{C}$, and so for every $C \in \mathcal{C}$. Taking $C = \Omega$ implies $c = \mathbb{P}(B)$, so $\mathcal{B}, \mathcal{C}$ are $\mathbb{P}$-indep.

Corollary 24

$(X_i)_{i=1}^N$ with values in $\{H, T\}$ are $\mathbb{P}$-independent $\iff$

$$\mathbb{P}(X_{k+1} = H|\sigma(X_1, \ldots, X_k)) \quad \textit{is constant for each } k = 1, \ldots, N-1.$$

Math finance: an intro to Option Pricing

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A motivating example

Pricing a call option fast

1. Let us compute the price V a call option with expiry $N \gg 1$.
2. Suppose that S is *Markov*, i.e., to estimate its future behavior, knowing its whole past = knowing just its present value.
3. Since $\overline{V}_N = v_N(S_N)$ for $v_N(x) := (x-K)^+/(1+r)^N$, the RNPF gives, for $n = N$:
$$\overline{V}_{n-1} = \mathbb{E}^{\mathbb{Q}}(v_n(S_n)|\mathcal{F}_{n-1}) \overbrace{=}^{S \text{ Markov}} \mathbb{E}^{\mathbb{Q}}(v_n(S_n)|S_{n-1}) \tag{37}$$
4. So $\overline{V}_{N-1} = v_{N-1}(S_{N-1})$ for some function v_{N-1}.
5. So (37) also holds for $n = N-1$, etc: if $\overline{V}_n = v_n(S_n)$ then $\overline{V}_{n-1} = v_{n-1}(S_{n-1})$, so $\overline{V}_n = v_n(S_n)$ for all $0 \leq n \leq N$.
6. This way we can compute $\overline{V}_n = v_n(S_n)$ just knowing the (few!) values of $S_n(\omega)$, instead of ω.

Example of pricing a call fast

1. In the N-period binomial model with constant parameters:
2. S is Markov, since

$$\mathbb{E}^{\mathbb{Q}}(f(S_n)|\mathcal{F}_{n-1}) = \tilde{p}f(S_{n-1}u) + (1-\tilde{p})f(S_{n-1}d) = g(S_{n-1}).$$

3. (37) gives $v_{n-1}(S_{n-1}) = \mathbb{E}^{\mathbb{Q}}(v_n(S_n)|S_{n-1})$, and so

$$v_{n-1}(s) = \tilde{p}v_n(su) + (1-\tilde{p})v_n(sd), \qquad \text{for } s \in \operatorname{Im}(S_n).$$

4. The replicating strategy is $G_n = g_n(S_n)$ for

$$g_n(s) := (1+r)\frac{v_{n+1}(su) - v_{n+1}(sd)}{su - sd}, \qquad \text{for } s \in \operatorname{Im}(S_n).$$

5. $\operatorname{Im}(S_n) = \{S_0 u^k d^{N-k} : k = 0, \ldots, n\}$ has only $n+1$ elements.

Markov Processes

Markov Processes

Definition

If $\mathbb{T} \subseteq [-\infty, \infty]$, $X = (X_t)_{t \in \mathbb{T}}$ adapted on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ is *Markov* if

$$\forall f \text{ Borel } \forall s \leq t, \; s, t \in \mathbb{T}, \quad \mathbb{E}(f(X_t)|\mathcal{F}_s) = \mathbb{E}(f(X_t)|X_s). \tag{38}$$

Equivalently:

1. Replace $X_t, s \leq t$ with $(X_{t_1}, \ldots, X_{t_n}), s \leq t_1 \leq \ldots \leq t_n$.
2. Ask only that $\forall f$ Borel $\forall s \leq t, \; s, t \in \mathbb{T}$
$$\mathbb{E}(f(X_t)|\mathcal{F}_s) \quad \text{is } \sigma(X_s)\text{-measurable} \tag{39}$$
3. If $\mathbb{T} = \{0, \cdots, N\}$, we could also have equivalently considered only the $s \leq t$ of the form $t = s + 1$ in eqs. (38) and (39).

Example of a process which is NOT Markov

Let us show that the process

$$M_n := \max_{1 \le k \le n} S_k, \quad n \ge 0$$

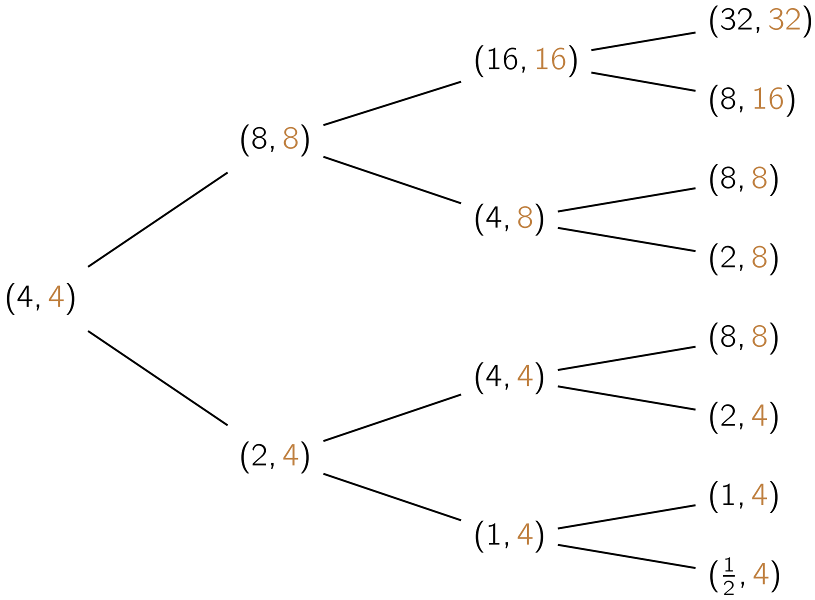
is NOT $\mathbb{Q}$-Markov if (B, S) follows the binomial model with

$$S_0 = 4, \quad u = 2, \quad d = \frac{1}{2}, \quad r = \frac{1}{4},$$

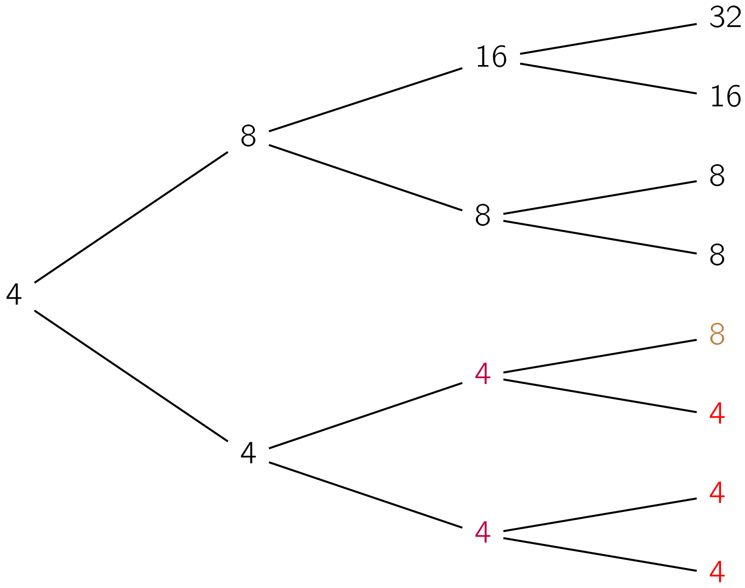
so that

$$\mathbb{Q}(X_{n+1} = H | \mathcal{F}_n) = \tilde{p}_n = \frac{1 + \frac{1}{4} - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{\frac{3}{4}}{\frac{3}{2}} = \frac{1}{2}.$$

Tree of (S, M)



Tree of M



Proof that M is not Markov

The above tree shows that M is NOT Markov, since for some f

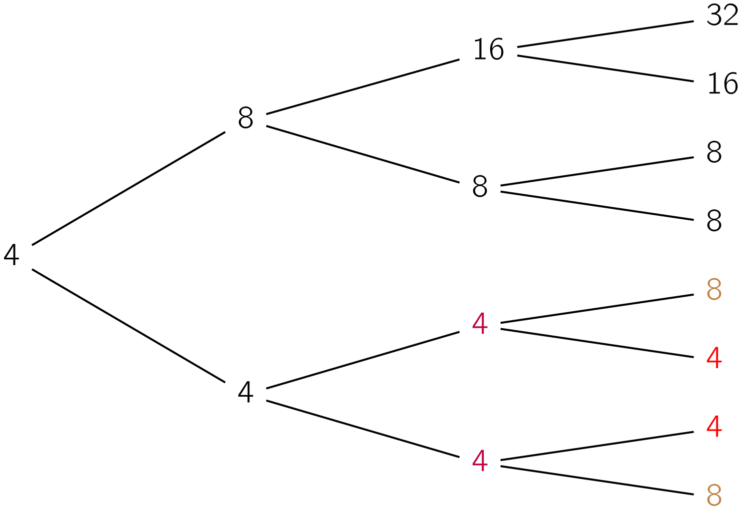
$$\mathbb{E}_2^{\mathbb{Q}} f(M_3)(TH) \neq \mathbb{E}_2^{\mathbb{Q}} f(M_3)(TT). \tag{40}$$

Indeed if $f(8) \neq f(4)$ then

$$\mathbb{E}_2^{\mathbb{Q}} f(M_3)(TH) = \tilde{p} f(8) + (1-\tilde{p}) f(4) \quad \text{does not equal}$$
$$\mathbb{E}_2^{\mathbb{Q}} f(M_3)(TT) = \tilde{p} f(4) + (1-\tilde{p}) f(4)$$

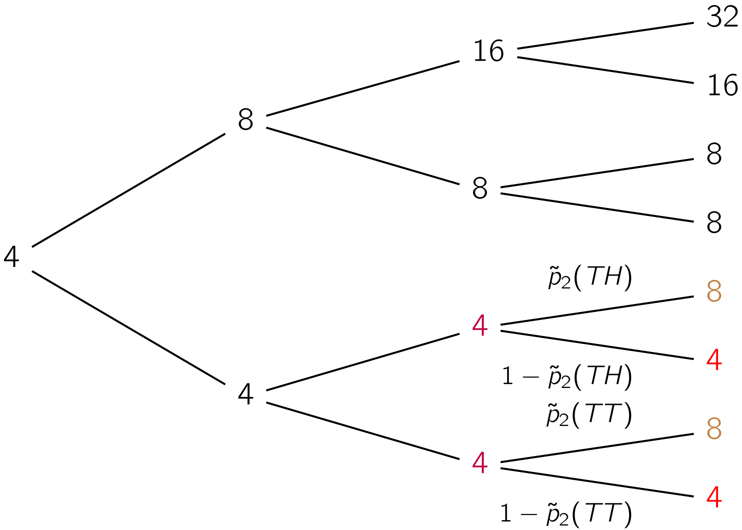
Thus M is not Markov, since (40) shows that $\mathbb{E}_2^{\mathbb{Q}} f(M_3)$ is not $\sigma(M_2)$ measurable (not constant on $\{M_2 = c\}$ for some c).

This process is $\mathbb{Q}$-Markov



$$\tilde{p}f(8) + (1-\tilde{p})f(4) = \tilde{p}f(4) + (1-\tilde{p})f(8) \text{ since } \tilde{p} = \tfrac{1}{2} = 1-\tilde{p}.$$

This process is NOT $\mathbb{Q}$-Markov



$\tilde{p}_2(TH)f(8)+(1-\tilde{p}_2(TH))f(4) \neq \tilde{p}_2(TT)f(8)+(1-\tilde{p}_2(TT))f(4)$
for some f, if $\tilde{p}_2(TH) \neq \tilde{p}_2(TT)$.

How to compute conditional expectations

Independence Lemma

If X, Y random vectors on $(\Omega, \mathcal{A}, \mathbb{P})$, $f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ Borel, $\mathcal{G} \subseteq \mathcal{A}$ σ-algebra, X $\mathcal{G}$-meas., Y indep. of $\mathcal{G}$, then

$$\mathbb{E}(f(X,Y)|\mathcal{G}) = g(X) \quad \text{for} \quad g(x) := \mathbb{E}(f(x,Y)), \quad g : \mathbb{R}^k \to \mathbb{R}.$$

Example: If X, Y IID $\sim U[0,1]$, compute $\mathbb{E}(X \wedge Y|X)= g(X)$ for

$$\begin{aligned} g(x) &:= \mathbb{E}(x \wedge Y) = \mathbb{E}(Y 1_{\{Y \le x\}} + x 1_{\{Y > x\}}) \\ &= \int_0^x y dy + \int_x^1 x dy = x - \frac{1}{2}x^2, \quad \forall x \in [0,1]. \end{aligned}$$

How to prove a process is Markov

Corollary

Assume W $\mathcal{F}$-adapted and $W_{k+1} = f_k(W_k, X_{k+1})$ for all k, f_k Borel and X_{k+1} is indep. of $\mathcal{F}_k$. Then W is Markov and $\mathbb{E}(f(W_{k+1})|\mathcal{F}_k) = g(W_k)$ for all k, where

$$g(w) := \mathbb{E}[f(f_k(w, X_{k+1}))] \tag{41}$$

Proof: For every k we have that

$$\mathbb{E}(f(W_{k+1})|\mathcal{F}_k) = \mathbb{E}((f \circ f_k)(W_k, X_{k+1})|\mathcal{F}_k) = g(W_k). \qquad \square$$

1. In binomial model $S_{n+1} = f_n(S_n, X_{n+1})$, S is $\mathcal{F}$-adapted and X_{n+1} is indep. of $\mathcal{F}_n$ under $\mathbb{P}$ for all n, so S is $\mathbb{P}$-Markov
2. S is also $\mathbb{Q}$-Markov if X_{n+1} is indep. of $\mathcal{F}_n$ under $\mathbb{Q}$ for all n, i.e. $\tilde{P}_n$ is deterministic for all n

Pricing and hedging fast using Markov Processes

Pricing fast using Markov Processes

1. Consider complete market $(B_t, S_t)_{t\in\mathbb{T}}$, $\mathcal{M}(\overline{S}) = \{\mathbb{Q}\}$, and payoff $\overline{Y}_N = f_N(W_N)$ for some $\mathbb{Q}$-Markov W
2. By the RNPF $(Y_n)_n$ satisfies, for $n = N-1$,

$$\overline{Y}_n = \mathbb{E}^{\mathbb{Q}}[\overline{Y}_{n+1}|\mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[f_{n+1}(W_{n+1})|\mathcal{F}_n] = f_n(W_n) \tag{42}$$

3. By backward induction get $\overline{Y}_n = f_n(W_n)$ *for every n*
4. To compute $\overline{Y}_n, f_n$ from $\overline{Y}_{n+1}$, we just need W_n, W_{n+1}
5. To find $(\overline{Y}_n)_n$ we need to *explicitly* compute $(f_n)_n$ by backward induction, starting from f_N
6. Normally $Y_N = f_N(W_N)$ and $W := (S, C)$ is $\mathbb{Q}$-Markov.
7. If $Y_N = f_N(C_N)$ and C not Markov, can still use $W := (S, C)$.

Hedging fast using Markov Processes

1. In binomial model, assume $\exists W$ adapted s.t.

$$\overline{Y}_N = f_N(W_N),\ W_{k+1} = h_k(W_k, X_{k+1}),\ \overline{S}_k = s_k(W_k) \quad \forall k \leq N$$

   for some f_N, h_k, s_k Borel.

2. If $\tilde{P}_n$ is deterministic, $(X_n)_n$ are $\mathbb{Q}$-indep., so W is $\mathbb{Q}$-Markov, so $\overline{Y}_k = f_k(W_k)$ for all $k \leq N$ for some $(f_k)_k$.
3. $G_n(\omega) = G_n(\omega(n)) = \frac{\overline{Y}_{n+1}(\omega(n)H) - \overline{Y}_{n+1}(\omega(n)T)}{\overline{S}_{n+1}(\omega(n)H) - \overline{S}_{n+1}(\omega(n)T)}$
4. Since $W_{n+1}(\omega(n)z) = h_n(W_n(\omega(n)), z), z \in \{H, T\}$, get that $G_n(\omega(n)) = g_n(W_n(\omega(n))$ for

$$g_n(w) := \frac{f_{n+1}(h_n(w, H)) - f_{n+1}(h_n(w, T))}{s_{n+1}(h_n(w, H)) - s_{n+1}(h_n(w, T))}.$$

5. Also $G_n = g_n(W_n)$ can be computed by looking only at W_n

Example of Markov pricing

Markov pricing a lookback option

1. Let us compute the price V a (floating) lookback put option, which has payoff $V_N := M_N - S_N$, where $M_n := \max_{i=0,\dots,n} S_i$.
2. We work in the N-period binomial model for $S = (S_n)_{n=0}^N$, with parameters $S_0, u, d, r > 0$ which satisfy $0 < d < 1 + r < u$.
3. Write $\overline{V}_N = f_N(S_N, M_N)$ for $f_N(s, m) := \frac{1}{(1+r)^N}(m - s)$.
4. Is $W := (S, M)$ $\mathbb{Q}$-Markov? $\exists h_n$ s.t. $W_{n+1} = h_n(W_n, X_{n+1})$?
5. If $\frac{S_{n+1}}{S_n} = q(X_{n+1})$, where $q(x) := \begin{cases} u, & \text{if } x = H \\ d, & \text{if } x = T \end{cases}$, then

$$S_{n+1} = S_n q(X_{n+1}), \quad M_{n+1} = M_n \vee S_{n+1} = M_n \vee (S_n q(X_{n+1}))$$

and so we can take $h_n(s, m, x) := (sq(x), m \vee (sq(x)))$.

Pricing and replication equations

1. Since $\overline{V}_n = \mathbb{E}^{\mathbb{Q}}[\overline{V}_{n+1}|\mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[f_{n+1}(W_{n+1})|\mathcal{F}_n] = f_n(W_n)$ we conclude that $\overline{V}_n = f_n(W_n)$ *for every* n.

2. By the independence lemma $\mathbb{E}^{\mathbb{Q}}[f(W_{n+1})|\mathcal{F}_n] = g(W_n)$, where

$$g(s,m) := \mathbb{E}^{\mathbb{Q}}[f(h_n(s,m,X_{n+1}))] = \tilde{p}f(su, m\vee(su)) + \tilde{q}f(sd, m\vee(sd))$$

   where $\tilde{p} := \tilde{p}_n := \mathbb{Q}(X_{n+1} = H|\mathcal{F}_n) = \frac{(1+r)-d}{u-d}$, $\tilde{q} := 1 - \tilde{p}$.

3. The pricing functions are given by $f_N(s,m) := \frac{1}{(1+r)^N}(m-s)$,

$$f_n(s,m) := \tilde{p}f_{n+1}(su, m\vee(su)) + \tilde{q}f_{n+1}(sd, m\vee(sd)), n = 0, \dots, N-1$$

4. The replicating strategy is $G_n = g_n(S_n, M_n)$ for

$$g_n(s,m) := (1+r)\frac{f_{n+1}(su, m\vee(su)) - f_{n+1}(sd, m\vee(sd))}{su - sd}.$$

Pricing the call in the binomial model

Pricing the call in the binomial model

1. In the arb.-free binomial model $(B_t, S_t)_{t=0,1,\dots,N}$ with constant coef. u, d, r, the price C of the call option satisfies

$$C_n = \mathbb{E}^{\mathbb{Q}}[\frac{B_n}{B_N}(S_N - K)^+|\mathcal{F}_n] = \frac{\mathbb{E}^{\mathbb{Q}}[\left(S_n \frac{S_N}{S_n} - K\right)^+ |\mathcal{F}_n]}{(1+r)^{N-n}}.$$

2. $R_i := S_i/S_{i-1}$ are IID, so $S_N/S_n = \Pi_{i=n+1}^N R_i$ is indep. of $\mathcal{F}_n^S = \mathcal{F}_n^R$ (under $\mathbb{Q}$). Trivially S_n is $\mathcal{F}_n^S$-measurable.

3. The indep. lemma gives $C_n = c(n, S_n)$ where

$$c(n,x) := \frac{1}{(1+r)^{N-n}} \mathbb{E}^{\mathbb{Q}}[(x\Pi_{n+1}^N R_i - K)^+]. \tag{43}$$

4. Since the $(R_i)_i$ are IID under $\mathbb{Q}$ we get

$$c(n,x) = \frac{1}{(1+r)^{N-n}} \sum_{j=0}^{N-n} \binom{N-n}{j} \tilde{p}^j (1-\tilde{p})^{N-n-j} (xu^j d^{N-n-j} - K)^+.$$

Hedging the call in the binomial and BS models

Hedging the call in the binomial and BS models

1. In the binomial mode, the hedging strategy G_n is given by the delta-hedging formula

$$G_n = \frac{c(n+1, S_n u) - c(n+1, S_n d)}{S_n u - S_n d}, \tag{44}$$

2. eq. (43) shows that $c(n, \cdot)$ is increasing, so $G_n \geq 0$.
3. eq. (44) suggests that the hedging strategy G in the BS model should be

$$G_t = \left(\frac{\partial c}{\partial x}\right)(t, S_t). \tag{45}$$

Math finance: an intro to Option Pricing

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Table of Contents of Week 10

THIS WEEK'S CONTENTS ARE NON-EXAMINABLE

From the Binomial to the Black and Scholes model

Using a model in continuous-time

1. We want to consider models with price movements at arbitrary dates and of arbitrary size
2. We want to work in continuous time to use the power of calculus
3. How should we model $(B_t, S_t)_{t \in [0,T]}$?
4. It is quite intuitive that B, S should be continuous processes
5. Let us take $\lim_{N \to \infty}$ of binomial models (B^N, S^N) with constant coefficients r^N, u^N, d^N to find the BS model

The bond in the BS model

1. Consider the binomial model $(B_t^N, S_t^N)_{t \in \pi^N}$, where
$$\pi^N := \{t_i^N\}_{i=0}^N, \quad \text{where } t_i^N := i\Delta, \Delta := T/N.$$
2. If r is the un-compounded interest rate *per unit time*, then investing 1 at time t_i^N returns $(1 + r\Delta)$ at time t_{i+1}^N
3. Thus we take $r^N := r\Delta = rT/N$, and compounding we get
$$B_T^N = (1 + rT/N)^N \to \exp(rT) = B_T \text{ as } N \to \infty.$$
4. We should consider a bank account $B_t = \exp(rt)$ with constant short-rate r.
5. Another reason: in discrete time $B_{t+1} = B_t(1 + r)$, i.e. $\Delta B_t = B_t r \Delta t$, so in continuous time take $dB_t = B_t R_t dt$.

The stock in the BS model

1. Since $R_i^N := S^N_{t_i^N}/S^N_{t_{i-1}^N}$ are IID and $S^N_{t_k^N} = S_0 \Pi_{i=1}^k R_i^N$, the $Y_i^N := \ln(R_i^N)$ are IID, and $X^N := \ln(S^N)$ satisfies
$$X^N_{t_k^N} = X_0^N + \textstyle\sum_{i=1}^k Y_i^N \quad \text{ with } X_0^N := \ln(S_0).$$
2. For appropriate u^N, d^N, apply slight extensions of CLT to get
$$X_t^N - X_s^N - \mu(t-s) \longrightarrow \sigma\cdot(W_t - W_s), \quad W_t - W_s \sim \mathcal{N}(0, t-s),$$
Thus $S_t^N \to S_t := S_0 \exp(\mu t + \sigma W_t)$ with $W_t \sim \mathcal{N}(0,t)$
3. If $r \leq s \leq t$ then $X_s^N - X_r^N$ and $X_t^N - X_s^N$ are independent, so their limit $X_t - X_s$ and $X_s - X_r$ are independent, so $W_t - W_s$ and $W_s - W_r$ are independent.
4. Analogously can prove that W has *independent increments*

Brownian Motion

Definition

$W = (W_t)_{t \geq 0}$ is a BM (*Brownian Motion*) if:

1. W has independent increments.
2. $W_0 = 0$, $W_t - W_s \sim \mathcal{N}(0, (t-s))$ for $0 \leq s < t$
3. W is continuous.

Generalisations of BM:

1. $W = (W^i)_{i=1}^n$ is a n-dim. BM if $W^1, \ldots, W^n$ are indep. BMs.
2. Given filtration $\mathcal{F}$, W is a $\mathcal{F}$-BM if item 1 is replaced by

$$W \text{ is } \mathcal{F}\text{-adapted}, \quad W_t - W_s \text{ is indep. of } \mathcal{F}_s \text{ for } 0 \leq s \leq t. \quad (1')$$

3. If $W_0 =$ an arbitrary rv X, we call W a BM started at X.

The Black-Scholes model

Definition

The BS (*Black-Scholes*) model (B, S) is

$$B_t := \exp(rt), \quad S_t := S_0 \exp(\mu t + \sigma W_t), \quad \mathcal{F} = \mathcal{F}^W \tag{43}$$

where W is a BM and $S_0, \sigma > 0, r > -1, \mu \in \mathbb{R}$.
Such S is called a GBM (*Geometric Brownian Motion*).

1. Take $s < t < u$. Since

$$\log\left(S_t/S_s\right) = \mu(t-s) + \sigma(W_t - W_s),$$

   we get that $\dfrac{S_t}{S_s}$ and $\dfrac{S_u}{S_t}$ are independent

2. Warning: $S_t - S_s$ and $S_u - S_t$ are **not** independent

Some Brownian martingales and Markov processes

W and $(W_t^2 - t)_t$ are martingales

Theorem 26

W is $\mathcal{F}$ BM $\Longrightarrow$ W and $(W_t^2 - t)_{t\geq 0}$ are $\mathcal{F}$-martingales.

We will write $\mathbb{E}_s(Y)$ instead of $\mathbb{E}(Y|\mathcal{F}_s)$, as common.

Proof: Add and subtract W_s to get

$$\mathbb{E}_s W_t = \mathbb{E}_s[W_t - W_s] + \mathbb{E}_s W_s = \mathbb{E}[W_t - W_s] + W_s = W_s$$

then do the same to compute $\mathbb{E}_s(W_t^2 - t)$ as

$$\begin{aligned}
&\mathbb{E}_s(W_t - W_s)^2 + \mathbb{E}_s[2W_s(W_t - W_s)] + \mathbb{E}_s[W_s^2] - t \\
&= \mathbb{E}(W_t - W_s)^2 + 2W_s\mathbb{E}[W_t - W_s] + W_s^2 - t \\
&= t - s + 2W_s 0 + W_s^2 - t = W_s^2 - s \qquad \square
\end{aligned}$$

Markov processes

Theorem 30

If W is $\mathcal{F}$-BM, $X_t := a(t, W_t), W_t := b(t, X_t), t \geq 0$, a, b Borel $\implies X$ is a $\mathcal{F}$-Markov process; in particular, BM and GBM are $\mathcal{F}$-Markov processes.

Proof: Add and subtract W_s from W_t to get that

$$\mathbb{E}_s[f(X_t)] = \mathbb{E}_s[f \circ a(t, W_t)] = g(W_s),$$

for $g(w) := \mathbb{E}[f \circ a(t, W_t - W_s + w)]$, so X is Markov since

$$\mathbb{E}_s[f(X_t)] = h(X_s), \qquad \text{for } h(x) := g(b(s, x)).$$

In particular, BM is Markov, and such is GBM S since

$$S_t := S_0 \exp(\mu t + \sigma W_t), \quad W_t = \frac{1}{\sigma}\left(\log\left(S_t/S_0\right) - \mu t\right).$$

The EMM in the BS model

Convergence under the risk-neutral measure

1. We showed binomial model (B^N, S^N) converges *in law* to the BS model (B, S)

2. Convergence in law with respect to proba $\mathbb{P}$ describing world events (*physical measure*), it gives the law of (B, S) under $\mathbb{P}$

3. Look at the convergence in law of (B^N, S^N) under its EMM $\mathbb{Q}^N$ to determine the law of (B, S) under its EMM $\mathbb{Q} =: \widetilde{\mathbb{P}}$.

4. Each R_i^N takes values u^N, d^N under $\mathbb{Q}^N$, and the $(R_i^N)_i$ are IIDs also under $\mathbb{Q}^N$, since each

$$\tilde{p}^N = \frac{(1 + r^N) - d^N}{u^N - d^N}$$

is constant.

The BS model under the risk-neutral measure

1. Thus, we still get $S_t := S_0 \exp(\tilde{\mu} t + \tilde{\sigma} \tilde{W}_t)$, where $\tilde{W}$ is a BM under $\mathbb{Q}$, but $\tilde{\mu}, \tilde{\sigma}$ may differ from μ, σ.

2. It turns out that $\tilde{\sigma} = \sigma$.

3. $\tilde{\mu}$ is determined by asking that $\overline{S} = S/B$ is a $\mathbb{Q}$-martingale. Applying the independence lemma shows that this holds iff $\tilde{\mu} = r - \sigma^2/2$.

4. So, in the BS model, the law of S under the EMM $\mathbb{Q}$ is

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma \widetilde{W_t} \right), \tag{50}$$

   where $\widetilde{W}$ is a BM under the EMM $\mathbb{Q}$.

Risk-neutral pricing in the BS model

The Feynman-Kac formula

1. The BS model (B, S) has a unique EMM $\mathbb{Q}$. By the RNPF, the price V of a derivative with payoff $V_T = f(S_T)$ satisfies

$$V_t/B_t = \mathbb{E}^{\mathbb{Q}}\left[f(S_T)/B_T \middle| \mathcal{F}_t\right].$$

2. Since $B_t = e^{rt}, S_t/B_t = S_0\mathcal{E}_t(\sigma W^{\mathbb{Q}})$, so we get that

$$S_T/S_t = \exp((r - \sigma^2/2)(T - t) + \sigma(W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}})),$$

so S_T/S_t is indep. of $\mathcal{F}_t$ (under $\mathbb{Q}$), and $B_t/B_T = e^{-r(T-t)}$.

3. The independence lemma yields the Feynman-Kac formula

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T} f\left(S_t \frac{S_T}{S_t}\right) \middle| \mathcal{F}_t\right] = v(t, S_t)$$

$$\text{where} \quad v(t, x) := \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T} f\left(x \frac{S_T}{S_t}\right)\right].$$

Markov Pricing in the BS model

1. We write $\mathbb{E}^{\mathbb{Q}}(g(S_T)|S_t = x) := h(x)$, where h is s.t. $\mathbb{E}^{\mathbb{Q}}(g(S_T)|\mathcal{F}_t) = h(S_t)$ (it exists since S is $\mathbb{Q}$-markov).
2. More generally, if a derivative has payoff $V_T = f(X_T)$ for some $\mathbb{Q}$-markov process X then

   $$V_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T} f(X_T) \middle| \mathcal{F}_t\right] = v(t, X_t)$$

   for some v which can be calculated either by evaluating an integral, or by solving a PDE (as we will see).
3. In the BS and binomial models many processes are $\mathbb{Q}$-Markov.
4. The proof that a process is $\mathbb{Q}$-Markov in BS model is normally very similar to the one in binomial model.

Pricing the binaries in the BS model

Pricing the cash binary in BS model

- The payoff of the *cash* binary option is $M_T = 1_{\{S_T \geq K\}}$, so its price is $M_t = m(t, S_t)$, for

$$m(t,x) := e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[1_{\{xS_T/S_t \geq K\}}\right]. \tag{51}$$

- Define $Y := -\frac{W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}}}{\sqrt{T-t}} \underset{\text{under } \mathbb{Q}}{\sim} \mathcal{N}(0,1)$, $\tau := T - t$ and write

$$xS_T/S_t = x\exp\left(-\sigma\sqrt{\tau}\,Y + (r - \sigma^2/2)\,\tau\right) =: h(x,t,Y) =: h(Y)$$

- It follows that $h(Y) \geq K$ is equivalent to

  1. $-\sigma\sqrt{\tau}\,Y + (r - \frac{\sigma^2}{2})\tau = \ln\left(\frac{S_T}{S_t}\right) \geq \ln\left(\frac{K}{x}\right) = -\ln\left(\frac{x}{K}\right)$
  2. i.e. $Y \leq d_- := d_-(\tau, x) := \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau\right)$

- Thus $m(t,x) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[1_{\{h(Y)\geq K\}}\right] = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[1_{\{Y \leq d_-(\tau,x)\}}\right]$.

Pricing the asset binary in BS model

1. In terms of the CDF $\mathcal{N}(y) = \mathbb{Q}(Y \leq y)$ of Y we can write

$$m(t, x) = e^{-r\tau}\mathbb{Q}(Y \leq d_-(\tau, x)) = e^{-r\tau}\mathcal{N}(d_-(\tau, x)).$$

2. The payoff of the *asset* binary option is $A_T = S_T 1_{\{S_T \geq K\}}$, so its price is $A_t = a(t, S_t)$, for

$$a(t, x) := e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[h(Y)1_{\{h(Y)\geq K\}}\right]. \tag{52}$$

3. Using the density of Y we get that $a(t, x)$ equals

$$e^{-r\tau}\int_{-\infty}^{d_-} x \exp\left(-\sigma\sqrt{T-t}y + (r - \tfrac{\sigma^2}{2})(T-t)\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy.$$

Binaries and $d_\pm$

1. Change variable $z := y + \sigma\sqrt{\tau}$, so $dz = dy$ and get

$$a(t,x) = \int_{-\infty}^{d_+} x \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = x\mathcal{N}(d_+),$$

   for

$$d_+(\tau, x) := d_-(\tau, x) + \sigma\sqrt{\tau}.$$

2. Both d_+ and d_- can be conveniently expressed by the formula

$$d_\pm(T-t, x) := \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right).$$

Pricing the call in the BS model

The BS pricing formula

1. The call option has payoff $C_T = (S_T - K)^+$.
2. Since $X^+ = X1_{\{X \geq 0\}}$, it follows that $C_T = A_T - KM_T$, where

$$A_T = S_T 1_{\{S_T \geq K\}}, \quad M_T = 1_{\{S_T \geq K\}},$$

3. By the law of one price $C_t = c(t, S_t)$, where
   $c(t,x) = a(t,x) - Km(t,x)$, so

$$c(t,x) = x\mathcal{N}(d_+) - Ke^{-r(T-t)}\mathcal{N}(d_-). \tag{53}$$

The Greeks

The Greeks

1. The partial derivatives of c are called the *Greeks*, they describe how c changes when its arguments change.

2. The dependence on S is measured by $\Delta := \partial_x c$. If we sell one call option, to hedge we hold $H_t = \partial_x c(t, S_t)$ shares; this is called *delta-hedging*.

3. For the call $c(t,x) = x\mathcal{N}(d_+) - Ke^{-r(T-t)}\mathcal{N}(d_-)$, so $\Delta = \mathcal{N}(d_+) > 0$.

4. Second-order effects on x involve $\Gamma := \partial^2_{x^2} c > 0$.

5. The time-dependence is given by $\Theta := \partial_t c$.

6. Volatility dependence is given by $\nu := \partial_\sigma c$, called *Vega*. This allows to calculate the *implied volatility surface*.

7. The sensitivity to interest rates is given by $\rho := \partial_r c$.

Delta-neutral and long gamma portfolio

1. At time t buy one call, hedge it shorting $\partial_x c(t, S_t)$ shares.
2. We have in the bank

$$M := M(t, S_t) := S_t \cdot \partial_x c(t, S_t) - c(t, S_t) = e^{-r(T-t)} K \mathcal{N}(d_-) > 0.$$

3. If $x := S_t$, our total wealth is

$$V(x) := V(t, x) = c(x) - x \partial_x c(x) + M(x) = 0.$$

4. If S_t were to jump to y, then $V(x)$ would become

$$V(y) := c(y) - y \partial_x c(x) + M(x) = c(y) - (c(x) + (y - x) \partial_x c(x)).$$

5. Thus $(\partial_y V)(x) = 0, \quad \partial^2_{y^2} V \geq 0.$
6. Such portfolio is called *delta-neutral*, *long gamma*