# Learning and Equilibrium Structure in Discrete Colonel Blotto: A Hart (2008) Deep Dive with Regret-Matching Implementation

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Department / Course: Dynamics of Iterated Games (Project 3)

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**Abstract**

This report studies a discrete Colonel Blotto game and connects two complementary perspectives: (i) Hart's analytic characterisation of equilibrium structure in symmetric discrete Blotto games via Colonel Lotto, General Lotto, and feasibility constructions; and (ii) regret-based learning as a computational method to approximate equilibrium play. We focus on the finite instance with $S = 5$ soldiers and $N = 3$ battlefields, which yields a $21 \times 21$ zero-sum matrix game. We implement regret-matching in self-play and verify convergence in a set-robust way using exploitability (Nash inequalities), value convergence, and the induced marginal troop distribution, which can be compared to Hart's predicted equilibrium family in the $(m, m + 1)$ regime.

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# 1 Roadmap and alignment with the project guideline

This report is structured to address the project guideline questions:

- **Q1–Q2:** Define the discrete Colonel Blotto game and represent the $(S, N) = (5, 3)$ instance as a finite matrix game.
- **Q4 (Hart):** Present Hart's reduction pipeline (Blotto $\to$ Lotto $\to$ General Lotto), the discrete General Lotto solution, and the feasibility mechanism that constructs optimal Blotto strategies in symmetric games.
- **Q5:** Describe regret-matching and the convergence notions relevant for set-valued Nash equilibria.
- **Q6:** Implement regret-matching for $(5, 3)$ and verify approximate Nash behaviour using exploitability, value convergence, and Hart-consistent marginal structure.

# 2 Page-by-page writing plan (20 pages total)

This section is a tight page-by-page checklist for writing the report in approximately 20 pages (excluding references and any appendix). Each "page" below corresponds to roughly one page of final PDF output.

### Page 1: Title + Abstract ($\frac{1}{2}$) + Roadmap ($\frac{1}{2}$)

- Write a 4–6 sentence abstract: (i) define the Blotto instance, (ii) summarise Hart's reduction and equilibrium characterisation, (iii) implement regret-matching on $(S, N) = (5, 3)$, (iv) verify value and set-valued equilibrium via exploitability and marginals.
- Include a short roadmap mapping sections to Q1/Q2/Q4/Q5/Q6.

### Page 2: Q1 — Colonel Blotto definition (formal)

- Define discrete Colonel Blotto with $S$ soldiers, $N$ battlefields.
- Pure strategy: integer allocation $x \in \mathbb{Z}_{\geq 0}^N$ with $\sum_{i=1}^N x_i = S$.
- Payoff rule: win/loss per battlefield (state your tie-breaking convention) and note zero-sum structure.
- Give one concrete example allocation for $(S, N) = (5, 3)$.

### Page 3: Q1 — Symmetry and why equilibrium is a set

- Explain exchangeability across battlefields: relabellings do not change strategic meaning.
- Explain why Nash equilibria need not be unique; equilibrium is naturally set-valued.
- Bridge to Hart's symmetrisation idea: random permutation "washes out" labels (no heavy proof yet).

### Page 4: Q2 — Enumerating pure strategies for $(5, 3)$

- Derive $K = \binom{S+N-1}{N-1} = \binom{7}{2} = 21$ using stars-and-bars.
- List the 21 allocations (or place full list in an appendix; show representative subset here).
- Define indexing $s_1, \ldots, s_{21}$.
- Deliverable: a small table snippet mapping *ID* $\leftrightarrow$ *allocation*.

### Page 5: Q2 — Constructing the payoff matrix $A$

- Define the matrix game: $A_{ij} = u(s_i, s_j)$ where $u$ is your Blotto payoff.
- Explain how to compute $A_{ij}$ from battlefield comparisons.
- Optionally show a heatmap/figure of $A$; state "we have a $21 \times 21$ finite zero-sum game".

### Page 6: Q4 — Hart pipeline overview (1-page map)

- Include a diagram/step list:

  Discrete Blotto $B(A, B; K) \rightarrow$ Colonel Lotto $L(A, B; K) \rightarrow$ General Lotto $\Gamma(a, b) \rightarrow$ Feasibility $\Rightarrow$ Blo

- State what you will use from Hart: (i) value and form of optimal marginals in the $(m, m+1)$ regime, (ii) feasibility constraints enabling implementation back in Blotto.

### Page 7: Q4 — Colonel Blotto vs Colonel Lotto (definitions + intuition)

- Define Hart's Blotto payoff (average sign across battlefields) and Colonel Lotto payoff (compare random battlefield draws).
- Explain intuition: Lotto collapses a high-dimensional partition into a 1D marginal RV.
- Place Hart's symmetrisation idea here (the $\sigma$-construction / permutation mixing).

### Page 8: Q4 — General Lotto game $\Gamma(a, b)$

- Define General Lotto: players choose integer-valued RVs $X, Y$ with fixed means $\mathbb{E}[X] = a$, $\mathbb{E}[Y] = b$.
- Define payoff: $H(X, Y) = \mathbb{P}(X > Y) - \mathbb{P}(X < Y)$.
- Explain: this is a relaxation; later we ask which solutions are feasible under partitions.
- Optional (short): mention the continuous analogue for intuition (keep brief).

### Page 9: Q4 — Hart's discrete General Lotto solution: case split table

- Present a clean "case table" of Hart's equilibrium/value across regimes: symmetric $a = b$, $\lfloor a \rfloor < \lceil b \rceil$, and remaining "same integer part" cases.
- Reference Hart's main theorem statements (no need to reprove; explain and interpret).

## Page 10: Q4 — Zoom into your regime: $(m, m+1)$ interval structure

- Compute your parameters for $(S, N) = (5, 3)$:

$$a = \frac{A}{K} = \frac{5}{3} \in (1, 2), \qquad m = 1, \qquad \alpha = a - m = \frac{2}{3}.$$

- Explain Hart's predicted structure in this regime (often expressed as mixtures of structured distributions / parity-restricted uniforms) and emphasise set-valuedness (e.g. convex families such as $\mathrm{conv}\{U_o^m, U_e^m\}$ when applicable).
- This is your "Hart $\to$ testable predictions" page.

## Page 11: Q4 — Feasibility (implementing marginals as partitions)

- Define $(A, K)$-feasible random variables: marginals induced by a distribution over $K$-partitions.
- Explain why General Lotto is a relaxation: some RVs with mean $a$ are not feasible under partitions.
- Give one short infeasibility intuition example (parity / support constraints).

## Page 12: Q4 — Hart's feasibility engine (Proposition 6) + what it does

- State Proposition 6 at a high level: existence of a matrix whose rows are partitions and whose columns have prescribed values.
- Explain the matrix construction idea and how it guarantees feasibility.
- Include a tiny toy illustration (e.g. $3 \times 3$) showing "columns correspond to marginals".

## Page 13: Q4 — From feasibility to symmetric discrete Blotto optimality (Theorem 7)

- State Theorem 7: Proposition 6 yields optimal strategies for every symmetric discrete Blotto game $B(A, A; K)$.
- Interpret: once the Lotto-optimal marginals are feasible, Hart constructs an optimal Blotto mixture.
- Connect explicitly to your case $A = 5$, $K = 3$.

## Page 14: Q5 — Regret definitions and regret-matching update rule

- Define external regret.
- Present regret-matching update: regrets $R_i(t)$, probabilities proportional to $\max\{R_i(t), 0\}$, uniform fallback if all non-positive.
- Emphasise time-averaging: compare $\bar{p}_T$ rather than the last iterate.
- Cite Neller–Lanctot for algorithmic framing.

### Page 15: Q5 — Convergence notions for set-valued NE

- General: no-regret ⇒ empirical play approaches the correlated equilibrium set.
- Zero-sum specialisation: low regret implies near-minimax value; exploitability provides a practical NE certificate.
- Key conceptual point: we do not require convergence to a single point; we verify Nash conditions and Hart-consistent structure.

### Page 16: Q6 — Implementation details (enumeration, payoff, RM loop)

- Enumerate the 21 strategies; build payoff matrix $A$.
- Implement regret-matching self-play.
- State experimental choices: iterations $T$, averaging scheme, seeds.
- Include a pseudocode box (about half a page).

### Page 17: Q6 — Convergence diagnostics (regret + payoff)

- Plot 1: average external regret vs $T$.
- Plot 2: running average payoff vs $T$ (symmetric case $\to 0$).
- Explain what each plot indicates.

### Page 18: Q6 — Nash verification robust to non-uniqueness (exploitability)

- Define exploitability / Nash-gap for $(\bar{p}_T, \bar{q}_T)$:
$$\varepsilon_A(\bar{p}_T, \bar{q}_T) = \max_i (A\bar{q}_T)_i - \bar{p}_T^\top A\bar{q}_T, \qquad \varepsilon_B(\bar{p}_T, \bar{q}_T) = \bar{p}_T^\top A\bar{q}_T - \min_j (\bar{p}_T^\top A)_j.$$
- Plot 3: $\varepsilon_A + \varepsilon_B$ vs $T$.
- Interpret: near-zero exploitability ⇒ near Nash equilibrium (as a set).

### Page 19: Q6 — Verifying Hart specifically (marginals + equilibrium family)

- Compute induced marginal distribution of soldiers in a random battlefield under $\bar{p}_T$.
- Plot histogram of the marginal $\widehat{X}$ and compare against Hart's predicted family/shape for the $(m, m+1)$ regime (e.g. mixtures / parity-restricted structures when applicable).
- Multi-seed experiment: show different runs converge to different points but share: (i) value near Hart's value, (ii) low exploitability, (iii) similar Hart-consistent marginals.

### Page 20: Conclusion + limitations + next steps

- Summarise what you established: Hart's structure (set-valued) and regret-matching empirical verification.
- Limitations: discretisation vs Hart's analytic setting; convergence rate; scalability.
- Next steps (brief): larger $S$, approximate best responses, or CFR comparison (optional).

## References