Stability of Oscillatory Rotating Disk Boundary Layers

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CARDIFF UNIVERSITY
PRIFYSGOL CAERDYDD

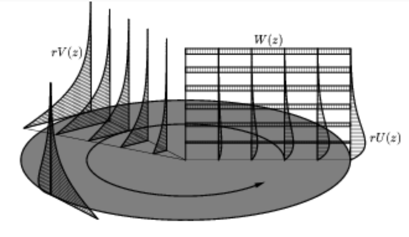
Structure

- **Part 1:** What?
- **Part 2:** Why?
- **Part 3:** How?

What?

What?

~~Stability of Oscillatory~~ **Rotating Disk Boundary Layers**



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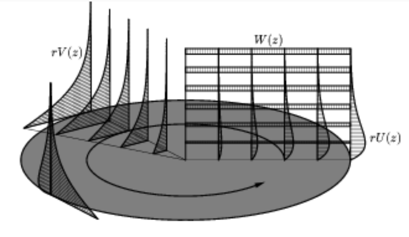

- Lengths scaled on constant boundary layer thickness:

$$\delta = \sqrt{\nu/\Omega_0}$$

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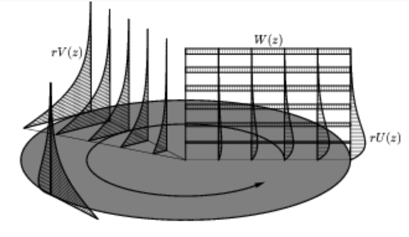

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- $R \equiv r_L$

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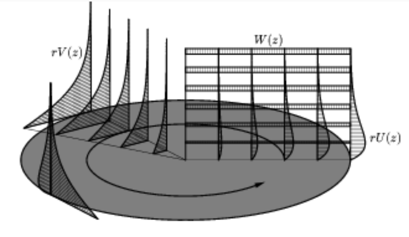

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- Convectively unstable for stationary disturbances:
    - Type I - $R_c \approx 286$
    - Type II - $R_c \approx 440$

Malik (1986)

What?

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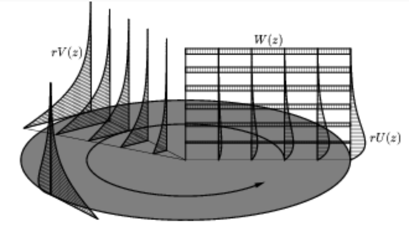

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Malik (1986)

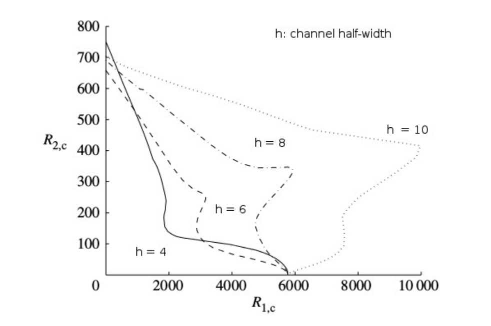
- Absolutely unstable for travelling disturbances:
  - $R_c \approx 507$

Lingwood (1995)

What?

Stability of Oscillatory ~~Rotating Disk~~ Boundary Layers

Adding oscillation to *channel* flow can be stabilising



Thomas et. al. (2011)

What?

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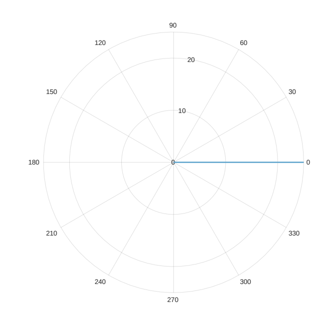

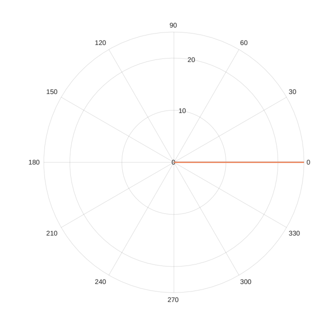

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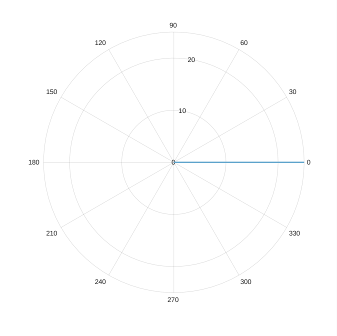

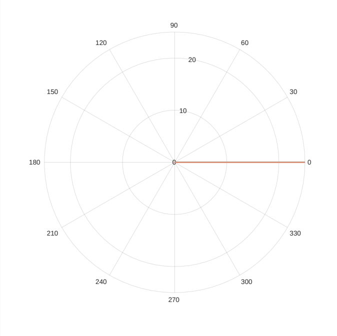

Dominant behaviour is Stokes layer for high-frequency, low amplitude oscillations.

Why?

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- Rotating disks $\approx$ Swept wings

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    - *Crossflow vortex suppression - focus on stationary disturbances.*

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Why rotating disks?

- Rotating disks $\approx$ Swept wings
  - *Crossflow vortex suppression - focus on stationary disturbances.*
- Other applications - atmospheric, oceanic, chemical deposition, mixing

How?

Setup

- Three-dimensional base flow

$$\mathbf{U}_B = (U, V, W)$$

Setup

- Three-dimensional base flow

$$\mathbf{U}_B = (U, V, W)$$

- Boundary conditions:

$$\begin{aligned} V(r, z=0, t) &= r\Omega(t) \\ &= r\left(\Omega_0 + \epsilon\phi\cos(\phi t)\right) \end{aligned}$$

Ω_0 - constant rotation rate
ϵ - angular displacement
ϕ - oscillation frequency

Scalings

- Retain steady scalings:

$$\delta^* = \sqrt{\frac{\nu^*}{\Omega_0^*}}, \qquad \tau = \frac{r_L^* \Omega_0^* t^*}{\delta^*}$$

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$\varphi = \frac{\phi}{\Omega_0}$ - number of oscillations per disk rotation period

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- Similarity structure:

$$\mathbf{U}_B = \left(\frac{U^*}{r^*\Omega_0^*}, \frac{V^*}{r^*\Omega_0^*}, \frac{W^*}{\delta^*\Omega_0^*} \right)$$

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$$\mathbf{U}_B = \left(\frac{r}{R_k} F, \frac{r}{R_k} G, \frac{1}{R_k} H \right)$$

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- Three parameters:

$$(R_k, R_s, \varphi)$$

($Rs \to 0$ recovers steady case)

$\varphi = \frac{\phi}{\Omega_0}$ - number of oscillations per disk rotation period

Approaches

- Three approaches:

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    1. Floquet Theory

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- Three approaches:
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    3. ~~*Frozen Flow Analysis*~~
- Solve Navier-Stokes equations using velocity-vorticity formulation.

Approaches

1. **Floquet Theory**

Floquet Theory

- Normal mode approximation:

$$u(r, \theta, z, \tau) \sim \hat{u}(z, \tau) e^{i\alpha r} e^{\mu\tau} e^{in\theta}$$

  - $\hat{u}(z, \tau)$ periodic

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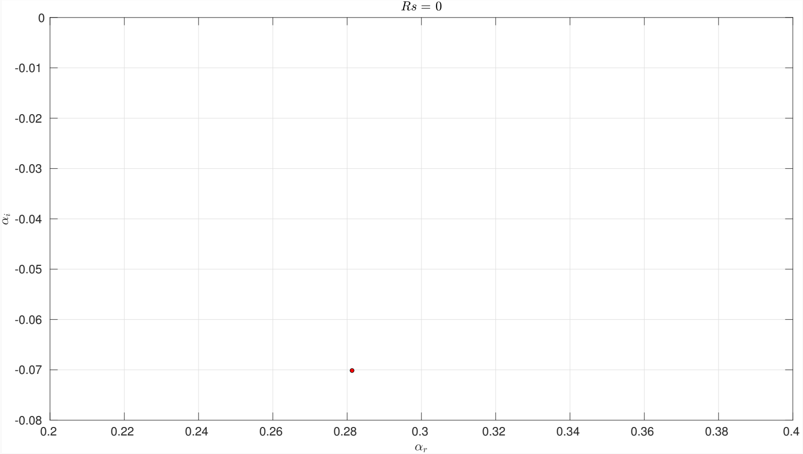
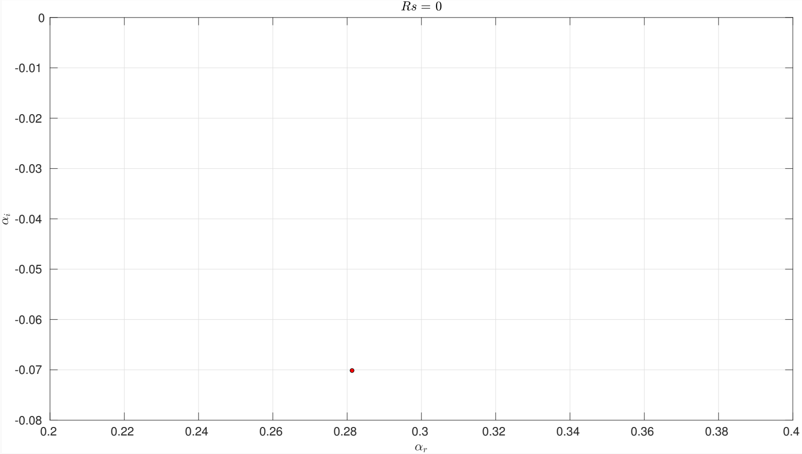
- Specify μ or α as real and solve for the other.

Floquet Results



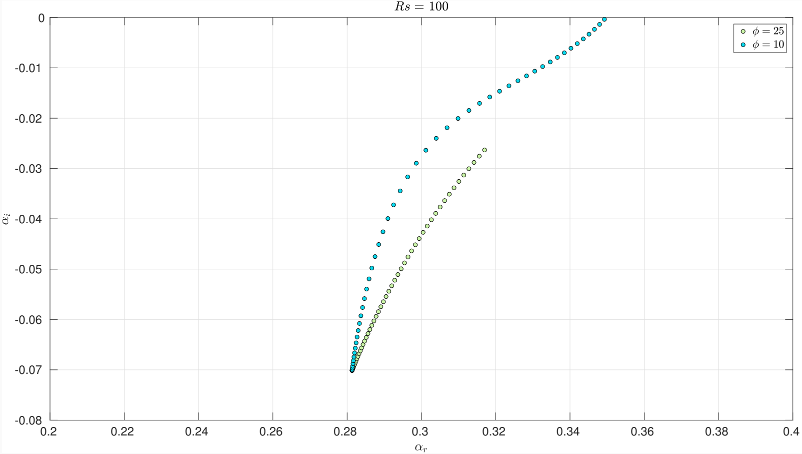
Steady case: $R_k = 500$, $n = 32$, $\alpha_i \approx -0.07$

Floquet Results



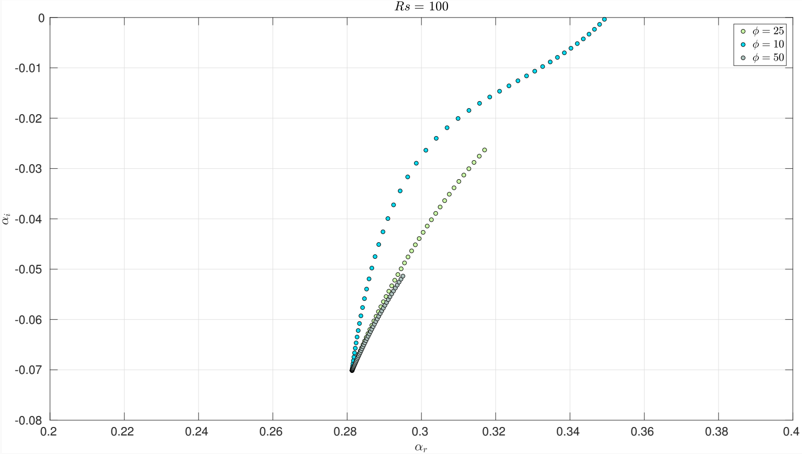
$R_k = 500,\ n = 32,\ \varphi = 25,\ R_s \in [0, 100]$

Floquet Results



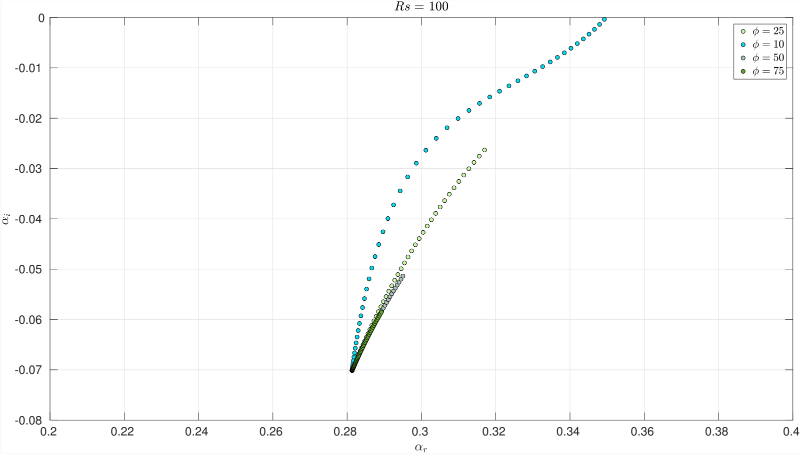
$$R_k = 500,\ n = 32,\ \varphi \in \{10, 25\},\ R_s \in [0, 100]$$

Floquet Results



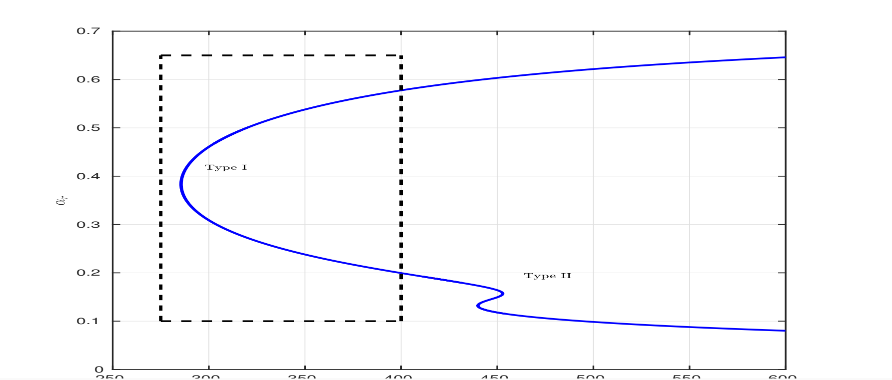
$R_k = 500,\ n = 32,\ \varphi \in \{10, 25, 50\},\ R_s \in [0, 100]$

Floquet Results



$R_k = 500,\ n = 32,\ \varphi \in \{10, 25, 50, 75\},\ R_s \in [0, 100]$

Neutral Curves



Steady

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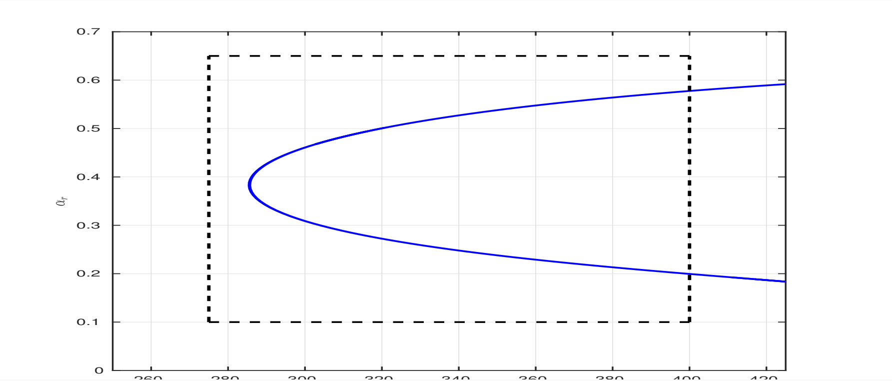
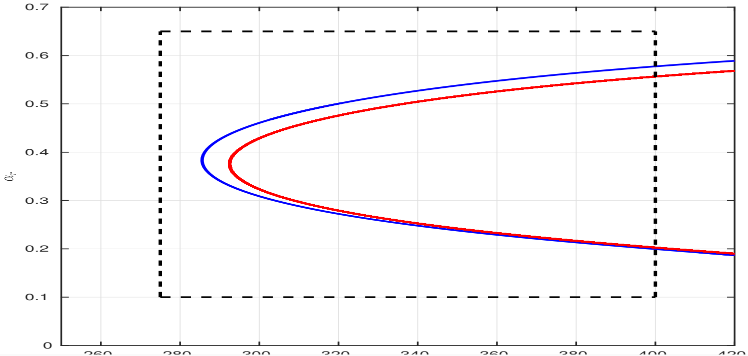


Steady Type I

Neutral Curves



Type I: $Rs = 20$, $\varphi = 25$

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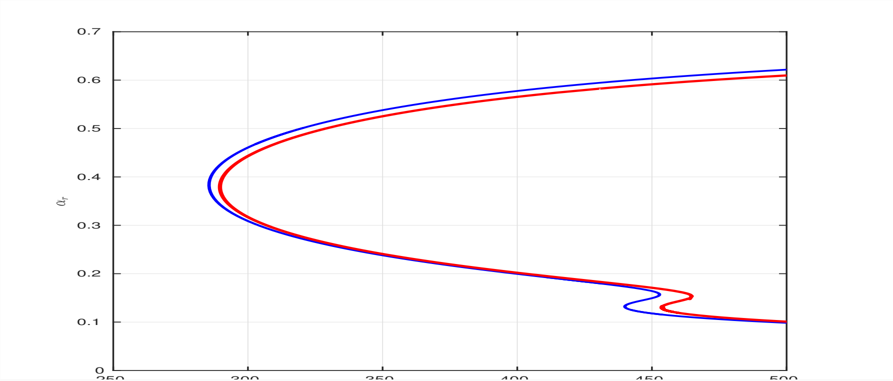
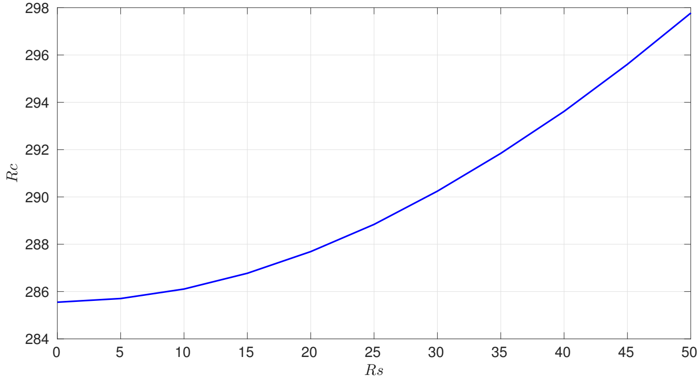


Full curve: $Rs = 20$, $\varphi = 25$

Neutral Curves



Critical Type I R_c for $\varphi = 25$ and $Rs \in [0, 50]$

Floquet Results

- Clear reduction in spatial growth rates for *small Rs*.

Floquet Results

- Clear reduction in spatial growth rates for *small Rs*.
- *Larger Rs being explored currently - preliminary results indicate intricate behaviour in neutral curves when* $U_w > 1$ *if*

$$G(0,\tau) = 1 + U_w \cos\left(\frac{\varphi}{R_k}\right)$$

Approaches

2. **Linear DNS**

Linear DNS

- Measure the response of the flow to a *stationary* disturbance.

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- Measure the response of the flow to a *stationary* disturbance.
- Forcing stationary with respect to modulated disk:

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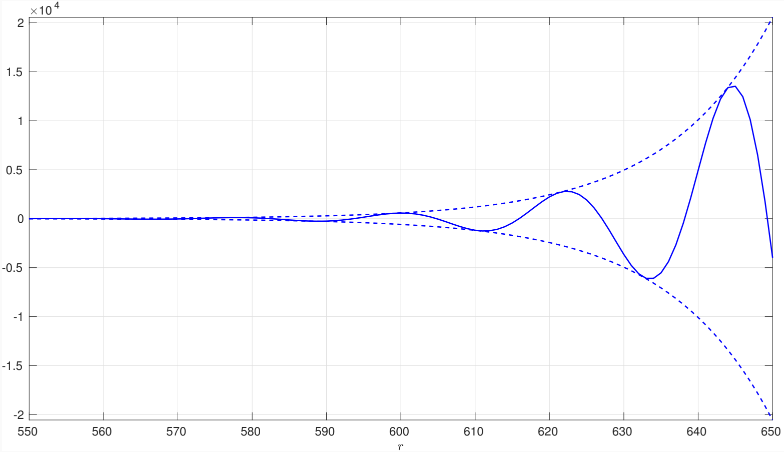

$u(r, z = 0, \tau > T_c)$ - *steady:* $R_k = 500$, $n = 32$

Linear DNS

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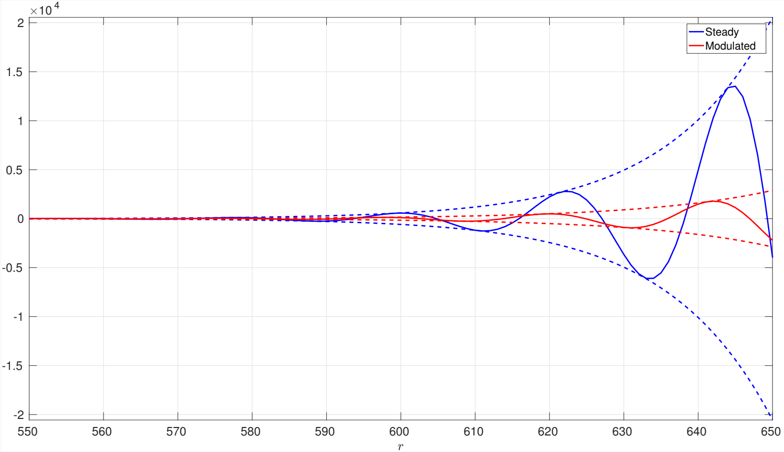

$u(r, z = 0, \tau > T_c)$ - *modulated:* $R_k = 500,\ n = 32,\ Rs = 50,\ \varphi = 25$

DNS

- Normal mode approximation:

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- $-\alpha_i$ gives radial growth rate.

DNS vs. Floquet

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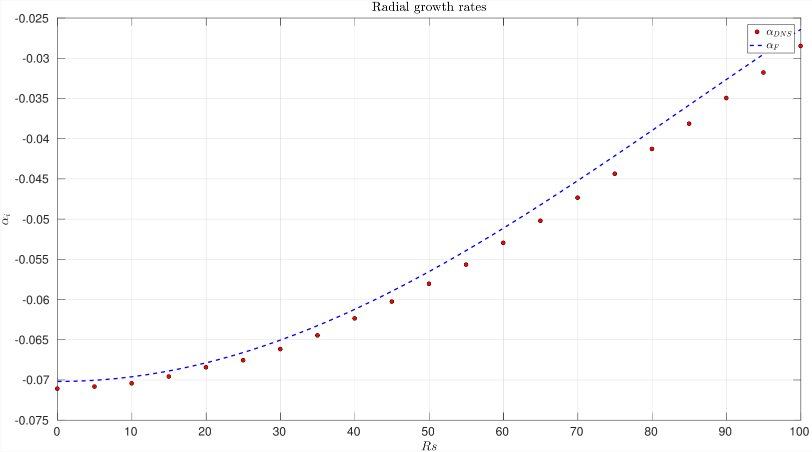
DNS vs. Floquet



Radial Growth Rates: $R_k = 500$, $n = 32$, $\varphi = 25$, $Rs \in \{0, 100\}$

Current & Future Work

- Look at parallels between oscillation and surface roughness.

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- Experimental confirmation.

Current & Future Work

- Look at parallels between oscillation and surface roughness.
- Experimental confirmation.
- Explore torsional oscillations.

Thank You

Approaches

3 **Frozen Flow Analysis**

Frozen Flow Analysis

- Freeze flow, treat τ as parameter:

$$p(r, \theta, z, \tau) = \hat{p}(z; \tau) e^{i\alpha r} e^{in\theta} e^{-i \int^{\tau} \omega}$$

where $\hat{\phi}$ is slowly varying.

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- Dispersion relation:

$$\mathcal{D}(\alpha, \omega; n, R_k, R_s, \varphi, \tau) = 0 \tag{1}$$

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- Specify α and solve (1) for $\omega(\tau)$ at each time-step.

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Frozen Flow Analysis

- $\frac{1}{T} \int_0^T \alpha(\tau) \approx \alpha_F$

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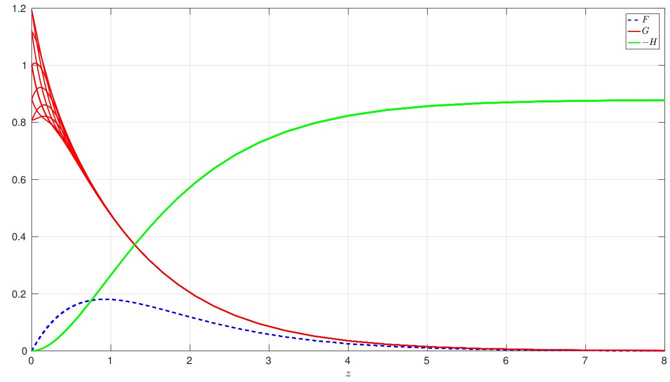
- $\frac{1}{T}\int_0^T \alpha(\tau) \approx \alpha_F$
- $\frac{1}{T}\int_0^T \omega(\tau) \approx \omega_F$

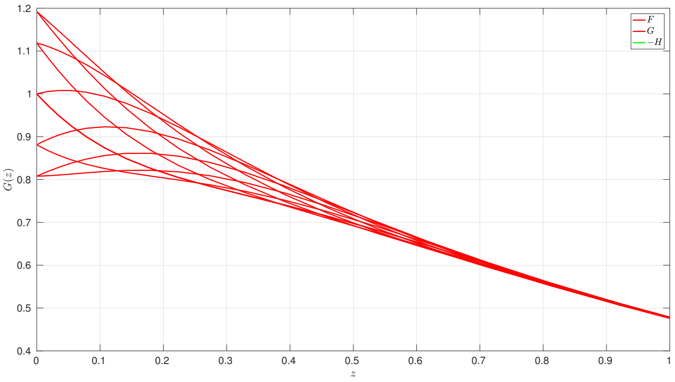
Typical Mean Flow Variation



Base flow variation for $Rs = 10$, $\varphi = 50$

- Zero average deviation from steady state across a period. $\int_0^T \mathbf{U} = 0$

Typical Mean Flow Variation



Azimuthal variation near wall for $Rs = 10$, $\varphi = 50$

- Zero average deviation from steady state across a period. $\int_0^T \mathbf{U} = 0$

Connections with the Stokes Layer

- Write

$$\begin{aligned}\mathbf{U}^T &= \mathbf{U}^S + \mathbf{U}^M \\ &= \mathbf{U}^S + \epsilon \mathbf{U}_1 + \mathcal{O}(\epsilon^2)\end{aligned}$$

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$$\delta_s = \sqrt{\frac{1}{\varphi}}$$

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- High frequency modulation suggests primary lengthscale:

$$\delta_s = \sqrt{\frac{1}{\varphi}}$$

- Rescale

$$\tilde{z} = \frac{z}{\delta_s}, \qquad \tilde{\tau} = \frac{\varphi}{R}\tau$$

Connections with the Stokes Layer

- Near the wall we have

$$\left(\frac{1}{\delta^2}\right)\frac{\partial F}{\partial \tilde{\tau}} = \left(\frac{1}{\delta^2}\right)F'' + \mathcal{O}\left(\delta^{-1}\right)$$
$$\left(\frac{1}{R\delta^2}\right)\frac{\partial G}{\partial \tilde{\tau}} = \left(\frac{1}{\delta^2}\right)G'' + \mathcal{O}\left(\delta^{-1}\right)$$
$$H \sim \delta F$$

with

$$F(0,\tilde{\tau}) = H(0,\tilde{\tau}) = 0, \qquad G(0,\tilde{\tau}) = \cos(\tilde{\tau})$$
$$F \to 0 \qquad G \to 0 \quad \text{as} \quad \tilde{z} \to \infty$$

Connections with the Stokes Layer

- To dominant balance:

$$\frac{\partial F}{\partial \tilde{\tau}} = F'', \quad \text{with} \quad F(0) = F(\tilde{z} \to \infty) = 0$$
$$\frac{1}{R}\frac{\partial G}{\partial \tilde{\tau}} = G'', \quad \text{with} \quad G(0) = cos(\tilde{\tau}), \;\; G(\tilde{z} \to \infty) = 0$$

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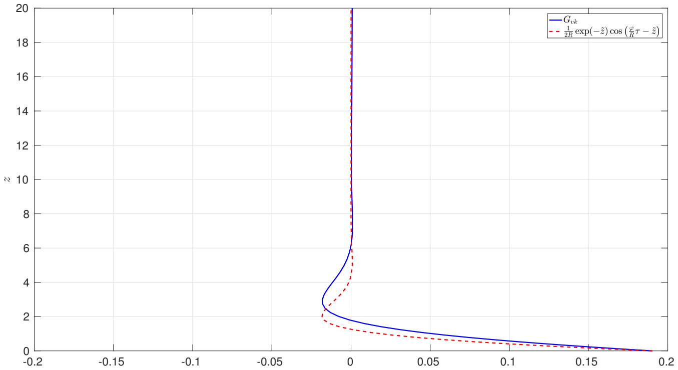
- Which gives Stokes layer:

$$F = 0$$
$$G = \frac{1}{R}\exp(-\tilde{z})\cos(\tilde{\tau} - \tilde{z})$$

Connections with the Stokes Layer



Comparison between Stokes layer profile and base flow variation

DNS - Wall Motion

- Wall displacement for stationary forcing (steady, rotating frame):

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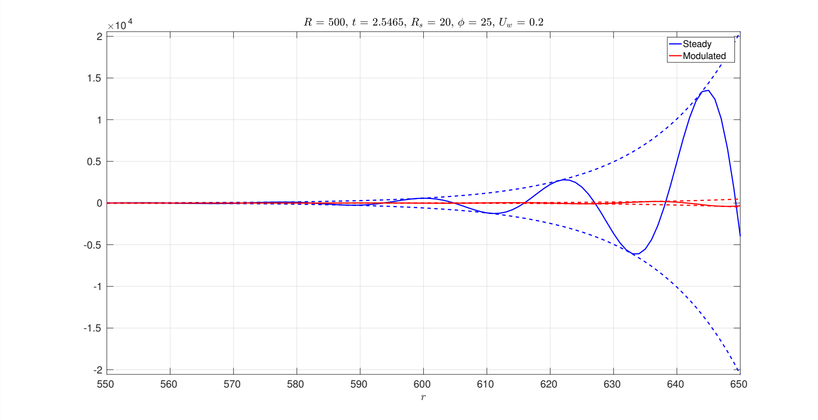
DNS - Stationary Forcing

- Prescribe wall motion $\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{in \int^{\tau} \Omega(\tilde{\tau})} e^{in\theta_0}$

DNS - Stationary Forcing

- Prescribe wall motion $\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{in \int^{\tau} \Omega(\tilde{\tau})} e^{in\theta_0}$

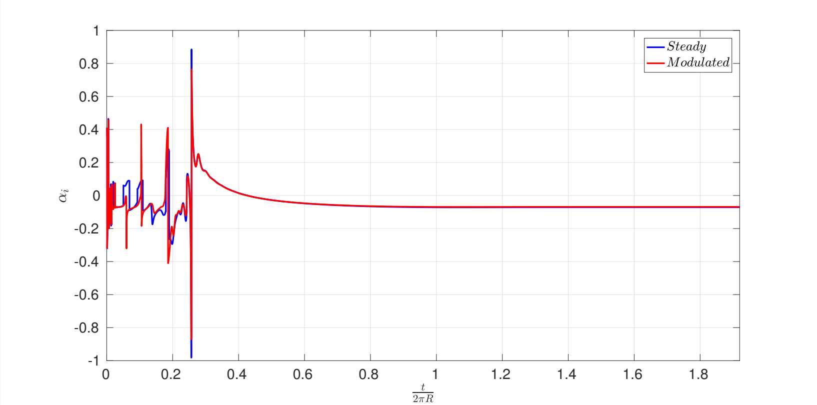


Radial evolution: $R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

- Receptivity issues.

DNS - Stationary Forcing

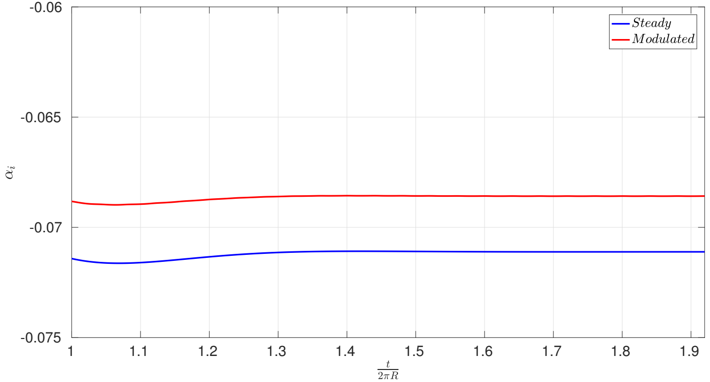
- Calculate $\alpha = -\frac{i}{A}\frac{\partial A}{\partial t}$ at fixed r.



$R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

DNS - Stationary Forcing

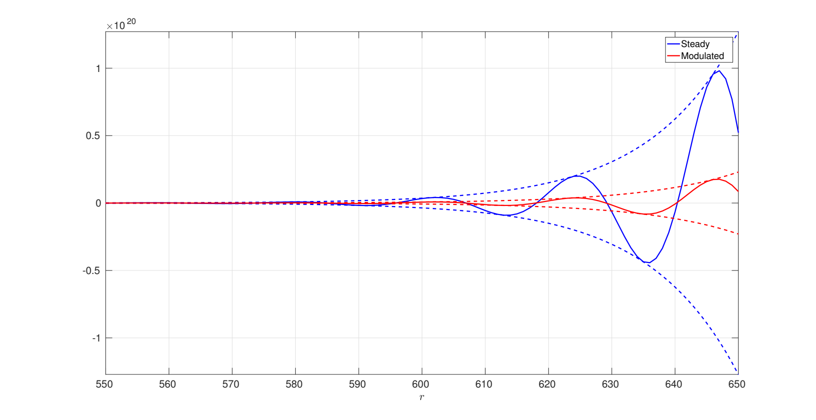
- Calculate $\alpha = -\frac{i}{A}\frac{\partial A}{\partial t}$ at fixed r.



$R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

DNS - Stationary Forcing

- Exponential growth reconstructed from $e^{i\alpha r}$



$R = 500,\ n = 32,\ Rs = 20,\ \varphi = 25$

DNS - Results

- Exactly prescribe (α, μ, ψ) from Floquet theory at inflow.

$$\psi(r, \theta, z, \tau) = \hat{\psi}(z, \tau) e^{\mu \tau} e^{i \alpha r} e^{i n \theta}$$

DNS - Results

- Exactly prescribe (α, μ, ψ) from Floquet theory at inflow.

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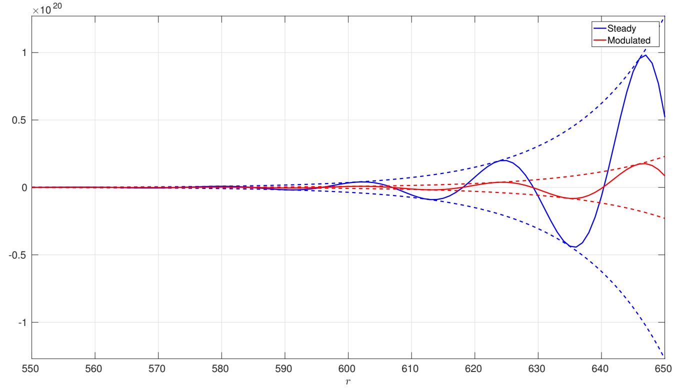


Radial evolution: $R = 500$, $n = 32$, $Rs = 20$, $\varphi = 25$

DNS - Results

- Calculate $\alpha = -\frac{i}{A}\frac{\partial A}{\partial t}$ for fixed r.

Stationary Forcing		
R_s	φ	α
0	N/A	0.2821 - 0.0712i
10	$\varphi = 25$	0.2820 - 0.0701i
	$\varphi = 50$	0.2820 - 0.0709i
20	$\varphi = 25$	0.2821 - 0.0686i
	$\varphi = 50$	0.2817 - 0.0702i

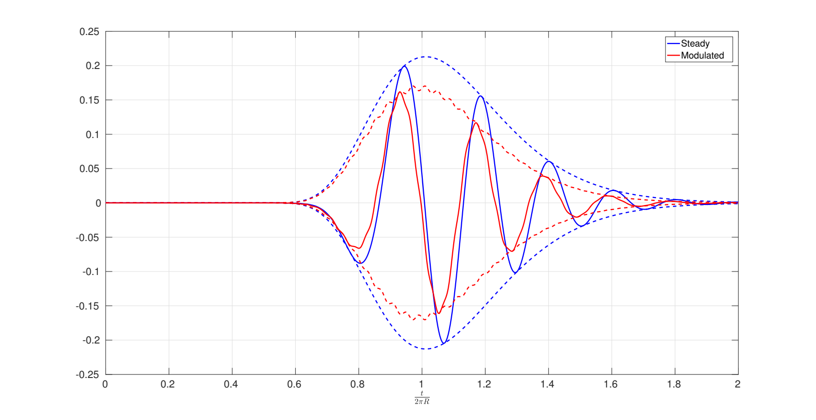
Inflow Prescribed Forcing		
R_s	φ	α
0	N/A	0.2821 - 0.0712i
10	$\varphi = 25$	0.2820 - 0.0701i
	$\varphi = 50$	0.2820 - 0.0709i
20	$\varphi = 25$	0.2821 - 0.0685i
	$\varphi = 50$	0.2818 - 0.0702i

DNS - Impulsive Forcing

- Prescribe impulsive wall motion $\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{-\sigma t^2}$

DNS - Impulsive Forcing

- Prescribe impulsive wall motion $\zeta(r, \theta_0, \tau) = e^{\lambda r^2} e^{-\sigma t^2}$



Temporal evolution: $R = 350$, $n = 32$, $Rs = 20$, $\varphi = 25$