Problem #1. Quadratic models in 1-D Optimization.

1(a) Locally, optimization methods consider a local linear or quadratic model. Consider the quadratic model:

$$f(x) = a \cdot x^2 + b \cdot x + c$$

Compute a general expression for the extreme point.

Solution: Extreme points occur where the derivative equals zero. Differentiate $f(x) = ax^2 + bx + c$:

$$f'(x) = 2ax + b.$$

Set $f'(x) = 0$ and solve:

$$x^* = -\frac{b}{2a}.$$

1(b) When is f convex?

Solution: A function is convex when its second derivative is nonnegative. Differentiate twice:

$$f''(x) = 2a.$$

Set $f''(x) \geq 0$ and solve:

$$a \geq 0.$$

1(c) When is f concave?

Solution: By flipping the inequality from the convex case, f is concave when $a \leq 0$.

1(d) When is the extreme point an actual minimum?

Solution: The extreme point is a minimum when $f'(x) = 0$ and $f''(x) > 0$, i.e., when $a > 0$.

1(e) When is the extreme point a maximum?

Solution: The extreme point is a maximum when $f'(x) = 0$ and $f''(x) < 0$, i.e., when $a < 0$.

1(f) Consider the constraint optimization problem:

$$\min_x f(x) \quad \text{subject to: } d \leq x \leq e.$$

where $-\infty < d < e < \infty$. Based on the KKT conditions, we know that the solution is either at $x = d$ or $x = e$ or at the extremum point. Suppose that $a < 0$. Show that the solution is either at $x = d$ or $x = e$. In this negative curvature example, the solution is always at the boundary.

Solution: For $a < 0$, f is concave, so any interior critical point is a maximum. From that peak the function decreases toward both ends, so the lowest value must occur at one of the endpoints:

$$x^* \in \{d, e\}.$$

1(g) For $a > 0$, show that all three cases are possible in (f).

Solution: For $a > 0$, f is convex and has a minimum at $x^* = -\dfrac{b}{2a}$. Compare x^* with the interval $[d, e]$:

$$\begin{array}{ll} \text{If } d \le x^* \le e, & \text{the minimum is at } x^*. \\ \text{If } x^* < d, & \text{the minimum is at } x = d. \\ \text{If } x^* > e, & \text{the minimum is at } x = e. \end{array}$$

Since all three positional relationships can occur for suitable coefficients, every case is possible.

Notes: A function f is concave if $-f$ is convex. Use the fact that a function is convex if

$$\frac{\partial^2 f(x)}{\partial x^2} > 0$$

everywhere. Furthermore, note the property of convex functions that

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2), \quad 0 \le t \le 1.$$

This property implies that convex functions stay below a line that connects the end-points at x_1 and x_2.

Problem #2. Quadratic models in 2-D Optimization.

The goal of this problem is to make the connection between max pooling and optimization. Consider the constrained optimization problem

$$\max_x \ \tfrac{1}{2}x^T Ax + b^T x \quad \text{subject to} \quad l_1 \le x_1 \le u_1,\ l_2 \le x_2 \le u_2.$$

2(a) Reformulate the problem as a constrained minimization problem that we studied in class.

Solution: Maximizing is equivalent to minimizing the negative. Define

$$g(x) \ := \ -\Big(\tfrac{1}{2}x^\top Ax + b^\top x\Big) = \tfrac{1}{2}x^\top(-A)x + (-b)^\top x.$$

Then 2(a) becomes

$$\min_x \ g(x) \quad \text{subject to} \quad \begin{cases} x_1 - u_1 \le 0, \\ l_1 - x_1 \le 0, \\ x_2 - u_2 \le 0, \\ l_2 - x_2 \le 0. \end{cases}$$

2(b) Compute the Hessian and the gradient of the function that we are minimizing.

Solution: Starting from

$$g(x) = -\Big[\tfrac{1}{2}x^\top Ax + b^\top x\Big],$$

and using that $\nabla_x\big(\tfrac{1}{2}x^\top Ax\big) = Ax$ when A is symmetric, we obtain

$$\nabla g(x) = -(Ax + b), \qquad \nabla^2 g(x) = -A.$$

2(c) Derive conditions so as to have a unique solution inside the constrained region. **Solution:** For an interior solution (no active constraints), KKT reduces to stationarity:

$$\nabla g(x^*) = 0 \implies -(Ax^* + b) = 0 \implies Ax^* + b = 0,$$

so

$$x^* = -A^{-1}b.$$

Uniqueness inside the box holds if g is *strictly convex*, i.e.

$$\nabla^2 g(x) = -A \succ 0 \quad (\text{equivalently } A \prec 0),$$

and the unconstrained minimizer lies strictly within the bounds:

$$l_1 < x_1^* < u_1, \qquad l_2 < x_2^* < u_2.$$

2(d) Derive four KKT-based conditions so as to have a unique solution at any of the four corners.
Solution: Let $g(x) = -\left(\frac{1}{2}x^\top Ax + b^\top x\right)$ with box constraints $x_1 \in [l_1, u_1],\ x_2 \in [l_2, u_2]$. Introduce KKT multipliers $\lambda_i \geq 0$ for $x_i - u_i \leq 0$ and $\mu_i \geq 0$ for $l_i - x_i \leq 0$. The Lagrangian is

$$\mathcal{L}(x, \lambda, \mu) = g(x) + \lambda_1(x_1 - u_1) + \mu_1(l_1 - x_1) + \lambda_2(x_2 - u_2) + \mu_2(l_2 - x_2).$$

Stationarity gives

$$\nabla g(x) + \begin{bmatrix} \lambda_1 - \mu_1 \\ \lambda_2 - \mu_2 \end{bmatrix} = 0, \qquad \text{and} \qquad \nabla g(x) = -(Ax + b) \text{ (for symmetric } A).$$

Complementary slackness at a corner forces the two active constraints to have positive multipliers and the inactive ones to be zero. Evaluated at each corner, uniqueness (with strict convexity $-A \succ 0$) follows if the gradient components point *outward* of the feasible box, i.e. the signs below hold (strict for uniqueness):

$$\textbf{Corner } (u_1, u_2): \quad \partial_{x_1} g(u_1, u_2) \leq 0, \ \ \partial_{x_2} g(u_1, u_2) \leq 0 \iff (Ax + b)\big|_{x=(u_1,u_2)} \succeq \mathbf{0}.$$

$$\textbf{Corner } (l_1, l_2): \quad \partial_{x_1} g(l_1, l_2) \geq 0, \ \ \partial_{x_2} g(l_1, l_2) \geq 0 \iff (Ax + b)\big|_{x=(l_1,l_2)} \preceq \mathbf{0}.$$

$$\textbf{Corner } (l_1, u_2): \quad \partial_{x_1} g(l_1, u_2) \geq 0, \ \ \partial_{x_2} g(l_1, u_2) \leq 0 \iff \begin{cases} (Ax + b)_1\big|_{x=(l_1,u_2)} \leq 0, \\ (Ax + b)_2\big|_{x=(l_1,u_2)} \geq 0; \end{cases}$$

$$\textbf{Corner } (u_1, l_2): \quad \partial_{x_1} g(u_1, l_2) \leq 0, \ \ \partial_{x_2} g(u_1, l_2) \geq 0 \iff \begin{cases} (Ax + b)_1\big|_{x=(u_1,l_2)} \geq 0, \\ (Ax + b)_2\big|_{x=(u_1,l_2)} \leq 0. \end{cases}$$

With $-A \succ 0$ (strict convexity of g), the strict versions of these inequalities ensure a *unique* minimizer located at the stated corner (the corresponding active multipliers are $\lambda_i = -\partial_{x_i} g > 0$ for upper bounds and $\mu_i = \partial_{x_i} g > 0$ for lower bounds).

2(e) Derive four KKT-based conditions so as to have a unique solution at any of the four sides.
Solution: Assume $g(x) = -\left(\frac{1}{2}x^\top Ax + b^\top x\right)$ with symmetric A, so $\nabla g(x) = -(Ax + b)$, and $-A \succ 0$ (strict convexity) for uniqueness. On a *side*, one bound is active and the other coordinate is interior; KKT gives a normal sign condition and tangential stationarity:

$$\begin{aligned}
\text{Side } x_1 = u_1 \ (l_2 < x_2 < u_2): &\quad \partial_{x_2} g(u_1, x_2) = 0, \quad \partial_{x_1} g(u_1, x_2) \leq 0. \\
\text{Side } x_1 = l_1 \ (l_2 < x_2 < u_2): &\quad \partial_{x_2} g(l_1, x_2) = 0, \quad \partial_{x_1} g(l_1, x_2) \geq 0. \\
\text{Side } x_2 = u_2 \ (l_1 < x_1 < u_1): &\quad \partial_{x_1} g(x_1, u_2) = 0, \quad \partial_{x_2} g(x_1, u_2) \leq 0. \\
\text{Side } x_2 = l_2 \ (l_1 < x_1 < u_1): &\quad \partial_{x_1} g(x_1, l_2) = 0, \quad \partial_{x_2} g(x_1, l_2) \geq 0.
\end{aligned}$$

With $-A \succ 0$, the strict versions of these inequalities (and interiority of the free coordinate) yield a *unique* minimizer on the specified side (and not at a corner).

2(f) Suppose that the function is strictly convex. Show that the maximum is on the boundary.
Solution: Let f be strictly convex on the convex, compact box $[l_1, u_1]\times[l_2, u_2]$. If a maximizer were interior, then $\nabla f(x^*) = 0$ and strict convexity ($\nabla^2 f \succ 0$) would make x^* a *minimum*, a contradiction. Hence any maximum over the box must occur on the boundary.

2(g) Suppose that the function is strictly concave. Show that the maximum is in the interior.
Solution: If f is strictly concave ($\nabla^2 f \prec 0$), it has a unique stationary point

$$x^* \text{ with } \nabla f(x^*) = 0 \quad (\text{for } f(x) = \tfrac{1}{2}x^\mathsf{T} Ax + b^\mathsf{T} x, \quad A \text{ neg. definite: } x^* = -A^{-1}b).$$

If this x^* lies in the open box $(l_1, u_1) \times (l_2, u_2)$, then by KKT (all multipliers zero) it is the unique maximizer and is *interior*. If $x^* \notin [l_1, u_1] \times [l_2, u_2]$, the maximizer occurs on the boundary. Thus, the maximum is in the interior *iff* x^* lies inside the box.

2(h) Suppose that the function is both concave and convex. Show that the function is linear. Also, show that we can always find a solution at one of the corners. This is a basic result used in Linear Programming that holds true for all problems.
Solution: If f is both convex and concave on a convex set, then it is *affine*:

$$\nabla^2 f \equiv 0 \implies f(x) = c^\mathsf{T} x + c_0 \quad (\text{i.e., linear up to a constant}).$$

Maximizing a linear function over the box $[l_1, u_1] \times [l_2, u_2]$ attains an optimum at an extreme point (corner). Equivalently, with $c = (c_1, c_2)$,

$$x_1^* = \begin{cases} u_1, & c_1 > 0, \\ l_1, & c_1 < 0, \\ \text{any in } [l_1, u_1], & c_1 = 0, \end{cases} \qquad x_2^* = \begin{cases} u_2, & c_2 > 0, \\ l_2, & c_2 < 0, \\ \text{any in } [l_2, u_2], & c_2 = 0, \end{cases}$$

so at least one corner is optimal (fundamental LP result).

Problem #3. Solutions to specific problems.

3(a) Consider the following equation:

$$\max_{x} \; 5(x_1-2)^2 + 100(x_2-3)^2 + 100 \quad \text{subject to} \quad 0 \le x_1 \le 4, \;\; 1 \le x_2 \le 5.$$

- Compute x that solves this problem.
- Is the function convex, concave, or neither? Explain.

Solution Let

$$f(x_1,x_2) = 5(x_1-2)^2 + 100(x_2-3)^2 + 100, \quad 0 \le x_1 \le 4, \quad 1 \le x_2 \le 5.$$

Since $\nabla^2 f = \mathrm{diag}(10,\, 200) \succ 0$, f is (strictly) *convex* (not concave). Maximizing a convex function over the convex box occurs on the boundary; here the objective increases with the squared distance from $(2,3)$, so we push each coordinate as far from 2 and 3 as allowed. Both ends are equally far in each coordinate, hence all four corners are maximizers:

$$x^* \in \{(0,1),\; (0,5),\; (4,1),\; (4,5)\}, \qquad f(x^*) = 5 \cdot 2^2 + 100 \cdot 2^2 + 100 = 520.$$

3(b) Consider the point $(x_1,x_2) = (0,0)$ for the function

$$f(x_1,x_2) = 5x_1^2 - 100x_2^2.$$

- Is the function increasing or decreasing as $x_1 \to \infty$ for $x_2 = 0$?
- Is the function increasing or decreasing as $x_2 \to \infty$ for $x_1 = 0$?
- Is the function convex, concave, or neither? Explain.
- Compute the eigen-decomposition of the Hessian of f and rewrite f in terms of that eigen-decomposition.

Note that the results described here generalize along the eigenvectors, which are orthogonal.
Solution Given

$$f(x_1,x_2) = 5x_1^2 - 100x_2^2.$$

Monotonicity along axes. For $x_2 = 0$, $f = 5x_1^2 \to +\infty$ as $x_1 \to \infty$ (increasing). For $x_1 = 0$, $f = -100x_2^2 \to -\infty$ as $x_2 \to \infty$ (decreasing).

Convex/concave.

$$\nabla^2 f = \begin{bmatrix} 10 & 0 \\ 0 & -200 \end{bmatrix}$$

has eigenvalues 10 and -200 (one > 0, one < 0) $\Rightarrow$ Hessian is indefinite $\Rightarrow$ f is **neither** convex nor concave (saddle).

Eigen-decomposition and rewrite. With $Q = I$ and $\Lambda = \mathrm{diag}(10, -200)$, we have

$$\nabla^2 f = Q\Lambda Q^\mathsf{T}, \qquad f(x) = \tfrac{1}{2} x^\mathsf{T} (\nabla^2 f)\, x = \tfrac{1}{2}\big(10x_1^2 - 200x_2^2\big) = 5x_1^2 - 100x_2^2,$$

so along the orthogonal eigenvectors e_1, e_2, the curvatures are $+10$ and -200, respectively.

3(c) Consider the point $(x_1, x_2) = (0, 0)$ for the function

$$f(x_1, x_2) = -5x_1^2 - 100x_2^2.$$

- Is the function increasing or decreasing as $x_1 \to \infty$ for $x_2 = 0$?
- Is the function increasing or decreasing as $x_2 \to \infty$ for $x_1 = 0$?
- Is the function convex, concave, or neither? Explain.

Solution Given

$$f(x_1, x_2) = -5x_1^2 - 100x_2^2.$$

Monotonicity along axes. For $x_2 = 0$, $f = -5x_1^2 \to -\infty$ as $x_1 \to \infty$ (decreasing). For $x_1 = 0$, $f = -100x_2^2 \to -\infty$ as $x_2 \to \infty$ (decreasing).

Convex/concave.

$$\nabla^2 f = \begin{bmatrix} -10 & 0 \\ 0 & -200 \end{bmatrix}$$

is negative definite (both eigenvalues < 0), hence f is **strictly concave**.