Mathematical Formulation

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector of variables, (unknowns or parameters).

$f(x) = f(x_1, x_2, \dots, x_n)$ is the objective function.

Optimization problem:

Compute x that minimizes f:

$$\min_{x \in \mathbb{R}} f(x)$$

subject to constraints:

$$\begin{cases} c_i(x) = 0, & i \in \mathcal{E}, \text{ (a set of indices)} \\ c_i(x) \geq 0, & i \in I, \text{ (a set of indices).} \end{cases}$$

$c_i(x)$ are scalar-valued: $c_i(x) \in \mathbb{R}$.

Note:
- allow only zero constraints and
- non-negative constraints.

Other constraints must be transformed to this form.

A first example:

Minimize $(x_1-2)^2 + (x_2-1)^2$

subject to:

$$\begin{cases} x_1^2 - x_2 \leq 0 \\ x_1 + x_2 \leq 2 \end{cases}$$

Formulation:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(x_1, x_2) = (x_1-2)^2 + (x_2-1)^2$$

$$c(x) = \begin{bmatrix} c_1(x) \\ c_2(x) \end{bmatrix} \leftarrow$$ two constraint functions allowing two indices: 1, 2.

There are no equality constraints $\Longrightarrow \mathcal{E} = \emptyset$.

We have $\mathcal{I} = \{1, 2\}$ and we must tranform the constraints to non-negative form:

$$x_1^2 - x_2 \leq 0$$

Multiply both sides by -1:

$$-x_1^2 + x_2 \geq 0$$

which gives: $c_1(x) = -x_1^2 + x_2$.

Also, from:

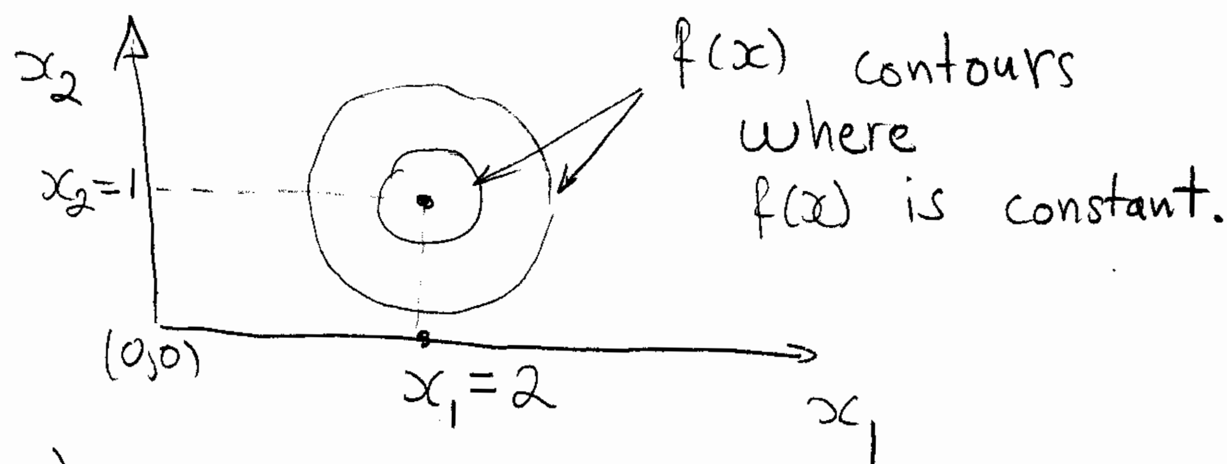
$$x_1 + x_2 \leq 2,$$

transform to:

$$0 \leq 2 - x_1 - x_2$$

which gives: $\underline{c_2(x) = 2 - x_1 - x_2}$.

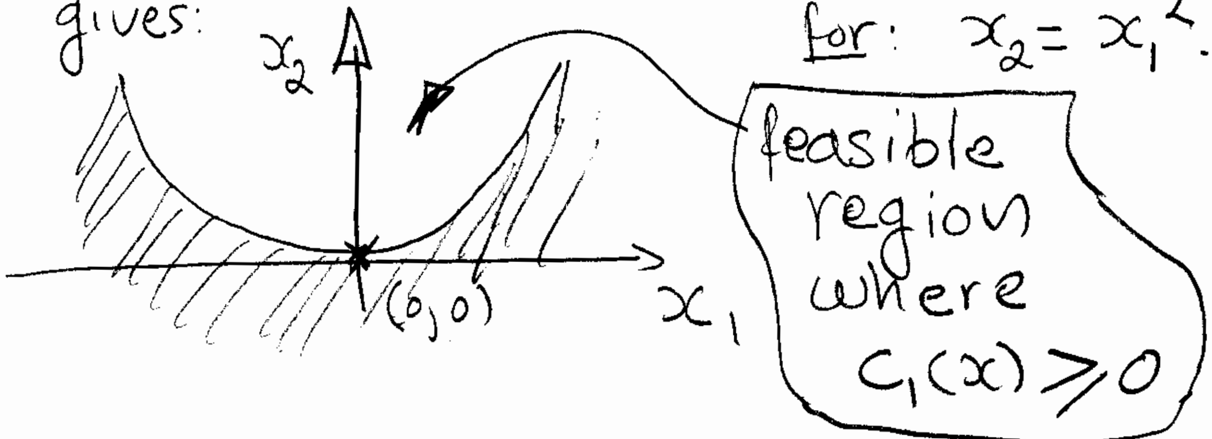
To see what is happening, we can plot everything:



For $c_1(x)$:

$$-x_1^2 + x_2 \geqslant 0 \Rightarrow c_1(x) = 0$$

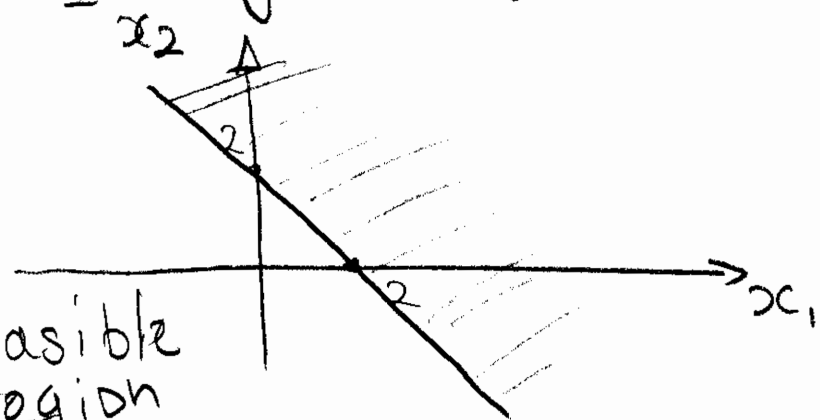
which gives: <u>for</u>: $x_2 = x_1^2$.



For $C_2(x)$:

$$c_2(x) = 2 - x_1 - x_2 \geqslant 0.$$

For equality: $x_2 = 2 - x_1$



Feasible region

(from $(x_1, x_2) = (0, -\infty)$ satisfying $\geqslant$).

Putting the three together: