Problem #1. Quadratic models in 1-D Optimization.

1(a) Locally, optimization methods consider a local linear or quadratic model. Consider the quadratic model:

$$f(x) = a \cdot x^2 + b \cdot x + c$$

Compute a general expression for the extreme point.

Solution: Extreme points occur where the derivative equals zero. Differentiate $f(x) = ax^2 + bx + c$:

$$f'(x) = 2ax + b.$$

Set $f'(x) = 0$ and solve:

$$x^* = -\frac{b}{2a}.$$

1(b) When is f convex?

Solution: A function is convex when its second derivative is nonnegative. Differentiate twice:

$$f''(x) = 2a.$$

Set $f''(x) \geq 0$ and solve:

$$a \geq 0.$$

1(c) When is f concave?

Solution: By flipping the inequality from the convex case, f is concave when $a \leq 0$.

1(d) When is the extreme point an actual minimum? **Solution:** The extreme point is a minimum when $f'(x) = 0$ and $f''(x) > 0$, i.e., when $a > 0$.

1(e) When is the extreme point a maximum?

Solution: The extreme point is a maximum when $f'(x) = 0$ and $f''(x) < 0$, i.e., when $a < 0$.

1(f) Consider the constraint optimization problem:

$$\min_x f(x) \quad \text{subject to: } d \leq x \leq e.$$

where $-\infty < d < e < \infty$. Based on the KKT conditions, we know that the solution is either at $x = d$ or $x = e$ or at the extremum point. Suppose that $a < 0$. Show that the solution is either at $x = d$ or $x = e$. In this negative curvature example, the solution is always at the boundary.

Solution: For $a < 0$, f is concave, so any interior critical point is a maximum. From that peak the function decreases toward both ends, so the lowest value must occur at one of the endpoints:

$$x^* \in \{d, e\}.$$

1(g) For $a > 0$, show that all three cases are possible in (f).

Solution: For $a > 0$, f is convex and has a minimum at $x^* = -\dfrac{b}{2a}$. Compare x^* with the interval $[d, e]$:

$$\begin{array}{ll} \text{If } d \le x^* \le e, & \text{the minimum is at } x^*. \\ \text{If } x^* < d, & \text{the minimum is at } x = d. \\ \text{If } x^* > e, & \text{the minimum is at } x = e. \end{array}$$

Since all three positional relationships can occur for suitable coefficients, every case is possible.

Notes: A function f is concave if $-f$ is convex. Use the fact that a function is convex if

$$\frac{\partial^2 f(x)}{\partial x^2} > 0$$

everywhere. Furthermore, note the property of convex functions that

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2), \quad 0 \le t \le 1.$$

This property implies that convex functions stay below a line that connects the end-points at x_1 and x_2.