ME 596 Spacecraft Attitude Dynamics and Control

Review of Particle Dynamics

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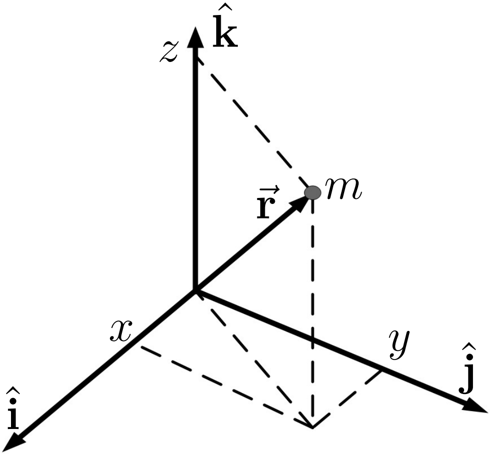
$$\vec{\mathbf{f}} = m\vec{\mathbf{a}}$$

- Applicable to mass particles with *constant* mass and *zero* volume, and is extensible to finite-size, constant-mass rigid bodies
- Not as simple as it looks
  - Determination of the force usually requires a free-body diagram, which can be extremely complicated
  - Determination of the acceleration sometimes involves complicated kinematics
  - Acceleration is second derivative of position vector with respect to time
  - Position vector is sometimes expressed in a non-inertial reference frame

Reference Frame and Position Vector

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$



Position vector must be from inertial origin to mass particle

If expressed in terms of inertial frame components, then differentiation is easy

Inertial Origin

An inertial origin is a point that is not accelerating with respect to any other inertial origin

- Alternatively, an inertial origin is a point for which Newton's laws are applicable
- There is no known inertial origin, but for most problems an origin can be found that is "inertial enough"
- For some problems, an Earth-fixed reference point is inertial enough, whereas for others, the rotation of the Earth must be taken into account

Inertial Reference Frame

An inertial reference frame is a set of three unit vectors that are mutually perpendicular, with their origins at a single inertial origin, whose directions remain fixed with respect to inertial space

- Alternatively, an inertial reference frame is a frame for which Newton's laws are applicable
- Usually the dynamics and control analyst must determine the simplest frame that is "inertial enough"
- What reference frames would be "inertial enough" for these problems?
  - Paper airplane, Cessna, B2, container ship, Falcon Heavy, Voyager, Rama, Oumuamua

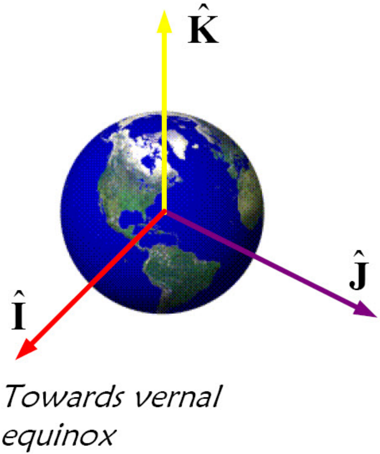
- A reference frame is a set of three mutually perpendicular (orthogonal) unit vectors
- Typical notations include

$$\hat{\mathbf{i}}\,\hat{\mathbf{j}}\,\hat{\mathbf{k}},\quad \hat{\mathbf{I}}\,\hat{\mathbf{J}}\,\hat{\mathbf{K}},\quad \hat{\mathrm{e}}_1\,\hat{\mathrm{e}}_2\,\hat{\mathrm{e}}_3,\quad \hat{\mathrm{b}}_1\,\hat{\mathrm{b}}_2\,\hat{\mathrm{b}}_3,\quad \hat{\mathrm{o}}_1\,\hat{\mathrm{o}}_2\,\hat{\mathrm{o}}_3$$

- Typical reference frames of interest for vehicles include
  - ECI (Earth-centered inertial)
  - Perifocal (Earth-centered, orbit-based inertial)
  - ECEF (Earth-centered, Earth-fixed, rotating)
  - Orbital (Earth-centered, orbit-baed, rotating)
  - Body (spacecraft-fixed, rotating)

Earth-Centered Inertial Frame (ECI)

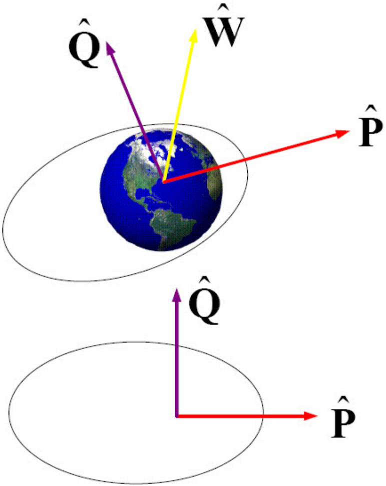
- The $\hat{\mathbf{I}}$-axis is in the vernal equinox direction (i.e., points to the Sun on the first day of Spring)
- The $\hat{\mathbf{K}}$-axis is Earth's rotation axis, perpendicular to the equatorial plane
- The $\hat{\mathbf{J}}$-axis is in the equatorial plane and completes the "triad" of mutually perpendicular unit vectors



Note that $\hat{\mathbf{I}} \times \hat{\mathbf{J}} = \hat{\mathbf{K}}$ by the "right-hand rule"

Perifocal Frame

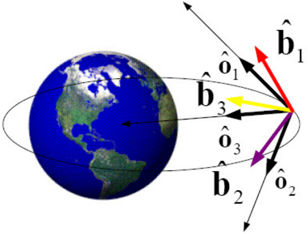
- Earth-centered, orbit-based, inertial
- The $\hat{\mathbf{P}}$-axis is in the periapsis direction
- The $\hat{\mathbf{W}}$-axis is perpendicular to the orbital plane (i.e., direction of orbit angular momentum vector, $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$)
- The $\hat{\mathbf{Q}}$-axis is in the orbital plane and completes the "triad" of mutually perpendicular unit vectors

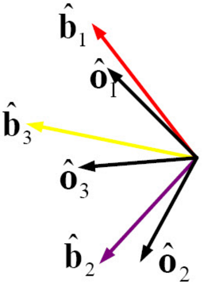


Note that $\hat{\mathbf{P}} \times \hat{\mathbf{Q}} = \hat{\mathbf{W}}$ by the "right-hand rule"

Orbital Frame

- Similar to "roll-pitch-yaw" frame, for spacecraft
- Spacecraft-centered, orbit-based, rotating (non-inertial)
- The $\hat{\mathbf{o}}_3$-axis is in the nadir direction (i.e., direction of Earth, $-\vec{\mathbf{r}}$)
- The $\hat{\mathbf{o}}_2$-axis is in the negative orbital normal direction (i.e., in the $-\hat{\mathbf{W}}$ direction)
- The $\hat{\mathbf{o}}_1$-axis completes the "triad" of mutually perpendicular unit vectors, and is in the velocity vector direction for circular orbits



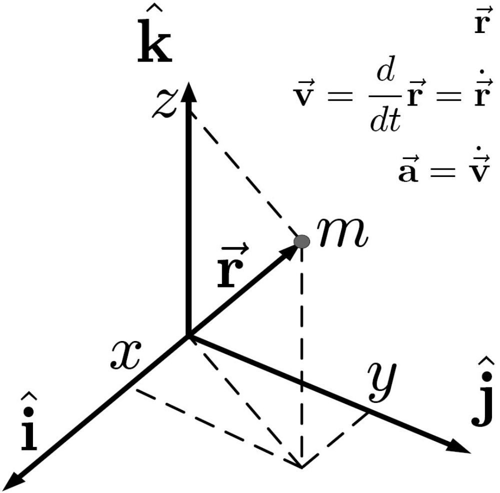


Note that $\hat{\mathbf{o}}_1 \times \hat{\mathbf{o}}_2 = \hat{\mathbf{o}}_3$ by the "right-hand rule"

Vector, Frame, and Matrix Notation (preview)

Symbol	**Meaning**
$\vec{\mathbf{v}}$	vector, an abstract mathematical object with direction and length
$\left\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\right\}$	the three unit base vectors of a reference frame
$\mathcal{F}_i$	the reference frame with base vectors $\left\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\right\}$
$\left\{\hat{\mathbf{i}}\right\}$	a column matrix whose 3 elements are the unit vectors of $\mathcal{F}_i$
$\mathbf{v}_i$	a column matrix whose 3 elements are the components of the vector $\vec{\mathbf{v}}$ expressed in $\mathcal{F}_i$
$\mathbf{v}_b$	a column matrix whose 3 elements are the components of the vector $\vec{\mathbf{v}}$ expressed in $\mathcal{F}_b$
$\mathbf{R}^{bi}$	rotation matrix that transforms vectors from $\mathcal{F}_i$ to $\mathcal{F}_b$
θ	a column matrix whose 3 elements are the Euler angles θ_1, θ_2, θ_3
$\vec{\omega}$	an angular velocity vector
$\vec{\omega}^{bi}$	angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$
ω_b^{bi}	angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ expressed in $\mathcal{F}_b$
ω_a^{bi}	angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ expressed in $\mathcal{F}_a$

Position, Velocity, and Acceleration Vectors



$$\begin{aligned} \vec{\mathbf{r}} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \\ \vec{\mathbf{v}} = \frac{d}{dt}\vec{\mathbf{r}} = \dot{\vec{\mathbf{r}}} &= \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \\ \vec{\mathbf{a}} = \dot{\vec{\mathbf{v}}} &= \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \end{aligned}$$

Derivatives are simple because unit vectors are constant, in direction and magnitude

Application of $\vec{\mathrm{f}} = m\vec{\mathrm{a}}$

Need to know the components of the force vector

$$\begin{aligned}
\vec{\mathbf{f}} &= f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}} + f_z\hat{\mathbf{k}} \\
&= m\left(\ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}\right)
\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}
f_x &= m\ddot{x} \\
f_y &= m\ddot{y} \\
f_z &= m\ddot{z}
\end{aligned}$$

Three second-order ordinary differential equations.

Linearity and coupling depend on nature of forces.

Simple Feedback Control Forces

Suppose the forces take the following form

$$\begin{aligned} f_x &= -k_{px}x - k_{dx}\dot{x} \\ f_y &= -k_{py}y - k_{dy}\dot{y} \\ f_z &= -k_{pz}z - k_{dz}\dot{z} \end{aligned}$$

Here the "k" terms are feedback gains, whose values are selected by the d&c analyst.

The k_p terms are ***proportional*** to the position errors, and the k_d terms are proportional to the velocity errors, or to the ***derivatives*** of the position errors, hence the controller is a "PD" controller

Equations of Motion with PD Control

First, define some new variables, termed the *states*

$$
\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= z \\
x_4 &= \dot{x} \\
x_5 &= \dot{y} \\
x_6 &= \dot{z} \\
&\downarrow \\
\mathbf{x} &\in \mathbf{R}^6
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= f_x/m = \frac{1}{m}(-k_{px}x_1 - k_{dx}x_4) \\
\dot{x}_5 &= f_y/m = \frac{1}{m}(-k_{py}x_2 - k_{dy}x_5) \\
\dot{x}_6 &= f_z/m = \frac{1}{m}(-k_{pz}x_3 - k_{dz}x_6) \\
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}; \mathbf{x}_0, \mathbf{k})
\end{aligned}
$$

$\boxed{\mathbf{x} \text{ is the state vector}}$

Linear System of Equations (PD Control)

Because of the unique form of these equations, they can be written in matrix form

- Each "right-hand side" appears as a summation of constants multiplying states, with the states always appearing linearly

- Thus, the equations comprise a system of linear, constant-coefficient, ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

Linear System (PD) (continued)

Simple rearrangement of the equations of motion leads to the block matrix form:

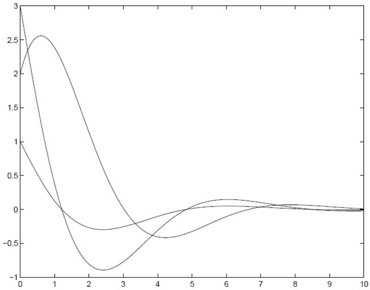
$$
\begin{aligned}
\dot{\mathbf{x}} &= \left[\begin{array}{ccc:ccc}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\hdashline
-k_{px}/m & 0 & 0 & -k_{dx}/m & 0 & 0 \\
0 & -k_{py}/m & 0 & 0 & -k_{dy}/m & 0 \\
0 & 0 & -k_{pz}/m & 0 & 0 & -k_{dz}/m
\end{array}\right] \mathbf{x} \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\operatorname{diag}(\mathbf{k}_p)/m & -\operatorname{diag}(\mathbf{k}_d)/m \end{bmatrix} \mathbf{x} \\
&= \mathbf{A}\mathbf{x}
\end{aligned}
$$

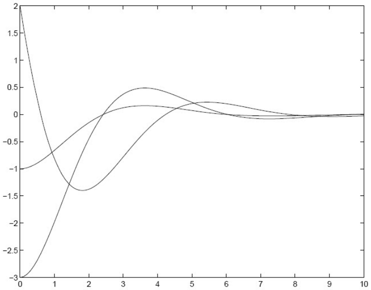
Because $\mathbf{A}$ is constant, the system is easily solved:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

Linear System (PD) (continued)

The nature of the PD controller is that it causes all of the states to approach zero asymptotically; thus it is a stable controller

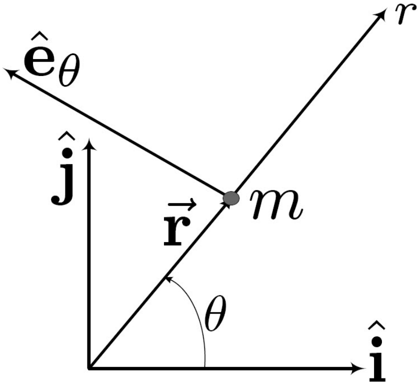




Note poor quality of these graphs. Not acceptable for technical presentations. How to improve?

Polar Coordinates, Rotating Reference Frame

$$
\begin{aligned}
\vec{\mathbf{r}} &= r\hat{\mathbf{e}}_r \\
\vec{\mathbf{v}} &= \dot{r}\hat{\mathbf{e}}_r + r\dot{\hat{\mathbf{e}}}_{\mathbf{r}}
\end{aligned}
$$



Convenient to visualize frame origin at particle, but position vector must be from inertial origin.

Unit vectors of rotating reference frame are constant in length, ***but not in direction***

Unit vectors ***are*** *orthogonal, in spite of distortion that appears in figure!*

Rate of Change of the Unit Vectors

- The unit vectors change because the angle θ changes
- Denote small change in time t and corresponding change in angle θ with Δt and $\Delta\theta$
- Consequently, the various derivatives can be written as

$$
\begin{aligned}
\dot{\hat{\mathbf{e}}}_{\mathbf{r}} &= \lim_{\Delta t \to 0} \frac{\Delta \hat{\mathbf{e}}_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \hat{\mathbf{e}}_\theta = \dot{\theta} \hat{\mathbf{e}}_\theta \\
\dot{\hat{\mathbf{e}}}_\theta &= -\dot{\theta} \hat{\mathbf{e}}_r \\
\vec{\mathbf{v}} &= \dot{r} \hat{\mathbf{e}}_r + r \dot{\theta} \hat{\mathbf{e}}_\theta \\
\vec{\mathbf{a}} &= \left(\ddot{r} - r \dot{\theta}^2 \right) \hat{\mathbf{e}}_r + \left(r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right) \hat{\mathbf{e}}_\theta
\end{aligned}
$$

Differentiating Vectors Expressed in Rotating Frame

If the position vector $\vec{\mathbf{r}}$ is expressed in a reference frame that has angular velocity $\vec{\omega}$ with respect to an inertial reference frame, then the components of the velocity vector $\dot{\vec{\mathbf{r}}}$ expressed in the rotating frame are found using

$$\dot{\mathbf{r}} + \omega^{\times}\mathbf{r}$$

where $\omega^{\times}$ denotes the 3×3 skew-symmetric matrix

$$\omega^{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

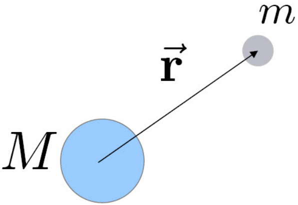
Differentiating again gives the components of the acceleration vector, $\ddot{\mathbf{r}}$, expressed in the rotating reference frame:

$$\ddot{\mathbf{r}} + 2\omega^{\times}\dot{\mathbf{r}} + \dot{\omega}^{\times}\mathbf{r} + \omega^{\times}\left(\omega^{\times}\mathbf{r}\right)$$

The first two equations on this slide will be used throughout this course in using vectors expressed in rotating reference frames.

Orbital Mechanics Application

- When the net force is easily decomposed into radial and transverse components, the preceding system is quite useful
- The most common example is the simple two-body problem, where the force is the inverse-square universal gravitational law



$$\vec{\mathbf{f}} = -\frac{GMm}{r^2}\hat{\mathbf{e}}_r$$

where G is the universal gravitational constant

Simple Orbital Example

Suppose we select units such that $GM = 1$; then the force is

$$\vec{\mathbf{f}}_r = -\frac{m}{r^2}\hat{\mathbf{e}}_r$$

Application of Newton's 2nd law leads to the two scalar nonlinear ordinary differential equations:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{1}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0 \end{aligned}$$

Why are these equations *nonlinear?*

Simple Orbital Example (continued)

Rewrite the nonlinear differential equations as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} \in \mathbf{R}^4$, the set of 4×1 real matrices.

Define new state variables:

$$
\begin{aligned}
x_1 &= r \\
x_2 &= \theta \\
x_3 &= \dot{r} \\
x_4 &= \dot{\theta} \\
&\downarrow \\
\mathbf{x} &\in \mathbf{R}^4
\end{aligned}
\qquad \Rightarrow \qquad
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= x_1 x_4^2 - \frac{1}{x_1^2} \\
\dot{x}_4 &= -\frac{2 x_3 x_4}{x_1} \\
&\downarrow \\
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x})
\end{aligned}
$$

Note that the state differential equation is a 4×1 system of coupled, nonlinear, ordinary differential equations.

Simple Orbital Example (continued)

- We can identify a special solution to the equations of motion for circular orbits
- Suppose $x_1 =$ constant (circular orbit), then $x_3 = 0$, and $x_4 =$ constant (constant angular rate, or mean motion)
- From the differential equation for x_3, which equals zero in this case, we can deduce that

$$x_1^3 x_4^2 = 1$$

- This result is a form of Kepler's Third Law, which relates the orbital period to the orbial radius
- The "1" is a result of the choice of canonical units with gravitational parameter $\mu = 1$

Orbital Mechanics Integrator

```
%(name  file  TwoBodyExampleDriver.m)
%   Two-Body  Example  Driver

x1=1;  x2=0;  x3=0;  x4=1;           % initial conditions
x0 = [x1;  x2;  x3;  x4];
tspan=linspace(0,2*pi,100);          % circular orbit period is 2pi

% setting up to use ode45
options  = odeset('abstol',1e-8,  'reltol',1e-8);
rhsfn  = 'TwoBodyExa';

% calling ode45 to do integration
[t,x]  = ode45(rhsfn,tspan,x0,options);

% graph results
figure;  hold  on
hg=plot(x(:,1).*cos(x(:,2)),  x(:,1).*sin(x(:,2)));
set(hg,'linestyle','-','linewidth',2);
```

Orbital Mechanics Integrator (2)

```
% changing dtheta/dt initial condition, period not 2pi
x0(4)=  1.1;
E=(x0(1)*x0(4))^2/2  - 1/x0(1);
a=-1/(2*E);
TP=2*pi*sqrt(a^3/1);
tspan=linspace(0,TP,100);
[t,x]  = ode45(rhsfn,tspan,x0,options);
hg=plot(x(:,1).*cos(x(:,2)),  x(:,1).*sin(x(:,2)));
set(hg,'linestyle',':','linewidth',2);
x0(4)=  0.8;
E=(x0(1)*x0(4))^2/2  - 1/x0(1); a=-1/(2*E);
TP=2*pi*sqrt(a^3/1); tspan=linspace(0,TP,100);
[t,x]  = ode45(rhsfn,tspan,x0,options);
hg=plot(x(:,1).*cos(x(:,2)),  x(:,1).*sin(x(:,2)));
set(hg,'linestyle','--','linewidth',2);
hg=xlabel('x  = r cos \theta'); set(hg,'fontsize',18);
hg=ylabel('y  = r sin  \theta');  set(hg,'fontsize',18);
legend('Circular','Large  Ellipse','Small  Ellipse');
set(gca,'fontsize',18);

% END OF TwoBodyExampleDriver.m
```

Orbital Mechanics Integrator (3)

The `Matlab` script `TwoBodyExampleDriver` calls the built-in function `ode45` and the following "right-hand side" function `TwoBodyExa`.

The "right-hand side" function simply returns the 4×1 matrix of the values of $\mathbf{f}(\mathbf{x})$ calculated with the current value of $\mathbf{x}$.

```
%   (name  file  TwoBodyExa.m)
function  xdot  = TwoBodyExa(t,x,options)
xdot=[x(3);
x(4);
x(1)*x(4)^2-1/x(1)^2;
-2*x(3)*x(4)/x(1)];
```

Using `TwoBodyExampleDriver` and `TwoBodyExa`, we can integrate the equations of motion and generate graphs of the orbits.

This `Matlab` file structure can be used to set up numerical integration for other systems of differential equations.

Orbital Mechanics Integrator (4)