## Specification

| | |
|---|---|
| Further partial fractions. | To include denominators $(ax+b)(cx+d)(ex+f)$ and $(ax+b)(cx^2+d)$. |
| Further integration. | To include the use of partial fractions. |
| Basic properties of the definite integral. | To include differentiation of an integral with respect to a variable limit. |
| Integration of $\dfrac{1}{\sqrt{a^2-x^2}}$ and $\dfrac{1}{a^2+x^2}$. | |

## Formulae and notes

**Partial Fractions :** If integrating $\int \frac{P(x)}{Q(x)}dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

| Factor in $Q(x)$ | Term in P.F.D | Factor in $Q(x)$ | Term in P.F.D |
|---|---|---|---|
| $ax+b$ | $\dfrac{A}{ax+b}$ | $(ax+b)^k$ | $\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$ |
| $ax^2+bx+c$ | $\dfrac{Ax+B}{ax^2+bx+c}$ | $(ax^2+bx+c)^k$ | $\dfrac{A_1x+B_1}{ax^2+bx+c}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$ |

**Ex.** $\int \frac{7x^2+13x}{(x-1)(x^2+4)}dx$

$$\int \frac{7x^2+13x}{(x-1)(x^2+4)}dx = \int \frac{4}{x-1}+\frac{3x+16}{x^2+4}dx$$

$$= \int \frac{4}{x-1}+\frac{3x}{x^2+4}+\frac{16}{x^2+4}dx$$

$$= 4\ln|x-1|+\tfrac{3}{2}\ln\left(x^2+4\right)+8\tan^{-1}\left(\tfrac{x}{2}\right)$$

Here is partial fraction form and recombined.

$$\frac{7x^2+13x}{(x-1)(x^2+4)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

Set numerators equal and collect like terms.

$$7x^2+13x = (A+B)x^2+(C-B)x+4A-C$$

Set coefficients equal to get a system and solve to get constants.

$$A+B=7 \qquad C-B=13 \qquad 4A-C=0$$

$$A=4 \qquad B=3 \qquad C=16$$

An alternate method that *sometimes* works to find constants. Start with setting numerators equal in previous example : $7x^2+13x = A\left(x^2+4\right)+\left(Bx+C\right)\left(x-1\right)$. Chose *nice* values of $x$ and plug in. For example if $x=1$ we get $20=5A$ which gives $A=4$. This won't always work easily.

**Trig Substitutions :** If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

| | | |
|---|---|---|
| $\sqrt{a^2-b^2x^2} \;\Rightarrow\; x=\frac{a}{b}\sin\theta$ | $\sqrt{b^2x^2-a^2} \;\Rightarrow\; x=\frac{a}{b}\sec\theta$ | $\sqrt{a^2+b^2x^2} \;\Rightarrow\; x=\frac{a}{b}\tan\theta$ |
| $\cos^2\theta=1-\sin^2\theta$ | $\tan^2\theta=\sec^2\theta-1$ | $\sec^2\theta=1+\tan^2\theta$ |

**Ex.** $\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx$

$x=\frac{2}{3}\sin\theta \;\Rightarrow\; dx=\frac{2}{3}\cos\theta\,d\theta$

$\sqrt{4-9x^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2\left|\cos\theta\right|$

Recall $\sqrt{x^2}=\left|x\right|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute $\theta$'s and remove absolute value bars based on that and,

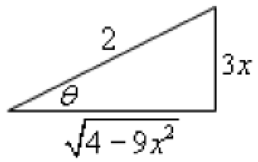
$$\left|x\right|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0 \end{cases}$$

In this case we have $\sqrt{4-9x^2}=2\cos\theta$.

$$\int \frac{16}{\frac{4}{9}\sin^2\theta\,(2\cos\theta)}\left(\tfrac{2}{3}\cos\theta\right)d\theta=\int\frac{12}{\sin^2\theta}\,d\theta$$

$$=\int 12\csc^2 d\theta=-12\cot\theta+c$$

Use Right Triangle Trig to go back to $x$'s. From substitution we have $\sin\theta=\frac{3x}{2}$ so,




From this we see that $\cot\theta=\frac{\sqrt{4-9x^2}}{3x}$. So,

$$\int\frac{16}{x^2\sqrt{4-9x^2}}\,dx=-\frac{4\sqrt{4-9x^2}}{x}+c$$

## Fundamental Theorem of Calculus

**Part I :** If $f(x)$ is continuous on $[a,b]$ then $g(x)=\int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x)=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$.

**Part II :** $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x)=\int f(x)\,dx$) then $\int_a^b f(x)\,dx=F(b)-F(a)$.

**Variants of Part I :**

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt=u'(x)\,f\left[u(x)\right]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt=-v'(x)\,f\left[v(x)\right]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt=u'(x)\,f\left[u(x)\right]-v'(x)\,f\left[v(x)\right]$$

## Properties

$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int cf(x)\,dx = c\int f(x)\,dx$, $c$ is a constant

$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, $c$ is a constant

$\int_a^a f(x)\,dx = 0$

$\int_a^b c\,dx = c(b-a)$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any value of $c$.

If $f(x) \ge g(x)$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $f(x) \ge 0$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m \le f(x) \le M$ on $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

## Common Integrals

$\int k\,dx = kx + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \ne -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$\int \ln u\,du = u\ln(u) - u + c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \cos u\,du = \sin u + c$

$\int \sin u\,du = -\cos u + c$

$\int \sec^2 u\,du = \tan u + c$

$\int \sec u \tan u\,du = \sec u + c$

$\int \csc u \cot u\,du = -\csc u + c$

$\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$

The first one is $\int \frac{dx}{a^2+x^2}$

This integral requires a substitution.

Let $x = a\tan\theta$ so that $dx = a\sec^2\theta\,d\theta$

$$
\begin{aligned}
\text{Then } \int \frac{dx}{a^2+x^2} &= \int \frac{a\sec^2\theta\,d\theta}{a^2+a^2\tan^2\theta} \\
&= \int \frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta} \\
&= \int \frac{1}{a}\,d\theta \\
&= \frac{1}{a}\theta + c \\
&= \frac{1}{a}\tan^{-1}\frac{x}{a} + c
\end{aligned}
$$

The second integral is $\int \frac{dx}{\sqrt{a^2 - x^2}}$

This interval also requires a substitution

$$\text{Let } x = a\sin\theta \qquad dx = a\cos\theta\, d\theta$$

$$\text{Then } \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta}{a\cos\theta}\, d\theta$$

$$= \theta + c$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + c$$

Question 1

*(a)* Express

$$\frac{x}{(x+2)(x^2+4)}$$

in partial fractions. [4]

*(b)* Hence evaluate the integral

$$\int_2^3 \frac{x}{(x+2)(x^2+4)}\, dx,$$

giving your answer correct to three decimal places. [6]

Question 2

The function $f$ is defined by

$$f(x) = \frac{1}{(x+1)(x+2)(x+3)} .$$

*(a)* Express $f(x)$ in partial fractions. [4]

*(b)* Evaluate the integral

$$\int_0^5 f(x)\, dx ,$$

giving your answer in the form $\ln\left(\frac{m}{n}\right)$ where $m$, $n$ are integers. [5]

Question 3

Let

$$f(x) = \frac{(x+1)(x+2)}{(x-1)(x^2+1)}.$$

*(a)* Express *f(x)* in partial fractions. [5]

*(b)* Find $\int f(x)\,dx$. [4]

Question 4

Use the substitution $x = y^2$ to evaluate the integral

$$\int_1^4 \frac{dx}{\sqrt{x(9-x)}},$$

giving your answer correct to two significant figures. [6]

Question 5

*(a)* Using the substitution $u = x^2$, evaluate the integral

$$\int_0^{\sqrt{3}} \frac{x\,dx}{(9+x^4)},$$

giving your answer in the form $\frac{\pi}{k}$, where $k$ is an integer. [5]

*(b)* Evaluate the integral

$$\int_0^1 \frac{dx}{\sqrt{25-9x^2}}.$$ [4]

Question 6

Using the substitution $u = x\sqrt{x}$, evaluate the integral

$$\int_0^2 \frac{\sqrt{x}}{\sqrt{9-x^3}}\,dx.$$

Give your answer correct to three decimal places. [5]

Question 7

Using the substitution $u = \tan x$, evaluate the integral

$$\int_0^{\frac{\pi}{6}} \frac{\sec^2 x}{\sqrt{3-\sec^2 x}}\,dx \quad .$$

Explain briefly why the integral could not be evaluated if the upper limit were changed to $\frac{\pi}{3}$ . [7]

Question 8

Using the substitution $u = \sqrt{x}$, evaluate the integral

$$\int_1^4 \frac{1}{(9+x)\sqrt{x}}\,\mathrm{d}x.$$

Give your answer correct to four decimal places. [5]

Question 9

*(a)* Differentiate the following integral with respect to $x$.

$$\int_0^x \sin(\mathrm{e}^t)\,\mathrm{d}t$$ [1]

*(b)* By putting $u = x^2$ and using the chain rule, differentiate the following integral with respect to $x$.

$$\int_0^{x^2} \sin(\mathrm{e}^t)\,\mathrm{d}t$$ [2]