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Module 9: Fourier Transform

Learning Objectives

- Learn discrete Fourier transform (DFT)
- Learn fast Fourier transform (FFT)
- Learn trigonometric interpolation

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Sources

- Textbook (Chapter 10: Trigonometric Interpolation and the FFT)
- Ref: J.Cooley and J.Tukey (1965), "An algorithm for the Machine Calculation of Complex Fourier Series", *Mathematics of Computation* **19**, 297-301.
- Wiki: Fourier Transform https://en.wikipedia.org/wiki/Fourier_transform
- Wiki: Fast Fourier Transform https://en.wikipedia.org/wiki/Fast_Fourier_transform
- Wiki: JPEG https://en.wikipedia.org/wiki/JPEG

Outline

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Introduction

- Sines and Cosines have great impact on modern technology.
- Trigonometric functions of multiple frequencies are natural interpolating functions for periodic data.
- Fourier transform is efficient at carrying out the interpolation and plays an essential role in signal processing.
- Fast Fourier transform makes discrete Fourier transform computationally cheap.

Outline

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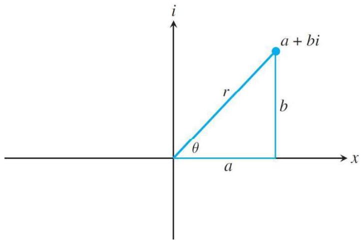
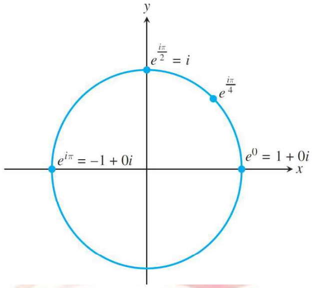
Complex arithmetic

- **Complex number:** $z = a + bi = re^{i\theta}$
  - $i = \sqrt{-1}$
  - magnitude $r = \sqrt{a^2 + b^2}$
  - angle $\theta = \arctan\dfrac{b}{a}$



- **Euler formula:** $e^{i\theta} = \cos\theta + i\sin\theta$



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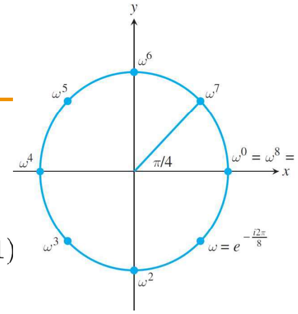
Roots of unity



- **nth root of unity:** solution of $z^n = 1$

  $$\omega_n = e^{-i\frac{2\pi}{n}k}: \quad e^{-i\frac{2\pi}{n}}, e^{-i\frac{4\pi}{n}}, e^{-i\frac{6\pi}{n}}, \cdots, e^{-i\frac{2n\pi}{n}} (= 1)$$

- **Primitive root of unity:** an nth root of unity is primitive if it is not a kth root of unity for any $0 < k < n$. E.g., $\omega = e^{-i\frac{2\pi}{n}}$.

- **Identity:** If ω is a primitive nth root of unity $\omega = e^{-i\frac{2\pi}{n}}$ for $n > 1$, then

$$\begin{aligned}
1 + \omega + \omega^2 + \omega^3 + \cdots + \omega^{n-1} &= 0 \\
1 + \omega^2 + \omega^4 + \omega^6 + \cdots + \omega^{2(n-1)} &= 0 \\
1 + \omega^3 + \omega^6 + \omega^9 + \cdots + \omega^{3(n-1)} &= 0 \\
&\vdots \\
1 + \omega^{n-1} + \omega^{2(n-1)} + \omega^{3(n-1)} + \cdots + \omega^{(n-1)(n-1)} &= 0 \\
1 + \omega^n + \omega^{2n} + \omega^{3n} + \cdots + \omega^{(n-1)n} &= n
\end{aligned}$$

Outline

Discrete Fourier Transform (DFT)

- Let $\omega = e^{-i\frac{2\pi}{n}}$.

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \xrightarrow[\text{Discrete Fourier Transform}]{y_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} x_k \omega^{jk}} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

- In matrix form,

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 + ib_0 \\ a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_{n-1} + ib_{n-1} \end{bmatrix} = \underbrace{\frac{1}{\sqrt{n}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \cdots & \omega^{n-1} \\ \omega^0 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \omega^0 & \omega^3 & \omega^6 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}}_{\text{Fourier Matrix: } F_n} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Property

- Let $\{y_j\}$ be the DFT of $\{x_j\}$. If all x_j are real number, then

  (1). y_0 is real.

  (2). for $j = 1, \cdots, n-1$, $\boxed{y_{n-j} = \bar{y}_j,}$ i.e., $\boxed{a_{n-j} + ib_{n-j} = a_j - ib_j.}$

  This is because

$$y_{n-j} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} x_k (\omega^{n-j})^k = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} x_k (\overline{\omega^j})^k = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \overline{x_k (\omega^j)^k} = \overline{y_j}.$$

- For example, if $x_0, \cdots, x_7$ are real numbers, then

$$F_8 \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 + ib_1 \\ a_2 + ib_2 \\ a_3 + ib_3 \\ a_4 \\ a_3 - ib_3 \\ a_2 - ib_2 \\ a_1 - ib_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_{\frac{n}{2}-1} \\ y_{\frac{n}{2}} \\ \overline{y}_{\frac{n}{2}-1} \\ \vdots \\ \overline{y}_1 \end{bmatrix}$$

Example

- Find the DFT of the vector x = $[1,0,-1,0]^T$.

Let ω be the 4th root of unity, or $\omega = e^{-i\pi/2} = \cos(\pi/2) - i\sin(\pi/2) = -i$. Applying the DFT, we get

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Inverse Discrete Fourier Transform

- DFT can be written in compact form as $y = F_n x$.
- The **Inverse Discrete Fourier Transform** of the vector y can be written as $x = F_n^{-1} y$, where the inverse matrix is

$$F_n^{-1} = \frac{1}{\sqrt{n}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \omega^0 & \omega^{-3} & \omega^{-6} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix}$$

Explanation of DFT

DFT can be viewed as the solution (i.e., the coefficients) of the following interpolating problem:

Given sample data $x_0, x_1, x_2, \cdots, x_{n-1}$ corresponding to parameter values $t = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \cdots, \frac{2(n-1)\pi}{n}$, find a complex function:

$$Q(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} y_k (e^{it})^k$$

such that

$$Q(t_j) = Q(\frac{2\pi j}{n}) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} y_k (e^{i\frac{2\pi j}{n}})^k = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} y_k (\omega^{-j})^k = x_j.$$

In fact, the interpolating conditions are equivalent to $F_n^{-1} y = x$.

Outline

Fast Fourier Transform (FFT)

- The naive DFT method applied to an n-vector requires $O(n^2)$ operations.
- Cooley and Tukey [1965] found the *Fast Fourier Transform (FFT)* to compute the DFT requires only $O(n \log n)$ opertations, which makes the field of signal processing convert from primarily analog to digital.
- The key to the FFT is the fact that

$$F_{2n}[x_0, x_1, x_2, x_3, \cdots, x_{2n-2}, x_{2n-1}]_j = \underline{F_n[x_0, x_2, \cdots, x_{2n-2}]_j} + e^{-i\frac{j\pi}{n}} \underline{F_n[x_1, x_3, \cdots, x_{2n-1}]_j}$$

$$F_{2n}[x_0, x_1, x_2, x_3, \cdots, x_{2n-2}, x_{2n-1}]_{n+j} = \underline{F_n[x_0, x_2, \cdots, x_{2n-2}]_j} - e^{-i\frac{j\pi}{n}} \underline{F_n[x_1, x_3, \cdots, x_{2n-1}]_j}$$

for $j = 0, 1, \cdots, n-1$, where

- $F_n[\cdots]$ represents the vector of scaled DFT of vector $[\cdots]$ with the scale $\sqrt{n}$.
- the sub-indices j simply refer to the j-th entry of the vector.

FFT

That is, the scaled DFT of the $2n$ data $x_0, x_n, x_1, x_{n+1}, \cdots, x_{n-1}, x_{2n-1}$ can be obtained by combining the scaled DFT's of the two n data sets $x_0, x_1, \cdots, x_{n-1}$ and $x_n, x_{n+1}, \cdots, x_{2n-1}$ componentwise.

For example, consider the DFT of vector $[x_0, x_1, x_2, \cdots, x_7]$. We can perform the following calculations in an inverse order:

- $F_8[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7]$
- $F_4[x_0, x_2, x_4, x_6]$ and $F_4[x_1, x_3, x_5, x_7]$
- $F_2[x_0, x_4]$, $F_2[x_2, x_6]$ and $F_2[x_1, x_5]$, $F_2[x_3, x_7]$
- $F_1[x_0], F_1[x_4], F_1[x_2], F_1[x_6], F_1[x_1], F_1[x_5], F_1[x_3], F_1[x_7]$.

Note that $F_1[x_j] = x_j$. After the 3 rounds of computation, we scale the obtained coefficients with $\frac{1}{\sqrt{8}}$, which gives the DFT of vector $[x_0, x_1, \cdots, x_7]$.

FFT: example

To verify the FFT formula by calculating the scaled DFT of vector $[x_0, x_1, x_2, x_3]$.

Let $\omega_4 = e^{-i\frac{2\pi}{4}} = -i$ and $\omega_2 = e^{-i\frac{2\pi}{2}} = -1$.

- First, calculate the scaled DFT of $[x_0, x_2]$.

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} \omega_2^0 & \omega_2^0 \\ \omega_2^0 & \omega_2^1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_0 + x_2 \\ x_0 - x_2 \end{bmatrix}$$

- Second, calculate the scaled DFT of $[x_1, x_3]$.

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} \omega_2^0 & \omega_2^0 \\ \omega_2^0 & \omega_2^1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_1 - x_3 \end{bmatrix}$$

FFT: example (cont)

- Third, calculate the scaled DFT of $[x_0, x_1, x_2, x_3]$.

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \omega_4^0 & \omega_4^0 & \omega_4^0 & \omega_4^0 \\ \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ \omega_4^0 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ \omega_4^0 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_0 + x_1 + x_2 + x_3 \\ x_0 - ix_1 - x_2 + ix_3 \\ x_0 - x_1 + x_2 - x_3 \\ x_0 + ix_1 - x_2 - ix_3 \end{bmatrix}$$

$$= \begin{bmatrix} (x_0 + x_2) + (x_1 + x_3) \\ (x_0 - x_2) - i(x_1 - x_3) \\ (x_0 + x_2) - (x_1 + x_3) \\ (x_0 - x_2) + i(x_1 - x_3) \end{bmatrix} = \begin{bmatrix} u_0 + v_0 \\ u_1 - iv_1 \\ u_0 - v_0 \\ u_1 + iv_1 \end{bmatrix}$$

That is,

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \\ z_{2+0} \\ z_{2+1} \end{bmatrix} = \begin{bmatrix} u_0 + v_0 \\ u_1 + e^{-i\frac{\pi}{2}} v_1 \\ u_0 - v_0 \\ u_1 - e^{-i\frac{\pi}{2}} v_1 \end{bmatrix}$$

Outline

Trigonometric interpolation

Problem: The set of functions $\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \cdots\}, \theta \in [0, 2\pi]$ spans a space. Given n data points $x_0, x_1, \cdots, x_{n-1}$, let $\Delta\theta = \frac{2\pi}{n}$ and $\theta_j = j\Delta\theta$ for $j = 0, 1, \cdots, n-1$. Now we want to find a function $P_n(\theta)$ as a linear combination of the first n basis functions in the set, such that $P_n(\theta_j) = x_j$.

(1) Denote the DFT of $[x_0, x_1, \cdots, x_{n-1}]$ by $y_j = a_j + ib_j, j = 0, 1, \cdots, n-1$. Construct the complex function:

$$
\begin{aligned}
Q(\theta) &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (a_k + ib_k) e^{ik\theta} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (a_k + ib_k)(\cos k\theta + i \sin k\theta) \\
&= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (a_k \cos k\theta - b_k \sin k\theta) + i \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (b_k \cos k\theta + a_k \sin k\theta) \\
&\triangleq P(\theta) + iI(\theta)
\end{aligned}
$$

We have already known that $Q(\theta_j) = x_j$.

Trigonometric interpolation (cont)

(2) If all the x_j are real, the real function

$$P(\theta) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (a_k \cos k\theta - b_k \sin k\theta)$$

satisfies $P(\theta_j) = x_j$ for $j = 0, 1, \cdots, n-1$. However, $P(\theta)$ contains $2n$ terms.

(3) Assume that n is even. Note that for $j = 0, 1, \cdots, n-1$,

$$\cos k\theta_j = \cos(n-k)\theta_j, \quad \sin k\theta_j = -\sin(n-k)\theta_j, \quad \text{for} \quad k = 1, 2, \cdots, \frac{n}{2} - 1$$

$$0,\ 1,\ 2,\ \cdots,\ \tfrac{n}{2}-1,\ \tfrac{n}{2},\ \tfrac{n}{2}+1, \cdots, n-2,\ n-1$$

Hence construct

$$P_n(\theta) = \frac{1}{\sqrt{n}} a_0 + \frac{1}{\sqrt{n}} \sum_{k=1}^{\frac{n}{2}-1} [(a_k + a_{n-k}) \cos k\theta - (b_k - b_{n-k}) \sin k\theta] + \frac{1}{\sqrt{n}} a_{\frac{n}{2}} \cos \frac{n}{2}\theta$$

satisfying $P_n(\theta_j) = P(\theta_j) = x_j$.

Trigonometric interpolation (cont)

(4) For real x_j, we have $a_{n-j} = a_j, b_{n-j} = -b_j$. Thus

$$P_n(\theta) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{\frac{n}{2}-1} [a_k \cos k\theta - b_k \sin k\theta] + \frac{a_{\frac{n}{2}}}{\sqrt{n}} \cos \frac{n}{2}\theta$$

(5) Similar development for odd n.

(6) By reparameterization, we obtain general trigonometric interpolation scheme. That is, given an interval $[c, d]$ and positive integer n, let $t_j = c + j(d-c)/n$ for $j = 0, \cdots, n-1$. The vector $x = (x_0, \cdots, x_{n-1})$ consists of n *real* numbers. Let $\{a_j + ib_j\}$ denote the DFT $F_n x$ of x. Then an *oder* n *trigonometric interpolating function* is

$$P_n(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{\frac{n}{2}-1} \left[a_k \cos \frac{2\pi(t-c)}{d-c} k - b_k \sin \frac{2\pi(t-c)}{d-c} k \right] + \frac{a_{\frac{n}{2}}}{\sqrt{n}} \cos \frac{\pi(t-c)}{d-c} n$$

for $t \in [c, d]$.

Extensions

- Extension to least squares fitting with trigonometric functions
- Extension to Discrete Cosine Transform (DCT)
- Extension to 2-dimensional DCT
- Applications in image compression (such as JPEG compression)
- …

Outline

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Homework

Find an order 8 trigonometric interpolant for data $[0, 1, 0, -1, 0, 1, 0, -1]$ on the interval $[0, 2\pi]$.

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Outline

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§4. Trigonometric interpolation
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Summary

- Discrete Fourier transform
  - provide a tool for many applications such as trigonometric interpolation
- Fast Fourier transform
  - provide a fast way to calculate DFT
- Trigonometric interpolation

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End

$$\{1, \cos(2\pi u), \sin(2\pi u), \cos(4\pi u), \sin(4\pi u), \cos(6\pi u), \sin(6\pi u), \cos(8\pi u)\}$$