Integration

Scott Morgan

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scott3142.com — @Scott3142

Topics

- Integration by completing the square
- Integration by substitution

- Integration by completing the square

Completing the Square

Examples:

Complete the square for the following quadratics:

$$x^2 + 6x + 1$$
$$5 - 8x - x^2$$
$$2x^2 - 8x + 11$$

Integration by Completing the Square

Examples:

Find

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx$$

$$\int \frac{1}{x^2 + 5x + 7} dx$$

$$\int \frac{1}{1 - 4x - 2x^2} dx$$

Integration by Completing the Square

Examples:

Find

$$\int \frac{1}{\sqrt{x^2 - 2x + 3}}\, dx$$

$$\int \frac{1}{\sqrt{1 - 4x - x^2}}\, dx$$

$$\int \frac{1}{\sqrt{x^2 - 2x - 5}}\, dx$$

$$\int \frac{1}{\sqrt{2x^2 + 5x + 1}}\, dx$$

$$\int \frac{1}{\sqrt{3 - x - x^2}}\, dx$$

Topics

- Integration by substitution

Trigonometric Substitutions

Expressions of the form $\sqrt{a^2 - x^2}$ can be reduced to the square root of a single term by a substitution either of the form

$$x = a\sin(\theta)$$

or of the form

$$x = a\cos(\theta)$$

Examples:

By making a substitution write the following as a single trigonometric term in terms of θ

$$\sqrt{9 - x^2}$$
$$\sqrt{25 - x^2}$$
$$\sqrt{1 - 4x^2}$$
$$\sqrt{4 - 9x^2}$$
$$\sqrt{25 - 16x^2}$$

Trigonometric Substitutions

Expressions of the form $\sqrt{a^2 - x^2}$ can be reduced to the square root of a single term by a substitution of the form

$$x = a \sinh(\theta)$$

Expressions of the form $\sqrt{x^2 - a^2}$ can be reduced to the square root of a single term by a substitution of the form

$$x = a \cosh(\theta)$$

Integration by Completing the Square

Examples:

By making a substitution write the following as a single trigonometric term in terms of θ

$$\sqrt{x^2 - 25}$$
$$\sqrt{x^2 + 16}$$
$$\sqrt{4x^2 - 1}$$
$$\sqrt{25x^2 - 1}$$
$$\sqrt{9x^2 - 4}$$
$$\sqrt{4x^2 + 25}$$

Integration by Completing the Square

Example:

By making the substitution $x = 2\cosh(u)$, find

$$\int \frac{x^2}{\sqrt{x^2 - 4}} \, dx$$

Integration by Completing the Square

Example:

By making the substitution $x = \sinh(u)$, find

$$\int (x^2 + 1)^{-\frac{3}{2}} dx$$

Integration by Completing the Square

Exam Question:

(a) Assuming the derivatives of $\sinh(\theta)$ and $\cosh(\theta)$, use the quotient rule to find the derivative of $\coth(\theta)$ [3]

(b) Use the substitution $x = \cosh(\theta) - 2$ to evaluate the integral

$$\int_1^2 \frac{dx}{(x^2 + 4x + 3)^{\frac{3}{2}}}$$

giving your answer correct to three significant figures. [8]

Integration by Completing the Square

Exam Question:

(a) By expressing $\text{sech}(x)$ and $\tanh(x)$ in terms of exponential functions, show that

$$\text{sech}^2(x) + \tanh^2(x) = 1$$

(b) Use the substitution $x = \sinh(u)$ to evaluate the integral

$$\int_0^1 \frac{x^2}{(1 + x^2)^{\frac{3}{2}}}\,dx$$

giving your answer correct to three significant figures.