Dual Quaternion Slerp

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1 What's a slerp?

The killer app for dual quaternions is called the slerp. Spherical Linear Interpolation allows you to smoothly go from one set of rotations/translations to another. It also helps with the dreaded "candy-wrapper effect" as you can see in figure 1:

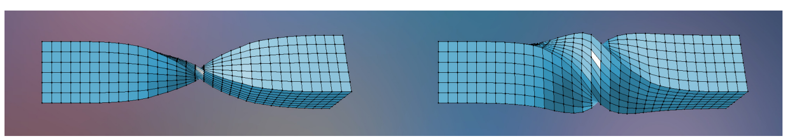

Figure 1: Linear vs Dual Quaternion skinning [Vai]

The slerp allows you to smoothly go from a starting to an ending dual quaternion using a parameter t. The number t goes from 0 to 1, corresponding to the start and the end.

Here is how you calculate the slerp in 6 easy steps [Tod]:

1. Determine the Start and End dual quaternions.
2. Calculate $\frac{\text{End}}{\text{Start}}$ which gives us the dual quaternion that goes from start to end.
3. Extract the axis/angle and translation from this new dual quaternion.
4. Multiply the angle and scale the translation vector by the parameter t.
5. Build a dual quaternion that contains this new angle and translation vector.

6. Your answer is this new dual quaternion multiplied by the Start.

Look away if you are allergic to math, but the steps above actually corresponds to this formula:

$$\boxed{e^{t\log(\text{End}/\text{Start})}\text{Start}}$$

where e is known as Euler's number $\approx 2.71828\ldots$

2 Slerp example

Imagine you start with a translation of $(2, 0, 0)$ and end with the same translation followed by a rotation of $180°$ about the z axis, as shown in figure 2:

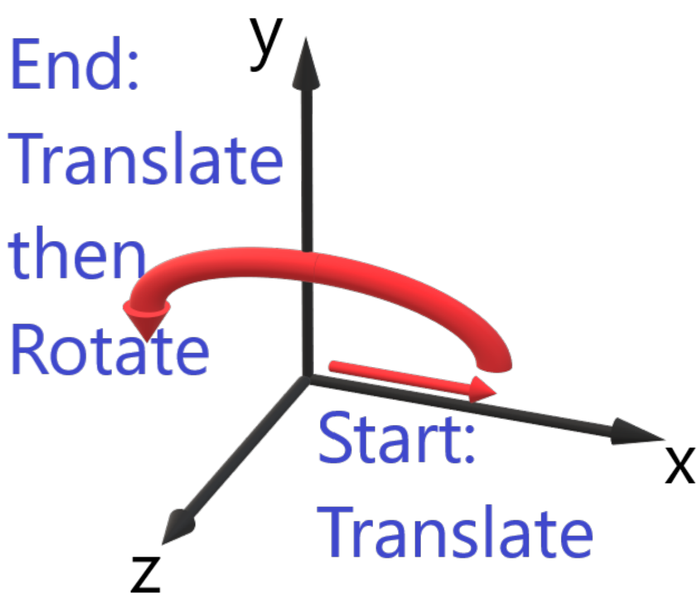

Figure 2: The start and ending dual quaternion motors

Imagine a point at the origin (0, 0, 0). If you slerped the point from start to end with $t = \frac{1}{2}$ where would the point go? Halfway from start to end should leave us straight up at position $(0, 2, 0)$. Let's see if we can work this out using the slerp steps!

2.1 Step 1: Determine start and end dual quaternions

The starting dual quaternion motor is a translation T of $[2, 0, 0]$ which is written as:

$$\text{Start} = T = 1 - \frac{2}{2}e_{01} = 1 - e_{01} \tag{1}$$

The ending dual quaternion is the translation T then a rotation R of 180° about the z axis, where R is written as:

$$\begin{aligned} R &= \cos(\frac{180^\circ}{2}) - \sin(\frac{180^\circ}{2})e_{12} \\ &= -e_{12} \end{aligned}$$

Perform the translation T followed by the rotation R to find the ending dual quaternion:

$$\begin{aligned} \text{End} &= RT \\ &= -e_{12}(1 - e_{01}) \\ &= -e_{12} + e_{12}e_{01} \\ &= -e_{12} - e_{02} \end{aligned} \tag{2}$$

2.2 Step 2: Calculate $\frac{\textbf{End}}{\textbf{Start}}$

The dual quaternion $\frac{\text{End}}{\text{Start}}$ gives us the one that goes from start to end. Remember we can divide by a dual quaternion by multiplying by it's reverse:

$$\frac{\text{End}}{\text{Start}} = \text{End}(\tilde{\text{Start}}) \tag{3}$$

Substituting in equations 1 and 2 into equation 3 we get:

$$\begin{aligned} \frac{\text{End}}{\text{Start}} &= (-e_{12} - e_{02})(1 - e_{10}) \\ &= (-e_{12} - e_{02})(1 + e_{01}) \\ &= -e_{12} - e_{12}e_{01} - e_{02} - e_{02}e_{01} \\ &= -e_{12} - e_{20} - e_{02} \\ &= -e_{12} \end{aligned} \tag{4}$$

This answer makes sense for our example, as it should be just a rotation of 180 ° about z, which is $-e_{12}$ written in geometric algebra.

2.3 Step 3: Extract the axis/angle and translation

Extracting the axis/angle and translation is pretty easy for $\frac{\text{End}}{\text{Start}} = -e_{12}$. The axis is $[0, 0, 1]$ and the angle is 180 °. There is no translation.

For a more complicated dual quaternion, you could follow these steps:

1. Extract the rotation axis and angle by remembering the first four elements of the dual quaternion $(w, e_{23}, e_{31}, e_{12})$ corresponds to a regular quaternion $(w, -i, -j, -k)$.

2. Extract the translation from the e_{01}, e_{02}, e_{03} parts of $t * \tilde{r} * 2$, where t refers to the infinite parts of $\frac{\text{End}}{\text{Start}}$ and $\tilde{r}$ refers to the rotation extracted above, but reversed [Ken12].

2.4 Step 4: Multiply the angle/translation vector by t

The quickest step! For our example we have $t = \frac{1}{2}$, so the new angle is $\frac{180}{2}^{\circ} = 90^{\circ}$ about the z axis, and zero translation.

2.5 Step 5: Build dual quaternion with new angle/translation

The rotation of 90° about the z-axis, corresponds to:

$$\begin{aligned} R &= \cos(\frac{90^{\circ}}{2}) - \sin(\frac{90^{\circ}}{2})e_{12} \\ &\approx 0.7071 - 0.7071e_{12} \end{aligned} \tag{5}$$

It would be cooler to write equation 5 with exact values of $\frac{1}{\sqrt{2}}$ instead of approximating to 0.7071, but I know some students get nervous looking at all the radical signs.

For completeness, why don't we say a translation of zero is:

$$T = 1 \tag{6}$$

Build a new dual quaternion using equations 5 and 6:

$$TR = 0.7071 - 0.7071e_{12} \tag{7}$$

2.6 Step 6: Multiply by the Start

Here's the final step. Multiply the dual quaternion we derived using the t value in equation 7 with the starting motor in equation 1 to find:

$$\begin{aligned}
\text{Slerped dual quaternion} &= (0.7071 - 0.7071e_{12})(1 - e_{01}) \\
&= 0.7071 - 0.7071e_{01} - 0.7071e_{12} + 0.7071e_{12}e_{01} \\
&= 0.7071 - 0.7071e_{01} - 0.7071e_{12} - 0.7071e_{02} \\
&= 0.7071(1 - e_{01} - e_{12} - e_{02})
\end{aligned} \tag{8}$$

Now let's see if this works!

2.7 Applying slerped motor to a point

Remembering in our example that Start is a translation of $[2, 0, 0]$ and End is the same translation followed by a rotation of 180° about the z axis. We used a value of $t = \frac{1}{2}$ to slerp between the dual quaternions. So that should transform the origin $(0, 0, 0)$ to the point straight up $(0, 2, 0)$. Let's see if it does!

We can write the position $(0, 0, 0)$ in geometric algebra as e_{123}. Apply the slerped motor M to this point using the sandwich product:

$$P = M(e_{123})\tilde{M} \tag{9}$$

Substitute equation 8 into 9 to find the transformed point:

$$\begin{aligned}
P &= 0.7071(1 - e_{01} - e_{12} - e_{02})(e_{123})(0.7071(1 + e_{01} + e_{12} + e_{02})) \\
&= 0.5(1 - e_{01} - e_{12} - e_{02})(e_{123})(1 + e_{01} + e_{12} + e_{02}) \\
&= 0.5(e_{123} + e_{032} + e_3 + e_{013})(1 + e_{01} + e_{12} + e_{02}) \\
&= 0.5(e_{123} + e_{032} - e_3 + e_{013} + e_{032} + e_{013} + e_3 + e_{013} + e_{123} - e_{032} \\
&+ e_{013} - e_{032}) \\
&= 0.5(2e_{123} + 4e_{013}) \\
&= e_{123} + 2e_{013}
\end{aligned} \tag{10}$$

The point in equation 10 corresponds to the 3D point (0, 2, 0), which is exactly what we hoped for! Hooray!

References

[Ken12] Ben Kenwright. "A beginners guide to dual-quaternions: What they are, how they work, and how to use them for 3D character hierarchies". English. In: *20th International Conference in Central Europe on Computer Graphics, Visualization and Computer Vision*. 20th International Conference in Central Europe on Computer Graphics, Visualization and Computer Vision 2012, WSCG 2012 ; Conference date: 26-06-2012 Through 28-06-2012. 2012, pp. 1–10. ISBN: 9788086943794.

[Tod] Hamish Todd. *Math in Game Development Summit: Quaternions to Homogeneous Points, Lines, and Planes*. URL: `https://www.gdcvault.com/play/1029237/`.

[Vai] Rodolphe Vaillant. *Dual Quaternions Skinning Tutorial*. URL: `http://rodolphe-vaillant.fr/entry/29/dual-quaternions-skinning-tutorial-and-c-codes`.