3.6 Calculus of variations in "Introduction to Applied Mathematics" by Gilbert Strang.

Topics:

(1) One-dimensional Problems:

$$P = \int F(u, u')\,dx, \text{ not just quadratic}$$

(2) Constraints, linear / non-linear

(3) Two-dimensional problems:

$$P = \iint F(u, u_x, u_y)\,dx\,dy$$

(4) Time-dependent problems with $u' = du/dt$

One-dimensional Problems

$$P(u) = \int_0^1 F(u, u')\,dx, \quad u(0) = a, \quad u(1) = b$$

Consider:

$F(u+v, u'+v')$ where v is a "test function" that perturbs $u(x)$, and also satisfies:

$$v(0) = v(1) = 0.$$

We assume that:

v, v' are "small"

This leads to the approximation:

$$F(u+v, u'+v') = F(u,u') + v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'} + \cdots$$

derivatives w.r.t u, u',
multiply by v, v'.

$$P(u+v) = \int_0^1 F(u+v, u'+v')\,dx$$

$$= \int_0^1 F(u,u')\,dx + \int_0^1 \left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right)dx$$

$$+ \cdots$$

$$= P(u) + \int_0^1 \left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right)dx + \cdots$$

The <u>weak form</u> requires

$\delta P = 0$ for every v.
(instead of $\delta P/\delta u = 0$ which does not make sense!)

This weak form requires:

$$\int_0^1 \left(v \frac{\partial F}{\partial u} + v' \frac{\partial F}{\partial u'} \right) dx = 0$$

After integrating by parts using:

$$\int u\,dv = uv - \int v\,du,$$

$$\int_0^1 \left(v \frac{\partial F}{\partial u} + v' \frac{\partial F}{\partial u'} \right) dx = \int_0^1 v \frac{\partial F}{\partial u} dx$$

$$+ \underbrace{\cancel{v \frac{\partial F}{\partial u'} \Big|_0^1}}_{0} - \int_0^1 v \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) dx$$

since $v(0) = v(1) = 0$.

$$= \int_0^1 \left[v \frac{\partial F}{\partial u} - v \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right] dx = 0$$

$$\Rightarrow \boxed{\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) = 0},$$ the strong form

20. Calculus of Variations

from: "Mathematical Methods for Physics and Engineering" by K.F. Riley, M.P. Hobson and S.J. Bence.

20.1 Consider

$$I = \int_a^b F(y, y', x)\, dx$$

where I is a <u>functional</u> (a function of functions) of some curve $y(x)$ to make $I[y(x)]$ stationary.

Consider:

$$y(x) \longrightarrow y(x) + \alpha \eta(x)$$

where α is small and $\eta(x)$ has "amenable mathematical properties" (diff'ble)

We require:

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0 \,, \quad \text{all } \eta(x)$$

$$I(y,\alpha) = \int_a^b F(y+\alpha\eta,\, y'+\alpha\eta';\, x)\,dx$$

$$= \int_a^b F(y,y',x)\,dx + \int_a^b \left(\frac{\partial F}{\partial y}\alpha\eta + \frac{\partial F}{\partial y'}\alpha\eta' \right) dx + O(\alpha^2)$$

For all $\eta(x)$, we require $\delta I = 0$:

$$\boxed{\delta I = \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right) dx = 0.} \quad (*)$$

(Gilbert Strang calls this the "weak form")

Recall integration by parts:

$$\int u\,dv = uv - \int v\,du \quad (**)$$

Apply $(**)$ to $(*)$, $\left(u = \frac{\partial F}{\partial y'},\ dv = \eta'\right)$

$$\int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right) dx = \int_a^b \frac{\partial F}{\partial y}\eta\,dx + \left[\frac{\partial F}{\partial y'}\eta\right]_a^b$$

$$- \int_a^b \eta \cdot \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] dx = 0$$

which becomes:

$$\left[\eta(x)\frac{\partial F}{\partial y'}\right]_a^b + \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]\right]\eta(x)\,dx = 0 \quad (1)$$

If $y(a), y(b)$ are known and fixed, no variation is allowed at the endpoints

$$\eta(a) = \eta(b) = 0,$$

which simplifies (1) to:

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx = 0$$

for any $\eta(x)$. This leads to:

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

the Euler-Lagrange equation.

20.2 Special cases

20.2.1 $\frac{\partial F}{\partial y} = 0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ (F does not contain y explicitely)

$$\Rightarrow \frac{\partial F}{\partial y'} = \text{constant.}$$

(see example showing that the shortest path joining two points is a straight line).

20.2.2 F does not contain x explicitly.

From: $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$, multiply by y'

Constrained Problems [Gilbert Strang]

Minimize $P(u) = \int F(u')\,dx$

with $u(0)=a$, $u(1)=b$,

subject to: $\int_0^1 u\,dx = A$ ~ ⊛

The constrain ⊛ is braught into the optimization problem using:

$$\lambda \left(\int_0^1 u\,dx - A \right)$$ added to $P(u)$:

Minimize $L = P(u) + \lambda \left(\int_0^1 u\,dx - A \right)$ is

$$L = \int_0^1 (F + \lambda u)\,dx - \lambda A.$$

The method of "undetermined Lagrange Multipliers" is extensible to any finitely many dimensions. For the minimum or maximum of:

$$F(a, b, \ldots, z)$$

with respect to:

$$a, b, \ldots, z$$

subject to:

$$G_1(a, b, \ldots, z) = C_1$$

$$G_2(a, b, \ldots, z) = C_2$$

$$\vdots$$

$$G_N(a, b, \ldots, z) = C_N$$

where $C_1, \ldots, C_N$ are <u>given</u>, Solve

$$\boxed{\frac{\partial F^*}{\partial x} = \frac{\partial F^*}{\partial y} = \cdots = \frac{\partial F^*}{\partial z} = 0}$$

where $\boxed{F^* = F + \sum_{i=1}^{N} \lambda_i G_i}$

3N The first variation of:

$$P = \int_0^1 F(u, u', u'', x)\, dx \text{ is}$$

$$\int_0^1 \left(v \frac{\partial F}{\partial u} + v' \frac{\partial F}{\partial u'} + v'' \frac{\partial F}{\partial u''} \right) dx = 0,$$

and the Euler equation is

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) = 0.$$

If there are constraints, simply use Lagrange multipliers to incorporate the constraints.

The spaces $C^\infty(\Omega)$, D_K

Consider $\alpha = (\alpha_1, \dots, \alpha_n)$
of non-negative integers α_i.

Define D^α by:

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

The order is:

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

We can have odd order: $|\alpha| = 1, 3, 5, \dots$,
or even order: $|\alpha| = 0, 2, 4, \dots$.

$$D^0 f = f.$$

A function $f : \Omega \longrightarrow \mathbb{C}$ belongs to $C^\infty(\Omega)$ if $D^\alpha f \in C(\Omega)$, every α.

(page 34, "Functional Analysis", 2nd ed, Rudin)

Nonlinear Equations

$$F = F(x, y, u, D_1 u, D_2 u, \ldots)$$

Let $u = D_0 u$.

Write:

$$F(u+v) = F(u) + \sum \frac{\partial F}{\partial D_i u} \cdot D_i v + \cdots$$

$$\iint \left(\frac{\partial F}{\partial D_i u}\right)(D_i v)\, dx\,dy \to \iint \left[D_i^T \left(\frac{\partial F}{\partial D_i u}\right)\right] v\, dx\,dy$$

after integration, where:

$$\boxed{D_i^T = \begin{cases} -D_i, & \text{for 1st, 3rd, 5th, ... derivatives} \\ +D_i, & \text{for 0th, 2nd, 4th, ... derivative} \end{cases}}$$

Each problem in the calculus of variations has three equivalent forms:

Variational form: $E(u) = \iint_S F(u)\,dx\,dy$

Weak form: $\dfrac{\partial E(u)}{\partial u} = \iint_S \left(\sum \dfrac{\partial F}{\partial D_i u}\right)(D_i v)\,dx\,dy = 0$

for all v.

Euler Equation: $\boxed{\sum_i D_i^T \left(\dfrac{\partial F}{\partial D_i u}\right) = 0}$

If F is a quadratic function of:

$D_0 u,\ D_1 u,\ D_2 u,\ \ldots,$

the expressions:

$\dfrac{\partial F}{\partial D_i u}$ are linear, and so

is the Euler equation.

Example 1

$$F(u) = u^2 + u_x^2 + u_y^2 + u_{xx}^2 + u_{xy}^2 + u_{yy}^2$$

where $u = u(x,y)$.

Evaluate:

$$\frac{\partial F}{\partial u} = 2u, \quad \frac{\partial F}{\partial u_x} = 2u_x, \quad \frac{\partial F}{\partial u_y} = 2u_y,$$

$$\frac{\partial F}{\partial u_{xx}} = 2u_{xx}, \quad \frac{\partial F}{\partial u_{xy}} = 2u_{xy}, \quad \frac{\partial F}{\partial u_{yy}} = 2u_{yy}.$$

To minimize

$$\boxed{E(u) = \iint_S F(u)\, dx\, dy} \quad (*)$$

the weak form is:

$$\boxed{2\iint \left[uv + u_x v_x + u_y v_y + u_{xx} v_{xx} + u_{xy} v_{xy} + u_{yy} v_{yy} \right] dx\, dy = 0} \quad (**)$$

For the strong form:

$D_0^T\left(\frac{\partial F}{\partial D_0 u}\right) = 2u$, D_0 has no effect

$D_1^T\left(\frac{\partial F}{\partial D_1 u}\right)$ has two forms:

- "1" $= x$ for the x-variable:

$$D_x^T\left(\frac{\partial F}{\partial D_x u}\right) = \frac{-\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) = (2u_x)_x \cdot (-1)$$

$$= \underline{\underline{-2u_{xx}}}.$$

- "1" $= y$ for the y-variable:

$$D_y^T\left(\frac{\partial F}{\partial D_y u}\right) = \frac{-\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = -2u_{yy}$$

$D_2^T\left(\frac{\partial F}{\partial D_2 u}\right)$ has three forms:

$$D_{xx},\ D_{xy} = D_{yx},\ D_{yy}$$

<u>Then:</u>

$$D_{xx}^T\left(\frac{\partial F}{\partial D_{xx}u}\right) = \left(2u_{xx}\right)_{xx} = 2u_{xxxx}.$$

$$D_{xy}^T\left(\frac{\partial F}{\partial D_{xy}u}\right) = \left(2u_{xy}\right)_{xy} = 2u_{xyxy}$$

$$D_{yy}^T\left(\frac{\partial F}{\partial D_{yy}u}\right) = \left(2u_{yy}\right)_{yy} = 2u_{yyyy}$$

The Euler <u>equation</u>:

$$\boxed{\sum D_i^T\left(\frac{\partial F}{\partial D_i u}\right) = 2\left[u - u_{xx} - u_{yy} + u_{xxxx} + u_{yyyy} + u_{xyxy}\right] = 0}$$

Generalization for the fixed end-point problem for n unknown functions

(From "Calculus of Variations" by Gelfand and Fomin, chapter 2, section 9).

Let $F(x, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n)$ have continuous first and second derivatives for all its arguments.

For an extremum of:

$$J[y_1, y_2, \ldots, y_n] = \int_a^b F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx$$

subject to:

$$y_1(a) = A_1,\ y_2(a) = A_2, \ldots, y_n(a) = A_n,$$

and

$$y_1(b) = B_1,\ y_2(b) = B_2, \ldots, y_n(b) = B_n,$$

we must find necessary (yet not sufficient) conditions.

Consider varying $y_1(x), \ldots, y_n(x)$ by considering

$y_1(x) + h_1(x),\ y_2(x) + h_2(x), \ldots, y_n(x) + h_n(x)$.

$$\Delta J = J[y_1 + h_1, \ldots, y_n + h_n] - J[y_1, \ldots, y_n]$$

$$= \int_a^b \Big[F(x, \ldots, y_i + h_i, \ldots, y_i' + h_i', \ldots, y_n' + h_n') - F(x, \ldots, y_i, \ldots, y_i', \ldots, y_n') \Big] dx$$

$$= \int_a^b \sum_{i=1}^{n} \left(F_{y_i} h_i + F_{y_i'} h_i' \right) dx + \cdots$$

where ... denotes terms of orders higher than h_i, h_i'.

We get:

$$\delta J = \int_a^b \sum_{i=1}^{n} \left(F_{y_i} h_i + F_{y_i'} h_i' \right) dx = 0 \quad (*)$$

Now (*) must also apply for:

$$0 = h_1(x) = h_2(x) = \cdots = h_{i-1}(x) = h_{i+1}(x) = \cdots = h_n(x),$$

$$h_i(x) \neq 0$$

which yields:

$$\boxed{\begin{array}{c} F_{y_1} - \dfrac{d}{dx}\left[F_{y_1'} \right] = 0 \\ \vdots \\ F_{y_n} - \dfrac{d}{dx}\left[F_{y_n'} \right] = 0 \end{array}} \quad (**)$$

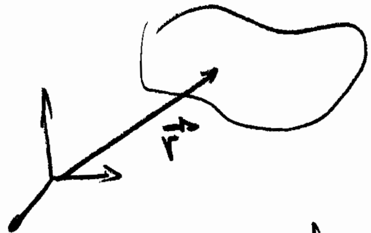
which is a necessary condition for

$y_1(x), \ldots, y_n(x)$ to be an extremum of:

$$\int_a^b F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx$$

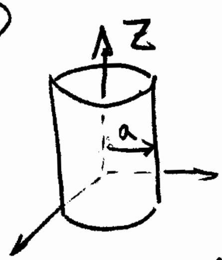
Geodesics (Example 2, p37 in "Calculus of Variations" by Gelfand and Fomin).

Suppose that a surface is given by $\vec{r} = \vec{r}(u,v)$



For a cylinder:

$$\vec{r}(\phi, z) = (a\cos\phi, a\sin\phi, z)$$



A curve lying on the surface is parametrized by $u = u(t)$, $v = v(t)$. The arclength between two points: $\vec{r}(u(t_1), v(t_1))$ and $\vec{r}(u(t_2), v(t_2))$ is given by:

$$J[u,v] = \int_{t_0}^{t_1} \sqrt{E u'^2 + 2F u' v' + G v'^2}\, dt$$

where: $E = \vec{r}_u \cdot \vec{r}_u$, $F = \vec{r}_u \cdot \vec{r}_v$, $G = \vec{r}_v \cdot \vec{r}_v$

We have:

$$F_u - \frac{d}{dt}\left[F_{u'}\right] = 0 \sim (1)$$

$$F_v - \frac{d}{dt}\left[F_{v'}\right] = 0 \sim (2)$$

(not to confuse F with the F in the expression for $J[u,v]$).

We get:

$$+(1/2)\left(Eu'^2+2Fu'v'+Gv'^2\right)^{-1/2}\cdot\left(E_u u'^2+2F_u u'v' + G_u v'^2\right) - \frac{d}{dt}\left(Eu'^2+2Fu'v'+Gv'^2\right)^{1/2}_{u'} =$$

$$= \frac{E_u u'^2+2F_u u'v'+G_u v'^2}{2\sqrt{Eu'^2+2Fu'v'+Gv'^2}} - \frac{d}{dt}\left[\frac{2(Eu'+Fv')}{\sqrt{Eu'^2+2Fu'v'+Gv'^2}}\right]\cdot\frac{1}{2}$$

$$= 0$$

$\underline{E(u,v),\ F(u,v),\ \text{only}}$

Similarly, and finally ...

$$\frac{E_u u'^2 + 2F_u u'v' + G_u v'^2}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} - \frac{d}{dt}\frac{2(Eu' + Fv')}{\sqrt{Eu' + 2Fu'v' + Gv'^2}} = 0$$

$$\frac{E_v u'^2 + 2F_v u'v' + G_v v'^2}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} - \frac{d}{dt}\frac{2(Fu' + Gv')}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} = 0$$

Final Notes on the Calculus of Variations

A very careful presentation on the subject can be found in:

"Calculus of Variations" by Gelfand and Fomin, Prentice-Hall, 1963. Chapter 1 is a must, especially for those interested in the functional analytic approach.

The book treats the problem of finding minima and maxima in some depth.

Another excellent mathematical treatment (older) can be found in:

"Calculus of Variations" by Caratheodory, 1935, re-published in 1989 by Chelsea Publishing Company, New York.