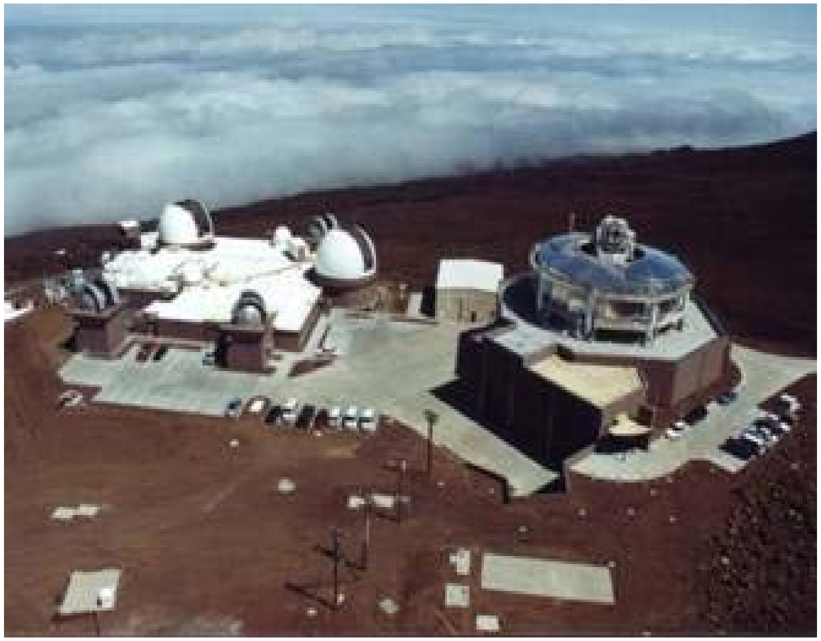
Orbit Determination I

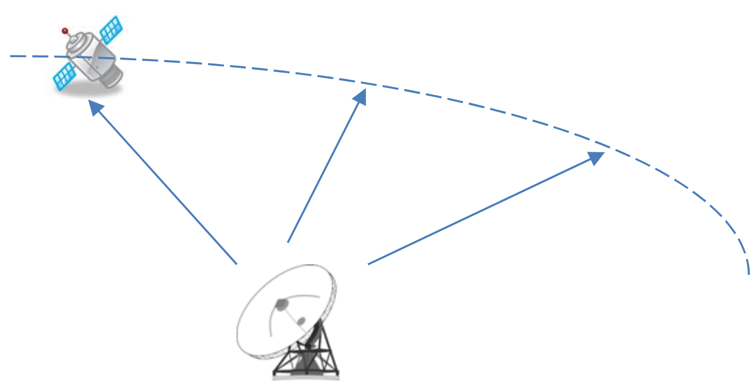


What is Orbit Determination?

- Now that we understand the process of sensing space objects as part of a passive tracking campaign, we'll focus on orbit determination

- In short, orbit determination (OD) involves using observation data of a space object obtained from one or more sensors to accurately calculate & predict the object's orbit

- OD draws on a branch of mathematics known as state estimation



What is State Estimation?

- State estimation is a technique whereby the current and past behavior of a dynamic system is discerned, based on observations of measurable quantities related to the system, in order that its future behavior may be predicted

- The dynamics of a system (i.e. its behavior over time) is often represented by a vector of "states"
  - Example would be a particle in 3-D motion → states consist of its position components ($[x\ y\ z]^T$) & velocity components ($[v_x\ v_y\ v_z]^T$)
  - Together these 6 qty's form the state vector

- While a particle is an idealization, many practical system are modeled in this way, e.g. aircraft, ships, missiles, & space objects

What is State Estimation?

- Change in the state vector with time is often described by a set of ordinary differential equations (ODEs) known as "state equations":

$$\dot{\overline{\mathrm{X}}}(\mathrm{t}) = \overline{\mathrm{F}}(\overline{\mathrm{X}}(\mathrm{t}))$$

- Both vectors in the above equation are of dimension n x 1, where n is the number of states

- In the case where the states consist of an object's position & velocity, $n = 6 \rightarrow \overline{\mathrm{F}}$ is the vector of "state rates":

- The system's motion can be discerned by propagating (i.e. numerically integrating forward in time) the ODEs from some initial conditions $\bar{X}_0$

- So how can the initial conditions be determined in a given scenario?

What is State Estimation?

- Unfortunately, in the case of most systems, their states can't be measured directly

- Qty's that can be measured, which are functions of the states, are often called "measurements," "outputs," or "observations"

- The functional relationship between measurements & states are usually governed by static (non-ODE) equations known as "measurement equations":

$$\overline{\mathrm{Y}}(\mathrm{t}) = \overline{\mathrm{G}}(\overline{\mathrm{X}}(\mathrm{t}))$$

$\bar{Y}$ and $\bar{G}$ are p x 1 vectors, where p is the total # of measurements

What is State Estimation?

- A good example of a measurement-state relationship is a Pitot-static tube on an aircraft

- Total pressure in the atmosphere is the sum of static & dynamic pressure:

$$p_t = p + \frac{1}{2}\rho v^2$$

- Aircraft velocity can't be measured directly, but a Pitot-static tube measures static & total pressure

- If air density is also known, velocity can be solved for:

$$v = \sqrt{\frac{2}{\rho}(p_t - p)}$$

Applications to Orbit Determination

- Let's extend these concepts of states & measurements to orbital mechanics of space objects

- A practical state vector for an Earth-orbiting space object is:
  - The object's position vector from Earth's center, expressed in ECI coordinates (denote this $[x\ y\ z]^T$)
  - The object's inertial velocity vector, also expressed in ECI coordinates (denote this $[v_x\ v_y\ v_z]^T$)

- For such objects, the 2-body ODEs of motion are well known:

$$\ddot{r} = -\frac{\mu\bar{r}}{r^3}$$

- where μ is Earth's gravitational constant & r is the object's instantaneous orbit radius (i.e. magnitude of its position vector from Earth's center): $r = \sqrt{x^2 + y^2 + z^2}$

Applications to Orbit Determination

- Assuming 2-body motion, the state equations for an Earth-orbiting space object can then be written as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -\frac{\mu x}{r^3} \\ -\frac{\mu y}{r^3} \\ -\frac{\mu z}{r^3} \end{bmatrix}$$

$\dot{\overline{\mathrm{X}}}(\mathrm{t})$ $\quad$ $\overline{\mathrm{F}}(\overline{\mathrm{X}}(\mathrm{t}))$

Applications to Orbit Determination

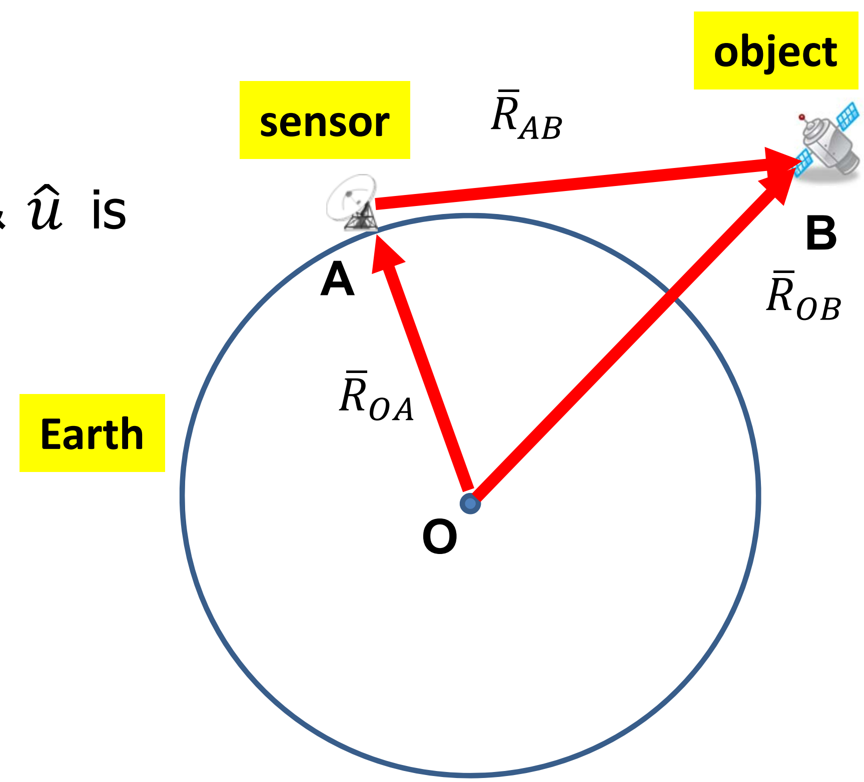
- Let us now derive the measurement equations

- Recall last week we examined sensor-to-object range & sensor-to-object line-of-sight (see sketch below)

- Let us write the vector $\bar{R}_{AB}$ as $\rho\hat{u}$

where ρ is sensor-to-object range & $\hat{u}$ is sensor-to-object LOS:

$$\begin{bmatrix} u_{xTH} \\ u_{yTH} \\ u_{zTH} \end{bmatrix} = \begin{bmatrix} cosElcosAz \\ cosElsinAz \\ sinEl \end{bmatrix}$$



Applications to Orbit Determination

- We can then rewrite the vector equation $\bar{R}_{OB} = \bar{R}_{OA} + \bar{R}_{AB}$ as

$$\bar{r} = \bar{R}_{OA} + \rho\hat{u} \text{ OR } \bar{r} - \bar{R}_{OA} = \rho\hat{u}$$

- Where $\bar{r}$ is the object's position vector from Earth's center (recall this comprises the first 3 states of the object's state vector)

- Typically $\bar{r}$ is expressed in ECI coordinates

- Note that, if the sensor's lat/long location & sidereal time are known, $\bar{R}_{OA}$ and $\hat{u}$ can also be expressed in ECI coordinates (using coordinate transformations we've learned); write their ECI expressions as

$$\bar{R}_{OA} = \begin{bmatrix} R_{OAx} \\ R_{OAy} \\ R_{OAz} \end{bmatrix} \qquad \hat{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

Applications to Orbit Determination

- Therefore, ρ is the magnitude of $\bar{R}_{AB}$:

$$\rho = |\bar{r} - \bar{R}_{OA}| = \sqrt{(x - R_{OAx})^2 + \left(y - R_{OAy}\right)^2 + (z - R_{OAz})^2}$$

- And $\hat{u}$ is the unit vector in the direction of $\bar{R}_{AB}$:

$$\hat{u} = \frac{\bar{r} - \bar{R}_{OA}}{|\bar{r} - \bar{R}_{OA}|} \Rightarrow \begin{aligned} u_x &= \frac{x - R_{OAx}}{\sqrt{(x - R_{OAx})^2 + \left(y - R_{OAy}\right)^2 + (z - R_{OAz})^2}} \\ u_y &= \frac{y - R_{OAy}}{\sqrt{(x - R_{OAx})^2 + \left(y - R_{OAy}\right)^2 + (z - R_{OAz})^2}} \end{aligned}$$

- Note that, because LOS is a unit vector, if any 2 of the 3 components (u_x, u_y, u_z) are known, the 3rd component is automatically known
  - Thus, we only obtain 2 (not 3) independent equations from a LOS measurement
  - Any 2 of the 3 component equations will suffice; here we arbitrarily choose the u_x & u_y equations

Applications to Orbit Determination

- Thus, we obtain one measurement equation for each radar measurement taken:

$$\rho = \sqrt{(x - R_{OAx})^2 + \left(y - R_{OAy}\right)^2 + (z - R_{OAz})^2}$$

y(t)

$g(\bar{X}(t))$

- & two measurement equations for each optical sensor measurement taken:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \dfrac{x - R_{OAx}}{\sqrt{(x - R_{OAx})^2 + \left(y - R_{OAy}\right)^2 + (z - R_{OAz})^2}} \\ \dfrac{y - R_{OAy}}{\sqrt{(x - R_{OAx})^2 + \left(y - R_{OAy}\right)^2 + (z - R_{OAz})^2}} \end{bmatrix}$$

$\overline{\mathrm{Y}}(\mathrm{t})$

$\overline{\mathrm{G}}(\overline{\mathrm{X}}(\mathrm{t}))$

Applications to Orbit Determination

- Note that when we display the one range measurement equation, y is a scalar qty & g is a scalar function
- Whereas, when we display the two LOS measurement equations, Y & G are vectors
- What is the actual dimension of Y & G , & will they contain range equations, LOS equations, or both?
  - The answer depends on the total # and types of measurements to be processed
  - Some OD scenarios involve data from a single sensor, or multiple sensors of the same type (all RF or all optical) → these are "range-only" or "angles-only" scenarios
  - Other scenarios involve data from multiple sensors of both types (RF & optical) → "range & angle" scenarios
  - This will become important later on

Applications to Orbit Determination

- We now have the fundamental building blocks for orbit determination: a model of the space object's dynamics & the measurement-state relationships

- OD is typically a 2-step process
  - Initial orbit determination (IOD) involves obtaining a decent (& fairly quick) solution from a minimal # of measurements
  - Precise orbit determination (POD) involves processing (usually) a large # of measurements, using an IOD solution as an initial guess (or "starter" solution)

- How do these 2 steps differ?
  - IOD normally considered a "deterministic" process, assuming no measurement or modeling error & simply fitting the measurements to the assumed dynamics
  - Whereas POD has the ability to accommodate error sources (e.g. sensor & modeling error) to find a statistical "best guess" of the object's orbit

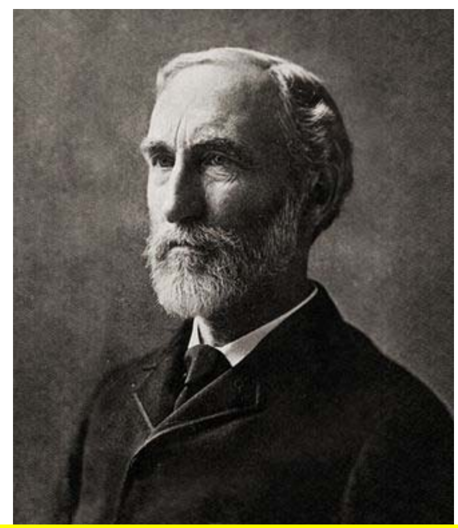
This week we will focus on IOD

Initial Orbit Determination

- IOD has a centuries-long history
- Early mathematicians & astronomers devised IOD methods to determine orbits of the Moon & deep space objects (e.g. planets, comets, asteroids)
- Based primarily on naked-eye observations (or with rudimentary telescopes)

Pierre-Simon Laplace

Carl Friedrich Gauss



Josiah Willard Gibbs

Initial Orbit Determination

- Since the dawn of the Space Age (Sputnik, 1957), IOD has become increasingly important for SSA

- Basically, IOD is necessary whenever a new object is detected
  - New launch of a satellite
  - Debris created from a breakup/collision event
  - Occasionally an object may be "lost" from the catalog, then "return"

- Some IOD methods based only on range measurements ("range-only"), while others based only on LOS measurements ("angles-only")
  - A few methods require both range & angles

Initial Orbit Determination

- Range-only vs angles-only IOD? Here are some considerations:
  - Radar sensors typically have sufficient signal power to sense objects in LEO, but very few can “see” beyond LEO
  - Telescopes can see celestial bodies in deep space (of course those bodies are massively large), but also mid- to large-size Earth-orbiting objects out to GEO
  - Telescopes require slewing (rotating) to track any non-GEO object as it moves across the sky; to track an object whose pass is particularly fast (e.g. LEO) may require unreasonably large slew rates
  - Result, in terms of SSA, is that low-altitude objects (e.g. LEO) tend to be sensed via radar, & high-altitude objects (e.g. GEO) tend to be sensed optically

- In this course, we will focus on angles-only OD

- We will now explore Laplace’s angles-only IOD method, which he developed in 1780

Laplace IOD

- Requires 3 LOS measurements
  - Makes sense because each LOS measurement yields 2 equations → 3 measurements yield 6 equations to solve 6 unknowns (object's position & velocity states)

- Begin by taking the 1st & 2nd time derivatives of $\bar{r} = \bar{R}_{OA} + \rho\hat{u}$ to obtain:

$$\dot{\bar{r}} = \dot{\bar{R}}_{OA} + \dot{\rho}\hat{u} + \rho\dot{\hat{u}}$$

$$\ddot{\bar{r}} = \ddot{\bar{R}}_{OA} + \ddot{\rho}\hat{u} + 2\dot{\rho}\dot{\hat{u}} + \rho\,\ddot{\hat{u}}$$

- Assuming 2-body motion, we can substitute $\ddot{\bar{r}} = -\frac{\mu\bar{r}}{r^3}$ into the left-hand side:

$$-\frac{\mu\bar{r}}{r^3} = \ddot{\bar{R}}_{OA} + \ddot{\rho}\hat{u} + 2\dot{\rho}\dot{\hat{u}} + \rho\,\ddot{\hat{u}}$$

Laplace IOD

- Now substituting $\bar{r} = \bar{R}_{OA} + \rho\hat{u}$ into the left-hand side:

$$-\frac{\mu(\bar{R}_{OA}+\rho\hat{u})}{r^3} = \ddot{\bar{R}}_{OA} + \ddot{\rho}\hat{u} + 2\dot{\rho}\dot{\hat{u}} + \rho\,\ddot{\hat{u}}$$

- Collecting the terms involving ρ & its derivatives on one side:

$$\ddot{\rho}\hat{u} + 2\dot{\rho}\dot{\hat{u}} + \rho\left(\ddot{\hat{u}} + \frac{\mu}{r^3}\hat{u}\right) = -\ddot{\bar{R}}_{OA} - \frac{\mu\bar{R}_{OA}}{r^3}$$

- These derived relationships are true at any time; but at which specific time(s) are we interested in evaluating the relationships?

- We will decide this later; for now, we will simply refer to this specific time as "$t*$"

Laplace IOD

- Let us assume for now that ρ & its derivatives are the only unknowns at t^*, & all other qty's are known

- We can then write the previous relationship in matrix form as:

$$\begin{bmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & \ddot{\hat{u}} + \frac{\mu}{r^3}\hat{u} \\ | & | & | \end{bmatrix} \begin{bmatrix} \ddot{\rho} \\ \dot{\rho} \\ \rho \end{bmatrix} = \begin{bmatrix} | \\ -\ddot{\bar{R}}_{OA} - \frac{\mu \bar{R}_{OA}}{r^3} \\ | \end{bmatrix}$$

3x3 matrix **3x1 vector** **3x1 vector**

(Note that $\hat{u}$, $2\dot{\hat{u}}$, & $\ddot{\hat{u}} + \dfrac{\mu}{r^3}\hat{u}$ are each 3x1 vectors, so

concatenating them together produces a 3x3 matrix)

Laplace IOD

- We can use Cramer's rule to solve ρ: $\rho = \frac{D_2}{D_1}$

where D_1 is the determinant of
$$\begin{bmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & \ddot{\hat{u}} + \frac{\mu}{r^3}\hat{u} \\ | & | & | \end{bmatrix}$$

And D_2 is the determinant of
$$\begin{bmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & -\ddot{\bar{R}}_{OA} - \frac{\mu \bar{R}_{OA}}{r^3} \\ | & | & | \end{bmatrix}$$

Laplace IOD

- Note that D_1 can be simplified to

$$D_1 = \begin{vmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & \ddot{\hat{u}} + \frac{\mu}{r^3}\hat{u} \\ | & | & | \end{vmatrix} = \begin{vmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & \ddot{\hat{u}} \\ | & | & | \end{vmatrix} + \cancel{\begin{vmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & \frac{\mu}{r^3}\hat{u} \\ | & | & | \end{vmatrix}}$$

Here I use bars to denote the determinant

= 0 because Rows 1 & 3 are proportional

$$= \begin{vmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & \ddot{\hat{u}} \\ | & | & | \end{vmatrix} = 2\begin{vmatrix} | & | & | \\ \hat{u} & \dot{\hat{u}} & \ddot{\hat{u}} \\ | & | & | \end{vmatrix}$$

Laplace IOD

- And D_2 can be rewritten as

$$D_2 = \begin{vmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & -\ddot{\bar{R}}_{OA} - \dfrac{\mu \bar{R}_{OA}}{r^3} \\ | & | & | \end{vmatrix}$$

$$= \begin{vmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & -\ddot{\bar{R}}_{OA} \\ | & | & | \end{vmatrix} + \begin{vmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & -\dfrac{\mu \bar{R}_{OA}}{r^3} \\ | & | & | \end{vmatrix}$$

$$= -2 \begin{vmatrix} | & | & | \\ \hat{u} & \dot{\hat{u}} & \ddot{\bar{R}}_{OA} \\ | & | & | \end{vmatrix} - 2\frac{\mu}{r^3} \begin{vmatrix} | & | & | \\ \hat{u} & \dot{\hat{u}} & \bar{R}_{OA} \\ | & | & | \end{vmatrix}$$

Laplace IOD

- If we define

$$D_2^* = \begin{vmatrix} | & | & | \\ \hat{u} & \dot{\hat{u}} & \ddot{\bar{R}}_{OA} \\ | & | & | \end{vmatrix} \text{ and } D_2^{**} = \begin{vmatrix} | & | & | \\ \hat{u} & \dot{\hat{u}} & \bar{R}_{OA} \\ | & | & | \end{vmatrix}$$

- Then D_2 becomes $D_2 = -2D_2^* - 2\frac{\mu}{r^3}D_2^{**}$

- Thus, ρ can be expressed as $\boxed{\rho = \frac{D_2}{D_1} = -2\frac{D_2^*}{D_1} - 2\frac{\mu}{r^3}\frac{D_2^{**}}{D_1}}$

- ρ is then a function of $\hat{u}, \dot{\hat{u}}, \ddot{\hat{u}}, r$, $\bar{R}_{OA}$, and $\ddot{\bar{R}}_{OA}$

- We will now derive expressions for some of these qty's

Laplace IOD

- Recall that this method requires LOS measurements at 3 times; call these LOS vectors $\hat{u}_1, \hat{u}_2, \hat{u}_3$ at times t_1, t_2, t_3

- Thus, if we wish to evaluate our expression for ρ at $t^* = t_1$, t_2, or t_3, $\hat{u}$ at these times is known; what about $\dot{\hat{u}}$ and $\ddot{\hat{u}}$?

- We can use Lagrange interpolation to fit the LOS history over the span of measurements (between t_1 & t_3) to a 2^{nd}-order polynomial:

$$\hat{u}(t) = \frac{(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)}\hat{u}_1 + \frac{(t - t_1)(t - t_3)}{(t_2 - t_1)(t_2 - t_3)}\hat{u}_2 + \frac{(t - t_1)(t - t_2)}{(t_3 - t_1)(t_3 - t_2)}\hat{u}_3$$

2^{nd}-order (quadratic) expression in *t*

Note that evaluating this formula at t_1 yields $\hat{u}_1$, at t_2 yields $\hat{u}_2$, & at t_3 yields $\hat{u}_3$ (as expected)

Laplace IOD

- Taking the 1st & 2nd time derivatives of this expression yields:

$$\dot{\hat{u}}(t) = \frac{2t - t_2 - t_3}{(t_1 - t_2)(t_1 - t_3)}\hat{u}_1 + \frac{2t - t_1 - t_3}{(t_2 - t_1)(t_2 - t_3)}\hat{u}_2 + \frac{2t - t_1 - t_2}{(t_3 - t_1)(t_3 - t_2)}\hat{u}_3$$

$$\ddot{\hat{u}}(t) = \frac{2}{(t_1 - t_2)(t_1 - t_3)}\hat{u}_1 + \frac{2}{(t_2 - t_1)(t_2 - t_3)}\hat{u}_2 + \frac{2}{(t_3 - t_1)(t_3 - t_2)}\hat{u}_3$$

- The expectation is that these formulas should provide a decent approximation for $\dot{\hat{u}}$ and $\ddot{\hat{u}}$ in the time range from t_1 to t_3
- If we insert the values for our measurements times t_1, t_2, & t_3 into these formulas, we see that $\ddot{\hat{u}}$ is constant & $\dot{\hat{u}}$ is linear in time
- Thus, $\dot{\hat{u}}$ & $\ddot{\hat{u}}$ can be approximated at any time from t_1 to t_3 by the above formulas

Laplace IOD

- Now what about $\bar{R}_{OA}$ and $\ddot{\bar{R}}_{OA}$?

- $\bar{R}_{OA}$ is the position vector from Earth's center to the sensor
  - Generally, a sensor's location on Earth is precisely known (often expressed in latitude & longitude)
  - From latitude & longitude, we know how to calculate the sensor's ECEF position vector, then if sidereal time at a particular instant is known, we can calculate the sensor's ECI position vector as well

- $\ddot{\bar{R}}_{OA}$ is the acceleration vector of this point on the Earth
  - Assuming Earth is a rotating sphere, kinematics tells us that this acceleration is $\bar{\omega}_E\ x\ (\bar{\omega}_E\ x\ \bar{R}_{OA})$, where $\bar{\omega}_E$ is the rotational velocity vector of the Earth about its axis

Laplace IOD

- All vectors in these calculations should be expressed in ECI coordinates; so how do we express $\ddot{\bar{R}}_{OA}$ in ECI?
- ECI expression of $\bar{\omega}_E$ is $\begin{bmatrix} 0 \\ 0 \\ \omega_E \end{bmatrix}$, where $\omega_E = 360^\circ$ per sidereal day
- ECI expression of $\bar{R}_{OA}$ is $\begin{bmatrix} R_{OAx} \\ R_{OAy} \\ R_{OAz} \end{bmatrix}$ (discussed on Slide 10 how to calculate this)
- Inserting into $\bar{\omega}_E \; x \; (\bar{\omega}_E \; x \; \bar{R}_{OA})$ will yield a vector in ECI

Laplace IOD

- Recall that ρ at a particular time has been shown to be a function of $\hat{u}, \dot{\hat{u}}, \ddot{\hat{u}}, r$, $\bar{R}_{OA}$, and $\ddot{\bar{R}}_{OA}$

- If we choose to cacluate ρ at one of the measurement times (t_1, t_2, or t_3), all of these qty's are known (i.e. we know how to cacluate them) except r

- Let us now take the formula $\bar{r} = \bar{R}_{OA} + \rho\hat{u}$ & dot it with itself:

$$\bar{r} \cdot \bar{r} = (\bar{R}_{OA} + \rho\hat{u}) \cdot (\bar{R}_{OA} + \rho\hat{u}) = \rho^2 \hat{u} \cdot \hat{u} + 2\rho\bar{R}_{OA} \cdot \hat{u} + \bar{R}_{OA} \cdot \bar{R}_{OA}$$

$$\rightarrow \; r^2 = \rho^2 + 2\rho\bar{R}_{OA} \cdot \hat{u} + R_{OA}{}^2$$

Laplace IOD

- Note that, for a spherical Earth, $R_{OA} = R_E$ (Earth's radius)

- Also, because $\bar{R}_{OA}$ & $\hat{u}$ are known at t_1, t_2, & t_3, $\bar{R}_{OA} \cdot \hat{u}$ is known → refer to this qty as "C":

$$r^2 = \rho^2 + 2C\rho + {R_E}^2$$

- Inserting $\rho = -2\frac{D_2^*}{D_1} - 2\frac{\mu}{r^3}\frac{D_2^{**}}{D_1}$ into this formula yields:

$$r^2 = 4\left(\frac{D_2^*}{D_1} + \frac{\mu}{r^3}\frac{D_2^{**}}{D_1}\right)^2 - 4C\left(\frac{D_2^*}{D_1} + \frac{\mu}{r^3}\frac{D_2^{**}}{D_1}\right) + {R_E}^2$$

Laplace IOD

- Expanding, multiplying through to eliminate r from the denominator, & grouping powers of r yields:

$$r^8 + \left(4C\frac{D_2^*}{D_1} - 4\frac{{D_2^*}^2}{{D_1}^2} - {R_E}^2\right)r^6 + \mu\left(4C\frac{D_2^{**}}{D_1} - 8\frac{D_2^* D_2^{**}}{{D_1}^2}\right)r^3 - 4\mu^2\frac{{D_2^{**}}^2}{{D_1}^2} = 0$$

- This is an 8th-order polynomial in r (object's orbit radius, i.e. magnitude of its position vector)

- Once we decide the time at which we wish to evaluate the polynomial, we insert all the known qty's at that time to calculate the coefficients → we can then solve for the 8 roots

- Depending on the accuracy of the method, one of these roots should be at least fairly close to the true value of r at that time, & it is this root that we choose to be the "solution" value of r

Laplace IOD

- So how does this value of r yield the orbit solution?

- Insert r into $\rho = -2\frac{D_2^*}{D_1} - 2\frac{\mu}{r^3}\frac{D_2^{**}}{D_1}$ to yield ρ, then insert ρ into $\bar{r} = \bar{R}_{OA} + \rho\hat{u}$ to yield $\bar{r}$, whose coordinates are (x, y, z)

- We've now solved 3 of the object's 6 orbital states; what about the other 3 states v_x , v_y , v_z?

Laplace IOD

- First we solve $\dot{\rho}$ from

$$\begin{bmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & \ddot{\hat{u}} + \frac{\mu}{r^3}\hat{u} \\ | & | & | \end{bmatrix} \begin{bmatrix} \ddot{\rho} \\ \dot{\rho} \\ \rho \end{bmatrix} = \begin{bmatrix} | \\ -\ddot{\bar{R}}_{OA} - \frac{\mu \bar{R}_{OA}}{r^3} \\ | \end{bmatrix}$$

- Use Cramer's rule as before:

$$D_1 = \begin{vmatrix} | & | & | \\ \hat{u} & 2\dot{\hat{u}} & \ddot{\hat{u}} + \frac{\mu}{r^3}\hat{u} \\ | & | & | \end{vmatrix} = 2\begin{vmatrix} | & | & | \\ \hat{u} & \dot{\hat{u}} & \ddot{\hat{u}} \\ | & | & | \end{vmatrix},\ D_2 = \begin{vmatrix} | & | & | \\ \hat{u} & -\ddot{\bar{R}}_{OA} - \frac{\mu \bar{R}_{OA}}{r^3} & \ddot{\hat{u}} + \frac{\mu}{r^3}\hat{u} \\ | & | & | \end{vmatrix} \rightarrow \dot{\rho} = \frac{D_2}{D_1}$$

This determinant may be simplified if desired (as in the case where we solved for ρ)

Laplace IOD

- Next we solve $\dot{\bar{R}}_{OA}$, the velocity vector of the point on the Earth where the sensor lies
  - Assuming Earth is a rotating sphere, kinematics tells us that this velocity is $\bar{\omega}_E \ x \ \bar{R}_{OA}$ (be sure to calculate in ECI coordinates, as discussed previously)

- Insert $\dot{\rho}$ & $\dot{\bar{R}}_{OA}$ into $\dot{\bar{r}} = \dot{\bar{R}}_{OA} + \dot{\rho}\hat{u} + \rho\dot{\hat{u}}$ to yield $\dot{\bar{r}}$, whose coordinates are (v_x , v_y , v_z)

- Thus ends the Laplace method!

Laplace IOD

A FEW ISSUES…

1) Which of the 8 roots is the "correct" value of *r* to choose?

- Begin by process of elimination:
  - Eliminate complex roots (come in conjugate pairs)
  - Eliminate values too small ($< R_E$) or large (beyond the radius of any reasonable Earth orbit)

- If more than 1 candidate value remain, these values will likely be significantly different from one another
  - If we have at least a vague *a priori* idea of the object's orbit radius (e.g. LEO, GPS, GEO), we should be able to select the correct value of *r*

Laplace IOD

2) What accuracy do we expect from Laplace's method?

- There are 2 approximations employed by the method:
  - 2-body dynamics assumed for the object's motion
  - Lagrange interpolation to fit the LOS history (& its derivatives) between t_1 & t_3

- Therefore, the more accurately these approximations represent the object's true orbital motion, the more accurate the Laplace orbit solution will be

- Example scenarios where we expect more accuracy are:
  - Shorter time spans → more accurate interpolation, & less time for propagation error to build up
  - Orbits with less perturbation force (GEO vs LEO)

Laplace IOD

3) At what time(s) in the object's orbit is it appropriate to evaluate the polynomial (& therefore solve the object's orbit)?

- Technically, we can evaluate the polynomial at any time
  - Coefficients are functions of $\hat{u}$, $\dot{\hat{u}}$, $\ddot{\hat{u}}$, $\bar{R}_{OA}$, and $\ddot{\bar{R}}_{OA}$
  - Lagrange interpolation provides formulas for $\hat{u}$, $\dot{\hat{u}}$, and $\ddot{\hat{u}}$, which can be used to evaluate these qty's at any time
  - $\bar{R}_{OA}$ and $\ddot{\bar{R}}_{OA}$ can also be easily determined at any time

- Practically, it is best to evaluate the polynomial at the time(s) where the Lagrange interpolation provides the best approximation to LOS
  - This approximation is generally much better between t_1 & t_3 than outside of this range
  - Most IOD implementations choose t_2 to evaluate the polynomial because it is between the endpoints of the approximation (t_1 & t_3)

These 3 issues have been the subject of extensive research → the answer in each case tends to be very scenario-dependent

IOD Wrap-up

- Though Laplace’s method is centuries old, it is still used today as a practical means of optical IOD (for both deep-space & Earth-orbiting objects)

- Other angles-only IOD methods include:
  - Gauss’ method (1800)
  - Double-R method (Escobal, 1965)
  - Taff’s method (1983)
  - Gooding’s method (1993)