From:

"Chapter 2. The Simplex Algorithm",

Combinatorial Optimization: Algorithms and Complexity,

by C. H. Papadimitriou and K. Steiglitz.

(p.29) 2.2 Basic Feasible Solutions

Consider:

$$\min \ c^T x$$
$$Ax = b, \quad A \text{ is } m \times n, \ m < n.$$
$$x \geq 0$$

Assume A is of rank m.

Let $B = \{A_{j_1}, \ldots, A_{j_m}\}$ be a basis for A.

Eg. If $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix}$, then A is of rank = 3.

$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for A.

A basic solution x is one for which:

$x_j = 0$, for $A_j \in B$

$x_{J_K} =$ kth component of $B^{-1}b$,

$k = 1, \ldots, m$, $A_{J_K} \notin B$.

To find a solution to $Ax = b$, do:

1. Choose a set B of linearly independent columns of A.
2. Set all components of x corresponding to columns not in B equal to zero.
3. Solve the m resulting equations to determine the remaining components of x. These are the <u>basic variables</u>.

In our example.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

2. $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix}$ $\begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{corresponding to columns 1, 2, and 3.} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{remaining two.} \end{matrix}$

3. Solve.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$$

for x_1, x_2, x_3. basic variables.

bfs if also a solution to (*)

(p.42) 2.4 Moving from bfs to bfs

Let x_0 be a bfs of an instance of LP with matrix A, with:

$$\mathcal{B} = \{ A_{B(i)} : i = 1, \dots, m \}$$

If the basic components of x_0 are:

$$x_{i_0}, \quad i_0 = 1, 2, \dots, m,$$

then:

$$\sum_{i=1}^{m} x_{i_0} A_{B(i)} = b, \qquad x_{i_0} \geq 0.$$

(*)

Note that for the non-basic columns, we have to be able to express them in-terms of the basis:

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$$\sum_{i=1}^{m} x_{ij} A_{B(i)} = A_j \quad (**)$$

Multiply (**) by $\theta > 0$ and subtract from (*) to get:

$$\sum_{i=1}^{m} \left(x_{i0} - \theta x_{ij} \right) A_{B(i)} + \theta A_j = b$$

To change x and remain feasible, we must have the new x still remains positive. Thus, the maximum value for x must be:

$$\theta_0 = \min_{\substack{i \\ \text{such that} \\ x_{ij} > 0}} \frac{x_{i0}}{x_{ij}}$$

(from $x_{i0} - \theta x_{ij} = 0$)

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In term of a bfs, we have the cost:

$$z_0 = \sum_{i=1}^{m} x_{i0} c_{B(i)}$$

As we know, other columns can be expressed in terms of the basis columns:

$$A_j = \sum_{i=1}^{m} x_{ij} A_{B(j)}$$

Thus, when x_j enters, x_{ij} from $x_{B(i)}$ **must leave**, reducing cost to

cost of entering $x_j = 1$

$$c_j - \underbrace{\sum_{i=1}^{m} x_{ij} c_{B(i)}}_{\text{definition of } z_j} = \bar{c}_j$$

where $\bar{c}_j$ is hereby defined as the **relative cost** of column j.

It is thus **profitable** (since it will reduce cost) to bring-in column j, provided $\bar{c}_j < 0$.

If all $\bar{c}_j \geq 0$, then there is no hope of reducing cost:

$\Rightarrow$ We are at global minimum

Now return to (for unboundedness):

$$\sum_{i=1}^{m} x_{ij} A_{B(i)} = A_j .$$

If all the $x_{ij} \leq 0$, all i, some j, then clearly if we also assume:

$$c_j - \sum_{i=1}^{m} x_{ij} c_{B(i)} < 0$$

$$\Rightarrow c_j + (-x_{1j}) c_{B(1)} + \cdots + (-x_{mj}) c_{B(m)} < 0$$

all of them are positive.

$\Rightarrow$ At least one of:

$c_{B(1)}, \ldots, c_{B(m)}, c_j$ is negative,

and setting that component $x_k = \infty$ grows unbounded to $-\infty$.

Pseudocode for the simplex algorithm is given on p. 49:

Procedure simplex

begin

opt := 'no', unbounded := 'no';

(if either becomes 'yes', we terminate)

While opt == 'no' **and** unbounded = 'no' **do**

if $\bar{c}_j \geq 0$ for all j,

then opt := 'yes' (terminate)

else

begin

Choose any j with $c_j < 0$;

If $x_{ij} \leq 0$ for all i

then unbounded := 'yes'

else

find $\theta_0 = \min\limits_{\substack{i \\ x_{ij} > 0}} \left[\dfrac{x_{i0}}{x_{ij}} \right] = \dfrac{x_{k0}}{x_{kj}}$

and pivot on x_{kj}

end

end

p49. Example 2.6

Set $c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and:

$$3x_1 + 2x_2 + x_3 = 1$$
$$5x_1 + x_2 + x_3 = 3$$
$$2x_1 + 5x_2 + x_3 + x_5 = 4$$

In "tableau form":

		x_1	x_2	x_3	x_4	x_5	
	0	1	1	1	1	1	– row 0
	1	3	2	1	0	0	– row 1
	3	5	1	1	1	0	– row 2
	4	2	5	1	0	1	– row 3

1. Identify the basis for columns 3, 4, 5.

2. To compute the values for x_3, x_4, x_5, "convert to identity":

    1. Subtract row 1 from rows 2+3:

    2. Subtract them from row 0 to get the relative costs of the non-basic variables (see book for proof)

After these two steps, we are left with:

$-z =$	-6	$\underbrace{-3 \quad -3}$		0	0	0
$x_3 =$	1	3	2	1	0	0
$x_4 =$	2	2	-1	0	1	0
$x_5 =$	3	-1	3	0	0	1

So, it is profitable to consider x_1, x_2
$(\bar{c}_1 = -3,\ \bar{c}_2 = -3)$.

For column 2:

$$\Theta_0 = \min_{\substack{i \\ x_{i2} > 0}} \left[\frac{x_{i0}}{x_{i2}} \right] = \min\left\{ \tfrac{1}{2},\ \tfrac{3}{3} \right\} = \tfrac{1}{2}$$

$J = 2$

Next, pivot on 2, making -3 zero:

		x_1	x_2	x_3	x_4	x_5
$-z =$	$-9/2$	$3/2$	0	$3/2$	0	0
New var $\to x_2 =$	$1/2$	$3/2$	1	$1/2$	0	0
$x_4 =$	$5/2$	$7/2$	0	$1/2$	1	0
$x_5 =$	$3/2$	$-11/2$	0	$-3/2$	0	1

Final answer: $\tfrac{1}{2} + \tfrac{5}{2} + \tfrac{3}{2} = \tfrac{9}{2}$.

$\bar{c} \geq 0 \Rightarrow$ stop