Assignment #6 Daily Returns

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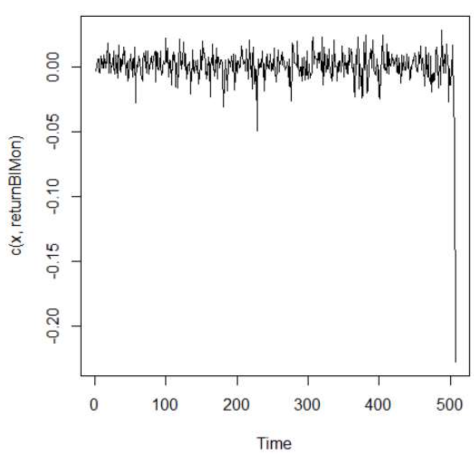
PART 1

Solve textbook Exercise 8 on page 448-449. Part a is worth 10 points. Each of parts b-d is worth 5 points.

Hint: for part a, review Section 5.5.2.

8. On Black Monday, the return on the S&P 500 was −22.8 %. Ouch! This exercise attempts to answer the question, "what was the conditional probability of a return this small or smaller on Black Monday?" "Conditional" means given the information available the previous trading day. Run the following R code:

```
library(rugarch)
library(Ecdat)
library(fGarch)
data(SP500,package="Ecdat")
returnBlMon = SP500$r500[1805] ; returnBlMon # [1] -0.2280063
x = SP500$r500[(1804-2*253+1):1804]
ts.plot(c(x,returnBlMon))
spec = ugarchspec(mean.model=list(armaOrder=c(1,0)),
        variance.model=list(garchOrder=c(1,1)),
        distribution.model = "std") #conditional distrib. of white noise
results1 = ugarchfit(data=x, spec=spec)
dfhat = coef(results1)[6] #Parameter estimate ->6th is degrees of freedom
forecast = ugarchforecast(results1, data=x, n.ahead=1) #b.ahead = 1 = 1 day ahead forecast
```



a)Use the information above to calculate the conditional probability of a return less than or equal to −0.228 on Black Monday. (10 pts)

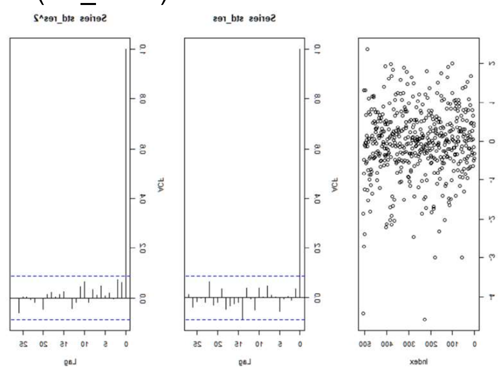
The conditional probability of a return less than or equal to -0.228 on Black Monday is 1.98e-05.

```
sp500mean=forecast@forecast$seriesFor;sp500mean #conditional expected return
sp500sigma=forecast@forecast$sigmaFor;sp500sigma #conditional std of Black Monday return
probBM=pstd(returnBlMon, mean = sp500mean , sd = sp500sigma,nu=dfhat); round(probBM,7)
#1.98e-05
```

b.)Compute and plot the standardized residuals. Also plot the ACF of the standardized residuals and their squares. Include all three plots with your work. Do the standardized residuals indicate that the AR(1)+GARCH(1,1) model fits adequately? (5 points)

The ACF plots are below and show no correlation in the standardized residuals, suggesting that an AR(1) model is appropriate for the conditional mean. Lag 14 almost touches a confidence internval but it generally not a concern. Additionally, there is no autocorrelation in the squared standardized residuals, suggesting the GARCH (1,1) model is appropriate for conditional variance.

```
std_res=results1@fit$residuals/results1@fit$sigma
par(mfrow=c(1,3))
plot(std_res)
acf(std_res)
acf(std_res^2)
```

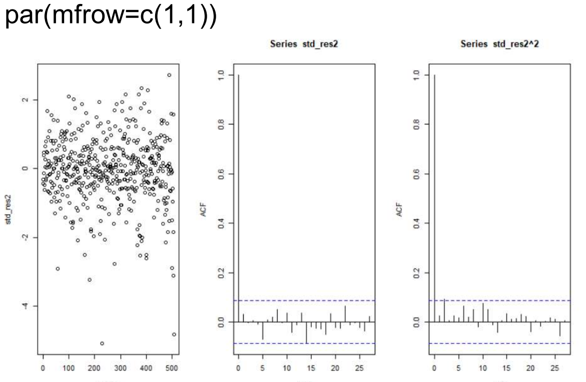


(c) Would an AR(1)+ARCH(1) model provide an adequate fit?

The AR(1) + ARCH(1) model appears to provide an adequate fit but the AR(1) + GARCH(1,1) shows marginal superiority.

The normalized AIC values are -6.5205 and -6.5177 for the AR(1) + GARCH(1,1) and AR(1) + ARCH(1) models, respectively. Multiplying by the sample size, the AIC values are -3299 and -3298. Based on this criteria, the AR(1) + GARCH (1,1) model is slightly better. The ACF plots of the AR(1) + ARCH(1) model show no major issue. The only noteworth item to highlight is that lag 2 of the squared standardized residuals touches a bound whereas in the AR(1) + GARCH (1,1) model the autocorrelation at that lag is much smaller.

```
# Fitting a AR(1)/ARCH(1) model:
spec2 = ugarchspec(mean.model=list(armaOrder=c(1,0)),
            variance.model=list(garchOrder=c(1,0)),
            distribution.model = "std") #conditional distrib. of white noise
results2 = ugarchfit(data=x, spec=spec2)
dfhat2 = coef(results2)[6] #Parameter estimate ->6th is degrees of freedom
forecast2 = ugarchforecast(results2, data=x, n.ahead=1) #b.ahead = 1 = 1 day ahead forecast

show(results1)
#AR(1) + GARCH(1,1)
aic1=-6.5205

show(results2)
#AR(1) + ARCH(1)
aic2= -6.5177

aic1 * length(x)
#-3299

aic2 * length(x)
#-3298


#plots
std_res2=results2@fit$residuals/results2@fit$sigma
par(mfrow=c(1,3))
plot(std_res2)
acf(std_res2)
acf(std_res2^2)
par(mfrow=c(1,1))
```



(d) Does an AR(1) model with a Gaussian conditional distribution provide an adequate fit? Use the arima() function to fit the AR(1) model. This function only allows a Gaussian conditional distribution.

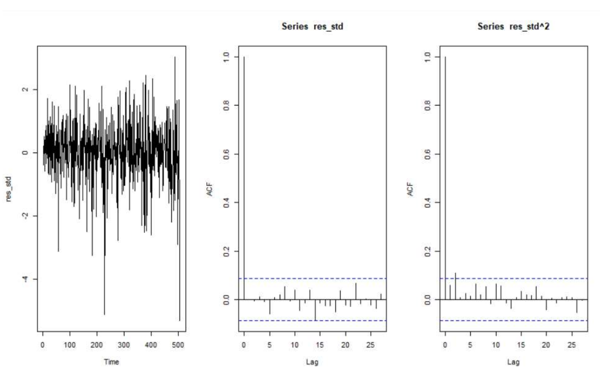
The fit of a simple AR(1) model with a Gaussian conditional distribution is worse than the fit of the AR(1) + GARCH(1,1) above. The AIC of -3247 for the AR(1) model is notably larger than the AIC of -3299 for the AR(1) + GARCH(1,1) model. The squared standardized residuals of AR(1) also show some autocorrelation at lag 2.

The inferior fit could be due to the missing GARCH modification and the Gaussian conditional distribution assumption.

```
#Fit AR(1)
fitAR = arima( x, order=c(1,0,0) )
fitAR
```

```
Call:
arima(x = x, order = c(1, 0, 0))

Coefficients:
        ar1  intercept
      0.127      0.001
s.e.  0.045      0.000

sigma^2 estimated as 0.0000945:  log likelihood = 1627,  aic = -3247
```

```
res = residuals(fitAR)
res_std = res / sqrt(fitAR$sigma2) # standardized residuals
par(mfrow=c(1,3))
plot(res_std)
acf(res_std)
acf(res_std^2)
par(mfrow=c(1,1))
```



PART 2

The conditional variance of an AR(1)+GARCH(1, 1) process

$Y(t)-\mu=\varphi(Y(t-1)-\mu)+a(t)$

$a(t)=\sigma(t)\epsilon(t)$

Is

$$\sigma(t)^2 = \omega + \alpha\, a(t-1)^2 + \beta\, \sigma(t-1)^2$$

where $\mu,\varphi,\omega,\alpha,\beta$ are constant parameters.

What is E[$Y(t)-\mu$] ? (1 pt)

Assuming the expected return on an asset is given by:

$Y_t = \mu + \sigma_t \epsilon_t$

where ϵ_t is a sequence of N(0,1) i.i.d. random variables then E[$Y(t)-\mu$] is the residual return at time t, E[$Y(t)-\mu$] is a mean-centered process where μ denotes the mean over the period.

$Y_t - \mu = \sigma_t \epsilon_t$

$a_t = \sigma_t \epsilon_t$

Given that the parameters AR(1) are time invariant, E[Y_t]= $\mathrm{E}[Y_{t-1}] = \mu$

What is E[$\epsilon(t)$] ? (1 pt)

E[$\epsilon(t)$] denotes the expected conditional error or white noise at time t. In AR(1) / GARCH(1,1) processes, the mean is modelled by a first order AR(1), with a GARCH(1,1) error:

$Y_t = \mu_t + a_t$

Where the stochastic component evolves according to an AR(1) process described as:

$\mu_t = \rho_t u_{t-1} + a_t$

$\mu_t = \alpha_0 + \alpha_1 u_{t-1} + a_t$

With the error term given by:

$a_t = \sigma_t \epsilon_t,$

Assuming the GARCH(1,1) process is stationary, ϵ_t is a random noise term satisfying E[ϵ_t] = 0 and ϵ_t is weak white noise.

What is E[ϵ(t)2]? (1 pt)

E[ϵ_t^2] is the squared of the estimated conditional error terms. Assuming the process is stationary, E[ϵ_t^2] = Var(ϵ_t) =1 .

d) What is E[$\epsilon(t)\epsilon(t-1)$] ? (1 pt)

E[$\epsilon(t)$ $\epsilon(t-1)$] are adjacent error terms and can be used to measure the covariance. The error terms $\epsilon(t)$ still have mean 0 and constant variance. These are fed into the below equation to calculate the autocorrelation parameter:

$$\mathrm{Cov}(\epsilon_t, \epsilon_{t-1}) = \rho\left(\frac{\sigma^2}{1-\rho^2}\right),$$

$$\mathrm{Corr}(\epsilon_t, \epsilon_{t-1}) = \frac{\mathrm{Cov}(\epsilon_t, \epsilon_{t-1})}{\sqrt{\mathrm{Var}(\epsilon_t)\mathrm{Var}(\epsilon_{t-1})}} = \rho,$$

e) What is the *unconditional* variance of the process? Show your derivation in addition to the answer. (20 pts)

When α + β < 1, the unconditional variance of a a_t GARCH(1,1) process is:

$$E\,[\sigma^2] = \frac{a_0}{1 - \alpha_1 - \beta_1}$$

Re-writing GARCH(1,1) we get the following:

$\sigma_t^2 = a_0 + \alpha_1\, a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$

$\sigma_t^2 = (1 - \alpha_1 - \beta_1)\, E\,[\sigma^2] + \alpha_1\, a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$

Written this way, it is easy to see that next period's conditional variance is a weighted combination of the unconditional variance of returns $E\,[\sigma^2]$, last period's squared residuals a_{t-1}^2, and last period's conditional varianceσ_{t-1}^2 with weights $(1 - \alpha_1 - \beta_1)$ that sum to one.

The entire AR(1) + GARCH (1,1) function would look like this:

$$Y_t = \alpha_0 + \alpha_1 u_{t-1} + \in_t \sqrt{(1 - \alpha_1 - \beta_1)\, E\,[\sigma^2] + \alpha_1\, a_{t-1}^2 + \beta_1 \sigma_{t-1}^2}$$

$E\,[\sigma^2]$ can also be written as γ(h). GARCH tries to account for if and when the variance strays from a long-term average and the function should be persistent to long-term average.

f) Show that this correctly reduces to Ruppert Eq. 12.8 for the unconditional variance of an AR(1) process. (1 pt)

$\mu_t = \alpha_0 + \alpha_1 u_{t-1} + a_t$

Recall the stochastic process above where μ_t is a AR(1) process, $\alpha_0 + \alpha_1 u_{t-1}$ and a_t is a random process with mean zero and constant variance. $\beta_1 = \emptyset$, which cancel out.

Because $E[a_t] = 0$ and $E[u_t] = E[u_{t-1}] = u$ for all t we can write:

$E[u_t] = \mu = \frac{\emptyset}{1-\alpha}$

$$COV[\mu_t, \mu_t - h] = E[(\mu_t - \frac{a_o}{1-\alpha})\ \ (\mu_{t-h}\ \frac{a_o}{1-\alpha})$$

$$= \alpha^{\tau}\, E[(u_{t-\tau} + \alpha u_{t-\tau-1} + \alpha^2 u_{t-\tau-2} + \dots)^2]\,.$$

$$\mathrm{Cov}\,[x_t, x_{t-\tau}] = \alpha^{\tau} V[x_{t-\tau}] = \alpha^{\tau} \frac{\sigma^2}{1-\alpha^2}$$