Hyperbolics

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Topics

- Hyperbolic Identities
- Calculus with Hyperbolics - Differentiation & Integration
- Inverse Hyperbolic Functions

Topics

- Hyperbolic Identities

Hyperbolic Functions

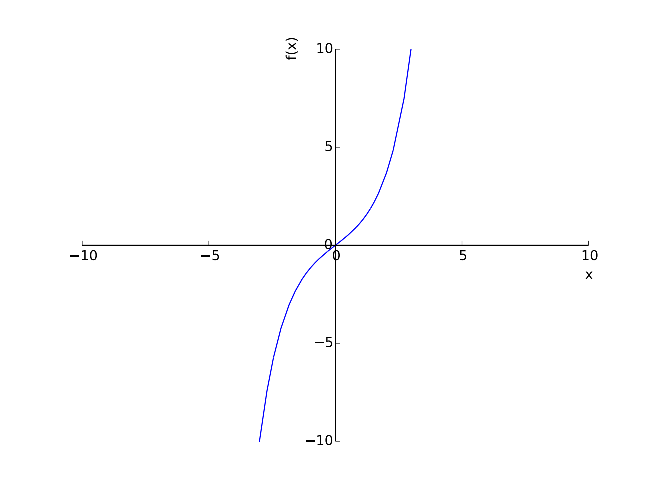
$$\sinh(x)$$
$$\cosh(x)$$

Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

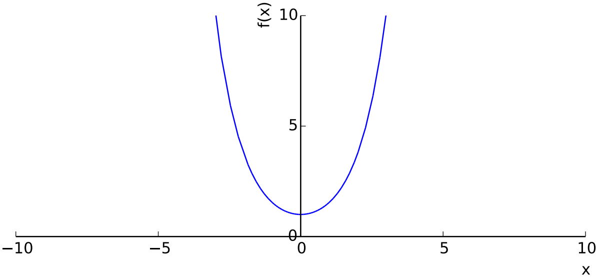
Hyperbolic Functions

$$y = \sinh(x)$$



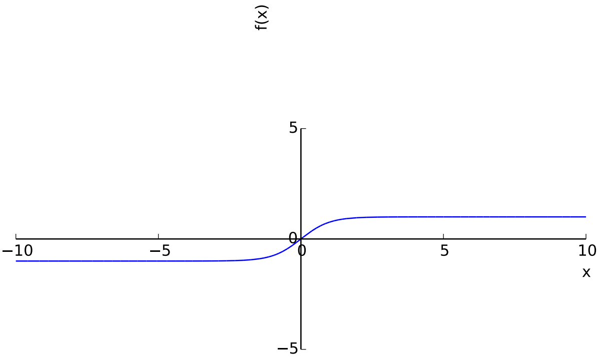
Hyperbolic Functions

$$y = \cosh(x)$$



Hyperbolic Functions

$$y = \tanh(x)$$



Hyperbolic Functions

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$
$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)}$$
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$
$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

Hyperbolic Functions

$$
\begin{aligned}
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\operatorname{cosech}(x) &= \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}} \\
\operatorname{sech}(x) &= \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}} \\
\coth(x) &= \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}
\end{aligned}
$$

Examples:

Prove the following identities, using the exponential definitions of the hyperbolic functions.

- $\cosh^2(x) - \sinh^2(x) = 1$
- $\cosh(2x) = 1 + 2\sinh^2(x)$
- $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

Trigonometric vs Hyperbolic Identities

Trigonometric Identity	Hyperbolic Identity
$\cos^2(x) + \sin^2(x) = 1$	$\cosh^2(x) - \sinh^2(x) = 1$
$1 + \tan^2(x) = \sec^2(x)$	$1 - \tanh^2(x) = \operatorname{sech}^2(x)$
$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$	$\coth^2(x) - 1 = \operatorname{cosech}^2(x)$

Trigonometric vs Hyperbolic Identities

Trigonometric Identity	Hyperbolic Identity
$\sin(2x) = 2\sin(x)\cos(x)$	$\sinh(2x) = 2\sinh(x)\cosh(x)$
$\cos(2x) = \cos^2(x) - \sin^2(x)$	$\cos(2x) = \cos^2(x) + \sin^2(x)$
$\cos(2x) = 2\cos^2(x) - 1$	$\cosh(2x) = 2\cosh^2(x) - 1$
$\cos(2x) = 1 - 2\sin^2(x)$	$\cosh(2x) = 1 + 2\sinh^2(x)$
$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$	$\tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)}$

Trigonometric vs Hyperbolic Identities

Trigonometric Identity	Hyperbolic Identity
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$	$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$
$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$	$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$

Equations with Hyperbolic Functions

Examples:

Solve the following equations:

- $3\sinh(x) - \cosh(x) = 1$
- $\tanh(x) + 4\,\mathrm{sech}(x) = 4$
- $12\cosh^2(x) + 7\sinh(x) = 24$

Topics

- Calculus with Hyperbolics - Differentiation & Integration

Calculus with Hyperbolic Functions - Differentiation

$$\frac{d(\sinh(x))}{dx}$$

$$\frac{d(\cosh(x))}{dx}$$

Calculus with Hyperbolic Functions - Differentiation

$$\frac{d(\sinh(x))}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{d(\cosh(x))}{dx}$$

Calculus with Hyperbolic Functions - Differentiation

$$\frac{d(\sinh(x))}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{d(\cosh(x))}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

Examples:

Find the derivatives of the following:

- $\tanh(x)$
- $\operatorname{cosech}(x)$
- $\operatorname{sech}(x)$
- $\coth(x)$

Calculus with Hyperbolic Functions - Differentiation

Examples:

Find the derivatives of the following:

- $\tanh(2x)$
- $\operatorname{sech}^2(x)$
- $\sinh(4x)$
- $\cosh^3(x)$
- $x\sinh(x)$
- $e^x\sinh(x)$
- $\sqrt{\cosh(5x)}$
- $e^{\cosh(x)}$
- $\ln(\sinh(x))$

Calculus with Hyperbolic Functions - Integration

$$\int \sinh(x)dx$$

$$\int \cosh(x)dx$$

Calculus with Hyperbolic Functions - Integration

$$\int \sinh(x)dx = \cosh(x) + c$$

$$\int \cosh(x)dx$$

Calculus with Hyperbolic Functions - Integration

$$\int \sinh(x)dx = \cosh(x) + c$$
$$\int \cosh(x)dx = \sinh(x) + c$$

Calculus with Hyperbolic Functions - Integration

Examples:

Find the following:

- $\int x \sinh(2x) dx$
- $\int \sinh^2(x) dx$
- $\int \cosh^2(x) dx$
- $\int \sinh(3x) dx$
- $\int \cosh(5x) dx$
- $\int 3x \cosh(4x) dx$
- $\int \sinh(x) \cosh(x) dx$

Topics

- Inverse Hyperbolic Functions

Inverse Hyperbolic Functions

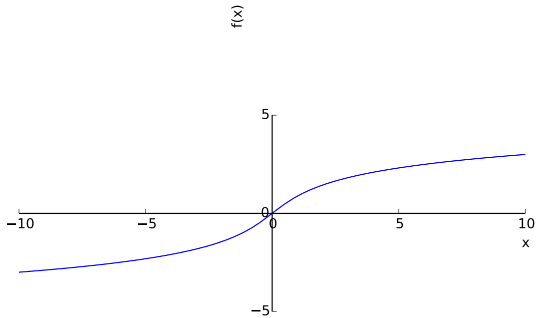
Example:

Using the exponential definition for $\sinh(x)$, show that

$$\begin{aligned}
\sinh^{-1}(x) &= \ln\left(x + \sqrt{x^2+1}\right) \\
\cosh^{-1}(x) &= \ln\left(x + \sqrt{x^2-1}\right) \\
\tanh^{-1}(x) &= \frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)
\end{aligned}$$

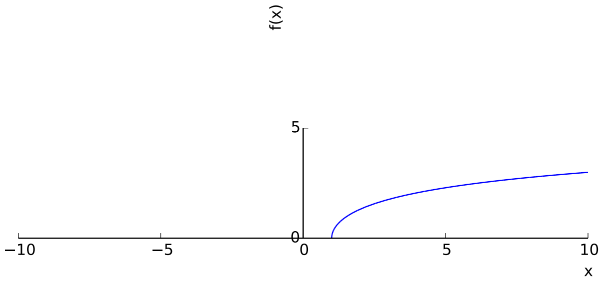
Inverse Hyperbolic Functions

$$y = \sinh^{-1}(x)$$



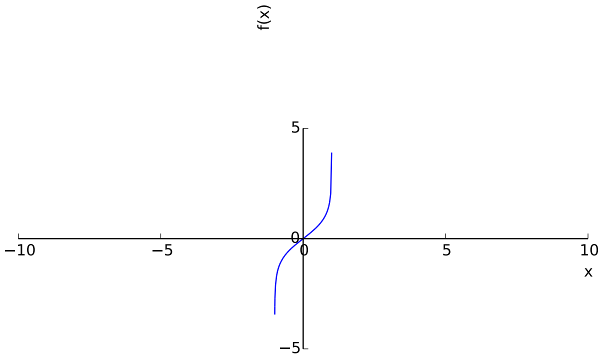
Inverse Hyperbolic Functions

$$y = \cosh^{-1}(x)$$



Inverse Hyperbolic Functions

$$y = \tanh^{-1}(x)$$



Example:

Using implicit differentiation, find:

$$\frac{d}{dx}\left(\sinh^{-1}(x)\right)$$
$$\frac{d}{dx}\left(\cosh^{-1}(x)\right)$$

Example:

Using implicit differentiation, find:

$$\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\cosh^{-1}(x)\right)$$

Example:

Using implicit differentiation, find:

$$\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}\left(\cosh^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

Standard Integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \quad (|x| < a)$$

$$\int \frac{1}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left(x + \sqrt{x^2 - a^2}\right) \quad (|x| < a)$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\int \frac{1}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| \quad (|x| < a)$$

$$\int \frac{1}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right|$$

Calculus with Hyperbolic Functions - Integration

Examples:

Find the following:

$$\int \frac{1}{\sqrt{16+x^2}}\,dx$$

$$\int \frac{1}{16+x^2}\,dx$$

$$\int \frac{1}{x^2-25}\,dx$$

$$\int \frac{1}{9-x^2}\,dx$$

$$\int \frac{1}{\sqrt{9-x^2}}\,dx$$

$$\int \frac{1}{\sqrt{x^2-25}}\,dx$$