ECE 506: Homework #1: Basic Optimization

To get help with the homework, please join the Saturday morning discussion sessions starting at 9am at `https://unm.zoom.us/j/99977790315`.

Problem #1. An Introduction to Linear Programming

This problem is focused on manipulating the basic Linear Programming equation:

$$\min_x \; c^\top x \quad \text{subject to } Ax = b \text{ and } x \geq 0. \tag{1}$$

(Here, $x \geq 0$ is understood componentwise.)

1(a) We begin with the simplest possible example. Consider the 1D problem:

$$\min_x \; c \cdot x \quad \text{subject to } ax = b \text{ and } x \geq 0. \tag{2}$$

From this case, answer the following:

i) **Example with no solution.** With the constraints $ax = b$ and $x \geq 0$, if $a \neq 0$ then the only candidate is $x^\star = \frac{b}{a}$. If $b/a < 0$, the nonnegativity constraint is violated, so the problem is infeasible; e.g., $a = 1,\ b = -1 \Rightarrow x^\star = -1$ (infeasible). (Also infeasible when $a = 0,\ b \neq 0$ since $0 = b$ cannot hold.)

ii) **Example with a simple solution.** Take $a = 2,\ b = 0$. Then $x^\star = \frac{b}{a} = 0$, which satisfies $x \geq 0$, and the objective value is $c\,x^\star = 0$.

iii) **Did you minimize anything? Explain.** No. When $a \neq 0$, the equality constraint pins down a single feasible point $x^\star$; if it is feasible, it is automatically optimal—there is no tradeoff to optimize over.

1(b) Invertible case. If A is invertible, the constraint $Ax = b$ has the unique solution $x^\star = A^{-1}b$. If $x^\star \geq 0$ (componentwise), it is the only feasible—and thus optimal—point with value $c^\top x^\star$; otherwise the problem is infeasible. No minimization needed.

1(c) Underdetermined case. The only case that is interesting is when we have many solutions to $Ax = b$. We then get to pick the one that minimizes $c^\top x$. This can only happen when the number of equations is smaller than the number of unknowns. Here is an example:

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2.$$

Note that we have one equation in two unknowns. We have more unknowns than we have equations! It may be possible to set up a proper optimization problem.

To have a proper solution, we must also satisfy $x_1, x_2 \geq 0$. These are called *feasible solutions*. They satisfy the constraints, and the optimal solution needs to satisfy them.

Task: Plot all possible solutions of $Ax = b$ satisfying $x_1, x_2 \geq 0$ for this case.

1(d) Optimization over the feasible set. For the case when $Ax = b$ described in 1(c), solve the proper optimization problem. For this case, solve:

$$\min_x \, [\,1 \;\; 1\,]\, x \quad \text{subject to } Ax = b \text{ and } x \geq 0. \tag{3}$$

Is the solution at the endpoints? Explain.