ECE 506: Homework #3: Fundamentals of 1D Unconstrained Optimization Methods

To get help with the homework, please join me for the Saturday morning discussion sessions starting at 9am at https://unm.zoom.us/j/99977790315.

Reference:
Numerical Methods for Unconstrained Optimization and Nonlinear Equations by J.E. Dennis, Jr. and R.B. Schnabel, Classics in Applied Mathematics, SIAM 1996.

Matlab code:
Download the code from `https://github.com/pattichis/opt/blob/main/Code-for-Hwk-from-2012-Opt-1D.zip`.

Coding examples:
For all of your homework solutions, you must provide (1) documented source code, (2) plots, and (3) discussion. In the discussion, I sketch examples. For your solutions, I would like to see coding examples. This is always true, unless I specifically ask for a "sketch".

Problem #1. Bisection.
1(a) Root finding for linear functions.

i. Provide a coding example that demonstrates everything working correctly.

ii. Provide a coding example that demonstrates failure if possible. If not possible, discuss why failure is not possible.

1(b) Root finding for quadratics.

i. Provide a coding example that demonstrates everything working correctly.

ii. Provide a coding example that demonstrates failure if possible. If not possible, discuss why failure is not possible.

1(c) Root finding for continuous functions.

i. What are the minimum requirements for bisection to work?

ii. Sketch an example.

1(d) Convergence property for root finding.
Assume that all conditions are satisfied. Derive an expression for the root interval after n steps of the algorithm.

Problem # 2 Repeat problem #1 for solving $f'(x) = 0$. For this problem, success or failure refers to the problem of minimizing $f(x)$.

Problem #3. Newton's Method.
3(a) Root finding for linear functions.

i. Give an example where everything works.

ii. Give an example where if possible. If not possible, discuss why failure is not possible.

3(b) Root finding for quadratics.

i. Give a coding example where everything works.

ii. Give a coding example where it fails if possible. If not possible, discuss why failure is not possible.

3(c) Root finding for continuously differentiable functions.

i. Based on Theorem 2.4.3 of page 22, give a coding example where everything works.

ii. Give a coding example that does not work.

iii. Does the "globally convergent method" given in `ds_method.m` work here? Document your results.

Problem # 4 Repeat problem #3 for solving $f'(x) = 0$.

Problem # 5 Prepare an example that converges for solving $f'(x) = 0$ for all algorthims provided. For this problem, success or failure refers to the problem of minimizing $f(x)$.

5(a) Which algorithm converged faster? For this problem, speed is measured in terms of the number of function evaluations. All other overhead is ignored.
5(b) What are the advantages and disadvantages of each algorithm?

Notes:

Definition 2.4.1 A function g is Lipschitz continuous with constant γ in a set X, written $g \in \mathrm{Lip}_\gamma(X)$, if for every $x, y \in X$,

$$|g(x) - g(y)| \le \gamma |x - y|.$$

LEMMA 2.4.2 For an open interval D, let $f: D \to \mathbb{R}$ and let $f' \in \mathrm{Lip}_\gamma(D)$. Then for any $x, y \in D$,

$$|f(y) - f(x) - f'(x)(y - x)| \le \frac{\gamma(y - x)^2}{2}. \tag{2.4.1}$$

THEOREM 2.4.3 Let $f: D \to \mathbb{R}$, for an open interval D, and let $f' \in \text{Lip}_\gamma(D)$. Assume that for some $\rho > 0$, $|f'(x)| \geq \rho$ for every $x \in D$. If $f(x) = 0$ has a solution $x_* \in D$, then there is some $\eta > 0$ such that: if $|x_0 - x_*| < \eta$, then the sequence $\{x_k\}$ generated by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, \ldots$$

exists and converges to x_*. Furthermore, for $k = 0, 1, \ldots,$

$$|x_{k+1} - x_*| \leq \frac{\gamma}{2\rho} |x_k - x_*|^2. \tag{2.4.3}$$