ME 596 Spacecraft Attitude Dynamics and Control

Satellite Dynamics

Professor Christopher D. Hall
Mechanical Engineering

University of New Mexico

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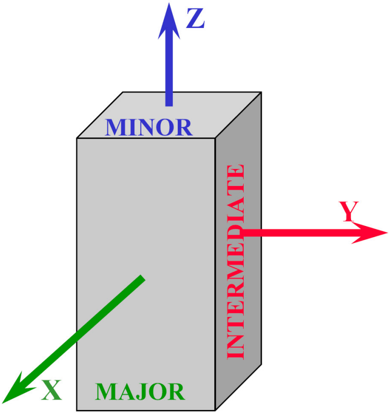
Rigid Body / “Real” Body Spin Stability

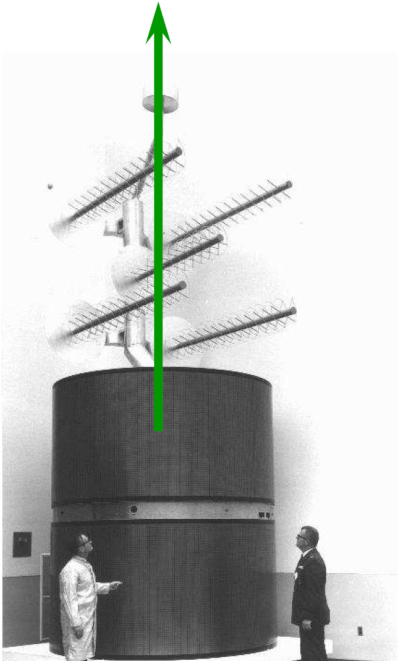


- $I_{xx} > I_{yy} > I_{zz}$
- Major axis spin is stable
- Minor axis spin is stable
- Intermediate axis spin is unstable
- Energy dissipation changes these results
  $\rightarrow$ Minor axis spin becomes unstable

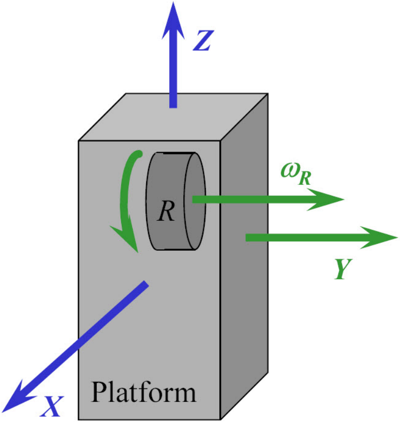
This behavior is called the Major Axis Rule

Gyrostats and Dual-Spin Spacecraft

- The term "Gyrostat" denotes a body that has an internal source of angular momentum, but which has a constant moment of inertia tensor.
- A rigid body containing one or more axisymmetric flywheels is a gyrostat.
- A rigid body containing one or more completely filled propellant tanks is a gyrostat.
- The gyrostat model is useful for studying the attitude dynamics of dual-spin spacecraft, but energy dissipation must also be included using an energy-sink analysis (just as with the rigid body model)



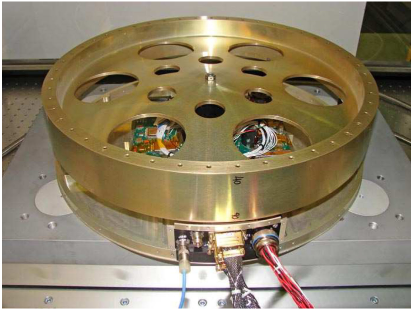
Effect of Rotor on Spin Stability



- A spinning rotor can stabilize the intermediate axis, destabilize other axes
- Stability condition
  $I_R \omega_R > (I_{xx} - I_{yy})\omega_y$
- As with rigid body, energy dissipation changes stability results
  $\rightarrow$ some stable spins become unstable

Satellites with Reaction Wheels

- Typically a three-axis stabilized spacecraft will have 3 or 4 reaction wheels (or "reaction wheel assemblies" = RWAs)
- The terms RW and "Momentum Wheel" (MW) denote nearly identical functions, and certainly denote the same technological concept
- RWs spin with nearly zero angular momentum and "react" to external torques
- MWs spin with some nominal value of angular momentum and modify that momentum as needed



Reaction Wheel
Lunar Reconnaissance Orbiter
(from NASA Goddard's website)

Gravity Gradient Satellite Dynamics

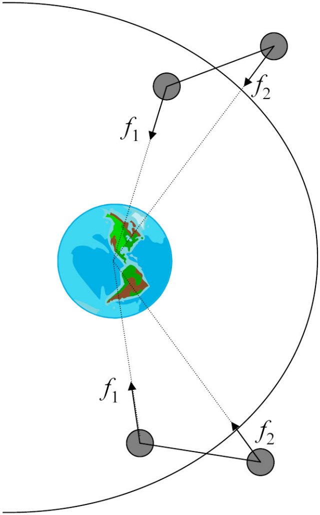
- Rotational equations of motion for a rigid body are:

$$\begin{aligned}\mathbf{I}\dot{\boldsymbol{\omega}} &= -\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{g} \\ \dot{\boldsymbol{\omega}} &= -\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{I}^{-1}\mathbf{g}\end{aligned}$$

- Rewrite as Euler's Equations for principal axes:

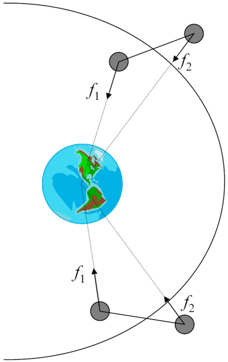
$$\begin{aligned}\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{g_1}{I_1} \\ \dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 + \frac{g_2}{I_2} \\ \dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{g_3}{I_3}\end{aligned}$$

- We know how to derive moments of inertia
- Next we derive gravitational torques for a satellite in a circular orbit



Gravity Gradient Satellite Dynamics (2)

- Simple gravity-gradient satellite dumbbell model
- Rigid massless rod with two point masses
- As satellite moves in circular orbit, it swings like a pendulum
- Consider only the motion in the orbital plane
- Force f_1 on body $>$ force f_2 on upper body
- Creates "restoring" torque



Is it clear why the restoring torque causes the dumbbell to swing like a pendulum?

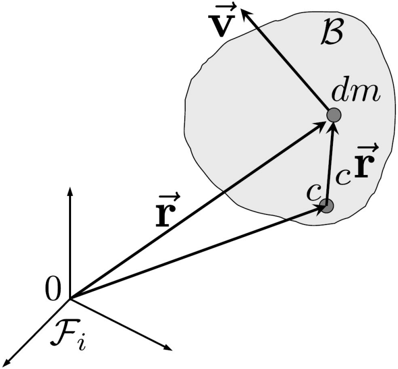
Gravity Acting on Rigid Satellite In Circular Orbit

- Every differential mass element of the body subject to Newton's Universal Gravitational Law:

$$d\vec{\mathbf{f}}_g = -\frac{GM\,dm}{r^2}\hat{\mathbf{e}}_r$$

- $\hat{\mathbf{e}}_r$ is unit vector in the negative $\vec{\mathbf{r}}$ direction
- For mass center $\hat{\mathbf{e}}_r = -\hat{\mathbf{o}}_3$ (recall $\mathcal{F}_o$ vector definitions)
- Position vector from primary to differential mass element is sum of position vector from primary to mass center and position vector from mass center to differential mass element:

$$\vec{\mathbf{r}} = {}_O\vec{\mathbf{r}} = {}^c_O\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}}$$



The r^2 in denominator of $d\vec{\mathbf{f}}_g$ is $\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}$, which we develop in the next couple of slides, as we want to integrate

$$\vec{\mathbf{f}}_g = \int_{\mathcal{B}} d\vec{\mathbf{f}}_g$$

Gravity Acting on Rigid Satellite In Circular Orbit (2)

- Integrate over the body:

$$\vec{\mathbf{f}}_g = -\int_{\mathcal{B}} \frac{GM}{r^2}\hat{\mathbf{e}}_r \, dm$$

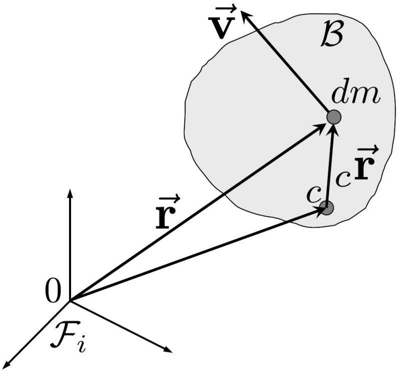
  which can be written as

$$\vec{\mathbf{f}}_g = -\int_{\mathcal{B}} \frac{GM}{r^3}\vec{\mathbf{r}} \, dm$$

- Expanding the position vector

$$\vec{\mathbf{f}}_g = -\int_{\mathcal{B}} \frac{GM}{|{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}|^3}({}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}) \, dm$$

- Assume $|{}^{c}_{O}\vec{\mathbf{r}}| \gg |{}_{c}\vec{\mathbf{r}}|$
- Expand the integrand in ________________ series



Gravity Acting on Rigid Satellite In Circular Orbit (3)

- Expand and simplify the denominator

$$\begin{aligned}
\frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{|{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}|^{3}} &= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{\left[({}^{c}_{O}\vec{\mathbf{r}} + \vec{\mathbf{r}}^{c}) \cdot ({}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}})\right]^{3/2}} \\
&= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{\left[{}^{c}_{O}\vec{\mathbf{r}} \cdot {}^{c}_{O}\vec{\mathbf{r}} + 2\,{}^{c}_{O}\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}^{c} + {}_{c}\vec{\mathbf{r}} \cdot {}_{c}\vec{\mathbf{r}}\right]^{3/2}} \\
&= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{\left[{}^{c}_{O}r^{2} + 2\,{}^{c}_{O}\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}^{c} + {}_{c}r^{2}\right]^{3/2}} \\
&= \frac{{}^{c}_{O}\vec{\mathbf{r}} + {}_{c}\vec{\mathbf{r}}}{{}^{c}_{O}r^{3}\left[1 + 2\,{}^{c}_{O}\vec{\mathbf{r}} \cdot {}_{c}\vec{\mathbf{r}}/{}^{c}_{O}r^{2} + {}_{c}r^{2}/{}^{c}_{O}r^{2}\right]^{3/2}}
\end{aligned}$$

- So far, this expression is exact, and the next step is to assume that spacecraft is much smaller than the size of the orbit
- Assume orbit radius $|{}^{c}_{O}\vec{\mathbf{r}}|$ is $\gg$ position vector from mass center to any differential element $|{}_{c}\vec{\mathbf{r}}|$, and expand in Taylor series

Gravity Acting on Rigid Satellite In Circular Orbit (4)

- Since $|{}^c_O\vec{\mathbf{r}}| \gg |{}_c\vec{\mathbf{r}}|$, Taylor series expansion leads to (Exercise!)

$$
\begin{aligned}
\frac{{}^c_O\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}}}{|{}^c_O\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}}|^3} &= \frac{{}^c_O\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}}}{{}^c_O r^3}\left(1 - 3\frac{{}^c_O\vec{\mathbf{r}} \cdot {}_c\vec{\mathbf{r}}}{{}^c_O r^2} + H.O.T.\right) \\
&\approx \frac{{}^c_O\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}}}{{}^c_O r^3}\left(1 - 3\frac{{}^c_O\vec{\mathbf{r}} \cdot {}_c\vec{\mathbf{r}}}{{}^c_O r^2}\right) \\
&\approx \frac{{}^c_O\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}}}{{}^c_O r^3}
\end{aligned}
$$

- Substituting the last of these approximations into the volume integral for $\vec{\mathbf{f}}_g$ leads to

$$
\begin{aligned}
\vec{\mathbf{f}}_g &= -\int_{\mathcal{B}} \frac{GM\,({}^c_O\vec{\mathbf{r}} + {}_c\vec{\mathbf{r}})}{{}^c_O r^3}\,dm \\
&= -\frac{GMm}{{}^c_O r^3}{}^c_O\vec{\mathbf{r}} = -\frac{GMm}{r^3}\vec{\mathbf{r}}
\end{aligned}
$$

What happened to ${}_c\vec{\mathbf{r}}$?

Gravity Acting on Rigid Satellite In Circular Orbit (5)

- Using $\vec{\mathbf{r}}$ to denote the position vector from primary to mass center of orbiting body, and applying Newton's Second Law, we obtain:

$$\ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^3}\vec{\mathbf{r}} = \vec{\mathbf{0}}$$

  which is the two-body problem eq. of motion from astrodynamics

- Two important facts:
  1) μ/r^3 is the mean motion squared, or n^2, and for circular orbits we use ω_c to represent n
  2) $\vec{\mathbf{r}} = -r\vec{\mathbf{o}}_3$, where $\vec{\mathbf{o}}_3$ is the nadir vector, the Earth-pointing unit vector of the orbital frame $\mathcal{F}_o$

- Note that two-body problem equation is an **approximation**, based on the reasonable assumption that the dimension of the satellite is small compared with the dimension of the orbit

- The next step is to develop the torque about the mass center, and use it in Euler's equations:

$$\vec{\mathbf{g}}_g^c = \int_{\mathcal{B}} {}_c\vec{\mathbf{r}} \times d\vec{\mathbf{f}}_g$$

Gravity Acting on Rigid Satellite In Circular Orbit (6)

- The torque due to gravity is

$$\vec{\mathbf{g}}_g^c = \int_{\mathcal{B}} {}_c\vec{\mathbf{r}} \times d\vec{\mathbf{f}}_g$$

- Applying the same assumptions as those used to approximate the force, we obtain the approximate moment about the mass center:

$$\vec{\mathbf{g}}_g^c = 3\frac{GM}{r^3}\hat{\mathbf{o}}_3 \times \vec{\mathbf{I}}^c \cdot \hat{\mathbf{o}}_3$$

- Expressed in a body-fixed reference frame, this gravity-gradient torque is

$$\mathbf{g}_{gb}^c = 3\frac{GM}{r^3}\mathbf{o}_{3b}^{\times}\mathbf{I}_b^c\mathbf{o}_{3b}$$

  or, simply

$$\mathbf{g} = 3\frac{\mu}{r^3}\mathbf{o}_3^{\times}\mathbf{I}\mathbf{o}_3 = 3\omega_c^2\mathbf{o}_3^{\times}\mathbf{I}\mathbf{o}_3$$

  Note that the $\hat{\mathbf{o}}_3$ terms arise from the fact that $\vec{\mathbf{r}} = -r\hat{\mathbf{o}}_3$

Gravity Acting on Rigid Satellite In Circular Orbit (7)

- So, the gravity gradient torque for a rigid spacecraft in a circular orbit, expressed in $\mathcal{F}_b$, is

$$\mathbf{g} = 3\omega_c^2 \mathbf{o}_3^\times \mathbf{I} \mathbf{o}_3$$

  This result is an **approximation** based on assumption that spacecraft size $\ll$ orbit radius; however it is a nonlinear function of attitude. Can you explain why?

- Euler's equations for this system are

$$\mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega} + 3\omega_c^2 \mathbf{o}_3^\times \mathbf{I}\mathbf{o}_3$$

- Recall that $\mathbf{o}_3$ is expressed in $\mathcal{F}_b$, and so it is the third column of $\mathbf{R}^{bo}$
- We need to keep track of the attitude of $\mathcal{F}_b$ with respect to $\mathcal{F}_o$

  What two frames are related by the angular velocity in Euler's equations?

  Can you now explain how the torque acting on a satellite can depend on position?

Equilibrium Motion for Rigid Satellite In Circular Orbit

- We are interested in finding special solutions where $\dot{\boldsymbol{\omega}} = \mathbf{0}$.
- Euler's equations are

$$\mathbf{I}\dot{\boldsymbol{\omega}} \;=\; -\boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega} + 3\omega_c^2 \mathbf{o}_3^\times \mathbf{I}\mathbf{o}_3$$

- If $\boldsymbol{\omega}$ is about a principal axis, then $\boldsymbol{\omega}^\times \mathbf{I}\boldsymbol{\omega} = \mathbf{0}$ (Convince yourself!)
- If $\mathcal{F}_b$ is aligned with $\mathcal{F}_o$, then the angular velocity of the body is about the "2" axis, $\boldsymbol{\omega} = [0 \quad -\omega_c \quad 0]^T$ (Convince yourself!)
- If $\mathcal{F}_b$ is aligned with $\mathcal{F}_o$, then $\mathbf{o}_3 = [0 \quad 0 \quad 1]^T$, and $\mathbf{o}_3^\times \mathbf{I}\mathbf{o}_3 = \mathbf{0}$ (Convince yourself!)
- Thus, if $\mathcal{F}_b$ is aligned with $\mathcal{F}_o$, then $\dot{\boldsymbol{\omega}} = \mathbf{0}$, which means that this attitude is an equilibrium or steady motion.

  But, is it a *stable* equilibrium motion?

  What mathematical tools are used to determine stability?

Equilibrium Motion for Rigid Satellite In Circular Orbit (2)

- Body frame nearly aligned with orbital frame
- Use 1-2-3 Euler angle sequence (roll-pitch-yaw) to relate $\mathcal{F}_b$ to $\mathcal{F}_o$

$$\mathbf{R}^{bo} = \begin{bmatrix} c_2c_3 & s_1s_2c_3 + c_1s_3 & s_1s_3 - c_1s_2c_3 \\ -s_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix}$$

- Linearize about $\theta_i = 0, i = 1, 2, 3$ (recall Roll, Pitch & Yaw material from *Attitude Kinematics*)

$$\mathbf{R}^{bo} \approx \mathbf{1} - \boldsymbol{\theta}^{\times} \Rightarrow \mathbf{o}_3 \approx [-\theta_2 \quad \theta_1 \quad 1]^T$$

- Thus the linear approximation for gravity gradient torque acting on a rigid body whose principal axes are "close" to being aligned with the orbital frame axes is

$$\mathbf{g}_{gg} = 3\omega_c^2 \mathbf{o}_3^{\times} \mathbf{I} \mathbf{o}_3 \approx 3\omega_c^2 \begin{bmatrix} (I_3 - I_2)\theta_1 \\ (I_3 - I_1)\theta_2 \\ 0 \end{bmatrix}$$

- The next step is to put this torque into Euler's equations and linearize the equations of motion.

Linearize about Equilibrium Motion

- Recall that Euler's equations are

$$\mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + 3\omega_c^2 \mathbf{o}_3^{\times}\mathbf{I}\mathbf{o}_3$$

where the angular velocity $\boldsymbol{\omega}$ is a 3×1 matrix of the elements of $\vec{\boldsymbol{\omega}}^{bi}$ expressed in $\mathcal{F}_b$.

- Write the angular velocity components in terms of Euler angles and their rates

$$\begin{aligned}
\boldsymbol{\omega} &= \boldsymbol{\omega}_b^{bi} = \boldsymbol{\omega}^{bo} + \boldsymbol{\omega}^{oi} = \boldsymbol{\omega}_b^{bo} + \mathbf{R}^{bo}\boldsymbol{\omega}_o^{oi} \\
&= \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\omega_c \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \dot{\theta}_1 - \omega_c\theta_3 \\ \dot{\theta}_2 - \omega_c \\ \dot{\theta}_3 + \omega_c\theta_1 \end{bmatrix}
\end{aligned}$$

- This approximation for $\boldsymbol{\omega}$ leads to a linearized version of Euler's equations written in terms of Euler angles
- Can you explain what approximations have been made here?

Substitute into Euler's Equations

- Euler's equations:

$$\mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + 3\omega_c^2\mathbf{o}_3^{\times}\mathbf{I}\mathbf{o}_3$$

- Angular velocity:

$$\boldsymbol{\omega} = \begin{bmatrix}\dot{\theta}_1 - \omega_c\theta_3 & \dot{\theta}_2 - \omega_c & \dot{\theta}_3 + \omega_c\theta_1\end{bmatrix}^{\mathsf{T}}$$

- Orbital frame "3" vector in $\mathcal{F}_b$:

$$\mathbf{o}_3 = \begin{bmatrix}-\theta_2 & \theta_1 & 1\end{bmatrix}^{\mathsf{T}}$$

  (Remember that the Euler angles relate $\mathcal{F}_b$ to $\mathcal{F}_o$)

- Make the substitution, neglect nonlinear terms (products of θ's)

$$\begin{aligned} I_1\ddot{\theta}_1 + (I_2 - I_3 - I_1)\omega_c\dot{\theta}_3 - 4(I_3 - I_2)\omega_c^2\theta_1 &= 0 \\ I_2\ddot{\theta}_2 + 3\omega_c^2(I_1 - I_3)\theta_2 &= 0 \\ I_3\ddot{\theta}_3 + (I_3 + I_1 - I_2)\omega_c\dot{\theta}_1 + (I_2 - I_1)\omega_c^2\theta_3 &= 0 \end{aligned}$$

  This result is a system of 3 second-order coupled constant-coefficient linear differential equations.

Stability of Pitch Motion

- Euler's equations after linearizing:

$$\begin{aligned} I_1\ddot{\theta}_1 + (I_2 - I_3 - I_1)\omega_c\dot{\theta}_3 - 4(I_3 - I_2)\omega_c^2\theta_1 &= 0 \\ I_2\ddot{\theta}_2 + 3\omega_c^2(I_1 - I_3)\theta_2 &= 0 \\ I_3\ddot{\theta}_3 + (I_3 + I_1 - I_2)\omega_c\dot{\theta}_1 + (I_2 - I_1)\omega_c^2\theta_3 &= 0 \end{aligned}$$

- Note that the θ_2 equation is decoupled from the θ_1 and θ_3 equations.
- The "2" axis is the pitch axis and is in the orbit normal direction.
- The θ_2 equation is in the form $\ddot{x} + kx = 0$, which is the equation for a mass-on-a-spring, and is stable if $k > 0$, and unstable if $k < 0$.
- Thus the pitch axis motion is only stable if $I_1 > I_3$, or if the roll axis inertia is greater than the yaw axis inertia.
- However, if the coupled roll-yaw motion is unstable, then the pitch motion stability is suspect, since we assumed that *all* of the Euler angles are small when we did the linearization.

Stability of Coupled Roll-Yaw Motion

- Coupled linearized roll-yaw equations:

$$
\begin{aligned}
I_1\ddot{\theta}_1 + (I_2 - I_3 - I_1)\omega_c\dot{\theta}_3 - 4(I_3 - I_2)\omega_c^2\theta_1 &= 0 \\
I_3\ddot{\theta}_3 + (I_3 + I_1 - I_2)\omega_c\dot{\theta}_1 + (I_2 - I_1)\omega_c^2\theta_3 &= 0
\end{aligned}
$$

- We can rewrite these equations as a matrix system of linear constant-coefficient differential equations for $\mathbf{x} = [\theta_1 \quad \theta_3]^\mathsf{T}$:

$$
\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}
$$

where

$$
\begin{aligned}
\mathbf{M} &= \begin{bmatrix} I_1 & 0 \\ 0 & I_3 \end{bmatrix} \\
\mathbf{G} &= (I_1 + I_3 - I_2)\omega_c \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
\mathbf{K} &= \omega_c^2 \begin{bmatrix} 4(I_2 - I_3) & 0 \\ 0 & (I_2 - I_1) \end{bmatrix}
\end{aligned}
$$

Convince yourself!

Linear Stability Analysis

- Coupled linearized roll-yaw equations in matrix form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

- Just as with *scalar* linear constant-coefficient differential equations, we can look for solutions in the form

$$\mathbf{x} = e^{\lambda t}\mathbf{c}$$

  where λ is the eigenvalue, and $\mathbf{c}$ is the constant $(2 \times 1$ matrix$)$ depending on the initial conditions.

- Differentiate the assumed solution twice:

$$\begin{aligned} \mathbf{x} &= e^{\lambda t}\mathbf{c} \\ \dot{\mathbf{x}} &= \lambda e^{\lambda t}\mathbf{c} \\ \ddot{\mathbf{x}} &= \lambda^2 e^{\lambda t}\mathbf{c} \end{aligned}$$

- Substituting into the differential equations yields:

$$e^{\lambda t}\left[\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}\right]\mathbf{c} = \mathbf{0}$$

Solving the Eigenvalue Problem

- Assuming a solution of the form

$$\mathbf{x} = e^{\lambda t}\mathbf{c}$$

leads to

$$e^{\lambda t}\left[\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}\right]\mathbf{c} = \mathbf{0}$$

- The value of $e^{\lambda t}$ is never zero, and we are not interested in the case where the initial state is $\mathbf{x}(0) = \mathbf{c} = \mathbf{0}$
- Thus, we require the 2×2 matrix in brackets to be *singular*, and that $\mathbf{c}$ be in the null space of that matrix (i.e., $[\cdot]\mathbf{c} = \mathbf{0}$)
- For the matrix to be singular, its determinant has to be zero
- The characteristic polynomial that results from setting det$[\cdot] = 0$ is

$$\hat{\lambda}^4 + (1 + 3k_1 + k_1k_3)\hat{\lambda}^2 + 4k_1k_3 = 0$$

where $\hat{\lambda} = \lambda/\omega_c$, $k_1 = (I_2 - I_3)/I_1$, and $k_3 = (I_2 - I_1)/I_3$

Solving the Eigenvalue Problem (2)

- The characteristic polynomial is

$$\hat{\lambda}^4 + (1 + 3k_1 + k_1k_3)\hat{\lambda}^2 + 4k_1k_3 \quad = \quad 0$$

  where $\hat{\lambda} = \lambda/\omega_c$, $k_1 = (I_2 - I_3)/I_1$, and $k_3 = (I_2 - I_1)/I_3$
- This polynomial can be written as

$$s^2 + b_1 s + b_0 \quad = \quad 0$$

  where $\hat{\lambda}^2 = s \Rightarrow \lambda = \pm\omega_c\sqrt{s}$.
- The condition for stability of the coupled roll-yaw motion is that the eigenvalues (λ) do not have positive real parts
- Thus the best we can do is to have both roots of the polynomial for s be negative and then all four eigenvalues will be of the form

$$\begin{aligned} \lambda &= \pm\omega_c\sqrt{s} \\ &= \pm\omega_c\sqrt{|s|}i \end{aligned}$$

- If s is positive, then the roll-yaw motion will be unstable

Routh-Hurwitz Conditions

- The characteristic polynomial in terms of s is

$$s^2 + b_1 s + b_0 \quad = \quad 0$$

- The Routh-Hurwitz conditions for s to be negative are

$$b_0 > 0 \qquad b_1 > 0 \qquad b_1^2 - 4b_0 > 0$$

- These conditions, along with the pitch stability condition can be rearranged to be

$$\begin{aligned} k_1 &> k_3 \text{ (I)}^* \\ k_1 k_3 &> 0 \text{ (II)} \\ 1 + 3k_1 + k_1 k_3 &> 0 \text{ (III)} \\ (1 + 3k_1 + k_1 k_3)^2 - 16 k_1 k_3 &> 0 \text{ (IV)} \end{aligned}$$

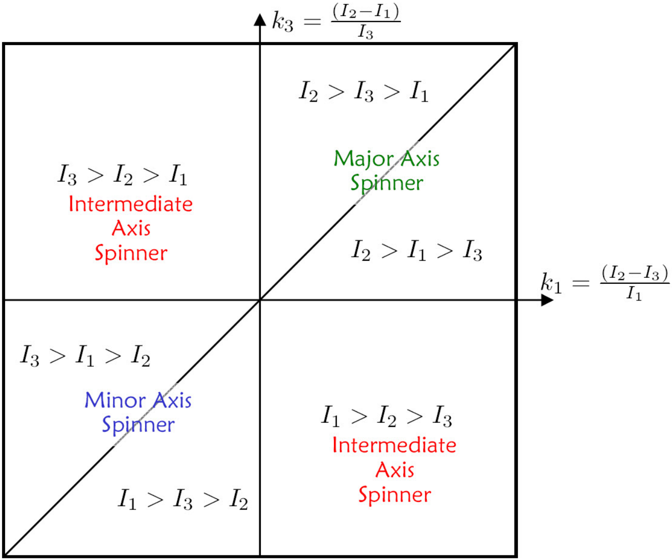
* I–IV denote region boundaries in subsequent plot

- The parameters k_1 and k_3 are called the "Smelt" parameters (DeBra and Delp, *Journal of the Astronautical Sciences*, 1961)

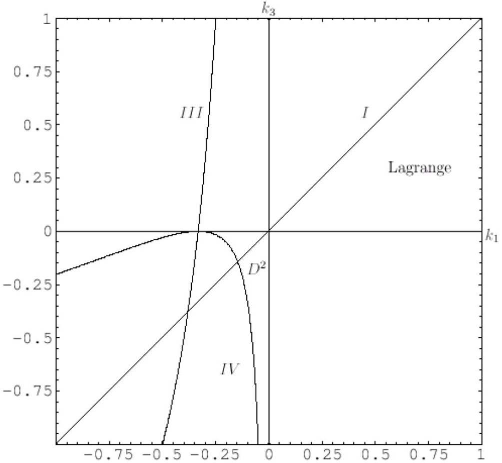
Smelt Parameter Plane

$|k_1| \leq 1$

$|k_3| \leq 1$



Smelt Parameter Plane



In practice, the Lagrange region is the only region with practical application. In fact, the nonlinear stability of the DeBra-Delp region remains an open research problem.

Gyrostats and Dual-Spin Spacecraft

- A dual-spin spacecraft is a special case of the more general "gyrostat"
- A gyrostat is a rigid body with moving parts arranged so that the moment of inertia is constant
- A rigid body with axisymmetric spinning wheels is the most common gyrostat
- The spinning wheels provide an internal source of angular momentum, $\vec{\mathbf{h}}_s$
- Can be used for attitude stabilization, or for performing attitude maneuvers
- Some spacecraft use one wheel, and some use four wheels
- Fourth wheel is typically for redundancy in the event of failure

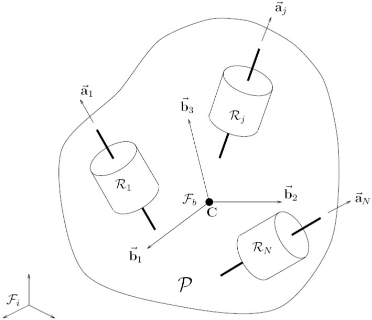


Figure 1: Gyrostat with N momentum wheels

- GPS satellites use momentum wheels for attitude stabilization
- Hubble Space Telescope uses momentum wheels for attitude stabilization as well as for large attitude maneuvers

Gyrostat Equations of Motion

- Angular momentum principle:

$$\begin{aligned} \vec{\mathbf{h}}^c &= \vec{\mathbf{I}}^c \cdot \vec{\boldsymbol{\omega}}^{bi} + \vec{\mathbf{h}}_s \\ \dot{\vec{\mathbf{h}}}^{\mathbf{c}} &= \vec{\mathbf{g}}^c \end{aligned}$$

  where $\vec{\mathbf{h}}_s$ is the spin angular momentum of the wheel(s) (not necessarily constant)

- How do we express these equations in a rotating reference frame?
- Recall formula for differentiating vector $\vec{\mathbf{a}}$ expressed in rotating frame $\mathcal{F}_b$:

$$\frac{d}{dt}\left[\left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}} \mathbf{a}\right] = \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}} \left[\dot{\mathbf{a}} + \boldsymbol{\omega}^{\times}\mathbf{a}\right]$$

  where $\boldsymbol{\omega} = \boldsymbol{\omega}^{bi}$

- Note that $\boldsymbol{\omega}^{bi}$ plays two distinct roles in these equations:
  1) forms part (but not all) of angular momentum
  2) is angular velocity of rotating frame
- In the body frame, then,

$$\begin{aligned} \mathbf{h}_b^c &= \mathbf{I}_b^c \boldsymbol{\omega}_b^{bi} + \mathbf{h}_{sb} \\ \dot{\mathbf{h}}_b^c &= -\left[\boldsymbol{\omega}_b^{bi}\right]^{\times} \left[\mathbf{I}_b^c \boldsymbol{\omega}_b^{bi} + \mathbf{h}_{sb}\right] \\ &\quad + \mathbf{g}_b^c \end{aligned}$$

- Note the importance of subscripts and superscripts, and compare with equivalent equations for rigid body

Gyrostat Equations of Motion (2)

- Dropping the b subscripts and c superscripts, we have

$$\begin{aligned} \mathbf{h} &= \mathbf{I}\boldsymbol{\omega} + \mathbf{h}_s \\ \dot{\mathbf{h}} &= -\boldsymbol{\omega}^\times \left[\mathbf{I}\boldsymbol{\omega} + \mathbf{h}_s\right] + \mathbf{g} \end{aligned}$$

- Now, let us consider the case of a single wheel, spinning about the principal axis $\hat{\mathbf{b}}_2$, with symmetry axis moment of inertia I_s and angular speed, *relative to the body*, Ω_s
- The term $\mathbf{h}_s$ (in $\mathcal{F}_b$), can be written as

$$\mathbf{h}_s = \begin{bmatrix} 0 & I_s\Omega_s & 0 \end{bmatrix}^\mathsf{T}$$

- And the total angular momentum can be written as

$$\mathbf{h} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 \\ I_s\Omega_s \\ 0 \end{bmatrix} = \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 + I_s\Omega_s \\ I_3\omega_3 \end{bmatrix}$$

- Again, compare with the similar rigid body angular momentum

Gyrostat Equations of Motion (3)

- Assuming that the relative spin rate of the wheel (as measured by a tachometer) is constant, the term $\dot{\mathbf{h}}$ is simply

$$\dot{\mathbf{h}} = \begin{bmatrix} I_1\dot{\omega}_1 & I_2\dot{\omega}_2 & I_3\dot{\omega}_3 \end{bmatrix}^\mathsf{T}$$

- Euler's equations for this case becomes

$$\begin{bmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{bmatrix} = -\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 + I_s\Omega_s \\ I_3\omega_3 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

- This matrix equation expands to the three scalar equations:

$$\begin{aligned} \dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{I_s}{I_1}\Omega_s\omega_3 + \frac{g_1}{I_1} \\ \dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 + \frac{g_2}{I_2} \\ \dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 - \frac{I_s}{I_3}\Omega_s\omega_1 + \frac{g_3}{I_3} \end{aligned}$$

- As with rigid body, we can obtain useful results for the torque-free motion, as well as for the axisymmetric torque-free case

Gyrostat Torque-Free Motion

- In the case of $\mathbf{g} = \mathbf{0}$, the gyrostat equations can be integrated in the same way that the rigid body equations are integrated, in terms of elliptic functions
- Exercise: review the integration of the asymmetric rigid body case, and carry out the same steps for the gyrostat.
- Torque-free gyrostat with constant-speed rotor spinning parallel to $\hat{\mathbf{b}}_2$:

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{I_s}{I_1}\Omega_s\omega_3 \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 - \frac{I_s}{I_3}\Omega_s\omega_1
\end{aligned}
$$

- As with rigid body, let us look for special solutions, such as steady spins
- Thus, we set the "dots" equal to zero and see whether there are values of ω_i that satisfy all three equations
- *Spoiler alert:* There are!

Gyrostat Torque-Free Motion (2)

- Looking for equilibria, set $\dot{\omega}_i = 0$:

$$\begin{aligned} 0 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{I_s}{I_1}\Omega_s\omega_3 \\ 0 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 \\ 0 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 - \frac{I_s}{I_3}\Omega_s\omega_1 \end{aligned}$$

- One possibility is $\omega_1 = \omega_3 = 0$, $\omega_2 = \Omega$
- That is, a steady spin about the same axis that $\hat{\mathbf{b}}_2$ axis, which is parallel to wheel spin axis
- **Exercise:** What are the other two possibilities? Hint: this is a simple algebra problem, which you can expect to see again.
- Determine whether this equilibrium is stable or unstable, and determine the relationship between inertias and wheel speed that stabilizes the motion (if possible)
- Whenever you are presented with a question such as "*Is this equilibrium stable or unstable*?" you should immediately think ________________ .

Gyrostat Torque-Free Motion (3)

- Linearize about $\boldsymbol{\omega} = [0 \quad \Omega \quad 0]^{\mathsf{T}}$, by setting $\omega_1 = 0 + \delta\omega_1$, $\omega_2 = \Omega + \delta\omega_2$, and $\omega_3 = 0 + \delta\omega_3$:

$$
\begin{aligned}
\dot{\delta\omega_1} &= \frac{I_2 - I_3}{I_1} (\Omega + \delta\omega_2)\, \delta\omega_3 + \frac{I_s}{I_1} \Omega_s\, \delta\omega_3 \\
\dot{\delta\omega_2} &= \frac{I_3 - I_1}{I_2} \delta\omega_1\, \delta\omega_3 \\
\dot{\delta\omega_3} &= \frac{I_1 - I_2}{I_3} \delta\omega_1\, (\Omega + \delta\omega_2) - \frac{I_s}{I_3} \Omega_s\, \delta\omega_1
\end{aligned}
$$

- *IF* the $\delta\omega_i$'s are small, then their products are negligible, and we drop them as "higher order terms"

$$
\begin{aligned}
\dot{\delta\omega_1} &= \frac{I_2 - I_3}{I_1} \Omega\delta\omega_3 + \frac{I_s}{I_1} \Omega_s\, \delta\omega_3 \\
\dot{\delta\omega_2} &= 0 \\
\dot{\delta\omega_3} &= \frac{I_1 - I_2}{I_3} \Omega\delta\omega_1 - \frac{I_s}{I_3} \Omega_s\, \delta\omega_1
\end{aligned}
$$

- Compare these equations to the linearization of Euler's equations for the torque-free rigid body
- Keep in mind that *IF* we determine the motion to be unstable, then that means the $\delta\omega_i$'s do not remain small, and these equations are no longer valid

Gyrostat Torque-Free Motion (4)

- The linearized equations simplify to:

$$
\begin{aligned}
\delta\dot{\omega}_1 &= \left(\frac{I_2 - I_3}{I_1}\Omega + \frac{I_s}{I_1}\Omega_s\right)\delta\omega_3 \\
\delta\dot{\omega}_3 &= \left(\frac{I_1 - I_2}{I_3}\Omega - \frac{I_s}{I_3}\Omega_s\right)\delta\omega_1
\end{aligned}
$$

- Recall that the inertias, the rotor spin rate relative to the body, Ω_s, and the platform angular velocity about $\hat{\mathbf{b}}_2$ are all constant
- As we did with the asymmetric, torque-free rigid body, we can differentiate the first equation and substitute in the second equation to obtain:

$$
\delta\ddot{\omega}_1 + \left(\frac{I_2 - I_3}{I_1}\Omega + \frac{I_s}{I_1}\Omega_s\right)\left(\frac{I_2 - I_1}{I_3}\Omega + \frac{I_s}{I_3}\Omega_s\right)\delta\omega_1 \quad = \quad 0
$$

- This equation is in the form $\ddot{x} + kx = 0$, so $k > 0$ is required for marginal stability, and if $k < 0$, the motion is unstable.
- Where have you seen $\ddot{x} + kx = 0$ before?
- Now we need to determine the conditions for $k > 0$

Gyrostat Torque-Free Motion (5)

- The linear second-order ODE is:

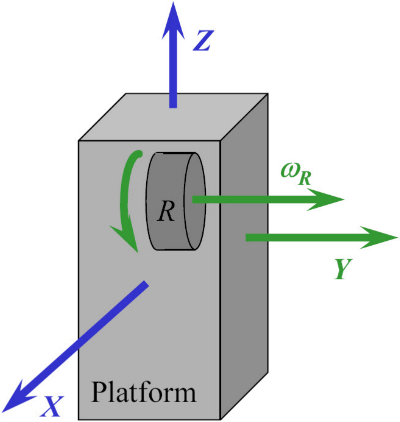
$$\ddot{\delta\omega}_1 + \left(\frac{I_2 - I_3}{I_1}\Omega + \frac{I_s}{I_1}\Omega_s\right)\left(\frac{I_2 - I_1}{I_3}\Omega + \frac{I_s}{I_3}\Omega_s\right)\delta\omega_1 \quad = \quad 0$$

- Evidently, the relationship between the inertias, Ω and Ω_s is pretty complicated, so let us consider the simple case where we know that $\hat{\mathbf{b}}_2$ is the intermediate axis, and choose the other two inertias so that $I_1 > I_2 > I_3$
- If Ω and Ω_s are both positive, then the first term in k is positive, so we can try to pick the wheel speed to make the second term positive:

$$\left(\frac{I_2 - I_1}{I_3}\Omega + \frac{I_s}{I_3}\Omega_s\right) > 0 \Rightarrow \Omega_s > \frac{I_1 - I_2}{I_s}\Omega$$

- Compare this result with the stability condition presented at the beginning of the Satellite Dynamics module, and repeated on the next slide.

Effect of Rotor on Spin Stability

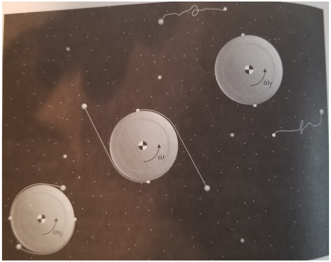


- A spinning rotor can stabilize the intermediate axis, destabilize other axes
- Stability condition
  $I_R \omega_R > (I_{xx} - I_{yy})\omega_y$
- As with rigid body, energy dissipation changes stability results
  $\rightarrow$ some stable spins become unstable

What are your ideas on how energy dissipation might affect these conditions?

Yo-Yo Despin Device

- Many spacecraft use an “upper stage” rocket motor to transfer from initial parking orbit to higher operational orbit
- Spin-stabilization makes steering of the thrust trajectory possible
- Despinning of the satellite in its operational orbit may be necessary
- The yo-yo concept makes despin easy
- Angular momentum is transferred from spacecraft to yo-yo masses

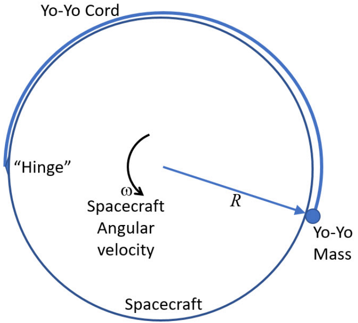


See, for example, Section 5.7 of *Spaceflight Dynamics*, by W. E. Wiesel, McGraw-Hill, 2nd edition, 1997, as well as the Wikipedia article
`https://en.wikipedia.org/wiki/Yo-yo_de-spin`

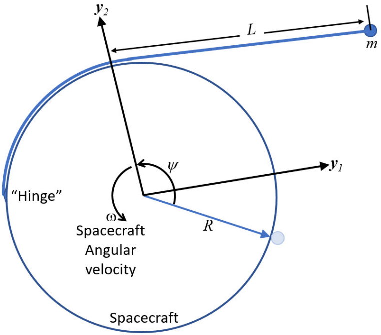
Yo-Yo Despin Mathematical Model

Simple model for despin analysis

- Axisymmetric rigid body with radius R, angular velocity ω
- Yo-yo consists of cord attached at "hinge," which can detach after deployment, and mass m initially attached



Only one yo-yo device shown for simplicity



- Despin maneuver begins when mass is released from spacecraft
- Yo-yo frame, $\mathcal{F}_y$ is defined so that $\hat{\mathbf{y}}_1$ is parallel to deployed segment of cord, and $\hat{\mathbf{y}}_2$ points to where cord is tangent to spacecraft

Yo-Yo Despin Analysis

- Length of deployed yo-yo cord:

$$L = R\psi \quad \Rightarrow \quad \dot{L} = R\dot{\psi}$$

- Position vector of yo-yo mass:

$$\vec{\mathbf{r}} = L\,\hat{\mathbf{y}}_1 + R\,\hat{\mathbf{y}}_2 = \mathbf{r}^{\mathsf{T}}\{\hat{\mathbf{y}}\}$$

- Expressed in rotating $\mathcal{F}_y$:

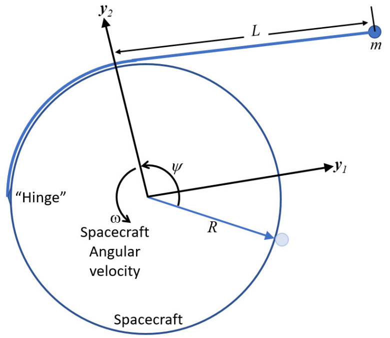
$$\mathbf{r} = [L \quad R \quad 0]^{\mathsf{T}}$$

- Angular velocity of $\mathcal{F}_y$ with respect to $\mathcal{F}_i$:

$$\begin{aligned}\mathbf{\Omega} &= [0 \quad 0 \quad \omega + \dot{\psi}]^{\mathsf{T}} \\ &= [0 \quad 0 \quad \omega + \dot{L}/R]^{\mathsf{T}}\end{aligned}$$

- Differentiate with respect to $\mathcal{F}_i$:

$$\mathbf{v} = \dot{\mathbf{r}} + \mathbf{\Omega}^{\times}\mathbf{r}$$



- Matrix $\dot{\mathbf{r}} + \mathbf{\Omega}^{\times}\mathbf{r}$ contains elements of $\vec{\mathbf{v}} = \dot{\vec{\mathbf{r}}}$ expressed in rotating $\mathcal{F}_y$
- Simplify this expression and use it in the expressions for energy and angular momentum

Yo-Yo Despin Analysis (2)

- Velocity of mass with respect to $\mathcal{F}_i$ expressed in rotating $\mathcal{F}_y$:

$$\frac{d}{dt}\left[\{\hat{\mathbf{y}}\}^{\mathsf{T}}\,\mathbf{r}\right] = \{\hat{\mathbf{y}}\}^{\mathsf{T}}\left[\dot{\mathbf{r}} + \boldsymbol{\Omega}^{\times}\mathbf{r}\right]$$

- Simplify the matrix (Exercise):

$$\dot{\mathbf{r}}+\boldsymbol{\Omega}^{\times}\mathbf{r} = \begin{bmatrix}-R\omega & L(\omega+\dot{L}/R) & 0\end{bmatrix}^{\mathsf{T}}$$

- Kinetic energy of two yo-yo masses is $T_m = 2\times\frac{1}{2}mv^2$:

$$T_m = m\left(R^2\omega^2 + L^2(\omega+\dot{L}/R)^2\right)$$

- Angular momentum of two yo-yo masses is $\vec{\mathbf{h}}_m = 2m\,\vec{\mathbf{r}}\times\vec{\mathbf{v}}$

$$\vec{\mathbf{h}}_m = 2m\left(R^2\omega + L^2(\omega+\dot{L}/R)\right)\,\hat{\mathbf{y}}_3$$

- Add T_m and $\vec{\mathbf{h}}_m$ to T_s and $\vec{\mathbf{h}}_s$ for spacecraft body

- Spacecraft rotational kinetic energy:

$$T_s = \frac{1}{2}C\omega^2$$

- Total kinetic energy:

$$\begin{aligned} T &= \frac{1}{2}C\omega^2 \\ &+ m\left(R^2\omega^2 + L^2(\omega+\dot{L}/R)^2\right) \end{aligned}$$

- Total kinetic energy is conserved, so

$$T = \left(\frac{1}{2}C + mR^2\right)\omega_0^2$$

- We will use the fact that the two total kinetic energy expressions must be equal

Yo-Yo Despin Analysis (3)

- Spacecraft angular momentum: $\vec{\mathbf{h}}_s = C\omega\,\hat{\mathbf{y}}_3$
- Total angular momentum magnitude:

$$h = C\omega + 2m\left(R^2\omega + L^2(\omega + \dot{L}/R)\right)$$

- Angular momentum is conserved, so

$$h = \left(C + 2mR^2\right)\omega_0$$

- The next few steps are not "obvious," and I recommend you write them out
- Divide T by mR^2 and h_m by $2mR^2$
- Define $K = (C/2 + mR^2)/(mR^2)$
- Set the two $T/(mR^2)$ expressions equal to each other and simplify

$$K(\omega_0^2 - \omega^2) = \frac{L^2}{R^2}(\omega + \dot{L}/R)^2$$

- Set the two $h/(2mR^2)$ expressions equal and obtain

$$K(\omega_0 - \omega) = \frac{L^2}{R^2}(\omega + \dot{L}/R)$$

Yo-Yo Despin Analysis (4)

- Factor $(\omega_0^2 - \omega^2) = (\omega_0 - \omega)(\omega_0 + \omega)$, and divide the T and h expressions to obtain
$$\dot{L} = R\omega_0 = \text{ constant}$$
- Substitute this result into the expression for T to obtain
$$\omega = \omega_0 \frac{K - L^2/R^2}{K + L^2/R^2}$$
- Despin implies that we want $\omega \to 0$, which happens if
$$L = R\sqrt{K} = \sqrt{R^2 + C/(2m)}$$
- Note that this result for the length of the yo-yo cords for despin is independent of the initial angular velocity
- That means that a slight error in the spin rate won't affect the peformance of the despin mechanism
- Also, note that if the cord length is $L > R\sqrt{K}$, then the final angular velocity will be negative, thus reversing the spacecraft's spin direction
- Finally, note that in the $\lim L \to \infty$, the final angular velocity is $\omega = -\omega_0$