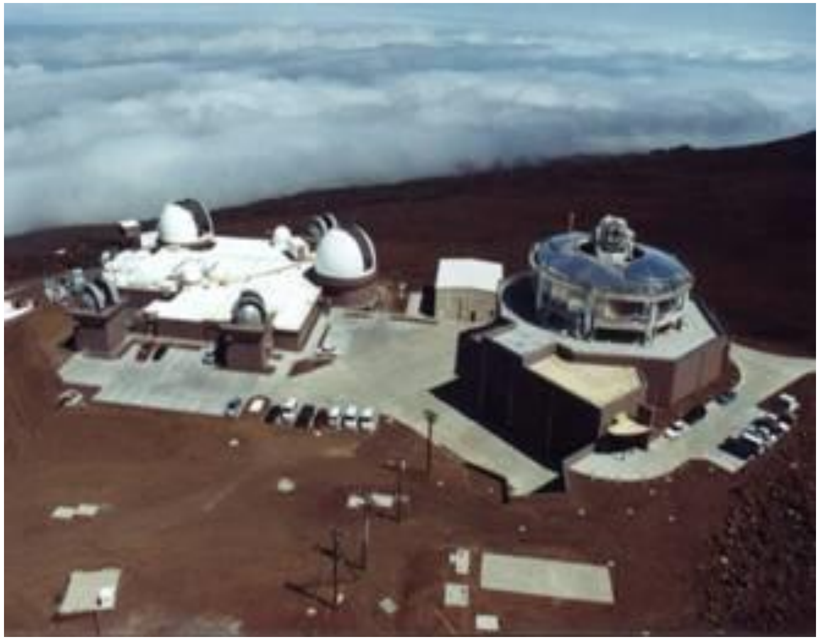
MECHANICAL ENGINEERING

Orbit Determination II



Precise Orbit Determination

- We talked last week about how OD is typically a 2-step process
  - Initial orbit determination (IOD) involves obtaining a decent (& fairly quick) solution from a minimal # of measurements
  - Precise orbit determination (POD) involves processing (usually) a large # of measurements, using an IOD solution as an initial guess
- We covered IOD last week; this week we focus on POD
- IOD is based on the idea that, if there were no error at all in the tracking process, a system's states could be determined exactly from a number of independent measurements equal to the number of states
  - For an Earth-orbiting space object, this number is 6
  - The Laplace method, for example, would solve the object's orbit exactly → 6 measurements to solve the 6 states

Precise Orbit Determination

- In reality, there are multiple error sources in play:
  - Model error → the dynamics we choose to model the object's motion will never exactly match the real motion
  - Measurement error → the sensor will have finite resolution, there will be noise (turbulence, interference) in the signal received by the sensor, etc
  - Sensor location error → sensor lat/long usually known very accurately, but not with 100% precision (with space-based sensors, this is even more problematic!)

- POD attempts to refine an IOD solution (called the "nominal" orbit), usually by processing many more measurements than were used in IOD

- The essence of POD is to make use of ALL available observation data to obtain the "best" estimate → but how do we define "best"?

Least Squares

- One example of a best estimate is the least-squares estimate

- Before applying this idea directly to orbital mechanics, let's start by considering an *n*-state system with the following 2 assumptions:
  - It is static, i.e. its states are constant & don't change with time:

$$\underbrace{\dot{\bar{X}}(t)}_{nx1} = \underbrace{\bar{0}}_{nx1} \rightarrow \underbrace{\bar{X}(t)}_{nx1} = \underbrace{\bar{X}}_{nx1}$$

  - Our measurements of the system are some linear function (i.e. linear combination) of the states:

$$\underbrace{y_i}_{1x1} = h_{i1}x_1 + h_{i2}x_2 + \cdots + h_{in}x_n = \underbrace{h_i}_{1xn} \ \underbrace{\bar{X}}_{nx1}$$

Least Squares

- If we collect n measurements y_1 - y_n , we can arrange these measurement equations in matrix-vector form:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} - & h_1 & - & - \\ - & h_2 & - & - \\ - & \vdots & - & - \\ - & h_n & - & - \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \rightarrow \quad \bar{Y} = H\bar{X}$$

nx1 **nxn** **nx1**

- If the measurements were perfect, we could solve the states exactly by

$$\bar{X} = H^{-1}\bar{Y}$$

Least Squares

- In reality, each measurement incurs error, which we'll call ε:

$$y_i = h_i \ \bar{X} + \ \varepsilon_i$$

- Since each ε_i is unknown, our $H^{-1}\bar{Y}$ solution isn't exact, but it is our "best" estimate of the states with the information we have

- If we have more than *n* measurements ("*p*" where *p* > *n*), we can again arrange the measurement equations in matrix-vector form (although now our *H* matrix is non-square):

$$\underset{\mathbf{px1}}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ y_p \end{bmatrix}} = \underset{\mathbf{pxn}}{\begin{bmatrix} - & h_1 & - & - \\ - & h_2 & - & - \\ - & \vdots & - & - \\ - & \vdots & - & - \\ - & \vdots & - & - \\ - & h_p & - & - \end{bmatrix}} \underset{\mathbf{nx1}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \quad \rightarrow \quad \bar{Y} = H\bar{X}$$

Least Squares

- Our system of equations is overdetermined (more measurements than unknowns)

- If the measurements were perfect, we could choose any *n* of the equations to solve the states by $\bar{X} = H^{-1}\bar{Y}$
  - No matter which *n* of the equations we chose, the solution would be the same (& would be the exact values of $\bar{X}$)

- But because each measurement is imperfect, we can utilize the extra measurements to (hopefully) improve on our "best" estimate obtained with *n* measurements

Least Squares

- Let's begin with $\bar{Y} = H\bar{X}$ & pre-multiply both sides by H^T:

$$\underbrace{H^T}_{\text{nxp}}\underbrace{\bar{Y}}_{\text{px1}} = \underbrace{H^T}_{\text{nxp}}\underbrace{H}_{\text{pxn}}\underbrace{\bar{X}}_{\text{nx1}}$$

- Because H^TH is square (n x n), we can invert it (provided it is invertible)

- Therefore, if we pre-multiply both sides by $(H^TH)^{-1}$, we obtain

$$(H^TH)^{-1}H^T\bar{Y} = (H^TH)^{-1}(H^TH)\bar{X} = \bar{X}$$

- This is the least-squares solution for $\bar{X}$
  - Utilizes all available measurement data to obtain the "best" estimate
  - $(H^TH)^{-1}H^T$ is called the "pseudo-inverse" of H

Least Squares

- Is the least-squares estimate really the "best" estimate we can obtain?

- The short answer is YES if the measurements all came from the same sensor

- The longer answer has to do with statistics:
  - The measurement errors ε_i can be modeled as a random distribution, whose parameters are based on the sensor's accuracy
  - Each ε_i can be viewed as a sample from this distribution
  - If we were to add one measurement at a time to our system of equations, recomputing $\bar{X} = (H^T H)^{-1} H^T \bar{Y}$ each time, the error between our solution for $\bar{X}$ & the true values of $\bar{X}$ may not decrease monotonically; but we should see a gradual asymptotic decrease to some minimum value

Nonlinear (Iterative) Least Squares

- The previous procedure is often called linear least-squares

- Let's now consider a new scenario similar to the previous one, but relaxing our 2nd assumption, i.e. the measurements are nonlinear functions of the states:

$$y_i = g_i(\bar{X})$$

- So our measurement equations cannot be written in matrix-vector form, but simply as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} g_1(\bar{X}) \\ g_2(\bar{X}) \\ \vdots \\ g_p(\bar{X}) \end{bmatrix}$$

Nonlinear (Iterative) Least Squares

- Consider a first-order Taylor expansion of $g_i(\bar{X})$:

$$g_i(\bar{X}) \approx g_i(\bar{X}_0) + \frac{\partial g_i}{\partial \bar{X}}|_{\bar{X}_0}(\Delta\bar{X}) \quad where \;\; \Delta\bar{X} = \bar{X} - \bar{X}_0$$

(Here the $\frac{\partial g_i}{\partial \bar{X}}|_{\bar{X}_0}$ notation means the partial derivative of g_i with respect to $\bar{X}$, evaluated at $\bar{X}_0$)

- So if we had an initial guess of the state values $\bar{X}_0$, we could define $\Delta \boldsymbol{y}_i = g_i(\bar{X}) - g_i(\bar{X}_0)$

- The Taylor expansion above approximates the relationship between $\Delta \boldsymbol{y}_i$ and $\Delta\bar{X}$:

$$\underbrace{\Delta \boldsymbol{y}_i}_{\text{1x1}} \approx \underbrace{\frac{\partial g_i}{\partial \bar{X}}}_{\text{1xn}}|_{\bar{X}_0}\underbrace{(\Delta\bar{X})}_{\text{nx1}}$$

- Thus, while the measurements are nonlinear functions of the states, the “delta-measurements” are linear functions of the “delta-states”

Nonlinear (Iterative) Least Squares

- The "delta-measurements" are found by evaluating $y_{i0} = g_i(\bar{X}_0)$ & subtracting these values from the actual measurement values:

$$\Delta\bar{Y} = \bar{Y} - \bar{Y}_0$$

- We then use the equation at bottom of the previous slide to solve $\Delta\bar{X}$:

$$\underbrace{\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \vdots \\ \vdots \\ \Delta y_p \end{bmatrix}}_{\text{px1}} = \underbrace{\begin{bmatrix} \frac{\partial g_1}{\partial x_1}\Big|_{\bar{X}_0} & \cdots & \frac{\partial g_1}{\partial x_n}\Big|_{\bar{X}_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1}\Big|_{\bar{X}_0} & \cdots & \frac{\partial g_p}{\partial x_n}\Big|_{\bar{X}_0} \end{bmatrix}}_{\text{pxn}} \underbrace{\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}}_{\text{nx1}} \quad \rightarrow \Delta\bar{Y} = H\Delta\bar{X}$$

- If $p = n$ (H is square), we solve $\Delta\bar{X} = H^{-1}\Delta\bar{Y}$; if $p > n$ (H is non-square), we solve $\Delta\bar{X} = (H^T H)^{-1} H^T \Delta\bar{Y}$ (the least-squares solution)

- This practice of solving linear equations to correct an initial guess is known as differential corrections

Nonlinear (Iterative) Least Squares

- Will solving $\Delta\bar{X}$ in this way & adding it to $\bar{X}_0$ then yield the correct value of $\bar{X}$?

- Typically no, because this is only a first-order correction (the Taylor expansion contains an infinite # of terms)

- But we can perform differential corrections iteratively as follows (described at bottom of p. 125 of Bate, Mueller, & White textbook):
  - Given an initial guess $\bar{X}_0$, calculate $\bar{Y}_0$ then $\Delta\bar{Y} = \bar{Y} - \bar{Y}_0$
  - Solve $\Delta\bar{X}$ from $\Delta\bar{Y} = H\Delta\bar{X}$, as detailed on the previous slide
  - Update our guess as $\bar{X}_1 = \bar{X}_0 + \Delta\bar{X}$ & repeat the above steps until a chosen stopping criterion is reached

Nonlinear (Iterative) Least Squares

- How do we choose a stopping criterion? (i.e. how many iterations are enough?)

- If the iterative process is properly converging, both $\Delta\bar{X}$ & $\Delta\bar{Y}$ should decrease slightly with each iteration

- A common stopping criterion is when the % change in $\Delta\bar{X}$ from one iteration to the next is below a chosen threshold (say, 5%), or similarly the % change in $\Delta\bar{Y}$

- The quantity $\Delta\bar{Y}$ is referred to as the residual; it represents the difference between the predicted measurement values (based on the current guess of $\bar{X}$) & the actual measurement values recorded
  - Our actual *ith* measurement value y_i is known
  - Our predicted *ith* measurement value is $g_i(\bar{X}_n)$, where $\bar{X}_n$ is our guess of $\bar{X}$ after *n* iterations in our process

Nonlinear (Iterative) Least Squares

- Is this iterative process guaranteed to converge?

- The short answer is no
  - The approach described here is a multi-dimensional linear gradient-based root-solving routine
  - With such routines, convergence depends strongly on the initial guess
  - One could utilize a second-order Taylor expansion of $g_i(\bar{X})$:

$$g_i(\bar{X}) \approx g_i(\bar{X}_0) + \frac{\partial g_i}{\partial \bar{X}}|_{\bar{X}_0}(\Delta \bar{X}) + \frac{1}{2}\frac{\partial^2 g_i}{\partial \bar{X}^2}|_{\bar{X}_0}(\Delta \bar{X})^2$$

  - This would be a more accurate approximation of $g_i(\bar{X})$, but leads to a system of coupled 2nd-order polynomials (rather than linear eqn's)

- Likelihood of convergence is driven mainly by (1) how close the initial guess is to the true state values (2) how much (or little) error the measurements contain

Nonlinear (Iterative) Least Squares

EXAMPLE:

- Consider a system with 3 states (x_1, x_2, & x_3), with 3 measurements of the system recorded (y_1, y_2, & y_3)

- The measurements are represented by the following equations:

$$y_1 = g_1(\bar{X}) = x_1 + sin x_2 + \ {x_3}^2$$

$$y_2 = g_2(\bar{X}) = ln x_1 + tan x_2$$

$$y_3 = g_3(\bar{X}) = \frac{x_1}{x_2} + \ {x_3}^3$$

- & suppose the measurements y_i are y_1 = 5.0998, y_2 = 0.1003, y_3 = 18

- How do we solve for x_1, x_2, & x_3?

Nonlinear (Iterative) Least Squares

EXAMPLE (cont'd):

- Let's start with a guess $\bar{X}_0$ of x_{10} = 0.9144, x_{20} = 0.0949, x_{30} = 1.9879; our predicted measurement values are then:

$$y_{10} = g_1(\bar{X}_0) = x_{10} + sinx_{20} + \ {x_{30}}^2 = 4.9609$$

$$y_{20} = g_2(\bar{X}_0) = lnx_{10} + tanx_{20} = 0.0057$$

$$y_{30} = g_3(\bar{X}_0) = \frac{x_{10}}{x_{20}} + \ {x_{30}}^3 = 17.4911$$

- So our residual values are $\Delta\bar{Y}_0 = \bar{Y} - \bar{Y}_0 = \begin{bmatrix} 0.1389 \\ 0.0946 \\ 0.5089 \end{bmatrix}$

Nonlinear (Iterative) Least Squares

EXAMPLE (cont'd):

- Next we calculate $H = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & cosx_2 & 2x_3 \\ \frac{1}{x_1} & sec^2x_2 & 0 \\ \frac{1}{x_2} & -\frac{x_1}{{x_2}^2} & 3{x_3}^2 \end{bmatrix}$

- And evaluate the elements at $\bar{X}_0$ → $H_0 = \begin{bmatrix} 1 & 0.9955 & 3.9759 \\ 1.0936 & 1.0091 & 0 \\ 10.5374 & -101.5322 & 11.8552 \end{bmatrix}$

- Then our corrections to the states are $\Delta\bar{X}_0 = H_0^{-1}\Delta\bar{Y}_0 = \begin{bmatrix} 0.0819 \\ 0.0050 \\ 0.0131 \end{bmatrix}$

Nonlinear (Iterative) Least Squares

EXAMPLE (cont'd):

- So our state update is $\bar{X}_1 = \bar{X}_0 + \Delta\bar{X}_0 = \begin{bmatrix} 0.9963 \\ 0.0999 \\ 2.0010 \end{bmatrix}$
- This completes one iteration; to iterate again, calculate:

$$\bar{Y}_1 = \mathrm{G}(\bar{X}_1)$$

$$\Delta\bar{Y}_1 = \bar{Y} - \bar{Y}_1$$

$$H_1 = \frac{\partial \bar{G}}{\partial \bar{X}}|_{\bar{X}_1}$$

$$\Delta\bar{X}_1 = H_1^{-1}\Delta\bar{Y}_1$$

$$\bar{X}_2 = \bar{X}_1 + \Delta\bar{X}_1$$

- This yields $\bar{X}_2 = \begin{bmatrix} 1.0000 \\ 0.1000 \\ 2.0000 \end{bmatrix}$; the residual $\Delta\bar{Y}_1$ was $\begin{bmatrix} -0.00017 \\ 0.0038 \\ 0.0167 \end{bmatrix}$

Nonlinear (Iterative) Least Squares

EXAMPLE (cont'd):

- If we calculate the residual $\Delta\bar{Y}_2$ corresponding to $\bar{X}_2$, all 3 of its values are 10^{-5} or below (same is true of $\Delta\bar{X}_2$)

- Regardless whether we base our stopping criterion on the values of $\Delta\bar{X}$ or the values of $\Delta\bar{Y}$, either way, the process is clearly converging

- I simulated the above measurements (y_i) using "true" state values of $x_1 = 1$, $x_2 = 0.1$, $x_3 = 2$; we've already achieved those values to 4 decimal places in only 2 iterations

- But what if we add error to our measurements? Let $y_1 = 5.1158$, $y_2 = 0.1160$, $y_3 = 17.9568$ & start with the same guess of $x_{10} = 0.9144$, $x_{20} = 0.0949$, $x_{30} = 1.9879$

Nonlinear (Iterative) Least Squares

<u>EXAMPLE (cont'd):</u>

- Applying the same algorithm & iterating 5 times yields $\bar{X}_5 = \begin{bmatrix} 1.0139 \\ 0.1018 \\ 2.0001 \end{bmatrix}$ with all values of $\Delta\bar{X}_5$ & $\Delta\bar{Y}_5$ on the order of 10^{-15} or less

- So again we converge, but to a state solution corresponding to the wrong (imperfect) measurement values
  - Our solution is less accurate than before (though still fairly close to the true $\bar{X}$)
  - If we add higher measurement error & repeat the process, we'll likely converge to a solution even further from the true $\bar{X}$

- It can be shown that adding enough measurement error and/or starting with a guess far enough from the true state values may cause the problem to diverge (try it yourself!)

Nonlinear (Iterative) Least Squares

EXAMPLE (cont'd):

- What if we had a 4th measurement? Say, y_4 = -0.4008, where

$$y_4 = g_4(\bar{X}) = x_1 x_2 + cos\ x_3$$

- Note that if we evaluate this function at the true state values x_1 = 1, x_2 = 0.1, x_3 = 2, we get -0.3162; so this measurement too contains error

- How do we incorporate this additional measurement into our solution process?

Nonlinear (Iterative) Least Squares

EXAMPLE (cont'd):

- The process is basically the same as before, but we calculate the least-squares solution at each iteration:
    - $\bar{Y}$, $\bar{Y}_i$, & $\Delta\bar{Y}_i$ are each 4x1 vectors (not 3x1)
    - *H* matrix is 4x3 (not 3x3)
    - Solve by calculating the pseudo-inverse of *H* (not the inverse)

- Will the process converge as before?
    - Generally yes, if the error on the 4th measurement isn't too high

- How will processing 4 measurements affect our state sol'n?
    - A single additional measurement may not move the sol'n closer to the true states (in fact, it may be further from the true states than with 3 measurements)
    - But generally the more measurements we process, the more accurate our sol'n should be

Precise Orbit Determination

- We're now one step away from connecting the preceding ideas to POD

- Let's consider another scenario, where now we relax our 1st assumption as well, i.e. the system is dynamic, whereby the states are governed by the following ODEs:

$$\dot{\bar{X}}(t) = \bar{F}(\bar{X})$$

- Here, since the states change with time, the time of each measurement is important
- The measurement equations are then written as

$$y(t_i) = g(\bar{X}(t_i))$$

Precise Orbit Determination

- Another implication here is that we're not solving a set of static states $\bar{X}$; so what form does the state estimate take?

- Typical practice is to solve the states at a particular initial or "epoch" time before the first measurement → call this t_0

- The Taylor expansion is then written in terms of $\bar{X}(t_{\mathbf{0}})$, so the correction term looks like:

$$\frac{\partial g(\bar{X}(t_i))}{\partial \bar{X}(t_{\mathbf{0}})}\Big|_{\bar{X}(t_{\mathbf{0}})_0}(\Delta\bar{X}(t_{\mathbf{0}}))$$

This "0" subscript denotes the initial/epoch time

This "0" subscript denotes the initial guess for $\bar{X}(t_{\mathbf{0}})$

Precise Orbit Determination

- So how do we take the derivative of a measurement at t_i with respect to the states at t_0?

- We will explore both a numerical & analytical method to do this…

Precise Orbit Determination

<u>NUMERICAL METHOD</u>:

- The elements of $\frac{\partial g(\bar{X}(t_i))}{\partial \bar{X}(t_0)}$ are $[\frac{\partial g(\bar{X}(t_i))}{\partial x_1(t_0)} \frac{\partial g(\bar{X}(t_i))}{\partial x_2(t_0)} \ldots \frac{\partial g(\bar{X}(t_i))}{\partial x_n(t_0)}]$

- Approximate $\frac{\partial g(\bar{X}(t_i))}{\partial x_1(t_0)}$ by $\frac{\Delta g(\bar{X}(t_i))}{\Delta x_1(t_0)}$ as follows:
  - Consider a "reference state" at the epoch time → call it $\bar{X}(t_0)_0$
  - Propagate the state (i.e. numerically integrate the ODEs) from t_0 to t_i to obtain $\bar{X}(t_i)_0$, then evaluate $g(\bar{X}(t_i)_0)$
  - Next, perturb $x_1(t_0)_0$ by a small amount $\Delta x(t_0)_0$ (so that $x_1(t_0)_0 + \Delta x(t_0)_0 = x_1(t_0)_0{*}$) & leave all other state values unchanged → call the perturbed state vector $\bar{X}(t_0)_0{*}$
  - Propagate $\bar{X}(t_0)_0{*}$ from t_0 to t_i to obtain $\bar{X}(t_i)_0{*}$, then evaluate $g(\bar{X}(t_i)_0){*}$

  - Then $\frac{\partial g(\bar{X}(t_i))}{\partial x_1(t_0)} \approx \frac{\Delta g(\bar{X}(t_i))}{\Delta x_1(t_0)} = \frac{g(\bar{X}(t_i)_0){*} - g(\bar{X}(t_i)_0)}{\Delta x(t_0)_0}$ (similarly for $\frac{\partial g(\bar{X}(t_i))}{\partial x_2(t_0)}$, etc)

NUMERICAL METHOD (cont'd):

- How much should we perturb each state to accurately represent $\frac{\partial g(\bar{X}(t_i))}{\partial \bar{X}(t_{\mathbf{0}})}$ numerically?
- The short answer is "an infinitesimal amount," as each partial derivative is the derivative right at the $\bar{X}(t_{\mathbf{0}})_0$ values
- So perturb each state as little as possible without causing precision issue in the result (perhaps 0.1-1.0%)

Precise Orbit Determination

<u>ANALYTICAL METHOD</u>:

- The chain rule of differentiation tells us that $\frac{\partial g(\bar{X}(t_i))}{\partial \bar{X}(t_\mathbf{0})} = \frac{\partial g(\bar{X}(t_i))}{\partial \bar{X}(t_\mathbf{i})} \frac{\partial \bar{X}(t_\mathbf{i})}{\partial \bar{X}(t_\mathbf{0})}$

- Next, we recognize that $\frac{\partial \bar{X}(t_i)}{\partial \bar{X}(t_\mathbf{0})}$ for linear systems is the state transition matrix (STM) of the system: $\bar{X}(t_\mathbf{i}) = \Phi(t_i, t_0)\bar{X}(t_0)$

- So how do we calculate Φ at each measurement time?
  - We know that $\Phi(t_0, t_0)$ = I (identity matrix) & $\dot{\Phi} = A\Phi$
  - For nonlinear systems, A is approximated at time t_i by $\frac{\partial \bar{F}}{\partial \bar{X}}\Big|_{\bar{X}(t_\mathbf{i})}$
  - Φ can then be propagated (linearly) to each measurement time

Linear "state matrix" of the system

- Note that this method approximates the nonlinear system as a linear one!

Precise Orbit Determination

- Let's now apply these ideas to determining the orbit (i.e. estimating the motion) of an Earth-orbiting space object
- The state vector consists of the object's ECI position & velocity components: $\bar{X} = [\mathrm{x\ y\ z}\ v_x\ v_y\ v_z]^T$
- Assume 2-body motion for the object's dynamics; we noted last week that these ODEs are:

$$\dot{\bar{X}}(t) = \bar{F}(\bar{X}) \longrightarrow \frac{d}{dt}\underbrace{\begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix}}_{\mathbf{6x1}} = \underbrace{\begin{bmatrix} v_x \\ v_y \\ v_z \\ -\frac{\mu x}{r^3} \\ -\frac{\mu y}{r^3} \\ -\frac{\mu z}{r^3} \end{bmatrix}}_{\mathbf{6x1}}$$

Precise Orbit Determination

- Focusing on angles-only OD (i.e. optical sensor measurements), we saw last week that the LOS measurement equations are

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \dfrac{x - R_{OAx}}{\sqrt{(x - R_{OAx})^2 + \left(y - R_{OAy}\right)^2 + (z - R_{OAz})^2}} \\ \dfrac{y - R_{OAy}}{\sqrt{(x - R_{OAx})^2 + \left(y - R_{OAy}\right)^2 + (z - R_{OAz})^2}} \end{bmatrix}$$

- Because LOS yields two pieces of info at each measurement time t_i, we will refer to the 2 scalar equations as g_x & g_y:

$$g_x\left(\bar{X}(t_i)\right) = \frac{x(t_i) - R_{OAx}}{\sqrt{(x(t_i) - R_{OAx})^2 + \left(y(t_i) - R_{OAy}\right)^2 + (z(t_i) - R_{OAz})^2}}$$

$$g_y\left(\bar{X}(t_i)\right) = \frac{y(t_i) - R_{OAy}}{\sqrt{(x(t_i) - R_{OAx})^2 + \left(y(t_i) - R_{OAy}\right)^2 + (z(t_i) - R_{OAz})^2}}$$

Precise Orbit Determination

- If the number of measurement times is p, we have $2p$ total measurement equations

- The number of orbital states to be solved is 6 → thus, if $p \geq 3$, we have enough equations to determine the object's orbit via differential corrections

- We can now outline the series of steps to perform angles-only POD via the numerical method described above

Precise Orbit Determination

NUMERICAL METHOD:

Given LOS measurements u_x & u_y at times t_i, where i = 1 to p:

1. Begin with an initial guess of states at the epoch time $\bar{X}(t_0)_0$

2. Propagate $\bar{X}(t_0)_0$ to each measurement time t_i & calculate $g_x(\bar{X}(t_i)_0) = u_x$ & $g_y(\bar{X}(t_i)_0) = u_y$, thereby constructing $\bar{Y}_0$

3. Perturb the first state $x(t_0)_0$ by a small amount $\Delta x(t_0)_0$ to obtain

$\bar{X}(t_0)_0 + \begin{bmatrix} \Delta x(t_0)_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \bar{X}(t_0)_0{}^*$, propagate $\bar{X}(t_0)_0{}^*$ to each measurement time t_i, & calculate $g_x(\bar{X}(t_i)_0)* = u_x$ & $g_y(\bar{X}(t_i)_0)* = u_y$ at each t_i

4. Calculate $\frac{g_x(\bar{X}(t_i)_0)* - g_x(\bar{X}(t_i)_0)}{\Delta x(t_0)_0}$ & $\frac{g_y(\bar{X}(t_i)_0)* - g_y(\bar{X}(t_i)_0)}{\Delta x(t_0)_0}$ at each t_i; these *2p* values constitute the first column of H_0

Precise Orbit Determination

NUMERICAL METHOD (cont'd):

5. One at a time, perturb each of the other 5 states by a small amount & perform Steps 3-4, thereby constructing the remaining columns of H_0

6. Calculate $\Delta\bar{X}(t_0)_0 = H_0^{-1}(\bar{Y} - \bar{Y}_0)$ (if p = 3)

6x1 **6x6** **6x1**

or $\Delta\bar{X}(t_0)_0 = (H_0^T H_0)^{-1} H_0^T (\bar{Y} - \bar{Y}_0)$ (if p > 3)

6x1 **6x2p** **2px6** **6x2p** **2px1**

7. Update the state estimate as $\bar{X}(t_0)_1 = \bar{X}(t_0)_0 + \Delta\bar{X}(t_0)_0$

8. Repeat Steps 2-7 until a desired stopping criterion is reached

Precise Orbit Determination

- For the analytical POD method, recall that we calculate $\underbrace{\frac{\partial g(\bar{X}(t_i))}{\partial \bar{X}(t_i)}}_{\text{2px6}}$ & $\underbrace{\frac{\partial \bar{X}(t_i)}{\partial \bar{X}(t_0)}}_{\text{6x6}}$ separately

- So we write the partial derivatives $\frac{\partial g_x(\bar{X}(t_i))}{\partial \bar{X}(t_i)}$ as

$$\underbrace{\frac{\partial g_x(\bar{X}(t_i))}{\partial \bar{X}(t_i)}}_{\text{1x6}} = \left[\frac{\partial g_x(\bar{X}(t_i))}{\partial x(t_i)} \quad \frac{\partial g_x(\bar{X}(t_i))}{\partial y(t_i)} \quad \frac{\partial g_x(\bar{X}(t_i))}{\partial z(t_i)} \quad \cancelto{0}{\frac{\partial g_x(\bar{X}(t_i))}{\partial v_x(t_i)}} \quad \cancelto{0}{\frac{\partial g_x(\bar{X}(t_i))}{\partial v_y(t_i)}} \quad \cancelto{0}{\frac{\partial g_x(\bar{X}(t_i))}{\partial v_z(t_i)}}\right]$$

- Note that $g_x(\bar{X}(t_i))$ is a function only of x, y, & z (not a function of v_x, v_y, or v_z), so the 4th, 5th, & 6th partials are identically 0

Precise Orbit Determination

- From the LOS measurement equations defined earlier, we have:

$$\frac{\partial g_x(\bar{X}(t_i))}{\partial x(t_i)} = \frac{1}{R_{AB}} - \frac{(x(t_i) - R_{OAx})^2}{{R_{AB}}^3}$$

$$\frac{\partial g_x(\bar{X}(t_i))}{\partial y(t_i)} = \frac{(x(t_i) - R_{OAx})(y(t_i) - R_{OAy})}{{R_{AB}}^3}$$

$$\frac{\partial g_x(\bar{X}(t_i))}{\partial z(t_i)} = \frac{(x(t_i) - R_{OAx})(z(t_i) - R_{OAz})}{{R_{AB}}^3}$$

$$where\ R_{AB} = \sqrt{(x(t_i) - R_{OAx})^2 + \left(y(t_i) - R_{OAy}\right)^2 + (z(t_i) - R_{OAz})^2}$$

Precise Orbit Determination

- Similarly:

$$\frac{\partial g_y(\bar{X}(t_i))}{\partial \bar{X}(t_i)} = \left[\frac{\partial g_y(\bar{X}(t_i))}{\partial x(t_i)} \quad \frac{\partial g_y(\bar{X}(t_i))}{\partial y(t_i)} \quad \frac{\partial g_y(\bar{X}(t_i))}{\partial z(t_i)} \quad 0 \quad 0 \quad 0\right]$$

1x6

where:

$$\frac{\partial g_y(\bar{X}(t_i))}{\partial x(t_i)} = \frac{(x(t_i) - R_{OAx})(y(t_i) - R_{OAy})}{{R_{AB}}^3}$$

$$\frac{\partial g_y(\bar{X}(t_i))}{\partial y(t_i)} = \frac{1}{R_{AB}} - \frac{\left(y(t_i) - R_{OAy}\right)^2}{{R_{AB}}^3}$$

$$\frac{\partial g_y(\bar{X}(t_i))}{\partial z(t_i)} = \frac{(y(t_i) - R_{OAy})(z(t_i) - R_{OAz})}{{R_{AB}}^3}$$

Precise Orbit Determination

- For $\frac{\partial \bar{X}(t_i)}{\partial \bar{X}(t_0)}$ we want to propagate the STM from t_0 to each measurement time $t_i \rightarrow \Phi(t_i, t_0)$

- We do this by integrating $\dot{\Phi} = A\Phi$ where:

$$A = \frac{\partial \bar{F}}{\partial \bar{X}}\Big|_{\bar{X}(t_0)} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \cdots & \frac{\partial f_1}{\partial v_z} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_6}{\partial x} & \cdots & \frac{\partial f_6}{\partial v_z} \end{bmatrix}\Bigg|_{\bar{X}(t_0)}$$

6x6

Precise Orbit Determination

- Recall that $\bar{F}(\bar{X}) = [v_x \ v_y \ v_z \ -\frac{\mu x}{r^3} \ -\frac{\mu y}{r^3} \ -\frac{\mu z}{r^3}]^{\boldsymbol{T}}$
- So A = $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11}=A_{22} $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and

$$A_{21} = \begin{bmatrix} \frac{3\mu {x_0}^2}{{r_0}^5} - \frac{\mu}{{r_0}^3} & \frac{3\mu x_0 y_0}{{r_0}^5} & \frac{3\mu x_0 z_0}{{r_0}^5} \\ \frac{3\mu x_0 y_0}{{r_0}^5} & \frac{3\mu {y_0}^2}{{r_0}^5} - \frac{\mu}{{r_0}^3} & \frac{3\mu y_0 z_0}{{r_0}^5} \\ \frac{3\mu x_0 z_0}{{r_0}^5} & \frac{3\mu y_0 z_0}{{r_0}^5} & \frac{3\mu {z_0}^2}{{r_0}^5} - \frac{\mu}{{r_0}^3} \end{bmatrix}$$

where r_0=$\sqrt{{x_0}^2 + {\boldsymbol{y}_0}^2 + {\boldsymbol{z}_0}^2}$

- We can now outline the series of steps to perform angles-only POD via the analytical method described above

Precise Orbit Determination

ANALYTICAL METHOD:

Given LOS measurements u_x & u_y at times t_i, where i = 1 to p:

1. Begin with an initial guess of states at the epoch time $\bar{X}(t_{\mathbf{0}})_0$

2. Propagate $\bar{X}(t_0)_0$ to the first measurement time t_1 & calculate $g_x(\bar{X}(t_1)_0) = u_x$ & $g_y(\bar{X}(t_1)_0) = u_y$; these are the first 2 elements of $\bar{Y}_0$

3. Also at t_1 , calculate the partials $\frac{\partial g_x(\bar{X}(t_1))}{\partial x(t_1)}$, $\frac{\partial g_x(\bar{X}(t_1))}{\partial y(t_1)}$, $\frac{\partial g_x(\bar{X}(t_1))}{\partial z(t_1)}$, $\frac{\partial g_y(\bar{X}(t_1))}{\partial x(t_1)}$, $\frac{\partial g_y(\bar{X}(t_1))}{\partial y(t_1)}$, and $\frac{\partial g_y(\bar{X}(t_1))}{\partial z(t_1)}$

4. Calculate the elements of A detailed on the previous slide, inserting the values of $\bar{X}(t_{\mathbf{0}})_0$

5. Integrate $\dot{\Phi} = A\Phi$ from t_0 to t_1, initialized by $\Phi(t_0,t_0) = \mathrm{I}$, to get $\Phi(t_1,t_0)$

Precise Orbit Determination

ANALYTICAL METHOD (cont'd):

6. Calculate $\begin{bmatrix} \frac{\partial \boldsymbol{g_x}(\bar{X}(t_1))}{\partial \bar{X}(t_1)} \\ \frac{\partial \boldsymbol{g_y}(\bar{X}(t_1))}{\partial \bar{X}(t_1)} \end{bmatrix} \frac{\partial \bar{X}(t_i)}{\partial \bar{X}(t_0)}$, which amounts to $\begin{bmatrix} \frac{\partial \boldsymbol{g_x}(\bar{X}(t_i))}{\partial x(t_i)} & \frac{\partial \boldsymbol{g_x}(\bar{X}(t_i))}{\partial y(t_i)} & \frac{\partial \boldsymbol{g_x}(\bar{X}(t_i))}{\partial z(t_i)} \\ \frac{\partial \boldsymbol{g_y}(\bar{X}(t_i))}{\partial x(t_i)} & \frac{\partial \boldsymbol{g_y}(\bar{X}(t_i))}{\partial y(t_i)} & \frac{\partial \boldsymbol{g_y}(\bar{X}(t_i))}{\partial z(t_i)} \end{bmatrix} \Phi_{3x6}$,

2x6 **6x6** **2x3** **3x6**

where Φ_{3x6} represents the upper 3x6 half of Φ; this constitutes the first 2 rows of H_0

7. Repeat Steps 2-6 for each of the other measurement times, thereby building up $\bar{Y}_0$ and H_0

8. Calculate $\Delta\bar{X}(t_0)_0 = H_0^{-1}(\bar{Y} - \bar{Y}_0)$ (if p = 3)

6x1 **6x6** **6x1**

or $\Delta\bar{X}(t_0)_0 = (H_0^T H_0)^{-1} H_0^T (\bar{Y} - \bar{Y}_0)$ (if p > 3)

6x1 **6x2p** **2px6** **6x2p** **2px1**

9. Update the state estimate as $\bar{X}(t_0)_1 = \bar{X}(t_0)_0 + \Delta\bar{X}(t_0)_0$

10. Repeat Steps 2-9 until a desired stopping criterion is reached

Precise Orbit Determination

A quick note about the analytical method:

- In Step 4, because the matrix *A* is being calculated at a specific time (& with specific state values), its elements are constant

- Therefore, the $\dot{\Phi} = A\Phi$ matrix equation is linear time-invariant (LTI) → Φ may be solved directly rather than by numerically integrating

- A common method is $\Phi = \mathcal{L}^{-1}[(sI - A)^{-1}]$, where $\mathcal{L}$ indicates the Laplace operator

Precise Orbit Determination

- The POD method presented here is often called “Batch Least-Squares” (BLS) or “Batch Differential Corrections”

- The BLS sol’n is the sol’n that minimizes the root-mean-square (RMS) residual, defined as

$$RMS = \sqrt{\frac{(\bar{Y} - \bar{Y}_0)^T (\bar{Y} - \bar{Y}_0)}{2p}}$$

- BLS has been utilized in space tracking/surveillance operations for years, & is still in use today

- Operational versions of BLS proceed largely as presented here, with the exception that higher-fidelity propagation (vs 2-body) is usually employed
  - We will learn about this next week!