Integration

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Topics

- Integration by partial fractions
- Integration by substitution

Topics

- Integration by partial fractions

Lesson Objectives

To be able to represent the following algebraic functions as partial fractions:

$$\frac{2x+3}{(x^2-1)(x+3)}, \quad \frac{x^2+3x-1}{x^2(x+3)}, \quad \frac{x}{(x+4)(x+1)^2}, \quad \frac{x^2+3x}{x+4}$$

To use partial fractions to evaluate integrals such as:

$$\int \frac{1}{x^2+2x-3}\,dx, \quad \int_1^2 \frac{x+1}{x^3+2x^2}\,dx$$

Partial Fractions

Examples

$$\frac{2x+3}{(x^2-1)(x+3)} =$$

$$\frac{x^2+3x-1}{x^2(x+3)} =$$

$$\frac{x}{(x+4)(x+1)^2} =$$

$$\frac{x^2+3x}{x+4} =$$

Examples

$$\frac{2x+3}{(x^2-1)(x+3)} = -\frac{1}{4(x+1)} - \frac{3}{8(x+3)} + \frac{5}{8(x-1)}$$

$$\frac{x^2+3x-1}{x^2(x+3)} =$$

$$\frac{x}{(x+4)(x+1)^2} =$$

$$\frac{x^2+3x}{x+4} =$$

Partial Fractions

Examples

$$\begin{aligned}
\frac{2x+3}{(x^2-1)(x+3)} &= -\frac{1}{4(x+1)} - \frac{3}{8(x+3)} + \frac{5}{8(x-1)} \\
\frac{x^2+3x-1}{x^2(x+3)} &= -\frac{1}{3x^2} - \frac{1}{9(x+3)} + \frac{10}{9x} \\
\frac{x}{(x+4)(x+1)^2} &= \\
\frac{x^2+3x}{x+4} &=
\end{aligned}$$

Examples

$$\frac{2x+3}{(x^2-1)(x+3)} = -\frac{1}{4(x+1)} - \frac{3}{8(x+3)} + \frac{5}{8(x-1)}$$

$$\frac{x^2+3x-1}{x^2(x+3)} = -\frac{1}{3x^2} - \frac{1}{9(x+3)} + \frac{10}{9x}$$

$$\frac{x}{(x+4)(x+1)^2} = -\frac{4}{9(x+4)} + \frac{4}{9(x+1)} - \frac{1}{3(x+1)^2}$$

$$\frac{x^2+3x}{x+4} =$$

Partial Fractions

Examples

$$
\begin{aligned}
\frac{2x+3}{(x^2-1)(x+3)} &= -\frac{1}{4(x+1)} - \frac{3}{8(x+3)} + \frac{5}{8(x-1)} \\
\frac{x^2+3x-1}{x^2(x+3)} &= -\frac{1}{3x^2} - \frac{1}{9(x+3)} + \frac{10}{9x} \\
\frac{x}{(x+4)(x+1)^2} &= -\frac{4}{9(x+4)} + \frac{4}{9(x+1)} - \frac{1}{3(x+1)^2} \\
\frac{x^2+3x}{x+4} &= (x-1) + \frac{4}{x+4}
\end{aligned}
$$

Integration with Partial Fractions

Examples

$$\int \frac{1}{x^2+2x-3}\,dx =$$

$$\int_1^2 \frac{x+1}{x^3+2x^2}\,dx =$$

Integration with Partial Fractions

Examples

$$\int \frac{1}{x^2+2x-3}\,dx = \int \left(\frac{1}{4(x-1)} - \frac{1}{4(x+3)} \right) dx$$

$$\int_1^2 \frac{x+1}{x^3+2x^2}\,dx =$$

Examples

$$\begin{aligned}\int \frac{1}{x^2+2x-3}\,dx &= \int \left(\frac{1}{4(x-1)} - \frac{1}{4(x+3)}\right) dx \\ &= \frac{1}{4}\left(\ln(1-x) - \ln(x+3)\right) + \text{const.}\end{aligned}$$

$$\int_1^2 \frac{x+1}{x^3+2x^2}\,dx =$$

Integration with Partial Fractions

Examples

$$\int \frac{1}{x^2+2x-3}dx = \int \left(\frac{1}{4(x-1)} - \frac{1}{4(x+3)}\right) dx$$
$$= \frac{1}{4}\left(\ln(1-x) - \ln(x+3)\right) + \text{const.}$$

$$\int_1^2 \frac{x+1}{x^3+2x^2}dx = \int_1^2 \left(\frac{1}{2x^2} - \frac{1}{4(x+2)} + \frac{1}{4x}\right) dx$$

Integration with Partial Fractions

Examples

$$\int \frac{1}{x^2+2x-3}\,dx = \int \left(\frac{1}{4(x-1)} - \frac{1}{4(x+3)}\right) dx$$
$$= \frac{1}{4}\left(\ln(1-x) - \ln(x+3)\right) + \text{const.}$$

$$\int_1^2 \frac{x+1}{x^3+2x^2}\,dx = \int_1^2 \left(\frac{1}{2x^2} - \frac{1}{4(x+2)} + \frac{1}{4x}\right) dx$$
$$= \left[-\frac{1}{2x} + \frac{1}{4}\ln(x) - \frac{1}{4}\ln(x+2)\right]_1^2$$

Integration with Partial Fractions

Examples

$$\int \frac{1}{x^2+2x-3}\,dx = \int \left(\frac{1}{4(x-1)} - \frac{1}{4(x+3)}\right) dx$$
$$= \frac{1}{4}\left(\ln(1-x) - \ln(x+3)\right) + \text{const.}$$

$$\int_1^2 \frac{x+1}{x^3+2x^2}\,dx = \int_1^2 \left(\frac{1}{2x^2} - \frac{1}{4(x+2)} + \frac{1}{4x}\right) dx$$
$$= \left[-\frac{1}{2x} + \frac{1}{4}\ln(x) - \frac{1}{4}\ln(x+2)\right]_1^2$$
$$\approx 0.35137$$

Topics

- Integration by substitution

Trigonometric Substitutions

Expressions of the form $\sqrt{a^2 - x^2}$ can be reduced to the square root of a single term by a substitution either of the form

$$x = a\sin(\theta)$$

or of the form

$$x = a\cos(\theta)$$

Integration by Substitution

Examples:

By making a substitution write the following as a single trigonometric term in terms of θ

$$\sqrt{9 - x^2}$$
$$\sqrt{25 - x^2}$$
$$\sqrt{1 - 4x^2}$$
$$\sqrt{4 - 9x^2}$$
$$\sqrt{25 - 16x^2}$$

Integration by Substitution

Examples:

Solve the following integration problems:

$$\int \frac{1}{\sqrt{9-x^2}}$$

$$\int \frac{1}{\sqrt{25-x^2}}$$

$$\int \frac{2x+3}{x^2+4}$$

Integration by Substitution

Examples:

Use the substitution $y = x^2$ to evaluate the integral:

$$\int_1^4 \frac{dx}{\sqrt{x(9-x)}}$$

giving your answer correct to two significant figures.

Examples:

Use the substitution $u = x^{\frac{3}{2}}$ to evaluate the integral:

$$\int_1^4 \frac{\sqrt{x}}{1 + x^3}\,dx$$

giving your answer correct to two significant figures.

Integration with Variable Limits

Two formulae:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x)$$

Examples:

Evaluate:

$$\frac{d}{dx}\left(\int_1^{x^2} \sin\left(\frac{1}{t}\right) dt\right)$$

Hint: Make a substitution $u = x^2$