From:

"Chapter 2. The Simplex Algorithm",

Combinatorial Optimization: Algorithms and Complexity,

by C. H. Papadimitriou and K. Steiglitz.

(p.29) 2.2 Basic Feasible Solutions

Consider:

$$\min \; c^T x$$
$$Ax = b, \quad A \text{ is } m \times n, \; m < n.$$
$$x \geq 0$$

Assume A is of rank m.

Let $B = \{A_{j_1}, \dots, A_{j_m}\}$ be a basis for A.

Eg. If $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix}$, then A is of rank $= 3$.

$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for A.

A basic solution x is one for which:

$$x_j = 0, \text{ for } A_j \notin B$$
$$x_{j_k} = k\text{th component of } B^{-1}b,$$
$$k = 1, \dots, m, \; A_{j_k} \in B.$$

To find a solution to $Ax = b$, do:

1. Choose a set B of linearly independent columns of A.
2. Set all components of x corresponding to columns not in B equal to zero.
3. Solve the m resulting equations to determine the remaining components of x. These are the <u>basic variables</u>.

If x_1, x_2, x_3 not all ≥ 0, repeat

In our example:

1. $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

2. $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix}$ $\begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{corresponding to columns 1, 2, and 3.} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{remaining two.} \end{matrix}$

3. Solve:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$$

for x_1, x_2, x_3.

basic variables.

We must have $x_1, x_2, x_3 \geq 0$

bfs if also a solution to (*)

(p.42) 2.4 Moving from bfs to bfs

Let x_0 be a bfs of an instance of LP with matrix A, with:

$$B = \{ A_{B(i)} : i = 1, \ldots, m \}$$

If the basic components of x_0 are:

$$x_{i_0}, \quad i_0 = 1, 2, \ldots, m,$$

then:

$$\sum_{i=1}^{m} x_{i_0} A_{B(i)} = b, \qquad x_{i_0} \geq 0.$$

(*)

Note that for the non-basic columns, we have to be able to express them in-terms of the basis:

$$\sum_{i=1}^{m} x_{ij} A_{B(i)} = A_j \quad \text{——} \; (**)$$

Multiply (**) by $\theta > 0$ and subtract from (*) to get:

$$\sum_{i=1}^{m} \left(x_{i0} - \theta x_{ij} \right) A_{B(i)} + \theta A_j = b$$

To change x and remain feasible, we must have the new x still remains positive. Thus, the maximum value for x must be:

$$\theta_0 = \min_{\substack{i \\ \text{such that} \\ x_{ij} > 0}} \frac{x_{i0}}{x_{ij}}$$

(from $x_{i0} - \theta x_{ij} = 0$)

In term of a bfs, we have the cost:

$$z_0 = \sum_{i=1}^{m} x_{i0}\, c_{B(i)}$$

As we know, other columns can be expressed in terms of the basis columns:

$$A_j = \sum_{i=1}^{m} x_{ij}\, A_{B(j)}$$

Thus, when x_j enters, x_{ij} <u>must leave</u>, reducing from $x_{B(i)}$

cost to $$\underbrace{c_j}_{\text{cost of entering } x_j = 1} - \underbrace{\sum_{i=1}^{m} x_{ij}\, c_{B(i)}}_{\text{definition of } z_j} = \bar{c}_j$$

where $\bar{c}_j$ is hereby defined as the <u>relative cost</u> of column j.

It is thus <u>profitable</u> (since it will reduce cost) to bring-in column j, provided $\bar{c}_j < 0$.

If all $\bar{c}_j \geq 0$, then there is no hope of reducing cost:

$\Rightarrow$ We are at global minimum

Now return to (for unboundedness):

$$\sum_{i=1}^{m} x_{ij} A_{B(i)} = A_j \, .$$

If all the $x_{ij} \leq 0$, all i, some j, then clearly if we also assume:

$$c_j - \sum_{i=1}^{m} x_{ij} c_{B(i)} < 0$$

$$\Rightarrow c_j + (-x_{1j}) c_{B(1)} + \cdots + (-x_{mj}) c_{B(m)} < 0$$

all of them are positive.

$\Rightarrow$ At least one of:

$c_{B(1)}, \ldots, c_{B(m)}, c_j$ is negative,

and setting that component $x_k = \infty$ grows unbounded to $-\infty$.

Pseudocode for the simplex algorithm is given on p. 49:

<u>Procedure</u> simplex

<u>begin</u>

opt := 'no', unbounded:='no';

(if either becomes 'yes', we terminate)

<u>While</u> opt=='no' <u>and</u>

unbounded = 'no' <u>do</u>

<u>if</u> $\bar{c}_j \geq 0$ for all j,

<u>then</u> opt:='yes' (terminate)

<u>else</u>

<u>begin</u>

Choose any J with $\bar{c}_J < 0$;

<u>If</u> $x_{iJ} \leq 0$ for all i

<u>then</u> unbounded := 'yes'

<u>else</u>

<u>find</u> $\theta_0 = \min\limits_{\substack{i \\ x_{iJ} > 0}} \left[\frac{x_{i0}}{x_{iJ}} \right]$

$= \frac{x_{k0}}{x_{kJ}}$

<u>and</u> pivot on x_{kJ}

<u>end</u>

<u>end</u>

p. 44. 2.5 Organization of a Tableau

Start with $Ax=b$. Assume:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ 5x_1 + x_2 + x_3 + x_4 &= 3 \\ 2x_1 + 5x_2 + x_3 + x_5 &= 4 \end{aligned}$$

Prepare the basic tableau:

	x_1	x_2	x_3	x_4	x_5
1	3	2	1	0	0
3	5	1	1	1	0
4	2	5	1	0	1

Elementary row operations do not change the structure.

For minimizing $c^T x$ subject to $Ax=b$, we follow Strang (modified to agree with P&S):

$$T = \left[\begin{array}{c|c} 0 & c \\ \hline b & A \end{array}\right]$$

We then proceed to identify a basis in the columns of A.

In this case, the basis is

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

which corresponds to variables x_3, x_4, x_5.

We break x into: $x = [x_B \; x_N]^T$

where x_B contains the basic variables.

For $c = [1\;1\;1\;1\;1]^T$, $x_B = [x_3 \; x_4 \; x_5]$,

we end up with the new tableau:

	x_3	x_4	x_5	x_1	x_2
0	1	1	1	1	1
1	1	0	0	3	2
3	1	1	0	5	1
4	1	0	1	2	5

This is the basis matrix B. The new tableau is:

$$\left[\begin{array}{c|cc} 0 & c_B & c_N \\ \hline b & B & N \end{array}\right]$$

Next, apply elementary row operations to convert this to I.

After "pivoting", we have:

$$\left[\begin{array}{c|cc} 0 & c_B & c_N \\ \hline B^{-1}b & I & B^{-1}N \end{array}\right] \sim (*)$$

Next, we operate on the top row: we zero-out the c_B part by multiplying the second row by c_B and subtracting from the top:

$$\left[\begin{array}{c|cc} -c_B B^{-1} b & 0 & c_N - c_B B^{-1} N \\ \hline B^{-1}b & I & B^{-1}N \end{array}\right] \sim (**)$$

In our example, $(*)$ gives:

$$\begin{array}{cccccc} & x_3 & x_4 & x_5 & x_1 & x_2 \\ \end{array}$$
$$\left[\begin{array}{c|ccc|cc} 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 3 & 2 \\ 2 & 0 & 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

Next, we zero-out this portion by elementary row operations to arrive to $(**)$. This gives:

		x_3	x_4	x_5	x_1	x_2
	$-z = -6$	0	0	0	-3	$\boxed{-3}$
x_3	1	1	0	0	3	2
x_4	2	0	1	0	2	-1
x_5	3	0	0	1	-1	3

Now, the top row contains the relative costs. Any variable with negative relative costs can enter the solution space. If there is no variable with negative relative cost, then we are done.

Here, we do have x_1 and x_2 with negative relative costs. We choose x_2.

Form the positive ratios between column zero and that of x_2:

$$1/2, \quad \cancel{2/-1}, \quad 3/3$$

not considering negatives

The one with the minimum value is $1/2$. This means that x_3 must leave the solution, and x_2 must enter.

We interchange the columns of x_3 and x_2 and repeat the procedure ("pivot" to get the identity in the columns of x_2, x_4, x_5 and zero out the row below x_2, x_4, x_5's, see M).

Repeat until:

(i) all relative-costs are non-negative ≥ 0.

(ii) A column with negative cost has negative entries

(See algorithm on page 6)

Final aswer is:

Zeros   all positive

$$
\begin{array}{r|ccccc}
 & x_2 & x_4 & x_5 & x_1 & x_3 \\
-z = -9/2 & 0 & 0 & 0 & 3/2 & 3/2 \\
\hline
x_2 = 1/2 & 1 & 0 & 0 & 3/2 & 1/2 \\
x_4 = 5/2 & 0 & 1 & 0 & 7/2 & 1/2 \\
x_5 = 3/2 & 0 & 0 & 1 & -11/2 & -3/2
\end{array}
$$

all ≥ 0 identity here

<u>Solution:</u> $x_2 = 1/2$, $x_4 = 5/2$, $x_5 = 3/2$.

$$c^T x = 1/2 + 5/2 + 3/2 = 9/2$$