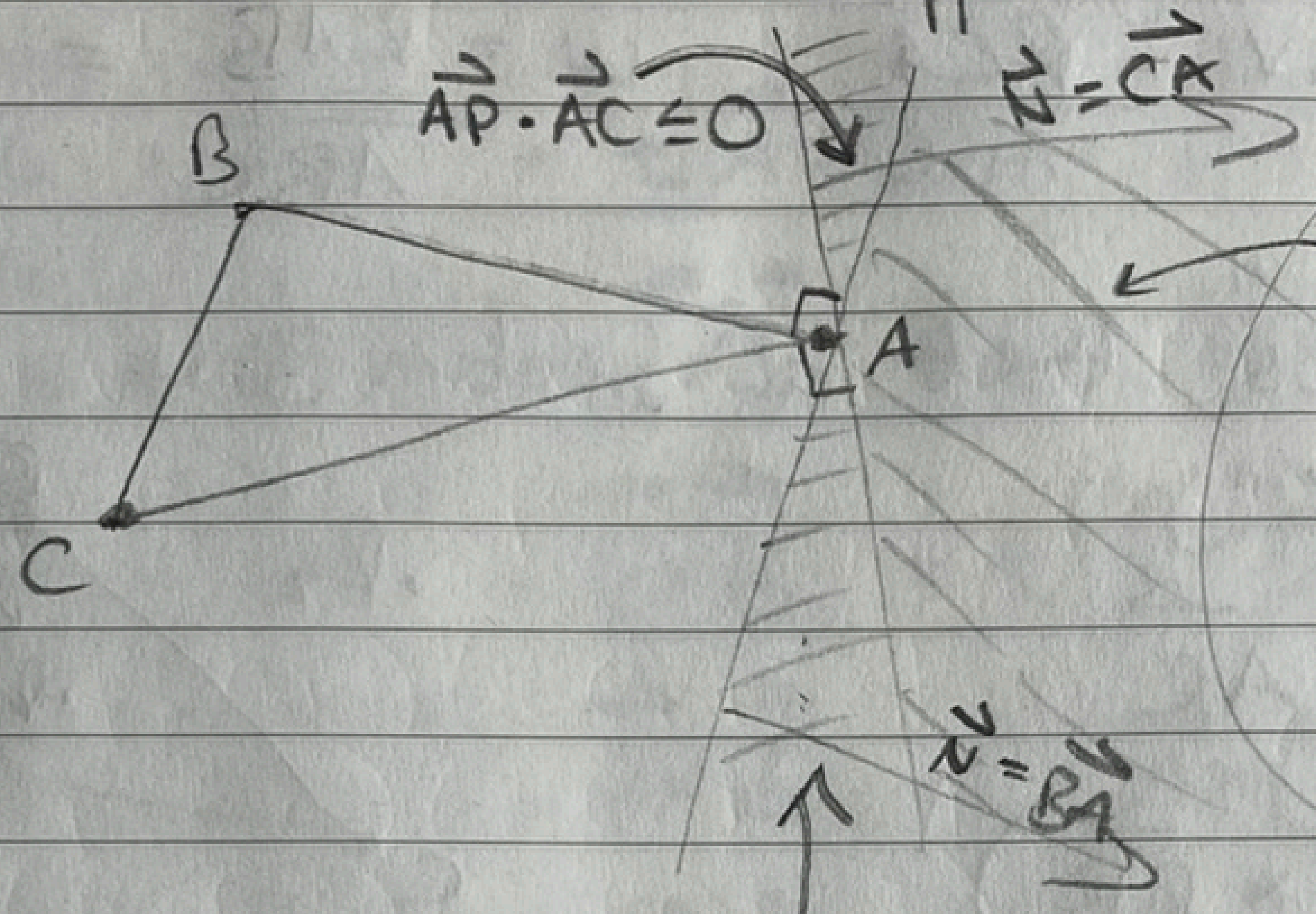
2024-27-02 UN

Scott wants to know how to detect collisions between a sphere and a triangle mesh. I'm looking at Ericson's collision detection book for "5.1.5 Closest Point on Triangle to Point"

The author recommends an approach that uses Voronoi features



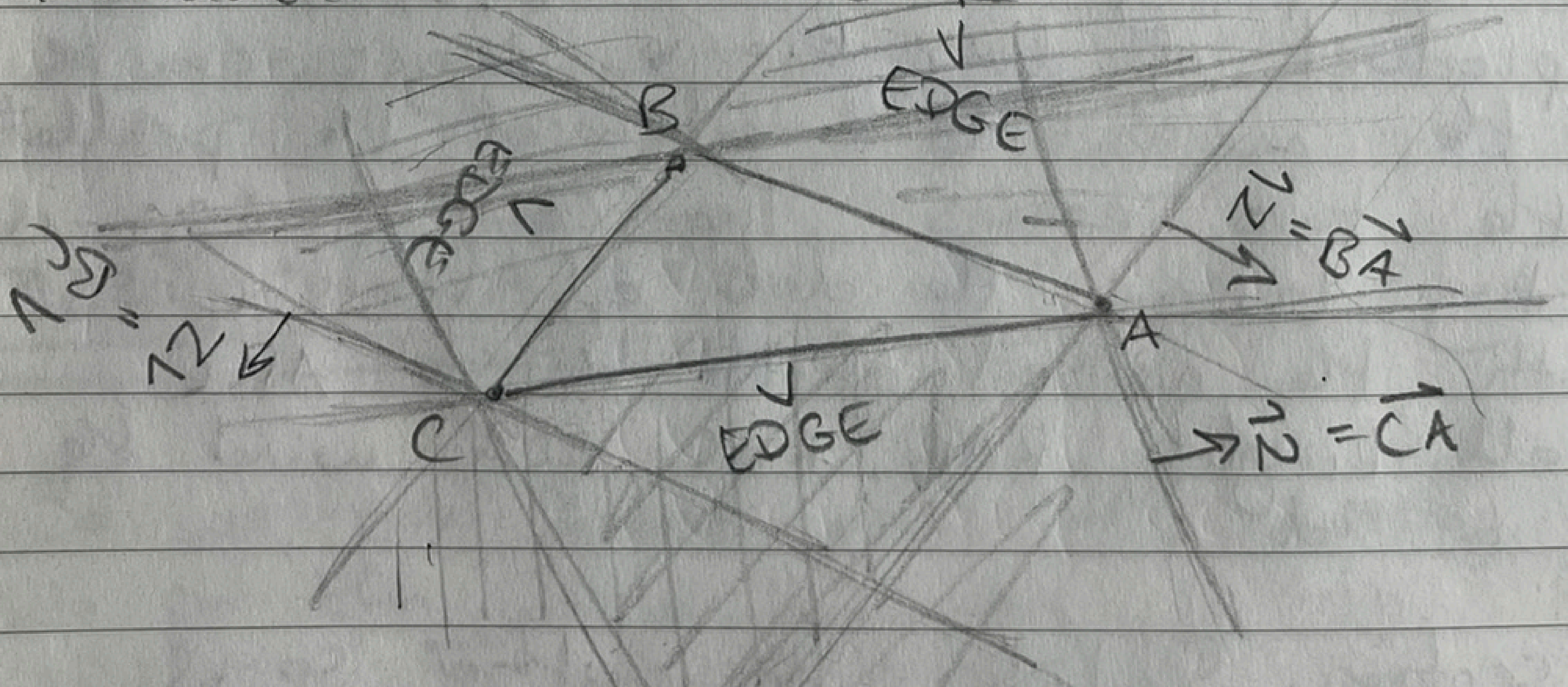
$$(P-A)\cdot(B-A) \le 0$$
$$\vec{AP}\cdot\vec{AB} \le 0$$

or in other words

the angle between $\vec{AP}$ & $\vec{AB} \geq 90°$

where P is a point

To figure out if point $\vec{P}$ lies in one of the edge Voronoi regions we gotta calculate the barycentric coordinates of the orthogonal projection of $\vec{P}$ onto ABC. The author calls that R



Recall that the Barycentric coordinates of R are given as the ratios of the signed areas of triangles RAB, RBC and RCA to the signed area of ABC

Let $\hat{n}$ be the normal of ABC and let $\vec{R} = \vec{P} - t\hat{n}$

orth. proj. of $\vec{P}$ onto ABC

point we are testing

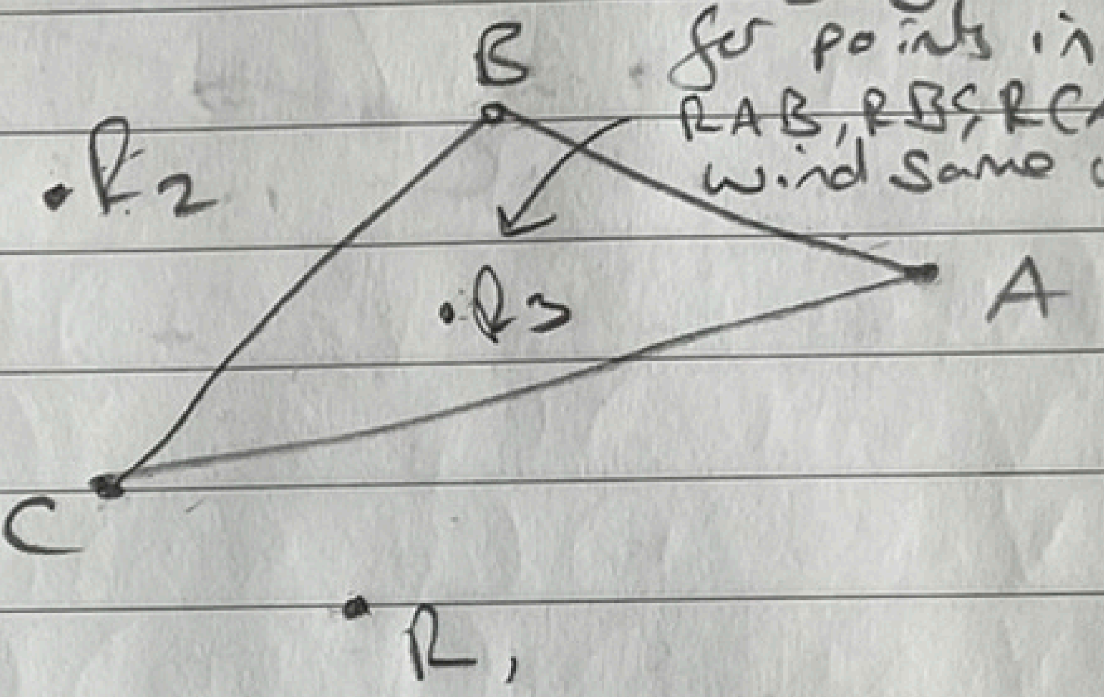
The Barycentric coordinates of $\vec{R}$ (u, v, w)

$\vec{R} = u\vec{A} + v\vec{B} + w\vec{C}$

Can be computed using

$$u = \frac{RBC}{ABC}, \quad v = \frac{RCA}{ABC}, \quad w = \frac{RAB}{ABC} = 1 - u - v$$

Author reminds us that in Section 3.4 we figured out the following



for points in triangle RAB, RBC, RCA wind same way

$$\vec{n} = \vec{AB} \times \vec{AC}$$

$$RAB = \vec{n} \cdot (\vec{RA} \times \vec{RB})$$

$$RBC = \vec{n} \cdot (\vec{RB} \times \vec{RC})$$

$$RCA = \vec{n} \cdot (\vec{RC} \times \vec{RA})$$

notice for R points outside the triangle one of the triangles RAB, RBC, RCA winds the other way

$$ABC = RAB + RBC + RCA$$

Reading section 3.4 again, the author says
→ Any function proportional to the triangle area can be used in computing the ratios u, v, w. So the cross product of two triangle edges can be used, as the magnitude = area of the parallelogram built by the two edges. The correct sign is maintained by taking the dot of this cross product with the normal of the triangle ABC, to see if all triangles RAB, RBC, RCA wind the same way or not

Back to section 5.1.5, the author says hold on, these expressions simplify so that you don't even need the orth. projection R

For example:

$$RAB = \vec{n} \cdot (\vec{RA} \times \vec{RB})$$
$$= \vec{n} \cdot ((\vec{A} - \vec{R}) \times (\vec{B} - \vec{R}))$$
$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \vec{R} - \vec{R} \times \vec{B} + \cancel{\vec{R} \times \vec{R}}) \leftarrow \vec{R} \times \vec{R} = 0$$
$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \underbrace{(\vec{P} - t\vec{n})} - \underbrace{(\vec{P} - t\vec{n})} \times \vec{B})$$

SUB IN $\vec{R} = \vec{P} - t\vec{n}$

$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \vec{P} + t\vec{A} \times \vec{n} - \vec{P} \times \vec{B} + t\vec{n} \times \vec{B})$$
$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \vec{P} - \vec{P} \times \vec{B} + t(\vec{n} \times \vec{B} + \vec{A} \times \vec{n}))$$
$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \vec{P} - \vec{P} \times \vec{B} + t(\vec{n} \times \vec{B} - \vec{n} \times \vec{A}))$$
$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \vec{P} - \vec{P} \times \vec{B} + t\vec{n} \times (\vec{B} - \vec{A}))$$
$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \vec{P} - \vec{P} \times \vec{B} + t\vec{n} \times \vec{AB})$$

Turns out $\vec{n}$ and $t\vec{n} \times \vec{AB}$ are neccessarily at 90° to each other. So $\vec{n} \cdot (t\vec{n} \times \vec{AB}) = 0$

$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \vec{P} - \vec{P} \times \vec{B})$$

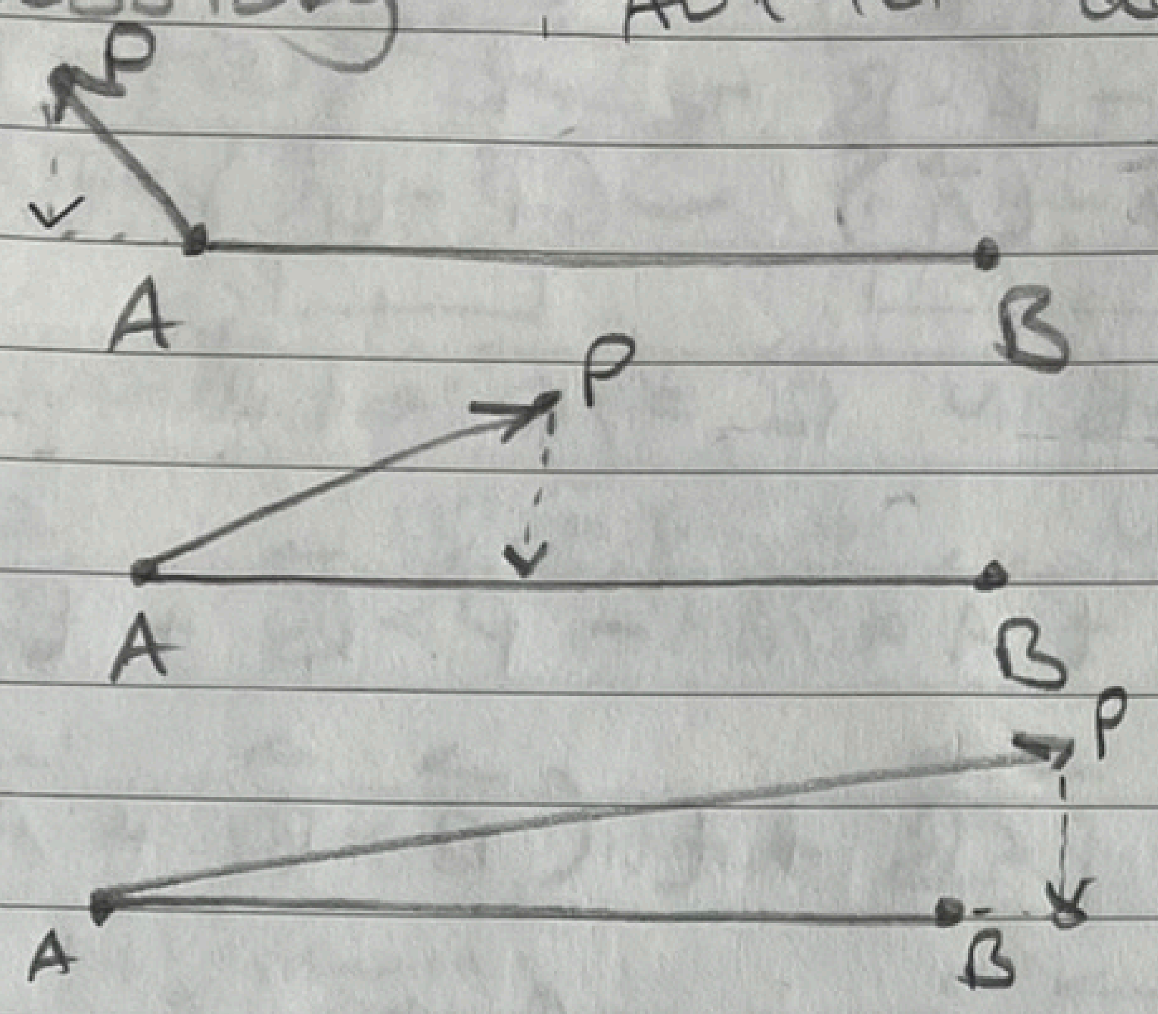
$\vec{P} \times \vec{P}$ is zero, so there is no harm adding it in

$$= \vec{n} \cdot (\vec{A} \times \vec{B} - \vec{A} \times \vec{P} - \vec{P} \times \vec{B} + \vec{P} \times \vec{P})$$
$$= \vec{n} \cdot ((\vec{A} - \vec{P}) \times (\vec{B} - \vec{P}))$$
$$= \vec{n} \cdot (\vec{PA} \times \vec{PB})$$

So we can figure out RAB directly from point P.
We can then write

$$RAB = \vec{n} \cdot (\vec{PA} \times \vec{PB})$$
$$RBC = \vec{n} \cdot (\vec{PB} \times \vec{PC})$$
$$RCA = \vec{n} \cdot (\vec{PC} \times \vec{PA})$$
$$ABC = RAB + RBC + RCA$$

Looking at section 5.1.2. we will need to find the closest point to the edge lines of the triangle possibly. Author describes it as



If the projected point is within the line segment, then that is the answer. Otherwise the endpoint closest to point P is the closest one

Parameterize the line using

$$P(t) = A + t(B - A)$$
$$= \vec{A} + t\vec{AB}$$

So what ya gotta do is project the point onto $\vec{AB}$

$$t = \frac{\vec{AP} \cdot \hat{AB}}{|AB|} = \frac{\vec{AP} \cdot \vec{AB}}{\vec{AB} \cdot \vec{AB}}$$

if $t < 0$, then set $t = 0$
$t > 1$, then set $t = 1$

$$\text{Position} = \vec{A} + t\vec{AB}$$

Ok that's for lines. Back to triangles, the author says

Compute parametric pos. s for projection P onto $\vec{AB}$

$$P' = \vec{A} + s\vec{AB}$$

where $s = \dfrac{\text{snom}}{\text{snom} + \text{sdenom}}$

$$= \frac{(P - A) \cdot \vec{AB}}{(P-A) \cdot \vec{AB} + (P - B) \cdot \vec{BA}}$$

$$S = \frac{\vec{AP} \cdot \vec{AB}}{\vec{AP} \cdot \vec{AB} + \vec{BP} \cdot \vec{BA}}$$

$$= \frac{\vec{AP} \cdot \vec{AB}}{(\vec{P} - \vec{A}) \cdot (\vec{B} - \vec{A}) + (\vec{P} - \vec{B}) \cdot (\vec{A} - \vec{B})}$$

$$= \frac{\vec{AP} \cdot \vec{AB}}{\cancel{\vec{P} \cdot \vec{B}} - \vec{P} \cdot \vec{A} - \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{A} + \cancel{\vec{P} \cdot \vec{A}} - \vec{P} \cdot \vec{B} - \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}}$$

$$= \frac{\vec{AP} \cdot \vec{AB}}{A \cdot A + B \cdot B - 2A \cdot B}$$

$$= \frac{\vec{AP} \cdot \vec{AB}}{(B - A) \cdot (B - A)}$$

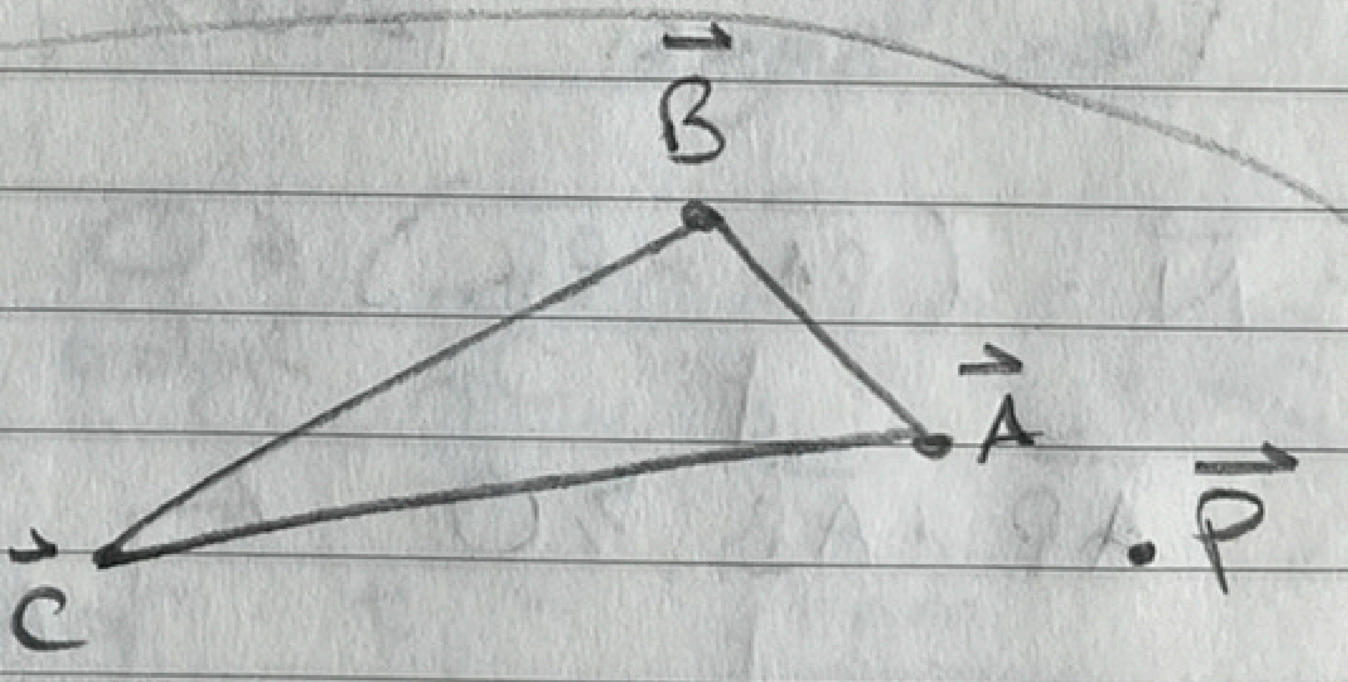
$$= \frac{\vec{AP} \cdot \vec{AB}}{\vec{AB} \cdot \vec{AB}} = \frac{\vec{AP} \cdot \hat{AB}}{|AB|} \checkmark$$

Wow, the author really doesn't like recalculating things, eh!

Ok so it does match the parameterization of a line
$P(t) = \vec{A} + t\vec{AB}$

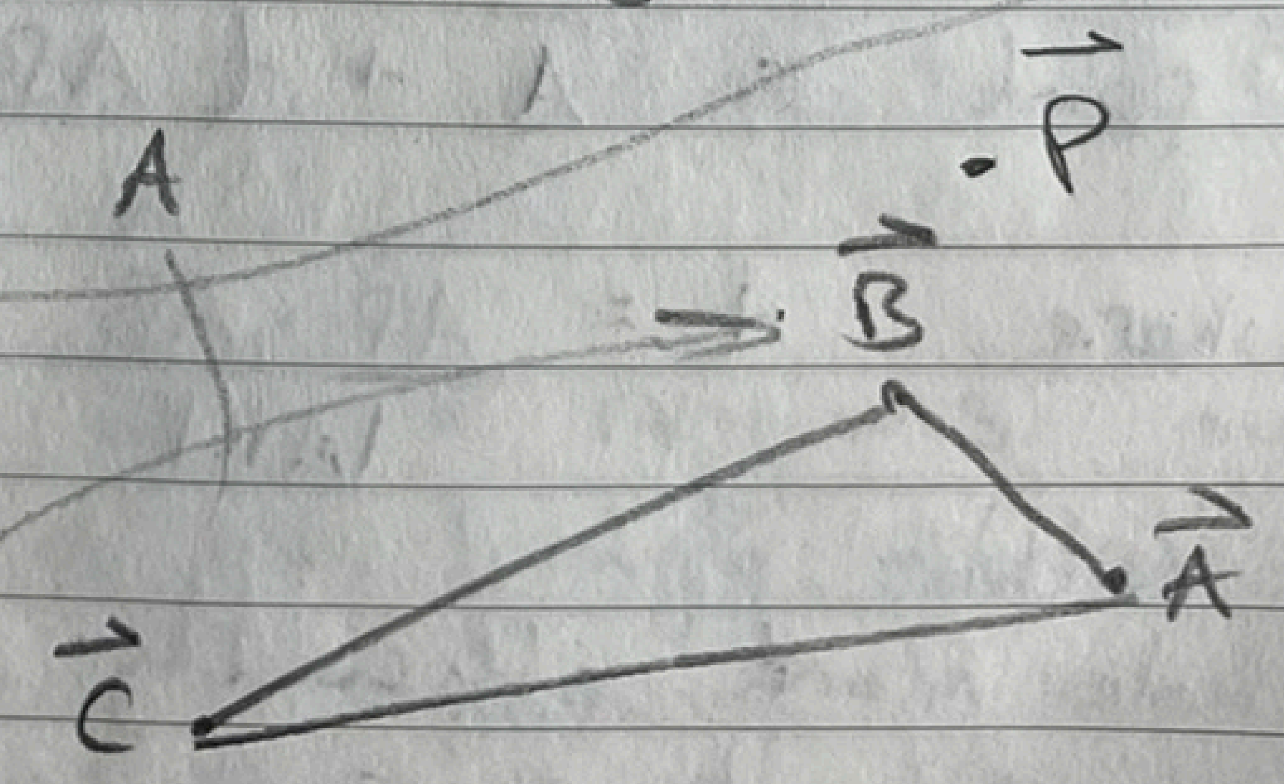
Author says if

$\vec{AP} \cdot \vec{AB} < 0$
&
$\vec{AP} \cdot \vec{AC} < 0$



we are at a vertex Voronoi region

So closest point is A



If
$\vec{BP} \cdot \vec{BC} < 0$
&
$\vec{BP} \cdot \vec{BA} < 0$

closest point is B

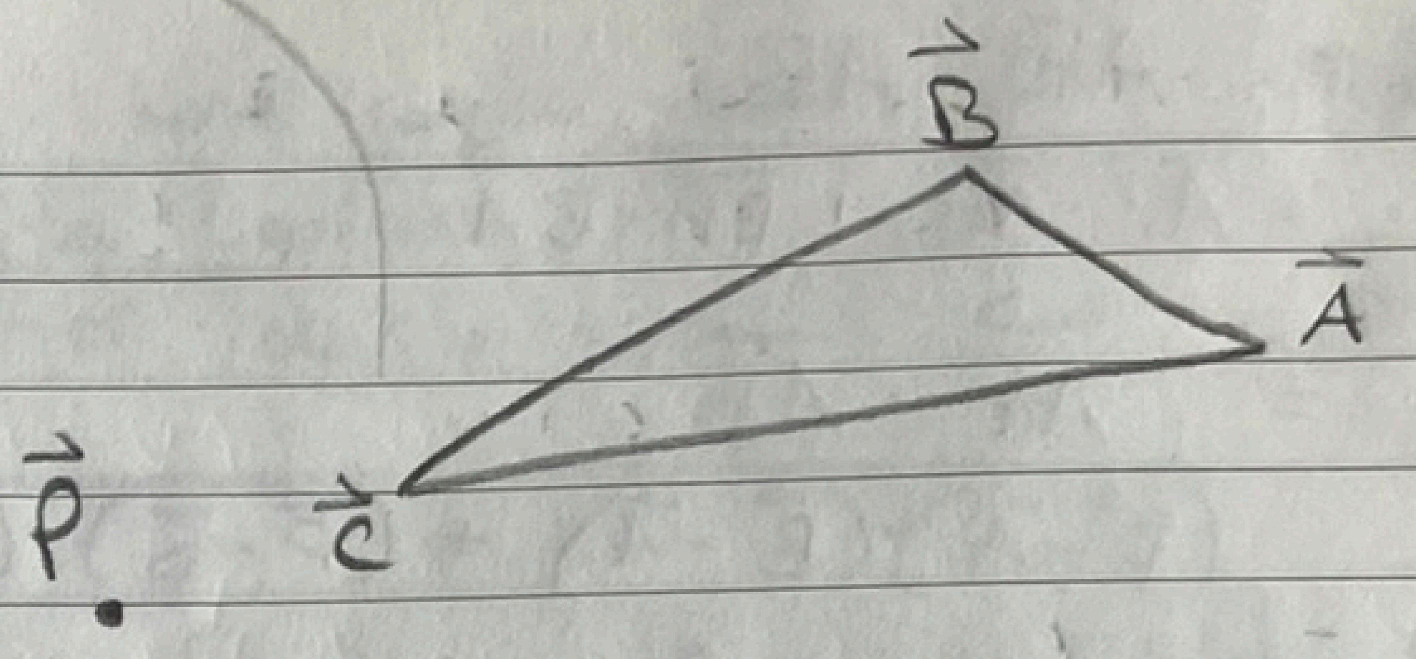
And lastly if

$\vec{CP} \cdot \vec{CB} < 0$

&

$\vec{CP} \cdot \vec{CA} < 0$

Closest point is $\vec{C}$



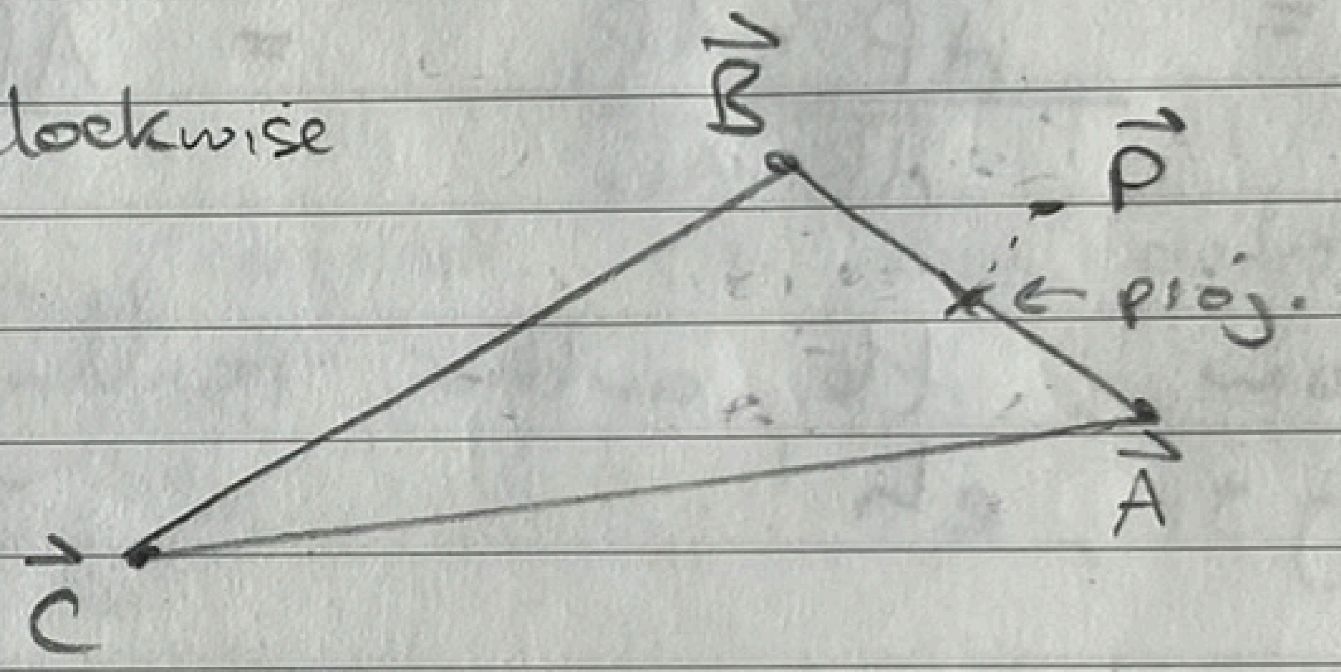
If none of the above is true then we are either in an edge Voronoi region or actually in the damn triangle

Author says we outside or on line $\vec{AB}$ if the triple scalar product $\vec{N}\ \vec{PA}\ \vec{PB} \le 0$

$\vec{N} = \vec{AB} \times \vec{AC}$ ← winds anti-clockwise

$vc = \vec{N} \cdot (\vec{PA} \times \vec{PB}) \le 0$

winds clockwise



Author then says return the projection of $\vec{P}$ onto $\vec{AB}$ is

$\vec{N} \cdot (\vec{PA} \times \vec{PB}) \le 0$

&

$\vec{AP} \cdot \vec{AB} \ge 0$

&

$\vec{BP} \cdot \vec{BA} \ge 0$

Angle between vectors $\le 90°$

Then

Closest point $= \vec{A} + t\,\vec{AB}$

where $t = \dfrac{\vec{AP} \cdot \hat{AB}}{|AB|}$

or the author has made more efficiently:

$$t = \frac{snom}{snom + sdenom} = \frac{\vec{AP} \cdot \vec{AB}}{\vec{AP} \cdot \vec{AB} + \vec{BP} \cdot \vec{BA}}$$

Similarly, we are outside or on $\vec{BC}$ and within the Voronoi region of $\vec{BC}$ if

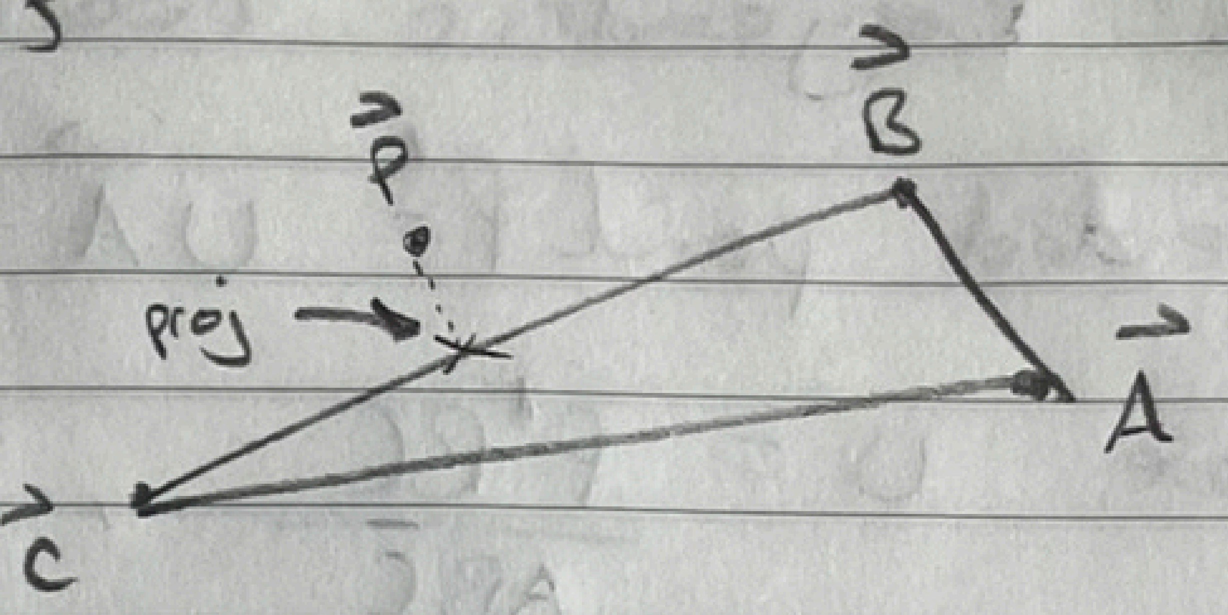
$\vec{N}\cdot(\vec{PB}\times\vec{PC}) \le 0$ author calls this va

&

$\vec{BP}\cdot\vec{BC} \geq 0$

&

$\vec{CP}\cdot\vec{CB} \geq 0$



Then closest point on line $\vec{BC} = \vec{B} + t\vec{BC}$

$t = \dfrac{\vec{BP}\cdot\hat{BC}}{|BC|}$ or author likes $t = \dfrac{\text{unom}}{\text{unom} + \text{udenom}}$

$= \dfrac{\vec{BP}\cdot\vec{BC}}{\vec{BP}\cdot\vec{BC} + \vec{CP}\cdot\vec{CB}}$

Last edge is $\vec{CA}$. We are outside $\vec{CA}$ & within the Voronoi region of $\vec{CA}$ if

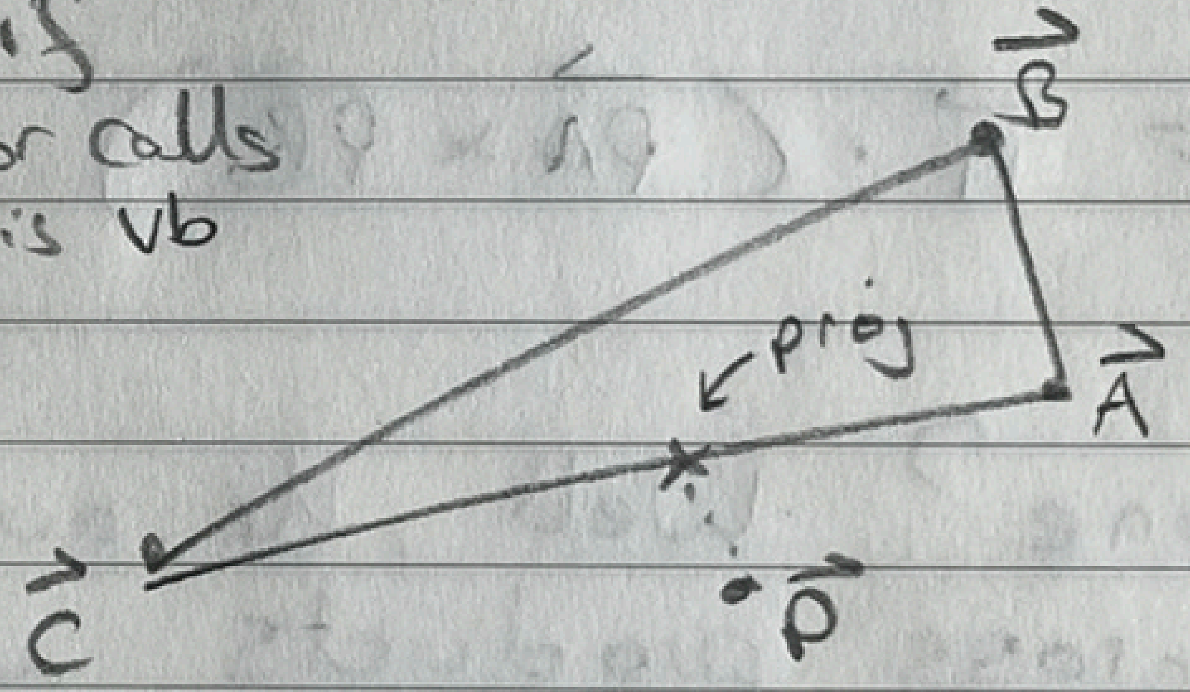
$\vec{N}\cdot(\vec{PC}\times\vec{PA}) \le 0$ author calls this vb

&

$\vec{AP}\cdot\vec{AC} \geq 0$

&

$\vec{CP}\cdot\vec{CA} \geq 0$



Then closest point on edge $\vec{AC} = \vec{A} + t\vec{AC}$

where $t = \dfrac{\vec{AP}\cdot\hat{AC}}{|AC|}$ or $t = \dfrac{\text{tnom}}{\text{tnom} + \text{tdenom}}$

$= \dfrac{\vec{AP}\cdot\vec{AC}}{\vec{AP}\cdot\vec{AC} + \vec{CP}\cdot\vec{CA}}$

Phew, we've checked all the regions outside the vertices & edges of the triangle. If all those checks returned false, then point P must project into the actual face region of the triangle!

So the orthogonal projection of P onto the triangle ABC has Barycentric coordinates U, V, W

Projected point $P' = U\vec{A} + V\vec{B} + W\vec{C}$

where $U = \frac{P'BC}{ABC}$, $V = \frac{P'CA}{ABC}$, $W = 1 - U - V$

$$U = \frac{va}{va + vb + vc}, \quad V = \frac{vb}{va + vb + vc}, \quad W = \frac{vc}{va + vb + vc} = 1 - U - V$$

where we derived va, vb & vc at the bottom of page 3 of these scribbles

$$va = \vec{n} \cdot (\vec{PB} \times \vec{PC})$$

$$vb = \vec{n} \cdot (\vec{PC} \times \vec{PA})$$

$$vc = \vec{n} \cdot (\vec{PA} \times \vec{PB})$$

we used these on pages 6 & 7 of these scribbles too, where $\vec{n} = \vec{AB} \times \vec{AC}$

All done? Well, the author mentions that all these cross products are more expensive to calculate than dot products. So he recommends reading up on the Lagrange identity:

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

That means

$$va = (\vec{AB} \times \vec{AC}) \cdot (\vec{PB} \times \vec{PC}) = (\vec{AB} \cdot \vec{BP})(\vec{AC} \cdot \vec{CP}) - (\vec{AB} \cdot \vec{CP})(\vec{AC} \cdot \vec{BP})$$

$$vb = (\vec{AB} \times \vec{AC}) \cdot (\vec{PC} \times \vec{PA}) = (\vec{AB} \cdot \vec{CP})(\vec{AC} \cdot \vec{AP}) - (\vec{AB} \cdot \vec{AP})(\vec{AC} \cdot \vec{CP})$$

$$vc = (\vec{AB} \times \vec{AC}) \cdot (\vec{PA} + \vec{PB}) = (\vec{AB} \cdot \vec{AP})(\vec{AC} \cdot \vec{BP}) - (\vec{AB} \cdot \vec{BP})(\vec{AC} \cdot \vec{AP})$$

where the author flipped the directions of $\vec{PA}$, $\vec{PB}$ & $\vec{PC}$ to reuse variables efficiently in code. Finally we are <u>DONE</u>!