Using Divergence Theorem to Compute Exact Rigid-Body Parameters of Objects Represented by Triangular Surface Meshes

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Introduction

In order to simulate dynamic behaviour of a rigid-body, one requires knowledge of a set of rigid-body parameters (RBP) such as the total mass of the rigid-body, center of mass, as well as the moments and products of inertia. The purpose of this document is to describe how to accurately compute RBPs of objects represented by triangular surface meshes using divergence theorem. Since all RBPs can be expressed in terms of 3D moments, in section A, a review of 3D moments and the formulas to compute them are provided. In section B, the RBPs are defined in terms of 3D moments. In section C, an example is provided demonstrating how to find closed-form expressions for 3D moments of objects represented by triangular surface meshes. In section D, formulas for finding geometric primitives, such as a cuboid or an ellipsoid, with specific RBPs are provided.

Part A: 3D Moments

Suppose that $\Omega \in \mathbb{R}^3$ is a region of space occupied by a rigid body and $\rho(x,y,z)\colon \Omega \to \mathbb{R}^+$ is scalar function that describes material density of this body at a particular point in space. 3D moment of order $p+q+r$ is defined as:

$$\tilde{m}_{p,q,r}(\Omega) = \int_{\Omega} \rho(x,y,z) x^p y^q z^r \, d\Omega \tag{1}$$

For an object with uniform density, $\rho(x,y,z)$ is constant, so (1) can be simplified:

$$m_{p,q,r}(\Omega) = \int_{\Omega} x^p y^q z^r \, d\Omega \tag{2}$$

3D moments as defined in (2) can also be evaluated by changing the domain of integration from the volume occupied by the region Ω to the surface enclosing this region; that is $m_{p,q,r}(\Omega) = m_{p,q,r}(\mathrm{S})$. This equivalence is enabled by the divergence theorem, which states that:

$$\int_{\Omega} (\vec{\nabla} \cdot \vec{\mathrm{F}}) d\Omega = \oint_{\mathrm{S}} \vec{\mathrm{F}} \cdot \vec{n} dS \tag{3}$$

where $\vec{\nabla} = \frac{\partial}{\partial \mathrm{x}}\hat{\mathbf{i}} + \frac{\partial}{\partial \mathrm{y}}\hat{\mathbf{j}} + \frac{\partial}{\partial \mathrm{z}}\hat{\mathbf{k}}$ and $\vec{\mathrm{F}} = {}^1F(x,y,z)\hat{\mathbf{i}} + {}^2F(x,y,z)\hat{\mathbf{j}} + {}^3F(x,y,z)\hat{\mathbf{k}}$

Let $\vec{\mathrm{F}} = \vec{\mathrm{F}}_{p,q,r}$, so by inspection of (3) and (2) we have:

$\vec{\nabla} \cdot \vec{\mathrm{F}} = \vec{\nabla} \cdot \vec{\mathrm{F}}_{p,q,r} = x^p y^q x^r$

Therefore:

$$x^p y^q x^r = \frac{\partial\{\,^1F_{p,q,r}\}}{\partial x} + \frac{\partial\{\,^2F_{p,q,r}\}}{\partial y} + \frac{\partial\{\,^3F_{p,q,r}\}}{\partial z} \qquad (4)$$

One of the solutions to this PDE is:

$$\begin{aligned} ^1F_{p,q,r} &= \frac{1}{3(p+1)} x^{p+1} y^q z^r \\ ^2F_{p,q,r} &= \frac{1}{3(q+1)} x^p y^{q+1} z^r \\ ^3F_{p,q,r} &= \frac{1}{3(r+1)} x^p y^q z^{r+1} \end{aligned} \qquad (5)$$

Since we are using a piece-wise linear surface (i.e., a mesh) composed from a union on N triangles (i.e., $\mathrm{S} = \bigcup \mathrm{T}_i$) to represent a rigid-body, moment of the region enclosed by this surface can be written as sum of the moments of individual triangles:

$$m_{p,q,r}(\mathrm{S}) = \oint_{\mathrm{S}} \vec{\mathrm{F}}_{p,q,r} \cdot \vec{n} dS = \sum_{i=1}^{N} \int_{\mathrm{T}_i} \vec{\mathrm{F}}_{p,q,r} \cdot \vec{n}_i dS = \sum_{i=1}^{N} m_{p,q,r}(\mathrm{T}_i) \qquad (6)$$

Combining (5) and (6) we get:

$$m_{p,q,r}(\mathrm{T}_i) = \sum_{j=1}^{3} n_{ij} \int_{\mathrm{T}_i} \,^jF_{p,q,r} dS \qquad (7)$$

The integrals in (7) can be rewritten in terms of barycentric coordinates (Figure 1) so that:

$$\int_{\mathrm{T}_i} \,^jF_{p,q,r} dS = 2A_i \int_0^1 \int_0^{1-u} \,^j_iF_{p,q,r}(u,v) dv du \qquad (8)$$

where A_i is the area of T_i. The integrand $^j_iF_{p,q,r}(u,v)$ is defined as:

$$^j_iF_{p,q,r}(u,v) = \frac{1}{3} \begin{cases} \frac{1}{(p+1)} x_i^{p+1}(u,v) y_i^q(u,v) z_i^r(u,v) & , \quad j = 1 \\ \frac{1}{(q+1)} x_i^p(u,v) y_i^{q+1}(u,v) z_i^r(u,v) & , \quad j = 2 \\ \frac{1}{(r+1)} x_i^p(u,v) y_i^q(u,v) z_i^{r+1}(u,v) & , \quad j = 3 \end{cases} \qquad (9)$$

where

$$x_i(u,v) = (x_{i2} - x_{i1})u + (x_{i3} - x_{i1})v + x_{i1}$$
$$y_i(u,v) = (y_{i2} - y_{i1})u + (y_{i3} - y_{i1})v + y_{i1} \tag{10}$$
$$z_i(u,v) = (z_{i2} - z_{i1})u + (z_{i3} - z_{i1})v + z_{i1}$$

Putting everything together we have:

$$m_{p,q,r}(\mathrm{T}_i) = 2A_i \sum_{j=1}^{3} n_{ij} \int_0^1 \int_0^{1-u} {}_i^jF_{p,q,r}(u,v)\,dv\,du \tag{11}$$

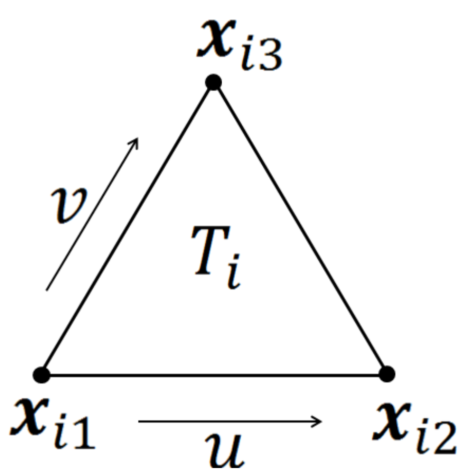

Figure 1: Barycentric coordinates (u,v) can be used specify location of a point inside a planar triangle. Note that $0 \leq u, v \leq 1$ and $0 \leq u + v \leq 1$.

Part B: Rigid Body Parameters and 3D Moments

Total volume of the region Ω occupied by the rigid-body is:

$$V = \int_{\Omega} d\Omega \tag{12}$$

By comparison with (2), total volume is equal to the zeroth moment:

$$V = m_{0,0,0} \tag{13}$$

Centroids of the rigid-body with respect to the x-, y- and z-axes are defined as:

$$\bar{x} = \int_{\Omega} x d\Omega \Big/ \int_{\Omega} d\Omega \qquad \bar{y} = \int_{\Omega} y d\Omega \Big/ \int_{\Omega} d\Omega \qquad \bar{z} = \int_{\Omega} z d\Omega \Big/ \int_{\Omega} d\Omega \tag{14}$$

Where once again comparison with (2) reveals that:

$$\bar{x} = \frac{m_{1,0,0}}{m_{0,0,0}} \qquad \bar{y} = \frac{m_{0,1,0}}{m_{0,0,0}} \qquad \bar{z} = \frac{m_{0,0,1}}{m_{1,0,0}} \tag{15}$$

The inertia tensor is a 2nd order symmetric tensor that looks like this:

$$\mathbf{I} = \begin{bmatrix} \bar{\mathrm{I}}_{xx} & -\bar{\mathrm{I}}_{xy} & -\bar{\mathrm{I}}_{xz} \\ -\bar{\mathrm{I}}_{yx} & \bar{\mathrm{I}}_{yy} & -\bar{\mathrm{I}}_{yz} \\ -\bar{\mathrm{I}}_{zx} & -\bar{\mathrm{I}}_{zy} & \bar{\mathrm{I}}_{zz} \end{bmatrix} \tag{16}$$

Due to symmetry $\bar{\mathrm{I}}_{xy} = \bar{\mathrm{I}}_{yx}$, $\bar{\mathrm{I}}_{xz} = \bar{\mathrm{I}}_{zx}$, and $\bar{\mathrm{I}}_{yz} = \bar{\mathrm{I}}_{zy}$.

The products and moments of inertia of a rigid-body in an arbitrary frame of reference are defined as:

$$\begin{aligned}
\mathrm{I}_{xx} &= \int_{\Omega} y^2 + z^2 d\Omega = m_{0,2,0} + m_{0,0,2} & \qquad \mathrm{I}_{xy} = \mathrm{I}_{yx} &= \int_{\Omega} yx d\Omega = m_{1,1,0} \\
\mathrm{I}_{yy} &= \int_{\Omega} x^2 + z^2 d\Omega = m_{2,0,0} + m_{0,0,2} & \qquad \mathrm{I}_{xz} = \mathrm{I}_{zx} &= \int_{\Omega} zx d\Omega = m_{1,0,1} \\
\mathrm{I}_{zz} &= \int_{\Omega} x^2 + y^2 d\Omega = m_{2,0,0} + m_{0,2,0} & \qquad \mathrm{I}_{yz} = \mathrm{I}_{zy} &= \int_{\Omega} yz d\Omega = m_{0,1,1}
\end{aligned} \tag{17}$$

By convention, however, the inertia tensor must defined with respect to the frame of reference where the centroid of the object coincides with the origin. Let (x', y', z') be the co-ordinates defined in such a frame of reference, then $x = x' + \bar{x}$, $y = y' + \bar{y}$, and $z = z' + \bar{z}$. Substituting these expressions into (17) and simplifying, we get:

$$
\begin{aligned}
\bar{\mathrm{I}}_{xx} &= \mathrm{I}_{xx} - (\bar{y}^2 + \bar{z}^2) m_{0,0,0} & \bar{\mathrm{I}}_{xy} &= \bar{\mathrm{I}}_{yx} = \mathrm{I}_{xy} - \bar{x}\bar{y} m_{0,0,0} \\
\bar{\mathrm{I}}_{yy} &= \mathrm{I}_{yy} - (\bar{x}^2 + \bar{z}^2) m_{0,0,0} & \bar{\mathrm{I}}_{xz} &= \bar{\mathrm{I}}_{zx} = \mathrm{I}_{xz} - \bar{x}\bar{z} m_{0,0,0} \\
\bar{\mathrm{I}}_{zz} &= \mathrm{I}_{zz} - (\bar{x}^2 + \bar{y}^2) m_{0,0,0} & \bar{\mathrm{I}}_{yz} &= \bar{\mathrm{I}}_{zy} = \mathrm{I}_{yz} - \bar{y}\bar{z} m_{0,0,0}
\end{aligned}
\tag{18}
$$

Here are two examples of how these formulas were obtained:

$$
\begin{aligned}
\mathrm{I}_{xx} &= \int_{\Omega} (y' + \bar{y})^2 + (z' + \bar{z})^2 d\Omega \\
&= \int_{\Omega} (y')^2 + (z')^2 d\Omega + 2\bar{y} \int_{\Omega} y' d\Omega + 2\bar{z} \int_{\Omega} z' d\Omega + (\bar{y}^2 + \bar{z}^2) \int_{\Omega} d\Omega \\
&= \int_{\Omega} (y')^2 + (z')^2 d\Omega + (\bar{y}^2 + \bar{z}^2) \int_{\Omega} d\Omega \\
&= \bar{\mathrm{I}}_{xx} + (\bar{y}^2 + \bar{z}^2) m_{0,0,0}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{I}_{xy} &= \int_{\Omega} (x' + \bar{x})(y' + \bar{y}) d\Omega \\
&= \int_{\Omega} x' y' d\Omega + \bar{x} \int_{\Omega} y' d\Omega + \bar{y} \int_{\Omega} x' d\Omega + \bar{x}\bar{y} \int_{\Omega} d\Omega \\
&= \int_{\Omega} x' y' d\Omega + \bar{x}\bar{y} \int_{\Omega} d\Omega \\
&= \bar{\mathrm{I}}_{xy} + \bar{x}\bar{y} m_{0,0,0}
\end{aligned}
$$

In these examples, $\int_{\Omega} x' d\Omega = \int_{\Omega} y' d\Omega = \int_{\Omega} z' d\Omega = 0$ because x', y', and z' are specified with respect to the reference frame where the object's centroid is situated at the origin.

Finally, substituting (15) and (17) into (18) we get the expressions for the elements of the inertia tensor (16) in terms of 3D moments:

$$\begin{aligned}
\bar{\mathrm{I}}_{xx} &= m_{0,2,0} + m_{0,0,2} - \left(\frac{m_{0,1,0}^2 + m_{0,0,1}^2}{m_{0,0,0}}\right) & \bar{\mathrm{I}}_{xy} &= \bar{\mathrm{I}}_{yx} = m_{1,1,0} - \frac{m_{1,0,0}m_{0,1,0}}{m_{0,0,0}} \\
\bar{\mathrm{I}}_{yy} &= m_{2,0,0} + m_{0,0,2} - \left(\frac{m_{1,0,0}^2 + m_{0,0,1}^2}{m_{0,0,0}}\right) & \bar{\mathrm{I}}_{xz} &= \bar{\mathrm{I}}_{zx} = m_{1,0,1} - \frac{m_{1,0,0}m_{0,0,1}}{m_{0,0,0}} \\
\bar{\mathrm{I}}_{zz} &= m_{2,0,0} + m_{0,2,0} - \left(\frac{m_{1,0,0}^2 + m_{0,1,0}^2}{m_{0,0,0}}\right) & \bar{\mathrm{I}}_{yz} &= \bar{\mathrm{I}}_{zy} = m_{0,1,1} - \frac{m_{0,1,0}m_{0,0,1}}{m_{0,0,0}}
\end{aligned} \tag{19}$$

Part C: Evaluating 3D Moments

As can be seen from (11), in order to calculate 3D moments we have to be able to evaluate the following double integral:

$$\int_0^1 \int_0^{1-u} {}_i^jF_{p,q,r}(u,v)dvdu \tag{20}$$

where ${}_i^jF_{p,q,r}(u,v)$ is defined in (9). For low-order moments, evaluation of (20) is trivial. For example, when $p = q = r = 0$, we have:

$$\begin{aligned}
\int_0^1 \int_0^{1-u} {}_i^1F_{0,0,0}(u,v)dvdu &= \frac{1}{3}\int_0^1 \int_0^{1-u} x_i(u,v)dvdu \\
&= \frac{1}{3}\int_0^1 \left[\int_0^{1-u} (x_{i2} - x_{i1})u + (x_{i3} - x_{i1})v + x_{i1}dv\right] du \\
&= \frac{1}{3}\int_0^1 (x_{i2} - x_{i1})u(1-u) + \frac{1}{2}(x_{i3} - x_{i1})(1-u)^2 + x_{i1}(1-u)du \\
&= \frac{1}{3}\left[(x_{i2} - x_{i1})\left(\frac{u^2}{2} - \frac{u^3}{3}\right) + \frac{1}{2}(x_{i3} - x_{i1})\left(u - u^2 + \frac{u^3}{3}\right) + x_{i1}\left(u - \frac{u^2}{2}\right)\right]_0^1 \\
&= \frac{1}{3}\left[\frac{x_{i2} - x_{i1}}{6} + \frac{x_{i3} - x_{i1}}{6} + \frac{x_{i1}}{2}\right] \\
&= \frac{x_{i1} + x_{i2} + x_{i3}}{18}
\end{aligned}$$

The integrals for $j = 2,3$ can be evaluated in the same manner, so:

$$\int_0^1 \int_0^{1-u} {}_i^2F_{0,0,0}(u,v)dvdu = \frac{y_{i1} + y_{i2} + y_{i3}}{18}$$

$$\int_0^1 \int_0^{1-u} {}_i^3F_{0,0,0}(u,v)dvdu = \frac{z_{i1} + z_{i2} + z_{i3}}{18}$$

From (11) we get:

$$\begin{aligned}
m_{0,0,0}(\mathrm{T}_i) &= 2A_i \sum_{j=1}^{3} n_{ij} \int_0^1 \int_0^{1-u} {}_i^jF_{0,0,0}(u,v)dvdu \\
&= \frac{1}{9}A_i[n_{i1}(x_{i1} + x_{i2} + x_{i3}) + n_{i2}(y_{i1} + y_{i2} + y_{i3}) + n_{i3}(z_{i1} + z_{i2} + z_{i3})] \\
&= \frac{1}{3}A_i\vec{\boldsymbol{n}}_i \cdot \frac{(\boldsymbol{x}_{i1} + \boldsymbol{x}_{i2} + \boldsymbol{x}_{i3})}{3} \\
&= \frac{1}{3}A_i\vec{\boldsymbol{n}}_i \cdot \overline{\boldsymbol{x}}_i
\end{aligned}$$

where $\vec{\boldsymbol{n}}_i$ and $\overline{\boldsymbol{x}}_i$ are the unit normal and centroid of the i-th triangle, respectively.

Finally, plugging this expression into (7), we get the formula for the total volume[1] (i.e., zeroth moment) of the object:

$$V = m_{0,0,0}(\Omega) = \sum_{i=1}^{N} m_{0,0,0}(\mathrm{T}_i) = \frac{1}{3}\sum_{i=1}^{N} A_i\vec{\boldsymbol{n}}_i \cdot \overline{\boldsymbol{x}}_i \tag{21}$$

Similar calculations must be performed to find remaining moments listed in (19).

Finding closed-form expressions for (20) becomes more laborious with increasing moment order. Fortunately, symbolic math packages (e.g., Symbolic Math Toolbox in Matlab) can be used facilitate this task. Explicit formulas for computing all moments (up to 3rd order) can be found in the accompanying M-file titled ‘RigidBodyParams’.

[1] Total mass of the object can be computed by multiplying the volume by density. In this communication, density is assumed to be equal to 1 unit of mass / 1 unit of volume.

Part D: Finding Geometric Primitives with Specified Inertial Parameters

Suppose we are given an arbitrary object and would like to find a geometric primitive such a cuboid or an ellipsoid with exactly the same inertial properties as the original object. Both the cuboid and the ellipsoid can be defined by three parameters $\{a, b, c\}$. In case of the former, $\{a, b, c\}$ represent length, width and height. And in case of the latter, $\{a, b, c\}$ represent the magnitude of the semi-axes. Without loss of generality, let's assume that $a \geq b \geq c$.

Suppose that $\mathbf{I}$ is the inertia tensor of the original object. Using singular value decomposition, $\mathbf{I}$ can be written as:

$$\mathbf{I} = \mathbf{R} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \mathbf{R}^{\mathrm{T}}$$

$$\mathbf{R} = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3] \tag{22}$$

where $\{\lambda_1, \lambda_2, \lambda_3\}$ are the magnitudes of the principal moments of inertia (s.t. $\lambda_1 \geq \lambda_2 \geq \lambda_3$) and $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ are the principal axes of inertia.

In order to find a geometric primitive (i.e., a cuboid or an ellipsoid) with the same $\{\lambda_1, \lambda_2, \lambda_3\}$ we have to solve the following system of equations:

$$\begin{aligned} \lambda_1 &= \alpha M(a^2 + b^2) \\ \lambda_2 &= \alpha M(a^2 + c^2) \\ \lambda_3 &= \alpha M(b^2 + c^2) \end{aligned} \tag{23}$$

where M is the total mass of the primitive we are looking for and α is a constant that varies depending on the primitive; $\alpha_{cuboid} = 1/12$ and $\alpha_{ellipsoid} = 1/5$. Since we previously assumed that density of the object is uniform and equal to one (see section A), $M = \rho V = V$. Recall that

$$\begin{aligned} V_{cuboid} &= abc \\ V_{eliipsoid} &= \frac{4\pi}{3} abc \end{aligned} \tag{24}$$

As first step to finding the solution, we rewrite (23) as:

$$\begin{aligned} 2\alpha V a^2 &= \lambda_1 + \lambda_2 - \lambda_3 = \beta_1 \\ 2\alpha V b^2 &= \lambda_1 + \lambda_3 - \lambda_2 = \beta_2 \\ 2\alpha V c^2 &= \lambda_2 + \lambda_3 - \lambda_1 = \beta_3 \end{aligned} \tag{25}$$

Simplifying further, we get:

$$\gamma(abc)a^2 = \beta_1$$
$$\gamma(abc)b^2 = \beta_2 \tag{26}$$
$$\gamma(abc)c^2 = \beta_3$$

where $\gamma_{cuboid} = 1/6$ and $\gamma_{ellipsoid} = 8\pi/15$.

Finally, the solution we are looking for is:

$$a^5 = \frac{\beta_1^2}{\gamma\sqrt{\beta_2\beta_3}}$$
$$b^5 = \frac{\beta_2^2}{\gamma\sqrt{\beta_1\beta_3}} \tag{27}$$
$$c^5 = \frac{\beta_3^2}{\gamma\sqrt{\beta_1\beta_2}}$$

where a is the length of the primitive along $\boldsymbol{r}_3$, b is the length of the primitive along $\boldsymbol{r}_2$, and c is the length of the primitive along $\boldsymbol{r}_3 \times \boldsymbol{r}_2$. Recall that when the sought primitive is an ellipsoid, $\{a, b, c\}$ represent the lengths of the <u>semi</u>-axes. The calculations described in this section are implemented in the accompanying m-files titled 'RBP_cuboid' and 'RBP_ellipsoid'.