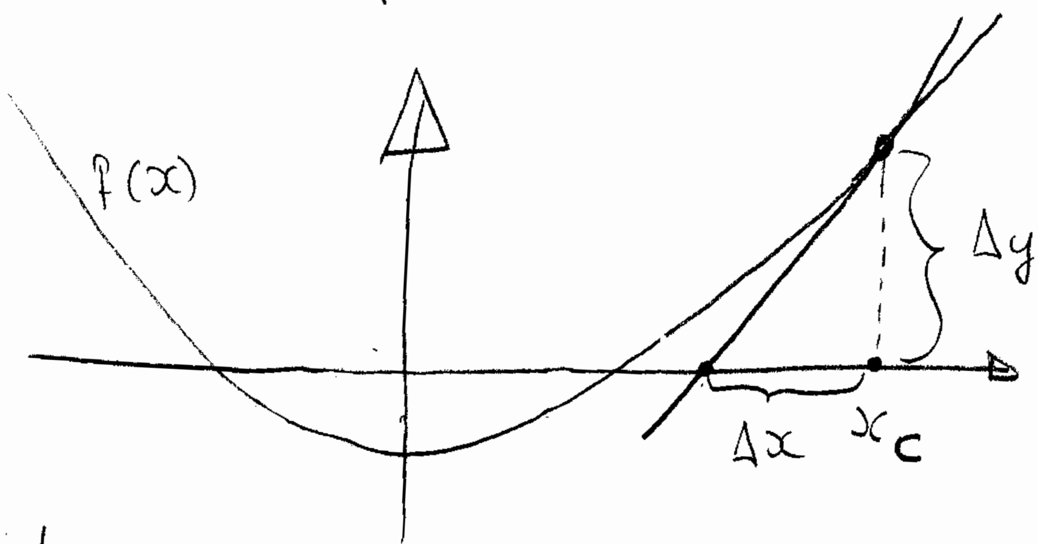
Newton's Method

(also: Section 2.2 of Dennis & Schnabel).



Basic idea: To solve $f(x)=0$, follow the tangent line from the current point x_c, to the intersection with $f(x)=0$.

Reason:
$$\frac{f(x_c) - f(x_*)}{x_c - x_*} \approx f'(x_c)$$

and $f(x_*)=0$. Thus:

$$f(x_c) \approx (x_c - x_*) \cdot f'(x_c)$$

$$\Rightarrow \quad x_* \approx x_c - \frac{f(x_c)}{f'(x_c)}$$

$\therefore$ Applies around x_* $(f'(x_c) \approx f'(x_*))$

Basic algorithm:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots$$

However, our iteration may fail to produce a reduction in $f(x)$.

If we fail, then $|f(x_{k+1})| > |f(x_k)|$.

In this case, use a mid-point formula: $x_{k+1} = \frac{x_{k+1} + x_k}{2}$ back, closer to x_k (backtracking).

Here is a summary of the modified Newton's method (p. 26 of D&J):

$$x_+ = x_c - \frac{f(x_c)}{f'(x_c)}$$

while $|f(x_+)| \geq |f(x_c)|$ do

$$x_+ \leftarrow \frac{x_+ + x_c}{2}$$

A general hybrid method is given by D & J as algorithm 2.5.1:

Given $f : \mathbb{R} \to \mathbb{R}$, x_0

for $k = 0, 1, 2, \ldots,$ do

1. Decide whether to stop or not. If not:

2. Make a local model of f around x_k, and use it to compute x_N that solves (or approximately solves) the model problem.

3(a) Decide whether to take $x_{k+1} = x_N$. If not:

3(b) Choose x_{k+1} using a global stategy

(Make more conservative use of x_N)

Sec. 2.6 Methods when derivatives are not Available (from D & S)

Approximate derivative by.

$$f'(x_c) \approx \frac{f(x_-) - f(x_c)}{x_- - x_c}$$

where x_- refers to the previous iterate. Eg. If we have:

$$x_1, \; x_2, \; x_3, \; x_4, \; \ldots$$
$$f(x_1), f(x_2), f(x_3), f(x_4), \ldots$$

use

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \quad \text{to get } f'(x_2)$$

$$\frac{f(x_2) - f(x_3)}{x_2 - x_3} \quad \text{to get } f'(x_3)$$

$$\vdots$$

Sec. 2.7 Minimization of a function of One Variable (from D & S)

Simply note that we want $f'(x)$.

So, apply Newton's algorithm to f':

$$\frac{f'(x_c) - f'(x_*)}{x_c - x_*} \simeq f''(x_c)$$

and $f'(x_*) = 0$.

$$\Rightarrow \quad f'(x_c) \simeq f''(x_c)\,(x_c - x_*)$$

$$\Rightarrow \quad x_* \approx x_c - \frac{f'(x_c)}{f''(x_c)}$$

Generate estimates using:

$$x_{k+1} = x_k - \frac{f'(x_c)}{f''(x_c)}$$