ME 596 Spacecraft Attitude Dynamics and Control

Rigid Body Dynamics

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Rigid Body Dynamics

Definition of rigid body. Linear momentum and extension of Newton's Second Law. Angular momentum and development of Euler's Law. Euler's equations, moments of inertia. Coupling between rotational and translational motion. Special cases: axisymmetric torque-free; asymmetric torque-free; Lagrange top.

Recall the fundamental dynamics equations

$$\begin{aligned} \vec{\mathbf{f}} &= \frac{d}{dt}\vec{\mathbf{p}} \\ \vec{\mathbf{g}} &= \frac{d}{dt}\vec{\mathbf{h}} \end{aligned}$$

where $\vec{\mathbf{f}}$ is applied force, $\vec{\mathbf{p}}$ is linear momentum, $\vec{\mathbf{g}}$ is applied torque, and $\vec{\mathbf{h}}$ is angular momentum.

In this module, we develop these equations for finite, rigid bodies.

Rigid Body Definition

- Point mass model is an idealization that is useful for describing vehicle motion for many applications; e.g.,
  - Aircraft performance
  - Spacecraft orbits
  - Only translational motion is of interest
- Rigid body model is an idealization that is useful for describing more complicated vehicle motions; e.g.,
  - Aircraft roll, pitch, and yaw
  - Spacecraft pointing
  - Marine vessel motion
  - Both translational and rotational motion are of interest

A rigid body is an idealized model of a solid body of finite dimension in which deformation is neglected. Alternatively, the distance between any two points in the body remains constant in any motion.

The motion of a rigid body is described by the position and velocity of any point in the body with respect to an inertial origin, and the orientation and angular velocity of a body frame with respect to an inertial frame.

Rigid Body Models

- Defining rigid body models takes two distinct approaches in the literature, both of which arrive at the basic equations of motion

$$
\begin{aligned}
\vec{\mathbf{f}} &= \frac{d}{dt}\vec{\mathbf{p}} && \textit{Newton's Second Law} \\
\vec{\mathbf{g}} &= \frac{d}{dt}\vec{\mathbf{h}} && \textit{Euler's Law}
\end{aligned}
$$

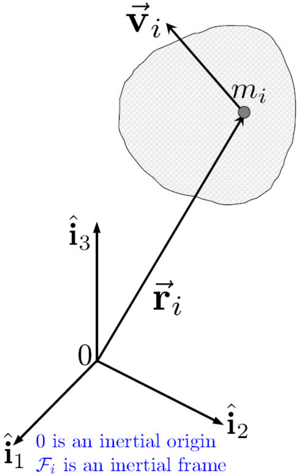
- First approach: begin with a system of n particles and take the limit as $n \to \infty$ to arrive at a continuous body
  - *Newton's Second Law* naturally extends to the system of particles and the resulting limit of a continuous body
  - *Euler's Law* results from application of Newton's Second Law
- Second approach: begin with a continuous body and assert that *Newton's Second Law* and *Euler's Law* are independent principles that apply to finite bodies, whether rigid or deformable

In the context of spacecraft dynamics, the end result is entirely the same for the two approaches.

Particle Model

- Model rigid body as system of n mass particles of mass $m_i, i = 1, ..., n$
- $\vec{\mathbf{r}}_i$ is position vector of m_i wrt inertial origin O
- Distance between any pair of mass particles is constant (*rigid*, not *flexible*)
- Velocity of each particle is $\vec{\mathbf{v}}_i$, and linear momentum is $\vec{\mathbf{p}}_i = m_i \vec{\mathbf{v}}_i$
- System linear momentum is $\vec{\mathbf{p}} = \sum_{i=1}^{n} m_i \vec{\mathbf{v}}_i$
- Angular momentum of system about O is

$$\vec{\mathbf{h}}^O = \sum_{i=1}^{n} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{v}}_i$$

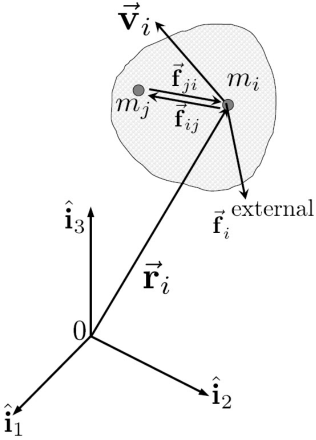


Each particle obeys Newton's Second Law, $\vec{\mathbf{f}}_i = \frac{d}{dt} \vec{\mathbf{p}}_i$

Particle Model (2)

- System linear momentum satisfies

$$
\begin{aligned}
\dot{\vec{\mathbf{p}}} &= \sum_{i=1}^{n} m_i \dot{\vec{\mathbf{v}}}_{\mathbf{i}} = \sum_{i=1}^{n} \dot{\vec{\mathbf{p}}}_{\mathbf{i}} \\
&= \sum_{i=1}^{n} \vec{\mathbf{f}}_i \\
&= \sum_{i=1}^{n} \vec{\mathbf{f}}_i^{\text{int}} + \sum_{i=1}^{n} \vec{\mathbf{f}}_i^{\text{ext}} \\
&= \sum_{i=1}^{n} \vec{\mathbf{f}}_i^{\text{ext}} = \vec{\mathbf{f}} \\
\dot{\vec{\mathbf{p}}} &= \vec{\mathbf{f}}
\end{aligned}
$$



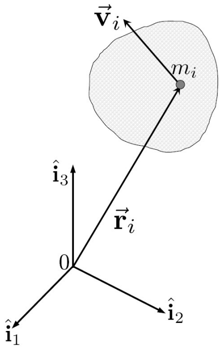
Internal forces cancel:
$\vec{\mathbf{f}}_{ij} = -\vec{\mathbf{f}}_{ji}$

So Newton's Second Law (using Third Law) extends to systems of particles, and hence to Particle Model for rigid bodies.

Particle Model (3)

- Angular momentum of system of particles satisfies

$$
\begin{aligned}
\dot{\vec{\mathbf{h}}}^O &= \sum_{i=1}^{n} \frac{d}{dt}\left[\vec{\mathbf{r}}_i \times m_i \vec{\mathbf{v}}_i\right] \\
&= \sum_{i=1}^{n} \left[\dot{\vec{\mathbf{r}}}_i \times m_i \vec{\mathbf{v}}_i + \vec{\mathbf{r}}_i \times m_i \dot{\vec{\mathbf{v}}}_i\right] \\
&= \sum_{i=1}^{n} \left[\vec{\mathbf{r}}_i \times \vec{\mathbf{f}}_i\right] = \sum_{i=1}^{n} \left[\vec{\mathbf{r}}_i \times \vec{\mathbf{f}}_i^{\text{ext}}\right] \\
&= \sum_{i=1}^{n} \vec{\mathbf{g}}_i^{O,\text{ext}} \\
\dot{\vec{\mathbf{h}}}^O &= \vec{\mathbf{g}}^{O,\text{ext}}
\end{aligned}
$$



So Newton's Second Law (using Third Law) leads to Euler's Law for Particle Model.

Continuous Model

- Model rigid body as a *continuum*, $\mathcal{B}$, that satisfies Newton's Second law and Euler's Law
- System linear momentum

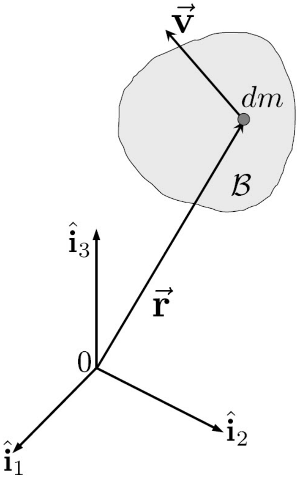
$$\vec{\mathbf{p}} = \int_{\mathcal{B}} \vec{\mathbf{v}}\, dm$$

- System angular momentum about point O

$$\vec{\mathbf{h}}^{O} = \int_{\mathcal{B}} \vec{\mathbf{r}} \times \vec{\mathbf{v}}\, dm$$

- These two vector quantities obey the linear and angular momentum principles as expressed by Newton and Euler

$$\dot{\vec{\mathbf{p}}} = \vec{\mathbf{f}} \quad \text{and} \quad \dot{\vec{\mathbf{h}}}^{O} = \vec{\mathbf{g}}^{O}$$



We use the continuous model in the remainder of the course.

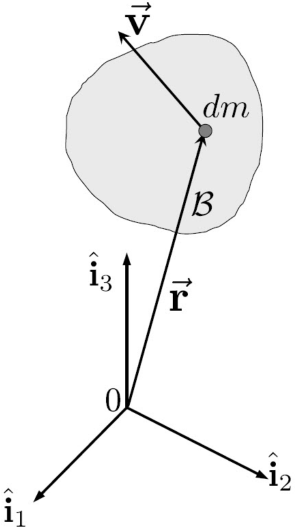
Continuous Model (2)

- What does the integral in $\vec{\mathbf{p}} = \int_{\mathcal{B}} \vec{\mathbf{v}}\, dm$ mean?
- It defines a three-dimensional volume integral over the volume of the continuous body $\mathcal{B}$; in general

$$\int_{\mathcal{B}} f(\vec{\mathbf{r}})\, dV =$$

$$\int_{r_1^-}^{r_1^+} \int_{r_2^-}^{r_2^+} \int_{r_3^-}^{r_3^+} f(\vec{\mathbf{r}})\, dr_3\, dr_2\, dr_1$$

- Three types of integrals arise in the development of rigid body equations of motion: scalar, vector, and tensor.
- Alternatively, these three mathematical objects can be called zeroth-rank, first-rank, and second-rank tensors.

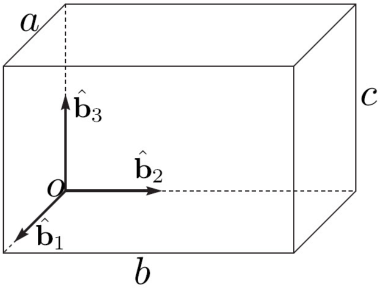


Continuous Model (3)

- For example, the total mass, m, is the integral over the body of the density, $\mu(\vec{\mathbf{r}}, t)$, which can in general vary from point to point within the body, or even with time.
- In this course, we assume that density is constant.

$$
\begin{aligned}
m &= \int_{\mathcal{B}} dm \\
&= \int_{\mathcal{B}} \mu \, dV \\
&= \int_0^a \int_0^b \int_0^c \mu \, dr_3 \, dr_2 \, dr_1 \\
&= \mu abc
\end{aligned}
$$

- Mass is a scalar, or zeroth-rank tensor, and is sometimes called the *zeroth moment of inertia*.

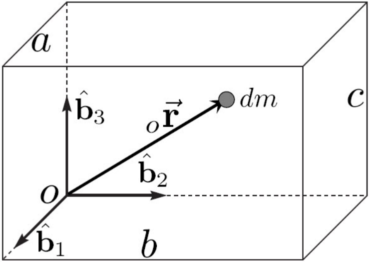


- The rectangular prism has sides of length a, b, and c.
- Note that the origin of the body frame shown *is not* the mass center.

Continuous Model (4)

- The *first moment of inertia about* o, $\vec{\mathbf{c}}^o$, is the integral over the body of the position vector from o to each differential mass element dm.
- The position vector is denoted ${}_o\vec{\mathbf{r}}$, where the left subscript indicates that it is the vector *from* o to dm.

$$
\begin{aligned}
\vec{\mathbf{c}}^o &= \int_{\mathcal{B}} {}_o\vec{\mathbf{r}}\, dm \quad \text{or} \\
\mathbf{c}^o_b &= \int_{\mathcal{B}} {}_o\mathbf{r}_b\, \mu\, dV \quad (\text{in } \mathcal{F}_b) \\
&= \int_0^a \int_0^b \int_0^c \mu [r_1 \;\; r_2 \;\; r_3]^\mathsf{T}\, dr_3\, dr_2\, dr_1 \\
&= \frac{\mu}{2} [a^2bc \;\; ab^2c \;\; abc^2]^\mathsf{T}
\end{aligned}
$$



- The first moment of inertia can be used to find the mass center, defined as the point about which $\mathbf{c}$ vanishes

Continuous Model (5)

- The mass center c, is defined as the point about which the first moment of inertia, $\vec{\mathbf{c}}^c$, vanishes*.

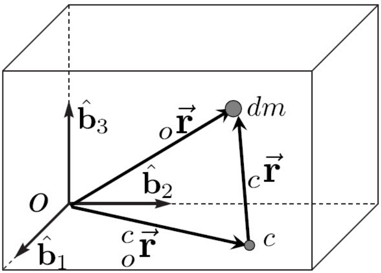
$$
\begin{aligned}
\vec{\mathbf{c}}^c &= \int_{\mathcal{B}} {}_c\vec{\mathbf{r}}\, dm = \vec{\mathbf{0}} \quad \text{or} \\
\mathbf{c}_b^c &= \int_{\mathcal{B}} {}_c\mathbf{r}_b\, \mu\, dV = \mathbf{0}
\end{aligned}
$$

- Position vector can be written as

$$
\begin{aligned}
{}_o\vec{\mathbf{r}} &= {}_o^c\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}} \quad \text{or} \\
{}_o\mathbf{r} &= {}_o^c\mathbf{r} + {}_c\mathbf{r}
\end{aligned}
$$

- Thus

$$
\begin{aligned}
\mathbf{c}^o &= \int_{\mathcal{B}} \left({}_o^c\mathbf{r} + {}_c\mathbf{r}\right)\, dm = \int_{\mathcal{B}} {}_o^c\mathbf{r}\, dm \\
&= {}_o^c\mathbf{r} \int_{\mathcal{B}} dm = m\, {}_o^c\mathbf{r}
\end{aligned}
$$



- Putting it all together, we obtain

$$
{}_o^c\mathbf{r} = \frac{1}{2}\,[a \;\; b \;\; c]^{\mathsf{T}}
$$

- That is, the position vector from o to c is halfway along each of the three axes from the origin, as expected.

*Be aware of abuse of notation here: c is used to identify the mass center, while in the prism example, it also denotes the length of one edge.

Continuous Model (6)

- Linear momentum has been defined as

$$\vec{\mathbf{p}} = \int_{\mathcal{B}} \vec{\mathbf{v}}\, dm$$

  where $\vec{\mathbf{v}}$ is time derivative of position vector $\vec{\mathbf{r}}$ of a differential mass element $dm = \mu\, dV$
- This position vector must be measured from an inertial origin, and the derivatives are taken with respect to inertial space
- The position vector can be written as ${}^o_O\vec{\mathbf{r}} + {}_o\vec{\mathbf{r}}$, where ${}^o_O\vec{\mathbf{r}}$ is the position vector from the inertial origin *to* o and ${}_o\vec{\mathbf{r}}$ is the vector *from* o to a point in the body
- *What is the velocity* $\vec{\mathbf{v}}$ *of a differential mass element?*
- The body frame motion has two *velocities*:
  • the velocity of its origin with respect to the inertial origin, ${}^o_O\vec{\mathbf{v}}$, and
  • the angular velocity of the frame with respect to an inertial frame, $\vec{\boldsymbol{\omega}}^{bi}$
- The velocity of a mass element dm is then

$$\vec{\mathbf{v}} = {}^o_O\vec{\mathbf{v}} + \vec{\boldsymbol{\omega}}^{bi} \times {}_o\vec{\mathbf{r}}$$

Continuous Model (7)

- Thus the linear momentum is

$$\vec{\mathbf{p}} = \int_{\mathcal{B}} \left({}^{o}_{O}\vec{\mathbf{v}} + \vec{\boldsymbol{\omega}}^{bi} \times {}_{o}\vec{\mathbf{r}} \right) \, dm$$

- The velocity of the origin and the angular velocity of the frame do not depend on the particular differential mass element, so they are constant with respect to the integration, and momentum becomes

$$\vec{\mathbf{p}} = m {}^{o}_{O}\vec{\mathbf{v}} + \vec{\boldsymbol{\omega}}^{bi} \times \int_{\mathcal{B}} {}_{o}\vec{\mathbf{r}} \, dm$$

- The right term is the first moment of inertia about the origin, so that

$$\vec{\mathbf{p}} = m {}^{o}_{O}\vec{\mathbf{v}} + \vec{\boldsymbol{\omega}}^{bi} \times \vec{\mathbf{c}}^{o}$$

- If o is the mass center c, then the linear momentum is simply

$$\vec{\mathbf{p}} = m {}^{c}_{O}\vec{\mathbf{v}}$$

or

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$$

- Thus linear momentum for a rigid body is equivalent to linear momentum of a particle of the same mass with the same velocity

Continuous Model (8)

- Newton's Second Law relates linear momentum to applied force:

$$\begin{aligned} \vec{\mathbf{p}} &= m_O^o \vec{\mathbf{v}} + \vec{\boldsymbol{\omega}}^{bi} \times \vec{\mathbf{c}}^o \\ \vec{\mathbf{f}} &= \dot{\vec{\mathbf{p}}} \end{aligned}$$

- How do we express these equations in a rotating reference frame?
- Recall the formula for differentiating a vector $\mathbf{a}$ expressed in a rotating frame $\mathcal{F}_b$:

$$\frac{d}{dt}\left[\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\mathbf{a}\right] = \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\left[\dot{\mathbf{a}} + \boldsymbol{\omega}^{\times}\mathbf{a}\right]$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}^{bi}$

- Applying the formula to $\vec{\mathbf{p}}$ and to $\vec{\mathbf{v}} = \dot{\vec{\mathbf{r}}}$, we obtain

$$\begin{aligned} \mathbf{p} &= m\left[\dot{\mathbf{r}} + \boldsymbol{\omega}^{\times}\mathbf{r}\right] + \boldsymbol{\omega}^{\times}\mathbf{c} \\ \mathbf{f} &= \dot{\mathbf{p}} + \boldsymbol{\omega}^{\times}\mathbf{p} \end{aligned}$$

which we can rearrange in the usual $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ form as

$$\begin{aligned} \dot{\mathbf{r}} &= \frac{1}{m}\left[\mathbf{p} - \boldsymbol{\omega}^{\times}\mathbf{c}\right] - \boldsymbol{\omega}^{\times}\mathbf{r} \\ \dot{\mathbf{p}} &= -\boldsymbol{\omega}^{\times}\mathbf{p} + \mathbf{f} \end{aligned}$$

- The $\dot{\mathbf{r}}$ equation is the kinematics equation, and is frequently written in an inertial frame instead of in a rotating frame

Continuous Model (9)

- The translational equations of motion in vector form are:

$$
\begin{aligned}
\vec{\mathbf{p}} &= m\vec{\mathbf{v}} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{c}} \\
\vec{\mathbf{f}} &= \dot{\vec{\mathbf{p}}}
\end{aligned}
$$

- The matrix translational equations of motion, expressed in a rotating frame and rearranged, are

$$
\begin{aligned}
\dot{\mathbf{r}} &= \frac{1}{m}\left[\mathbf{p} - \boldsymbol{\omega}^{\times}\mathbf{c}\right] - \boldsymbol{\omega}^{\times}\mathbf{r} \\
\dot{\mathbf{p}} &= -\boldsymbol{\omega}^{\times}\mathbf{p} + \mathbf{f}
\end{aligned}
$$

What subscripts and superscripts are omitted, and what meaning for them should we assume?

- Note that $\vec{\boldsymbol{\omega}}$ is generally obtained from the rotational equations of motion, and that $\vec{\mathbf{f}}$ may depend on the rotational motion; *i.e.*,

$$
\vec{\mathbf{f}} = \vec{\mathbf{f}}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, \bar{\mathbf{q}}, \vec{\boldsymbol{\omega}})
$$

or

$$
\vec{\mathbf{f}} = \vec{\mathbf{f}}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, \mathbf{R}, \vec{\boldsymbol{\omega}})
$$

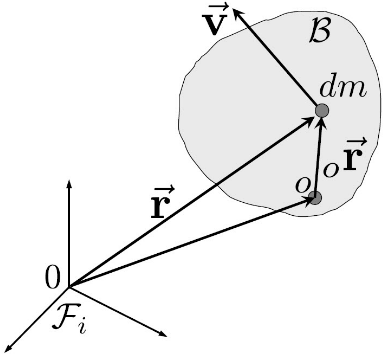
- That is, the force generally depends on the dynamic *state* of the body: its position, velocity, attitude, and angular velocity

Continuous Model (10)

- There are two forms of *angular momentum* in common use: *moment of momentum*, and *angular momentum*
- The *moment of momentum*, $\vec{\mathbf{H}}^o$ about o is

$$\vec{\mathbf{H}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times \vec{\mathbf{v}}\, dm$$

  where ${}_o\vec{\mathbf{r}}$ is the position vector of a differential mass element from point o, and $\vec{\mathbf{v}}$ is the velocity of the mass element with respect to inertial space.
- Usually choose o as c or 0
- The quantity $\vec{\mathbf{v}}\, dm$ is in fact the *linear momentum* of the differential mass element, and by taking the cross product with the moment arm ${}_o\vec{\mathbf{r}}$, we are forming the moment of momentum about o.

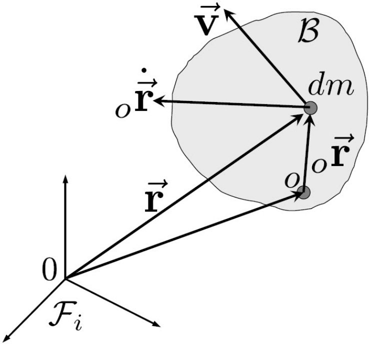


Continuous Model (11)

- The *angular momentum*, $\vec{\mathbf{h}}^o$ about o is

$$\vec{\mathbf{h}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times {}_o\dot{\vec{\mathbf{r}}}\, dm$$

  where ${}_o\vec{\mathbf{r}}$ is the position vector of a differential mass element from point o, and ${}_o\dot{\vec{\mathbf{r}}}$ is the velocity of the mass element with respect to o.
- The difference between the two definitions is the velocity terms that are used
- Angular momentum (as defined here) is more commonly used in spacecraft dynamics
- The quantity ${}_o\dot{\vec{\mathbf{r}}}\, dm$ is *not* the momentum of the differential mass element, and by taking the cross product with the moment arm ${}_o\vec{\mathbf{r}}$, we are forming the moment of a momentum-like quantity about o.



Continuous Model (12)

- We will work with the *angular momentum*, $\vec{\mathbf{h}}^o$ about o:

$$\vec{\mathbf{h}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times {}_o\dot{\vec{\mathbf{r}}}\, dm$$

- The *velocity* term is

$${}_o\dot{\vec{\mathbf{r}}} = \vec{\boldsymbol{\omega}} \times {}_o\vec{\mathbf{r}}$$

where $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}^{bi}$. Thus

$$\vec{\mathbf{h}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times \left[\vec{\boldsymbol{\omega}} \times {}_o\vec{\mathbf{r}}\right] \, dm$$

- Expand the integrand using the identity:

$$\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}\right) - \vec{\mathbf{c}}\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right)$$

- Also, we will drop the left subscript o on the position vector $\vec{\mathbf{r}}$, understanding that it is the position vector from the point o to the differential mass element dm.

$$\begin{aligned} \vec{\mathbf{h}}^o &= \int_{\mathcal{B}} \left[\vec{\mathbf{r}} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}\right)\right] \, dm \\ &= \int_{\mathcal{B}} \left[\vec{\boldsymbol{\omega}}\left(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}\right) - \vec{\mathbf{r}}\left(\vec{\boldsymbol{\omega}} \cdot \vec{\mathbf{r}}\right)\right] \, dm \end{aligned}$$

Rewrite the two terms in the integrand as

$$\begin{aligned} \vec{\boldsymbol{\omega}}\left(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}\right) &= \vec{\mathbf{r}} \cdot \vec{\mathbf{r}}\,\vec{\boldsymbol{\omega}} \\ \vec{\mathbf{r}}\left(\vec{\boldsymbol{\omega}} \cdot \vec{\mathbf{r}}\right) &= \vec{\mathbf{r}}\vec{\mathbf{r}} \cdot \vec{\boldsymbol{\omega}} \end{aligned}$$

Whenever we write three vectors in the form $\vec{\mathbf{a}}\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$, parentheses are implied around the two vectors with the dot product between them.

$$\vec{\mathbf{a}}\vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}}\left(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}\right)$$

and

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\vec{\mathbf{c}} = \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right)\vec{\mathbf{c}}$$

Continuous Model (13)

- Introduce a new mathematical object, the *identity tensor*, $\vec{\mathbf{1}}$, defined by its operation on vectors:

$$\vec{\mathbf{1}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{1}} = \vec{\mathbf{v}}$$

- Use the identity tensor to rewrite the first term in the integrand as

$$\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}\,\vec{\boldsymbol{\omega}} = \vec{\mathbf{r}} \cdot \vec{\mathbf{r}}\,\vec{\mathbf{1}} \cdot \vec{\boldsymbol{\omega}} = r^2 \vec{\mathbf{1}} \cdot \vec{\boldsymbol{\omega}}$$

- Combining all these expressions, we write $\vec{\mathbf{h}}^o$ as

$$\vec{\mathbf{h}}^o = \int_{\mathcal{B}} \left[r^2 \vec{\mathbf{1}} \cdot \vec{\boldsymbol{\omega}} - \vec{\mathbf{r}}\vec{\mathbf{r}} \cdot \vec{\boldsymbol{\omega}} \right] \, dm$$

Exercise: Verify that
$\vec{\mathbf{1}} = \left\{ \hat{\mathbf{b}} \right\}^{\mathsf{T}} \left\{ \hat{\mathbf{b}} \right\}$

- The angular velocity vector is independent of the position vector to each individual mass element, so

$$\vec{\mathbf{h}}^o = \int_{\mathcal{B}} \left[r^2 \vec{\mathbf{1}} - \vec{\mathbf{r}}\vec{\mathbf{r}} \right] \, dm \cdot \vec{\boldsymbol{\omega}}$$

- The integral is the *moment of inertia tensor*, and is *constant*, just as the total mass and first moment integrals are:

$$\vec{\mathbf{I}}^o = \int_{\mathcal{B}} \left[r^2 \vec{\mathbf{1}} - \vec{\mathbf{r}}\vec{\mathbf{r}} \right] \, dm$$

- If the point o is clear, omit the superscript:

$$\vec{\mathbf{I}} = \int_{\mathcal{B}} \left[r^2 \vec{\mathbf{1}} - \vec{\mathbf{r}}\vec{\mathbf{r}} \right] \, dm$$

Continuous Model (14)

- Angular momentum (about o) is then
$$\vec{\mathbf{h}} = \vec{\mathbf{I}} \cdot \vec{\boldsymbol{\omega}}$$
where
$$\vec{\mathbf{I}} = \int_{\mathcal{B}} \left[r^2 \vec{\mathbf{1}} - \vec{\mathbf{r}}\vec{\mathbf{r}} \right] \, dm$$
- How do we calculate $\vec{\mathbf{I}}$?
- Think of $\vec{\mathbf{I}}$ as you think of vectors: it is an abstract mathematical object that has numbers associated with it when you choose a particular reference frame.
- Specifically, $\vec{\mathbf{I}}$ is a second-rank tensor, and when it is expressed in a particular reference frame, it takes the form of a 3×3 symmetric matrix.
- We previously computed body integrals involving $\vec{\mathbf{r}}\,dm$, and expressed $\vec{\mathbf{r}}$ in a body-fixed frame:
$$\vec{\mathbf{r}} = \mathbf{r}^{\mathsf{T}} \left\{ \hat{\mathbf{b}} \right\} = \left\{ \hat{\mathbf{b}} \right\}^{\mathsf{T}} \mathbf{r}$$
Refresh your memory regarding notation!
- Substitute these expressions into the terms in the integrand $\vec{\mathbf{I}}$:
$$\begin{aligned} r^2 &= \vec{\mathbf{r}} \cdot \vec{\mathbf{r}} \\ &= \mathbf{r}^{\mathsf{T}} \left\{ \hat{\mathbf{b}} \right\} \cdot \left\{ \hat{\mathbf{b}} \right\}^{\mathsf{T}} \mathbf{r} \\ &= \mathbf{r}^{\mathsf{T}} \mathbf{1} \mathbf{r} \\ &= \mathbf{r}^{\mathsf{T}} \mathbf{r} \\ \vec{\mathbf{r}}\vec{\mathbf{r}} &= \left\{ \hat{\mathbf{b}} \right\}^{\mathsf{T}} \mathbf{r} \mathbf{r}^{\mathsf{T}} \left\{ \hat{\mathbf{b}} \right\} \end{aligned}$$
- Spend a couple minutes working out how these terms fit into $\vec{\mathbf{I}}$.

Continuous Model (15)

- Make the substitutions, recalling that

$$
\begin{aligned}
\vec{\mathbf{1}} &= \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\left\{\hat{\mathbf{b}}\right\} \\
\mathbf{1} &= \left\{\hat{\mathbf{b}}\right\}\cdot\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\mathbf{I}} &= \int_{\mathcal{B}} \left[r^2\vec{\mathbf{1}} - \vec{\mathbf{r}}\vec{\mathbf{r}}\right]\,dm \\
&= \int_{\mathcal{B}} \left[r^2\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\left\{\hat{\mathbf{b}}\right\}\right. \\
&\quad \left. - \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\mathbf{r}\mathbf{r}^{\mathsf{T}}\left\{\hat{\mathbf{b}}\right\}\right]\,dm \\
&= \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\int_{\mathcal{B}} \left[r^2\mathbf{1} - \mathbf{r}\mathbf{r}^{\mathsf{T}}\right]dm\left\{\hat{\mathbf{b}}\right\}
\end{aligned}
$$

- Take the dot product of both sides with $\left\{\hat{\mathbf{b}}\right\}$ from the left and $\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}$ from the right

- Taking the dot products gives

$$
\left\{\hat{\mathbf{b}}\right\}\cdot\vec{\mathbf{I}}\cdot\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}} = \int_{\mathcal{B}}\left[r^2\mathbf{1} - \mathbf{r}\mathbf{r}^{\mathsf{T}}\right]\,dm
$$

- In the same way that taking dot products of a vector $\vec{\mathbf{v}}$ with the unit vectors gives the *components* of the vector in the given frame, the expression on the left is the *components* of the inertia tensor in $\mathcal{F}_b$:

$$
\begin{aligned}
\mathbf{v}_b &= \vec{\mathbf{v}}\cdot\left\{\hat{\mathbf{b}}\right\} \\
\mathbf{I}_b &= \left\{\hat{\mathbf{b}}\right\}\cdot\vec{\mathbf{I}}\cdot\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}
\end{aligned}
$$

Continuous Model (16)

- The matrix version of the moment of inertia tensor is computed in similar fashion as the first moment of inertia vector

$$\mathbf{I}_b^o = \int_{\mathcal{B}} \left[r^2 \mathbf{1} - \mathbf{r}\mathbf{r}^\mathsf{T} \right] \mu \, dV$$

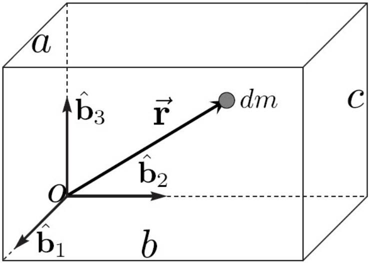
- The term in brackets is easily shown to be

$$\begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix}$$

- The required integration is

$$\int_0^a \int_0^b \int_0^c [\cdot] \, \mu \, dr_3 \, dr_2 \, dr_1$$

- Each element of the matrix is integrated independently of the rest

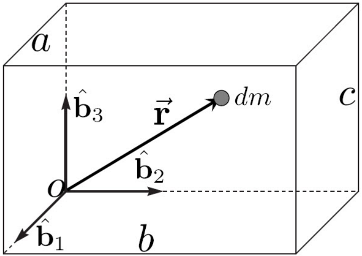


For example,

$$\begin{aligned} I_{11} &= \frac{\mu abc}{3} \left(b^2 + c^2 \right) \\ &= \frac{m}{3} \left(b^2 + c^2 \right) \end{aligned}$$

You should be able to carry out the required integrations.

Continuous Model (17)



- Carrying out the integrations leads to

$$\mathbf{I}_b^o = m \begin{bmatrix} \frac{1}{3}(b^2+c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2+c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2+b^2) \end{bmatrix}$$

- Note the pattern, especially that $\mathbf{I}^\mathsf{T} = \mathbf{I}$, and that the three diagonal elements are positive.
- We have computed the moment of inertia matrix *about* point o, and with respect to the $\hat{\mathbf{b}}$ frame. We need to be able to *translate* the origin, and *rotate* the frame.

Continuous Model (18) Summary

- Angular momentum about o:

$$\vec{\mathbf{h}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times {}_o\dot{\vec{\mathbf{r}}}\,\mathbf{dm}$$
$$\vec{\mathbf{h}}^o = \vec{\mathbf{I}}^o \cdot \vec{\boldsymbol{\omega}}$$

where

$$\vec{\mathbf{I}}^o = \int_{\mathcal{B}} \left[{}_o r^2 \vec{\mathbf{1}} - {}_o\vec{\mathbf{r}}\,{}_o\vec{\mathbf{r}}\right] dm$$

- The superscript o denotes that angular momentum, and inertia tensor, are taken *about* point o.
- The left subscript o denotes that the integral is computed with position vectors from o to differential mass elements dm.

- Express angular momentum vector, angular velocity vector, and moment of inertia tensor in $\mathcal{F}_b$:

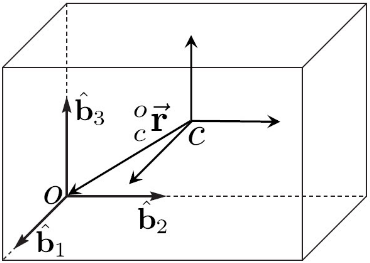
$$\begin{aligned}
\mathbf{h}_b^o &= \vec{\mathbf{h}}^o \cdot \{\hat{\mathbf{b}}\} \\
\boldsymbol{\omega}_b^{bi} &= \vec{\boldsymbol{\omega}}^{bi} \cdot \{\hat{\mathbf{b}}\} \\
\mathbf{I}_b^o &= \{\hat{\mathbf{b}}\} \cdot \vec{\mathbf{I}} \cdot \{\hat{\mathbf{b}}\}^{\mathsf{T}} \\
\mathbf{h}_b^o &= \mathbf{I}_b^o \boldsymbol{\omega}_b^{bi} \\
\mathbf{h} &= \mathbf{I}\boldsymbol{\omega}
\end{aligned}$$

Exercise: Carry out the steps in these equations

- Note that the angular velocity here is *the angular velocity of the body-fixed frame with respect to an inertial frame.*

Parallel Axis Theorem

- Suppose we know the moment of inertia matrix about the mass center, c, expressed in a body-fixed reference frame, and we want the moment of inertia matrix about a different point o, defined by ${}^{o}_{c}\mathbf{r}_b$, expressed in the same frame, but with translated origin.
- The former is denoted $\mathbf{I}_b^c$ and the latter is denoted $\mathbf{I}_b^o$
- We can safely omit the subscript b, since the reference frame is the same for all the matrices
- We retain the superscripts c and o, since we will be working with vectors and matrices with respect to both origins



- Technically, the two frames are the same, since their unit vectors are in the same directions
- Their origins, however, are displaced; hence the term *parallel axes*

Parallel Axis Theorem (2)

- Moment of inertia matrix about c:

$$\mathbf{I}^c = \int_{\mathcal{B}} \left[{}_c r^2 \mathbf{1} - {}_c\mathbf{r}\,{}_c\mathbf{r}^\mathsf{T} \right] dm$$

- We write ${}_c\mathbf{r}$ as

$${}_c\mathbf{r} = {}^o_c\mathbf{r} + {}_o\mathbf{r}$$

and obtain

$$\begin{aligned} \mathbf{I}^c &= \int_{\mathcal{B}} \left[{}_o r^2 \mathbf{1} - {}_o\mathbf{r}\,{}_o\mathbf{r}^\mathsf{T} \right] dm \\ &\quad - m \left[{}^o_c r^2 \mathbf{1} - {}^o_c\mathbf{r}\,{}^o_c\mathbf{r}^\mathsf{T} \right] \end{aligned}$$

Exercise: Carry out the intermediate steps, making use of the fact that $\mathbf{c}^c = \mathbf{0}$

- The integral in this expression is the moment of inertia matrix about the point o, in $\mathcal{F}_b$

- Thus we have derived the Parallel Axis Theorem:

$$\begin{aligned} \mathbf{I}^c &= \mathbf{I}^o - m \left[{}^o_c r^2 \mathbf{1} - {}^o_c\mathbf{r}\,{}^o_c\mathbf{r}^\mathsf{T} \right] \\ \mathbf{I}^o &= \mathbf{I}^c + m \left[{}^o_c r^2 \mathbf{1} - {}^o_c\mathbf{r}\,{}^o_c\mathbf{r}^\mathsf{T} \right] \end{aligned}$$

- Alternatively

$$\mathbf{I}^o = \mathbf{I}^c - m\,{}^o_c\mathbf{r}^\times\,{}^o_c\mathbf{r}^\times$$

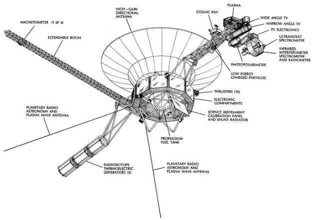
Exercise: Show the two expressions are equivalent using

$$-\mathbf{a}^\times \mathbf{a}^\times = \mathbf{1} - \mathbf{a}\mathbf{a}^\mathsf{T}$$

Question: Does it matter whether we use ${}^o_c\mathbf{r}$ or ${}^c_o\mathbf{r}$?

Importance of Parallel Axis Theorem

- Recalling the simple version of the PAT from undergraduate dynamics books, one might wonder why the development here is so complicated, and why it is important in spacecraft dynamics
- Spacecraft are fairly complex assemblages of a variety of components
- We can easily look up the moments of inertia of cylinders and plates and so forth
- We need to “add” all those inertia matrices together to obtain the important $\mathbf{I}$ for the entire spacecraft



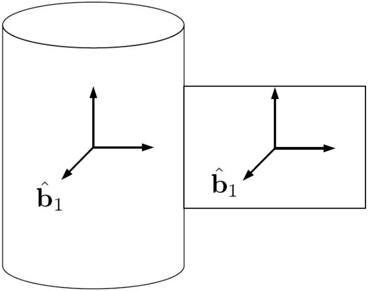
The Voyager Spacecraft is an example of a complicated space structure. How would you determine its inertia matrix?

- The Parallel Axis Theorem is one of the two tools that we use to combine all those simple inertia matrices into the spacecraft inertia matrix.
- The other tool is the Change Of Vector Basis Theorem, which we will develop after doing a simple PAT example

Steps in Application of PAT

1. Decompose the spacecraft into simple components
2. Look up or do the integral to obtain the principal moments of inertia of each component about its own mass center
3. Choose an origin and a base frame in which to add all the inertia matrices together (normally this might be the mass center and principal axes of the largest component of the spacecraft)
4. If the principal axes of the component are not parallel to the base frame, apply the Change Of Vector Basis Theorem (to be developed next)
5. Determine the position vector from the base frame origin to the mass center of the component
6. Apply the PAT to each component
7. Add all of the inertia matrices together
8. The result is $\mathbf{I}_b^o$, the inertia matrix of the spacecraft about the origin of the base frame and expressed in the base frame
9. Determine the mass center of the total spacecraft
10. Apply the PAT to the total spacecraft inertia to obtain $\mathbf{I}_b^c$, the inertia matrix of the spacecraft about its mass center
11. Find the principal axes and apply the COVBT to determine the principal moments of inertia

Application of Parallel Axis Theorem



Cylindrical spacecraft with rectangular solar panel

- Cylinder has height $h = 4$ m, radius $r = 1.5$ m, and mass $m_c = 100$ kg.
- Solar panel is in the $\hat{\mathbf{b}}_2 - \hat{\mathbf{b}}_3$ plane. It has height $c = 3$ m, length $b = 2$ m, and mass $m_p = 10$ kg.

- Look up the moments of inertia for a cylinder and a plate, which are usually given for *principal axes*, and *about the mass center* of the specific component

Cylinder

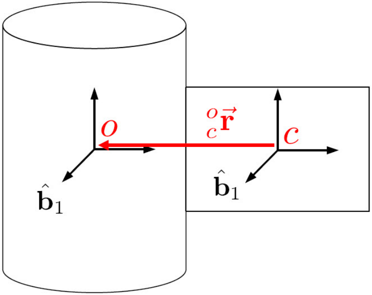
$$\mathbf{I}^c_{\text{cyl}} = m_c\,\text{diag}\begin{bmatrix} (3r^2 + h^2)/12 \\ (3r^2 + h^2)/12 \\ r^2/2 \end{bmatrix}$$

Panel

$$\mathbf{I}^c_{\text{pan}} = m_p\,\text{diag}\begin{bmatrix} (b^2 + c^2)/12 \\ c^2/12 \\ b^2/12 \end{bmatrix}$$

- To add them, we have to decide what origin to use; some possibilities: c of cylinder, c of panel, or c of composite body

Application of PAT (2)



- Compute the moment of inertia of the composite body about the mass center of the cylinder
- Apply PAT to panel

$$_c^o\vec{\mathbf{r}} = -(b/2 + r)\hat{\mathbf{b}}_2$$

We can drop the $_c^o$ on $\mathbf{r}$ – why?

For the panel

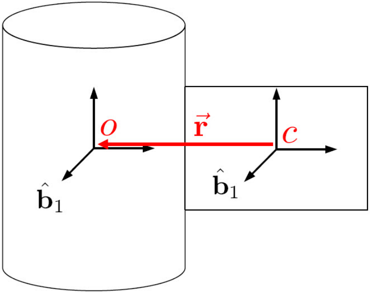
$$\mathbf{I}^c_{\text{pan}} = m_p \,\text{diag}\begin{bmatrix} (b^2+c^2)/12 \\ c^2/12 \\ b^2/12 \end{bmatrix}$$

and parallel axis theorem is:

$$\mathbf{I}^o = \mathbf{I}^c + m\left[r^2\mathbf{1} - \mathbf{r}\mathbf{r}^\mathsf{T}\right]$$

$$\begin{aligned}
\mathbf{r} &= [0 \quad -(b/2+r) \quad 0]^\mathsf{T} \\
\mathbf{r}^\mathsf{T}\mathbf{r} &= (b/2+r)^2 \\
\mathbf{r}\mathbf{r}^\mathsf{T} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & (b/2+r)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
[\cdot] &= \begin{bmatrix} (\frac{b}{2}+r)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{b}{2}+r)^2 \end{bmatrix}
\end{aligned}$$

Application of PAT (3)



- The moment of inertia of the panel about the cylinder mass center is clearly still diagonal, as there are no off-diagonal elements that arise in application of the PAT

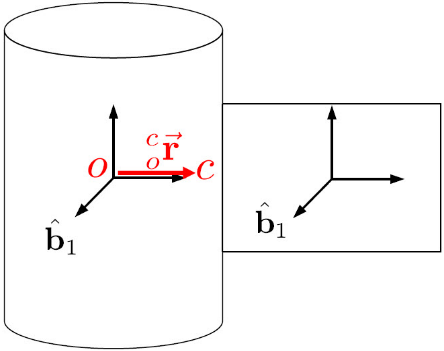
- For the panel (where o is the mass center of the cylinder):

$$\mathbf{I}^o_{\text{pan}} = m_p \,\text{diag} \begin{bmatrix} \frac{b^2+c^2}{12} + (\frac{b}{2}+r)^2 \\ c^2/12 \\ \frac{b^2}{12} + (\frac{b}{2}+r)^2 \end{bmatrix}$$

- The inertia about the $\hat{\mathbf{b}}_2$ axis did not change. Why?
- We finish the calculation, by adding together the two moment of inertia matrices, which are *both about the mass center of the cylinder*
- Note that we have not used any of the numbers that were given!

Strategy: Use symbols until the final calculations are required

Application of PAT (4)



- Next step: Calculate the moment of inertia of the composite body about *its mass center*

Some observations

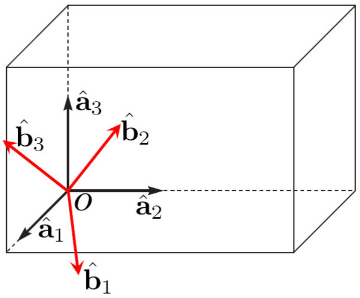
- We applied PAT to the panel to get $\mathbf{I}^{o}_{\text{pan}}$ about the cylinder mass center
- The cylinder mass center is *not* the mass center of the composite body
  How do we compute c for the composite body?
- When we add $\mathbf{I}^{o}_{\text{pan}}$ to $\mathbf{I}^{o}_{\text{cyl}}$, we get $\mathbf{I}^{o}_{\text{comp}}$, the moment of inertia matrix for the composite body, *about* the point o, which is not the mass center of the composite body
- The applicable form of the PAT is

$$\mathbf{I}^{c} = \mathbf{I}^{o} - m\left[\mathbf{r}^{\mathsf{T}}\mathbf{r}\mathbf{1} - \mathbf{r}\mathbf{r}^{\mathsf{T}}\right]$$

Why is this the correct form of the PAT, and what is $\mathbf{r}$ in this expression?

Change of Vector Basis Theorem

- Suppose we know the moment of inertia matrix about some point, o, expressed in a body-fixed reference frame, $\mathcal{F}_a$, and we want the moment of inertia matrix about o, expressed in *a different frame*, $\mathcal{F}_b$
- The former is denoted $\mathbf{I}_a^o$ and the latter is denoted $\mathbf{I}_b^o$
- We cannot safely omit the subscripts a and b
- Since the origin o is the same for both inertia matrices, we can omit the superscript



- The moment of inertia *tensor*, $\vec{\mathbf{I}}^o$ is the same mathematical object, but we are expressing it in a different frame, exactly as we might express a vector in two different frames

Change of Vector Basis Theorem (2)

- We have

$$\mathbf{I}_a = \int_{\mathcal{B}} \left[r_a^2 \mathbf{1} - \mathbf{r}_a \mathbf{r}_a^{\mathsf{T}}\right] dm$$

- The relationship between $\mathcal{F}_a$ and $\mathcal{F}_b$ is the rotation matrix $\mathbf{R}^{ba}$, and we use it to rotate $\mathbf{r}_a$ into $\mathcal{F}_b$:

$$\begin{aligned} \mathbf{r}_b &= \mathbf{R}^{ba}\mathbf{r}_a \quad \text{or} \\ \mathbf{r}_a &= \mathbf{R}^{ab}\mathbf{r}_b \end{aligned}$$

  What is $\mathbf{R}^{ab}$?

- Substitution into the integral for $\mathbf{I}_a$ leads to

$$\mathbf{I}_a = \mathbf{R}^{ab} \int_{\mathcal{B}} \left[r_b^2 \mathbf{1} - \mathbf{r}_b \mathbf{r}_b^{\mathsf{T}}\right] dm \, \mathbf{R}^{ba}$$

- The integral is obviously the moment of inertia expressed in $\mathcal{F}_b$, which is what we want

- Thus, pre-multiplying by $\mathbf{R}^{ba}$ and post-multiplying by $\mathbf{R}^{ab}$ leads to

$$\mathbf{R}^{ba}\mathbf{I}_a\mathbf{R}^{ab} = \mathbf{I}_b$$

  Why do these multiplications work?

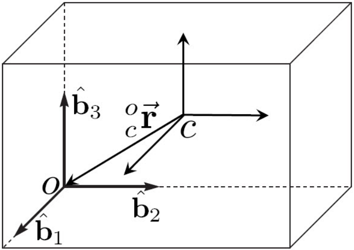
- This equation is the change of basis theorem, and is the equivalent of using a rotation matrix to rotate a vector from one frame to another

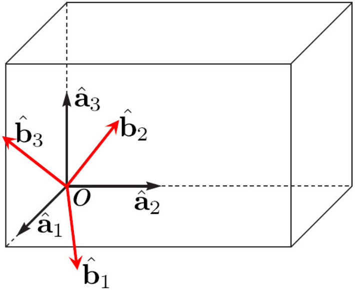
- Note that

$$\mathbf{R}^{ab}\mathbf{I}_b\mathbf{R}^{ba} = \mathbf{I}_a$$

- Thus, if we know $\mathbf{I}$ in any frame, and we know the rotation matrix from that frame to any other frame, we can easily obtain $\mathbf{I}$ in the other frame.

PAT and COVBT Summary





Parallel axis theorem:

$$\mathbf{I}^o = \mathbf{I}^c + m\left[\mathbf{r}^\mathsf{T}\mathbf{r}\mathbf{1} - \mathbf{r}\mathbf{r}^\mathsf{T}\right]$$

Alternatively

$$\mathbf{I}^o = \mathbf{I}^c - m\mathbf{r}^\times\mathbf{r}^\times$$

Change of vector basis theorem:

$$\mathbf{R}^{ba}\mathbf{I}_a\mathbf{R}^{ab} = \mathbf{I}_b$$

$$\mathbf{R}^{ab}\mathbf{I}_b\mathbf{R}^{ba} = \mathbf{I}_a$$

Used to compute principal axes and principal moments of inertia

What are the implied subscripts and superscripts in these expressions?

The most important application of the Change of Vector Basis Theorem is finding the principal axes for a rigid body

Some observations

- The inertia tensor is symmetric:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{I}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{I}} \cdot \vec{\mathbf{u}}$$

  Exercise: convince yourself, keeping in mind that the vectors and tensor are not matrices!

- The inertia matrix is symmetric:

$$\begin{aligned} \mathbf{I} &= \mathbf{I}^{\mathsf{T}} \\ \mathbf{u}^{\mathsf{T}}\mathbf{I}\mathbf{v} &= \mathbf{v}^{\mathsf{T}}\mathbf{I}^{\mathsf{T}}\mathbf{u} \end{aligned}$$

  Exercise: convince yourself

Fact:

`All eigenvalues of real, symmetric matrices are real.`

Fact:

`All eigenvectors of real, symmetric matrices are real and are mutually orthogonal.`

Conclusion:

`The eigenvectors of the inertia matrix comprise a set of base vectors for a body-fixed reference frame.`

Important:

`The moment of inertia matrix in that frame is diagonal.`

Principal Axes (2)

- Compute the eigenvalues and eigenvectors of $\mathbf{I}$, expressed in a non-principal reference frame, $\mathcal{F}_a$
- The eigenvalue problem is stated as
$$\mathbf{I}_a\mathbf{x} = \lambda\mathbf{x}$$
where λ and $\mathbf{x}$ are the unknowns
- If we find a $(\lambda, \mathbf{x})$ pair that solves the eigenvalue problem, then we have found an eigenvalue, λ, and an eigenvector, $\mathbf{x}$
- For an inertia matrix, the eigenvalues are the *principal moments of inertia*, and the eigenvectors are the *components of the principal frame base vectors, expressed in* $\mathcal{F}_a$
- In `Matlab`, for example, we can compute the eigensystem of a matrix `Ia`, using
`[V,D] = eig(Ia)`
where `D` is diagonal matrix containing the eigenvalues, and `V` is a matrix containing the eigenvectors
- Suppose we have calculated three eigenvalues and the associated eigenvectors:
$$\lambda_1, \lambda_2, \lambda_3$$
$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$$
- We can write the following:
$$\mathbf{I}_a \left[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3\right] = \left[\lambda_1\mathbf{x}_1 \quad \lambda_2\mathbf{x}_2 \quad \lambda_3\mathbf{x}_3\right]$$

Principal Axes (3)

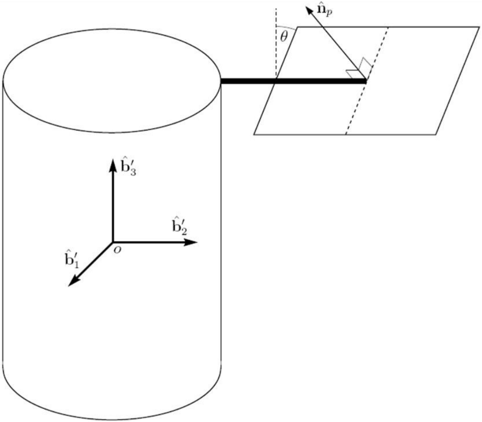
```
 I = % a fabricated example
1.9003    0.7171    1.0633
0.7171    1.7826    0.7806
1.0633    0.7806    1.6428
>> [V,D]=eig(I)
V =
0.5837    0.5207    0.6230
0.1809   -0.8314    0.5254
-0.7916    0.1940    0.5795
D =
0.6804         0         0
0    1.1513         0
0         0    3.4939
>> I*V
ans =
0.3972    0.5995    2.1769
0.1231   -0.9572    1.8357
-0.5386    0.2233    2.0246
```

```
>> V*D
ans =
0.3972    0.5995    2.1769
0.1231   -0.9572    1.8357
-0.5386    0.2233    2.0246
>> V'*I*V
ans =
0.6804    0.0000   -0.0000
0.0000    1.1513    0.0000
-0.0000    0.0000    3.4939
```

$$
\begin{aligned}
\mathbf{I}_a \left[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3\right] &= \left[\lambda_1 \mathbf{x}_1 \quad \lambda_2 \mathbf{x}_2 \quad \lambda_3 \mathbf{x}_3\right] \\
\mathbf{I}_a \mathbf{X} &= \mathbf{X} \,\mathsf{diag}\left[\lambda_1 \quad \lambda_2 \quad \lambda_3\right] \\
\mathbf{I}_a \mathbf{X} &= \mathbf{X} \boldsymbol{\Lambda} \\
\mathbf{X}^{-1} \mathbf{I}_a \mathbf{X} &= \boldsymbol{\Lambda} \\
\mathbf{R}^{ba} \mathbf{I}_a \mathbf{R}^{ab} &= \mathbf{I}_b
\end{aligned}
$$

$\mathcal{F}_b$ is the principal frame

A Comprehensive Example



- cylinder diameter $d = 1$ m
- height $h = 2$ m
- density $\mu = 100$ kg/m^3
- rod length $\ell = 2$ m
- rodd mass $m_r = 1$ kg
- panel length $L = 2$ m
- panel width $w = 0.5$ m
- panel mass $m_p = 4$ kg

The panel is rotated about the rod through angle θ; if $\theta = 0$, then the panel is in the $\hat{\mathbf{b}}'_2\hat{\mathbf{b}}'_3$ plane, and if $\theta = 90^\circ$, then the panel is parallel to the $\hat{\mathbf{b}}'_1\hat{\mathbf{b}}'_2$ plane.

What are the principal axes and principal moments of inertia?

A spacecraft is comprised of 3 rigid bodies: a *cylinder*, a *rod*, and a *panel*.
The $\mathcal{F}_{b'}$ frame is at the mass center of the cylinder with $\hat{\mathbf{b}}'_3$ axis parallel to the cylinder's symmetry axis, and the $\hat{\mathbf{b}}'_2$ axis parallel to the rod.

A Comprehensive Example (2)

Steps to solution:

- For each body, write $\mathbf{I}^c$ for the body in its own principal, centroidal reference frame
- For the panel, rotate the matrix from its principal frame to a frame parallel to $\mathcal{F}_{b'}$
- For the panel and the rod, apply the parallel axis theorem to obtain $\mathbf{I}^o$ (where o is the mass center of the cylinder)
- Add $\mathbf{I}^o_{\text{pan}}$ and $\mathbf{I}^o_{\text{rod}}$ to $\mathbf{I}^o_{\text{cyl}}$ to obtain $\mathbf{I}^o_{\text{comp}}$
- Compute first moment of inertia about o, $\mathbf{c}^o_{\text{comp}}$, and use to find mass center location, ${}^c_o\mathbf{r}$
- Apply parallel axis theorem to composite body to obtain $\mathbf{I}^c_{\text{comp}}$
- Compute eigensystem of $\mathbf{I}^c_{\text{comp}}$ to find principal moments of inertia and principal axes

A Comprehensive Example (3)

Carrying out all the calculations leads to $\mathbf{I}_{b'}^{c}$, where c denotes the spacecraft mass center, and we can use `eig` to get the principal moments of inertia and principal axes.

```
I=[95.12404742328032    0                   0.00435535263615
    0                  67.10644276869270 -11.14523611604115
    0.00435535263615  -11.14523611604115  47.81876889747119];
[AXES,MOIS]=eig(I)
AXES =
-0.00007559450357   0.00007898843123  -0.99999999402315
 0.41573134711738   0.90948746302016   0.00004041198337
 0.90948746077637  -0.41573134157769  -0.00010159022019
MOIS =
 42.72422438363204    0                   0
  0                  72.20098684007061    0
  0                   0                  95.12404786574156
```

We need to rearrange the order of these eigenvalues and eigenvectors. Why? And How?

A Comprehensive Example (4)

Why do we want to rearrange the eigenvalues and eigvenvectors?

- When we find the eigenvectors, we are looking for $\mathbf{R}^{ab}$, whose columns are the unit vectors of (principal) $\mathcal{F}_b$ expressed in (non-principal) $\mathcal{F}_a$.
- Note that here $\mathcal{F}_a$ is $\mathcal{F}_{b'}$, the frame in which we found the moment of inertia matrix.
- For practical reasons, such as visualization, we'd like for the principal axes to be "close" to the same directions as the original vectors
- Columns of `AXES` are directions of principal axes
- Column 1 of `AXES` is "close" to $[0\ 0\ 1]^\mathsf{T}$, so we want column 1 to be $\mathbf{b}_{3b'}$, or the third column of $\mathbf{R}^{b'b}$.
- Similarly, column 2 of `AXES` is "close" to $[0\ 1\ 0]^\mathsf{T}$, so we want it to be the second column of $\mathbf{R}^{b'b}$.
- Column 3 of `AXES` is "close" to $[1\ 0\ 0]^\mathsf{T}$, but it's in the negative direction, so we multiply it by -1 and put it in the first column of $\mathbf{R}^{b'b}$.

A Comprehensive Example (5)

How do we rearrange the eigenvalues and eigvenvectors?

- Based on observations and conclusions about directions of columns of the `AXES` matrix, we construct the rotation matrix as follows:

$$\mathbf{R}^{b'b} = \begin{bmatrix} 0.999999994 & 0.000078988 & -0.000075594 \\ -0.000040411 & 0.909487463 & 0.415731347 \\ 0.000101590 & -0.415731341 & 0.909487460 \end{bmatrix}$$

- The principal moments of inertia are the diagonal elements of `MOIS`
- We have to reorder them so that they remain associated with the reordered eigenvectors

$$\mathbf{I}_b = \text{diag}\ [95.124047865 \quad 72.200986840 \quad 42.724224383]$$

Exercise: Verify that $\mathbf{R}^{b'b}\mathbf{I}_b\mathbf{R}^{bb'} = \mathbf{I}_{b'}$.

Angular Momentum Principle (aka Euler's Law)

- Recall the definition

$$\vec{\mathbf{h}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times {}_o\dot{\vec{\mathbf{r}}}\,\mathbf{dm}$$

- Differentiate:

$$\dot{\vec{\mathbf{h}}}^{\mathbf{o}} = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times {}_o\ddot{\vec{\mathbf{r}}}\,dm$$

- Recall that ${}_o\vec{\mathbf{r}} = {}_O\vec{\mathbf{r}} - {}^o_O\vec{\mathbf{r}}$; therefore

$$\begin{aligned} \dot{\vec{\mathbf{h}}}^{\mathbf{o}} &= \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times {}_O\ddot{\vec{\mathbf{r}}}\,dm \\ &- \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times {}^o_O\ddot{\vec{\mathbf{r}}}\,dm \end{aligned}$$

- Define $\vec{\mathbf{g}}^o = \int_{\mathcal{B}} {}_o\vec{\mathbf{r}} \times \vec{\mathbf{f}}\,dm \Rightarrow$

$$\dot{\vec{\mathbf{h}}}^{\mathbf{o}} = \vec{\mathbf{g}}^o - \vec{\mathbf{c}}^o \times {}^o_O\ddot{\vec{\mathbf{r}}}$$

Observations:

- If $o = c$, then $\vec{\mathbf{c}}^c = \vec{\mathbf{0}}$, so

$$\dot{\vec{\mathbf{h}}}^{\mathbf{c}} = \vec{\mathbf{g}}^c$$

- If o is not accelerating with respect to the inertial origin O, then ${}^o_O\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{0}}$, so

$$\dot{\vec{\mathbf{h}}}^{\mathbf{o}} = \vec{\mathbf{g}}^o$$

- These observations motivate our usual selection of either an inertial origin or the mass center as the origin for the angular momentum principle

Angular Momentum Principle (2)

- Angular momentum principle:

$$\begin{aligned} \vec{\mathbf{h}}^c &= \vec{\mathbf{I}}^c \cdot \vec{\boldsymbol{\omega}}^{bi} \\ \dot{\vec{\mathbf{h}}}^{\mathbf{c}} &= \vec{\mathbf{g}}^c \end{aligned}$$

- How do we express these equations in a rotating reference frame?
- Recall the formula for differentiating a vector $\mathbf{a}$ expressed in a rotating frame $\mathcal{F}_b$:

$$\frac{d}{dt}\left[\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\mathbf{a}\right] = \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}\left[\dot{\mathbf{a}} + \boldsymbol{\omega}^{\times}\mathbf{a}\right]$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}^{bi}$

- Note that $\boldsymbol{\omega}^{bi}$ plays two distinct roles in these equations: angular velocity of the rotating reference frame, and angular velocity of the rigid body. We will see one example where this distinction is important.

- Expressing the two equations in $\mathcal{F}_b$ gives

$$\begin{aligned} \mathbf{h}_b^c &= \mathbf{I}_b^c \boldsymbol{\omega}_b^{bi} \\ \dot{\mathbf{h}}_b^c + \boldsymbol{\omega}_b^{bi\times}\mathbf{h}_b^c &= \mathbf{g}_b^c \end{aligned}$$

- The latter is usually written as

$$\dot{\mathbf{h}}_b^c = -\boldsymbol{\omega}_b^{bi\times}\mathbf{h}_b^c + \mathbf{g}_b^c$$

- Finally, assuming that the frame and origin are understood, we typically write

$$\begin{aligned} \mathbf{h} &= \mathbf{I}\boldsymbol{\omega} \\ \dot{\mathbf{h}} &= -\boldsymbol{\omega}^{\times}\mathbf{h} + \mathbf{g} \end{aligned}$$

- How would these equations differ if we had not chosen the mass center as the reference point?

Rotational Equations of Motion

- Recall previously developed kinematics equations

$$
\begin{aligned}
\dot{\bar{\mathbf{q}}} &= \frac{1}{2}\begin{bmatrix} \mathbf{q}^\times + q_4\mathbf{1} \\ -\mathbf{q}^\mathsf{T} \end{bmatrix}\boldsymbol{\omega} \\
&= \mathbf{Q}(\bar{\mathbf{q}})\boldsymbol{\omega}
\end{aligned}
$$

or

$$\dot{\boldsymbol{\theta}} = \mathbf{S}^{-1}(\boldsymbol{\theta})\boldsymbol{\omega}$$

- Solve $\mathbf{h} = \mathbf{I}\boldsymbol{\omega}$ for $\boldsymbol{\omega}$, and substitute into the kinematics equations to obtain the system

$$
\begin{aligned}
\boldsymbol{\omega} &= \mathbf{I}^{-1}\mathbf{h} \\
\dot{\bar{\mathbf{q}}} &= \mathbf{Q}(\bar{\mathbf{q}})\boldsymbol{\omega} \\
\dot{\mathbf{h}} &= -\boldsymbol{\omega}^\times\mathbf{h} + \mathbf{g}
\end{aligned}
$$

- Two matrix differential equations of total order 7, and an algebraic equation of order 3. There is also the implicit constraint $\bar{\mathbf{q}}^\mathsf{T}\bar{\mathbf{q}} = 1$.

- We can numerically integrate the equations for $\bar{\mathbf{q}}$ and $\mathbf{h}$
- Recall that force generally depends on the dynamic *state*: position, velocity, attitude, and angular velocity

$$\vec{\mathbf{f}} = \vec{\mathbf{f}}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, \mathbf{R}, \vec{\boldsymbol{\omega}})$$

- The same applies to the torque:

$$\mathbf{g} = \mathbf{g}(\mathbf{r}, \mathbf{v}, \mathbf{R}, \boldsymbol{\omega})$$

- Note that we have omitted the overarrows, thus implying that $\mathbf{g}$ and its vector arguments are 3×1 matrices expressed in $\mathcal{F}_b$

Coupled Equations of Motion

- The complete set of coupled translational and rotational equations of motion for a rigid body:

$$
\begin{aligned}
\boldsymbol{\omega} &= \mathbf{I}^{-1}\mathbf{h} \\
\mathbf{v} &= \frac{1}{m}\mathbf{p} \\
\mathbf{f} &= \mathbf{f}(\mathbf{r}, \mathbf{v}, \mathbf{R}, \boldsymbol{\omega}) \\
\mathbf{g} &= \mathbf{g}(\mathbf{r}, \mathbf{v}, \mathbf{R}, \boldsymbol{\omega})
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{r}} &= \frac{1}{m}\left[\mathbf{p} - \boldsymbol{\omega}^{\times}\mathbf{c}\right] - \boldsymbol{\omega}^{\times}\mathbf{r} \\
\dot{\mathbf{p}} &= -\boldsymbol{\omega}^{\times}\mathbf{p} + \mathbf{f} \\
\dot{\bar{\mathbf{q}}} &= \mathbf{Q}(\bar{\mathbf{q}})\boldsymbol{\omega} \\
\dot{\mathbf{h}} &= -\boldsymbol{\omega}^{\times}\mathbf{h} + \mathbf{g}
\end{aligned}
$$

- The first two equations provide *velocities* as functions of momenta
- The third and fourth equations define the force and moment in terms of the dynamic state
- The last four equations are the actual differential equations
- Note that there are 13 differential equations, so the state vector is $\mathbf{x} \in R^{13}$

Euler's Equations

- A common approach is to substitute $\mathbf{h} = \mathbf{I}\boldsymbol{\omega}$ into the angular momentum equation to obtain

$$
\begin{aligned}
(\dot{\mathbf{I}\boldsymbol{\omega}}) &= -\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{g} \\
\mathbf{I}\dot{\boldsymbol{\omega}} &= -\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{g} \\
\dot{\boldsymbol{\omega}} &= -\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{I}^{-1}\mathbf{g}
\end{aligned}
$$

- The latter equation is the matrix form of *Euler's equations*
- If a non-principal reference frame is used, then $\mathbf{I}^{-1}$ is a bit complicated; however, for a principal frame, $\mathbf{I}^{-1}$ is simple
- For a principal frame, we write

$$
\begin{aligned}
\mathbf{I} &= \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \\
&= \text{diag}\,[I_1 \;\; I_2 \;\; I_3]
\end{aligned}
$$

- The inverse of a full-rank (*i.e.*, non-singular) diagonal matrix is simply the diagonal matrix containing the scalar inverses:

$$
\begin{aligned}
\mathbf{I}^{-1} &= \begin{bmatrix} 1/I_1 & 0 & 0 \\ 0 & 1/I_2 & 0 \\ 0 & 0 & 1/I_3 \end{bmatrix} \\
&= \text{diag}\,[1/I_1 \;\; 1/I_2 \;\; 1/I_3]
\end{aligned}
$$

Euler's Equations (2)

- Assuming a principal frame, Euler's equations may be written as the three scalar equations

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{g_1}{I_1} \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 + \frac{g_2}{I_2} \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{g_3}{I_3}
\end{aligned}
$$

- Given initial conditions on $\boldsymbol{\omega}$ and a means of computing the torque $\mathbf{g}$, these equations can be integrated to obtain $\boldsymbol{\omega}(t)$, which can in turn be used to integrate the kinematics equations to obtain $\mathbf{R}(t)$
- As previously noted, the torque can depend on position, velocity, attitude, and angular velocity, so these equations are usually *coupled* with the translational dynamics and rotational kinematics equations.
- Some useful insight into rotational dynamics is possible by considering two **torque-free** cases:
  - **axisymmetric** $(I_1 = I_2)$
  - **asymmetric**
- We will address both these cases before moving on to satellite applications

Axisymmetric Torque-Free Rigid Body

Axisymmetric $\Rightarrow$ two principal moments of inertia are equal
Examples: circular cylinder or cone, square cross-section prism, tops
Spacecraft examples: Sputnik, Explorer I (not torque-free, though)

Begin with Euler's Equations:

$$\begin{aligned}\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{g_1}{I_1} \\ \dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 + \frac{g_2}{I_2} \\ \dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{g_3}{I_3}\end{aligned}$$

Set $I_1 = I_2 = A$ and $I_3 = C$

$$\begin{aligned}\dot{\omega}_1 &= \frac{A - C}{A}\omega_2\omega_3 + \frac{g_1}{A} \\ \dot{\omega}_2 &= \frac{C - A}{A}\omega_1\omega_3 + \frac{g_2}{A} \\ \dot{\omega}_3 &= \frac{g_3}{C}\end{aligned}$$

Set $\mathbf{g} = \mathbf{0}$

$$\begin{aligned}\dot{\omega}_1 &= \frac{A - C}{A}\omega_2\omega_3 \\ \dot{\omega}_2 &= \frac{C - A}{A}\omega_1\omega_3 \\ \dot{\omega}_3 &= 0\end{aligned}$$

Integrate $\dot{\omega}_3$

$$\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \omega_3(t_0) \equiv \Omega$$

The other two equations become:

$$\begin{aligned}\dot{\omega}_1 &= \frac{A - C}{A}\Omega\omega_2 \\ \dot{\omega}_2 &= \frac{C - A}{A}\Omega\omega_1\end{aligned}$$

Axisymmetric Torque-Free Rigid Body (2)

- Angular velocity about symmetry axis is constant, $\omega_3 = \Omega$, and other two components of $\boldsymbol{\omega}$ satisfy:

$$\begin{aligned} \dot{\omega}_1 &= \frac{A-C}{A}\Omega\omega_2 \\ \dot{\omega}_2 &= \frac{C-A}{A}\Omega\omega_1 \end{aligned}$$

- Since A, C, and Ω are constants, these two equations comprise a *linear*, *constant-coefficient*, *system* of *ordinary differential equations*:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \Rightarrow \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)$$

- The matrix exponential $e^{\mathbf{A}t}$ (letting $t_0 = 0$) is not the same as the scalar exponential.
- What is the $\mathbf{A}$ matrix, and what is $\mathbf{x}$ matrix?
- In Matlab, `x = expm(A*t)*x0` is the correct statement.
- Compare `expm(A)` and `exp(A)` – what's the difference?
- This system is so simple, we can solve it without the matrix exponential
- Define $\hat{\Omega} = \Omega(A-C)/A$:

$$\begin{aligned} \dot{\omega}_1 &= \hat{\Omega}\omega_2 \\ \dot{\omega}_2 &= -\hat{\Omega}\omega_1 \end{aligned}$$

- Differentiate the first and substitute in from the second:

$$\ddot{\omega}_1 + \hat{\Omega}^2\omega_1 = 0$$

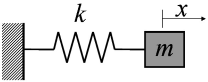
- Since $\hat{\Omega}$ is a constant, this equation is a *linear*, *constant-coefficient*, *second-order*, *ordinary differential equation*

Axisymmetric Torque-Free Rigid Body (3)

- Have $\omega_3 = \Omega$
- Defined $\hat{\Omega} = \Omega(A - C)/A$
- Derived

$$\ddot{\omega}_1 + \hat{\Omega}^2 \omega_1 = 0$$

Does this equation remind you of anything?
Hint:



- The solution to this equation can be written as

$$\omega_1 = c_1 \cos \hat{\Omega} t + c_2 \sin \hat{\Omega} t$$

where the two c_i's depend on initial conditions.

- Solution can also be written as

$$\omega_1 = a \sin \left(\hat{\Omega} t + \varphi \right)$$

where $a = \pm\sqrt{c_1^2 + c_2^2}$ is the amplitude and φ is the phase.

- Determination of φ in terms of c_1 and c_2 is left as an exercise.
- We can always choose φ so that $a > 0$
- In textbooks, this differential equation and its solution are referred to as the Simple Harmonic Oscillator
- Exercise: Given $\boldsymbol{\omega}(0) = [1 \;\; 2 \;\; 3]^\mathsf{T}$, determine c_1, c_2, a, φ.

Axisymmetric Torque-Free Rigid Body (4)

- We have integrated the kinetics equations for rotational motion of an axisymmetric rigid body with zero torque:

$$\begin{aligned}\omega_1 &= a\sin\left(\hat{\Omega}t+\varphi\right)\\ \omega_2 &= a\cos\left(\hat{\Omega}t+\varphi\right)\\ \omega_3 &= \Omega\end{aligned}$$

- Alternatively

$$\begin{aligned}\omega_1 &= c_1\cos\hat{\Omega}t + c_2\sin\hat{\Omega}t\\ \omega_2 &= -c_1\sin\hat{\Omega}t + c_2\cos\hat{\Omega}t\\ \omega_3 &= \Omega\end{aligned}$$

- Have you done the recommended exercise from previous slide?

- Another Exercise: Show that $\omega_1^2+\omega_2^2$ is constant.
- Knowing the angular velocity as a function of time, we can integrate the kinematics equations
- You could use $\boldsymbol{\omega}(t)$ on right hand side of either $\dot{\boldsymbol{\theta}}$, $\dot{\bar{\mathbf{q}}}$, or any other kinematics representation, and numerically integrate to obtain the attitude
- However, it turns out that we can solve the kinematics equations analytically, for a complete closed-form solution to the attitude equations of motion for an axisymmetric torque-free rigid body

Axisymmetric Torque-Free Rigid Body (5)

- We have solution for $\boldsymbol{\omega}(t)$

$$\begin{aligned}\omega_1 &= a\sin\left(\hat{\Omega}t+\varphi\right)\\ \omega_2 &= a\cos\left(\hat{\Omega}t+\varphi\right)\\ \omega_3 &= \Omega\end{aligned}$$

- We can integrate the kinematics using Euler angle formulation
- 3-1-3 Euler angles

$$\begin{aligned}\phi &= \theta_1 \quad \text{precession}\\ \theta &= \theta_2 \quad \text{nutation}\\ \psi &= \theta_3 \quad \text{spin}\end{aligned}$$

- Visualize a spinning top: the symmetry axis "precesses" about the vertical; the angle between the symmetry axis and the vertical is the nutation angle; the top "spins" about its symmetry axis

- For this set of Euler angles, the relationship $\boldsymbol{\omega} = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ reduces to

$$\begin{aligned}\omega_1(t) &= \dot{\phi}\sin\theta\sin\psi\\ \omega_2(t) &= \dot{\phi}\sin\theta\cos\psi\\ \omega_3(t) &= \dot{\psi}+\dot{\phi}\cos\theta\end{aligned}$$

Exercise!

- Applying the solution for $\boldsymbol{\omega}$,

$$\begin{aligned}a\sin\left(\hat{\Omega}t+\varphi\right) &= \dot{\phi}\sin\theta\sin\psi\\ a\cos\left(\hat{\Omega}t+\varphi\right) &= \dot{\phi}\sin\theta\cos\psi\\ \Omega &= \dot{\psi}+\dot{\phi}\cos\theta\end{aligned}$$

- Stare at this set of equations long enough and some patterns begin to emerge

Axisymmetric Torque-Free Rigid Body (6)

- Combining the kinetics solution and the kinematics ODEs gave:

$$
\begin{aligned}
a\sin\left(\hat{\Omega}t+\varphi\right) &= \dot{\phi}\sin\theta\sin\psi \\
a\cos\left(\hat{\Omega}t+\varphi\right) &= \dot{\phi}\sin\theta\cos\psi \\
\Omega &= \dot{\psi}+\dot{\phi}\cos\theta
\end{aligned}
$$

- Choose phase φ so that $a > 0$
- Choose $\hat{\mathbf{b}}_3$ direction so $\Omega > 0$
  Exercise: Given $\boldsymbol{\omega}(0) = [1 \ \ 2 \ \ -3]^{\mathsf{T}}$, what do you need to do to make $\Omega > 0$?
- Note that we can show $0 < \theta < \pi/2$, so that $\sin\theta$ and $\cos\theta$ are > 0
- Furthermore, we can prove that θ is constant
- Assume $\vec{\mathbf{h}}$ (constant, right?) is aligned with $\hat{\mathbf{i}}_3$, and $\mathbf{h} = \mathbf{I}\boldsymbol{\omega}$ is

$$
\mathbf{h} = [A\omega_1 \ \ A\omega_2 \ \ C\Omega]^{\mathsf{T}}
$$

- Component along $\hat{\mathbf{b}}_3$ is

$$
\vec{\mathbf{h}}\cdot\hat{\mathbf{b}}_3 = h\cos\theta = C\Omega
$$

- Solve for θ:

$$
\begin{aligned}
\cos\theta &= \frac{C\Omega}{h} \\
&= \frac{C\Omega}{\sqrt{A^2a^2+C^2\Omega^2}}
\end{aligned}
$$

  which proves that θ is constant, and $\theta \in [0, \pi/2]$
- This result also shows that $\dot{\phi} > 0$. Exercise. Hint: $\omega_1^2 + \omega_2^2$ is constant.

Axisymmetric Torque-Free Rigid Body (7)

- First two kinematics equations:

$$\begin{aligned} a\sin\left(\hat{\Omega}t+\varphi\right) &= \dot{\phi}\sin\theta\sin\psi \\ a\cos\left(\hat{\Omega}t+\varphi\right) &= \dot{\phi}\sin\theta\cos\psi \end{aligned}$$

- Divide first by second:

$$\tan\left(\hat{\Omega}t+\varphi\right) = \tan\psi$$

- Thus, either

$$\begin{aligned} (1) \quad \psi &= \hat{\Omega}t+\varphi \quad \text{or} \\ (2) \quad \psi &= \pi+\hat{\Omega}t+\varphi \end{aligned}$$

- In either case, the *spin rate* is

$$\dot{\psi} = \hat{\Omega} = \Omega\left(\frac{A-C}{A}\right)$$

which is > 0 is $A > C$ (prolate) and < 0 if $A < C$ (oblate)

- Recall that

$$\begin{aligned} \sin(\pi+x) &= -\sin x \\ \cos(\pi+x) &= -\cos x \end{aligned}$$

- If (1) is true, then

$$a = \dot{\phi}\sin\theta$$

- If (2) is true, then

$$a = -\dot{\phi}\sin\theta$$

(2) leads to a contradiction, since $\Omega - \dot{\psi} > 0$.

- Thus (1) must be true, and since $\sin\theta > 0$,

$$\dot{\phi} = a/\sin\theta > 0$$

Axisymmetric Torque-Free Rigid Body (8)

Summary of the Solution

- Solution of Euler's equations:

$$
\begin{aligned}
\omega_1 &= a \sin\left(\hat{\Omega} t + \varphi\right) \\
\omega_2 &= a \cos\left(\hat{\Omega} t + \varphi\right) \\
\omega_3 &= \Omega
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \sqrt{\omega_1^2(0) + \omega_2^2(0)} > 0 \\
\varphi &= \tan^{-1} \frac{\omega_1(0)}{\omega_2(0)} \\
\Omega &= \omega_3(0) > 0 \\
\hat{\Omega} &= \Omega \left(\frac{A - C}{A} \right)
\end{aligned}
$$

- Solution of kinematics equations:

$$
\begin{aligned}
\dot{\phi} &= a / \sin\theta = \texttt{const} \\
\dot{\theta} &= 0 \\
\dot{\psi} &= \hat{\Omega} = \texttt{const} \\
\theta &= \cos^{-1}\left(\frac{C\Omega}{\sqrt{A^2 a^2 + C^2 \Omega^2}} \right)
\end{aligned}
$$

- Precession rate is positive
- Nutation angle is constant
- Spin rate sign depends on whether the body is *minor* or *major* axis spinner (*i.e.*, *prolate* or *oblate*)

Much of the language of spacecraft dynamics is connected to the concepts developed in this solution.

Example: *nutation dampers* are energy dissipation devices designed to cause $\dot{\theta} < 0$ so that the nutation angle decreases over time.

Asymmetric Torque-Free Rigid Body

`The rotational motion of a torque-free` *`asymmetric`* `rigid body also admits analytical solutions.`

- Begin with Euler's equations:

$$\begin{aligned} \dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{g_1}{I_1} \\ \dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 + \frac{g_2}{I_2} \\ \dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{g_3}{I_3} \end{aligned}$$

- Like the axisymmetric case, these equations are coupled
- Unlike the axisymmetric case, the $\dot{\omega}_3$ equation is nontrivial, and the equations are *nonlinear*
- Nonlinearity is because there are *products* of the dependent variables on the right-hand sides

- $\vec{\mathbf{g}} = \mathbf{0} \left(\Rightarrow \vec{\mathbf{h}} = \text{const}\right)$:

$$\begin{aligned} \dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \\ \dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 \\ \dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \end{aligned}$$

- In matrix form:

$$\dot{\boldsymbol{\omega}} = -\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega}$$

The rotational kinetic energy is

$$T = \int_{\mathcal{B}} {}_c\dot{\vec{\mathbf{r}}} \cdot {}_c\dot{\vec{\mathbf{r}}}\, dm = \frac{1}{2}\boldsymbol{\omega}^{\mathsf{T}}\mathbf{I}\boldsymbol{\omega}$$

Exercise: Develop T

Asymmetric Torque-Free Rigid Body (2)

- From previous slide, we have

$$
\begin{aligned}
\dot{\boldsymbol{\omega}} &= -\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} \\
T &= \frac{1}{2}\boldsymbol{\omega}^{\mathsf{T}}\mathbf{I}\boldsymbol{\omega}
\end{aligned}
$$

- Show that T is constant:

$$
\begin{aligned}
\frac{dT}{dt} &= \frac{1}{2}\frac{d}{dt}\left(\boldsymbol{\omega}^{\mathsf{T}}\mathbf{I}\boldsymbol{\omega}\right) \\
&= \frac{1}{2}\left[\frac{d}{dt}(\boldsymbol{\omega}^{\mathsf{T}})\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\omega}^{\mathsf{T}}\mathbf{I}\frac{d}{dt}(\boldsymbol{\omega})\right] \\
&= \frac{1}{2}\left[\dot{\boldsymbol{\omega}}^{\mathsf{T}}\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\omega}^{\mathsf{T}}\mathbf{I}\dot{\boldsymbol{\omega}}\right] \\
&= \boldsymbol{\omega}^{\mathsf{T}}\mathbf{I}\dot{\boldsymbol{\omega}} \\
&= -\boldsymbol{\omega}^{\mathsf{T}}\mathbf{I}\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} \\
&= -\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{\omega}^{\times}(\mathbf{I}\boldsymbol{\omega}) \\
&= 0
\end{aligned}
$$

- In scalar form, the two conserved quantities T and h^2 are:

$$
\begin{aligned}
2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\
h^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2
\end{aligned}
$$

- So, we have two conserved quantities (aka *first integrals*) and three first-order differential equations; therefore, we're 2/3 of the way towards integrating Euler's equations
- Exercise: What is the distinction between stating $\vec{\mathbf{h}} = \texttt{const}$ and $\mathbf{h} = \texttt{const}$?
  Which statement is true for the torque-free rigid body?

What is the nature of the geometric objects described by the two first integrals given above?

Asymmetric Torque-Free Rigid Body (3)

- We have expressions for two *ellipsoids* in "$\boldsymbol{\omega}$-space:"

$$\begin{aligned} 2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\ h^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \end{aligned}$$

- Assume, without loss of generality*, that

$$I_1 > I_2 > I_3$$

- Thus (Exercise)

$$2I_1T - h^2 > 0$$

and

$$2I_3T - h^2 < 0$$

- * If chosen principal $\mathcal{F}_b$ does not satisfy this inequality, we can simply reorder the principal axes.

- Complete the solution of Euler's equations
- Solve for ω_1^2 and ω_3^2 in terms of ω_2^2, T, h^2, and $\mathbf{I}$ and substitute into $\dot{\omega}_2$ to obtain

$$\dot{\omega}_2 = f(\omega_2; \mathbf{I}, T, h^2)$$

semicolon in $f(\cdot;\cdot)$ is standard notation indicating that terms to right of semicolon are constants in the differential equation

- The resulting differential equation can be integrated to obtain

$$\omega_2 = a\,\mathrm{sn}(\tau; k)$$

where a is a constant amplitude, and $\tau = \kappa(t - t_0)$.

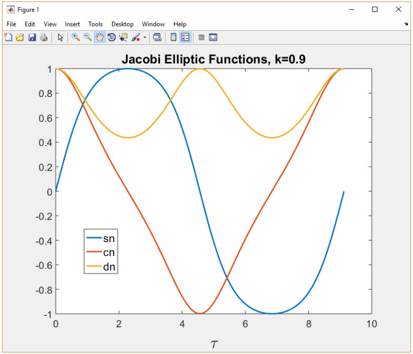
- The constants κ and k depend on $\mathbf{I}, T$, and h^2

Asymmetric Torque-Free Rigid Body (4)

- Jacobi elliptic functions are generalizations of familiar trig and hyperbolic functions
- Solution for ω_2:

$$\omega_2 = a \text{ sn } (\tau; k)$$

  uses elliptic function `sn`
- Similar solutions can be obtained for ω_1 and ω_2, and involve the three elliptic functions: `sn`, `cn`, and `dn`
- When $k = 0$, `sn` $=$ `sin`, `cn` $=$ `cos`, and `dn` $= 1$
- When $k = 1$, `sn` $=$ `tanh`, `cn` $=$ `sech`, and `dn` $=$ `sech`
- When $k \in (0, 1)$, elliptic functions are periodic, with period $4K(k)$, where $K(k)$ is "complete elliptic integral of the first kind"



- In Matlab, elliptic functions are computed using the function call `[sn,cn,dn]=ellipj(tau,m)` where $m = k^2$
- We will not have any exam questions on elliptic functions, but there will be a homework assignment using Matlab

Asymmetric Torque-Free Rigid Body (5)

- We have the general solution.
- Are there any *special* solutions?
- What about steady spins, in which all three $\dot{\omega}_j$'s are constant?

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2
\end{aligned}
$$

- Stare at the equations long enough and see that if any two of the $\omega_j (j = 1, 2, 3)$ are zero, then all three $\dot{\omega}_j (j = 1, 2, 3)$ are zero
- Thus steady spin about either *principal* axis is the only steady spin that is possible
- We want to determine the *stability* of these steady spins
- **Choice:** *Either*
  Prescribe an ordering of the principal moments of inertia (*i.e.*, $I_1 > I_2 > I_3$), and check all three possible steady spins for stability;
  or,
  Select a single steady spin, and determine the conditions on the moments of inertia for stability
- We will **Select**
  $\boldsymbol{\omega} = [0 \ \ \Omega \ \ 0]^{\mathsf{T}}, \quad \Omega > 0$
- Exercise: Verify that this choice of $\boldsymbol{\omega}$ makes all three $\dot{\omega}_j$'s equal to zero.

Asymmetric Torque-Free Rigid Body (6)

- One way to investigate stability is to *linearize* about the motion of interest
- Here we wish to linearize about $\boldsymbol{\omega} = [0 \;\; \Omega \;\; 0]^{\mathsf{T}}$
- By linearization, we mean *Add a small perturbation to the motion of interest, expand the equations of motion in a Taylor series, and determine whether the assumed small motion indeed remains small*
- Thus, we assume

$$\boldsymbol{\omega} = [0 \;\; \Omega \;\; 0]^{\mathsf{T}} + \delta\boldsymbol{\omega}$$

and we want to determine the conditions under which $\delta\boldsymbol{\omega}$ remains small, and the conditions under which it grows exponentially

- Adding small perturbation gives

$$\begin{aligned} \omega_1 &= 0 + \delta\omega_1 \\ \omega_2 &= \Omega + \delta\omega_2 \\ \omega_3 &= 0 + \delta\omega_3 \end{aligned}$$

- Substitute these expressions into Euler's equations

$$\begin{aligned} \dot{\delta\omega_1} &= \frac{I_2 - I_3}{I_1}(\Omega + \delta\omega_2)\delta\omega_3 \\ \dot{\delta\omega_2} &= \frac{I_3 - I_1}{I_2}\delta\omega_1\delta\omega_3 \\ \dot{\delta\omega_3} &= \frac{I_1 - I_2}{I_3}\delta\omega_1(\Omega + \delta\omega_2) \end{aligned}$$

- Expand and ignore the products of $\delta\omega_j$ terms (*i.e.*, ignore *higher order terms*)
- Exercise: Explain how this step is identical to taking Taylor series and retaining only linear terms.

Asymmetric Torque-Free Rigid Body (7)

- Ignoring higher order terms gives

$$\begin{aligned}\delta\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\Omega\delta\omega_3 \\ \delta\dot{\omega}_2 &= 0 \\ \delta\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\Omega\delta\omega_1\end{aligned}$$

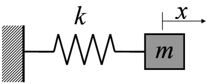
- These equations are *linear, constant-coefficient, coupled, ordinary differential equations*
- You should compare with Euler's equations for axisymmetric rigid body
- One difference is that the constants are not the same as for axisymmetric case (because $I_1 \neq I_3$ here)

- Differentiate $\delta\dot{\omega}_1$ and substitute $\delta\dot{\omega}_3$ to obtain

$$\delta\ddot{\omega}_1 + \frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}\Omega^2\delta\omega_1 = 0$$

- Clearly this is a *second-order, linear, constant-coefficient, ordinary differential equation* in the form

$$\ddot{x} + kx = 0$$



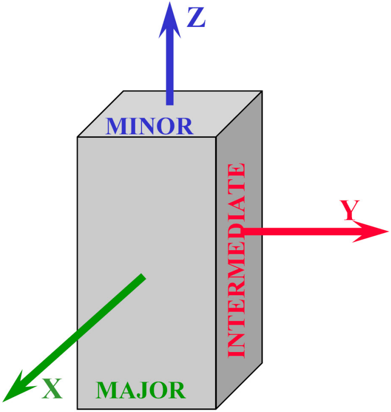
- $k > 0$ is required for stability, which translates to

$$I_2 > \{I_1, I_3\} \quad \text{or} \quad I_2 < \{I_1, I_3\}$$

Asymmetric Torque-Free Rigid Body (8)

Summary of the Results

- Euler's equations are coupled and nonlinear
- Two first integrals: rotational kinetic energy and angular momentum magnitude
- Each first integral defines an ellipsoid in $\boldsymbol{\omega}$-space
- Can solve analytically using Jacobi elliptic functions
- Steady spin about any principal axis is a solution
- Linear stability analysis reveals
  — Major axis spin is stable
  — Minor axis spin is stable
  — Intermediate axis spin is unstable



Previews: ellipsoids provide a useful visualization, and energy dissipation destabilizes minor axis spin

Energy and Angular Momentum

- Recall that energy and angular momentum are conserved (in the torque-free case):

$$\begin{aligned} 2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\ h^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \end{aligned}$$

- The two expressions describe ellipsoids in $\boldsymbol{\omega}$-space
- Solutions to Euler's equations must satisfy the two first integrals, and thus must be on the intersections of the two ellipsoids
- Assume, without loss of generality, that

$$I_1 > I_2 > I_3$$

- Exercise: Explain why we can always make this assumption.
- The principal axes of the ellipsoids are the ω_1, ω_2, and ω_3 axes.
- Given the assumed ordering, the ω_1 direction is the smallest dimension of each ellipsoid and the ω_3 direction is the largest dimension of each ellipsoid
- Exercise: Explain why this is true.

Energy and Angular Momentum (2)

- Transform from $\boldsymbol{\omega}$-space to $\mathbf{h}$-space: $h_1 = I_1\omega_1$, etc.

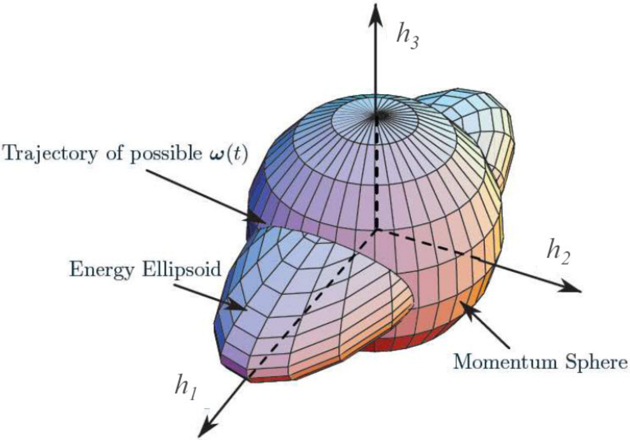
$$\begin{aligned} 2T &= h_1^2/I_1 + h_2^2/I_2 + h_3^2/I_3 \\ h^2 &= h_1^2 + h_2^2 + h_3^2 \end{aligned}$$

- The two expressions describe ellipsoids in $\mathbf{h}$-space
- Furthermore, the angular momentum ellipsoid is a sphere
- Rewrite as

$$\begin{aligned} 1 &= \frac{h_1^2}{2TI_1} + \frac{h_2^2}{2TI_2} + \frac{h_3^2}{2TI_3} \\ 1 &= \frac{h_1^2}{h^2} + \frac{h_2^2}{h^2} + \frac{h_3^2}{h^2} \end{aligned}$$

- These expressions are in the standard form for an ellipsoid and a sphere, respectively
- The principal axes of the ellipsoids are the h_1, h_2, and h_3 axes.
- Given the assumed ordering, the h_1 direction is the largest dimension of the energy ellipsoid and the h_3 direction is the smallest dimension of the energy ellipsoid
- Exercise: Explain why this is true.

Energy and Angular Momentum (3)

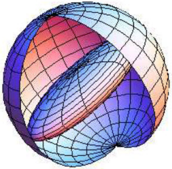


Energy and Angular Momentum (4)

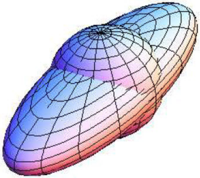
1) Fix the angular momentum and 2) Vary the energy from T_{min} to T_{max}

1) $\Rightarrow$ momentum sphere has constant radius

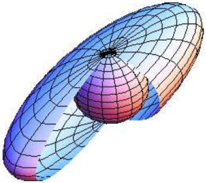
2) $\Rightarrow$ energy ellipsoid varies from "inside" the momentum sphere to "outside" the momentum sphere



Min Energy: Energy Ellipsoid is inside and tangent to Momentum Sphere



Intermediate Energy: Energy Ellipsoid is inside/outside and tangent to Momentum Sphere



Max Energy: Energy Ellipsoid is outside and tangent to Momentum Sphere

Energy Dissipation

- Real spacecraft are not rigid bodies
- Internal motion of propellant and flexible components dissipates energy, but has little (or no) effect on angular momentum
- **Energy Sink Hypothesis**: *Spacecraft internal motion is sufficiently small that the moments of inertia are constant, and the internal motion results in slow energy dissipation such that* $\dot{T} \leq 0$
- Thus, in the absence of external torques, angular momentum magnitude is constant and rotational kinetic energy decreases

Energy Dissipation (2)

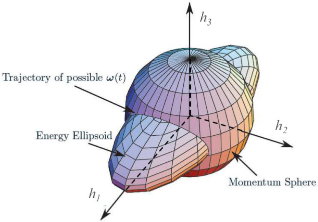
- Consider an axisymmetric torque-free body with energy dissipation

$$\begin{aligned} T &= \frac{1}{2} I_1 \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2 \\ &= \frac{1}{2} I_3 \omega_0^2 \left[1 + \left(\frac{I_3}{I_1} - 1\right) \sin^2\theta\right] \end{aligned}$$

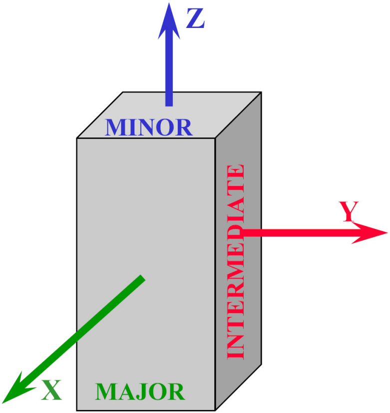
- Given the energy sink hypothesis, $\dot{T} \leq 0$
- Taking that derivative gives

$$\dot{T} = I_3 \omega_0^2 \left(\frac{I_3}{I_1} - 1\right) (\sin\theta\cos\theta)\,\dot{\theta}$$

- What can we deduce about $\dot{\theta}$?
- Note that $\sin\theta\cos\theta \geq 0$ (why?)
- What does the moment of inertia ratio imply?
- Explain the behavior of this motion in the context of the energy and momentum ellipsoids



Rigid Body / "Real" Body Spin Stability



- $I_{xx} > I_{yy} > I_{zz}$
- Major axis spin is stable
- Minor axis spin is stable
- Intermediate axis spin is unstable
- Energy dissipation changes these results
  $\rightarrow$ Minor axis spin becomes unstable

This behavior is called the Major Axis Rule