ECE 506: Homework #1: Basic Optimization

To get help with the homework, please join the Saturday morning discussion sessions starting at 9am at `https://unm.zoom.us/j/99977790315`.

Problem #1. An Introduction to Linear Programming
This problem is focused on manipulating the basic Linear Programming equation:

$$\min_x \; c^\top x \quad \text{subject to } Ax = b \text{ and } x \geq 0. \tag{1}$$

(Here, $x \geq 0$ is understood componentwise.)

1(a) Problem statement. We begin with the simplest possible example! Consider the 1D problem:

$$\min_x \; c \cdot x \quad \text{subject to } ax = b \text{ and } x \geq 0. \tag{2}$$

From this case, answer the following:

i) Give an example where there is no solution.

ii) Give an example with a simple solution.

iii) For your solution, did you minimize anything? Explain.

Solution. With the constraints $ax = b$ and $x \geq 0$, if $a \neq 0$ then the only candidate is $x^\star = \frac{b}{a}$.

i) If $b/a < 0$, the nonnegativity constraint is violated, so the problem is infeasible; e.g., $a = 1,\ b = -1 \Rightarrow x^\star = -1$ (infeasible). (Also infeasible when $a = 0,\ b \neq 0$ since $0 = b$ cannot hold.)

ii) Example with a simple solution: Take $a = 2,\ b = 0$. Then $x^\star = \frac{b}{a} = 0$, which satisfies $x \geq 0$, and the objective value is $c\,x^\star = 0$.

iii) Did we minimize anything? No. When $a \neq 0$, the equality constraint pins down a single feasible point $x^\star$; if it is feasible, it is automatically optimal—there is no tradeoff to optimize over.

1(b) Problem statement. More generally, consider $Ax = b$ for many dimensions. Suppose that A is invertible. In this case, show that there is no minimization! To show this, compute the solution without minimizing $c^\top x$.

Solution. If A is invertible, the constraint $Ax = b$ has the unique solution $x^\star = A^{-1}b$. If $x^\star \geq 0$ (componentwise), it is the only feasible—and thus optimal—point with value $c^\top x^\star$; otherwise the problem is infeasible. No minimization needed.

1(c) Problem statement. The only case that is interesting is when we have many solutions to $Ax = b$. We then get to pick the one that minimizes $c^\top x$. This can only happen when the number of equations is smaller than the number of unknowns. Here is an example:

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2.$$

Note that we have one equation in two unknowns. We have more unknowns than we have equations! It may be possible to set up a proper optimization problem.

To have a proper solution, we must also satisfy $x_1, x_2 \geq 0$. These are called *feasible solutions*. They satisfy the constraints, and the optimal solution needs to satisfy them.

Task: Plot all possible solutions of $Ax = b$ satisfying $x_1, x_2 \geq 0$ for this case.

Solution. The feasible set is

$$\{(x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid x_1 + 2x_2 = 2\},$$

which is the line segment in the first quadrant between the intercepts $(2, 0)$ and $(0, 1)$. Any feasible point can be written as

$$(x_1, x_2) = (2 - 2t,\ t), \quad t \in [0, 1].$$

A plot of this segment (restricted to $x_1, x_2 \geq 0$) depicts the feasible region.

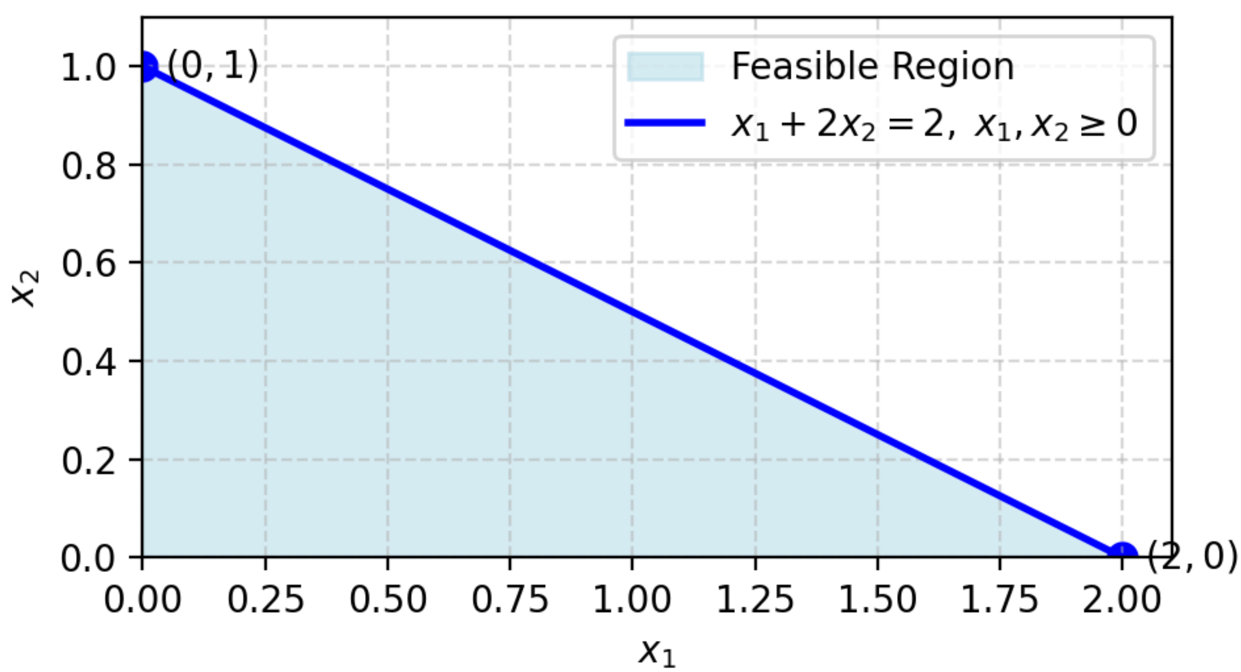

Figure 1: Feasible region for $x_1 + 2x_2 = 2$ with $x_1, x_2 \geq 0$.

1(d) Problem statement. For the case when $Ax = b$ described in 1(c), solve the proper optimization problem. For this case, solve:

$$\min_x \begin{bmatrix} 1 & 1 \end{bmatrix} x \quad \text{subject to } Ax = b \text{ and } x \geq 0. \tag{3}$$

Is the solution at the endpoints? Explain.

Solution. The feasible set is $\{(x_1, x_2) : x_1 + 2x_2 = 2,\ x_1, x_2 \geq 0\}$, i.e., the segment between $(2, 0)$ and $(0, 1)$. Along this line, substitute $x_1 = 2 - 2x_2$ into the objective $x_1 + x_2$ to get

$$x_1 + x_2 = (2 - 2x_2) + x_2 = 2 - x_2.$$

Over $x_2 \in [0, 1]$, this is minimized at $x_2 = 1$, giving the endpoint $(0, 1)$ with optimal value 1. Yes—the optimum lies at an endpoint.