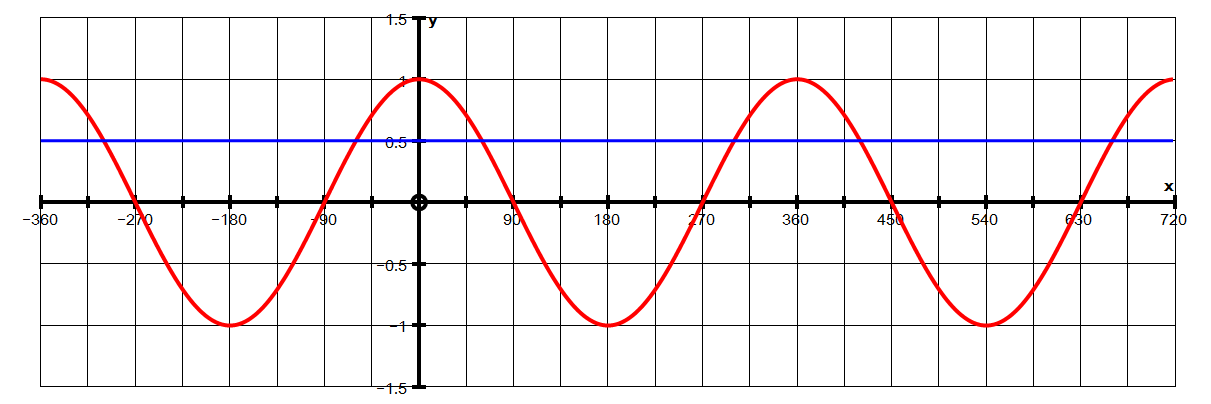
**FP2 : Trigonometry and Complex Numbers**

**1.1 General Solution of Trigonometric Equations**

Consider the equation

Using a calculator, the first solution is obtained:



*y* = cos *x*

Consideration of the symmetry of the cosine graph will enable us to find the other solutions within a specific domain. For example,

the solution of is

The roots of the equation that are in a similar position (just after the maximum point) on the cosine wave as 60° are

*θ* = ……., −660°, −300°, 60°, 420°, 780°, …….

and we can summarise this list by writing

where *k* is an integer.

The roots of the equation that are in a similar position (just after the maximum point) on the cosine wave as −60° are

*θ* = ……., −780°, −420°, −60°, 300°, 660°, …….

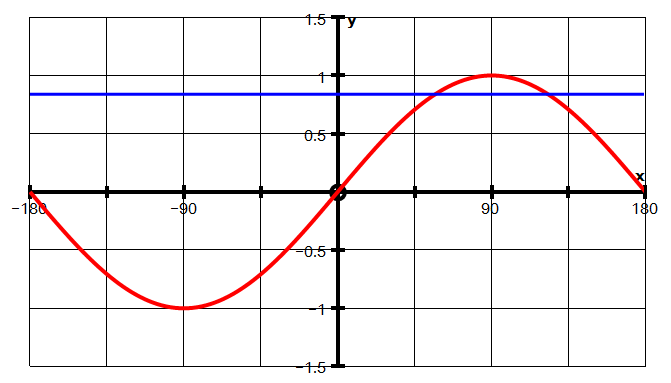
and we can summarise this list by writing

where *k* is an integer.

The **GENERAL SOLUTION** of a trigonometric equation is a rule that gives ALL the possible roots of the equation. The general solution of the equation can be written as

where *k* is an integer

If, instead, we were working in radians then the general solution of the equation would be written as where *k* is an integer

**Example 1**: Find the general solution (in degrees) of the equation

Clearly is a solution;

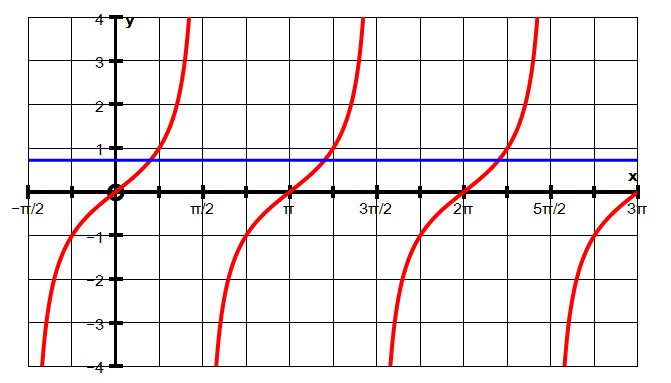
The symmetry of the sine graph gives another solution in a different position of the sine wave: ;

Other solutions will be of the form or

where *k* is an integer.

The general solution is where *k* is an integer.

**Example 2**: Find the general solution (in radians) of the equation

Clearly is a solution; other solutions are

π, 2π, … further to the left or right on the curve.

The general solution is where *k* is an integer.

|  |  |  |
| --- | --- | --- |
| **Equation** | **General Solution (degrees)** | **General Solution (Radians)** |
|  | Where *k* is an integer. | Where *k* is an integer. |
|  | Where *k* is an integer. | Where *k* is an integer. |
|  | Where *k* is an integer. | Where *k* is an integer. |

**Exercise 1**

1) Find the general solution (in degrees) of the following equations:

a) b) c) d)

2) Find the general solution (in radians) of the following equations:

a) b) c) d)

**Example 3**: Find the general solution (in degrees) of the equation

Since −0.5 = sin (−30°), the equation can be written as

* where *k* is an integer.

**Example 4**: Find the general solution (in radians) of the equation

We know that

⇒ where *k* is an integer;

so

⇒

* where *k* is an integer.

**Example 5**: Find the general solution (in radians) of the equation

First of all, remember that

so

where *k* is an integer.

Consideration of the compound angle formulae should rapidly convince you of the validity of the identities

and

which can prove useful in solving equations:

**Example 6**: Find the general solution (in degrees) of the equation

Using the fact that with ,

we know that .

⇒

⇒

⇒

⇒

⇒ where *k* is an integer.

**Exercise 2**

1) Find the general solution (in degrees) of the following equations:

a) b) c)

2) Find the general solution (in radians) of the following equations:

a) b) c)

3) Find the general solution (in degrees) of the following equations:

a) b)

**1.2 The Factor Formulae for Trigonometric Functions**

Recall the Compound Angle Formulae:

[1]

[2]

[3]

[4]

[1] + [2] gives

[5]

Similarly, [1] − [2] gives

[6]

[3] + [4] gives

[7]

and [3] − [4] gives

[8]

Equations [5] to [8] are useful when we wish to express a product of trigonometric functions as a sum or difference of trigonometric functions. One obvious application is in integration:

**Example 7 :**  Evaluate

Equation [7] tells us that

Putting gives

and remembering that

means that

So

The **Factor Formulae** are a immediate consequence of equations [5] to [8].

If we write

and

then and

so and

Equation [5] becomes [9]

Equation [6] becomes [10]

Equation [7]

becomes [11]

Equation [8]

becomes [12]

**The Factor Formulae**

These formulae are in the WJEC formula booklet !

The factor formulae are particularly useful in the solution of the equations to convert a sum or difference of trigonometric functions into a product of trigonometric functions.

**Example 8 :**  Solve the equation

Using the first factor formula

with *A* = *x* and *B* = 5*x* gives

So the equation

Don’t divide by cos 2*x*:

It might be 0 !

can be rewritten as

⇒

⇒

⇒

⇒

⇒

**Example 9 :** Find the general solution (in radians) of the equation

Using the factor formulae and remembering that cos(−θ) = cos θ, we could write

or

or

Only the second of these possibilities will really help solve the equation:

⇒

⇒

⇒

⇒

⇒

where *k* is an integer.

**Example 10 :** Provethat   
.

and hence find the general solution, in degrees, of the equation

The factor formulae give

as required.

⇒

⇒

⇒ ⇒ where *k* is an integer

**Exercise 3**

1) Find

a) b)

2) Use the factor formulae to solve the equations

a)

b)

c)

d)

3) Find the general solution, in radians, of the equation

4) Find the general solution, in degrees of the equation

5) Find the general solution, in radians, of the equation

6) Prove that . Hence find the value of

7) Prove that .

Hence find the general solution, in degrees, of the equation .

**1.3 The *t* = tan ½*x* substitution**

If

then

|  |  |
| --- | --- |
|  | and    The formulae for sin *x* and cos *x* are in the WJEC formula booklet ! |

So, we have the three results:

If then   
 ;

The formulae for sin *x* and cos *x* are in the WJEC formula booklet !

and these results may be used to solve a range of trigonometric equations.

**Example 11 :** Use the substitution to find the general solution (in degrees) of the equation

|  |  |
| --- | --- |
|  |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ | Where *k* is an integer. |

(NB : The R,α method of the C4 module could also be used to solve this equation.)

**Example 12 :** Use the substitution to find the general solution (in radians) of the equation

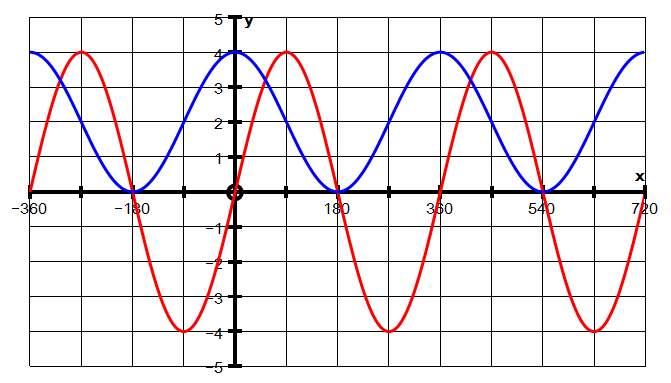
|  |  |
| --- | --- |
|  |  |
| ⇒ | Don’t cancel out the *t* s because, if you do you will lose the *t* = 0 solutions ! |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ | Where *k* is an integer. |

Now consider the equation

If we use the substitution to solve the equation we obtain

|  |  |
| --- | --- |
|  |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ |  |
| ⇒ | Where *k* is an integer. |

but if we consider the intersection of the graphs of and we can see that something is wrong:



**Our solution has lost the roots −180°, 180° , 540° etc.**

**What has gone wrong ??**

When *x* = −180°, 180° , 540° …..

then and for each of these values is not defined.

**How do overcome this problem ?**

**Once you have completed your solution using the substitution go back to the original equation and investigate whether or not 180° (or π radians) is a root of the equation: if it isn’t then your original solution is correct but if it is then you will need to amend your original solution by also including 180° + *k*360° (or π + 2*k*π radians).**

**The *t* = tan ½*x* substitution in Integration**

The substitution is useful in integrating functions of the form .

**Example 13 :** Evaluate 

Putting 

so  
 

**Exercise 4**

1) Use the substitution to find the general solution (in degrees) of the equation

2) Use the substitution to find the general solution (in radians) of the equation

3) Use the substitution to find the general solution (in degrees) of the equation

4) Use the substitution to find the general solution (in radians) of the equations

a) b)

5) Evaluate the following integrals :

a)  b) 

**2.1 Complex Numbers : de Moivre’s Theorem**

Consider  :

This result can be generalised :

**de Moivre’s Theorem:** If *n* is a positive integer then

**Proof :** The proof is by induction.

When *n* = 1 the statement is clearly true.

Suppose the statement is true for *n* = *k*. We aim to show that it is also true for *n* = *k* + 1:

The compound angle formulae tell us that

and that

so, if the de Moivre’s Theorem is true for *n* = *k* then it is also true for *n* = *k* + 1.

We know it is true for n = 1 so it is also true for *n* = 2; it is true for n = 2 so it is also true for *n* = 3;

is true for n = 3 so it is also true for *n* = 4; ………….. and this completes the proof by induction.

**Extension:** If and *n* is an integer then

**Proof :**

We know the statement is true for positive integers.

For *n* = 0 and

so the result is true for *n* = 0.

If *n* is a negative integer then *n* = *p* where *p* is positive so

but *p* = *n*, and remembering that

as required.

Suppose |*z*| = *r* and arg *z* = and *n* is an integer then we know that ⇒ so the modulus of is and the argument of is *n*.

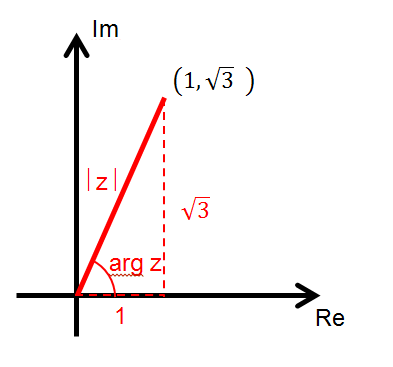
It is quite useful to have a shorthand for this:

If |*z*| = *r* and arg *z* = then we say that *z* = [ *r*, ].

We have just proved that

If *n* is an integer and then

de Moivre’s theorem gives a quick way of finding powers of complex numbers:

**Example 14 :** Find the value of  in the form *p* + *iq*.

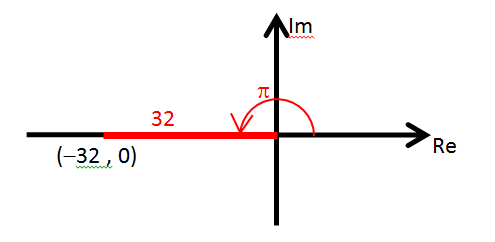
If then

and

⇒

but so an argument of is the same as an argument of .

We can also use de Moivre’s theorem to find roots of complex numbers.

**Example 15 :** Find all solutions of the equation . Prove that if the roots are plotted on an Argand diagram then they form the vertices of a regular pentagon.

From the argand diagram we know that

so

Suppose that

We want

is one solution of the equation.

However, we could equally well have written

For we require

which gives

but and are the same complex number; and are the same complex number; etc.

So, in fact, there are only **five** distinct solutions:

The solutions of the equation are

and

From the modulus and argument forms of these solutions we can see that all the solutions lie on the circle |*z*| = 2 and that the solutions are equally spaced around this circle. The solutions therefore are the vertices of a regular pentagon.

Notice also that *z*1 and *z*4 are a complex conjugate pair and that *z*2 and *z*3 are also a complex conjugate pair: this should not be a surprise since in FP1 you saw that the roots of a polynomial with **real coefficients are either real or appear in complex conjugate pairs.**

**Exercise 5**

1) Use de Moivre's theorem to write down alternative forms for

|  |  |  |
| --- | --- | --- |
| a) | b) | c) |
| d) | e) | f) |

2) Find the value of  expressing your answer in the form *p* + *iq .*

3)Find the value of  expressing your answer in the form *p* + *iq.*

4) Find all the solutions of the equation , giving your answers in the form . The solutions are plotted on an Argand diagram and form the vertices of a quadrilateral. What is the area of the quadrilateral ?

5) Find the modulus and argument of each of the roots of the equation .

6) Find the modulus and argument of the complex number .

Hence find the three cube roots of , giving your answers in the form .

7) Giving your answers in the form , find the fifth roots of the complex number .

8) Given that where , find the values of *r* and *θ*. Hence find the four fourth roots of in the form , where the values *x* and *y* are correct to three significant figures.

9) Given that ,

a) find the modulus and argument of *z*;

b) find the three cube roots of *z* in the form ;

c) prove that where *z* is a positive real number whose value should be determined.

**2.2 Using de Moivre’s Theorem to Obtain Trigonometric Identities**

de Moivre’s theorem can be used to quickly establish a range of trigonometric identities.

**Example 16 :** Find an expansion for cos 4 in terms of cos .

de Movre’s theorem tells us that

so we can write ,

where denotes the real part of the complex number z.

If we introduce the shorthand then we have

Using Binomial expansion or Pascal’s triangle

⇒

⇒

but

**Example 17 :** If , prove that

and hence solve the equation .

|  |  |
| --- | --- |
|  |  |
| ⇒ | where |
| ⇒ |  |
|  | Dividing top and bottom of this fraction by gives  where |
|  | and this is the required expression. |
| Now |  |
|  |  |
| ⇒ |  |
| If we now let we obtain | |
|  |  |
| The equation has general solution    ⇒  where *k* is an integer. | |
| ⇒ |  |
| ⇒ |  |
| ⇒ | but the tan function has period so there only four distinct solutions: |

We know that if

then [1]

and [2]

so [3]

and [4]

These last two results can be used to rewrite powers of trigonometric function as a sum of simple trigonometric functions. For example, consider

Using [3] with *n* =1 gives ⇒

Using [4] with *n* =1 gives ⇒

Combining these results:

⇒

⇒

⇒

⇒

⇒

⇒

then using [3] with *n* = 6, 4, and 2

⇒

⇒

This result could then be used to quickly integrate :

# To summarise:

If then (de Moivre’s Theorem) and

From these results we can deduce that

and

**Exercise 6**

|  |  |
| --- | --- |
| 1) | a) Use de Moivre’s theorem to prove that    b) By putting , find the six roots of the equation ,  expressing the answers in trigonometric form. |
| 2) | a) Use de Moivre's theorem to prove that  b) Find the general solution of the equation 2 cos 3 = 1.  c) Hence solve the cubic equation . |
| 3) | a) By using de Moivre’s theorem, show that |
|  |  |
|  | b) By putting into this equation, show that |
| 4) | a) Write down de Moivre’s theorem for the case when *n* = 7.  b) Hence show that for |
|  | Where *A, B, C* and *D* are constants to be determined. |
|  | c) Deduce the limiting value of as *θ* tends to zero. |
| 5) | a) Given that show that .  b) Prove that  c) Hence find the general solution, in degrees, of the equation |
| 6) | a) Given that show that and write down a similar  result for .  b) By considering , or otherwise, express in the  form where *A, B* and *C* are constants.  c) Hence find |
| 7) | a) Given that show that  b) Prove that  c) Hence find the general solution, in degees, of the equation |
| 8) | By expanding , where , show that  Where *A, B* and *C* are constants to be determined.  Hence find |
|  |  |