# Trigonometry, Sets & Functions

**Scott Morgan**

*Further Mathematics Support Programme - WJEC A-Level Further Mathematics*
*17th February 2018*

*scott3142.com — @Scott3142*

# Trigonometry, Sets & Functions

1. Sets & Functions

# Properties of Real Functions

- Even/Odd/Neither

# Properties of Real Functions

- Even/Odd/Neither
- Continuous/Discontinuous

# Properties of Real Functions

- Even/Odd/Neither
- Continuous/Discontinuous
- Increasing/Decreasing/Neither

# Properties of Real Functions

- Even/Odd/Neither
- Continuous/Discontinuous
- Increasing/Decreasing/Neither
- Bounded/Unbounded

# Real Functions - Even & Odd

- A function is **even** if its graph has the $y$ axis as a line of symmetry.

# Real Functions - Even & Odd

- A function is **even** if its graph has the $y$ axis as a line of symmetry.
  - For an even function, $f(-x) = f(x)$

# Real Functions - Even & Odd

- A function is **even** if its graph has the $y$ axis as a line of symmetry.
  - For an even function, $f(-x) = f(x)$
- A function is **odd** if its graph has *rotational symmetry of order 2* about the origin.

# Real Functions - Even & Odd

- A function is **even** if its graph has the $y$ axis as a line of symmetry.
  - For an even function, $f(-x) = f(x)$
- A function is **odd** if its graph has *rotational symmetry of order 2* about the origin.
  - For an odd function, $f(-x) = -f(x)$

# Real Functions - Even & Odd

- A function is **even** if its graph has the $y$ axis as a line of symmetry.
    - For an even function, $f(-x) = f(x)$
- A function is **odd** if its graph has *rotational symmetry of order 2* about the origin.
    - For an odd function, $f(-x) = -f(x)$
- A function can be *neither* even *nor* odd.

# Real Functions - Even & Odd

- A function is **even** if its graph has the $y$ axis as a line of symmetry.
  - For an even function, $f(-x) = f(x)$
- A function is **odd** if its graph has *rotational symmetry of order 2* about the origin.
  - For an odd function, $f(-x) = -f(x)$
- A function can be *neither* even *nor* odd.
- Can you think of any examples?

# June 2008 - Q1

- For each of the following functions state, with a reason, whether it is *even*, *odd* or *neither even nor odd*.

# June 2008 - Q1

- For each of the following functions state, with a reason, whether it is *even*, *odd* or *neither even nor odd*.

    1. $\frac{x}{x^2+1}$

# June 2008 - Q1

- For each of the following functions state, with a reason, whether it is *even*, *odd* or *neither even nor odd*.

    1. $\frac{x}{x^2+1}$

- $f(-x) = \frac{-x}{(-x^2)+1} = -\left(\frac{x}{x^2+1}\right) = -f(x)$
- Hence, the function is odd.

# June 2008 - Q1

- For each of the following functions state, with a reason, whether it is *even*, *odd* or *neither even nor odd*.

  2. $e^x + 1$

# June 2008 - Q1

- For each of the following functions state, with a reason, whether it is *even*, *odd* or *neither even nor odd*.

  2. $e^x + 1$

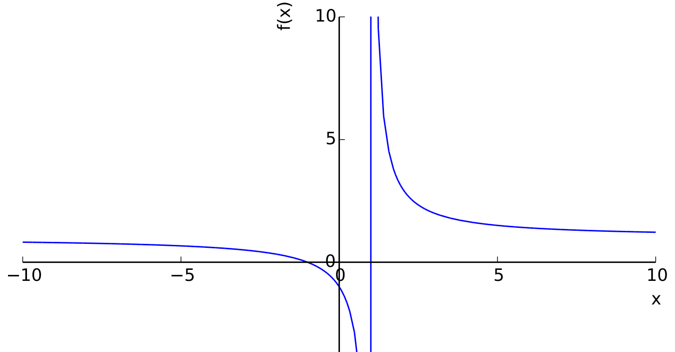
- $f(-x) = e^{-x} + 1$ which is neither $f(x)$ or $-f(x)$.
- Hence, the function is neither even nor odd.

# Continuous & Discontinuous

- A function $f(x)$ is called continuous if the graph of the function consists of a single unbroken line.

# Continuous & Discontinuous

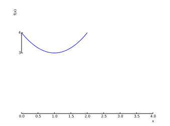
- A function $f(x)$ is called continuous if the graph of the function consists of a single unbroken line.
- The function $f(x) = \frac{x+1}{x-1}$ has a discontinuity at $x = 1$.



# Continuous & Discontinuous

- A function $f(x)$ is called continuous if the graph of the function consists of a single unbroken line.

- Consider the function $f(x) = \begin{cases} x^2 - 2x + 4, & x > 2 \\ -x^2 + 6x - 7, & x \leq 2 \end{cases}$



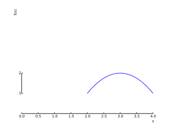


**Figure 1:** This function is made from two functions and has a jump at $x = 1$.

# Real Functions - Increasing/Decreasing

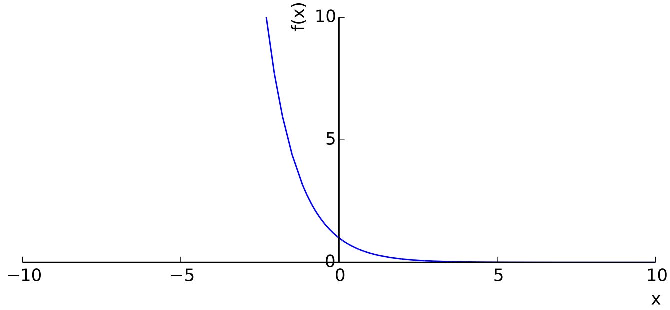
- A function $f(x)$ is called **strictly decreasing** if the value of $f(x)$ decreases as $x$ increases.

# Real Functions - Increasing/Decreasing

- A function $f(x)$ is called **strictly decreasing** if the value of $f(x)$ decreases as $x$ increases.
- This means that the gradient of the functions is negative (not including zero) for all values of $x$.

# Real Functions - Increasing/Decreasing

- A function $f(x)$ is called **strictly decreasing** if the value of $f(x)$ decreases as $x$ increases.
- This means that the gradient of the functions is negative (not including zero) for all values of $x$.



## Real Functions - Increasing/Decreasing

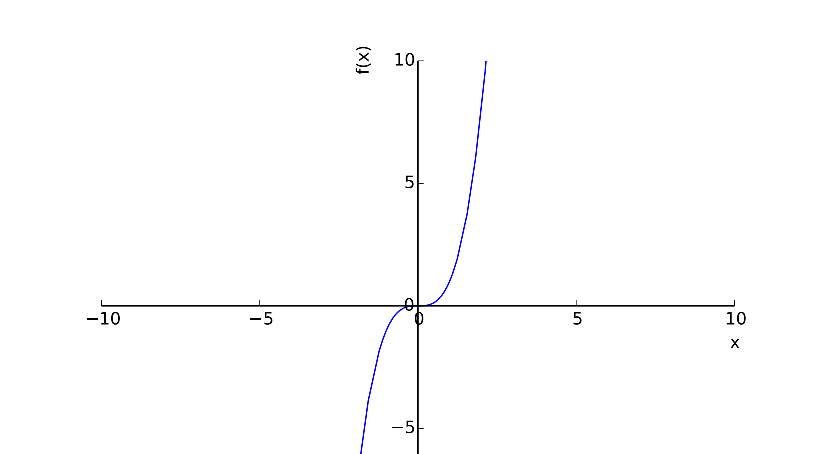
- A function $f(x)$ is called **strictly increasing** if the value of $f(x)$ increases as $x$ increases.

# Real Functions - Increasing/Decreasing

- A function $f(x)$ is called **strictly increasing** if the value of $f(x)$ increases as $x$ increases.

# Real Functions - Increasing/Decreasing

- A function $f(x)$ is called **strictly increasing** if the value of $f(x)$ increases as $x$ increases.
- This means that the gradient of the functions is positive (not including zero) for all values of $x$.



# Real Functions - Increasing/Decreasing

- Differentiating a function is often a good way to test these properties.

# June 2007 - Q8

- The function $f$ is defined on the domain $(0, 2)$ by

  $$f(x) = \begin{cases} 4x^2, & 0 < x < 1 \\ (x+1)^2, & 1 \leq x < 2 \end{cases}$$

  1. Determine whether or not $f$ is continuous when $x = 1$.
  2. Show that $f$ is a strictly increasing function.
  3. Obtain an expression for $f^{-1}(x)$ on each part of its domain.

# June 2011 - Q3

- The piecewise function $f$ is defined by
  $f(x) = \begin{cases} -x^2 + 6x - 7, & x \leq 2 \\ x^2 - 2x + 4, & x > 2 \end{cases}$
  1. Determine whether or not $f$ is continuous for all values of $x$.
  2. Determine whether or not $f$ is a strictly increasing function.
  3. The interval $[1, 3]$ is denoted by $A$. Determine $f(A)$.

2. Trigonometry

## Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.

# Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.
- We can use the **general solution formulae** to find them all.

# Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.
- We can use the **general solution formulae** to find them all.
- Let $p$ denote the *principle value* of the relevant trig inverse function at point $A$.

# Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.
- We can use the **general solution formulae** to find them all.
- Let $p$ denote the *principle value* of the relevant trig inverse function at point $A$.

$$\sin(\theta) = A$$

# Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.
- We can use the **general solution formulae** to find them all.
- Let $p$ denote the *principle value* of the relevant trig inverse function at point $A$.

$$\sin(\theta) = A \implies \theta = n\pi + (-1)^n p, \quad \text{for any integer } n$$

# Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.
- We can use the **general solution formulae** to find them all.
- Let $p$ denote the *principle value* of the relevant trig inverse function at point $A$.

$$\sin(\theta) = A \implies \theta = n\pi + (-1)^n p, \quad \text{for any integer } n$$

$$\cos(\theta) = A$$

# Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.
- We can use the **general solution formulae** to find them all.
- Let $p$ denote the *principle value* of the relevant trig inverse function at point $A$.

$$\sin(\theta) = A \implies \theta = n\pi + (-1)^n p, \quad \text{for any integer } n$$

$$\cos(\theta) = A \implies \theta = 2n\pi + p, \quad \theta = (2n-1)\pi - p, \quad \text{for any integer } n$$

# Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.
- We can use the **general solution formulae** to find them all.
- Let $p$ denote the *principle value* of the relevant trig inverse function at point $A$.

$$\sin(\theta) = A \implies \theta = n\pi + (-1)^n p, \quad \text{for any integer } n$$

$$\cos(\theta) = A \implies \theta = 2n\pi + p, \quad \theta = (2n-1)\pi - p, \quad \text{for any integer } n$$

$$\tan(\theta) = A$$

# Trigonometric Equations

- A trigonometric equation can have infinitely many solutions.
- We can use the **general solution formulae** to find them all.
- Let $p$ denote the *principle value* of the relevant trig inverse function at point $A$.

$$\sin(\theta) = A \implies \theta = n\pi + (-1)^n p, \quad \text{for any integer } n$$

$$\cos(\theta) = A \implies \theta = 2n\pi + p, \quad \theta = (2n-1)\pi - p, \quad \text{for any integer } n$$

$$\tan(\theta) = A \implies \theta = \pm p + 2n\pi, \quad \text{for any integer } n$$

# Examples

- Find the general solution, in degrees, to each of the following questions:
    1. $\sin(x) = 0.3$
    2. $\tan(x) = 1.5$
    3. $\cos(x) = -0.7$
    4. $\sin(x) = -0.6$

# Important Formulae

- These are in your formula booklet:
    1. *Little-t* formula:

# Important Formulae

- These are in your formula booklet:
  1. *Little-t* formula:
     - $t = \tan\left(\frac{1}{2}\theta\right)$

# Important Formulae

- These are in your formula booklet:
    1. *Little-t* formula:
        - $t = \tan\left(\frac{1}{2}\theta\right)$
        - $\sin(\theta) = \frac{2t}{1+t^2}$

# Important Formulae

- These are in your formula booklet:
    1. *Little-t* formula:
        - $t = \tan\left(\frac{1}{2}\theta\right)$
        - $\sin(\theta) = \frac{2t}{1+t^2}$
        - $\cos(\theta) = \frac{1-t^2}{1+t^2}$

# Important Formulae

- These are in your formula booklet:
  1. *Little-t* formula:
     - $t = \tan\left(\frac{1}{2}\theta\right)$
     - $\sin(\theta) = \frac{2t}{1+t^2}$
     - $\cos(\theta) = \frac{1-t^2}{1+t^2}$
  2. *Half-angle* formula:

# Important Formulae

- These are in your formula booklet:
  1. *Little-t* formula:
     - $t = \tan\left(\frac{1}{2}\theta\right)$
     - $\sin(\theta) = \frac{2t}{1+t^2}$
     - $\cos(\theta) = \frac{1-t^2}{1+t^2}$
  2. *Half-angle* formula:
     - $\sin(\alpha) + \sin(\beta) = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$

# Important Formulae

- These are in your formula booklet:
    1. *Little-t* formula:
        - $t = \tan\left(\frac{1}{2}\theta\right)$
        - $\sin(\theta) = \frac{2t}{1+t^2}$
        - $\cos(\theta) = \frac{1-t^2}{1+t^2}$
    2. *Half-angle* formula:
        - $\sin(\alpha) + \sin(\beta) = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$
        - $\sin(\alpha) - \sin(\beta) = 2\sin\left[\frac{1}{2}(\alpha-\beta)\right]\cos\left[\frac{1}{2}(\alpha+\beta)\right]$

# Important Formulae

- These are in your formula booklet:
  1. *Little-t* formula:
     - $t = \tan\left(\frac{1}{2}\theta\right)$
     - $\sin(\theta) = \frac{2t}{1+t^2}$
     - $\cos(\theta) = \frac{1-t^2}{1+t^2}$
  2. *Half-angle* formula:
     - $\sin(\alpha) + \sin(\beta) = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$
     - $\sin(\alpha) - \sin(\beta) = 2\sin\left[\frac{1}{2}(\alpha-\beta)\right]\cos\left[\frac{1}{2}(\alpha+\beta)\right]$
     - $\cos(\alpha) + \cos(\beta) = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$

# Important Formulae

- These are in your formula booklet:
  1. *Little-t* formula:
     - $t = \tan\left(\frac{1}{2}\theta\right)$
     - $\sin(\theta) = \frac{2t}{1+t^2}$
     - $\cos(\theta) = \frac{1-t^2}{1+t^2}$
  2. *Half-angle* formula:
     - $\sin(\alpha) + \sin(\beta) = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$
     - $\sin(\alpha) - \sin(\beta) = 2\sin\left[\frac{1}{2}(\alpha-\beta)\right]\cos\left[\frac{1}{2}(\alpha+\beta)\right]$
     - $\cos(\alpha) + \cos(\beta) = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$
     - $\cos(\alpha) - \cos(\beta) = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right]$

## Important Formulae - When to Use Them:

- Use the *little-t* formula when you are solving an equation with *both* cos and sin:

$$a\sin(\theta) + b\cos(\theta) = c$$

# Important Formulae - When to Use Them:

- Use the *little-t* formula when you are solving an equation with *both* cos and sin:

$$a\sin(\theta) + b\cos(\theta) = c$$

- Use the *half-angle* formula when you have *only* cos or sin:

$$\sin(a\theta) + \sin(b\theta) = c$$
$$\cos(a\theta) + \cos(b\theta) = c$$

# June 2006 - Q5

- By putting $t = \tan\left(\frac{\theta}{2}\right)$, find the general solution of the equation:
    1. $3\cos(\theta) + 4\sin(\theta) = 3 - \tan\left(\frac{\theta}{2}\right)$

## June 2009 - Q4

- Find the general solution to the equation:
    1. $\sin(\theta) + \sin(2\theta) + \sin(3\theta) = 0$