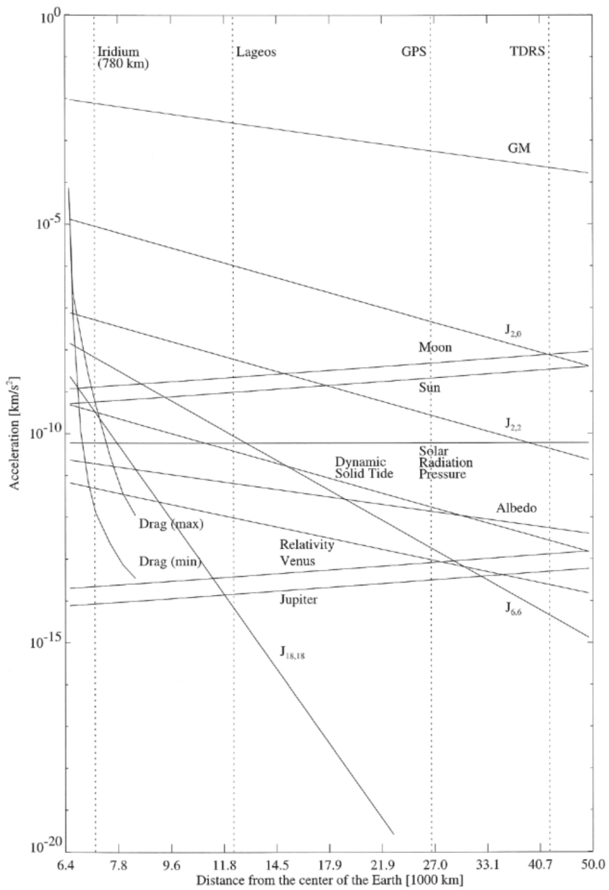
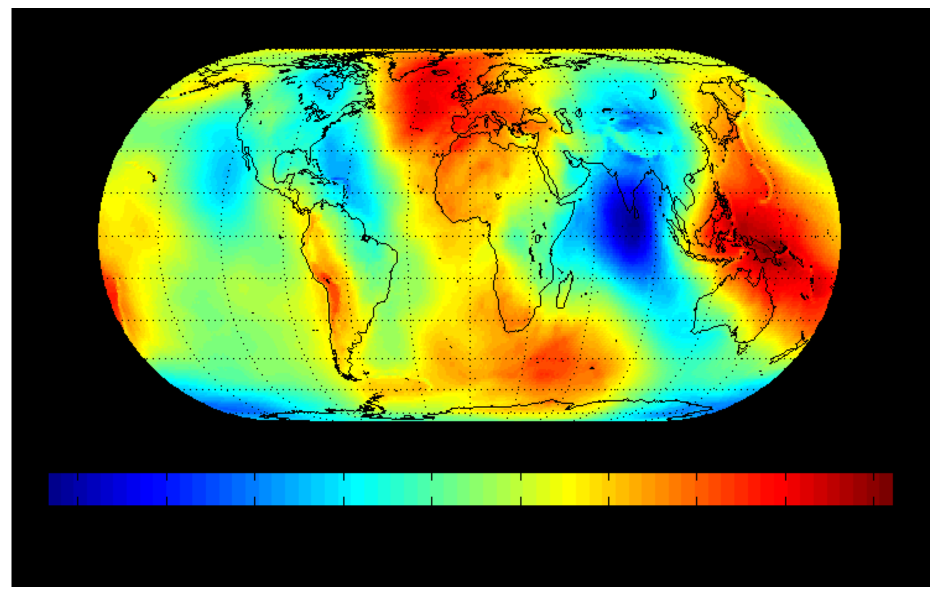
MECHANICAL ENGINEERING

Perturbation Force Modeling





Dynamic Model Fidelity

- Consider the 2-body/Keplerian motion model for the free motion of an Earth-orbiting space object:

$$\ddot{\bar{r}} = -\frac{\mu \bar{r}}{r^3}$$

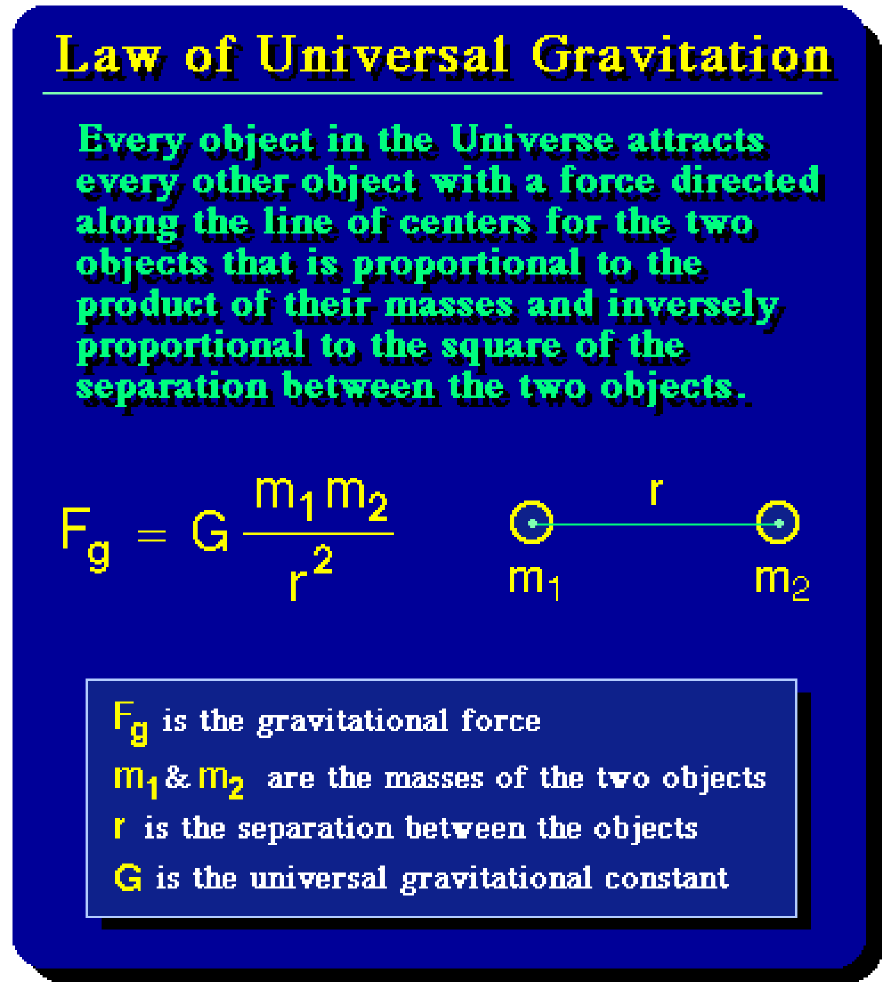
- While this is the most widely used model, it is also the lowest fidelity (i.e. most simplistic)

- While the conic section paradigm of 2-body motion approximates actual space object motion generally well over moderate time spans, various temporal & spatial errors exist

- 2-body motion based on Newton's Law of Gravitation

Earth’s Gravity

- Recall Newton’s Law of Gravitation from the orbital mechanics review in Module 1
- This law assumes the 2 bodies attracting each other are point masses (or equivalently, that the distance between them is much greater than either object’s radius)
- 2-body motion assumes Earth’s gravity is the only force acting on the object AND is acting according to this law



Earth's Gravity

- In reality, Earth-orbiting objects' motion is affected in other ways by Earth's gravity due mainly to 2 factors:
  - Earth (like all bodies) is a distributed mass
  - Earth is not a perfect sphere

- For this reason, the magnitude & direction of force due to "higher order gravity effects" varies as a function of position → the gravitational force an object experiences varies slightly throughout its orbit

- Many intricate models of Earth's gravity field have been developed, usually expressed in Cartesian or spherical coordinates → $\bar{\boldsymbol{F}}(x, y, z)$ or $\bar{\boldsymbol{F}}(r, \theta, \phi)$ where x, y, z or r, θ, ϕ are the coordinates of the position (i.e. point) at which we wish to calculate the force (or acceleration) due to Earth's gravity

Earth's Gravity

- One widely accepted model characterizes the potential U due to Earth's gravity
  - U often expressed as a function of ECI position: $U(x, y, z)$
  - Gradient of U is acceleration: $\frac{\partial U}{\partial x} = a_x \qquad \frac{\partial U}{\partial y} = a_y \qquad \frac{\partial U}{\partial z} = a_z$

- U is then modeled as a series expansion where each term U_k represents the acceleration due to a differential portion of the Earth → terms of the form:

$$U_k = \frac{\mu}{r}\left(\frac{R_E}{r}\right)^k P_k(sin\varphi)(C_m \cos\lambda + S_m \sin\lambda)$$

where r is distance of point from Earth's center, R_E is Earth radius, μ is Earth's gravitational constant, ϕ is latitude, λ is longitude

Earth's Gravity

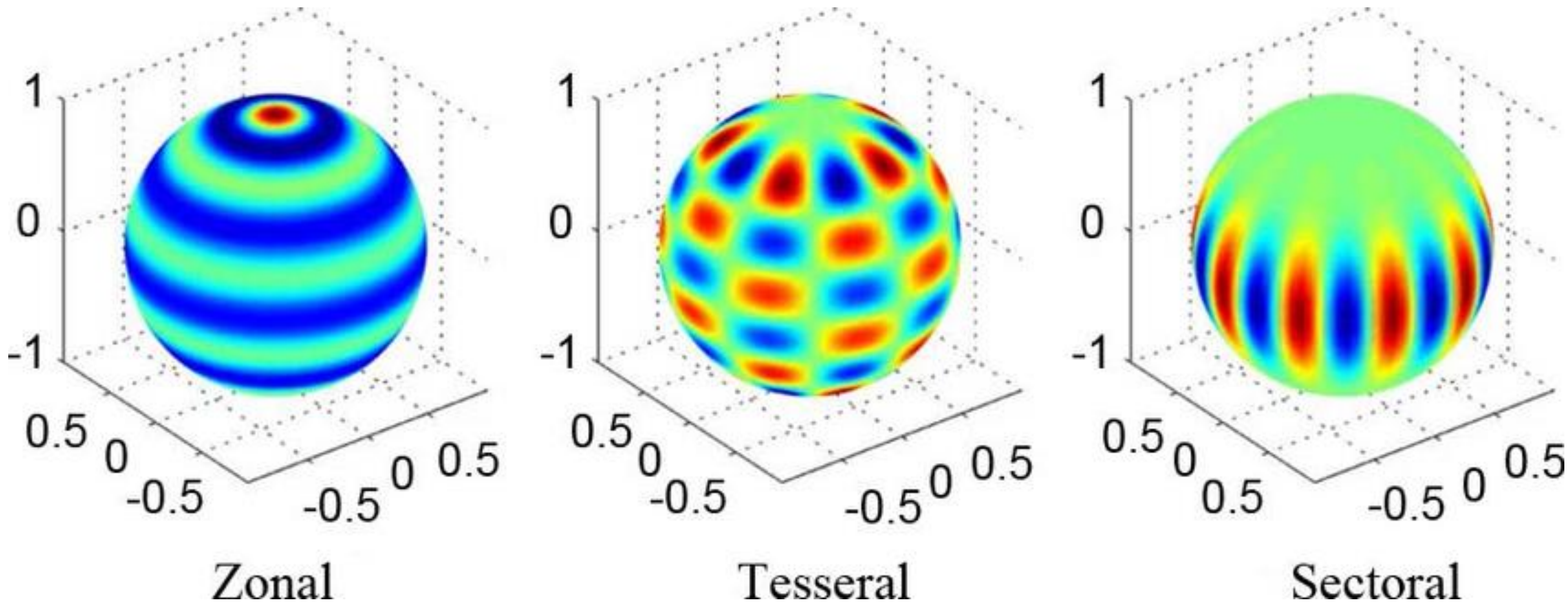
- So what are P_k, C_m, & S_m?

- P_k represents the series of Legendre polynomials (or Legendre functions)
  - P_k can be derived for any k (= 1, 2, 3, …) depending on how many terms one wishes to retain in the expansion of U
  - Some math references contain tables of the first several (5, 10, etc) Legendre polynomials
  - P_k are the latitude-dependent terms in U
- C_m, & S_m are constant coefficients representing the longitude-dependent terms in U

- Because we're modeling elements of a sphere (or near-sphere) & the terms involve trig functions, these are often called spherical harmonics

Earth's Gravity

- Harmonics are divided into the following categories:
  - Zonal: only latitude-dependent; symmetric about Earth's polar axis
  - Tesseral: divides sphere into a checkerboard array
  - Sectorial: only longitude-dependent; symmetric about Earth's equator



Earth's Gravity

- The dominant (& most commonly known) harmonic term is J_0, the 2-body term
- $J_1 = 0$ (as are several of the harmonic terms)
- 2nd most dominant harmonic term is J_2 → due to Earth's oblateness (fatter at the equator than at the poles)
- Acceleration due to J_2 expressed as a function of ECI position by

$$a_{J2,x} = \frac{3\mu J_2 R_E^2}{2r^5}\left(\frac{5z^2}{r^2} - 1\right)\mathrm{x}$$

$$a_{J2,y} = \frac{3\mu J_2 R_E^2}{2r^5}\left(\frac{5z^2}{r^2} - 1\right)\mathrm{y}$$

$$a_{J2,z} = \frac{3\mu J_2 R_E^2}{2r^5}\left(\frac{5z^2}{r^2} - 3\right)\mathrm{z}$$

- **J_2 : harmonic coefficient (= 0.00108 for Earth)**
- **R_E : Earth radius**
- **r = $\sqrt{x^2 + y^2 + z^2}$**

Earth's Gravity

- So if we wish to propagate an object's motion due to 2-body & J2 forces, our ODEs are

$$\ddot{x} = -\frac{\mu x}{r^3} + a_{J2,x} = -\frac{\mu x}{r^3} + \frac{3\mu J_2 R_E^2}{2r^5}\left(\frac{5z^2}{r^2} - 1\right)\mathrm{x}$$

$$\ddot{y} = -\frac{\mu y}{r^3} + a_{J2,y} = -\frac{\mu y}{r^3} + \frac{3\mu J_2 R_E^2}{2r^5}\left(\frac{5z^2}{r^2} - 1\right)\mathrm{y}$$

$$\ddot{z} = -\frac{\mu z}{r^3} + a_{J2,z} = -\frac{\mu z}{r^3} + \frac{3\mu J_2 R_E^2}{2r^5}\left(\frac{5z^2}{r^2} - 3\right)z$$

Effect of Earth's Gravity

- The primary effect of Earth's gravity perturbations on Earth-orbiting objects, from an SSA perspective, is the way J_2 force perturbs an object's motion

- Actual effect is fairly complex, but a simplified version is as follows:
  - Periodic oscillation (or "osculation") of all 6 orbit elements ($a, e, i, \Omega, \omega, \upsilon$)
  - Secular changes (i.e. changes that grow with time) to Ω, ω, υ
  - Secular changes to Ω are called nodal precession → gives rise to Sun-synchronous orbits
  - Secular changes to ω are called perigee precession → gives rise to Molniya orbits

These effects are detailed in the ME 595 course

Drag

- Drag equation for Earth-orbiting objects is the same as for aircraft:

$$\bar{a}_{drag} = -\frac{1}{2}\frac{C_D A}{m}\rho v_{rel}\bar{v}_{rel} \text{ where } \bar{v}_{rel} = \bar{v} - \bar{\omega}_E \text{x} \bar{r}$$

- **C_D : drag coefficient**
- **A : area normal to velocity vector**
- **m: mass**
- **ρ: density**
- **$\bar{r}$: position vector**
- **$\bar{v}$: velocity vector**
- **$\bar{\omega}_E$: Earth's rotation rate (= 360° /day)**

- 360° per sidereal day is approx. 7.292E-5 rad/sec, so Earth's rotational velocity vector may be expressed in the J2000 frame as $\bar{\omega}_E$ = 7.292E-5 $\hat{\boldsymbol{k}}$ rad/sec

- Drag is a non-conservative force → dissipates energy

- We talked last week about how drag decelerates a space object, hastening de-orbit → but how long does this take?

Drag

- So if we wish to propagate an object's motion due to 2-body & drag forces, our ODEs are

$$\ddot{x} = -\frac{\mu x}{r^3} + a_{drag,x}$$

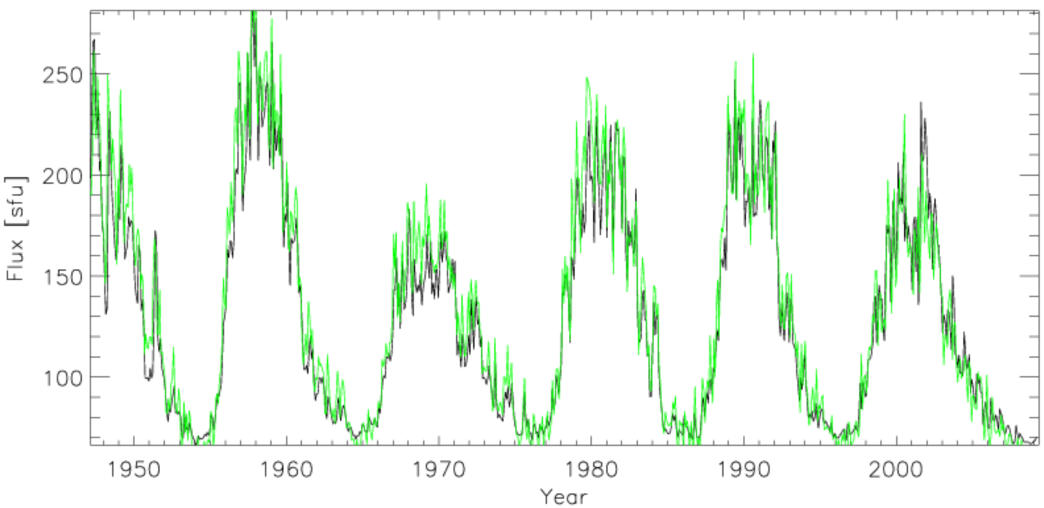
$$\ddot{y} = -\frac{\mu y}{r^3} + a_{drag,y}$$

$$\ddot{z} = -\frac{\mu z}{r^3} + a_{drag,z}$$

- Where the *x*, *y*, & *z* components of a_{drag} depend on the *x*, *y*, & *z* components of $\bar{v}$ - $\bar{\omega}_E$x$\bar{r}$ at each integration time step

Drag

- All qty's in the drag equation are uncertain to some degree
- The most uncertain (by far) is density, ρ
- Driven by multiple time-varying phenomena, including:
    - Geomagnetic planetary index (a_p)
    - Geomagnetic planetary index (k_p)
    - Solar flux ($F_{10.7}$) → driven by multiple temporal effects, e.g. 11-year solar cycle, 27-day solar rotation cycle



Solar flux plot shows peaks approx. every 11 years

- Scientists can model these phenomena to some extent, but much of this behavior is not well understood, i.e., considered random

Drag

- Numerous density models have been derived over several decades
- Numerical or empirical models make extensive use of lookup tables reflecting historical data
  - Examples are Jacchia, Harris-Priester, MSIS
- Analytical or physics models attempt to model every phenomenon mathematically
  - Examples are GITM, TIE-GCM
- Many of these models are extremely detailed & computationally intensive
- As with most scientific models, the choice of which to use boils down to a tradeoff between accuracy & computation time

Drag

- The simplest density model is the exponential "standard" atmosphere: $\rho = \rho_0 exp(-\frac{h-h_0}{H})$

- **ρ_0: reference density**
- **h: altitude**
- **h_0: reference altitude**
- **H: scale altitude**

Altitude h_{ellp} (km)	Base Altitude h_o (km)	Nominal Density ρ_o (kg/m^3)	Scale Height H (km)	Altitude h_{ellp} (km)	Base Altitude h_o (km)	Nominal Density ρ_o (kg/m^3)	Scale Height H (km)
0–25	0	1.225	7.249	150–180	150	2.070×10^{-9}	22.523
25–30	25	3.899×10^{-2}	6.349	180–200	180	5.464×10^{-10}	29.740
30–40	30	1.774×10^{-2}	6.682	200–250	200	2.789×10^{-10}	37.105
40–50	40	3.972×10^{-3}	7.554	250–300	250	7.248×10^{-11}	45.546
50–60	50	1.057×10^{-3}	8.382	300–350	300	2.418×10^{-11}	53.628
60–70	60	3.206×10^{-4}	7.714	350–400	350	9.518×10^{-12}	53.298
70–80	70	8.770×10^{-5}	6.549	400–450	400	3.725×10^{-12}	58.515
80–90	80	1.905×10^{-5}	5.799	450–500	450	1.585×10^{-12}	60.828
90–100	90	3.396×10^{-6}	5.382	500–600	500	6.967×10^{-13}	63.822
100–110	100	5.297×10^{-7}	5.877	600–700	600	1.454×10^{-13}	71.835
110–120	110	9.661×10^{-8}	7.263	700–800	700	3.614×10^{-14}	88.667
120–130	120	2.438×10^{-8}	9.473	800–900	800	1.170×10^{-14}	124.64
130–140	130	8.484×10^{-9}	12.636	900–1000	900	5.245×10^{-15}	181.05
140–150	140	3.845×10^{-9}	16.149	1000–	1000	3.019×10^{-15}	268.00

- Yields a rough approximation of density up to a few hundred km (actual drag does generally exhibit exponential behavior)

Effect of Drag

- The primary effect of drag, from an SSA perspective, is to de-orbit LEO objects → if there were no drag, objects would never naturally de-orbit
  - Magnitude of drag varies inversely with altitude
  - Effect on circular orbits is to follow a slow inward spiral toward Earth, until re-entry
  - For elliptical orbits, drag tends to slowly circularize the orbit as it spirals (b/c greatest effect of drag is at perigee of an orbit, which tends to lower its altitude at apogee)

- Due to the complexity (& "randomness") in so many phenomena contributing to drag, any sort of orbital lifetime prediction for an object is fraught with uncertainty
  - Prediction accuracy increases as object gets closer to re-entry

- Final note about drag: as with aerodynamic drag, side force components (e.g. "lift") exist in addition to the negative velocity force…but these components are often too small to warrant modeling

3^{rd}-Body Gravity

- Newton's Law of Gravitation tells us that every object in the universe exerts gravitational force on every other object

- For a given satellite, out of all the objects in the universe, most of these objects' mass is too small &/or their distance from the satellite too large for their gravity to have any effect on the satellite's motion

- Therefore, even in high-fidelity orbit propagators, only a few objects' gravity (other than Earth's) is included in such models

- The two most common "3^{rd} bodies" whose gravity is included in such models are the Sun & the Moon

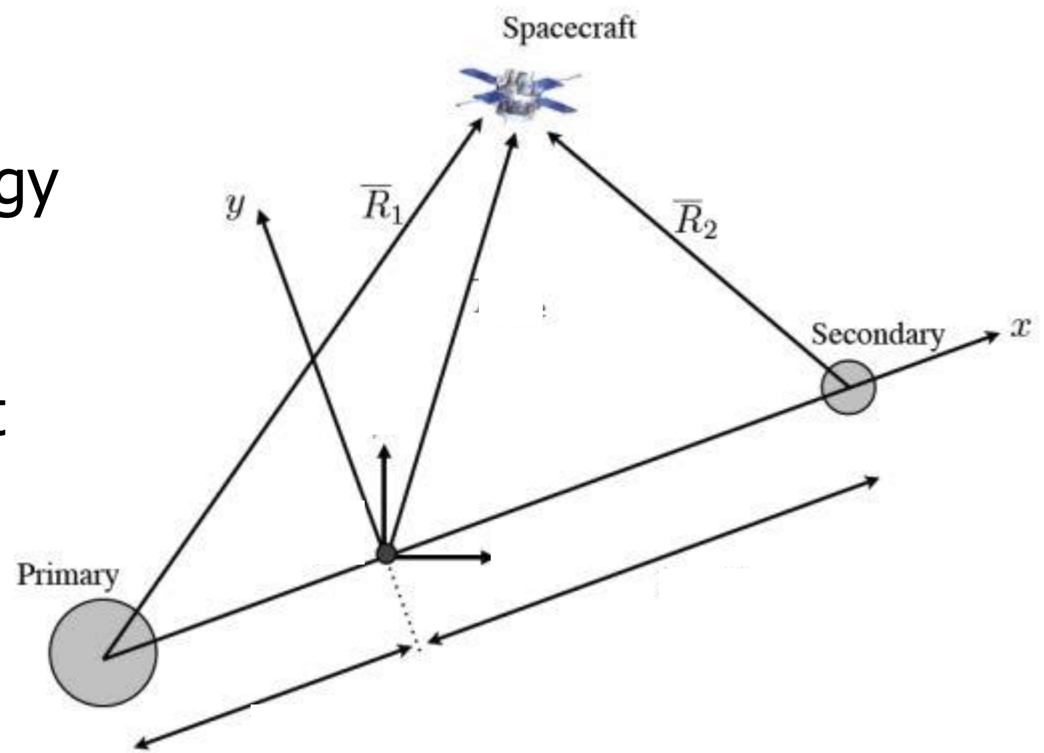
3rd-Body Gravity

- Acceleration due to a 3rd body is given by

$$a_{3rd} = -\mu_{3rd}\left[\frac{\bar{R}_2}{R_2^3} + \frac{\bar{R}_1 - \bar{R}_2}{(R_1 - R_2)^3}\right]$$

- **μ_{3rd}: gravitational constant of 3rd body**
- **$\bar{R}_1$: vector from Earth's center to object**
- **$\bar{R}_2$: vector from center of 3rd body to object**

- 3rd body forces are conservative (no energy dissipation), therefore they will not raise or lower an object's orbit



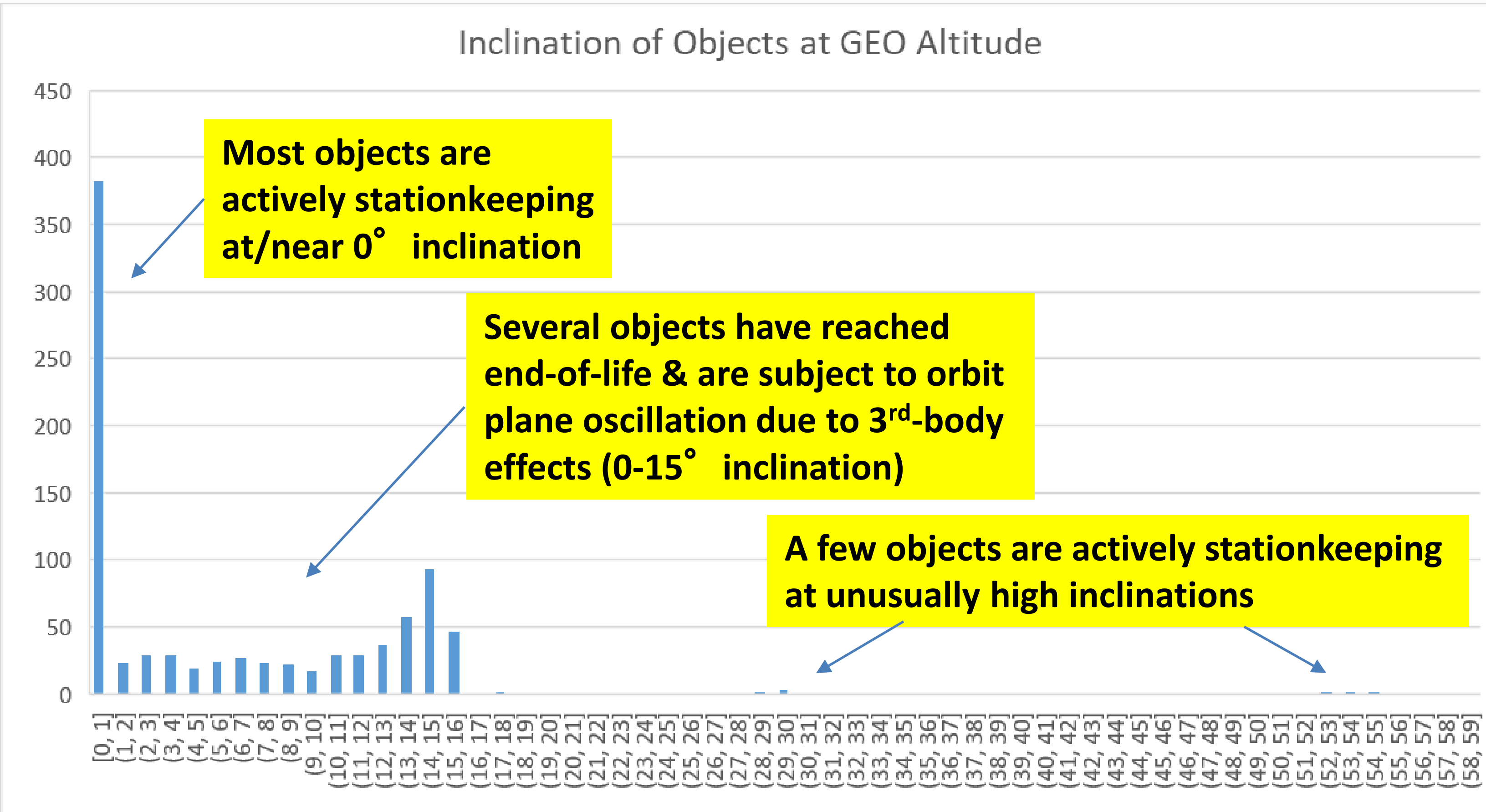
Effect of 3rd-Body Gravity

- The general effect of 3rd-body gravity is similar higher order gravity effects from Earth, i.e. periodic & secular changes to orbit elements

- One interesting 3rd-body effect, from an SSA standpoint, pertains to objects in/around GEO
  - Orbit plane oscillation of amplitude ~15°
  - Only occurs for objects in free motion (not maneuvering or stationkeeping)
  - Most GEO objects are actively stationkeeping → effect is only seen in objects that have reached end-of-life
  - Period of oscillation ~ 50yrs

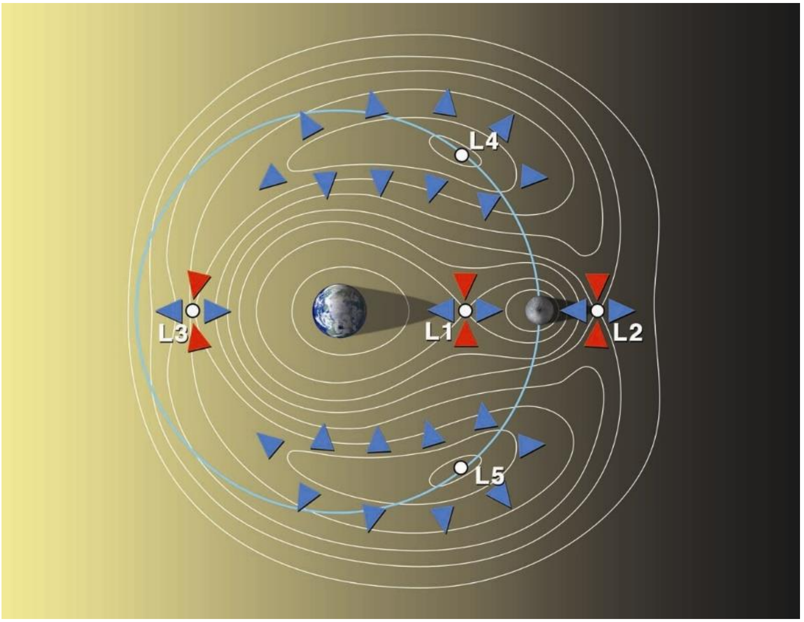
Effect of 3rd-Body Gravity

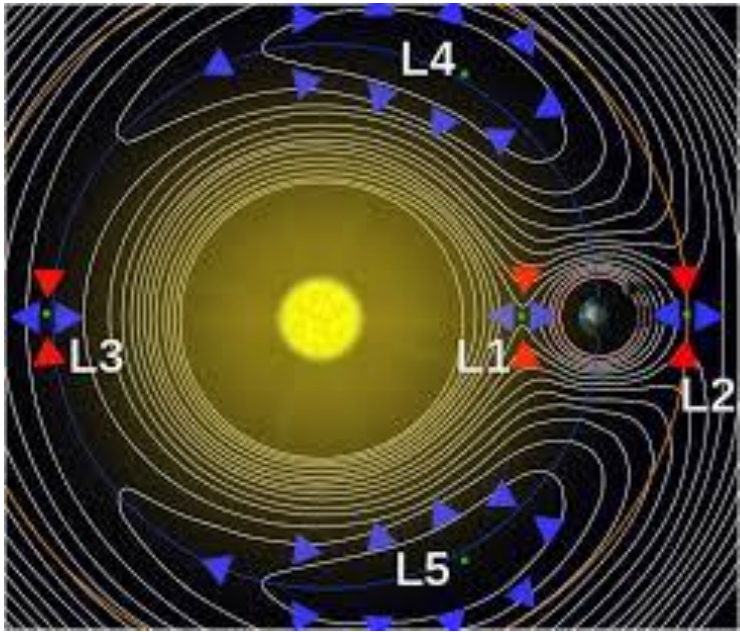
- Below is a histogram of inclination of objects in/around the GEO altitude (Source: space catalog downloaded from space-track.org)



Effect of 3rd-Body Gravity

- Another effect of 3rd-body gravity is the existence of Lagrange points
- These are points near 2 large attracting bodies where a smaller object will maintain its position relative to the large bodies
- Earth-Moon & Sun-Earth Lagrange points are shown below (5 pts each, labeled $L_1 - L_5$)





Effect of 3rd-Body Gravity

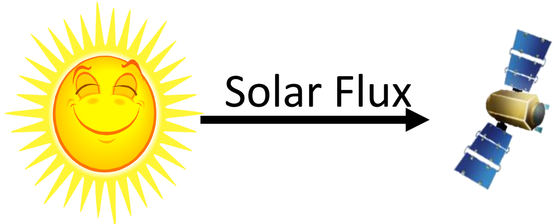
- Some interesting applications of Lagrange point missions include:
  - **Earth-Moon L_1**: Communication link at between Moon missions & ground stations on Earth
  - **Earth-Moon L_2**: Mission to explore the far side of the Moon
  - **Sun-Earth L_1**: Sun surveillance mission that is never eclipsed by Earth
  - **Sun-Earth L_2**: Mission whose objectives require eclipse conditions (would be in constant eclipse)

- For various reasons, objects in Lagrange point orbits aren't routinely tracked as part of SSA activities
  - Long distance from Earth would require large, expensive sensors
  - Very few objects currently in Lagrange point orbits → no threat of collisions, no "threatening" missions to be tracked

- However, 2 potential SSA implications of Lagrange point orbits are:
  - High-ground sensing: wide surveillance of the catalog from a strategic vantage point
  - If 1 or more Lagrange points eventually does get heavily populated, space-based sensors could be deployed in this vicinity

Solar Radiation Pressure

- Due to impingement of solar flux (i.e. photons) on an object's surface
- Many similarities between SRP & drag force
  - Both non-conservative
  - Both driven by solar phenomena
  - Both highly unpredictable
- Acceleration due to SRP is given by



$$\bar{a}_{SRP} = -\frac{\Phi}{c}\frac{A}{m}(\hat{u}_n \cdot \hat{u}_s)[(1-\epsilon)\hat{u}_s + 2\epsilon(\hat{u}_n \cdot \hat{u}_s)\hat{u}_n]$$

- **Φ: solar flux**
- **c : speed of light**
- **A : object area (facing Sun)**
- **m : object mass**
- **ε: reflectivity coefficient**
- **$\hat{u}_s$: unit vector from object to Sun**
- **$\hat{u}_n$: unit vector normal to object surface (facing Sun)**

Effect of Solar Radiation Pressure

- Unlike drag, SRP doesn't always act against object's motion → not even purely along-track (forward or backward orbital direction)
  - To some degree, effect of SRP on object during half of its orbital path canceled out during the other half → but still has potential to add or subtract energy overall (i.e. raise or lower orbit)
- SRP basically independent of orbital altitude (since the Sun is 93 million miles from Earth)
  - Dwarfed by other perturbations (e.g. J2, drag) at lower altitudes
  - Significant (by comparison) at higher altitudes → GEO & above
- High area-to-mass ratio (HAMR) objects in GEO very sensitive to SRP
  - Since SRP is area-dependent, its effect on HAMR objects can be significant & unpredictable
  - Tracking/OD of these objects can be quite difficult!
- Concept of solar sailing has garnered interest: using high area-to-mass ratio surfaces with precise attitude control to utilize SRP for propulsion
  - Found to be feasible from GEO out to deep space/interplanetary orbits
  - Solar sail missions would provide significant tracking/OD challenges

Relative Magnitudes of Perturbations

- Below is a chart developed by Montenbruck & Gill (2012) depicting the relative magnitudes of various perturbations as a function of altitude

- Some things to note:
  - All Earth gravity forces decrease with altitude (consistent with Newton's Law of Gravitation)
  - SRP & 3rd body forces increase slightly with altitude (as object gets closer to Moon or Sun)
  - Around 600-800km altitude, SRP & 3rd body forces become greater than drag