ME 596 Spacecraft Attitude Dynamics and Control

Overview of Astrodynamics

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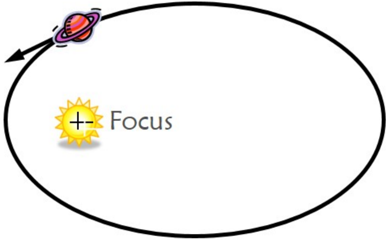
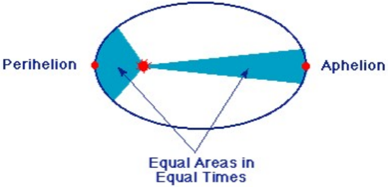
Kepler's Laws

First Law (1609): The orbit of each planet is an ellipse with the Sun at a focus.

Second Law (1609): The line joining the planet to the Sun sweeps out equal areas in equal time.

Third Law (1619): The square of the period of a planet is proportional to the cube of its mean distance from the Sun.





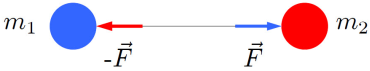
These laws were formed for motion of planets about the sun, but also apply to artificial satellites orbiting the sun, planets, or moons.

Newton's Laws (*Principia*, 1687)

First Law: Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

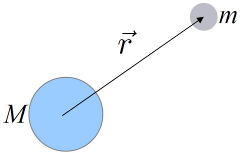
Second Law: The time rate of change of linear momentum is proportional to the applied force. $\vec{\mathbf{F}} \propto \frac{{}^i d(m^i \vec{\mathbf{v}})}{dt}$

Third Law: For every action, there is an equal and opposite reaction.



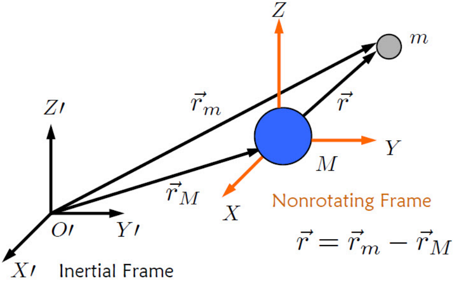
Universal Gravitational Law: Any two bodies attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

$$\vec{\mathbf{F}}_{gm} = -\frac{GMm}{r^2}\frac{\vec{\mathbf{r}}}{r}$$



Assumptions:

- The bodies are spherically symmetric (equivalent to assuming point masses).
- The only force is the gravitational force, along the line joining the centers of the two bodies.



$$
\begin{aligned}
m\ddot{\vec{\mathbf{r}}}_m &= -\frac{GMm}{r^3}\vec{\mathbf{r}} \Rightarrow \ddot{\vec{\mathbf{r}}}_m = -\frac{GM}{r^3}\vec{\mathbf{r}} \\
M\ddot{\vec{\mathbf{r}}}_M &= \frac{GMm}{r^3}\vec{\mathbf{r}} \Rightarrow \ddot{\vec{\mathbf{r}}}_M = \frac{Gm}{r^3}\vec{\mathbf{r}}
\end{aligned}
$$

$$
\underbrace{\ddot{\vec{\mathbf{r}}} = -\frac{G(M+m)}{r^3}\vec{\mathbf{r}}}_{\text{Differential equation of relative motion}}
$$

The differential equation of relative motion is the starting point for most of the work we do with the two-body problem.

Equation of Motion of Two-Body Problem

When $M \gg m$, as is the case for artificial satellites orbiting the Earth, as well as for space probes and sounding rockets, then we can simplify the differential equation of relative motion:

$$G(M+m) \approx GM$$

so

$$\ddot{\vec{\mathbf{r}}} + \frac{GM}{r^3}\vec{\mathbf{r}} = \vec{\mathbf{0}}$$

or, defining $\mu = GM$

$$\ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^3}\vec{\mathbf{r}} = \vec{\mathbf{0}}$$

There are two constants of the motion (aka first integrals):

- Conservation of mechanical energy, $E = T + V$, where T is the kinetic energy and V is the potential energy
- Conservation of angular momentum, $\vec{\mathbf{h}} = \vec{\mathbf{r}} \times \vec{\mathbf{v}}$

In the next few slides, we derive these two conservation laws.

Conservation of Mechanical Energy

Start with the equations of motion (EOM), a nonlinear, vector, second-order, ordinary differential equation:

$$\ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^3}\vec{\mathbf{r}} = \vec{\mathbf{0}}$$

Multiply (dot product) by $\dot{\vec{\mathbf{r}}}$:

$$\dot{\vec{\mathbf{r}}} \cdot \ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^3}\vec{\mathbf{r}} \cdot \dot{\vec{\mathbf{r}}} = 0$$

Use the facts that $\dot{\vec{\mathbf{r}}} \cdot \vec{\mathbf{r}} = \vec{\mathbf{r}} \cdot \dot{\vec{\mathbf{r}}} = \dot{r}r$, and that $\dot{\vec{\mathbf{v}}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \dot{\vec{\mathbf{v}}} = \dot{v}v$ to simplify to:

$$\dot{v}v + \frac{\mu}{r^2}\dot{r} = 0$$

which can be written as:

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) + \frac{d}{dt}\left(c - \frac{\mu}{r}\right) = \frac{d}{dt}\left(\frac{v^2}{2} + c - \frac{\mu}{r}\right) = 0$$

Since $\frac{d}{dt}(X) = 0$ means that "X" is constant, we finally arrive at:

$$\mathcal{E} = \frac{v^2}{2} + \left(c - \frac{\mu}{r}\right)$$

Conservation of Mechanical Energy

From the previous slide, we have:

$$\mathcal{E} \quad = \quad \frac{v^2}{2} + \left(c - \frac{\mu}{r}\right)$$

where

$\mathcal{E}$	is the specific mechanical energy
$v^2/2$	is the kinetic energy per unit mass of the orbiting body
$(c - \mu/r)$	is the potential energy per unit mass of the orbiting body
c	is is an arbitrary constant of integration

We are free to choose the constant c arbitrarily. Two possible choices are $c = \mu/r_M$ (where r_M denotes the radius of the central body) and $c = 0$.

If $c = \mu/r_M$, then the potential energy is zero at the surface of the central body.

If $c = 0$, then the potential energy is zero when the satellite is an infinite distance from the central body. Note that the potential energy is always ≤ 0 for this choice of c.

The "standard" choice is the latter, so that

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$$

Matlab Function: `rv2E.m`

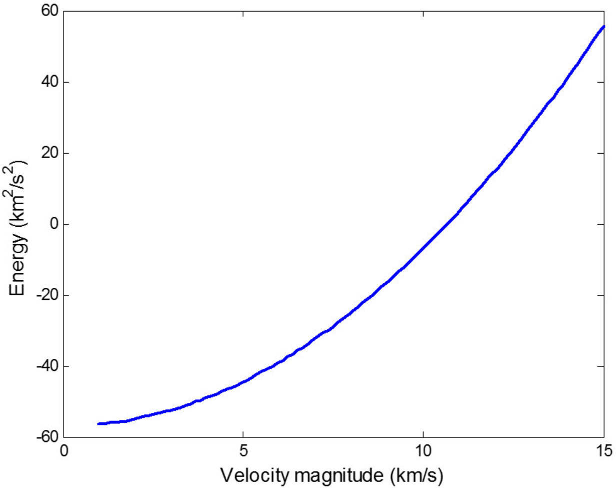
```
function E = rv2E (R,V,mu)
% RV2E  Function to calculate the specific mechanical energy of a Keplerian orbit.
%    The inputs of rv2E are the position vector of the satellite, R, the velocity
%    vector of the satellite, V, and the gravitational parameter of the central
%    body, mu.  The output is the specific mechanical energy, E.
%
%    The position and velocity vectors must be either 3x1 or 1x3 matrices, and
%    the function will resize both of them so that they are 3x1.  If a value
%    for mu is not inputed, the default value is that for Earth (3.986e5 km^3/s^2)

% Determine if input vectors are the correct size and resize them
if length (R)~=3 || length (V)~=3
error ('Input vectors are not the correct size.  Please input new vectors.')
end
R = [R (1); R (2); R (3)];
V = [V (1); V (2); V (3)];

% If no value of mu is given, use the default value for Earth
if nargin < 3
mu = 3.986e5;
end

% CALCULATE ORBIT RADIUS AND VELOCITY MAGNITUDE
r = norm (R);
v = norm (V);

% CALCULATE SPECIFIC MECHANICAL ENERGY OF THE ORBIT
E = v^2/2 - mu/r;
```

Matlab m-file using `rv2E.m`

```
% This program (an "m-file") calculates the energy while varying the magnitude
% of the velocity vector for a given radius vector, then makes a plot of E vs |V|

% Set parameters
mu = 3.986e5;
Npoints = 100;

% Initialize variables
R = [7000; 0; 0];
Vrange = linspace(1,15,Npoints); % each velocity magnitude will be used to compute E
Erange = zeros(1,Npoints);

for i=1:Npoints
V = [0; Vrange(i); 0];        % set V perpendicular to R, so either periapsis or apoapsis
Erange(i) = rv2E(R,V,mu);
end

figure
hg=plot(Vrange,Erange);   % use "handle graphics" to make professional graph
set(hg, 'linewidth',2)
xlabel('Velocity magnitude (km/s)','fontsize',12)
ylabel('Energy (km^2/s^2) ', 'fontsize',12)
```

Plot obtained using Matlab m-file and `rv2E.m`

- This plot is the result of running the Matlab “m-file” that calls the Matlab function rv2E.m
- Note that for small velocities the energy is negative, and for large velocities the energy is positive
- Think about that point where the energy is zero and ask what it might mean



Conservation of Angular Momentum

Start with the equations of motion (EOM), a nonlinear, vector, second-order, ordinary differential equation:

$$\ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^3}\vec{\mathbf{r}} = \vec{\mathbf{0}}$$

Multiply (cross product) by $\vec{\mathbf{r}}$:

$$\vec{\mathbf{r}} \times \ddot{\vec{\mathbf{r}}} + \frac{\mu}{r^3}\vec{\mathbf{r}} \times \vec{\mathbf{r}} = \vec{\mathbf{0}}$$

Use the facts that $\vec{\mathbf{r}} \times \vec{\mathbf{r}} = \vec{\mathbf{0}}$, and $\vec{\mathbf{r}} \times \ddot{\vec{\mathbf{r}}} = \frac{d}{dt}\left(\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}\right)$ to obtain:

$$\frac{d}{dt}\left(\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}\right) = \vec{\mathbf{0}}$$

And therefore:

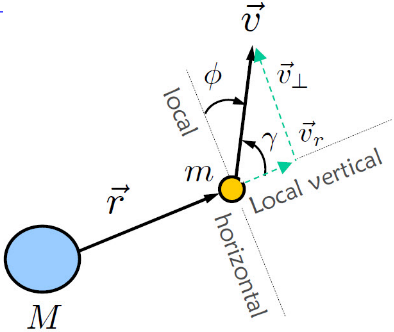
$$\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}} = \vec{\mathbf{h}} = \text{ a constant vector}$$

The angular momentum vector $\vec{\mathbf{h}}$ is constant in both magnitude and direction.

Conservation of Angular Momentum

ϕ = flight-path angle

γ = zenith angle



$\vec{\mathbf{h}}$ is perpendicular to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$

Thus, since $\vec{\mathbf{h}}$ is constant, the motion takes place in a plane that is fixed in inertial space.

The constant plane is called the "orbital plane."

Here are some useful relationships involving angular momentum, flight path angle, and zenith angle:

$$
\begin{aligned}
h &= rv \sin\gamma \\
h &= rv \cos\phi \\
\tan\phi &= \frac{v_r}{v_\perp}
\end{aligned}
$$

The Trajectory Equation

The next step in analyzing the two-body problem equation of motion is to "solve" it, obtaining the trajectory equation. Start with:

$$\ddot{\vec{\mathbf{r}}} = -\frac{\mu}{r^3}\vec{\mathbf{r}}$$

Multiply (cross product) times the constant vector $\vec{\mathbf{h}}$:

(a) $$\ddot{\vec{\mathbf{r}}} \times \vec{\mathbf{h}} = -\frac{\mu}{r^3}\vec{\mathbf{r}} \times \vec{\mathbf{h}} = \vec{\mathbf{h}} \times \frac{\mu}{r^3}\vec{\mathbf{r}} = \frac{\mu}{r^3}\vec{\mathbf{h}} \times \vec{\mathbf{r}}$$

The left-hand side can be written as

(b) $$\ddot{\vec{\mathbf{r}}} \times \vec{\mathbf{h}} = \frac{d}{dt}\left(\dot{\vec{\mathbf{r}}} \times \vec{\mathbf{h}}\right)$$

We can simplify the right-hand side, $\frac{\mu}{r^3}\vec{\mathbf{h}} \times \vec{\mathbf{r}}$, using the identity:

$$\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) \times \vec{\mathbf{C}} = \vec{\mathbf{B}}\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{C}}\right) - \vec{\mathbf{C}}\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right)$$

Thus,

(c) $$\frac{\mu}{r^3}\left(\vec{\mathbf{r}} \times \vec{\mathbf{v}}\right) \times \vec{\mathbf{r}} = \frac{\mu}{r^3}\left[\vec{\mathbf{v}}\left(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}\right) - \vec{\mathbf{r}}\left(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}\right)\right] = \frac{\mu}{r}\vec{\mathbf{v}} - \frac{\mu \dot{r}}{r^2}\vec{\mathbf{r}} = \mu\frac{d}{dt}\left(\frac{\vec{\mathbf{r}}}{r}\right)$$

The Trajectory Equation

Combine (b) and (c):

$$\frac{d}{dt}\left(\dot{\vec{\mathbf{r}}} \times \vec{\mathbf{h}}\right) = \mu \frac{d}{dt}\left(\frac{\vec{\mathbf{r}}}{r}\right)$$

Integrate to obtain:

$$\dot{\vec{\mathbf{r}}} \times \vec{\mathbf{h}} = \frac{\mu \vec{\mathbf{r}}}{r} + \vec{\mathbf{B}}$$

where $\vec{\mathbf{B}}$ is a constant of integration.

Multiply (dot product) by the position vector $\vec{\mathbf{r}}$:

$$\vec{\mathbf{r}} \cdot \dot{\vec{\mathbf{r}}} \times \vec{\mathbf{h}} = \frac{\mu \vec{\mathbf{r}} \cdot \vec{\mathbf{r}}}{r} + \vec{\mathbf{r}} \cdot \vec{\mathbf{B}}$$

Apply the identity:

$$\vec{\mathbf{A}} \cdot \left(\vec{\mathbf{B}} \times \vec{\mathbf{C}}\right) = \left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) \cdot \vec{\mathbf{C}}$$

Thus,

$$h^2 = \mu r + rB\cos\nu \Rightarrow r = \frac{h^2/\mu}{1 + (B/\mu)\cos\nu}$$

where ν is the angle between the radius vector $\vec{\mathbf{r}}$ and the constant vector $\vec{\mathbf{B}}$.

The Trajectory Equation

The equation for a conic section:

$$r = \frac{p}{1 + e\cos\nu}$$

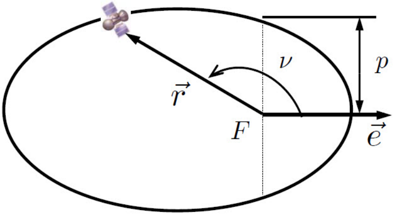
The TBP Trajectory Equation:

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos\nu}$$

Pattern recognition:

- The parameter or semi-latus rectum is $p = h^2/\mu$.
- The eccentricity is $e = B/\mu$

Eccentricity	Orbit or Trajectory
$e = 0$	Circle
$0 < e < 1$	Ellipse
$e = 1$	Parabola
$e > 1$	Hyperbola



Some Terminology

- **Satellite:** an object which travels in an elliptical orbit around a planet is called a satellite of that planet.
- **Space probe:** an object which travels in an open trajectory; i.e., a hyperbolic trajectory in a vicinity of a planet and in elliptical orbit around the sun is called a space probe.
- **Interstellar probe:** an object with a velocity greater than heliocentric escape velocity is called an interstellar probe.
- **Orbit or Trajectory:** path of a spacecraft or natural body in space. We use orbit for closed (circles and ellipses) and trajectory for open paths (parabolas and hyperbolas).
- **Barycenter:** the location of the center of mass of two bodies.

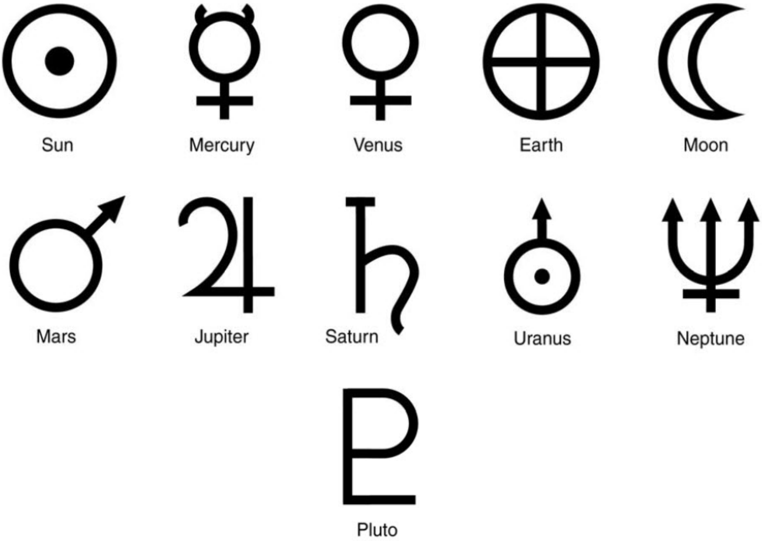
Some Important Orbit Terminology

- **"Peri"** means the closest point. "Periapsis" is the generic term. "Perigee" means the closest point to the earth. "Perihelion" means the closest point to the sun, and so forth.
- **"Apo"** means the farthest point. "Apoapsis" is the generic term. "Apogee" means the farthest point from the earth. "Aphelion" means the farthest point from the sun.
- **Keplerian orbit:** is an orbit in which
  1. The only force is gravity
  2. The central body is spherically symmetric
  3. The primary mass is much greater than secondary mass
  4. There are no other masses in the system

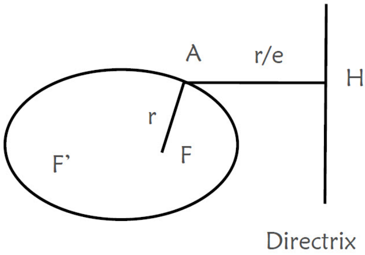
  Also called "two-body problem" orbit or "unperturbed" orbit

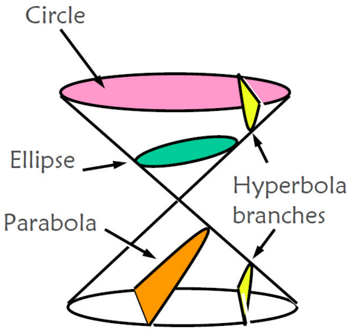
A memory aid: **a** b c d e f g h I j k l m n **o** **p** q r s t u v w x y z

Planet Symbols



Conic Sections





A conic section is the locus of points such that the ratio of the absolute distance from a given point (a focus) to the absolute distance from a given line (a directrix) is a positive constant e called the eccentricity.

The Circular Orbit

For a circular orbit, the eccentricity is $e = 0$. Thus

$$r = \frac{p}{1 + 0\cos\nu} = p = a = \frac{h^2}{\mu}$$

The circular velocity can be found using energy:

$$\frac{v_{cs}^2}{2} - \frac{\mu}{a} = -\frac{2\mu}{a} \Rightarrow v_{cs} = \sqrt{\frac{\mu}{a}}$$

The orbital period is

$$TP_{cs} = \frac{2\pi}{\omega} = \frac{2\pi a}{v_{cs}} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Pop Quiz!

An Earth satellite is in a circular orbit with altitude 400 km. What are the satellite's speed and orbital period?

Useful information:

$$
\begin{aligned}
\mu_\oplus &= 3.986 \times 10^5 \text{ km}^3/\text{s}^2 \\
R_\oplus &= 6378 \text{ km}
\end{aligned}
$$

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Answers

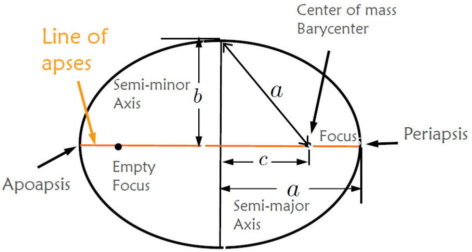
$$\begin{aligned} v_{cs} &= \sqrt{\mu/r} = \sqrt{3.986 \times 10^5/(6378+400)} = 7.669 \text{ km/s} \\ TP_{cs} &= 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{6778^3}{3.986 \times 10^5}} = 5,553 \text{ seconds} \\ &= 92.56 \text{ minutes} \end{aligned}$$

Common mistake: Using $r = 400$ km, i.e., forgetting to add the radius of the Earth.

The Elliptical Orbit

Eccentricity: $0 < e < 1$.
Eccentricity is related to orbit radius by

$$e = \frac{r_a - r_p}{r_a + r_p}$$



The orbital period is found by integrating an expression for angular momentum:

$$\begin{aligned} h &= \frac{r^2 d\nu}{dt} \Rightarrow dt = \frac{r^2}{h} d\nu \\ dA &= \frac{r^2}{2} d\nu \Rightarrow dt = \frac{2}{h} dA \\ TP &= \frac{2\pi ab}{h} = 2\pi \sqrt{\frac{a^3}{\mu}} \end{aligned}$$

Remember the memory aid: **a** b c d e f g h I j k l m n **o** **p** q r s t u v w x y z

Pop Quiz!

An Earth satellite is in an elliptical orbit with perigee altitude 400 km and apogee altitude 900 km. What are the satellite's speed at perigee and apogee, and orbital period?

Useful information:

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Answers

$$\begin{aligned} e &= 0.03557, a = 7028 \text{ km}, \mathcal{E} = -28.35 \text{ km}^2/s^2 \\ v &= \sqrt{2(\mathcal{E} + \mu/r)} \Rightarrow v_p = 7.804 \text{ km/s and } v_a = 7.268 \text{ km/s} \\ TP &= 97.72 \text{ minutes} \end{aligned}$$

Common mistake: Forgetting to add the radius of the Earth.

The Parabolic Trajectory

Eccentricity: $e = 1$.

Trajectory equation simplifies to

$$r = \frac{p}{1 + \cos 0} = \frac{p}{2}$$

Because a parabolic trajectory is the lowest energy open trajectory, the speed of an object at any point on a parabolic trajectory is called the *escape velocity.*

Using the energy equation (with $\mathcal{E} = 0$), we can solve for escape velocity:

$$\frac{v_{esc}^2}{2} - \frac{\mu}{r} = 0 \Rightarrow v_{esc} = \sqrt{\frac{2\mu}{r}} = \sqrt{2} v_{cs}$$

Note that as $r \to \infty$, the velocity on a parabolic trajectory $\to 0$.

The escape velocity concept is the principal application of parabolic trajectories in orbital mechanics.

Pop Quiz!

An earth satellite is in a circular orbit with altitude 400 km. What is the satellite's speed? What is the satellite's escape speed? Suppose the satellite's speed is changed to the escape speed, then what is the satellite's speed when the radius reaches infinity?

Pop Quiz!

An earth satellite is in a circular orbit with altitude 400 km. What is the satellite's speed? What is the satellite's escape speed? Suppose the satellite's speed is changed to the escape speed, then what is the satellite's speed when the radius reaches infinity?

Answers

$$\begin{aligned} v_{cs} &= 7.669 \text{ km/s} \\ v_{esc} &= 10.845 \text{ km/s} \\ v_{\infty} &= 0 \end{aligned}$$

Common mistake: Forgetting to add the radius of the Earth.

The Hyperbolic Trajectory

Eccentricity: $e > 1$

Semi-major axis: $a < 0$

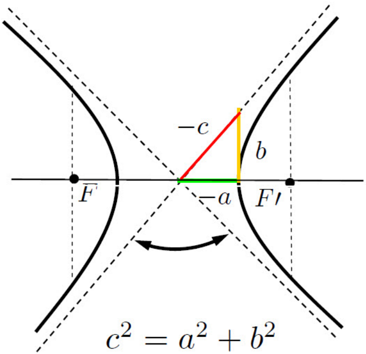
Energy: $\mathcal{E} > 0$

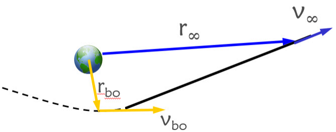
Turning angle: $\sin\frac{\delta}{2} = \frac{a}{c} = \frac{1}{e}$

Hyperbolic excess velocity:

$$\begin{aligned} \mathcal{E} &= \frac{v_{bo}^2}{2} - \frac{\mu}{r_{bo}} = \frac{v_\infty^2}{2} - \frac{\mu}{r_\infty} \\ v_\infty^2 &= v_{bo}^2 - \frac{2\mu}{r_{bo}} = v_{bo}^2 - v_{esc}^2 \end{aligned}$$

Note: v_∞^2 is denoted by C_3 and is called *departure energy*.





Pop Quiz!

An earth satellite is in a circular orbit with altitude 500 km. What is the satellite's speed? What is the satellite's escape speed? Suppose the satellite's speed is changed to twice the escape speed, then what is the eccentricity of the resulting trajectory?

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Answers

$$\begin{aligned} v_{cs} &= 7.613 \text{ km/s}, v_{esc} = 10.766 \text{ km/s}, v_{bo} = 21.532 \text{ km/s} \\ &\Rightarrow \mathcal{E} = 173.86 \text{ km}^2/\text{s}^2 \Rightarrow a = -1146.33 \text{ km} \\ &\ h = 148,096 \text{ km}^2/\text{s} \\ e &= 7 \end{aligned}$$

Common mistake: Forgetting to add the radius of the Earth.