Appendix A. Matrix Factorizations (p. 600 − 603)

Permutation matrices

Consider $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$ swaps x_1 & x_2

Now, for $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} z \\ x \\ y \end{bmatrix}$, we

will have: $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$

grab z for y.   get x.

Thus, for permutation matrices, we permute the rows of the identity matrix to the desired order. We use P to identify permutation matrices

Triangular Matrices.

Consider $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ ~ $(*)$

"non-zero triangle"

$\Rightarrow \quad x = b_1, \quad x + 2y = b_2 \Rightarrow y = \frac{1}{2}(b_2 - b_1),$

$x + 2y + 3z = b_3 \Rightarrow z = b_3 - 2y - x = \cdots$

(*) is an example of a system of form:

$$Lx = b,$$

where L is a <u>lower-triangular</u> matrix, one for which $Lx=b$ is particularly easy to solve.

<u>Similarly</u>, for

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \Rightarrow 6z = 3 \Rightarrow \boxed{z = 0.5}$$
$$4y + 5z = 2 \Rightarrow y = \dots$$

"triangle" are thus also easy to solve.

This an example of an upper triangular system.

<u>Factorizations</u>

The LU factorization is $PA = LU$.

To solve $Ax=b$, multiply by P on both sides:

$$PAx = Pb$$
$$\Rightarrow LUx = Pb$$
$$\Rightarrow Lz = Pb, \text{ for } Ux = z.$$

$\Rightarrow$ Soln is: (I) Solve $Lz=Pb$ for z, (II) Solve $Ux=z$ for x.

The new A is simply:

$$\begin{bmatrix} 6 & 7 & 8 \\ 0 & 4 & 5 \\ 0 & 5/6 & 5/3 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1/6 & 0 & 0 \end{bmatrix}$$

A_{31}/A_{11}

keep this lower-triangular.

* repeat with the second column, <u>ignoring</u> the first entry, since we are done with it.

Largest value is 4. It is in the second row, as we desire. No need to switch rows.

* To make A upper-triangular, we must zero out all the entries below the 4:

$$-\frac{(5/6)}{4}\begin{bmatrix} 0 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/6 & 5/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 15/24 \end{bmatrix}$$

desired zero here

* Then set L by:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/6 & 5/24 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 6 & 7 & 8 \\ 0 & 4 & 5 \\ 0 & 0 & 15/24 \end{bmatrix}$$

To compute $PA = LU$, we use algorithm A.1 to compute P, L, U from A.

<u>Basic idea:</u> ⇒ <u>Compute $PA=LU$ by converting A to</u>

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

Start with: $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $L = 0_{3\times3}$ matrix.

* Start with the first column of A, and find the largest element: 6.
* swap with the first row for both A & P

correct your book!

$A = \begin{bmatrix} 6 & 7 & 8 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

* Zero-out all the entries below:
  multiply the first row by A_{31}/A_{11} and subtract it from the third:

$$-[6 \quad 7 \quad 8] \cdot \left(\tfrac{1}{6}\right) + [1 \quad 2 \quad 3] = [0 \quad 5/6 \quad 5/3]$$

zeros out this entry

* Final output is simply:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/6 & 5/24 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 6 & 7 & 8 \\ 0 & 4 & 5 \\ 0 & 0 & 15/24 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The Gaussian elimination algorithm is given as A.1, but fix the 'swap step' to:

$\Rightarrow$ swap rows i and j of matrices A and P. *

The algorithm becomes significantly simpler if A is positive definite.

We have $A = LL^T$ or $\begin{pmatrix} P = I, U = L^T \\ \text{in } PA = LU \end{pmatrix}$.

Algorithm 4.2 (Cholesky factorization)

Given $A \in \mathbb{R}^{n\times n}$ symmetric positive definite:

for $i = 1, 2, \ldots, n$

$\quad L_{ii} = \sqrt{A_{ii}}$

$\quad$ **for** $j = i+1, i+2, \ldots, n$

$\quad\quad L_{ji} = A_{ji} / L_{ii}$ ← multiplier

$\quad\quad$ **for** $k = i+1, i+2, \ldots, j$ ← note the simplifications due to symmetry of A.

$\quad\quad\quad A_{jk} = A_{jk} - L_{ji} L_{ki}$

$\quad\quad$ **end**

$\quad$ **end**

end

To solve $Ax = b$:

$\Rightarrow LL^T x = b$ or $\begin{cases} \underline{\text{Solve}} \\ Lz = b \text{ for } z. \\ z = L^T x \text{ for } x. \end{cases}$