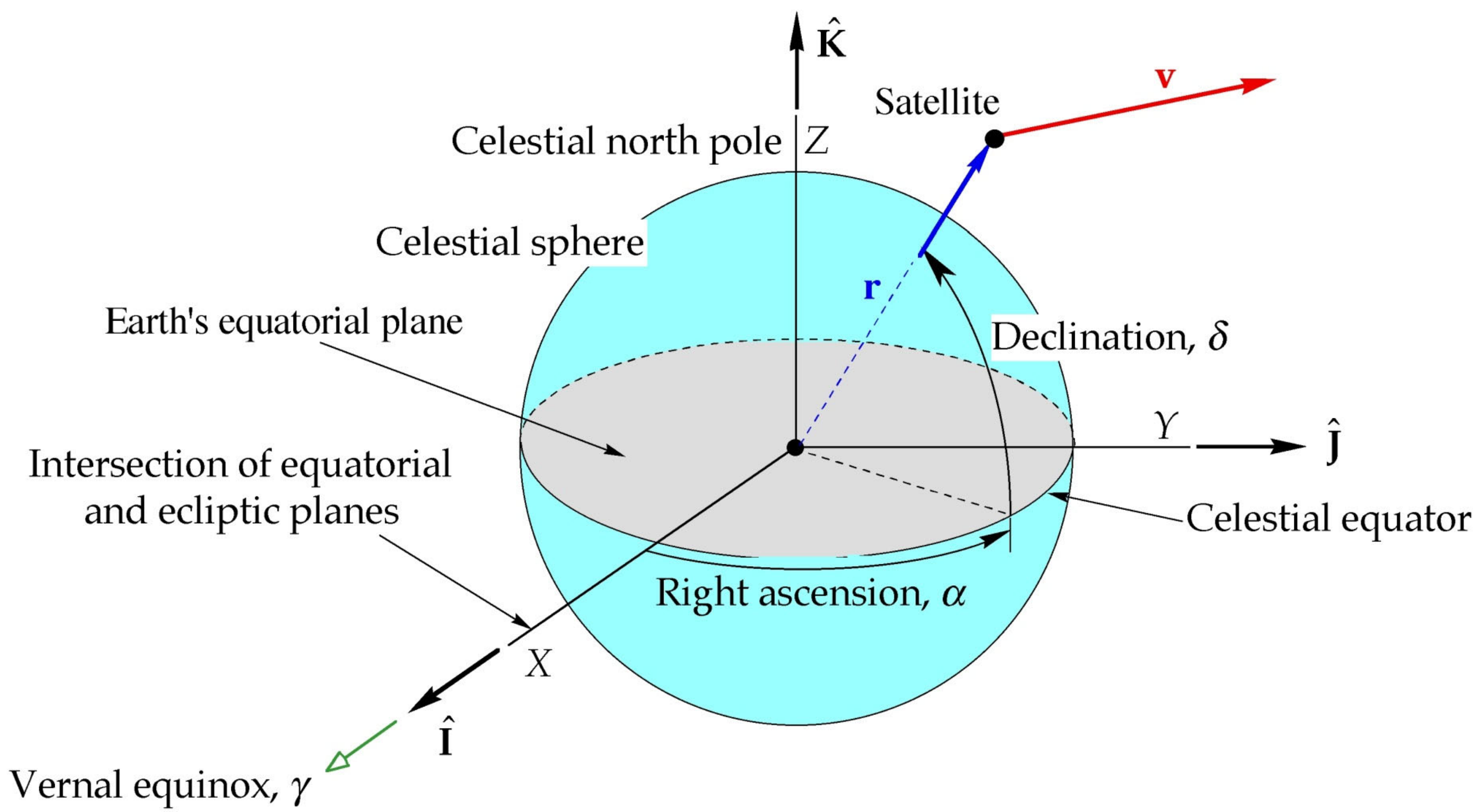
UNM MECHANICAL ENGINEERING

# Coordinate Systems, Time Measurement, and Transformations



# Coordinate Frames

•Why are coordinate frames important in orbit mechanics?

•To not only locate an object's instantaneous position ($X, Y, Z$), but its orbit as well (characterized by position & velocity vector at an epoch time: $X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}$)

•In Cartesian or rectilinear frames, the coordinates are entirely linear distances, e.g. $X, Y, Z$

•In curvilinear frames , the coordinates contain angles as well (e.g. $\rho$, $\theta$, $\phi$)

•In addition to its coordinatization, it is important to know whether a frame is inertial

•An inertial frame is defined as one that is neither accelerating nor rotating

•While the concept of a precisely inertial frame is somewhat elusive, several frames common to orbit mechanics are very nearly inertial (& for all practical purposes are taken to be inertial)

# Coordinate Frames

Here we will focus on the following coordinate frames:

- Celestial (Right Ascension-Declination) Frame
- Geocentric Equatorial Frame
- Earth-Centered Earth-Fixed frame
- Topocentric Horizon Frame

# Celestial (Right Ascension-Declination) Frame

A key facet of the Celestial Frame is the definition of the vernal equinox line:

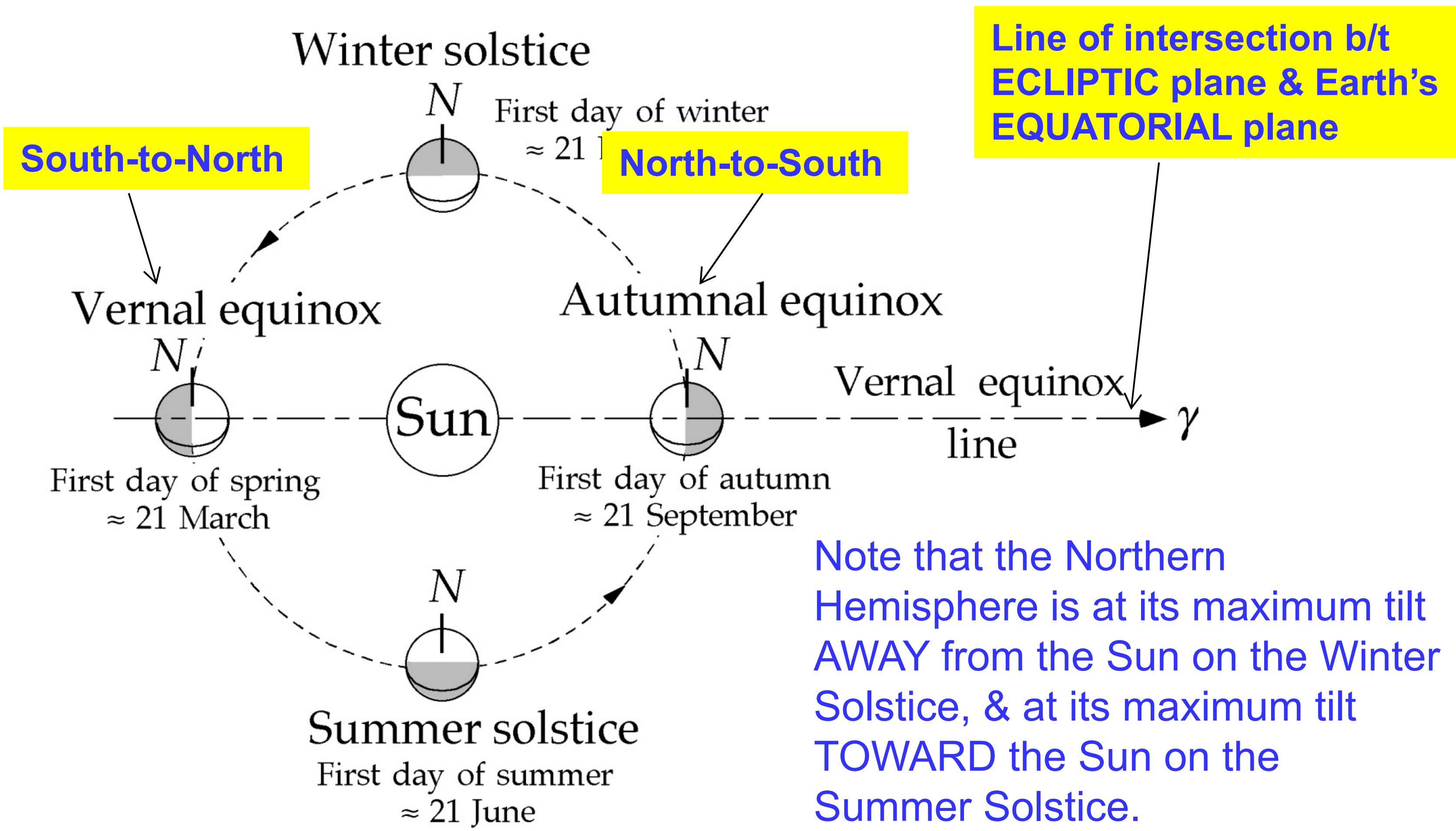
- The ecliptic is the plane of the earth's orbit around the sun

- The Earth's equatorial plane is not coincident with the ecliptic plane; i.e. Earth's axis of rotation, which passes through the north and south poles, is not perpendicular to the ecliptic

- It is tilted from "vertical" (i.e. normal to the ecliptic) by an angle known as the obliquity of the ecliptic $\varepsilon$, which is approximately 23.4 degrees (this is also the angle between the Earth's equatorial plane & the ecliptic plane)

- The line of intersection between the ecliptic plane & Earth's equatorial plane passes through the Sun twice a year: on the first day of Spring (vernal equinox) & the first day of Fall (autumnal equinox); the line pointing from Earth toward Sun on first day of Spring is the vernal equinox line

# Celestial (Right Ascension-Declination) Frame



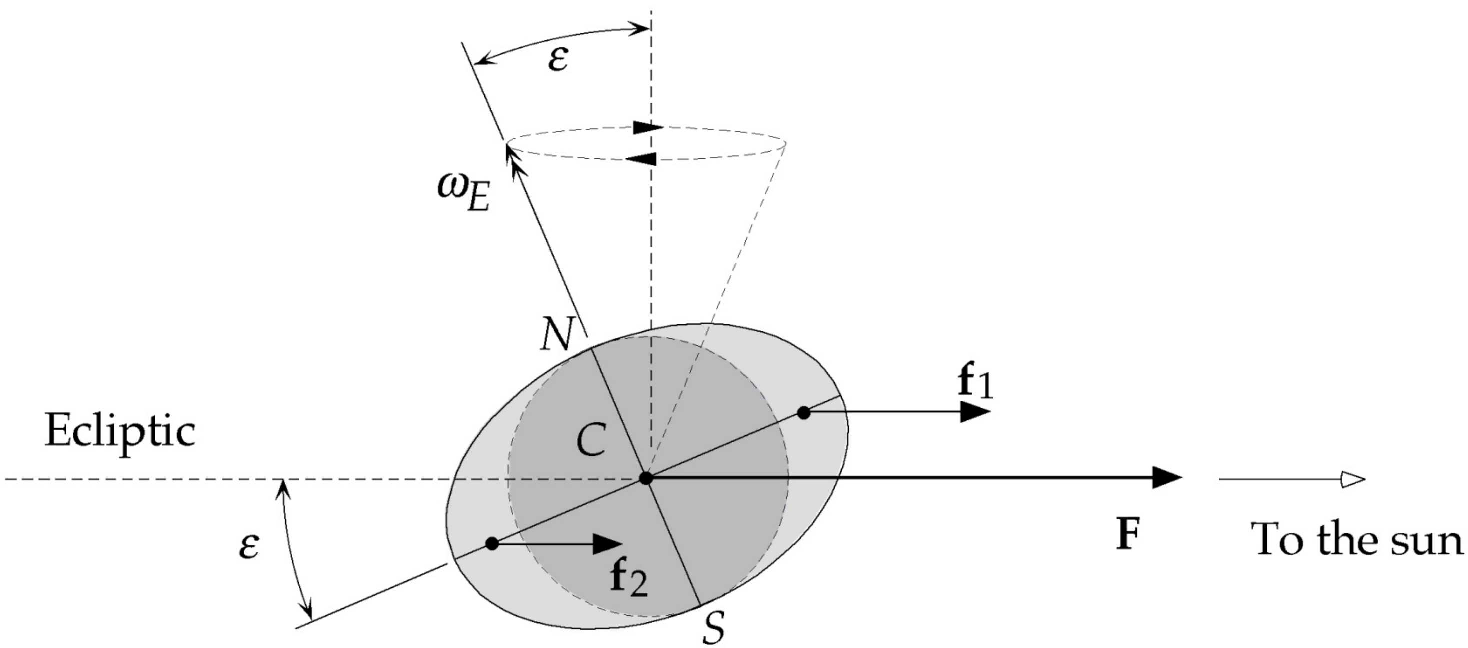

Note that the Northern Hemisphere is at its maximum tilt AWAY from the Sun on the Winter Solstice, & at its maximum tilt TOWARD the Sun on the Summer Solstice.

# Celestial (Right Ascension-Declination) Frame

•Earth's spin axis maintains its inertial orientation except for slight westward precession about the "vertical" (normal to the ecliptic), at the rate of about 1.4 degrees per century

•This causes the vernal equinox line to precess westward around the "vertical" as well

•Thus, the vernal equinox direction is not constant, but slowly changing




Secondary (perturbing) gravitational forces on the earth

# Celestial (Right Ascension-Declination) Frame

•The Celestial Frame has Earth's center as its origin; it is based on the concept of the celestial sphere, concentric with the Earth & of infinite radius

•The Celestial Frame has no capability to delineate an object's distance from Earth's center, but only its direction; think of it as a spherical coordinate system with no distance coordinate (i.e. "$\rho$, $\theta$, $\phi$ without the $\rho$")

•The two angles delineating direction are right ascension (RA or $\alpha$) and declination (Dec or $\delta$)

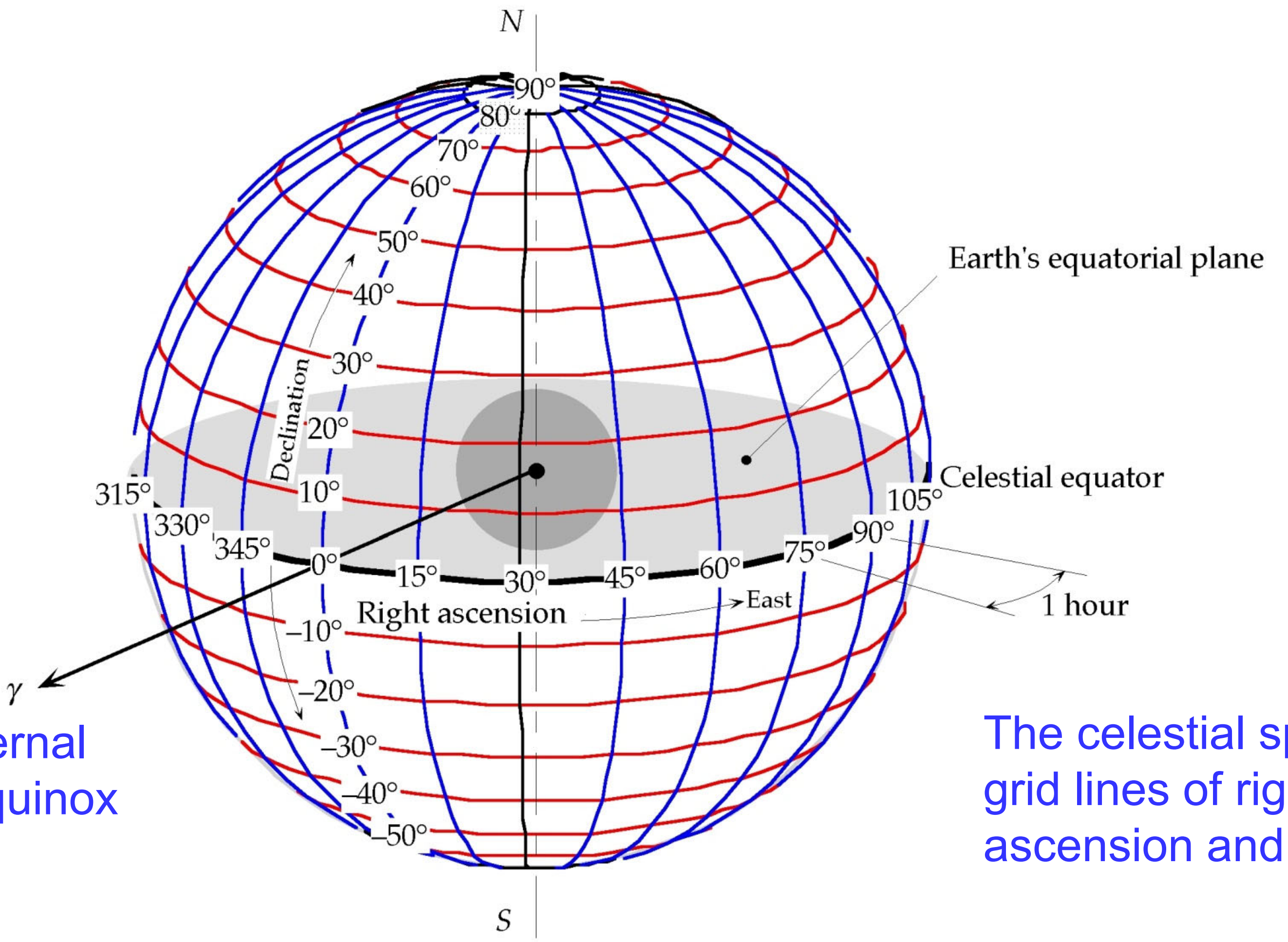
•The easiest way to visualize RA & Dec of an object is to consider a vector from Earth's center to the object

- Dec is the angle between the vector & Earth's equatorial plane (here called the celestial equator)
- RA is the angle between the projection of this vector onto the celestial equator & the vernal equinox (which always lies in the celestial equator)

•Dec is essentially a measure of latitude on the celestial sphere, positive to the north of the equator and negative to the south (ranging from -90° to 90°)

•RA is essentially a measure of longitude along the celestial equator east from the vernal equinox (ranging from 0° to 360°)

# Celestial (Right Ascension-Declination) Frame




Vernal equinox

The celestial sphere, with grid lines of right ascension and declination

# Celestial (Right Ascension-Declination) Frame

- We locate the position of stars, planets, satellites, etc., by giving their right ascension and declination

- A star catalog contains the RA, Dec, & magnitude of the known stars in the universe

- While stars are considered fixed in inertial space, because the vernal equinox direction isn't constant, the RA & Dec of each star changes (but extremely slowly)

- The RA & Dec of each planet in the solar system change more rapidly, given that planets are moving & are much closer to Earth than stars

- The RA & Dec of Earth-orbiting objects change even more rapidly, given that they are much closer to Earth than stars or planets (& therefore move faster than either, relative to Earth)

# Geocentric Equatorial Frame

- For Earth-orbiting objects, a convenient inertial frame is the Geocentric Equatorial Frame:
  - Origin at Earth's center
  - X-axis ($\hat{i}$) points toward vernal equinox
  - Z-axis ($\hat{k}$) points toward North Pole (Earth's axis of rotation)
  - Y-axis completes the orthogonal triad ($\hat{j} = \hat{k} \times \hat{i}$) so that Earth's equatorial plane is the ECI x-y plane

- Also known as the Earth-Centered Inertial (or ECI) Frame

- Closely related to the Celestial Frame in that:
  - Both frames defined by the vernal equinox direction
  - Earth's equatorial plane (i.e. celestial equator) is key to both frames

# Geocentric Equatorial Frame

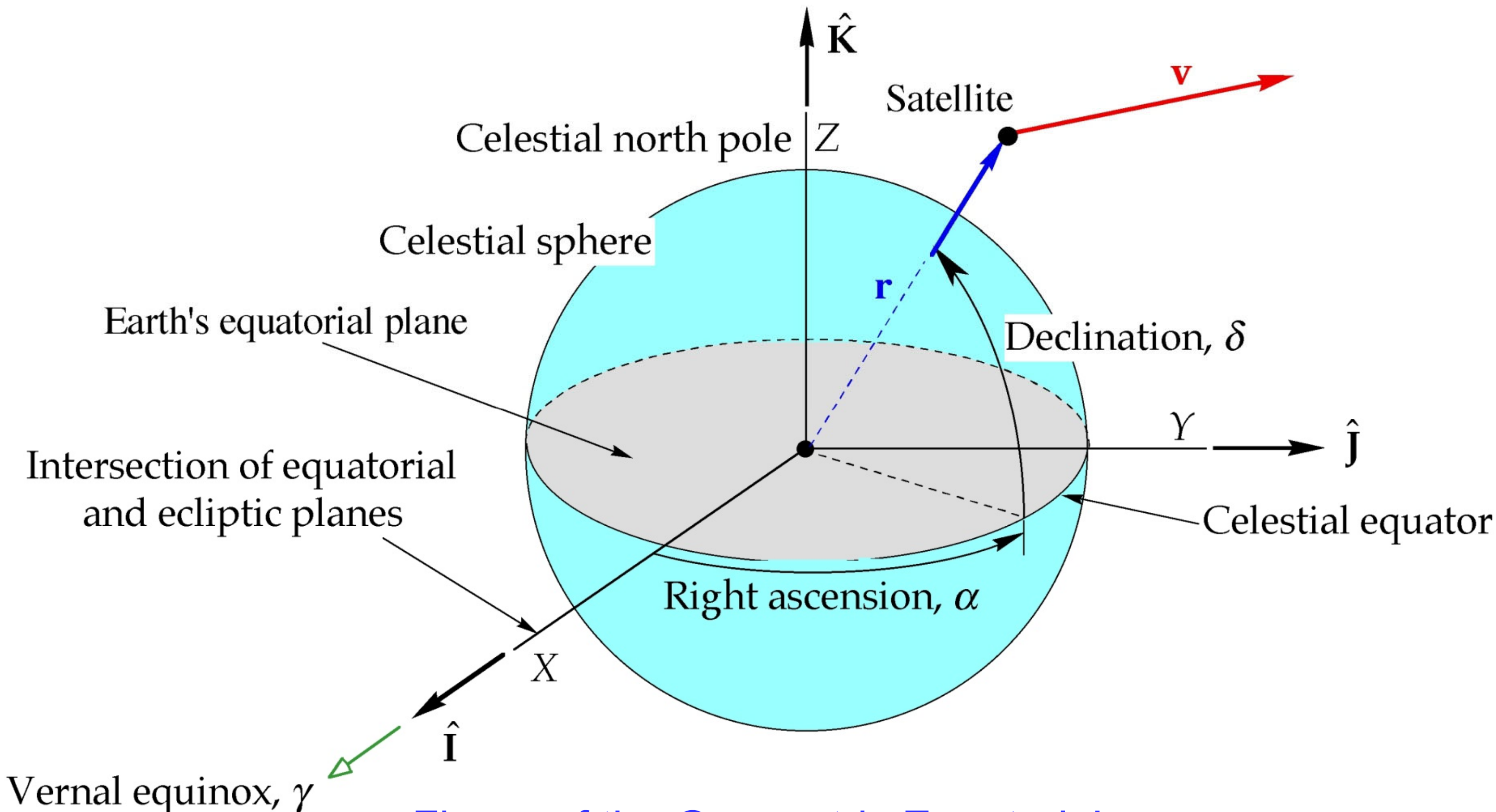



Figure of the Geocentric Equatorial (ECI) Frame, with components of the Celestial Frame depicted as well

# Geocentric Equatorial Frame

- Recall the definition of an inertial frame is one that neither accelerates nor rotates

- The ECI Frame does BOTH!
  - Its origin (Earth's center) is being accelerated by the Sun
  - It rotates due to precession of Earth's rotation axis (slowly changing vernal equinox direction)

- So how do we remedy this apparent conundrum?
  - For Earth-orbiting objects, the Sun's effect on their motion is far less than that of the Earth (Sun's gravity treated as a perturbation force)
  - So in a local sense, we can treat the ECI origin as if it's not moving
  - To alleviate rotation, we can choose the vernal equinox direction at ONE moment in time & align the ECI x axis with that direction → "J2000" standard is to choose the vernal equinox direction at exactly noon on 1 Jan 2000

# Earth-Centered Earth-Fixed Frame

- Earth-Centered Earth-Fixed (ECEF) Frame defined as follows:
  - Origin at Earth’s center
  - z axis points toward North pole
  - x & y axes in equatorial plane, with x pointing toward prime (Greenwich) meridian
  - y axis 90 degrees east of x axis
- Lat/long/height of a location on Earth can be easily converted to ECEF coordinates (& vice versa)
  - Can use geocentric or geodetic latitude (Bate, Mueller, & White textbook clarifies this on p. 94)
- Note that ECEF Frame is NOT inertial (x & y axes rotate with the Earth)
  - Vernal equinox direction & prime meridian align once per day as Earth rotates
  - At any given time, the angle between vernal equinox direction & prime meridian is called “sidereal time” (to be covered later)

# Topocentric Horizon Frame

- Topocentric-Horizon (TH) Frame is useful for ground sensors that observe space (e.g. radars & telescopes)

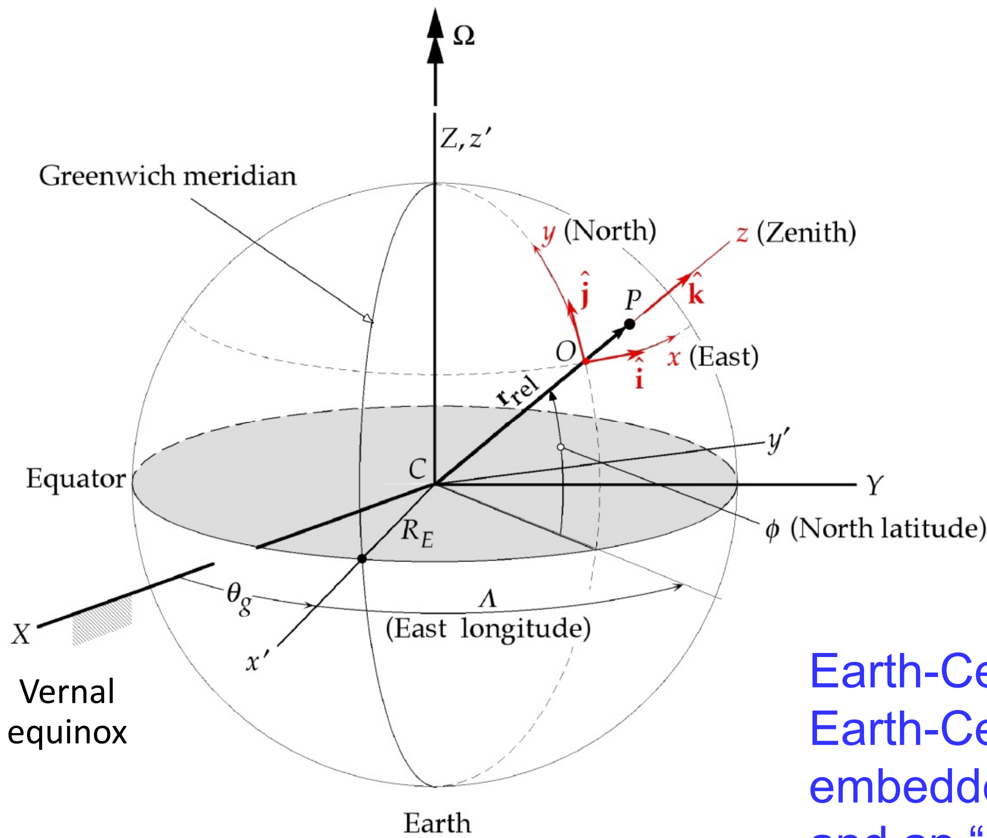
- TH Frame defined as follows:
  - Origin at a point on Earth's surface
  - z axis points "up" (direction from Earth's center to the point)
  - x axis points South
  - y axis points East

- (Some variants of the TH Frame exist, e.g. "East/North/Up" or "North/East/Down")

- TH origin may stay fixed on Earth if the point of interest is fixed (e.g. a fixed ground sensor) OR it may move along the Earth if the point of interest is moving (e.g. a mobile ground sensor)

- Either way, TH Frame is clearly NOT inertial---local South, East, & "up" directions change as Earth rotates

# ECI, ECEF, & TH Frames




Earth-Centered Inertial Frame (*XYZ*); Earth-Centered Earth-Fixed Frame (*x' y' z')* embedded in and rotating with the Earth; and an "East/North/Up" Topocentric-Horizon Frame *xyz* attached to a point *O* on the Earth's surface

# Other Coordinate Frames

Bate, Mueller, & White also mentions the following coordinate frames:

- Heliocentric-Ecliptic Frame (p. 54)

- Perifocal Frame (p. 57)

# Time Measurement

•The sidereal day is the time it takes the Earth to complete one rotation relative to inertial space (the fixed stars)

•The apparent synodic (or solar) day is the time it takes the sun to rotate once around the Earth, from high noon one day to high noon the next

•While the Earth makes one absolute rotation around its axis, it advances $2\pi/365.26$ radians along its orbit around the Sun

•Thus, the Earth must rotate an extra $2\pi/365.26$ radians to go from "high noon to high noon" (1 solar day)

•A solar day is then 1/365.26 longer than a sidereal day → 1 solar day = 1.002738 sidereal days

•Therefore, Earth rotates 360° in a sidereal day, but 360*1.002738° in a solar day

# Time Measurement

•Universal Time (UT), also known as Greenwich Mean Time (GMT), is the local solar time at the Greenwich meridian

•The ECI & ECEF Frames always share a common z axis direction & common x-y plane

•At any given time, the ECI x axis & ECEF x axis are separated by an angle known as Greenwich sidereal time, $\theta_g$ (somewhat of a misnomer, since this is an angle, not a time)

•If we knew $\theta_g$ at a particular time $t_0$ (call it $\theta_{g0}$ at this time), then $\theta_g$ at a later time $t_f$ can be calculated by $\theta_g = \theta_{g0} + 1.002738 * 360 * (t_f - t_0)$, where $t_0$ & $t_f$ are expressed in solar days & $\theta_g$ & $\theta_{g0}$ are expressed in degrees

•The value of $\theta_{g0}$ at several epochs in history is recorded & can be easily looked up

•Knowing $\theta_g$ is useful when transforming between certain coordinate frames (as we will see later)

# Coordinate Transformation

•A vector can be expressed in any coordinate frame

•If we start with the expression of a vector in frame “A” & have equations relating the unit vectors of frame “B” in terms of the unit vectors of frame “A”:

$$\hat{i}_B = Q_{11}\hat{i}_A + Q_{12}\hat{j}_A + Q_{13}\hat{k}_A$$
$$\hat{j}_B = Q_{21}\hat{i}_A + Q_{22}\hat{j}_A + Q_{23}\hat{k}_A$$
$$\hat{k}_B = Q_{31}\hat{i}_A + Q_{32}\hat{j}_A + Q_{33}\hat{k}_A$$

•…we can then express the vector in frame B; i.e. “transform” it from frame A to frame B

•Note that

$$\hat{i}_B \cdot \hat{i}_A = Q_{11},\ \ \hat{i}_B \cdot \hat{j}_A = Q_{12},\ \ etc$$

# Coordinate Transformation

•Suppose we have an arbitrary vector expressed in the coordinates of frame A, & we write it as a 3x1 column vector:

$$\bar{v}|_A = \begin{bmatrix} v_{1A} \\ v_{2A} \\ v_{3A} \end{bmatrix}$$

•Suppose we also write the expression of the vector in the coordinates of frame B as a 3x1 column vector:

$$\bar{v}|_B = \begin{bmatrix} v_{1B} \\ v_{2B} \\ v_{3B} \end{bmatrix}$$

•We can then write $\bar{v}|_B = Q\bar{v}|_A$

# Coordinate Transformation

- Let ***Q*** be the matrix that relates the unit vectors in an ***i',j',k'*** frame to the unit vectors in an ***i,j,k*** frame:

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

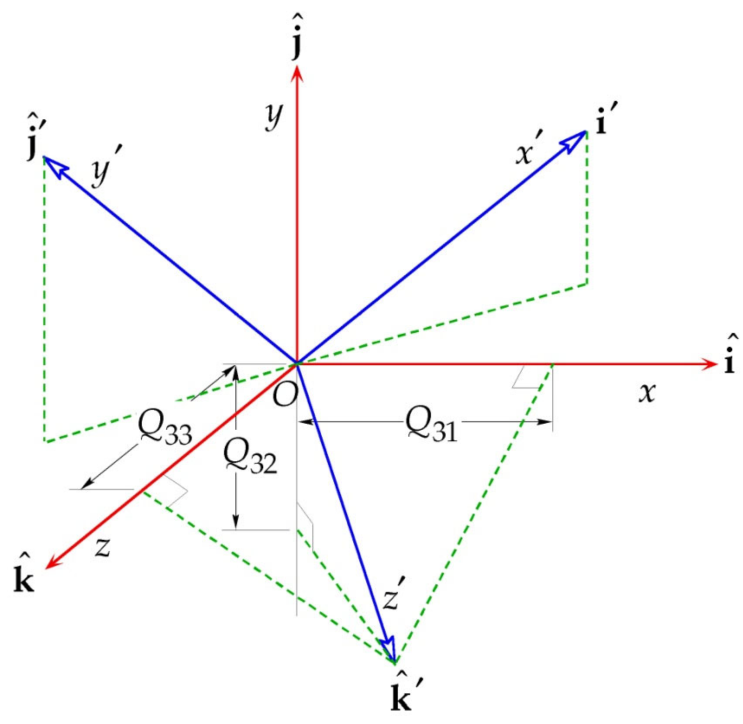
The rows of [**Q**] comprise the direction cosines of the *x'y'z'* axes along *xyz:*

$$Q_{ij} = \hat{\mathbf{u}}'_i \cdot \hat{\mathbf{u}}_j \qquad i \;\&\; j = 1,2,3$$

$$\hat{\mathbf{u}}_1 = \hat{\mathbf{i}} \qquad \hat{\mathbf{u}}_2 = \hat{\mathbf{j}} \qquad \hat{\mathbf{u}}_3 = \hat{\mathbf{k}}$$

$$\hat{\mathbf{u}}'_1 = \hat{\mathbf{i}}' \qquad \hat{\mathbf{u}}'_2 = \hat{\mathbf{j}}' \qquad \hat{\mathbf{u}}'_3 = \hat{\mathbf{k}}'$$

- Thus $Q_{11} = \hat{i}' \cdot \hat{i},\; Q_{12} = \hat{i}' \cdot \hat{j},\; etc.$




**A coordinate transformation matrix can also be called a direction cosine matrix (DCM)**

# Coordinate Transformation

•The matrix relating the unit vectors in the ***i,j,k*** frame to the unit vectors in the ***i',j',k'*** frame (call it *Q'*) is the inverse of the *Q* matrix:

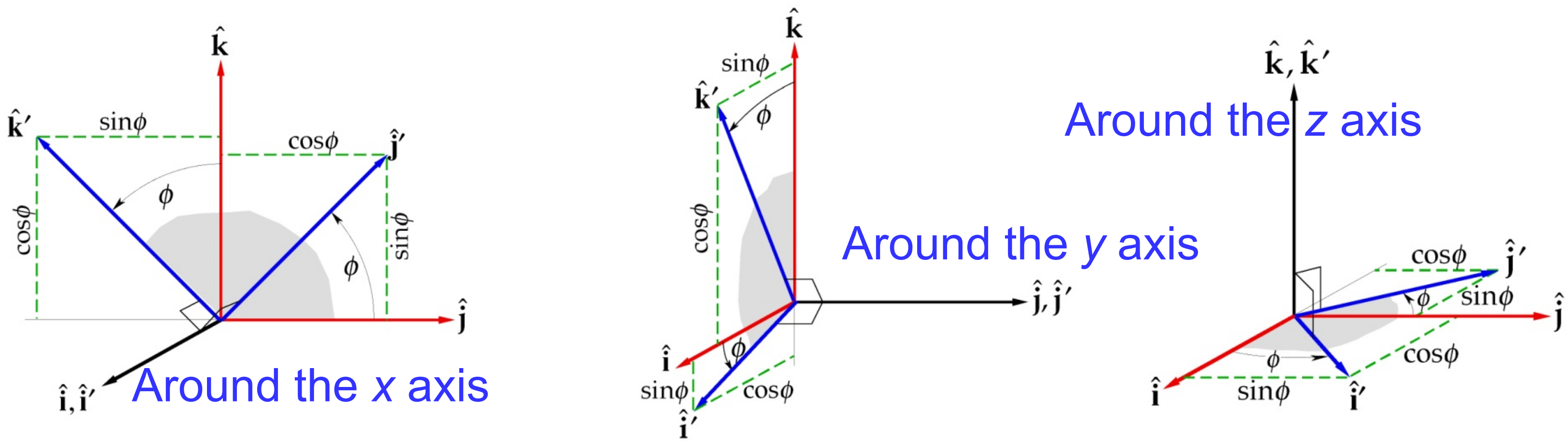
•A DCM is orthogonal, i.e. its transpose is its inverse: $Q^T = Q^{-1}$; so *Q'* is also the transpose of the *Q* matrix

•Therefore $\bar{v}' = Q\bar{v}$ and $\bar{v} = Q^T\bar{v}'$

•Remember that $\bar{v}$ and $\bar{v}'$ are the same vector, just expressed in different frames

# Coordinate Transformation

•Consider the case of a transformation involving a single rotation about one of the coordinate axes (x, y, or z); this is called an elementary rotation of the Cartesian frame:




•Designate each DCM as $R_1(\phi)$, $R_2(\phi)$, $R_3(\phi)$, where the subscript "1" signifies rotation about the x axis, "2" about y, & "3" about z, & $\phi$ signifies rotation angle

# Coordinate Transformation

•Using the $Q_{ij} = \hat{\mathbf{u}}'_i \cdot \hat{\mathbf{u}}_j$ formula, we can calculate the DCMs in each case

•For a rotation about the x axis:

$$\hat{i}' = \hat{i} = 1\hat{i} + 0\hat{j} + 0\hat{k}$$
$$\hat{j}' = \left(\hat{j}' \cdot \hat{i}\right)\hat{i} + \left(\hat{j}' \cdot \hat{j}\right)\hat{j} + \left(\hat{j}' \cdot \hat{k}\right)\hat{k} = 0\hat{i} + \cos\phi\hat{j} + \sin\phi\hat{k}$$
$$\hat{k}' = \left(\hat{k}' \cdot \hat{i}\right)\hat{i} + \left(\hat{k}' \cdot \hat{j}\right)\hat{j} + \left(\hat{k}' \cdot \hat{k}\right)\hat{k} = 0\hat{i} - \sin\phi\hat{j} + \cos\phi\hat{k}$$

•Thus: $$\left[\mathbf{R}_1(\phi)\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

•Likewise: $$\left[\mathbf{R}_2(\phi)\right] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \quad \left[\mathbf{R}_3(\phi)\right] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Coordinate Transformation

•The transformation between two coordinate frames of arbitrary orientation can be represented as a sequence of elementary (x, y, or z-axis) rotations

•This is called an Euler angle sequence (or Euler rotation sequence)

•In such a case, the $\boldsymbol{R_i}$ matrices in the sequence are multiplied together to comprise the overall transformation matrix (or DCM) that we have labeled $\boldsymbol{Q}$

•For example, suppose we rotate a frame XYZ 20° about its x axis to obtain an intermediate frame, then rotate the intermediate frame 30° about its y axis to obtain a frame X'Y'Z'

•We can then write $\bar{v}|_I = R_1(20°)\bar{v}|_{XYZ}$ and $\bar{v}|_{X'Y'Z'} = R_2(30°)\bar{v}|_I$

& therefore $\bar{v}|_{X'Y'Z'} = R_2(30°)R_1(20°)\bar{v}|_{XYZ}$

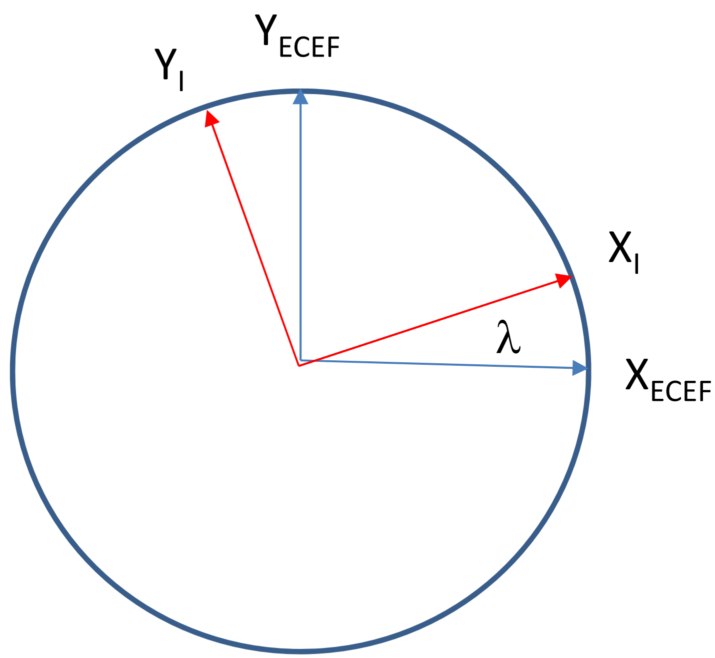
where the elements of each $\boldsymbol{R_i}$ matrix are as given on the previous slide

# Coordinate Transformation

•We can use these concepts to calculate the DCM between the ECEF Frame & the TH Frame (assume spherical Earth)

•Consider a ground sensor located at longitude $\lambda$ & latitude L

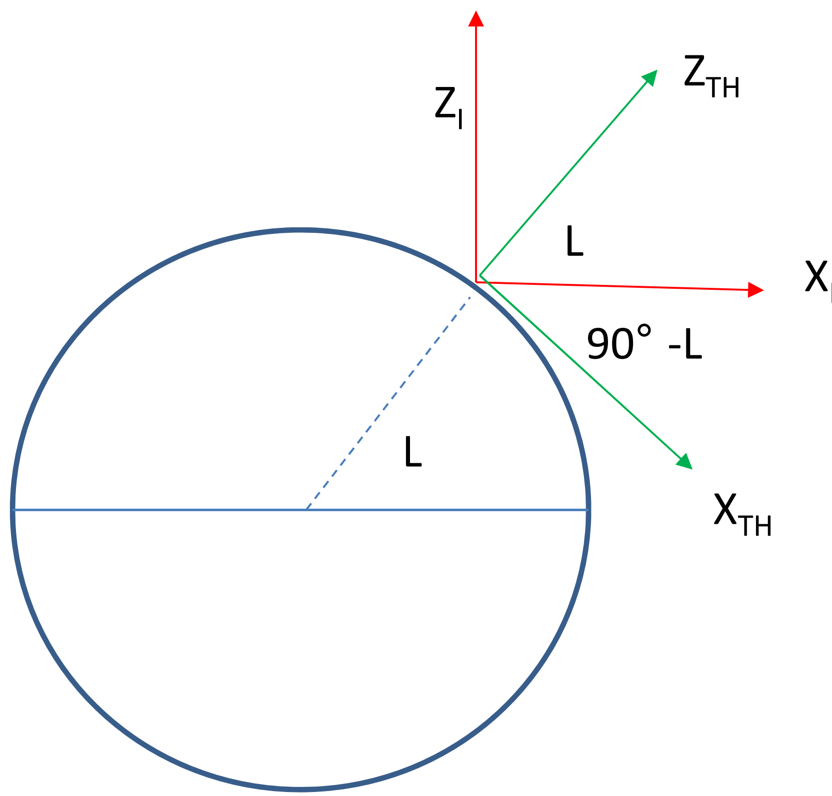
•First, rotate the ECEF Frame about its z axis by $\lambda$ to form an intermediate frame



# Coordinate Transformation

•Next, move the origin of the intermediate frame from Earth's center to the sensor location at (L, $\lambda$)

•Then rotate the intermediate frame about its y axis by (90° – L) to achieve the ground sensor's TH (South/East/Up) frame



# Coordinate Transformation

- Rotation from ECEF Frame to intermediate frame represented by $R_3(\lambda)$

- Rotation from intermediate Frame to TH frame represented by $R_2(90°\text{-}L)$

- Therefore

$$Q_{ECEF}^{TH} = R_2(90° - \mathrm{L})R_3(\lambda) =$$

$$\begin{bmatrix} cos(90° - \mathrm{L}) & 0 & -sin(90° - \mathrm{L}) \\ 0 & 1 & 0 \\ sin(90° - \mathrm{L}) & 0 & cos(90° - \mathrm{L}) \end{bmatrix} \begin{bmatrix} cos\lambda & sin\lambda & 0 \\ -sin\lambda & cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} sinL & 0 & -cosL \\ 0 & 1 & 0 \\ cosL & 0 & sinL \end{bmatrix} \begin{bmatrix} cos\lambda & sin\lambda & 0 \\ -sin\lambda & cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} sinLcos\lambda & sinLsin\lambda & -cosL \\ -sin\lambda & cos\lambda & 0 \\ cosLcos\lambda & cosLsin\lambda & sinL \end{bmatrix}$$

- So $\bar{v}|_{TH} = Q_{ECEF}^{TH}\bar{v}|_{ECEF}$ and $\bar{v}|_{ECEF} = \left(Q_{ECEF}^{TH}\right)^T \bar{v}|_{TH}$

# Coordinate Transformation

- Recall the ECI & ECEF Frames are related by a z axis rotation through an angle of $\theta_g$ (Greenwich sidereal time)

- So the DCM between the ECI Frame & the TH Frame is expressed by the same formula as on the previous slide, but replacing $\lambda$ with $\theta = \lambda + \theta_g$

- This agrees with the derivation of Sec 2.8.4 of Bate, Mueller, & White textbook

# Coordinate Transformation

EXAMPLE PROBLEM:

A sensor is located at sea level with coordinates of L = 20°N, $\lambda$ = 35°E. At an instant in time, a space object's position vector from Earth's center expressed in ECI coordinates is (5294.35, 3707.14, 2352.42) km. What is the vector from the sensor to the object, expressed in the sensor's TH (South/East/Up) coordinates? Assume spherical Earth of radius 6378km & $\theta_g$ at this instant = 0.

# Coordinate Transformation

SOLUTION:

First, calculate vector from Earth's center to the sensor in ECI coordinates:

- We know this vector in TH coordinates is $\begin{bmatrix} 0 \\ 0 \\ R_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6378 \end{bmatrix} km$
- So, sensor location in ECEF coordinates is

$$\bar{r}_{sensor}|_{ECEF} = \left(Q_{ECEF}^{TH}\right)^T \bar{r}_{sensor}|_{TH} = \begin{bmatrix} sinLcos\lambda & sinLsin\lambda & -cosL \\ -sin\lambda & cos\lambda & 0 \\ cosLcos\lambda & cosLsin\lambda & sinL \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 6378 \end{bmatrix}$$

$$= \begin{bmatrix} 6378 * \cos(20°) * \cos(35°) \\ 6378 * \cos(20°) * \sin(35°) \\ 6378 * \sin(20°) \end{bmatrix} = \begin{bmatrix} 4909.47 \\ 3437.65 \\ 2181.40 \end{bmatrix} km$$

- Because $\theta_g$ = 0 at this instant, ECI & ECEF Frames are aligned
- So $\bar{r}_{sensor}|_{ECI} = \bar{r}_{sensor}|_{ECEF} = \begin{bmatrix} 4909.47 \\ 3437.65 \\ 2181.40 \end{bmatrix} km$

# Coordinate Transformation

SOLUTION (cont'd):

- So, position vector from sensor to object in ECI coordinates is:

$$\bar{r}_{so}|_{ECI} = \bar{r}_{object}|_{ECI} - \bar{r}_{sensor}|_{ECI} = \begin{bmatrix} 5294.35 - 4909.47 \\ 3707.14 - 3437.65 \\ 2352.42 - 2181.40 \end{bmatrix} km = \begin{bmatrix} 384.88 \\ 269.49 \\ 171.01 \end{bmatrix} km = \bar{r}_{so}|_{ECEF}$$

because ECI & ECEF are aligned

- Therefore, the vector from the sensor to the object, expressed in the sensor's TH (South/East/Up) coordinates, is:

$$\bar{r}_{so}|_{TH} = Q_{ECEF}^{TH} \bar{r}_{so}|_{ECEF}$$

$$= \begin{bmatrix} sin(20°)cos(35°) & sin(20°)sin(35°) & -cos(20°) \\ -sin(35°) & cos(35°) & 0 \\ cos(20°)cos(35°) & cos(20°)sin(35°) & sin(20°) \end{bmatrix} \begin{bmatrix} 384.88 \\ 269.49 \\ 171.01 \end{bmatrix} km = \begin{bmatrix} 0 \\ 0 \\ 500 \end{bmatrix} km$$

- So at this instant, the object is directly overhead of the sensor (i.e. zenith)