Specification

Real functions.
The language of set theory, image and inverse image of a set under a function.

Notation to be used for image and inverse image:
$\mathrm{f}(A) = \{\mathrm{f}(x) : x \in A \text{ and } \mathrm{f}(x) \text{ is defined}\}$
$\mathrm{f}^{-1}(B) = \{x : \mathrm{f}(x) \text{ is defined and } \mathrm{f}(x) \in B\}$

Odd functions, even functions, strictly increasing functions, strictly decreasing functions, bounded functions.

Functions defined piecewise on their domain.

Informal treatment only.

The idea of continuity.

Sketching graphs of rational functions, including those in which the degree of the numerator exceeds that of the denominator.

For example:
$$y = \frac{9(x-3)}{(x+1)(x-2)}; \quad y = \frac{(x+1)^2}{(2x-3)}$$

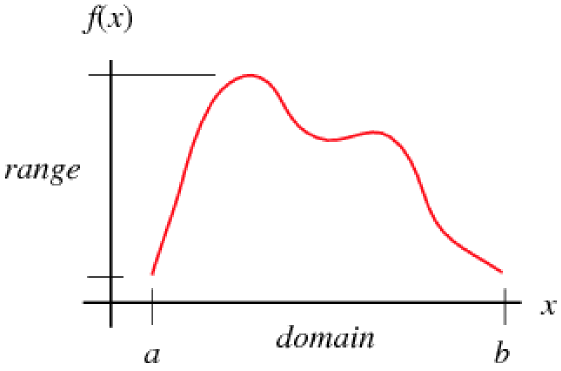
Asymptotes.

Including asymptotes which are not parallel to a coordinate axis.

Formulae and Notes

A function is a relation that uniquely associates members of one set with members of another set. More formally, a function from A to B is an object f such that every $a \in A$ is uniquely associated with an object $f(a) \in B$. A function is therefore a many-to-one (or sometimes one-to-one) relation. The set A of values at which a function is defined is called its domain, while the set $f(A) \subset B$ of values that the function can produce is called its range. Here, the set B is called the codomain of f.

Several notations are commonly used to represent (non-multivalued) functions. The most rigorous notation is $f : x \to f(x)$, which specifies that f is function acting upon a single number x (i.e., f is a univariate, or one-variable, function) and returning a value $f(x)$. To be even more precise, a notation like "$f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$" is sometimes used to explicitly specify the domain and codomain of the function. The slightly different "maps to" notation $f : x \mapsto f(x)$ is sometimes also used when the function is explicitly considered as a "map."



If $f: D \to Y$ is a map (a.k.a. function, transformation, etc.) over a domain D, then the range of f, also called the image of D under f, is defined as the set of all values that f can take as its argument varies over D, i.e.,

$$\text{Range}(f) = f(D) = \{f(\mathbf{X}) : \mathbf{X} \in D\}.$$

Note that among mathematicians, the word "image" is used more commonly than "range."

The range is a subset of Y and does not have to be all of Y.

Given a function $f(x)$, its inverse $f^{-1}(x)$ is defined by

$$f(f^{-1}(x)) = f^{-1}(f(x)) \equiv x.$$

Therefore, $f(x)$ and $f^{-1}(x)$ are reflections about the line $y = x$.

Let f be a function defined on a set A and taking values in a set B. Then f is said to be an injection (or injective map, or embedding) if, whenever $f(x) = f(y)$, it must be the case that $x = y$. Equivalently, $x \neq y$ implies $f(x) \neq f(y)$. In other words, f is an injection if it maps distinct objects to distinct objects. An injection is sometimes also called one-to-one.

Let f be a function defined on a set A and taking values in a set B. Then f is said to be a surjection (or surjective map) if, for any $b \in B$, there exists an $a \in A$ for which $b = f(a)$. A surjection is sometimes referred to as being "onto."

A map is called bijective if it is both injective and surjective. A bijective map is also called a bijection. A function f admits an inverse f^{-1} (i.e., "f is invertible") iff it is bijective.

Even functions [edit]

Let *f*(*x*) be a real-valued function of a real variable. Then *f* is **even** if the following equation holds for all *x* and -*x* in the domain of *f*:[1]

$$f(x) = f(-x),$$

or

$$f(x) - f(-x) = 0.$$

Geometrically speaking, the graph face of an even function is symmetric with respect to the *y*-axis, meaning that its graph remains unchanged after reflection about the *y*-axis.

Examples of even functions are |*x*|, x^2, x^4, cos(*x*), and cosh(*x*).

Odd functions [edit]

Again, let $f(x)$ be a real-valued function of a real variable. Then f is **odd** if the following equation holds for all x and $-x$ in the domain of f:[2]

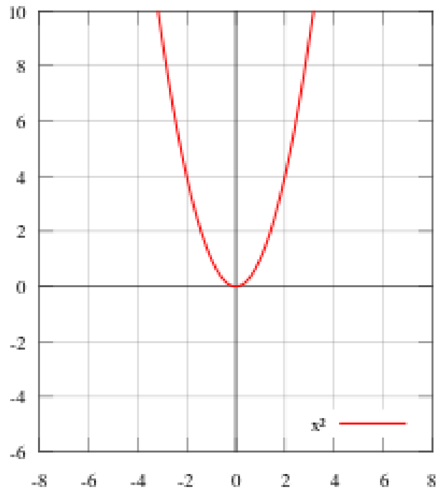
$$-f(x) = f(-x),$$

or

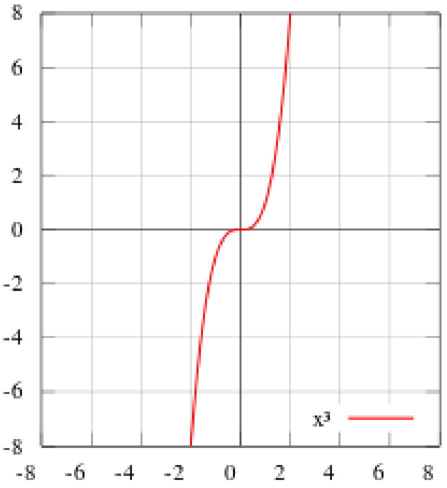
$$f(x) + f(-x) = 0.$$

Geometrically, the graph of an odd function has rotational symmetry with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin.

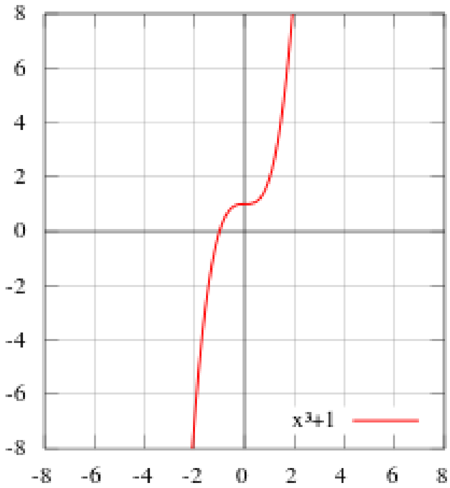
Examples of odd functions are x, x^3, $\sin(x)$, $\sinh(x)$, and $\mathrm{erf}(x)$.



$f(x) = x^2$ is an example of an even function.



$f(x) = x^3$ is an example of an odd function.



$f(x) = x^3 + 1$ is neither even nor odd.

Uniqueness properties [edit]

- If a function is even and odd, it is equal to 0 everywhere it is defined.
- If a function is odd, the absolute value of that function is an even function.

Properties involving addition and subtraction [edit]

- The sum of two even functions is even, and any constant multiple of an even function is even.
- The sum of two odd functions is odd, and any constant multiple of an odd function is odd.
- The difference between two odd functions is odd.
- The difference between two even functions is even.
- The sum of an even and odd function is neither even nor odd, unless one of the functions is equal to zero over the given domain.

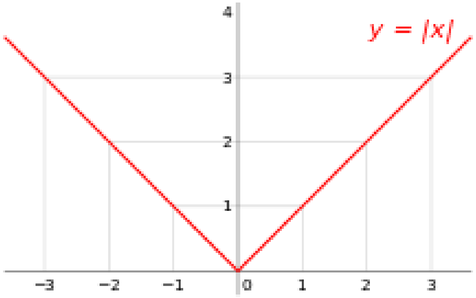
Properties involving multiplication and division [edit]

- The product of two even functions is an even function.
- The product of two odd functions is an even function.
- The product of an even function and an odd function is an odd function.
- The quotient of two even functions is an even function.
- The quotient of two odd functions is an even function.
- The quotient of an even function and an odd function is an odd function.

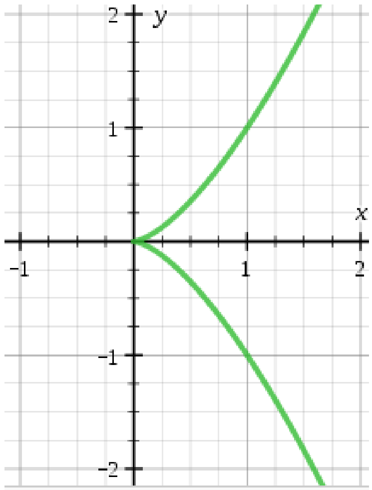
In calculus (a branch of mathematics), a **differentiable function** of one real variable is a function whose derivative exists at each point in its domain. As a result, the graph of a differentiable function must have a (non-vertical) tangent line at each point in its domain, be relatively smooth, and cannot contain any breaks, bends, or cusps.

More generally, if x_0 is a point in the domain of a function *f*, then *f* is said to be **differentiable at x_0** if the derivative $f'(x_0)$ exists. This means that the graph of *f* has a non-vertical tangent line at the point $(x_0, f(x_0))$. The function *f* may also be called **locally linear** at x_0, as it can be well approximated by a linear function near this point.

A differentiable function



The absolute value function is continuous (i.e. it has no gaps). It is differentiable everywhere *except* at the point $x = 0$, where it makes a sharp turn as it crosses the y-axis.



An ordinary cusp on the cubic curve (semicubical parabola) $x^3 - y^2 = 0$, which is equivalent to the multivalued function $f(x) = \pm x^{3/2}$. This relation is continuous, but is not differentiable at

If *f* is differentiable at a point x_0, then *f* must also be continuous at x_0. In particular, any differentiable function must be continuous at every point in its domain. *The converse does not hold*: a continuous function need not be differentiable. For example, a function with a bend, cusp, or vertical tangent may be continuous, but fails to be differentiable at the location of the anomaly.

In practice, it is convenient to use the following three conditions of continuity of a function $f(x)$ at point $x = a$:

1. Function $f(x)$ is defined at $x = a$;
2. Limit $\lim_{x \to a} f(x)$ exists;
3. It holds that $\lim_{x \to a} f(x) = f(a)$.

Continuity Theorems

Theorem 1.

Let the function $f(x)$ be continuous at $x = a$ and let C be a constant. Then the function $Cf(x)$ is also continuous at $x = a$.

Theorem 2.

Let the functions $f(x)$ and $g(x)$ be continuous at $x = a$. Then the sum of the functions $f(x) + g(x)$ is also continuous at $x = a$.

Theorem 3.

Let the functions $f(x)$ and $g(x)$ be continuous at $x = a$. Then the product of the functions $f(x)g(x)$ is also continuous at $x = a$.

Theorem 4.

Let the functions $f(x)$ and $g(x)$ be continuous at $x = a$. Then the quotient of the functions $\frac{f(x)}{g(x)}$ is also continuous at $x = a$ assuming that $g(a) \neq 0$.

Theorem 5.

Let $f(x)$ be differentiable at the point $x = a$. Then the function $f(x)$ is continuous at that point.

Remark: The converse of the theorem is not true, that is, a function that is continuous at a point is not necessarily differentiable at that point.

Theorem 6 (Extreme Value Theorem).

If $f(x)$ is continuous on the closed, bounded interval $[a, b]$, then it is bounded above and below in that interval. That is, there exist numbers m and M such that

$$m \leq f(x) \leq M$$

for every x in $[a, b]$ (see Figure 1).

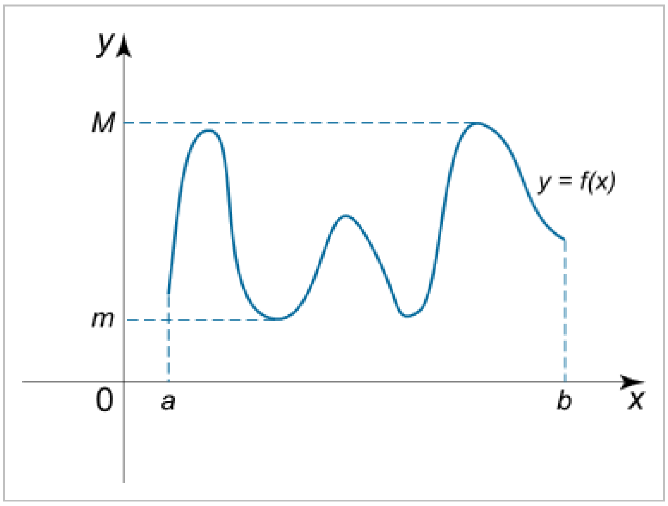


Figure 1.

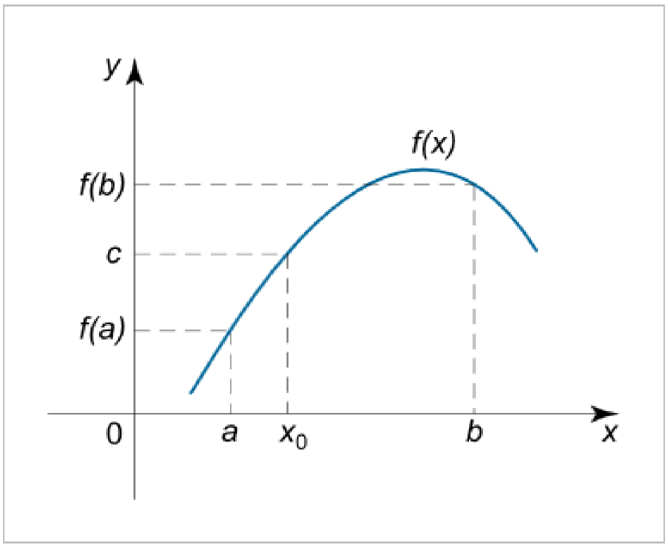


Figure 2.

Theorem 7 (Intermediate Value Theorem).

Let $f(x)$ be continuous on the closed, bounded interval $[a, b]$. Then if c is any number between $f(a)$ and $f(b)$, there is a number x_0 such that

$$f(x_0) = c.$$

The intermediate value theorem is illustrated in Figure 2.

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x = c$ is a critical point of $f(x)$ provided either

1. $f'(c) = 0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

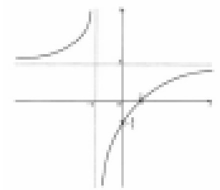
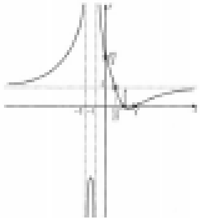
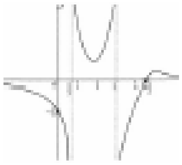
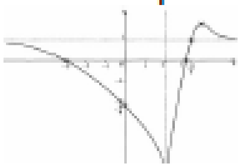
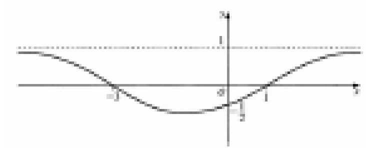
Partial Fractions : If integrating $\int \frac{P(x)}{Q(x)}dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\dfrac{A}{ax+b}$	$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Curve Sketching Procedure

Questions tend to lead you through some or all of the following steps:

General Procedure for $y = f(x)$	Rational Functions: $y = \frac{p(x)}{q(x)}$
1. Mark in VERTICAL ASYMPTOTES (usually caused by attempted division by 0). Consider behaviour for values of x close to the asymptotes	Look for where $q(x) = 0$
2. Find points where the curve meets the axes. 3. Consider behaviour of curve as $x \to \pm\infty$. This may give horizontal or slant (oblique) asymptotes).	If deg(p) < deg(q) then $y \to 0$ as $x \to \pm\infty$. If deg(p) = deg(q) then there will be a horizontal asymptote which may be found either by dividing the top and bottom of the fraction by x^n where n is the degree of p and q or by polynomial division. If deg(p) = deg(q) + 1 then there will be a slant asymptote whose equation can be found using polynomial division
4. If necessary, consider stationary points	
5. Use the information to sketch the curve.	

$y = \dfrac{4x - 8}{x + 3}$	Linear numerator and linear denominator 	1 horizontal asymptote 1 vertical asymptote
$y = \dfrac{(x - 3)(2x - 5)}{(x + 1)(x + 2)}$	**2 distinct linear factors in the denominator** – quadratic numerator 	2 vertical asymptotes 1 horizontal asymptote **The curve will usually cross the horizontal asymptote**
$y = \dfrac{2x - 9}{3x^2 - 11x + 6}$	2 distinct linear factors in the denominator – linear numerator 	2 vertical asymptotes 1 horizontal asymptote horizontal asymptote is $y = 0$
$y = \dfrac{(x - 3)(x + 3)}{(x - 2)^2}$	Quadratic numerator – quadratic denominator with equal factors 	1 vertical asymptote 1 horizontal asymptote
$y = \dfrac{x^2 + 2x - 3}{x^2 + 2x + 6}$	Quadratic numerator with no real roots for denominator (irreducible) 	The curve does not have a vertical asymptote

Question 1

For each of the following functions state, with a reason, whether it is even, odd or neither even nor odd.

(a) $\dfrac{x}{x^2+1}$ [2]

(b) $e^x + 1$ [2]

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Question 2

The functions f, g and h are defined as follows:

$$f(x) = \sin x$$

$$g(x) = |\,x\,|$$

$$h(x) = \frac{1}{x}$$

(a) State, with a reason, which one of the above functions is not continuous.

(b) State, with a reason, whether

(i) g is even or odd,

(ii) h is even or odd.

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Question 3

Let f be a function with domain $(-a, a)$ and define functions g and h as follows.

$$g(x) = f(x) + f(-x)$$
$$h(x) = f(x) - f(-x)$$

(a) Show that g is an even function and h is an odd function. Hence show that f can be expressed as the sum of an even function and an odd function. [3]

(b) Given that, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

$$f(x) = \ln(1 + \sin x),$$

(i) find and simplify an expression for $g(x)$,

(ii) show that

$$h(x) = 2\ln(\sec x + \tan x).$$ [7]

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Question 4

The function f is defined by

$$f(x) = 1 + ax^3 \quad \text{for } x < 2,$$
$$f(x) = bx^2 - 3 \quad \text{for } x \geqslant 2.$$

Given that both f and its derivative f' are continuous at $x = 2$, find the values of the constants a and b. [6]

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Question 5

The piecewise function f is defined by

$$f(x) = -x^2 + 6x - 7 \qquad (x \leqslant 2),$$

$$f(x) = x^2 - 2x + 4 \qquad (x > 2).$$

(a) Determine whether or not f is continuous for all values of x. [2]

(b) Determine whether or not f is a strictly increasing function. [4]

(c) The interval [1, 3] is denoted by A. Determine $f(A)$. [3]

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<u>Question 6</u>

The function f is defined as follows.

$$f(x) = x \qquad \text{for } x < 0,$$

$$f(x) = \sin x \qquad \text{for } x \geqslant 0.$$

Determine whether or not

(i) the function f,

(ii) its derivative f'

is continuous when $x = 0$.

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<u>Question 7</u>

The function f is defined on the domain $(0, 2)$ by

$$f(x) = 4x^2 \qquad \text{for } 0 < x < 1,$$

$$f(x) = (x+1)^2 \qquad \text{for } 1 \leqslant x < 2.$$

(a) Determine whether or not f is continuous when $x = 1$.

(b) Show that f is a strictly increasing function.

(c) Obtain an expression for $f^{-1}(x)$ on each part of its domain.

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Question 8

The function f is defined by

$$f(x) = \frac{x(x+3)}{x-1} \quad .$$

(a) Show that $f(x)$ can be written in the form

$$ax + b + \frac{c}{x-1}$$

where a, b, c are constants to be found.

(b) Find the coordinates of the stationary points on the graph of f.

(c) State the equation of each of the asymptotes on the graph of f and sketch the graph of f.

(d) Find $f^{-1}(A)$, where A is the interval $[0, 10]$.

Question 9

The function f is defined by

$$f(x) = \frac{x}{(x-1)^2}.$$

(a) Find the coordinates of the stationary point on the graph of f. [4]

(b) State the equation of each of the asymptotes of the graph of f. [2]

(c) Sketch the graph of f. [2]

(d) Find $f^{-1}(A)$, where A is the interval $[0, 2]$. [5]

Question 10

The function f is defined by

$$f(x) = \frac{(x+1)^2}{(x-1)(x-2)}.$$

(a) Prove that $f(x)$ can be written in the form

$$1 - \frac{4}{x-1} + \frac{9}{x-2}.$$

Hence find expressions for $f'(x)$ and $f''(x)$. [7]

(b) Find the coordinates of the stationary points on the graph of f and classify each point as a maximum or minimum. [6]

(c) State the equation of each of the asymptotes on the graph of f. [2]

(d) Sketch the graph of f. [3]