Linear Stability of the Rotating Disk Boundary Layer

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- Why study rotating disks?
- Why study oscillatory motion on the disk?

Introduction to the Rotating Disk - Setup

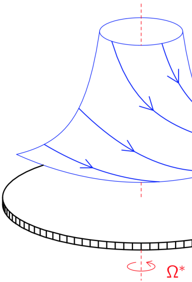


Figure 1 : Rotating Disk Profile

Introduction to the Rotating Disk - History

- Why is it an interesting problem?
  - Canonical example of a three-dimensional boundary layer.
  - Approximation to swept-wing flow.
  - More amenable to experiments.

Introduction to the Rotating Disk - Setup

- First studied in 1921 by Theodore von Kármán who derived an exact similarity solution to the Navier Stokes equations.

$$F(z) = \frac{U^*}{r^*\Omega^*}, \quad G(z) = \frac{V^*}{r^*\Omega^*}, \quad H(z) = \frac{W^*}{(\nu\Omega^*)^{\frac{1}{2}}}$$

  where $\mathbf{U} = \mathbf{U}^*(z)$ is the velocity profile in cylindrical polars, Ω^* is the rotation rate of the disk and ν is the kinematic viscosity.
- This gives system of ODEs to solve for the base flow.
- Worth noting - Reynolds number is equivalent to radial position on the disk.

Basic Stability Concepts

- Perturb base flow by adding infinitesimal disturbance such that

$$\mathbf{u} = \mathbf{U}^B + \epsilon \mathbf{u}_p$$

- Substitute into Navier-Stokes equations and equate terms of order ϵ.
- Assume disturbance is of travelling wave form and express as

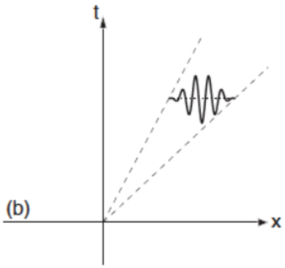
$$\mathbf{u}_p = \hat{u}(z) e^{i(\alpha r + n\theta - \omega t)}$$

- This gives eigenvalue problem

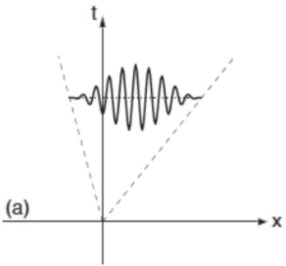
$$\mathcal{D}(\alpha, n, \omega; R) = 0$$

- Disturbance decays temporally if $\Im(\omega) < 0$ and spatially if $\Im(\alpha) > 0$.

Convective & Absolute instability



(a) Convective instability



(b) Absolute instability

Introduction to the Rotating Disk - Stability

- Disk is convectively unstable for $R > 286$. Instability mechanism is similar to swept wing flow.
- Disk is also *absolutely unstable* for $R > 507.3$. Discovered by Rebecca Lingwood in 1995, this is important because of its proximity to the experimentally observed critical Reynolds number for *transition* to turbulence.
- This absolute instability is not present in the swept-wing configuration due to the lack of periodicity.

Velocity-vorticity Formulation

$$\frac{\partial \xi_r}{\partial t} + \frac{1}{r}\frac{\partial N_r}{\partial \theta} - \frac{\partial N_\theta}{\partial z} - \frac{2}{R}\left(\xi_\theta + \frac{\partial w}{\partial r}\right) = \frac{1}{R}\left[\left(\nabla^2 - \frac{1}{r^2}\right)\xi_r - \frac{2}{r^2}\frac{\partial \xi_\theta}{\partial \theta}\right]$$

$$\frac{\partial \xi_\theta}{\partial t} + \frac{\partial N_r}{\partial z} - \frac{\partial N_z}{\partial r} + \frac{2}{R}\left(\xi_r - \frac{1}{r}\frac{\partial w}{\partial \theta}\right) = \frac{1}{R}\left[\left(\nabla^2 - \frac{1}{r^2}\right)\xi_\theta + \frac{2}{r^2}\frac{\partial \xi_r}{\partial \theta}\right]$$

$$\nabla^2 w = \frac{1}{r}\left(\frac{\partial \xi_r}{\partial \theta} - \frac{\partial (r\xi_\theta)}{\partial r}\right)$$

$$\mathbf{u} = (u_r, u_\theta, w), \quad \boldsymbol{\xi} = (\xi_r, \xi_\theta, \xi_z)$$

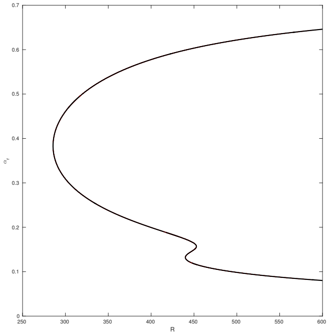
$$\mathbf{N} = (N_r, N_\theta, N_z) = (\nabla \times \mathbf{U}_B) \times \mathbf{u} + \boldsymbol{\xi} \times \mathbf{U}_B$$

$$u_r = -\int_z^\infty \left(\xi_\theta + \frac{\partial w}{\partial r}\right) dz, \quad u_\theta = \int_z^\infty \left(\xi_r - \frac{1}{r}\frac{\partial w}{\partial \theta}\right) dz$$

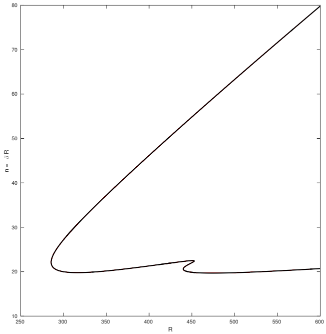
$$\xi_z = \frac{1}{r}\int_z^\infty \left(\frac{\partial (r\xi_r)}{\partial r} + \frac{\partial \xi_\theta}{\partial \theta}\right) dz$$

Local Eigenvalue Problem

Solving $\mathcal{D}(\alpha, n, \omega; R) = 0$.



(a) Neutral Curve for α_r



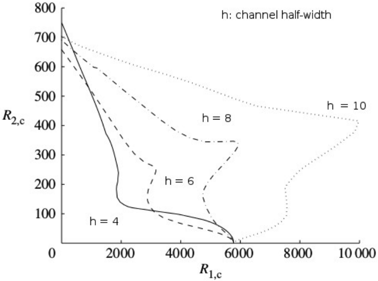
(b) Neutral Curve for $n = \beta R$

Periodic Modulation - Motivation

- We can adapt the steady problem to include a time-dependent part by way of oscillations of the disk.
- Adding oscillations to **channel** flow can be stabilising.

$$u = \gamma_1 U_1^S + \gamma_2 U_2^P$$

where U_1^S and U_2^P are the steady base flow profiles for Poiseuille channel flow ($\gamma_1 = 0$) and purely oscillatory channel flow ($\gamma_2 = 0$).



Periodic Modulation - Setup

- We can alter the von Kármán similarity variables to include a time-dependent structure

$$F(z,t) = \frac{U^*(z,t)}{r^*\Omega^*}, \quad G(z,t) = \frac{V^*(z,t)}{r^*\Omega^*}, \quad H(z,t) = \frac{W^*(z,t)}{(\nu\Omega^*)^{\frac{1}{2}}}$$

- System of ODEs becomes time-dependent

$$\begin{aligned} \frac{\partial F}{\partial t} &= F^2 - (G+1)^2 + F'H - F'' \\ \frac{\partial G}{\partial t} &= 2F(G+1) + G'H - G'' \\ H' &= -2F \end{aligned}$$

with

$$U(0,t) = W(0,t) = 0, \qquad V(0,t) = A\cos(\omega t)$$
$$U \to 0 \quad V \to -1 \quad \text{as} \quad z \to \infty$$

ROTATING FRAME

Periodic Modulation - Setup

- We can alter the von Kármán similarity variables to include a time-dependent structure

$$F(z,t)=\frac{U^*(z,t)}{r^*\Omega^*},\quad G(z,t)=\frac{V^*(z,t)}{r^*\Omega^*},\quad H(z,t)=\frac{W^*(z,t)}{(\nu\Omega^*)^{\frac{1}{2}}}$$

- System of ODEs becomes time-dependent

$$\frac{\partial F}{\partial t}=F^2-G^2+F'H-F''$$
$$\frac{\partial G}{\partial t}=2FG+G'H-G''$$
$$H'=-2F$$

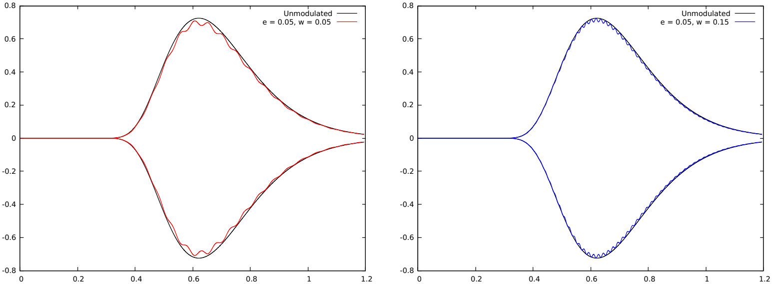
with

$$U(0,t)=W(0,t)=0,\qquad V(0,t)=1+A\cos(\omega t)$$
$$U\to 0\qquad V\to 0\quad\text{as}\quad z\to\infty$$

NON-ROTATING (LAB) FRAME

Preliminary Results

Imagine an impulsive forcing to the disk surface at some radially localised location $r = r_e$. Boundary conditions in the rotating frame are of the form $G(z = 0) = \epsilon \cos(\omega t)$.



(a) $\epsilon = 0.05$ and $\omega = 0.05$

(b) $\epsilon = 0.05$ and $\omega = 0.15$

Figure 4 : Wavepacket envelopes at $r = 450$ with azimuthal mode number $n = 28$ and an impulse excited at $r_e = 400$

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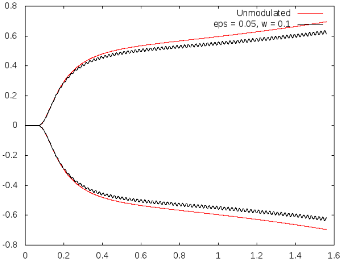
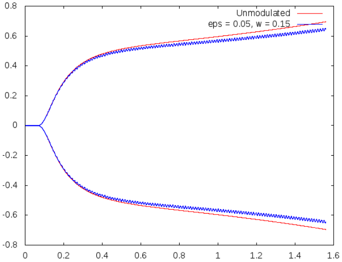
Preliminary Results

Imagine an impulsive forcing to the disk surface at some radially localised location $r = r_e$. Boundary conditions in the rotating frame are of the form $G(z = 0) = \epsilon \cos(\omega t)$.



(a) $\epsilon = 0.05$ and $\omega = 0.1$



(b) $\epsilon = 0.05$ and $\omega = 0.15$

Figure 5 : Wavepacket envelopes at $r = 540$ with azimuthal mode number $n = 67$ and an impulse excited at $r_e = 510$

Comparison with Garrett et. al. (2016) - Roughness

- Garrett et. al. (2016) use boundary conditions on G to approximate anisotropic roughness.
- They show that roughness component can be stabilising.
- Roughness component, *in some sense*, is similar to oscillatory motion.

Future Work

- Incorporate Floquet theory to further understand oscillatory component.
- Quantify any apparent effects and provide a physical reasoning.
- Is an oscillatory component stabilising for the rotating disk?

Floquet Theory

- Take normal mode approximation of the form

$$p(r,\theta,z,t) = \hat{p}(z,t)e^{\mu\tau}e^{i(\alpha r+\beta R\theta)}$$

  where $\hat{p}(z,t)$ is periodic in t and all of the exponential growth in time of p is factored into $e^{\mu t}$. Also $\tau = \omega t$ non-dimensionalises the time scale.
- Decompose time dependent base flow into

$$\mathbf{U}^{B}(z,t) = \mathbf{U}^{VK}(z) + \sum_{n=-\infty}^{\infty} u_n(z)e^{i\tau}$$

- Decompose $\hat{p}$ into harmonics such that

$$\hat{p} = \sum_{n=-\infty}^{\infty} \hat{p}_n(z)e^{in\tau}$$

  and substitute into equations.
- Gives system of perturbation equations to solve for μ.