Chapter 13: Linear Programming: The Simplex Method

Linear Programming

In <u>standard form</u>:

$$\min c^T x, \text{ subject to } Ax = b,\ x \geq 0. \quad (*)$$

where:

c, x are vectors in $\mathbb{R}^n$,

b is a vector in $\mathbb{R}^m$,

A is an $m \times n$ matrix.

Transforming to standard form:

Ex 1 Transform $\min c^T x$, subject to $Ax \geq b$ to standard form.

<u>step 1.</u> Rewrite using extra variables z:

$$\min c^T x, \text{ subject to: } Ax - z = b,\ z \geq 0.$$

But x can be positive or negative. This is addressed in step 2.

<u>step 2.</u> Define x by:

$$x = x^+ - x^-, \quad x^+ \geq 0, \quad x^- \geq 0$$

$$x^+ = \max(x, 0), \quad x^- = \max(-x, 0)$$

<u>step 3.</u>

$$\min \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}^T \begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} \text{ such that: } [A\ -A\ -I] \begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} = b$$

for z!

<u>and</u> $\begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} \geq 0.$

Notes:

① $c^T x = c^T(x^+ - x^-)$
$= c^T x^+ + (-c)^T x^-$

② $Ax - z = b \Rightarrow A(x^+ - x^-) - z = b$
or $Ax^+ - Ax^- - z = b$

$$[A \;\; -A \;\; -I]\begin{bmatrix} x^+ \\ x^- \\ z \end{bmatrix} = b.$$

Ex 2-1

For $x \leqslant u$, use:

$x + \omega = u, \quad \omega \geqslant 0.$

Use $x = x^+ - x^-$:

$x^+ - x^- + \omega = u$ (*)

Return to the standard form:

① $c^T x = c^T x^+ + (-c)^T x^-$

so $c^T x$ is now:

$$\min \; [c \;\; -c \;\; 0]\begin{bmatrix} x^+ \\ x^- \\ w \end{bmatrix}$$

② $$[A \;\; -A \;\; I \;\; -I \;\; I]\begin{bmatrix} x^+ \\ x^- \\ w \end{bmatrix} = \begin{bmatrix} b \\ u \end{bmatrix}, \quad \begin{bmatrix} x^+ \\ x^- \\ w \end{bmatrix} \geqslant 0.$$

Note: Once the solutions x^+, x^- have been computed, then:

$$x = x^+ - x^-.$$

Ex 2-2 $Ax \leq b$.

$\Rightarrow Ax + y = b, \quad y \geq 0$

$$\Rightarrow \begin{cases} \min & [c \;\; -c \;\; 0] \begin{bmatrix} x^+ \\ x^- \\ y \end{bmatrix}, \\ & [A \;\; -A \;\; I] \begin{bmatrix} x^+ \\ x^- \\ y \end{bmatrix} = b, \quad \begin{bmatrix} x^+ \\ x^- \\ y \end{bmatrix} \geq 0. \end{cases}$$

Ex 2-3 To do: $\max c^T x$,

use $\min (-c^T) x$.

Assume $m < n$ in $\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{n\times 1}{b}$

So that we have fewer equations than the number of unknowns.

Otherwise, eg for $Ax = b$, A $n\times n$, we could simply compute $x = A^{-1}b$, or there may not exist a solution.

13.1 Optimality and Duality

To form the Lagrangian, we note that we need: $Ax=b$, $x \geq 0$, multipliers: π, s:

Then: $\mathcal{L}(x,\pi,s) = c^T x - \pi^T(Ax-b) - s^T x$.

$$\mathcal{L}_x(x,\pi,s) = c^T - \pi^T A - s^T = 0$$

$$\Rightarrow \quad c^T = \pi^T A + s^T$$

$$\Rightarrow \begin{cases} c = A^T\pi + s \\ Ax = b \\ x \geq 0, \quad s \geq 0 \\ x_i s_i = 0, \quad i = 1, 2, \dots, n \end{cases}$$

$s \geq 0$ ← for inequality constraint.

s_i ← inequality constraint multiplier:

① x is zero for x on boundary, or

② s_i is zero for solution inside.

<u>Clearly</u>: $x^T s = 0$ also

Since $x_i, s_i \geq 0$, <u>then</u> $x^T s = 0 \Rightarrow x_i s_i = 0$.

Also note that:

$$c^T x^* = (A^T\pi^* + s^*)^T x^*$$

From: $c = A^T\pi^* + s^*$

$$= \left[(\pi^*)^T A + (s^*)^T\right] x$$

$$= (\pi^*)^T A x + \underbrace{(s^*)^T x}_{0}$$

$$= (\pi^*)^T A x$$

Note that this is a number.

So $\left[(\pi^*)^T A x\right]^T = x^T A^T (\pi^*)$

$$= (Ax)^T \pi^*$$

We thus get: $c^T x^* = \underbrace{(Ax^*)^T}_{Ax^* = b} \pi^* = \underline{b^T \pi^*}$

The dual Problem

$$\max \; b^T\pi \quad \text{subject to} \quad A^T\pi \leq c \quad (**)$$

which is the same as:

$$\min \; -b^T\pi \quad \text{subject to} \quad c - A^T\pi \geq 0.$$

to match our Lagrangian formulation.

Form the Lagrangian:

$$\bar{\mathcal{L}}(\pi, x) = \underbrace{-b^T \pi}_{\substack{\text{what we are}\\\text{trying to}\\\text{minimize}}} - x^T \underbrace{(c - A^T \pi)}_{\substack{\text{inequality}\\\text{constraint.}}}$$

Then:

$$\bar{\mathcal{L}}_\pi(\pi, x) = -b^T + x^T A^T = 0$$

$$\Rightarrow \begin{cases} Ax = b, \quad A^T\pi \leq c, \quad x \geq 0 \\ x_i (c - A^T\pi)_i = 0, \quad i = 1, 2, \ldots, n \end{cases}$$

which can be matched to the primal problem using $s = c - A^T \pi$ (see book).

<u>Thm 13.1</u> (Duality Thm of LP)

* if either the primal (*) or the dual problem (**) has a solution with finite optimal objective value, then so does the other, and the objective values are equal.
* if either problem has an unbounded objective, then the other problem has no feasible points.

Note that the Lagrange multipliers (π, s) for an optimal x^* is also called <u>sensitivity analysis</u>.

13.2 Geometry of the Feasible Set

<u>Assume:</u>

A has full row rank.

The assumption is that the rows are independent.

Suppose:

* x is <u>a feasible point</u>
* x has <u>at-most</u> m non-zero components (recall that A is $m \times n$).
* $\exists\, B(x)$, an index set of $\{1, 2, \ldots, n\}$ such that:
  - → $B(x)$ contains exactly m indices
  - → $i \notin B(x) \Rightarrow x_i = 0$
  - → the $m \times m$ matrix:

$$B = [A_i]_{i \in B(x)}, \quad A_i \text{ is the } i\text{-th column of } A$$

is non-singular.

Then x is called a <u>basic feasible point</u>.

The simplex method generates iterates x^k that are basic feasible points.

Thm 13.2 Fundamental Thm of LP

* if there is a feasible point, then there is a basic feasible point.
* if we have solutions, then at-least one of them is a basic optimal point.
* If the problem is feasible and bounded, then it has an optimal solution.

Thm 13.3 All basic feasible points are vertices of the feasible polytope:

$$\{ x \mid Ax = b, \ x \geq 0 \}.$$

and vice-versa.