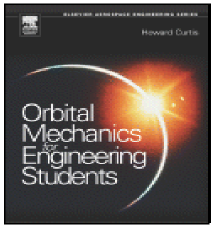
ORBITAL MECHANICS

Chapter 1 - Dynamics of Point Masses



Chapter Outline

1.2 Vectors

1.3 Kinematics

1.4 Mass, force and Newton’s law of gravitation

1.5 Newton’s law of motion

1.6 Time derivatives of moving vectors

1.7 Relative motion

1.2 Vectors

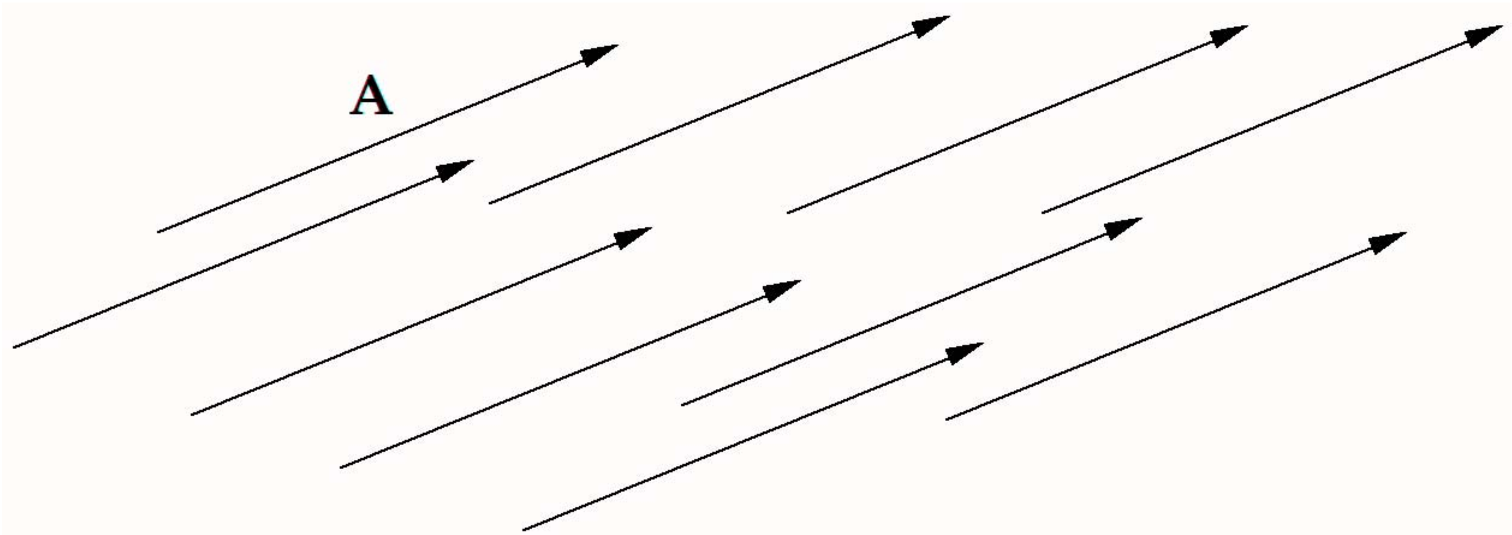
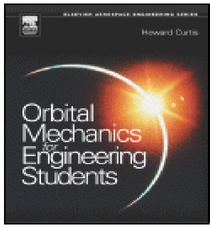


Figure 1.1 All of these vectors may be denoted **A** (or $\overline{A}$) since their magnitudes and directions are the same.



1.2 Vectors

Consider the vectors **A** $= A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and **B** $= B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

The sum or *resultant* **C** of **A** and **B** is defined by

C = A+B $= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$

By construction, vector addition is commutative, i.e. **A+B = B+A**

1.2 Vectors

The resultant can also be determined graphically using the parallelogram rule.

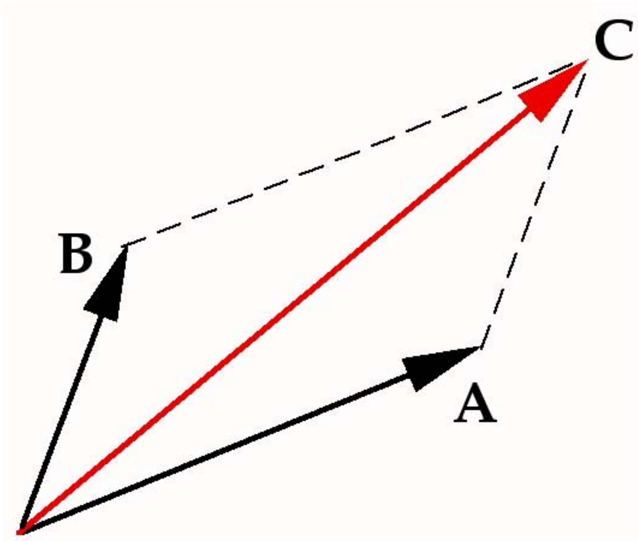
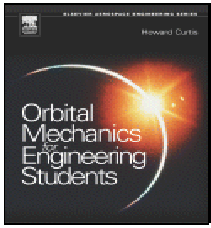


Figure 1.2 Parallelogram rule of vector addition.



1.2 Vectors

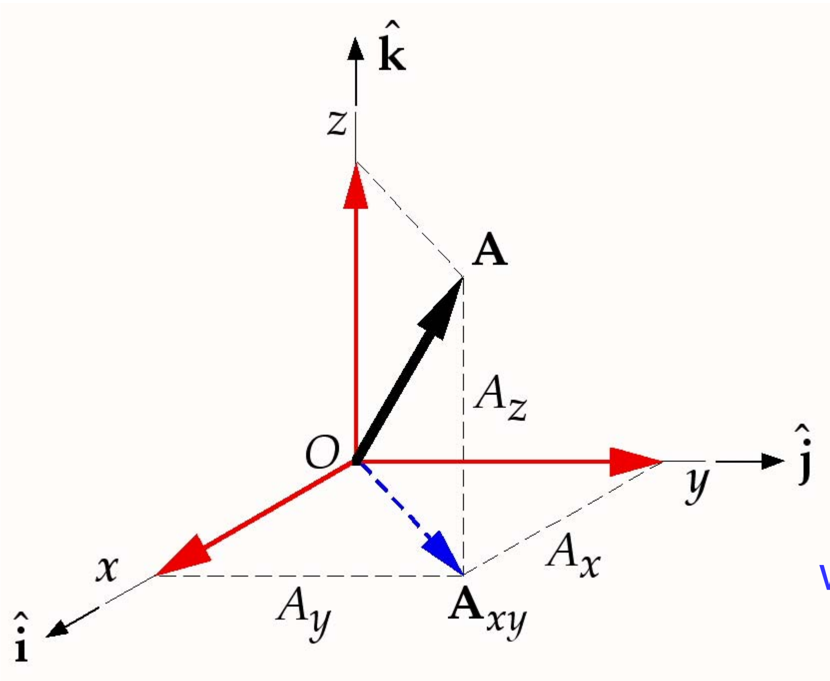


Figure 1.3 Three-dimensional, right-handed Cartesian coordinate system.

Consider the vector **A:** $\bar{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

Magnitude of **A**: $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector in the direction of **A**:

$$\hat{\mathbf{u}}_A = \cos\theta_x\hat{\mathbf{i}} + \cos\theta_y\hat{\mathbf{j}} + \cos\theta_z\hat{\mathbf{k}} = \frac{\bar{A}}{A}$$

where

$$\cos\theta_x = \frac{A_x}{A} \qquad \cos\theta_y = \frac{A_y}{A} \qquad \cos\theta_z = \frac{A_z}{A}$$

The magnitude of a unit vector is **ONE**

(hence the name)

1.2 Vectors

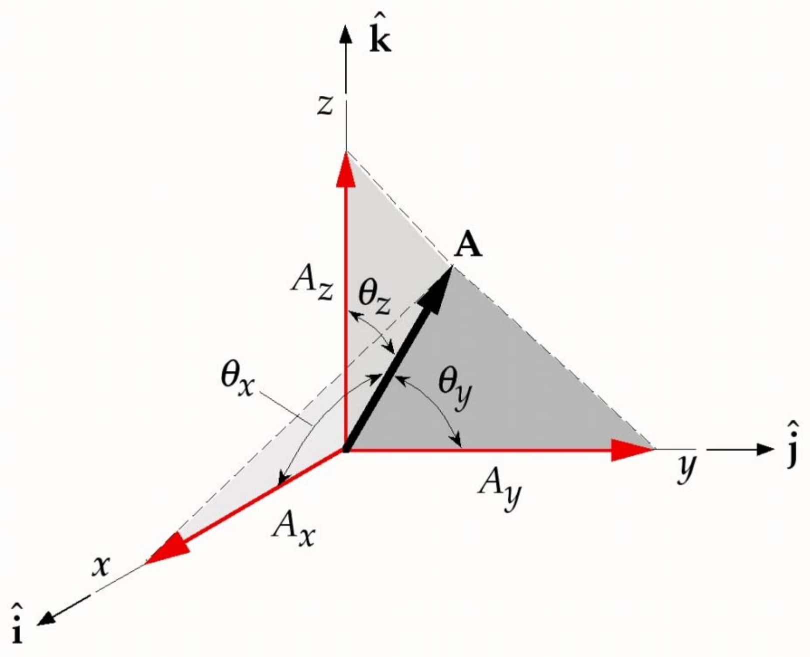
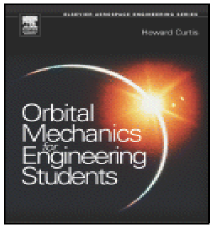


Figure 1.4 Direction angles in three dimensions (also called direction cosines).

The direction angles θ_x, θ_y and θ_z are measured between the vector and the positive coordinate axes.

The sum of θ_x, θ_y and θ_z is not in general known *a priori* and cannot be assumed to be, say, 180°.



1.2 Vectors

To express a 2-D vector requires TWO pieces of information:

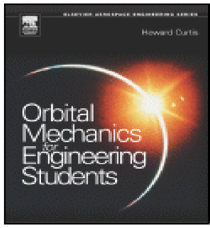
- its **i** and **j** components
- magnitude and ONE angle

To express a 3-D vector requires THREE pieces of information:

- its **i, j,** and **k** components
- magnitude and TWO angles

So why are there THREE direction angles??

- Only TWO angles are needed to uniquely define the vector (the 3rd angle can be determined given the values of the other two)



1.2 Vectors

The dot product of two vectors is a **scalar** defined as

$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$ ← "magnitude/direction" formula for dot product

where θ is the angle between the heads of the vectors.

$$\theta = \cos^{-1}\left(\frac{\overline{A} \bullet \overline{B}}{AB}\right)$$

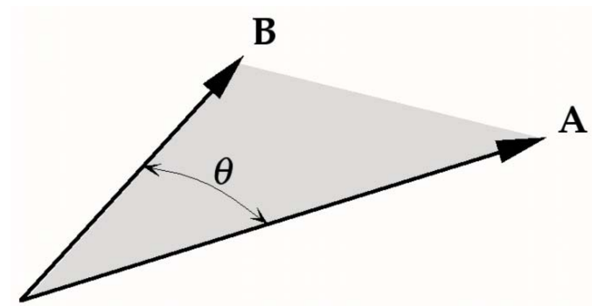


Figure 1.5 The angle between two vectors brought tail to tail by parallel shift**.**

If $\overline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\overline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

then $\overline{A} \bullet \overline{B} = A_xB_x + A_yB_y + A_zB_z$

("Cartesian" formula for dot product)

The dot product is zero if θ is 90^0.

The dot product is **commutative:** $\overline{A} \bullet \overline{B} = \overline{B} \bullet \overline{A}$

1.2 Vectors

- Scalar projection of **B** onto **A**

$$B_A = \mathbf{B} \cdot \hat{\mathbf{u}}_A = \mathbf{B} \cdot \frac{\mathbf{A}}{A} = B\cos\theta$$

(also called the component of **B** in the direction of **A**)

- Scalar projection of **A** onto **B**

$$A_B = \mathbf{A} \cdot \frac{\mathbf{B}}{B} = A\cos\theta$$

(also called the component of **A** in the direction of **B**)

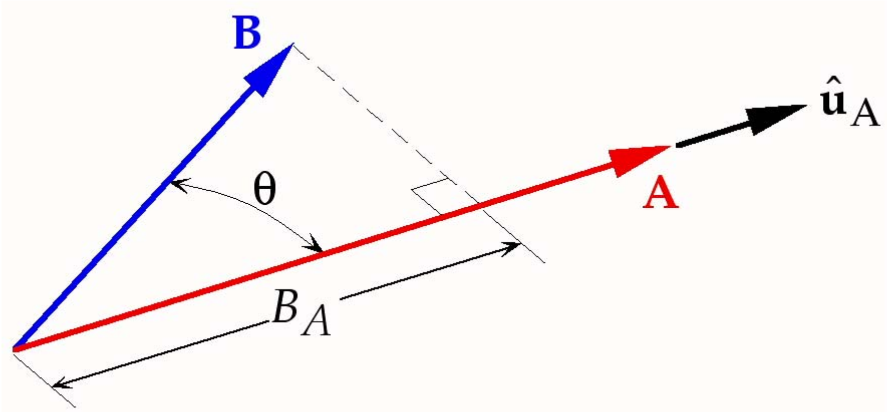


Figure 1.6 Projecting the vector **B** onto the direction of **A**.

1.2 Vectors

Cross product $\mathbf{A}\times\mathbf{B} = AB\sin\theta\,\hat{\mathbf{n}}_{AB}$, where θ is the angle between the heads of **A** and **B**.

“magnitude/direction” formula for dot product

$\hat{\mathbf{n}}_{AB}$ is the unit vector normal to the plane defined by the two vectors and is defined by $\hat{\mathbf{n}}_{AB} = \dfrac{\mathbf{A}\times\mathbf{B}}{\|\mathbf{A}\times\mathbf{B}\|}$

The cross product is not commutative. $\overline{A}\times\overline{B} = -\left(\overline{B}\times\overline{A}\right)$

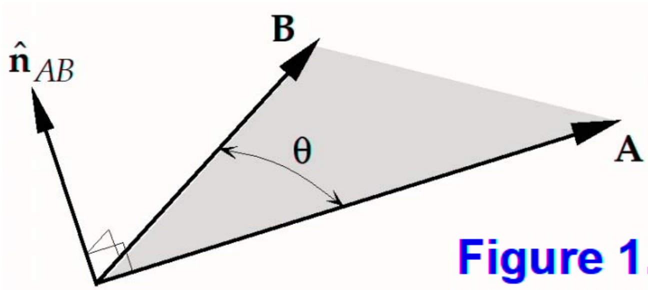
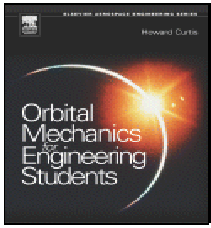


Figure 1.7 $\hat{\mathbf{n}}_{AB}$ is normal to both **A** and **B** and defines the direction of the cross product $\mathbf{A}\times\mathbf{B}$.



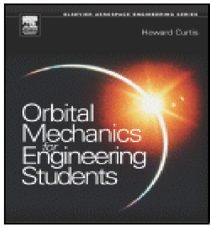
1.2 Vectors

Cartesian formula for cross product is best expressed as a determinant:

$$\overline{A} \times \overline{B} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right) \times \left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \left(A_yB_z - A_zB_y\right)\hat{i} + \left(A_zB_x - A_xB_z\right)\hat{j} + \left(A_xB_y - A_yB_x\right)\hat{k}$$

Direction of $\hat{n}_{AB}$ determined by the “right-hand rule” (What is this??)



1.2 Vectors

REMEMBER...

The dot product of two vectors yields a SCALAR

The cross product of two vectors yields a VECTOR that is perpendicular to both of the vectors

Be sure you know the “right-hand” rule, as it pertains to cross products! (Learn it, review it, study it)

Also useful are the “bac-cab” rule:

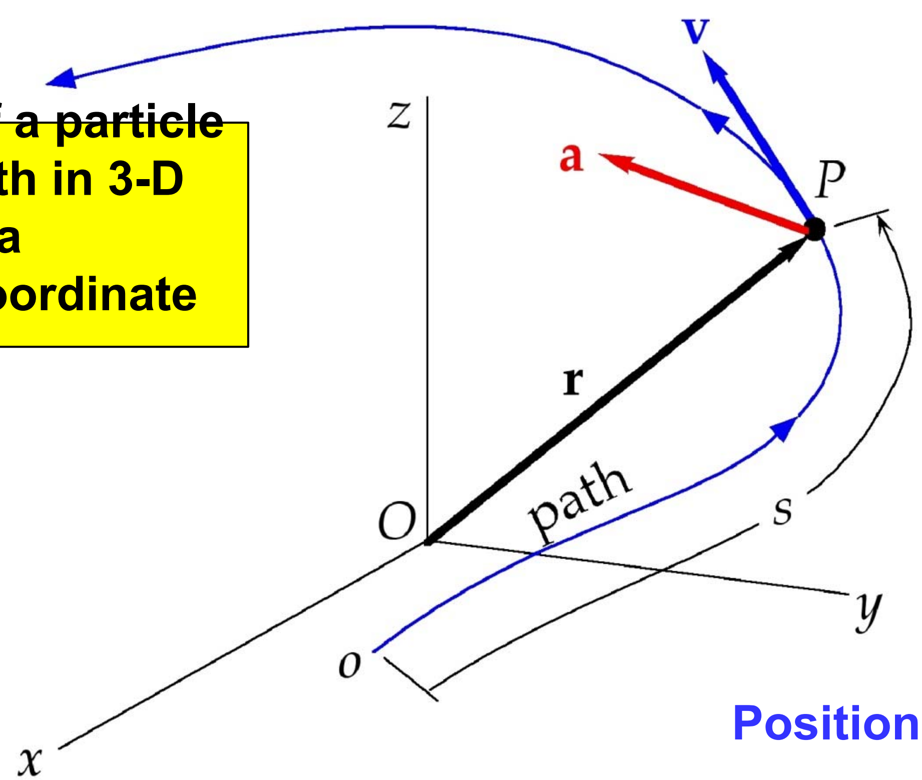
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

& the interchange of the dot and the cross:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$$

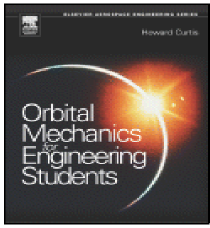
1.3 Kinematics

Depiction of a particle *P* and its path in 3-D space, with a reference coordinate frame



Position vector r $= x\hat{i} + y\hat{j} + z\hat{k}$

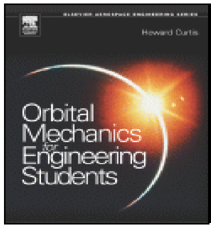
Figure 1.8 Position, velocity and acceleration vectors.



1.3 Kinematics

NOTE: from the preceding figure depicting the **particle at some instant in time**, its **path**, & the **reference frame**…

- One can draw the MAGNITUDE and DIRECTION of the position vector at that instant in time (because it connects the origin of the reference frame with the particle)
- One can draw the DIRECTION of the velocity vector at any instant in time (because it is tangent to the path of the particle)
- One cannot accurately draw the acceleration vector without further information



1.3 Kinematics

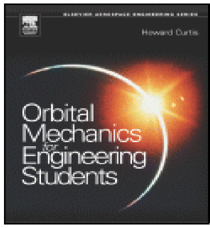
- The distance of *P* from the origin $\|\mathbf{r}\| = r = \sqrt{x^2 + y^2 + z^2}$
- The velocity **v** and acceleration **a** in the x-y-z frame are given by:

$$\mathbf{v}(t) = \frac{dx(t)}{dt}\hat{\mathbf{i}} + \frac{dy(t)}{dt}\hat{\mathbf{j}} + \frac{dy(t)}{dt}\hat{\mathbf{k}} = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}}$$

$$\mathbf{a}(t) = \frac{dv_x(t)}{dt}\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\hat{\mathbf{j}} + \frac{dv_z(t)}{dt}\hat{\mathbf{k}} = a_x(t)\hat{\mathbf{i}} + a_y(t)\hat{\mathbf{j}} + a_z(t)\hat{\mathbf{k}}$$

- It is convenient to represent the time derivative by means of an overhead dot. For example, $v_x = \dot{x}$

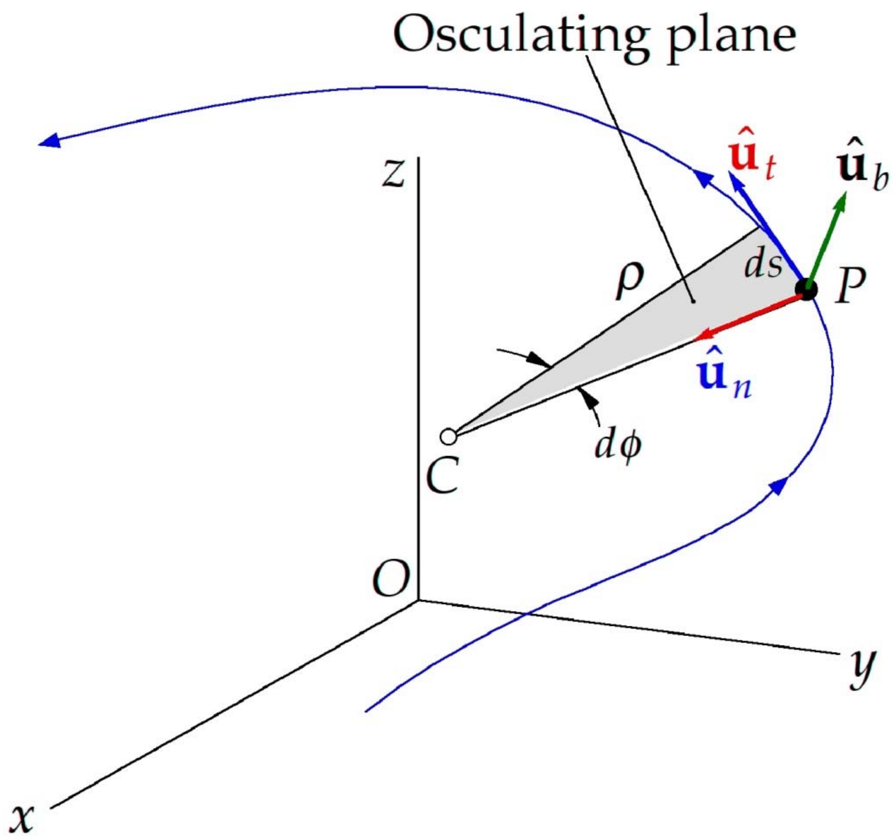
$$a_x = \dot{v}_x = \ddot{x}$$



1.3 Kinematics

- NOTE: the time derivative of a vector yields a vector quantity; the magnitude of a vector yields a scalar quantity!

1.3 Kinematics



$$\bar{v} = v\hat{u}_t$$

$$\bar{a} = a_t\hat{u}_t + a_n\hat{u}_n$$

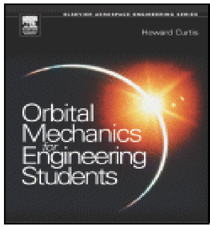
$$a_t = \dot{v}, \quad a_n = \frac{v^2}{\rho}$$

$$ds = \rho d\phi$$

So that $\dot{s} = \rho\dot{\phi}$

or $\dot{\phi} = \dfrac{v}{\rho}$

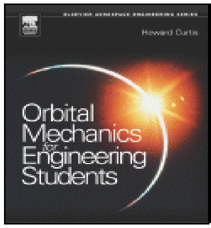
Figure 1.9 Orthogonal triad of unit vectors associated with the moving point *P*.



1.3 Kinematics

- **Q**: If I'm given $\mathbf{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ and $\mathbf{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$, how do I find $\hat{u}_t,\ \hat{u}_n,\ and\ \hat{u}_b$?

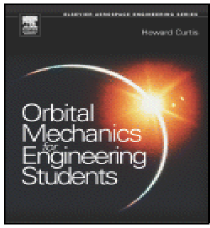
- **A**: $\hat{u}_t$ by Eqn (1.23), $\hat{u}_b$ by Eqn (1.28), and $\hat{u}_n$ by Eqn (1.29)



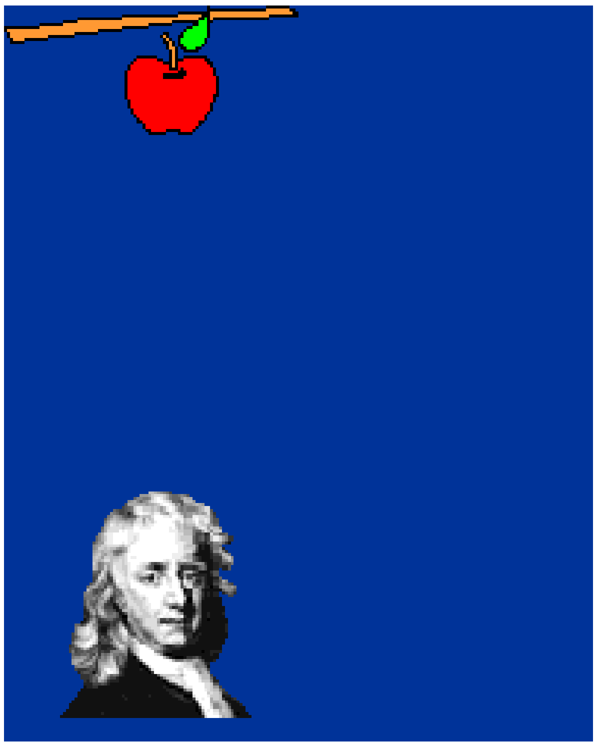
1.3 Kinematics

NOTE: errors in textbook Example 1.6

- **2nd Edition**:
  - Eqn (c): a_n should have units of $\frac{m}{s^2}$ (not $\left(\frac{m}{s}\right)^2$)
  - Eqn (f): **v** should have units of $\frac{m}{s}$
  - Eqn (k): a_n should have units of $\frac{m}{s^2}$ (not $\frac{m}{s}$)
  - At bottom: $\mathbf{r}_C$ should have units of *m* NOT *km*
- **3rd Edition**:
  - Eqn (c): a_n should have units of $\frac{m}{s^2}$ (not $\left(\frac{m}{s}\right)^2$)
  - Eqn (f): **v** should have units of $\frac{m}{s}$
  - Eqn (m): $\mathbf{r}_C$ should have units of *m*



1.4 Mass, Force and Newton’s Law of Gravitation

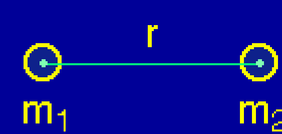


Sir Isaac Newton

Law of Universal Gravitation

Every object in the Universe attracts every other object with a force directed along the line of centers for the two objects that is proportional to the product of their masses and inversely proportional to the square of the separation between the two objects.

$$F_g = G\frac{m_1 m_2}{r^2}$$

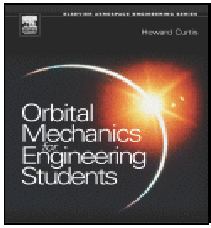


F_g is the gravitational force

m_1 & m_2 are the masses of the two objects

r is the separation between the objects

G is the universal gravitational constant



1.4 Mass, Force and Newton's Law of Gravitation

In particular, the magnitude of the force between the Earth & any other object is

$$F_g = \frac{G m_E m_{obj}}{r^2}$$

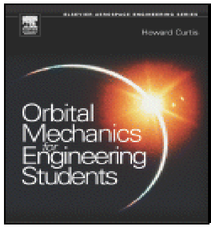
From Newton's 2nd Law of Motion $F = ma$

We see that that acceleration on the object (or ANY object) due to the gravity force is

$$a_{obj} = \frac{F_g}{m_{obj}} = \frac{G m_E}{r^2}$$

Setting r equal to Earth's radius at (R_E = 6378km) yields the "standard sea level gravitational acceleration"

$$g_0 = \frac{G m_E}{R_E^{\ 2}}$$



1.4 Mass, Force and Newton's Law of Gravitation

Textbook derives the relationship between acceleration of gravity *g* and altitude *z*:

$$g = g_0 \frac{R_E{}^2}{(R_E + z)^2} = \frac{g_0}{(1 + z/R_E)^2}$$

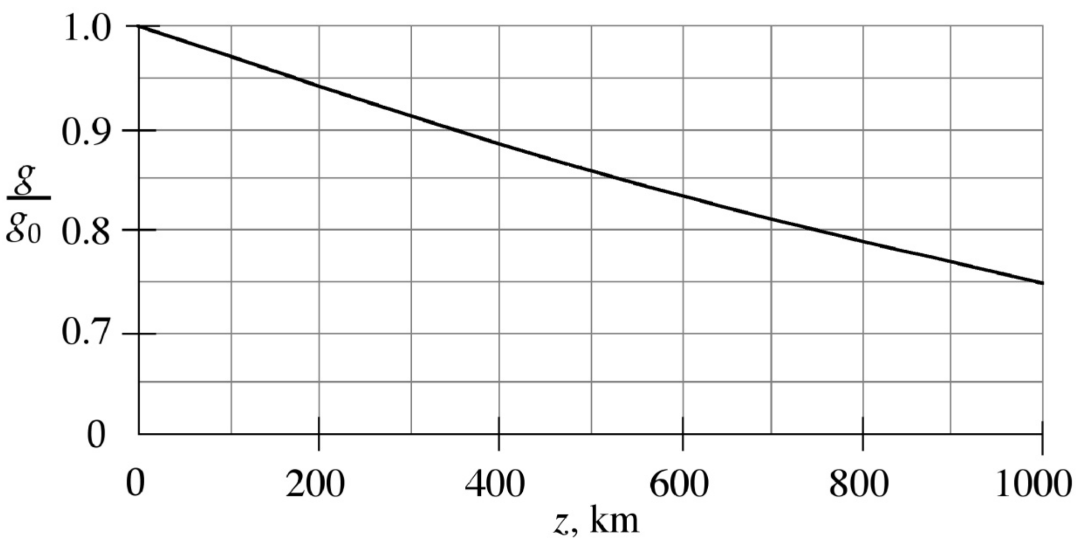
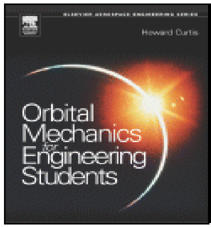
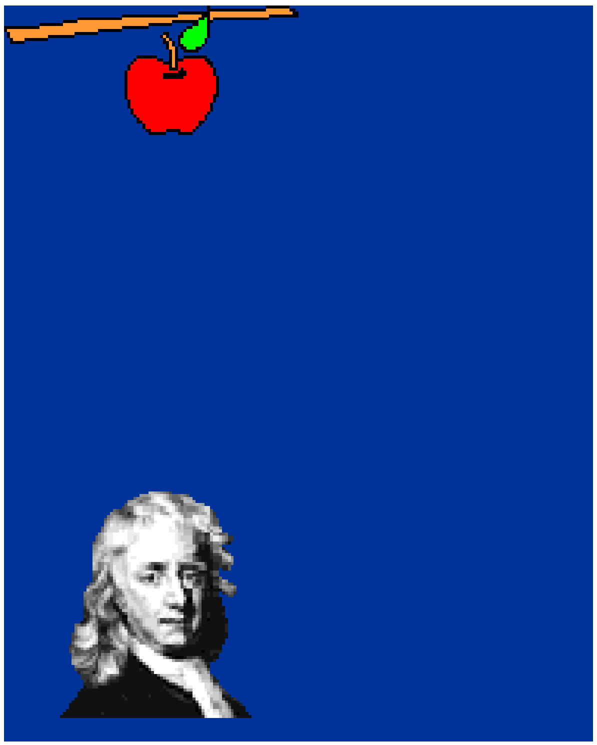


Figure 1.10 Variation of the acceleration of gravity with altitude.



1.4 Mass, Force and Newton's Law of Gravitation



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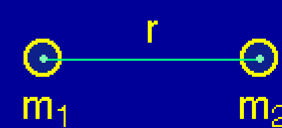

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m_1 & m_2 are the masses of the two objects

r is the separation between the objects

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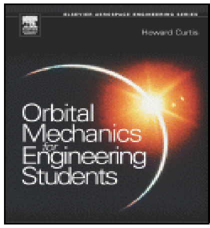

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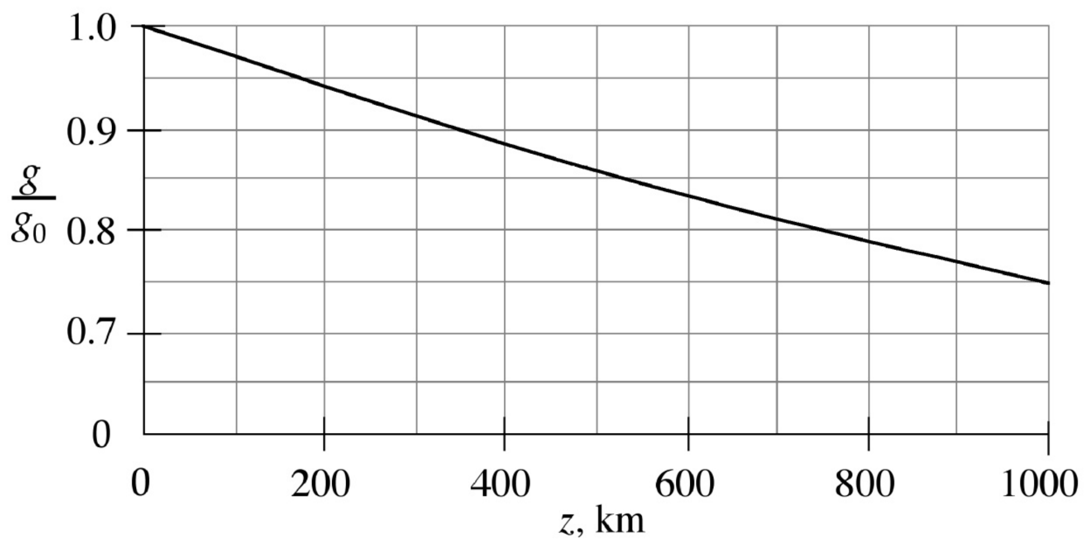
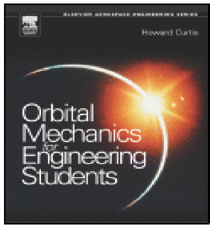
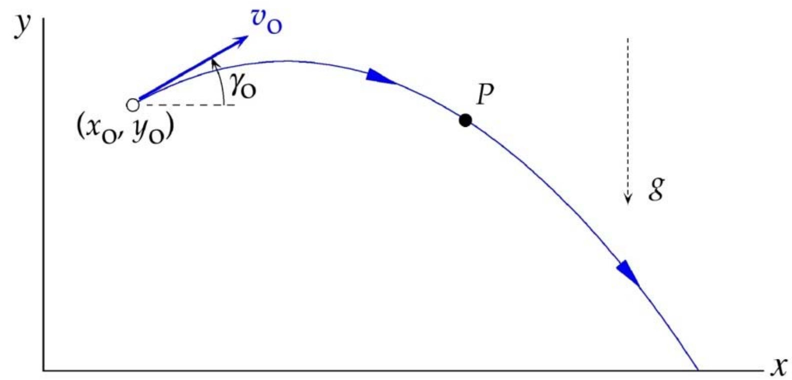


Figure 1.10 Variation of the acceleration of gravity with altitude.



1.4 Mass, Force and Newton's Law of Gravitation

Example 1.7: Show that in the absence of an atmosphere (i.e. accounting only for gravity force, NOT drag), the shape of a ballistic trajectory is a parabola.



- **Particle has initial position (x_0, y_0), initial velocity v_0 at flight path angle γ_0 (←note this is a vector expressed in "magnitude/direction" form)**

$$\ddot{x} = 0, \ddot{y} = -g$$

- **Equations are**

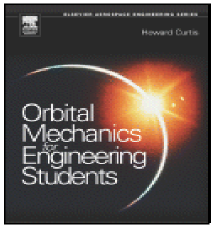
$$\dot{x} = C_1, \dot{y} = -gt + C_2$$

- **Integrating once with respect to time yields**

$$\dot{x}(0) = v_0 \cos\gamma_0, \dot{y}(0) = v_0 \sin\gamma_0$$

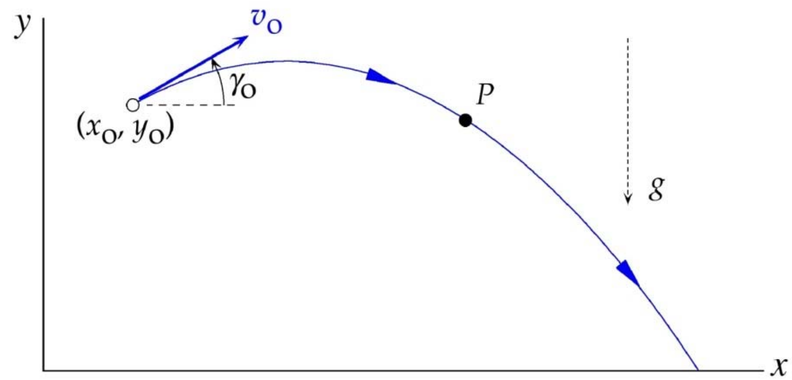
- **so velocity expressions become:**

$$\dot{x} = v_0 \cos\gamma_0, \dot{y} = v_0 \sin\gamma_0 - gt$$



1.4 Mass, Force and Newton's Law of Gravitation

Example 1.7 (cont'd):



- **Integrating once more with respect to time yields**

$$x = C_3 + (v_0 \cos\gamma_0)t, \; y = C_4 + (v_0 \sin\gamma_0)t - \frac{1}{2}gt^2$$

- $x(0) = x_0, y(0) = y_0$ **so position expressions become:**

$$x = x_0 + (v_0 \cos\gamma_0)t \qquad y = y_0 + (v_0 \sin\gamma_0)t - \frac{1}{2}gt^2$$

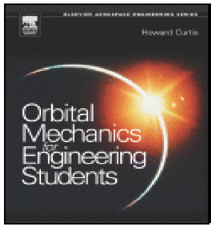
(NOTE error in 2nd Edition of book→missing a "t" in x solution)

- **Solving x equation for "t" yields**

$$t = \frac{x - x_0}{v_0 \cos\gamma_0}$$

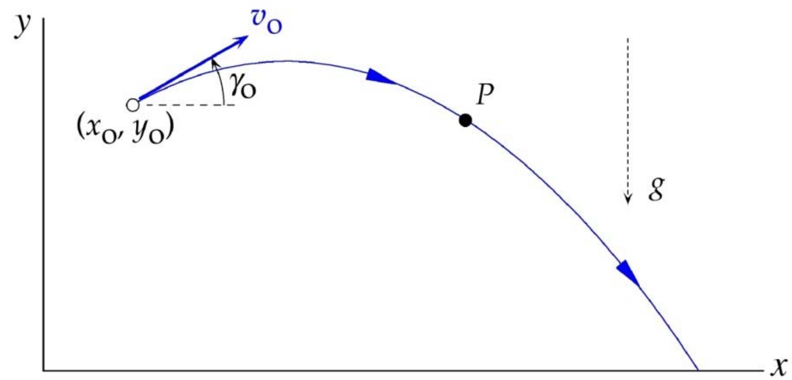
- **Inserting into y expression yields**

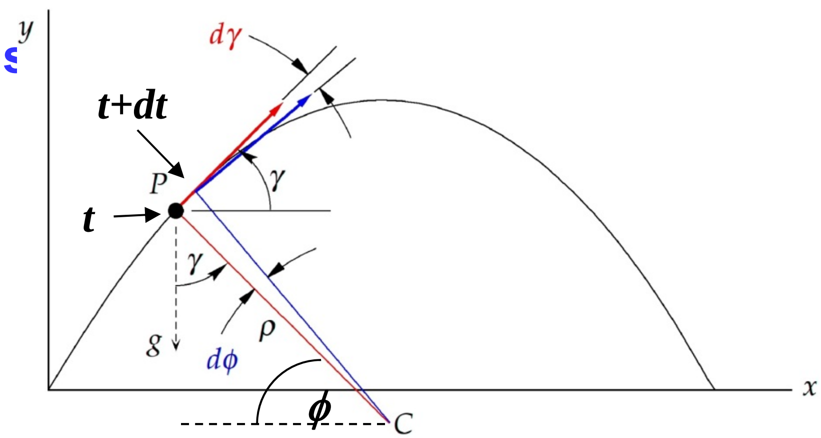
$$y = y_0 + (x - x_0)\tan\gamma_0 - \frac{1}{2}g\left(\frac{x - x_0}{v_0 \cos\gamma_0}\right)^2$$



1.4 Mass, Force and Newton's Law of Gravitation

Example 1.8: An aircraft flies on a parabolic trajectory. Derive the expression for flight path angle γ with respect to speed v.





- **First, notice that $d\gamma = -d\phi$ (Why?)**
- **This means** $\dot{\gamma} = -\dot{\phi}$
- **Substituting in Eqn (1.30) for** $\dot{\phi}$ **yields**
$$\dot{\gamma} = -\frac{v}{\rho}$$
- **Substituting in Eqn (1.25b) for a_n yields**
$$\dot{\gamma} = -\frac{a_n}{v}$$
- **Finally, notice from the figure that $a_n = g\cos\gamma$ (Why?) → So,**
$$\dot{\gamma} = -\frac{g\cos\gamma}{v}$$

1.5 Newton's (2nd) Law of Motion

Newton's second law of motion $\mathbf{F}_{net} = m\mathbf{a}$

where **a** is the absolute acceleration of the center of mass.

a is measured in a frame of reference which itself has neither translational nor rotational acceleration relative to the fixed stars. This reference is called an absolute or inertial frame of reference.

Note that the acceleration vector (**a**) & force vector (**F**$_{net}$) will always be in the same direction!

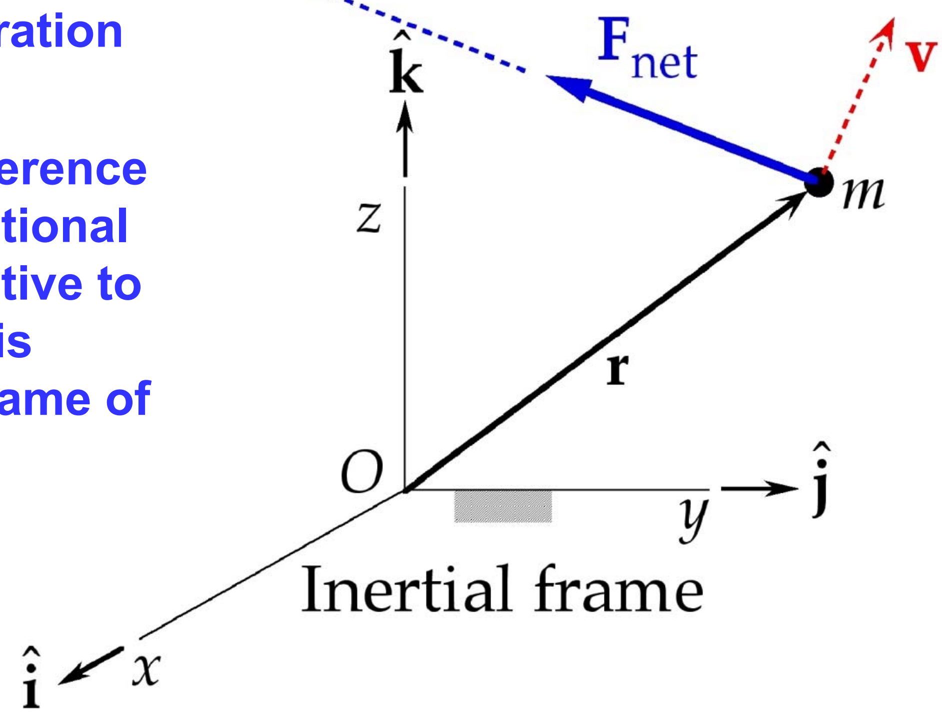
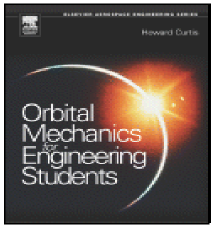


Figure 1.13 The absolute acceleration **a** of a particle is in the direction of the net force.



1.5 Newton's (2nd) Law of Motion

Impulse of a force:

$$\mathscr{I} = \int_{t_1}^{t_2} \mathbf{F}\,dt$$

Angular momentum (moment of momentum):

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$$

Angular impulse-momentum principle:

$$M_{O,net} = \dot{H}_O \quad \Rightarrow \quad \int_{t_1}^{t_2} \mathbf{M}_{O\mathrm{net}}\,dt = \mathbf{H}_{O2} - \mathbf{H}_{O1}$$

1.6 Time Derivatives of Moving Vectors

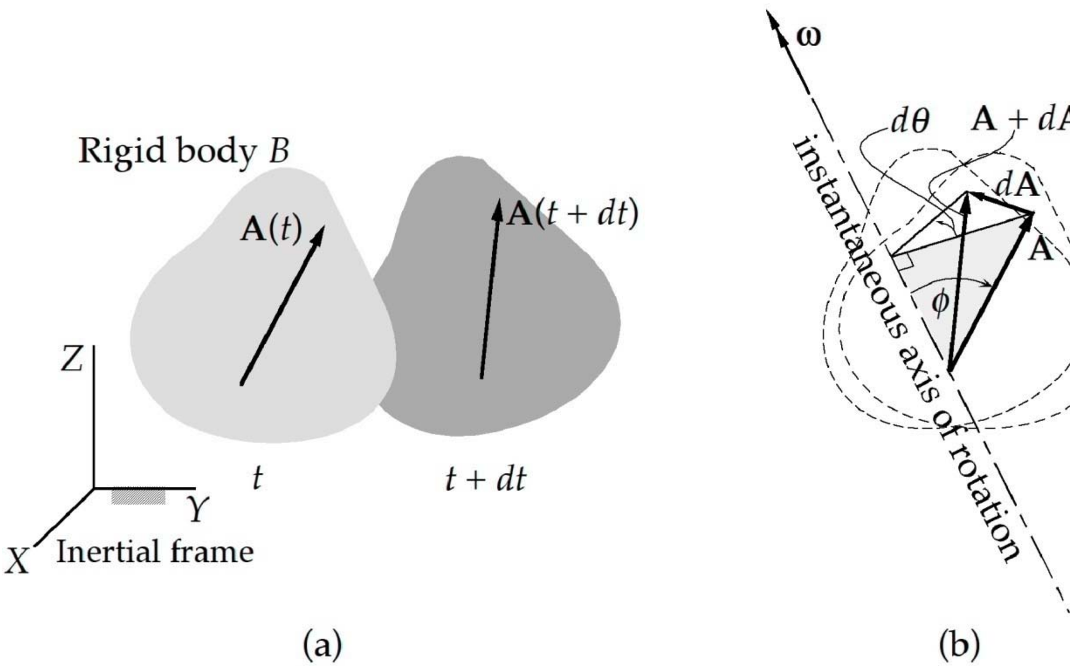


Figure 1.15 Displacement of a rigid body.

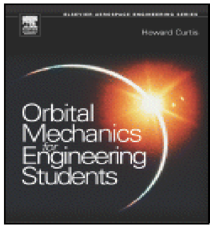
$$d\bar{A} = \|\bar{A}\|(\sin\phi)(d\theta)\hat{n}$$

$$\|\bar{\omega}\| = \frac{d\theta}{dt}$$

$$\bar{\alpha} = \frac{d\bar{\omega}}{dt}$$

Time derivative of a rotating vector of fixed magnitude:

$$\boxed{\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega}\times\mathbf{A}} \quad \left(\text{if } \frac{d}{dt}\|\mathbf{A}\| = 0\right)$$



1.6 Time Derivatives of Moving Vectors

- NOTE: $\overline{\omega}$ is called by several names
  - angular velocity
  - rotational velocity
  - angular rate
  - rotational rate
- …and its units expressed in various ways
  - deg/sec
  - rad/sec
  - RPMs

1.6 Time Derivatives of Moving Vectors

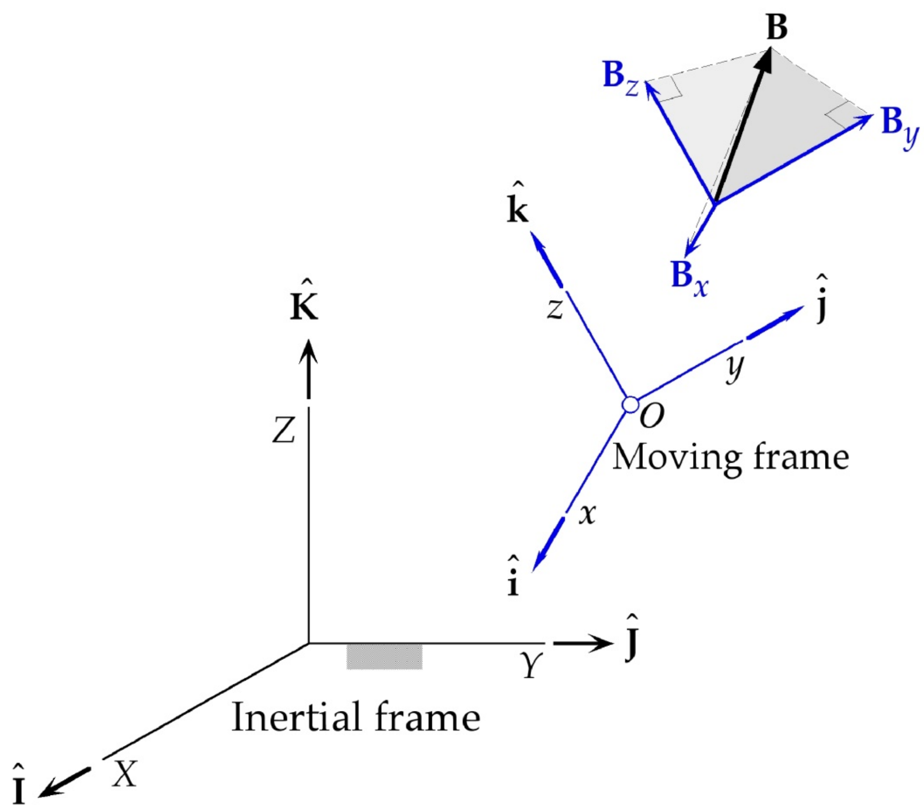
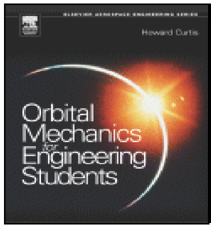


Figure 1.16 Fixed (inertial) and moving rigid frames of reference.



1.6 Time Derivatives of Moving Vectors

- Inertial frame unit vectors are $\hat{I}, \hat{J}, \hat{K}$ (constant in magnitude and direction)

- Moving frame has absolute angular velocity $\overline{\Omega}$

(i.e. its angular velocity with respect to inertial frame is $\overline{\Omega}$)

- Moving frame unit vectors are $\hat{i}, \hat{j}, \hat{k}$ (constant in magnitude but their direction changes with time)

1.6 Time Derivatives of Moving Vectors

- Consider the time-dependent vector $\overline{B} = B_X \hat{I} + B_Y \hat{J} + B_Z \hat{K}$

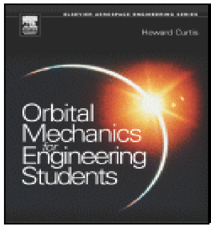
(time-dependent = magnitude and/or direction changes with time)

- The time derivative of this vector is

$$\frac{d\overline{B}}{dt} = \dot{B}_X \hat{I} + \dot{B}_Y \hat{J} + \dot{B}_Z \hat{K} + B_X \cancelto{0}{\dot{\hat{I}}} + B_Y \cancelto{0}{\dot{\hat{J}}} + B_Z \cancelto{0}{\dot{\hat{K}}}$$

$$= \dot{B}_X \hat{I} + \dot{B}_Y \hat{J} + \dot{B}_Z \hat{K}$$

- $\hat{I}, \hat{J}, \hat{K}$ derivatives are zero because the unit vectors of a coordinate frame have constant magnitude and direction as seen in that frame; thus, the time derivative of a unit vector taken in the unit vector's own frame is always zero



1.6 Time Derivatives of Moving Vectors

- $\overline{B}$ can also be expressed in the $\hat{i}, \hat{j}, \hat{k}$ frame:

$$\overline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

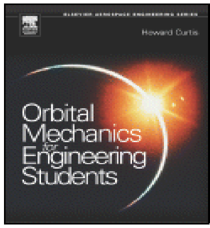
lower-case subscripts

- Time derivative of this vector is

$$\frac{d\overline{B}}{dt} = \dot{B}_x\hat{i} + \dot{B}_y\hat{j} + \dot{B}_z\hat{k} + B_x\dot{\hat{i}} + B_y\dot{\hat{j}} + B_z\dot{\hat{k}}$$

- What are the time derivatives of these unit vectors?

$$\frac{d\overline{A}}{dt} = \overline{\omega} \times \overline{A} \Rightarrow \frac{d\hat{i}}{dt} = \overline{\Omega} \times \hat{i}, \quad \frac{d\hat{j}}{dt} = \overline{\Omega} \times \hat{j}, \quad \frac{d\hat{k}}{dt} = \overline{\Omega} \times \hat{k}$$



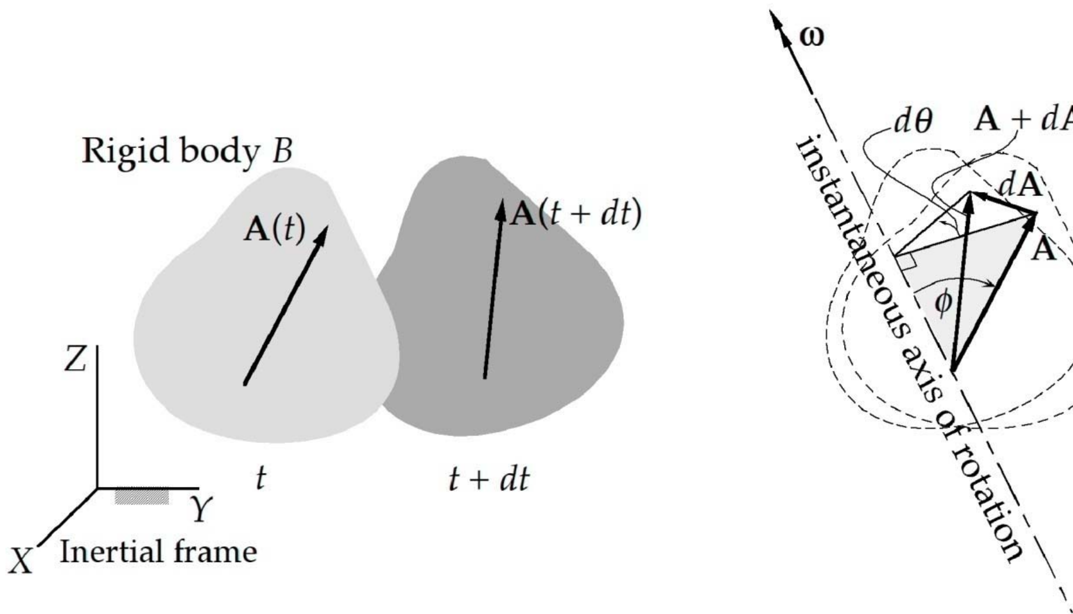
1.6 Time Derivatives of Moving Vectors

- After some manipulation & vector algebra, we arrive at Eqn 1.56: $$\frac{d\mathbf{B}}{dt} = \left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} + \mathbf{\Omega} \times \mathbf{B}$$

- Where $$\left.\frac{d\mathbf{B}}{dt}\right)_{\text{rel}} = \frac{dB_x}{dt}\hat{\mathbf{i}} + \frac{dB_y}{dt}\hat{\mathbf{j}} + \frac{dB_z}{dt}\hat{\mathbf{k}}$$

- This eqn illustrates the relationship between the absolute time derivative $\frac{d\bar{B}}{dt}$ & relative time derivative $\left.\frac{d\bar{B}}{dt}\right)_{rel}$

1.6 Time Derivatives of Moving Vectors



(a) (b)

Figure 1.15 Displacement of a rigid body.

$$d\bar{A} = \left\|\bar{A}\right\|(\sin\phi)(d\theta)\hat{n}$$

$$\left\|\bar{\omega}\right\| = \frac{d\theta}{dt}$$

$$\bar{\alpha} = \frac{d\bar{\omega}}{dt}$$

Time derivative of a rotating vector of fixed magnitude:

$$\boxed{\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega}\times\mathbf{A}} \qquad \left(\text{if } \frac{d}{dt}\left\|\mathbf{A}\right\| = 0\right)$$

1.6 Time Derivatives of Moving Vectors

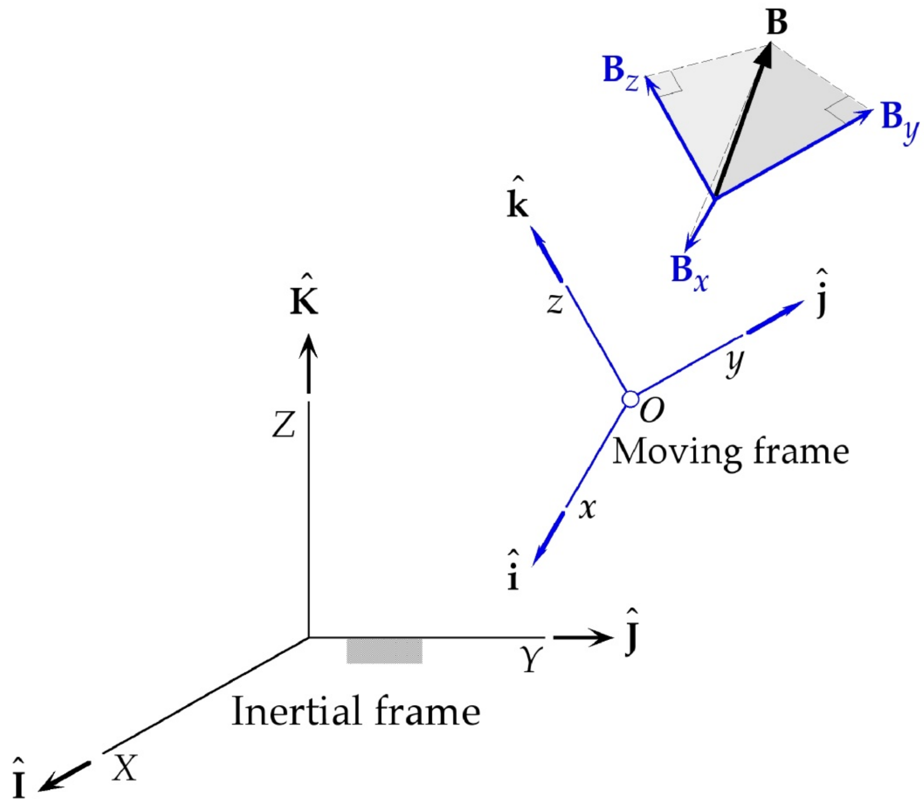
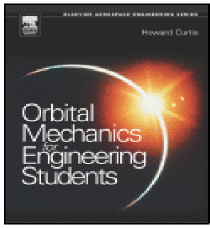


Figure 1.16 Fixed (inertial) and moving rigid frames of reference.



1.6 Time Derivatives of Moving Vectors

- Consider the time-dependent vector $\overline{B} = B_X \hat{I} + B_Y \hat{J} + B_Z \hat{K}$

(time-dependent = magnitude and/or direction changes with time)

- The time derivative of this vector is

$$\frac{d\overline{B}}{dt} = \dot{B}_X \hat{I} + \dot{B}_Y \hat{J} + \dot{B}_Z \hat{K} + B_X \cancelto{0}{\dot{\hat{I}}} + B_Y \cancelto{0}{\dot{\hat{J}}} + B_Z \cancelto{0}{\dot{\hat{K}}}$$

$$= \dot{B}_X \hat{I} + \dot{B}_Y \hat{J} + \dot{B}_Z \hat{K}$$

- $\hat{I}, \hat{J}, \hat{K}$ derivatives are zero because the unit vectors of a coordinate frame have constant magnitude and direction as seen in that frame; thus, the time derivative of a unit vector taken in the unit vector's own frame is always zero

1.6 Time Derivatives of Moving Vectors

- $\overline{B}$ can also be expressed in the $\hat{i}, \hat{j}, \hat{k}$ frame:

$$\overline{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

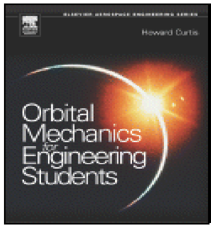
lower-case subscripts

- Time derivative of this vector is

$$\frac{d\overline{B}}{dt} = \dot{B}_x \hat{i} + \dot{B}_y \hat{j} + \dot{B}_z \hat{k} + B_x \dot{\hat{i}} + B_y \dot{\hat{j}} + B_z \dot{\hat{k}}$$

- What are the time derivatives of these unit vectors?

$$\frac{d\overline{A}}{dt} = \overline{\omega} \times \overline{A} \Rightarrow \frac{d\hat{i}}{dt} = \overline{\Omega} \times \hat{i}, \quad \frac{d\hat{j}}{dt} = \overline{\Omega} \times \hat{j}, \quad \frac{d\hat{k}}{dt} = \overline{\Omega} \times \hat{k}$$



1.6 Time Derivatives of Moving Vectors

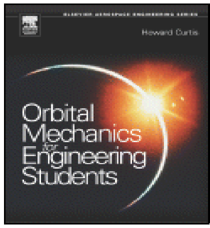
- After some manipulation & vector algebra, we arrive at Eqn 1.56:

$$\boxed{\frac{d\mathbf{B}}{dt} = \left.\frac{d\mathbf{B}}{dt}\right)_{\mathrm{rel}} + \boldsymbol{\Omega}\times\mathbf{B}}$$ where $$\left.\frac{d\mathbf{B}}{dt}\right)_{\mathrm{rel}} = \frac{dB_x}{dt}\hat{\mathbf{i}} + \frac{dB_y}{dt}\hat{\mathbf{j}} + \frac{dB_z}{dt}\hat{\mathbf{k}}$$

- Repeating the vector time derivative eqn from the previous slide:

$$\frac{d\overline{B}}{dt} = \underbrace{\dot{B}_x\hat{i} + \dot{B}_y\hat{j} + \dot{B}_z\hat{k}}_{\left.\frac{d\overline{B}}{dt}\right)_{rel}} + \underbrace{B_x\dot{\hat{i}} + B_y\dot{\hat{j}} + B_z\dot{\hat{k}}}_{\overline{\Omega}\times\overline{B}}$$

- Eqn 1.56 illustrates the relationship between the absolute time derivative of a vector & the relative time derivative of a vector



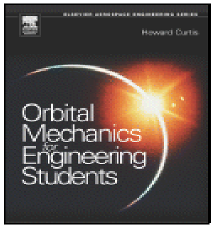
1.6 Time Derivatives of Moving Vectors

- Eqn 1.56 is called the Coriolis theorem

- It says the time derivative of any vector as seen in frame "F1" is different from the time derivative as seen in frame "F2", & this difference is based on the rotational velocity of F2 relative to F1:

$$\left.\frac{d\overline{B}}{dt}\right)_{F1} = \left.\frac{d\overline{B}}{dt}\right)_{F2} + \overline{\Omega}_{F2/F1} \times \overline{B}$$

- Neither of the frames has to be inertial (although in the cases we will explore, usually one of them is)

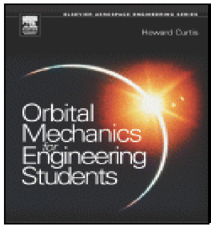


1.6 Time Derivatives of Moving Vectors

- NOTE: a vector is a vector is a vector; it can be written/expressed in any coordinate frame (inertial, relative, etc), but this does not change its magnitude or direction

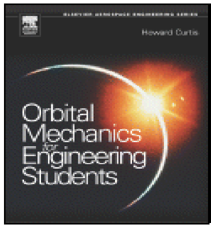
- When differentiating a vector (taking its time derivative), it is important to indicate the coordinate frame in which the derivative is taken (often with a subscript)
  - If no subscript, usually assume inertial

- The coordinate frame in which a vector's time derivative is computed & the coordinate frame in which that vector is expressed are TWO DIFFERENT CONCEPTS!



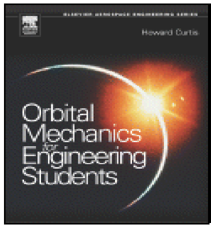
1.6 Time Derivatives of Moving Vectors

- EXAMPLE: given inertial frame “A” & rotating frame “B” & position vector $\bar{r}$
  - Can I take the time derivative of $\bar{r}$ in frame “A” (absolute velocity) & express the result in frame “A”? YES
  - Can I take the time derivative in frame “B” (relative velocity) & express the result in frame “B”? YES
  - Can I take the time derivative in frame “A” & express the result in frame “B”? YES
  - Can I take the time derivative in frame “B” & express the result in frame “A”? YES



1.6 Time Derivatives of Moving Vectors

- $\dfrac{d\overline{B}}{dt} = \dot{B}_X \hat{I} + \dot{B}_Y \hat{J} + \dot{B}_Z \hat{K}$ is an example of a time derivative of a vector taken in an inertial frame & expressed in that inertial frame (unit vectors $\hat{I}, \hat{J}, \hat{K}$)
- $\left.\dfrac{d\mathbf{B}}{dt}\right)_{\text{rel}} = \dfrac{dB_x}{dt}\hat{\mathbf{i}} + \dfrac{dB_y}{dt}\hat{\mathbf{j}} + \dfrac{dB_z}{dt}\hat{\mathbf{k}}$ is an example of a time derivative of a vector taken in a relative frame & expressed in that relative frame (unit vectors $\hat{i}, \hat{j}, \hat{k}$)
- Q: in what frame is Eqn (1.56) expressed?
- A: NONE, these vectors are not yet expressed in ANY coordinate frame
- Q: Can I write (express) the various terms of Eqn (1.56) $\dfrac{d\overline{B}}{dt}$, $\left.\dfrac{d\overline{B}}{dt}\right)_{rel}$, $\overline{\Omega} \times \overline{B}$ in different coordinate frames?
- A: Yes, but I wouldn't! NEVER try to add/subtract/dot/cross 2 vectors expressed in different frames



1.6 Time Derivatives of Moving Vectors

•How are Eqn 1.52 & Eqn 1.56 related?

$$\frac{d\bar{A}}{dt} = \bar{\omega} \times \bar{A} \qquad \frac{d\bar{B}}{dt} = \left.\frac{d\bar{B}}{dt}\right)_{rel} + \bar{\Omega} \times \bar{B}$$

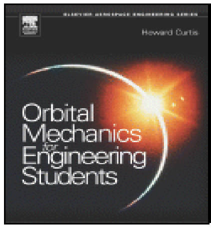
•In Eqn 1.52, $\bar{A}$ is a vector "inscribed" in a rigid body, $\frac{d\bar{A}}{dt}$ is the time derivative of $\bar{A}$ in an inertial frame, & $\bar{\omega}$ is the body's (inertial) angular rate about its axis of rotation

•In Eqn 1.56, $\bar{B}$ is a generic vector, $\frac{d\bar{B}}{dt}$ is the time derivative of $\bar{B}$ in an inertial frame, $\left.\frac{d\bar{B}}{dt}\right)_{rel}$ is the time derivative of $\bar{B}$ in a rotating frame, & $\bar{\Omega}$ is the angular rate of the rotating frame relative to the inertial frame

•The scenario of Eqn 1.56 is the same as that of Eqn 1.52 if $\bar{\omega}$ the angular rate of the rotating frame = the rigid body's angular rate about its axis of rotation

– Imagine defining a frame that rotates with the rigid body; is then the angular rate of this rotating frame

•So why does Eqn 1.56 contain a "$\left.\frac{d}{dt}\right)_{rel}$" term & Eqn 1.52 does not??



1.6 Time Derivatives of Moving Vectors

Quick note about expressing unit vectors of a particular frame in the coordinates of another frame:

- **Suppose Frame A has direction unit vectors** $\hat{I}, \hat{J}, \hat{K}$

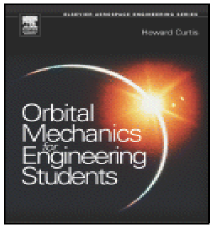
& Frame B has direction unit vectors $\hat{i}, \hat{j}, \hat{k}$

- **Suppose the unit vectors of Frame B are expressed in terms of the unit vectors of A as:**

$$\begin{aligned}\hat{i} &= M_{11}\hat{I} + M_{12}\hat{J} + M_{13}\hat{K} \\ \hat{j} &= M_{21}\hat{I} + M_{22}\hat{J} + M_{23}\hat{K} \\ \hat{k} &= M_{31}\hat{I} + M_{32}\hat{J} + M_{33}\hat{K}\end{aligned}$$

OR

$$\begin{bmatrix}\hat{i} \\ \hat{j} \\ \hat{k}\end{bmatrix} = \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}\begin{bmatrix}\hat{I} \\ \hat{J} \\ \hat{K}\end{bmatrix}$$



1.6 Time Derivatives of Moving Vectors

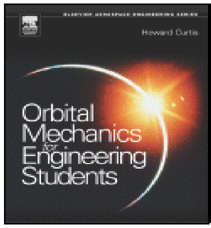
- Any such matrix M is known as a direction cosine matrix (DCM) & has the property the $M^{-1} = M^T$

- Thus, $$\begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} = \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}^{-1} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}^{T} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

- Then the unit vectors of Frame A are expressed in terms of the unit vectors of B as:

$$\hat{I} = M_{11}\hat{i} + M_{21}\hat{j} + M_{31}\hat{k}$$
$$\hat{J} = M_{12}\hat{i} + M_{22}\hat{j} + M_{32}\hat{k}$$
$$\hat{K} = M_{13}\hat{i} + M_{23}\hat{j} + M_{33}\hat{k}$$

- This is exemplified in Example 1.13, eqn (c) & eqn (e)

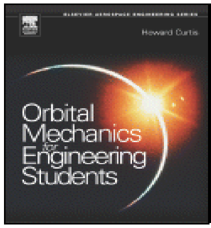


1.6 Time Derivatives of Moving Vectors

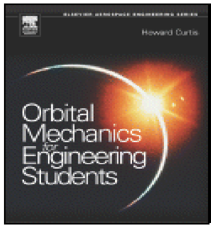
- Textbook p. 27-28 derives the expression for the second (i.e. double) absolute time derivative of a vector in terms of the time derivatives of that vector in a rotating frame
- "Recursive" method is to take the absolute time derivative of both sides of Eqn (1.56), substituting in Eqn (1.56) where necessary
- NOTE: un-numbered eqn above Eqn (1.58) involves the product rule of derivatives applied to the cross product between 2 vectors:

$$\frac{d}{dt}\left(\overline{\Omega}\times\overline{B}\right)=\frac{d\overline{\Omega}}{dt}\times\overline{B}+\overline{\Omega}\times\frac{d\overline{B}}{dt}$$

- Result is Eqn (1.60)



Chapter 2 - The Two-body Problem



Chapter Outline

2.2 Equations of motion in an inertial frame

2.3 Equations of relative motion

2.4 Angular momentum and the orbit formulas

2.5 The energy law

2.6 Circular orbits

2.7 Elliptical orbits

2.8 Parabolic trajectories

2.9 Hyperbolic trajectories

2.10 Perifocal frame

(2.11 The Lagrange coefficients)

(2.12 Restricted three-body problem)

2.2 Equations of Motion in an Inertial Frame

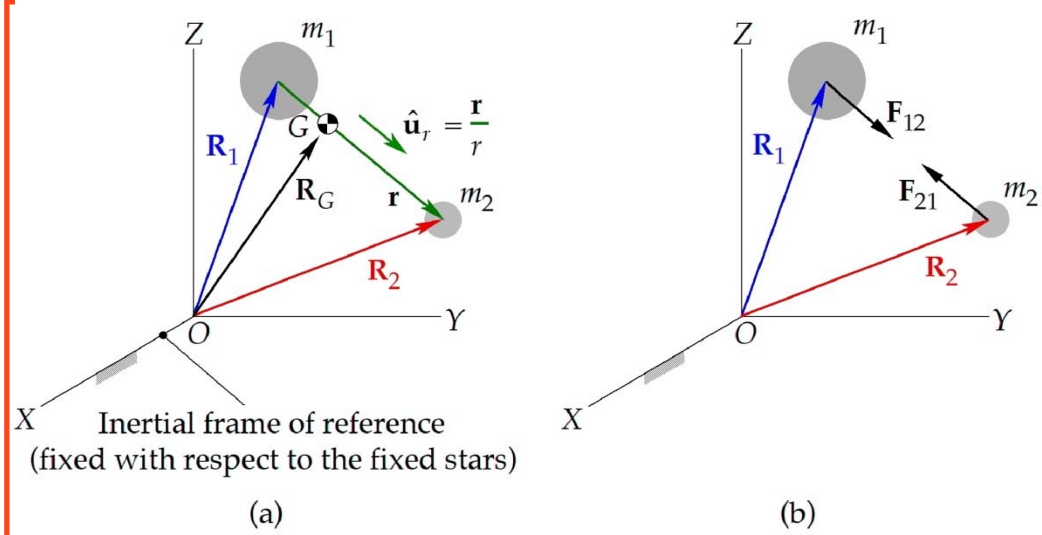


Figure 2.1: (a) Two masses located in an inertial frame. (b) Free-body diagram.

Book defines:

$$\mathbf{R}_1 = X_1\hat{\mathbf{I}} + Y_1\hat{\mathbf{J}} + Z_1\hat{\mathbf{K}}$$

$$\mathbf{R}_2 = X_2\hat{\mathbf{I}} + Y_2\hat{\mathbf{J}} + Z_2\hat{\mathbf{K}}$$

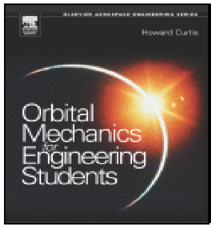
Position vector $\mathbf{R}_G$ of the center of mass *G* defined by:

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2}$$

The absolute velocity and the absolute acceleration of *G* defined by:

$$\mathbf{v}_G = \dot{\mathbf{R}}_G = \frac{m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2}{m_1 + m_2}$$

$$\mathbf{a}_G = \ddot{\mathbf{R}}_G = \frac{m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2}{m_1 + m_2}$$



2.2 Equations of Motion in an Inertial Frame

- **Book also defines position vector of m_2 relative to m_1 (i.e. vector pointing from m_1 to m_2):**

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 = (X_2 - X_1)\hat{\mathbf{I}} + (Y_2 - Y_1)\hat{\mathbf{J}} + (Z_2 - Z_1)\hat{\mathbf{K}}$$

$$r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \qquad \hat{u}_r = \frac{\bar{r}}{r}$$

- **Newton's Law of Gravitational Attraction yields:**

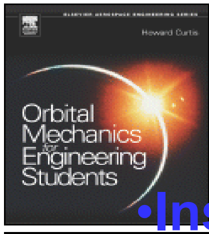
$$\bar{F}_{12} = \frac{Gm_1m_2}{r^2}\hat{u}_r, \quad \bar{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{u}_r$$

- **Newton's 2nd Law of Motion (F = ma) yields:**

$$m_1\ddot{\bar{R}}_1 = \frac{Gm_1m_2}{r^2}\hat{u}_r, \quad m_2\ddot{\bar{R}}_2 = -\frac{Gm_1m_2}{r^2}\hat{u}_r$$

- **Dividing by mass on both sides of each eqn yields:**

$$\ddot{\bar{R}}_1 = \frac{Gm_2}{r^2}\hat{u}_r, \quad \ddot{\bar{R}}_2 = -\frac{Gm_1}{r^2}\hat{u}_r$$



2.2 Equations of Motion in an Inertial Frame

- Inserting $\hat{u}_r = \frac{\bar{r}}{r}$ yields:

$$\ddot{\mathbf{R}}_1 = Gm_2 \frac{\mathbf{r}}{r^3} \qquad \ddot{\mathbf{R}}_2 = -Gm_1 \frac{\mathbf{r}}{r^3}$$

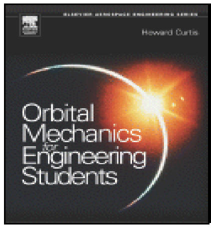
- These are the equations of motion of two bodies in inertial space

- The 2 vector ODE's in R_1 & R_2 together comprise 6 scalar ODE's

- The position vector **R** and velocity vector **V** of each body are referred to collectively as the state vector (12 states in all)

- NOTE: Inserting Eqns (2.11) &(2.12) into Eqns (2.4) yields $a_G = 0$

- Thus, the center of mass of a 2-body system may serve as the origin of an inertial frame



2.2 Equations of Motion in an Inertial Frame

- Eqns (2.19a) &(2.19b) are the 6 scalar ODE's

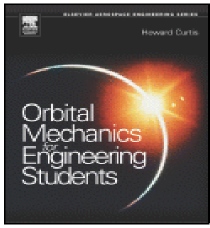
- These may be integrated numerically from any initial time t_0 to any final time t_f

- Initial conditions are the initial values of each body's position and velocity vector (12 initial states in all)

- Numerical integration also called "propagation"

- Algorithm 2.1 outlines the use of MATLAB to propagate the states

- Result is list of the values of each state at various chosen times between t_0 & t_f ("time histories" of the states)



2.2 Equations of Motion in an Inertial Frame

- Time histories may be plotted in various useful ways:
  - Plot each state vs. time
  - 3-D plot of position states (X-Y-Z) of each body (Figs 2.2 & 2.3)
  - 2-D plot of position states (X-Y, X-Z, or Y-Z) of each body (Figs 2.4 & 2.5)
  - 2-D/3-D plots of velocity states (called a hodograph)

1.7 Relative Motion

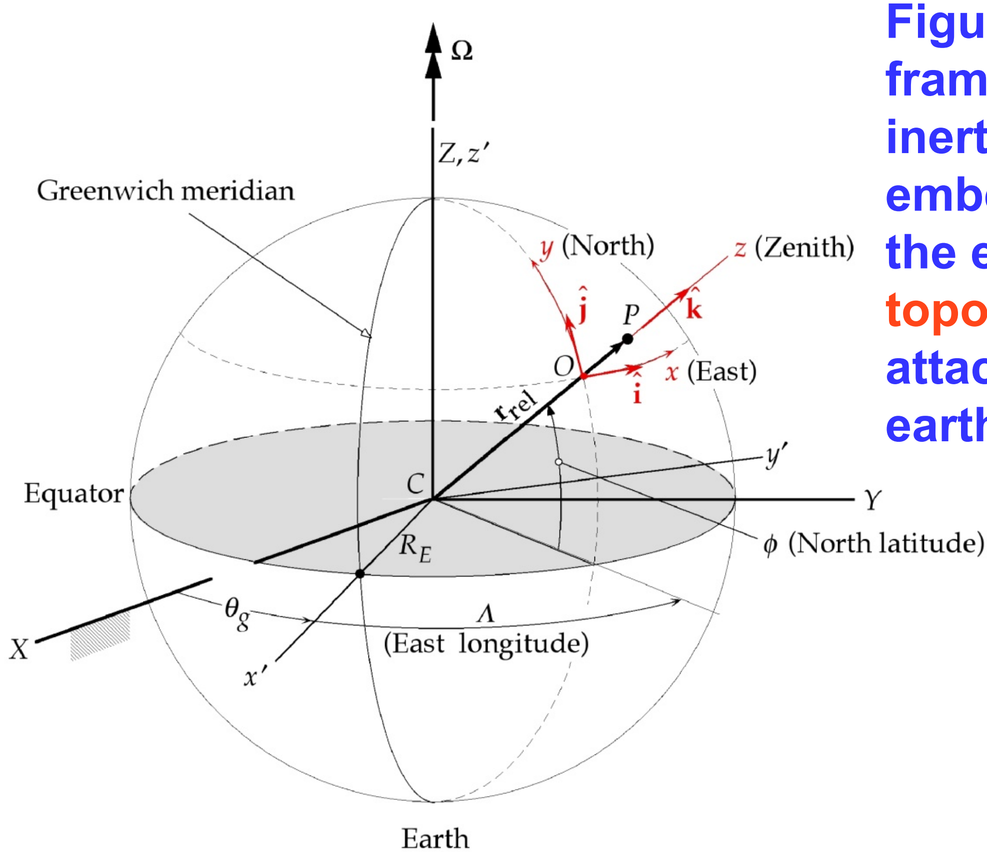
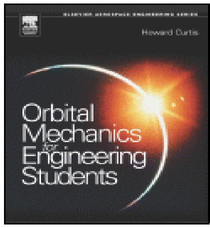


Figure 1.18 Earth-centered inertial frame (*XYZ*); earth-centered non-inertial *x' y' z'* frame embedded in and rotating with the earth; and a non-inertial, topocentric-horizon frame *xyz* attached to a point *O* on the earth's surface.



1.7 Relative Motion

- Eqn 1.73 given without derivation (based on spherical triangles)

- Author then uses relative velocity formula & relative acceleration formula to derive expressions for particle's absolute velocity (Eqn 1.83) & acceleration (Eqn 1.84) in terms of:
  - particle's relative position/velocity/acceleration components in TH frame: $x, y, z, \dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}$
  - particle's latitude/longitude: Λ, ϕ
  - Earth's rotation rate: Ω = 360°/day

- Note that these absolute velocity & acceleration vectors are **expressed** in the TH frame (not an inertial frame such as ECI!)

Figure for Example 1.14

NOTE: the coordinate frame shown, whose origin is at the center of the aircraft, is actually NOT a TH frame as defined in the book! TH frame is oriented the same way as this frame (East-North-Up), but its origin is on the ground below the aircraft.

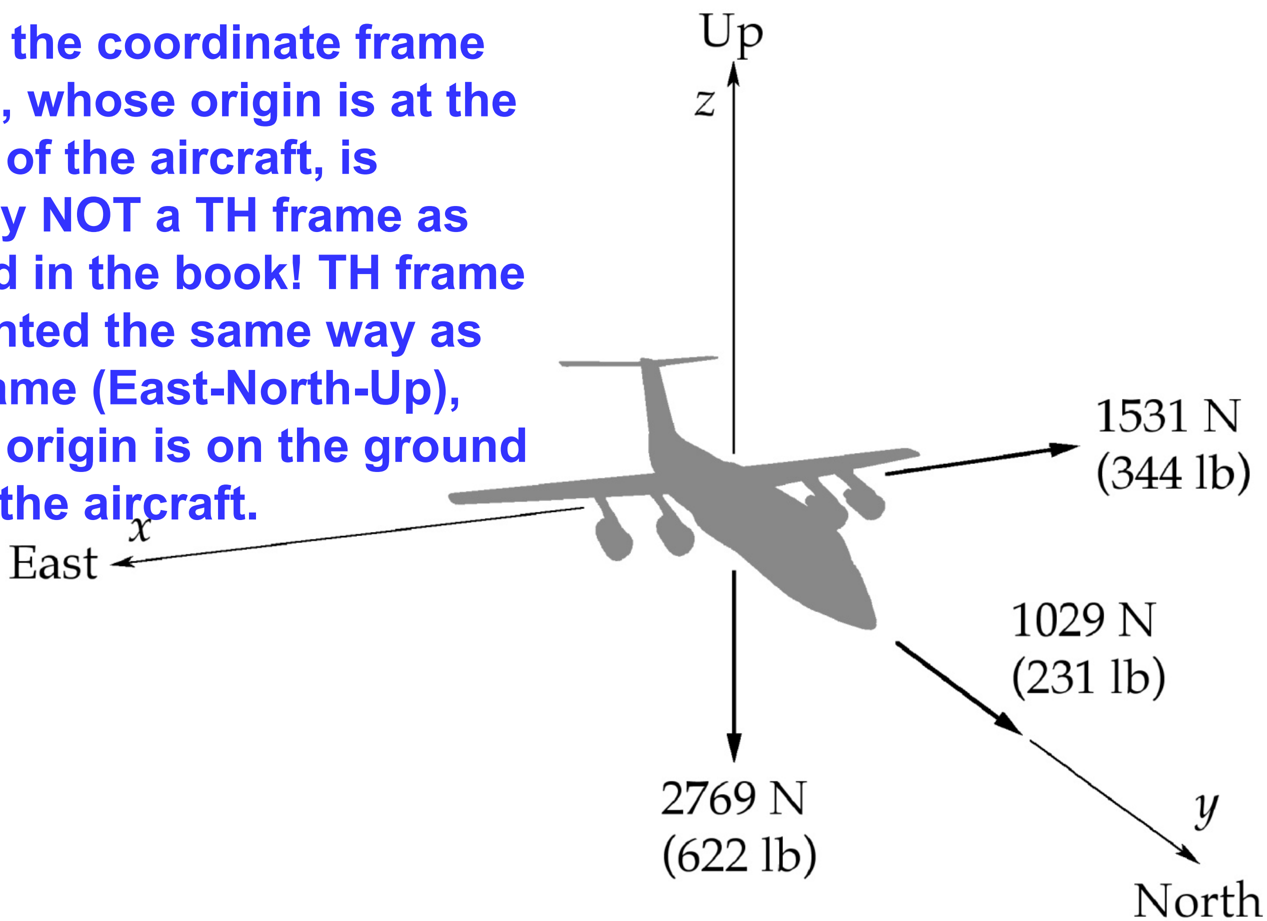
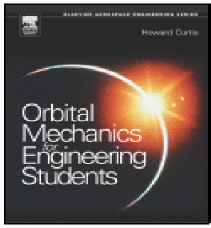
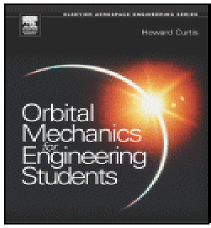


Figure 1.19 Components of the net force on the airplane.

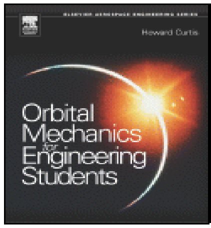


Chapter 1 Summary

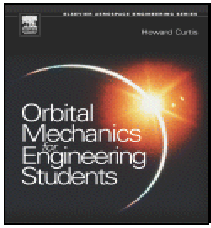
- A vector is a quantity that is specified by both a magnitude and a direction. Vector operations include dot product and cross product. Examples of vectors are velocity and acceleration.
- Formulas were developed for calculating the time derivatives of moving vectors & applied to the computation of absolute/inertial velocity and acceleration in terms of relative velocity and acceleration (& vice-versa).
- Newton's 2nd law of motion ($F = ma$) & Newton's law of gravitational motion are both vector equations that must be expressed in an inertial frame of reference.
- (NOTE: we will not cover Sec 1.8)



Chapter 2 - The Two-body Problem



Chapter Outline



2.2 Equations of Motion in an Inertial Frame

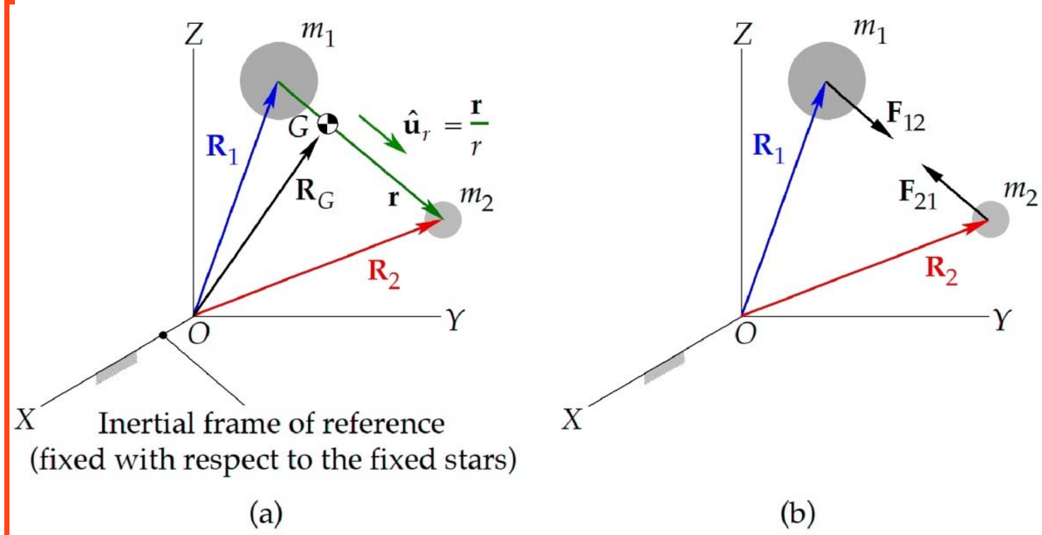


Figure 2.1: (a) Two masses located in an inertial frame. (b) Free-body diagram.

Book defines:

$$\mathbf{R}_1 = X_1\hat{\mathbf{I}} + Y_1\hat{\mathbf{J}} + Z_1\hat{\mathbf{K}}$$

$$\mathbf{R}_2 = X_2\hat{\mathbf{I}} + Y_2\hat{\mathbf{J}} + Z_2\hat{\mathbf{K}}$$

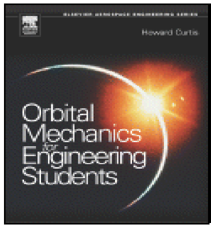
Position vector $\mathbf{R}_G$ of the center of mass *G* defined by:

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2}$$

The absolute velocity and the absolute acceleration of *G* defined by:

$$\mathbf{v}_G = \dot{\mathbf{R}}_G = \frac{m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2}{m_1 + m_2}$$

$$\mathbf{a}_G = \ddot{\mathbf{R}}_G = \frac{m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2}{m_1 + m_2}$$



2.2 Equations of Motion in an Inertial Frame

- **Book also defines position vector of m_2 relative to m_1 (i.e. vector pointing from m_1 to m_2):**

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 = (X_2 - X_1)\hat{\mathbf{I}} + (Y_2 - Y_1)\hat{\mathbf{J}} + (Z_2 - Z_1)\hat{\mathbf{K}}$$

$$r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \qquad \hat{u}_r = \frac{\bar{r}}{r}$$

- **Newton's Law of Gravitational Attraction yields:**

$$\bar{F}_{12} = \frac{Gm_1 m_2}{r^2}\hat{u}_r, \quad \bar{F}_{21} = -\frac{Gm_1 m_2}{r^2}\hat{u}_r$$

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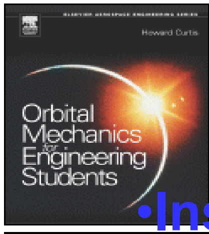

2.2 Equations of Motion in an Inertial Frame

- Inserting yields:

$$\hat{u}_r = \frac{\bar{r}}{r}$$

$$\ddot{\mathbf{R}}_1 = Gm_2 \frac{\mathbf{r}}{r^3} \qquad \ddot{\mathbf{R}}_2 = -Gm_1 \frac{\mathbf{r}}{r^3}$$

- These are the e… of m… o bodies in inertial space

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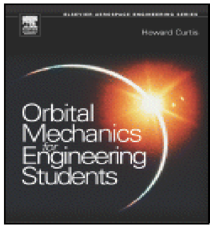
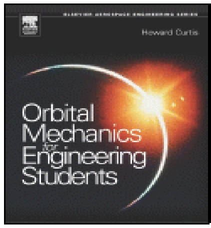
- Thus, the center of mass of a 2-body system may serve as the origin of an inertial frame



2.2 Equations of Motion in an Inertial Frame

- Eqns (2.19a) &(2.19b) are the 6 scalar ODE's
- These may be integrated numerically from any initial time t_0 to any final time t_f
- Initial conditions are the initial values of each body's position and velocity vector (12 initial states in all)
- Numerical integration also called "propagation"
- Algorithm 2.1 outlines the use of MATLAB to propagate the states
- Result is list of the values of each state at various chosen times between t_0 & t_f ("time histories" of the states)



2.2 Equations of Motion in an Inertial Frame

- Time histories may be plotted in various useful ways:
  - Plot each state vs. time
  - 3-D plot of position states (X-Y-Z) of each body (Figs 2.2 & 2.3)
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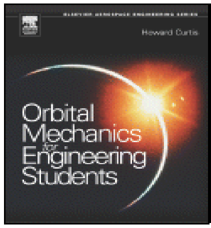
  - 2-D/3-D plots of velocity states (called a **hodograph**)



2.2 Equations of Motion in an Inertial Frame

- Eqns (2.19a) &(2.19b) are the 6 scalar ODE's of motion of two bodies in inertial space, under gravitational attraction to each other

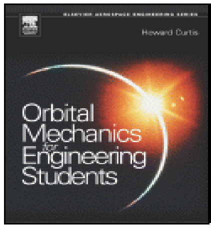
- These may be integrated numerically from any initial time t_0 to any final time t_f

- Initial conditions are the initial values of each body's position and velocity vector (12 initial states in all):

$$X_{10} \quad Y_{10} \quad Z_{10} \quad \dot{X}_{10} \quad \dot{Y}_{10} \quad \dot{Z}_{10} \quad X_{20} \quad Y_{20} \quad Z_{20} \quad \dot{X}_{20} \quad \dot{Y}_{20} \quad \dot{Z}_{20}$$

- Numerical integration also called "propagation"

- Algorithm 2.1 outlines the use of MATLAB to propagate the states



2.2 Equations of Motion in an Inertial Frame

- Result of propagation is a list of the values of each state at various chosen times between t_0 & t_f ("time histories" of the states)
- Time histories may be plotted in various useful ways:
    - Plot each state vs. time
    - 3-D plot of position states (X-Y-Z) of each body (Figs 2.2 & 2.3)
    - 2-D plot of position states (X-Y, X-Z, or Y-Z) of each body (Figs 2.4 & 2.5)
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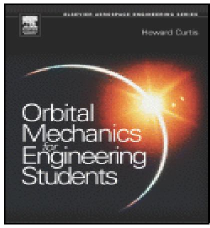


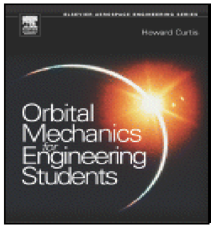
2.2 Equations of Motion in an Inertial Frame

A NOTE ABOUT NUMERICAL INTEGRATION (e.g. R-K, "ode" function in Matlab):

- **Every method boils down to a numerical way to calculate the area under a curve**

- **This leads to breaking this area up into discrete portions, i.e. time steps (usually adjustable)**

- **These time steps are independent of the time intervals at which we wish to record the data (for plotting, tabulating, etc)**

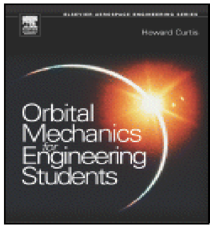


2.2 Equations of Motion in an Inertial Frame

- Example: suppose I want to integrate the ODEs of Eqn (2.19) (given some initial conditions) from $t_0 = 0$ to $t_f = 100$sec, & I want to record the states every 1sec in order to plot the 3D (X-Y-Z) trajectories of m_1 & m_2

- Numerical method will basically do the following:
  - Calculate the area under the curve from t = 0 to t = 1sec, given initial conditions at t_0 (i.e. t = 1sec temporarily plays the role of t_f)
  - Calculate the area under the curve from t = 1sec to t = 2sec given “initial conditions” at t = 1sec (i.e. t = 1sec temporarily plays the role of t_0 & t = 2sec temporarily plays the role of t_f)
  - This continues until the temporary t_f is the actual t_f (100sec)

- Each step in the above process involves breaking the area up into discrete intervals (which again, have nothing to do with the 1-sec time step desired for plotting the results)



2.2 Equations of Motion in an Inertial Frame

- Example 2.2 calls for the numerical integration of states for 480sec

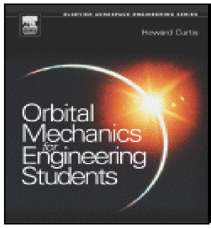
- Resulting X_1, Y_1, Z_1, X_2, Y_2, Z_2 position components at each time comprise R_1 & R_2

- Part (a) calls for 3-D plot of position states of each body (R_1 & R_2) (Fig 2.2)

- Part (b) calls for 3-D plot of $R_2 - R_1$ & $R_G - R_1$ → motion of m_2 & center of mass with respect to m_1 → m_1 at the origin (Fig 2.3a)

- Part (c) calls for 3-D plot of $R_1 - R_G$ & $R_2 - R_G$ → motion of m_1 & m_2 with respect to center of mass → c.o.m. at the origin (Fig 2.3b)

- Q: Is the frame in Fig 2.3a an inertial frame? What about Fig 2.3b?
- A: inertial frame cannot accelerate, so Fig 2.3b frame is inertial, Fig 2.3a frame is not



2.2 Equations of Motion in an Inertial Frame

- Note Eqns (2.19a) &(2.19b) are NONLINEAR ODE's

- This tends to make the resulting motion difficult to predict or characterize

- If similar ODE's were written for THREE bodies, the resulting motion would be even more extreme

2.2 Equations of Motion in an Inertial Frame

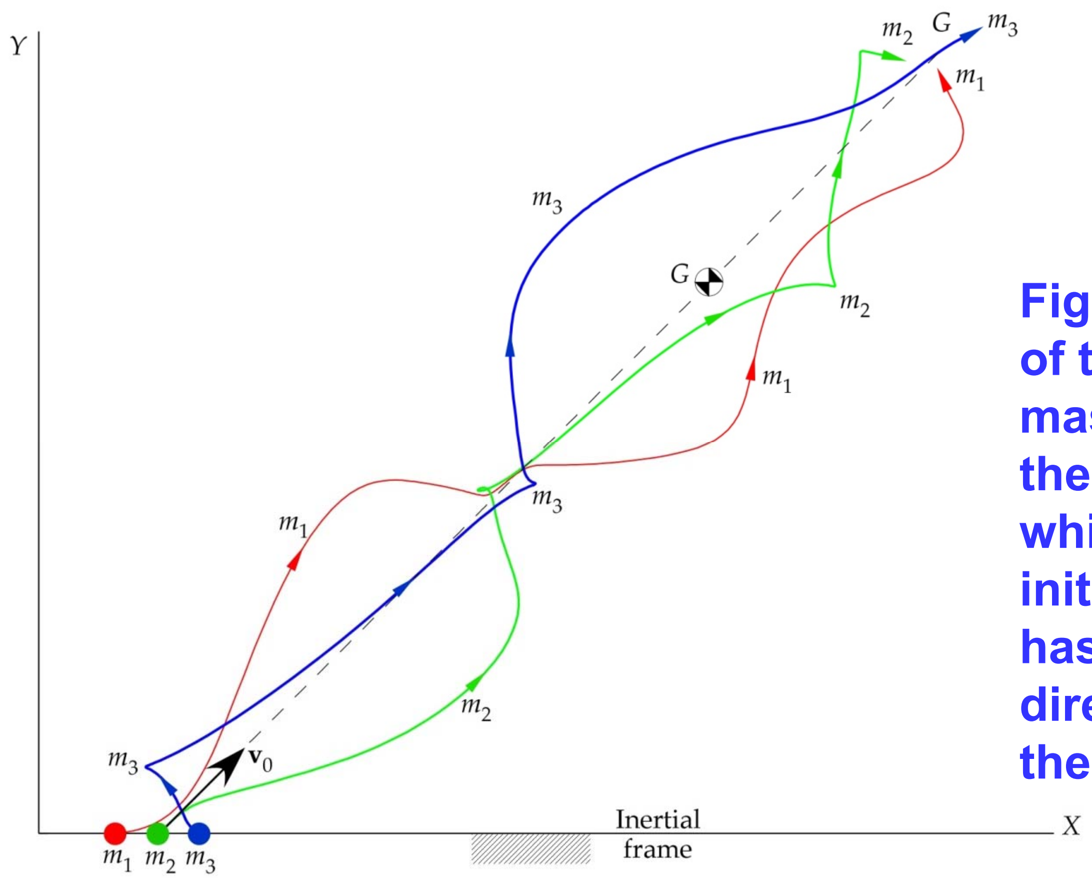


Figure 2.4: The motion of three identical masses as seen from the inertial frame in which m_1 and m_3 are initially at rest, while m_2 has an initial velocity $\mathbf{v}_0$ directed upwards and to the right, as shown.

2.2 Equations of Motion in an Inertial Frame

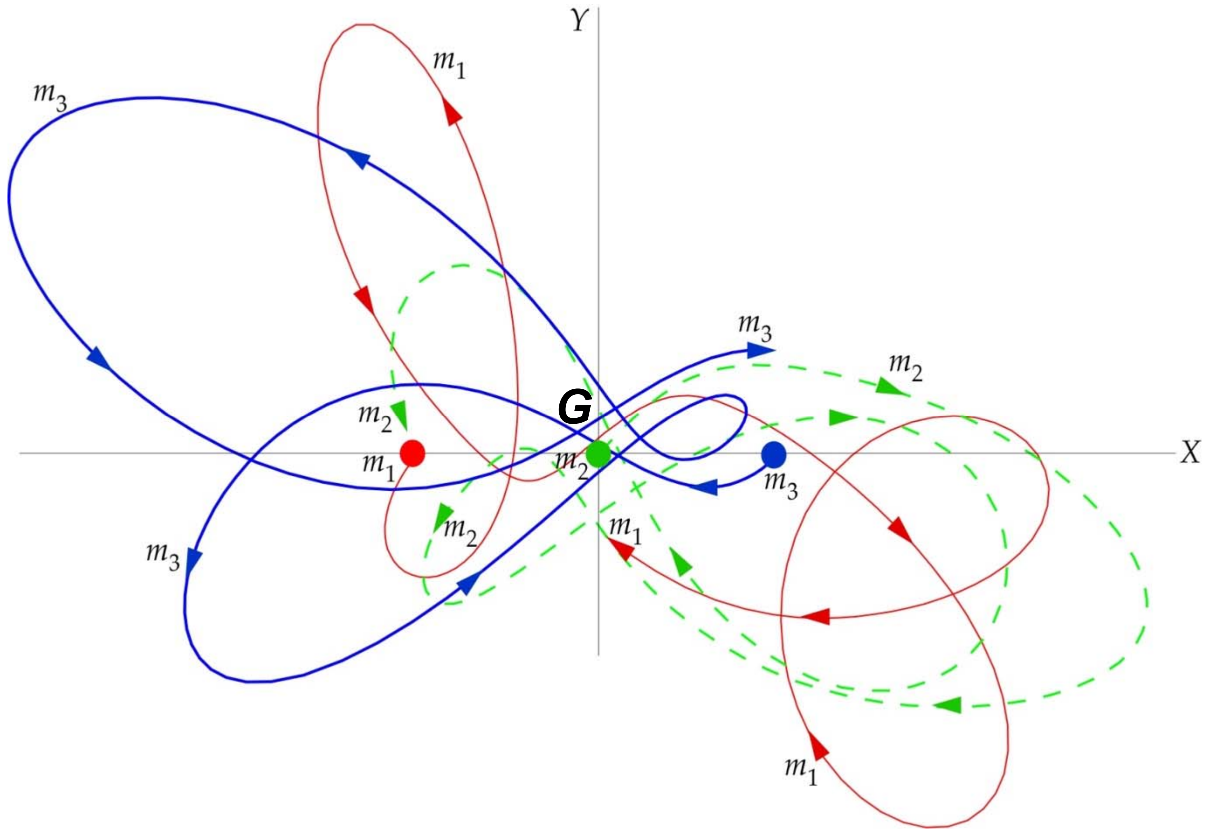
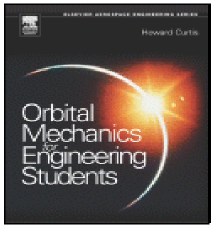
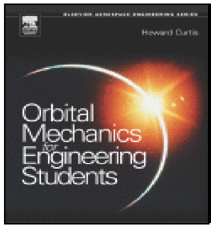


Figure 2.5: The same motion as Figure 2.4, as viewed from the inertial frame attached to the center of mass *G*.



2.2 Equations of Motion in an Inertial Frame

- A quick note about MATLAB:
  - Powerful computational software based on vector/matrix manipulation
  - Consists of the basic version plus numerous add-ons ("toolboxes")
  - Very useful for the kinds of calculations performed in this course
  - You are not required to know or use MATLAB
  - However, the student version of basic MATLAB is fairly cheap & easy to learn



2.3 Equations of Relative Motion

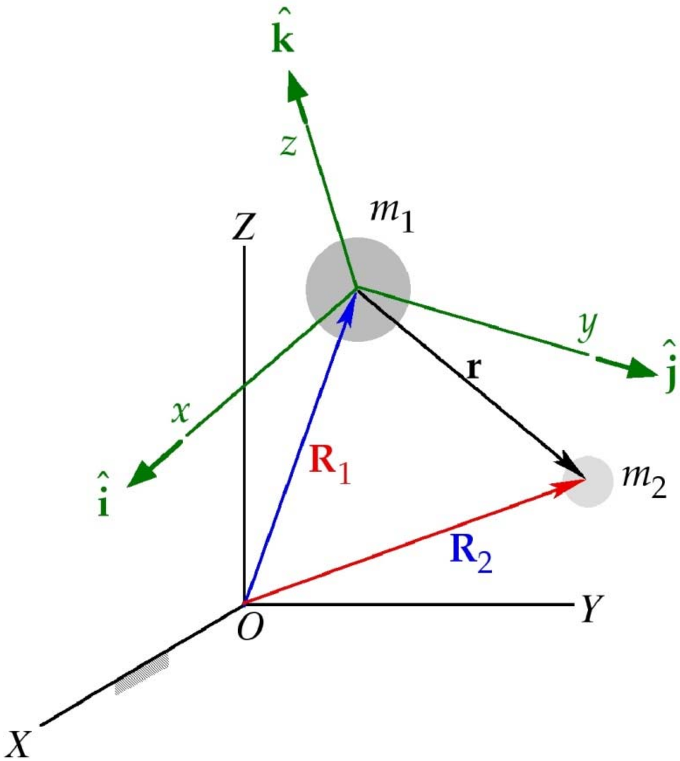


Figure 2.6: Reference frame ***xyz*** attached to point mass of ***m_1***.

Differentiating Eqn (2.5) twice & inserting Eqns (2.18a) & (2.18b) yields the equation of motion of ***m_2*** relative to ***m_1***:

$$\boxed{\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}}$$

Eqn 2.22: The fundamental equation of relative two-body motion

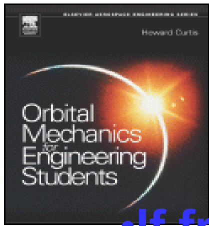
where $\bar{r} = \bar{R}_2 - \bar{R}_1$ (Eqn 2.5) and μ is the gravitational parameter:

$$\mu = G\left(m_1 + m_2\right) \quad (\mathrm{km^3/s^2})$$

Expressed in the inertial frame:

$$\bar{r} = \left(X_2 - X_1\right)\hat{I} + \left(Y_2 - Y_1\right)\hat{J} + \left(Z_2 - Z_1\right)\hat{K}$$

Expressed in the frame whose origin is at m_1 $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$



2.3 Equations of Relative Motion

•If frame xyz is non-accelerating and non-rotating (i.e. inertial), the acceleration on the left-hand side of Eqn (2.22) may be written as

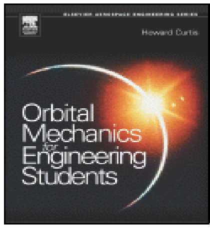
$$\ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}$$

•If frame xyz is NOT inertial, (e to be added based on the frame's translational &/or rotational motion → for now, assume inertial

•Eqn (2.22), a vector ODE, may be expressed as 3 scalar ODEs (Eqn (2.23))

•Algorithm 2.2 describes numerical integration of Eqn (2.22) from t_0 to t_f (ERROR: the phrase "m_1 relative to m_2" should read "m_2 relative to m_1")

•Also, Example 2.3 has an error in the initial state vector; it should read
y_0 = [8000km 0 6000km 0 7km/s 0]

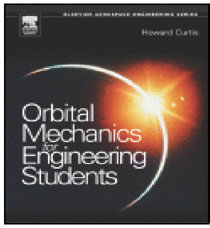


2.3 Equations of Relative Motion

- Because the center of mass G between m_1 and m_2 is not accelerating nor rotating, we can construct an inertial frame with origin at G and write the equations of motion for m_1 and m_2 in this frame

- Eqn (2.18a) & (2.18b) are the equations of motion written in an arbitrary inertial frame

- Eqn (2.24) is the same as (2.18b) but with $\ddot{\vec{R}}_2$ replaced with $\ddot{\vec{r}}_2$
  - **Why can we do this**? Because $\ddot{\vec{r}}_2$ and $\ddot{\vec{R}}_2$ are equal, due to the fact that neither G nor the origin of the arbitrary inertial frame are accelerating or rotating
  - That is, both the frame in which Eqn (2.18b) is written and the frame in which Eqn (2.24) is written are **inertial**

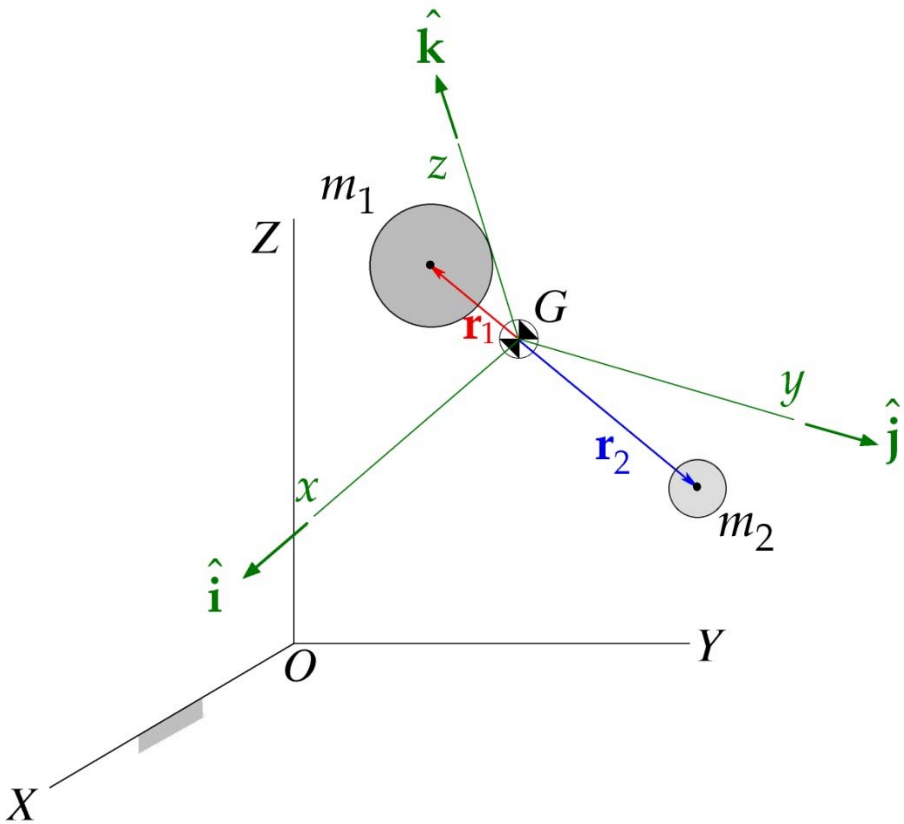


2.3 Equations of Relative Motion

- Left-hand side of Eqn (2.24) can be written in terms of r_2 instead of r (just as the right-hand side is)

- Result is the un-numbered equation above Eqn 2.27 (or Eqn 2.27 when expressed in terms of μ); equation of motion for m_1 can be written in similar fashion

2.3 Equations of Relative Motion



Non-rotating frame *xyz* attached to the center of mass ***G***.

The equation of motion of m_1 relative to the center of mass ***G*** is

$$\ddot{\mathbf{r}}_1 = -\frac{\mu''}{r_1^3}\mathbf{r}_1$$

where

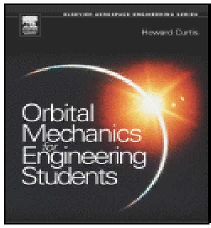
$$\mu'' = \left(\frac{m_2}{m_1+m_2}\right)^3 \mu$$

The equation of motion of m_2 relative to the center of mass ***G*** is

$$\ddot{\mathbf{r}}_2 = -\frac{\mu'}{r_2^3}\mathbf{r}_2$$

where

$$\mu' = \left(\frac{m_1}{m_1+m_2}\right)^3 \mu$$



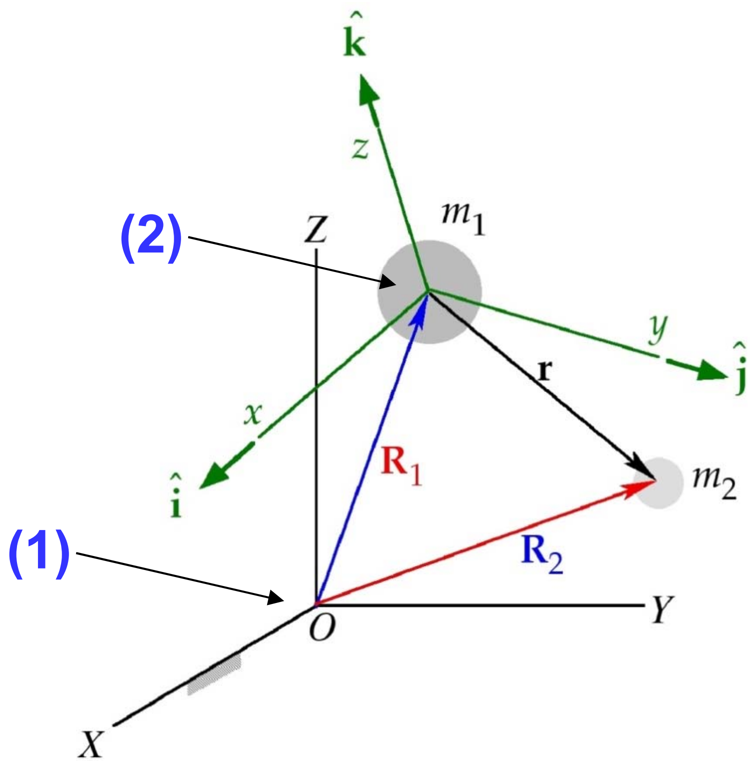
2.3 Equations of Relative Motion

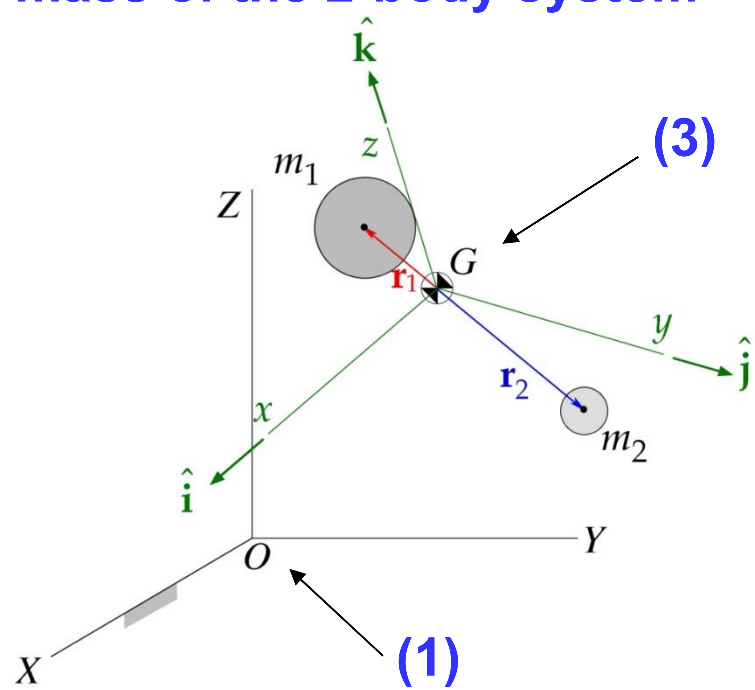
- **What is the main point of this section?** **That the equations of motion for two masses under gravitational attraction are of the same form whether they are:**
  - **Written relative to either of the masses (Eqn 2.22)**
  - **Written relative to the center of mass (un-numbered equations below Eqn 2.27)**

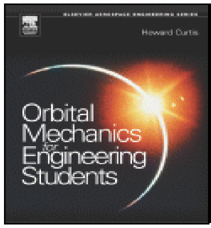
- **Therefore, a plot of the motion in each case above should look similar**
  - **Fig 2.3 shows the motion of one mass relative to the other mass AND the motion of one mass relative to the center of mass are elliptical in shape**
  - **We will see this verified mathematically in the next section**

2.3 Equations of Relative Motion

- In Sec 2.2, we looked at the 2-body ODEs from the perspective of (1) an arbitrary inertial point in space, (2) one of the masses, & (3) the center of mass of the 2-body system

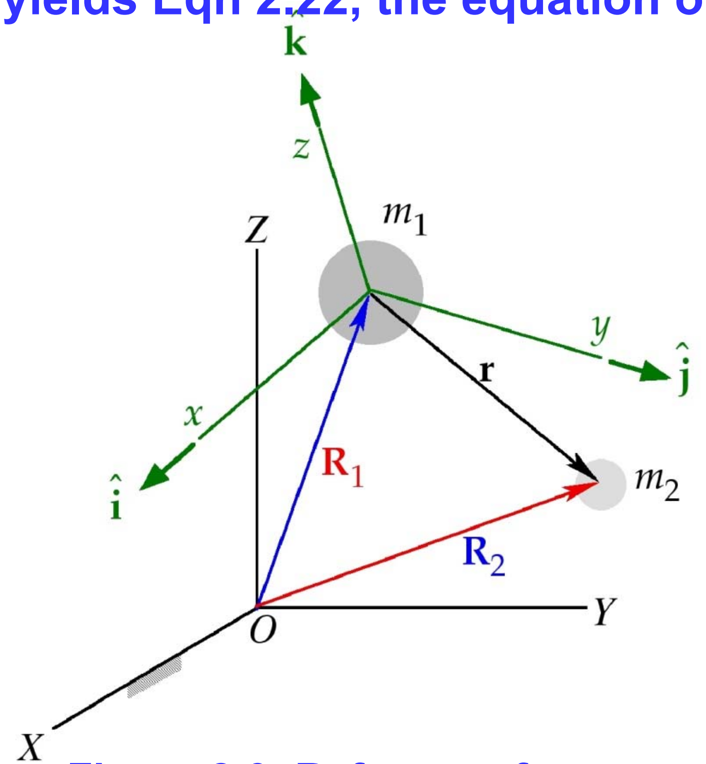






2.3 Equations of Relative Motion

•Differentiating Eqn (2.5) twice & inserting Eqns (2.18a) & (2.18b) yields Eqn 2.22, the equation of motion of m_2 relative to m_1:



$$\boxed{\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}}$$

The fundamental equation of relative two-body motion.

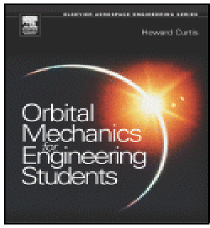
where r is as defined in Eqn (2.5) and μ is the gravitational parameter:

$$\mu = G\left(m_1 + m_2\right) \quad (\text{km}^3/\text{s}^2)$$

Although Eqn 2.22 is defined/derived in the inertial frame, r can be expressed in the frame whose origin is at m_1:

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

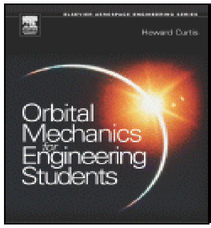
Figure 2.6: Reference frame *xyz* attached to point mass of m_1 (a "co-moving" frame)



2.3 Equations of Relative Motion

- In the un-numbered eqn above Eqn 2.20, the author refers to $\ddot{\bar{r}}$ as the “relative acceleration vector”…then on p. 68 refers to $\ddot{\bar{r}}_{rel}$ as the “relative acceleration” because it is acceleration as seen in the relative (co-moving) frame

- These are 2 totally different vectors, yet both referred to as “relative acceleration” → VERY confusing/misleading

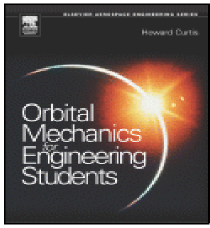


2.3 Equations of Relative Motion

- Because the center of mass G between m_1 and m_2 is not accelerating, we can construct an inertial frame with origin at G (call it frame xyz) and write the equations of motion for m_1 and m_2 in this frame

- Eqn (2.18a) & (2.18b) are the equations of motion written in an arbitrary inertial frame (call it frame XYZ)

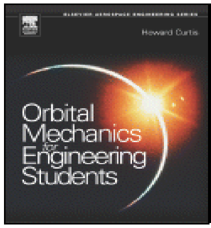
- Noting that $\hat{u}_r = \frac{\bar{r}}{r}$, Eqn (2.24) is the same as (2.18b) but with $\ddot{\bar{R}}_2$ replaced with $\ddot{\bar{r}}_2$
    - Why is this true? Because $\ddot{\bar{r}}_2$ and $\ddot{\bar{R}}_2$ are equal, due to the fact that neither G nor the origin of the arbitrary inertial frame are accelerating (see Eqn 1.70)
    - Another explanation is that both the frame in which Eqn (2.18b) is written and the frame in which Eqn (2.24) is written are inertial



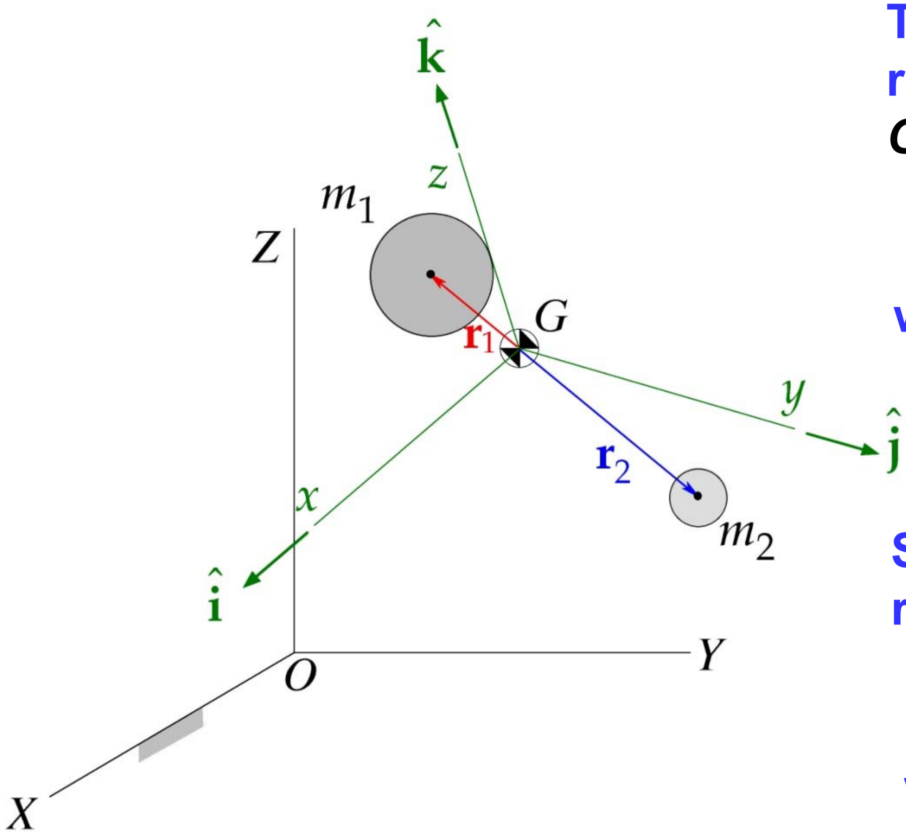
2.3 Equations of Relative Motion

- **Left-hand side of Eqn (2.24) can be written in terms of r_2 instead of r (just as the right-hand side is)**

- **Result is the un-numbered equation above Eqn 2.27 (or Eqn 2.27 when expressed in terms of μ); equation of motion for m_1 can be written in similar fashion**



2.3 Equations of Relative Motion



Non-rotating frame *xyz* attached to the center of mass *G*.

The equation of motion of m_2 relative to the center of mass *G* is then found to be

$$\ddot{\mathbf{r}}_2 = -\frac{\mu'}{r_2^3}\mathbf{r}_2$$

where

$$\mu'' = \left(\frac{m_2}{m_1 + m_2}\right)^3 \mu$$

Similarly, the equation of motion of m_1 relative to the center of mass *G* is

$$\ddot{\mathbf{r}}_1 = -\frac{\mu''}{r_1^3}\mathbf{r}_1$$

where

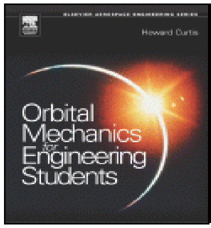
$$\mu' = \left(\frac{m_1}{m_1 + m_2}\right)^3 \mu$$

2.3 Equations of Relative Motion

- **Note that if we derive $\ddot{\mathbf{r}}_2 = -\frac{\mu'}{r_2^3}\mathbf{r}_2$, we can then derive $\ddot{\mathbf{r}}_1 = -\frac{\mu''}{r_1^3}\mathbf{r}_1$ by simply interchanging the roles of m_1 and m_2 (i.e. interchanging the "1" and "2" subscripts)**

- **That is, starting from Eqn 2.27, interchanging the "1" and "2" subscripts yields $\ddot{\mathbf{r}}_1 = -\frac{\mu''}{r_1^3}\mathbf{r}_1$ or**

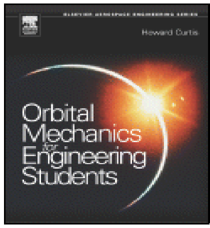
$$-\left(\frac{m_2}{m_2+m_1}\right)^3 \frac{\mu}{r_1^3}\vec{r}_1 = \ddot{\vec{r}}_1$$



2.3 Equations of Relative Motion

- **What is the main point of this section?** **That the equations of motion for two masses under gravitational attraction are of the same form whether they are:**
  - **Written relative to either of the masses (Eqn 2.22)**
  - **Written relative to the center of mass (un-numbered equations below Eqn 2.27)**

- **Therefore, a plot of the motion in each case above should look similar**
  - **Fig 2.3 shows the motion of one mass relative to the other mass AND the motion of one mass relative to the center of mass are elliptical in shape**
  - **We will see this verified mathematically in the next section**



2.3 Equations of Relative Motion

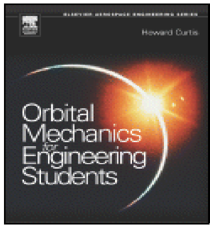
- Q: What happens if one mass is significantly greater than the other?

- A: As $m_1 >> m_2$ (i.e. as $m_2/m_1 \rightarrow 0$):
  - Position vector $\mathbf{R}_G$ of the center of mass definec so R_G becomes R_1

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2}$$

  - Thus, r_1 becomes 0 and r_2 becomes r
  - μ' becomes μ
  - μ'' becomes 0

- This means:
  - Center of mass migrates toward m_1
  - m_2's motion w.r.t. center of mass is identical to its motion w.r.t. m_1
  - m_1 doesn't move w.r.t. center of mass



2.4 Angular Momentum and the Orbit Formulas

- Section 1.5 introduced the concept of angular momentum of a mass about a point “O”:

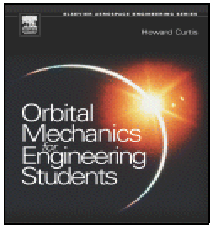
$H_O = r \times m_2 v$

- Section 2.4 introduces the concept of one mass having angular momentum relative to another mass (in this case m_2 relative to m_1):

$H_{2/1} = r \times m_2 v$

- Dividing this relative angular momentum by mass yields **specific relative angular momentum** (Eqn 2.28):

$$\boxed{\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}}$$



2.4 Angular Momentum and the Orbit Formulas

- The text below Eqn 2.28 proves the **Conservation of angular momentum** (Eqn 2.29):

$$\frac{d\mathbf{h}}{dt} = 0 \qquad \left(\text{or } \mathbf{r} \times \dot{\mathbf{r}} = \text{constant}\right)$$

- This implies that the **magnitude** and **direction** of h are constant, unless m_2 is acted upon by an external moment (torque)
  - Angular version of Newton's 1st Law

- Also implies that **the path of m_2 around m_1 lies in a plane** (the plane containing r and v) & that h is normal to this plane

- The magnitude of angular momentum depends only on the transverse component of the relative velocity (Eqn 2.31): $h = rv_\perp$

2.4 Angular Momentum and the Orbit Formulas

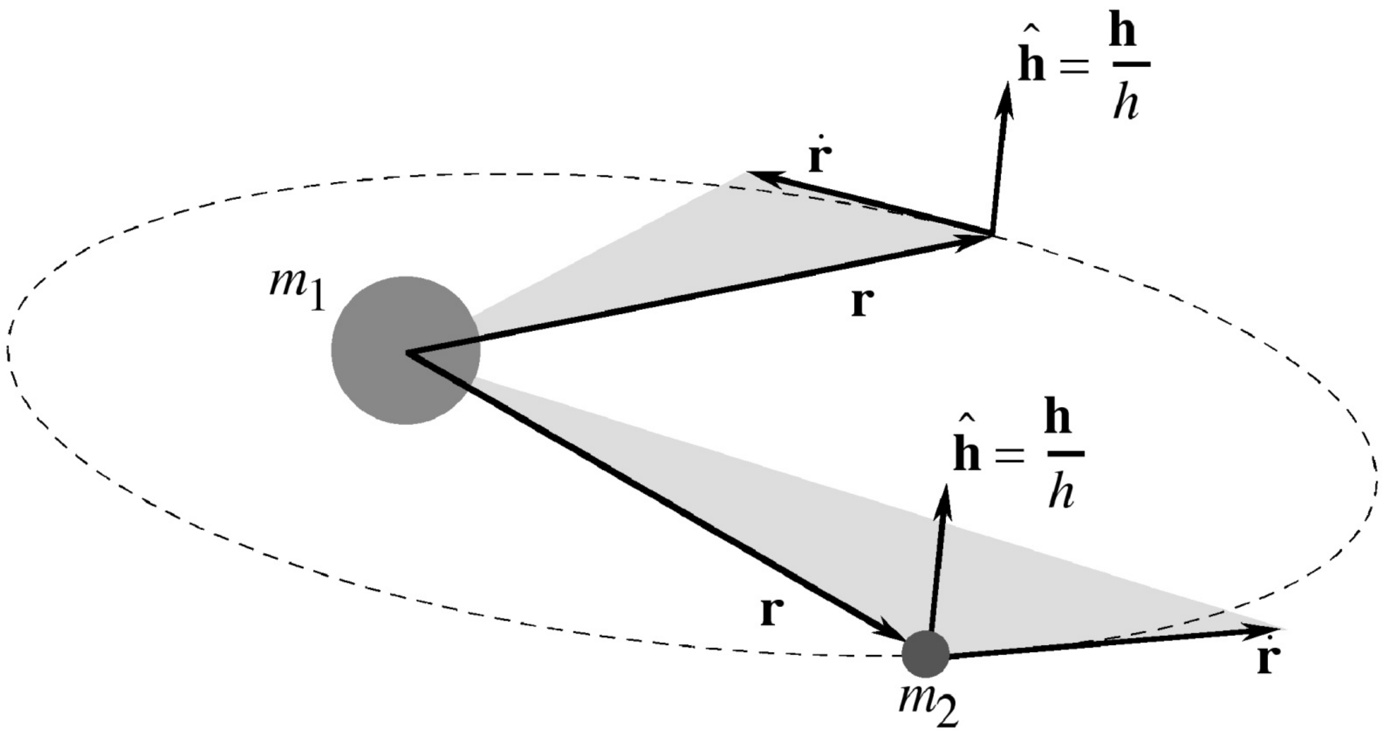
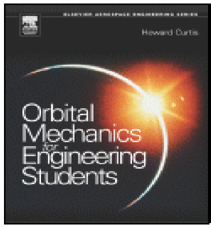


Figure 2.8: The path of m_2 around m_1 lies in a plane whose normal is defined by **h**.



2.4 Angular Momentum and the Orbit Formulas

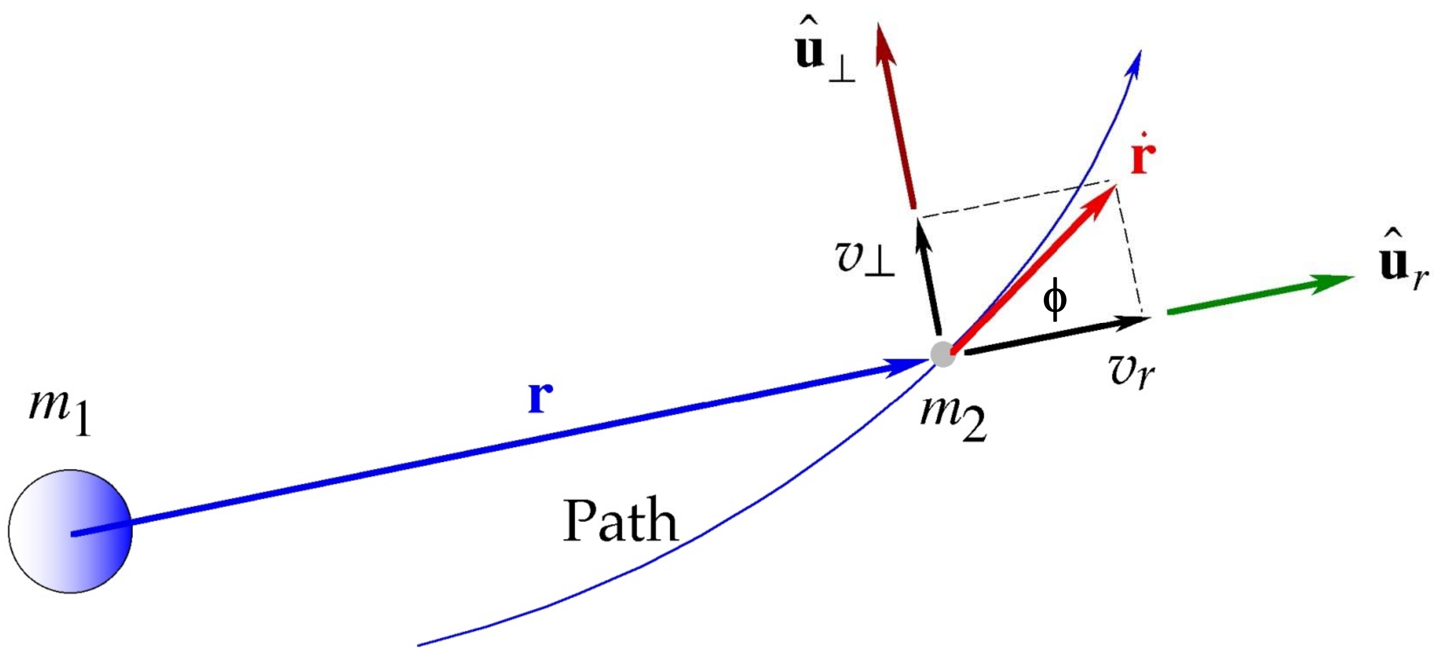
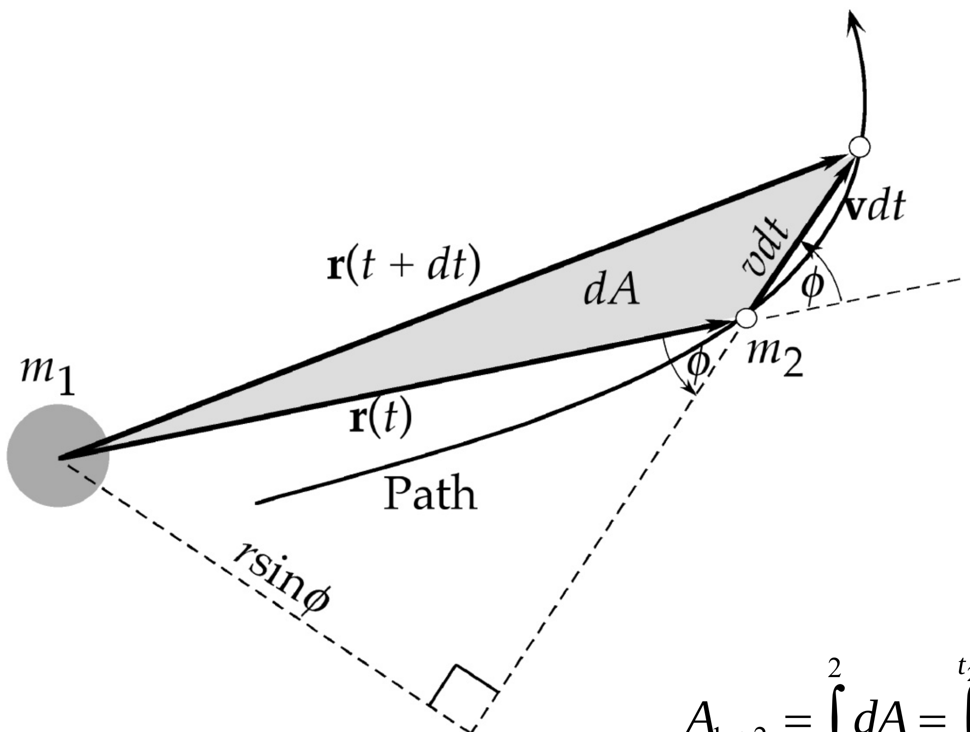


Figure 2.9: Components of the velocity of m_2, viewed above the plane of the orbit.

2.4 Angular Momentum and the Orbit Formulas



$$dA = \frac{1}{2} \times vdt \times r\sin\phi$$

$$= \frac{1}{2} \times r \times v\sin\phi \times dt = \frac{1}{2} r v_{\perp} dt = \frac{1}{2} h dt$$

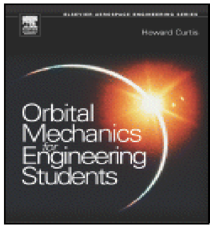
$$\frac{dA}{dt} = \frac{h}{2}$$

dA/dt **- areal velocity (time rate of change of area swept out)**

Kepler's second law - Equal areas are swept out in equal times.

$$A_{1\to 2} = \int_1^2 dA = \int_{t_1}^{t_2} \frac{dA}{dt} dt = \int_{t_1}^{t_2} \frac{h}{2} dt = \frac{h}{2} \int_{t_1}^{t_2} dt = \frac{h}{2}(t_2 - t_1)$$

Figure 2.10: Differential area *dA* swept out by the relative position vector r during time interval *dt*.



2.4 Angular Momentum and the Orbit Formulas

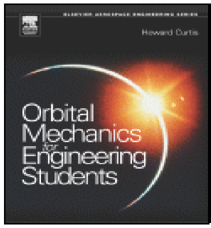
The first integral of $\ddot{\mathbf{r}} = -\mu \dfrac{\mathbf{r}}{r^3}$ is $\dot{\bar{r}} \times \bar{h} - \mu \dfrac{\bar{r}}{r} = \bar{C}$ (Eqn 2.39)

which can be rearranged as $\dfrac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} - \dfrac{\mathbf{r}}{r} = \mathbf{e}$ (Eqn 2.40)

e is the dimensionless eccentricity vector, which is a *constant* and lies in the orbital plane.

The line defined by the vector e is commonly called the apse line.

The vector **C** = μ**e** is called the Laplace vector.



2.4 Angular Momentum and the Orbit Formulas

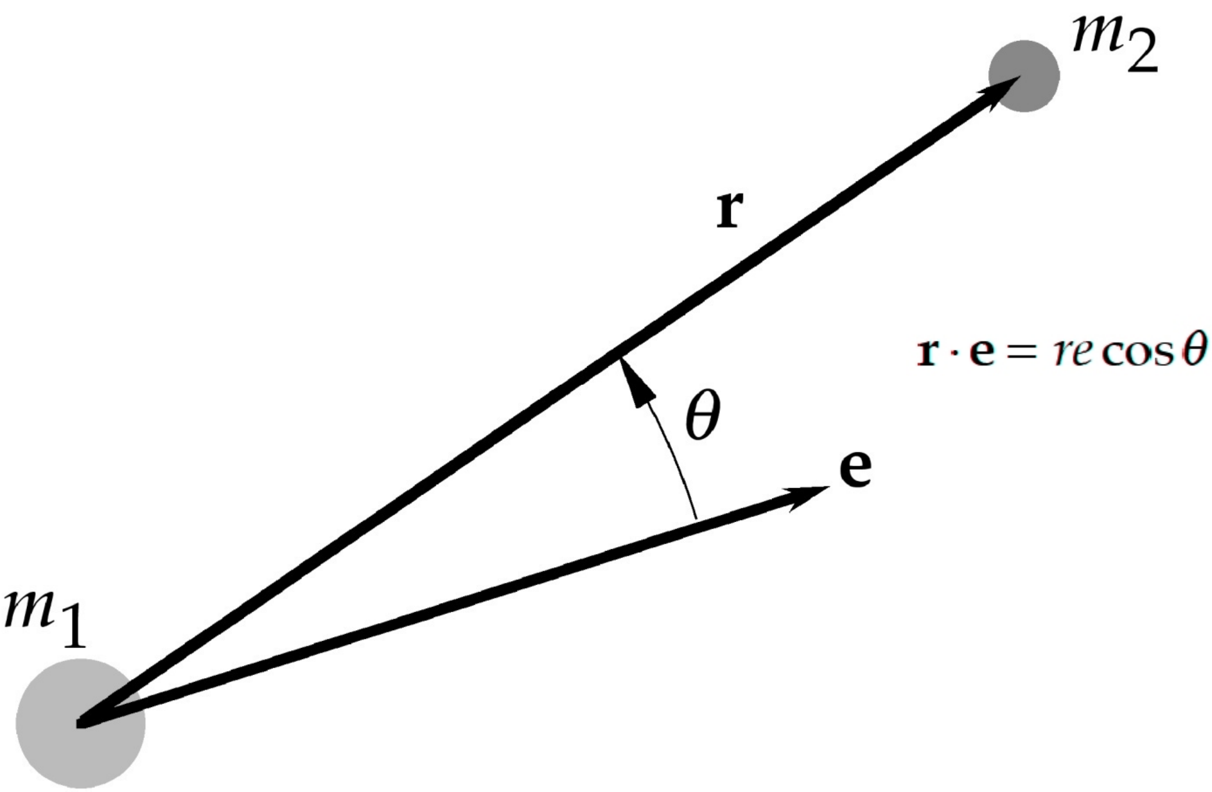
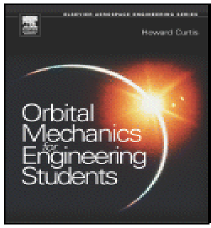


Figure 2.11: The true anomaly θ is the angle between the eccentricity vector e and the position vector r.



2.4 Angular Momentum and the Orbit Formulas

From $\frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r} = \mathbf{e}$ we obtain the orbit formula (Eqn 2.45):

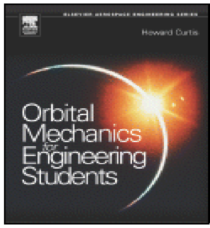
$$\boxed{r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta}}$$

On the right-hand side, h, e, & μ are constant; only θ changes. Thus, this equation shows how r changes with θ.

This is the equation of a conic section (circle, ellipse, parabola, or hyperbola) from which we get Kepler's First Law: The planets follow elliptical paths around the sun.

The point of closest approach ($\theta = 0$) lies on the apse line and is called periapsis.

Keplerian orbits: Two-body orbits



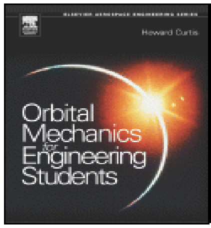
2.4 Angular Momentum and the Orbit Formulas

- **Over the next few pages of the book, the author emphasizes the motion of m_2 relative to m_1 in polar (r, θ) coordinates**

- **Begins with two fundamental formulas from dynamics:**

$$v_r = \dot{r} \; and \; v_\perp = r\dot{\theta}$$

- **And from there derives several quantities (orbital parameters) in terms of h, e, and θ**



2.4 Angular Momentum and the Orbit Formulas

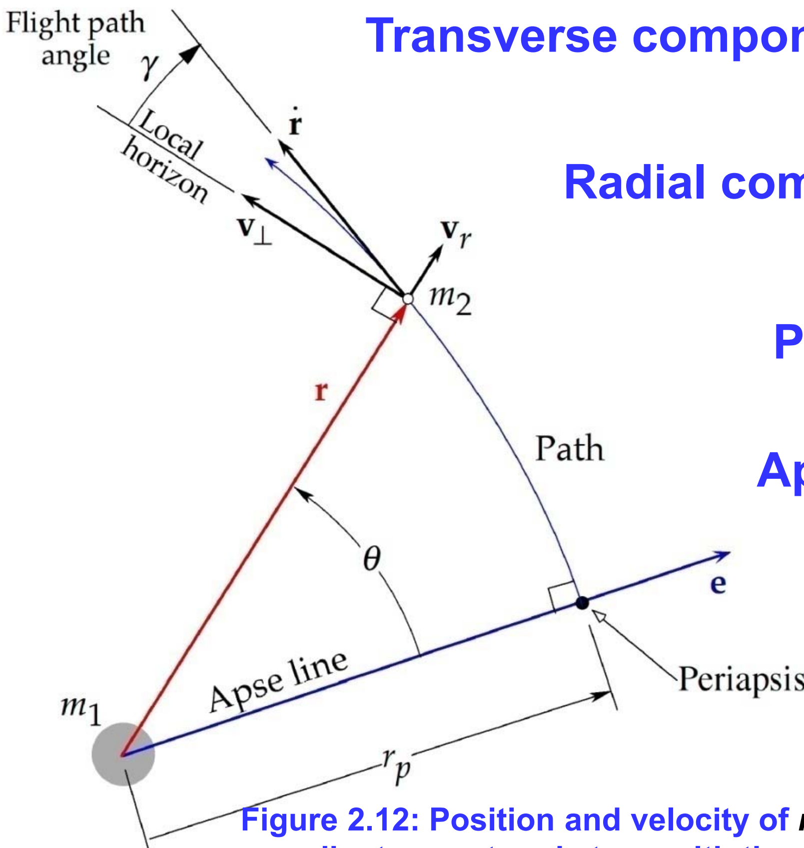


Figure 2.12: Position and velocity of m_2 in polar coordinates centered at m_1, with the eccentricity vector being the reference for true anomaly (polar angle θ)

Transverse component of v: $v_\perp = \frac{h}{r} = \frac{\mu}{h}\left(1 + e\cos\theta\right)$ **(Eqn 2.48)**

Radial component of v: $v_r = \dot{r} = \frac{\mu}{h} e \sin\theta$ **(Eqn 2.49)**

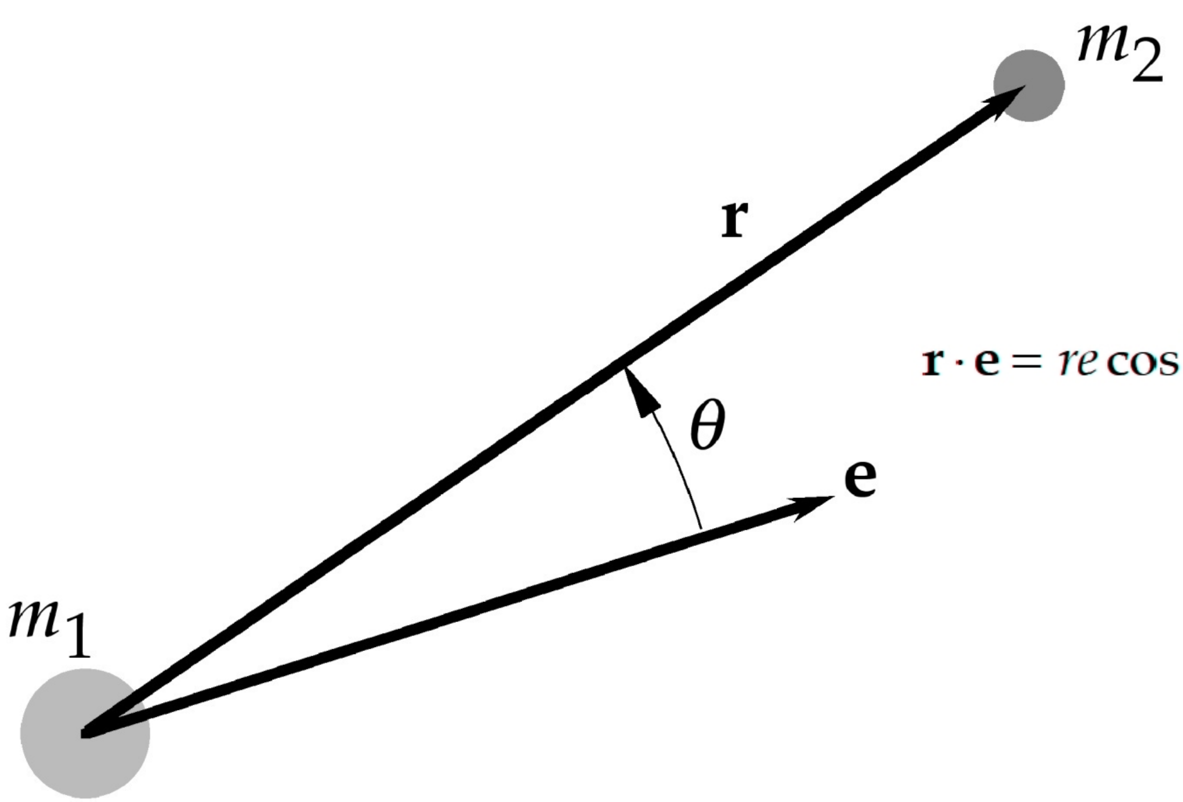
Periapsis radius: $r_p = \frac{h^2}{\mu}\frac{1}{1+e}$ **(Eqn 2.50)**

Apoapsis radius: $r_a = \frac{h^2}{\mu}\frac{1}{1-e} \longleftarrow$ **(NOT in the book!)**

Flight path angle: $\gamma = \tan^{-1}\left(\frac{e\sin\theta}{1+e\cos\theta}\right)$

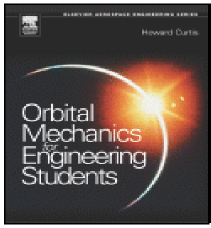
(Eqn 2.52)

2.4 Angular Momentum and the Orbit Formulas



$$\mathbf{r}\cdot\mathbf{e} = re\cos\theta$$

Figure 2.11: The true anomaly θ is the angle between the eccentricity vector e and the position vector r.



2.4 Angular Momentum and the Orbit Formulas

From $\dfrac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} - \dfrac{\mathbf{r}}{r} = \mathbf{e}$ **we obtain the orbit formula (Eqn 2.45):**

$$\boxed{r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta}}$$

On the right-hand side, h, e, & μ are constant; only θ changes. Thus, this equation shows how r changes with θ.

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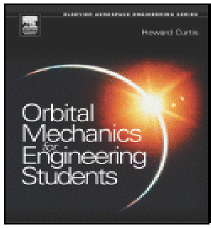
Keplerian orbits: Two-body orbits



2.4 Angular Momentum and the Orbit Formulas

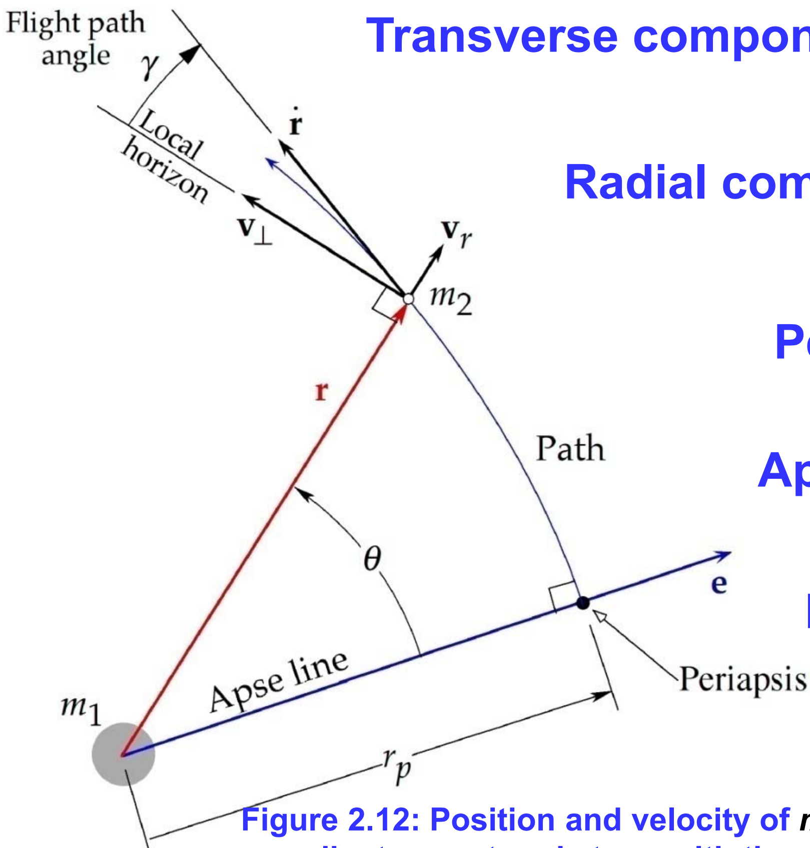


Figure 2.12: Position and velocity of m_2 in polar coordinates centered at m_1, with the eccentricity vector being the reference for true anomaly (polar angle θ)

Transverse component of v $v_\perp = \dfrac{h}{r} = \dfrac{\mu}{h}(1 + e\cos\theta)$ **(Eqn 2.48)**

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Apoapsis radius: $r_a = \dfrac{h^2}{\mu}\dfrac{1}{1-e} \longleftarrow$ **(NOT in the book!)**

Flight path angle: $\gamma = \tan^{-1}\left(\dfrac{e\sin\theta}{1 + e\cos\theta}\right)$

(Eqn 2.52)

2.4 Angular Momentum and the Orbit Formulas

Semi-latus rectum (parameter):

$$p = \frac{h^2}{\mu}$$

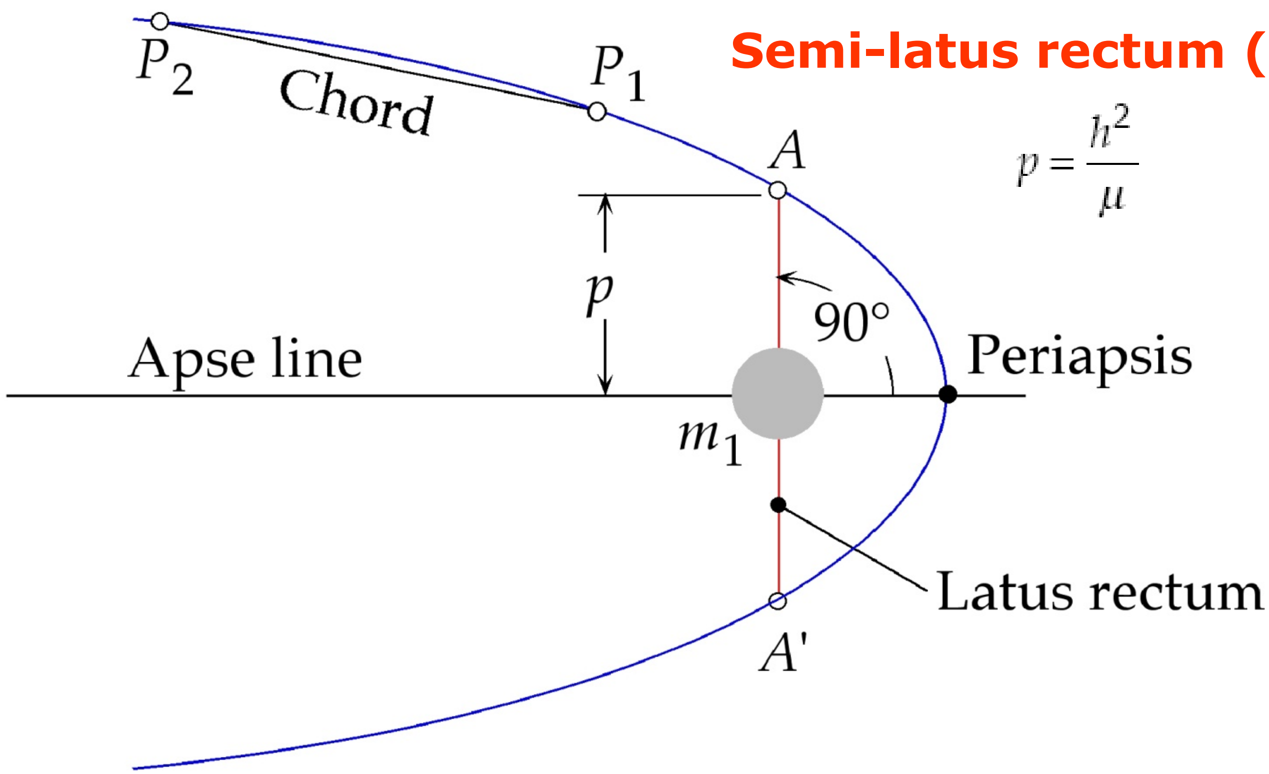
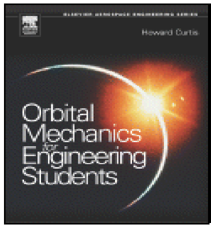


Figure 2.13: Illustration of latus rectum, semi-latus rectum *p*, and the chord between any two points on an orbit.



2.5 The Energy Law

The total energy per unit mass ε (or "specific energy") is the sum of the kinetic and potential energies per unit mass:

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \qquad \text{(Eqn 2.57)}$$

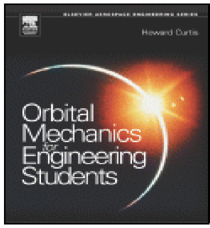
$v^2/2$ is the relative kinetic energy per unit mass

$(-\mu/r)$ is the potential energy per unit mass of the body m_2 in the gravitational field of m_1

Conservation of energy states that ε is constant for a body under gravitational attraction: as KE increases, PE decreases, or vice-versa

The total energy $\mathcal{E}$ of a satellite of mass m is obtained by multiplying specific energy by the mass: $\mathcal{E} = m\varepsilon$

Eqn 2.60 expresses ε in terms of μ, h, and e



2.5 The Energy Law

Some notes about potential energy ($-\mu/r$):

•It is always negative

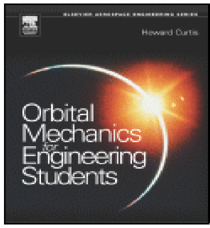
•As r decreases (m_2 getting closer to m_1), potential energy becomes more negative (i.e. PE decreases but its magnitude increases)

•As r increases (m_2 getting further from m_1), potential energy becomes less negative (i.e. PE increases but its magnitude decreases)

•In the limit as $r \rightarrow 0$, $PE \rightarrow -\infty$ and as $r \rightarrow \infty$, $PE \rightarrow 0$ (approaching from the negative side of zero)

•Therefore we can state that the larger r is, the higher (i.e. less negative) the potential energy, & the smaller r is, the lower (i.e. more negative) the potential energy

•…and COE tells us that that the larger r is, the lower the kinetic energy (i.e. the SLOWER m_2 moves in its orbit around m_1), & the smaller r is, the higher the kinetic energy (i.e. the FASTER m_2 moves in its orbit around m_1)



2.6 Circular Orbits (e = 0)

Velocity of a circular orbit: $v_{circ} = \sqrt{\frac{\mu}{r}}$ **(Eqn 2.63)**

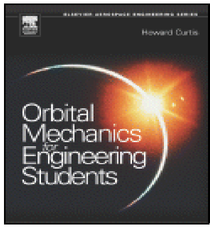
The time *T* required for one orbit is known as the period.

Period of a circular orbit: $T_{\text{circular}} = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}}$ **(Eqn 2.64)**

Total specific energy of a circular orbit: $\varepsilon_{\text{circular}} = -\frac{\mu}{2r}$ **(Eqn 2.65)**

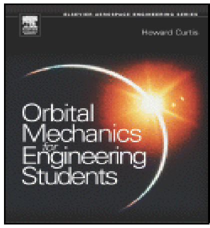
The larger the orbit is, the higher (i.e. less negative) is its energy.

Low earth orbit (LEO) is one whose altitude lies between about 150 km (100 miles) and about 1000 km (600 miles).



2.6 Circular Orbits (e = 0)

- The **sidereal day** is the time it takes the earth to complete one rotation relative to inertial space (the fixed stars)
- The **synodic day** is the time it takes the sun to apparently rotate once around the earth, from high noon one day to high noon the next
- If a satellite's orbit has a period equal to one sidereal day, it is called **geosynchronous**
- In this orbit is also circular & equatorial, it is called **geostationary**
- A satellite on a **geostationary Earth orbit (GEO)** always remains above the same point on the earth's equator, & appears to "hover" in the same location in the sky



2.6 Circular Orbits (e = 0)

While the earth makes one absolute rotation around its axis, it advances 2π/365.26 radians along its solar orbit.

Earth's inertial angular velocity: $$\omega_E = \frac{2\pi + \dfrac{2\pi}{365.26}}{(24 \times 3600)\,\mathrm{s}} = 72.9211 \times 10^{-6} \ \ \mathrm{rad/s}$$

(Eqn 2.67)

For a satellite in GEO, $\omega_{sat} = \omega_E$: $$\frac{\sqrt{\mu / r_{\mathrm{GEO}}}}{r_{\mathrm{GEO}}} = \omega_E \quad \Rightarrow \quad r_{\mathrm{GEO}} = \left(\frac{\mu}{{\omega_E}^2}\right)^{\frac{1}{3}}$$

GEO radius: $$r_{\mathrm{GEO}} = 42\,164 \ \mathrm{km} \quad (26\,205 \ \mathrm{mi})$$ (solved in Example 2.5)

GEO speed: $$v_{\mathrm{GEO}} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398\,600}{42\,164}} = 3.057 \ \ \mathrm{km/s}$$ (solved in Example 2.5)

2.7 Elliptical Orbits (0<*e*<1)

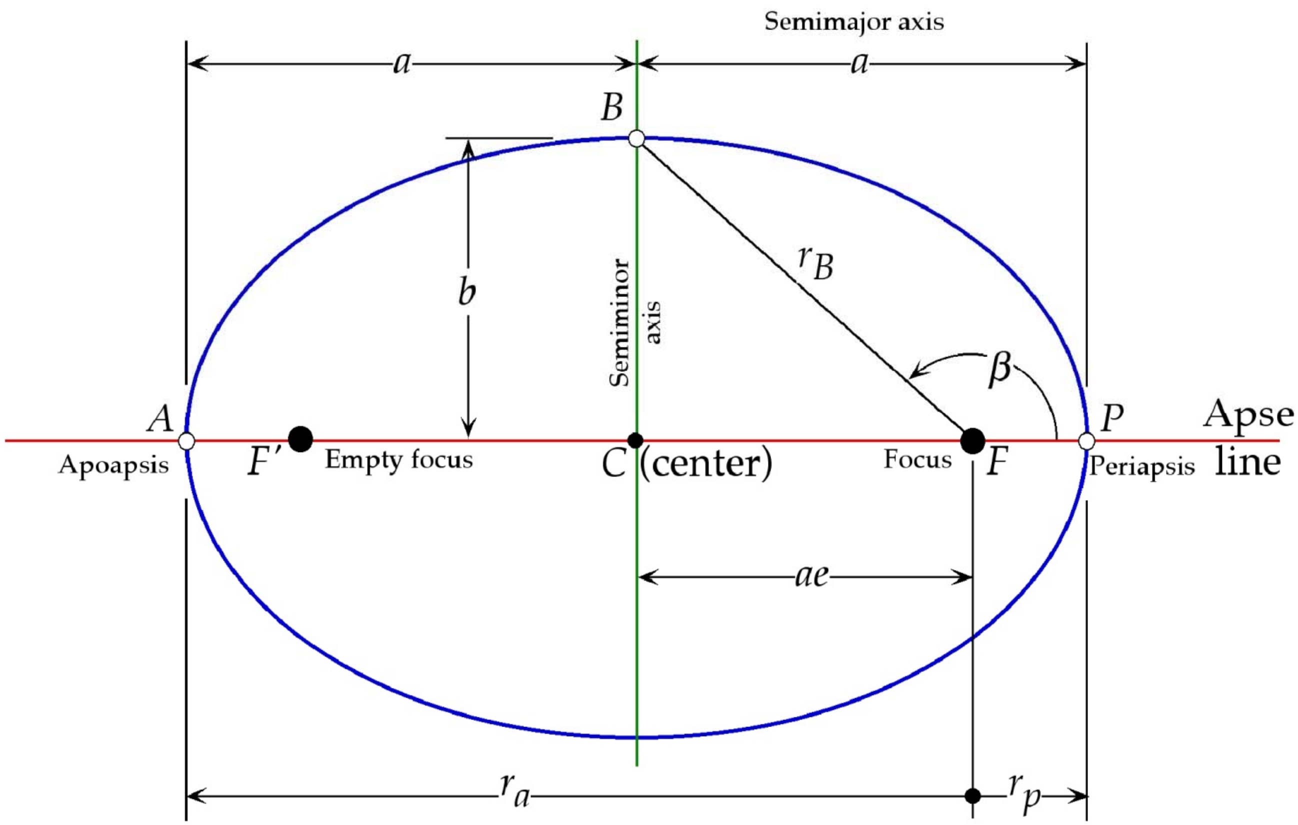
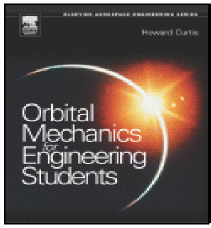
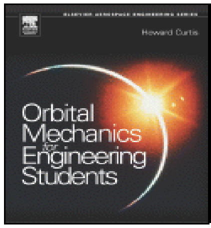


Figure 2.18: Elliptical orbit. *m_1* is at the focus *F*. *F′* is the unoccupied empty focus.



2.7 Elliptical Orbits (0<*e*<1)

- When 0 < e < 1 in Eqn 2.45, this guarantees *r* remains bounded
- Minimum value of ***r*** is reached when θ = 0° (periapsis), & maximum value is reached when θ = 180° (apoapsis)
- Inserting these θ values into Eqn 2.45 yields $$r_p = \frac{h^2}{\mu}\frac{1}{1+e}$$ (Eqn 2.50) & $$r_a = \frac{h^2}{\mu}\frac{1}{1-e}$$ (Eqn 2.70)
- Semimajor axis of the ellipse: $$r_p + r_a = 2a \rightarrow a = \frac{r_p + r_a}{2} = \frac{h^2}{\mu\left(1-e^2\right)}$$ (Eqn 2.71)
- Given that $$\frac{h^2}{\mu} = a\left(1-e^2\right)$$, we can now write:
  - $r = a\left(1-e^2\right)/1+e\cos\theta$ (Eqn 2.72)
  - $r_p = a\left(1-e\right)$ (Eqn 2.73)
  - $r_a = a\left(1+e\right)$
  -



2.7 Elliptical Orbits ($0<e<1$)

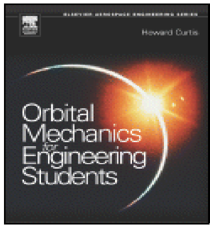
- The **semiminor axis** ***b*** is found in terms of the semimajor axis and the eccentricity of the ellipse:

$$b = a\sqrt{1-e^2} \quad \textbf{(Eqn 2.76)}$$

- Distance from center of the ellipse to m_1 (at the focus) is

$$a - r_p = a - a(1-e) = ae \quad \textbf{(below Eqn 2.73)}$$

- Author also derives $e = -\cos\beta$ (Eqn 2.75) & $r_B = a$ (below Eqn 2.75)



2.7 Elliptical Orbits (0<*e*<1)

- **Note that these eqn's for *r*, r_p, r_a, and *b* do NOT contain μ**
- **This is because the qty's involved in these eqn's (bascially all qty's displayed in Fig 2.18) are basic geometric properties of an ellipse**
- **There doesn't have to be an orbital motion scenario occurring (i.e. a small mass orbiting around a large attracting body) in order for these qty's to have meaning**
- ***h* and *v* on the other hand DO pertain specifically to orbital motion scenarios; notice each of these qty's *IS* a function of μ**
    - **Rearranging Eqn 2.71 yields** $h = \sqrt{\mu a\left(1-e^2\right)}$
    - **See Eqns 2.48 & 2.49 for components of *v***

2.7 Elliptical Orbits ($0<e<1$)

This is the eqn for an ellipse of semimajor axis a & seminimor axis b, centered at (0,0)

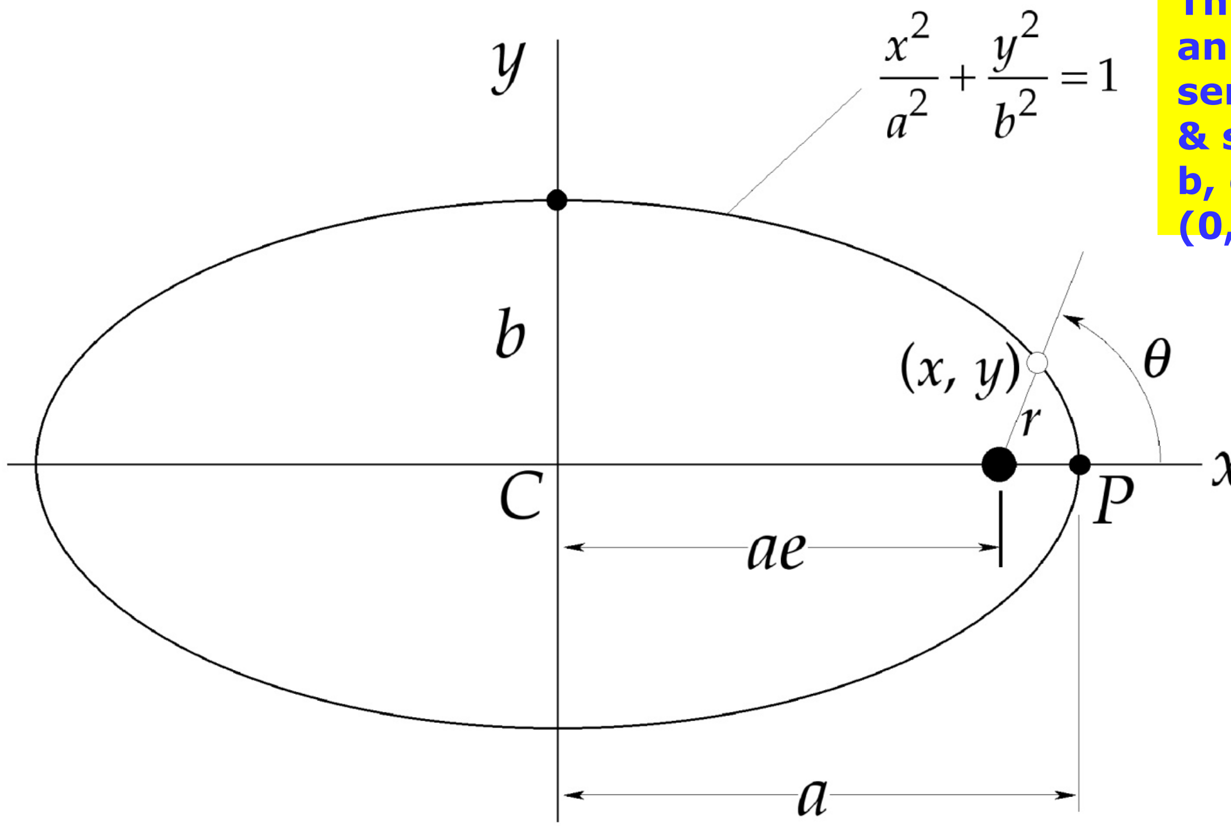
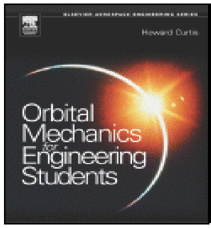


Figure 2.19: Cartesian coordinate description of the orbit.



2.7 Elliptical Orbits ($0<e<1$)

- Eqn (2.71) is substituted into Eqn (2.60) to show that **specific energy** of an elliptical orbit is a function ONLY of its semimajor axis:

$$\varepsilon = -\frac{\mu}{2a}$$

(Eqn 2.80 → compare to Eqn 2.65)

- Next, Kepler's 2nd law , along with the formula for area of an ellipse, is used to derive the period of an elliptical orbit (also a function only of semimajor axis):

$$A = \pi a b \quad \Longrightarrow \quad T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}$$

(Eqn 2.83 → compare to Eqn 2.64)

- This is known as Kepler's 3rd law

- A useful formula for calculating the eccentricity of an elliptical orbit is then derive

$$e = \frac{r_a - r_p}{r_a + r_p}$$

(Eqn 2.84)

2.7 Elliptical Orbits ($0<e<1$)

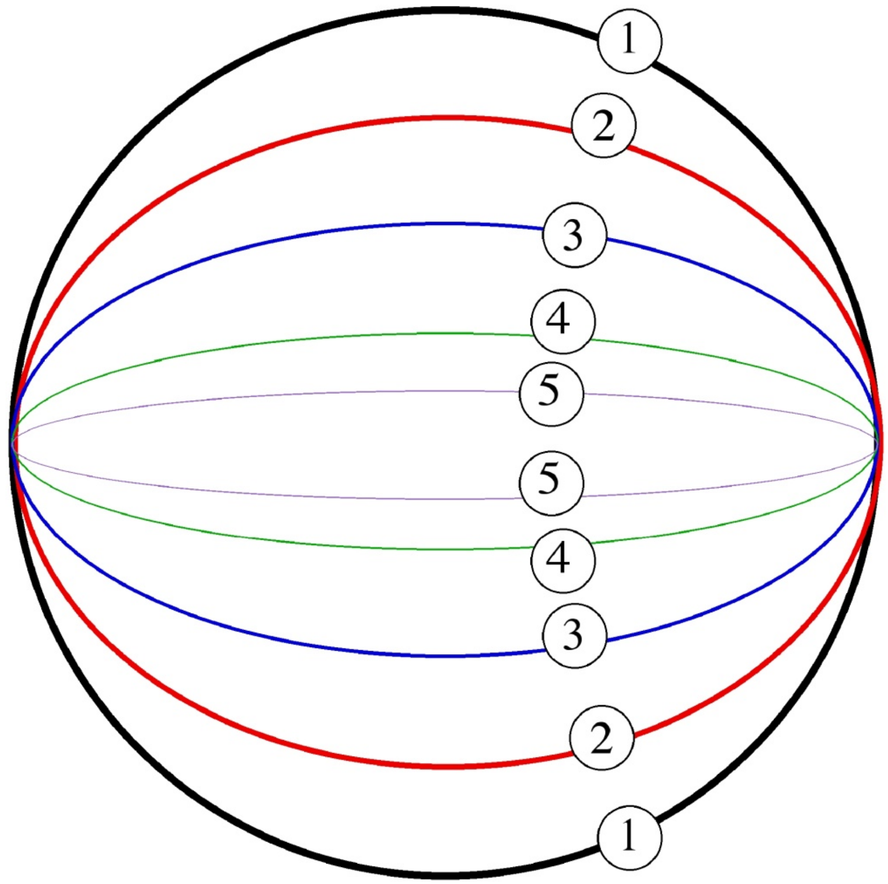


Figure 2.20: Since all five ellipses have the same major axis, their periods and energies are identical.

Figure for Example 2.7

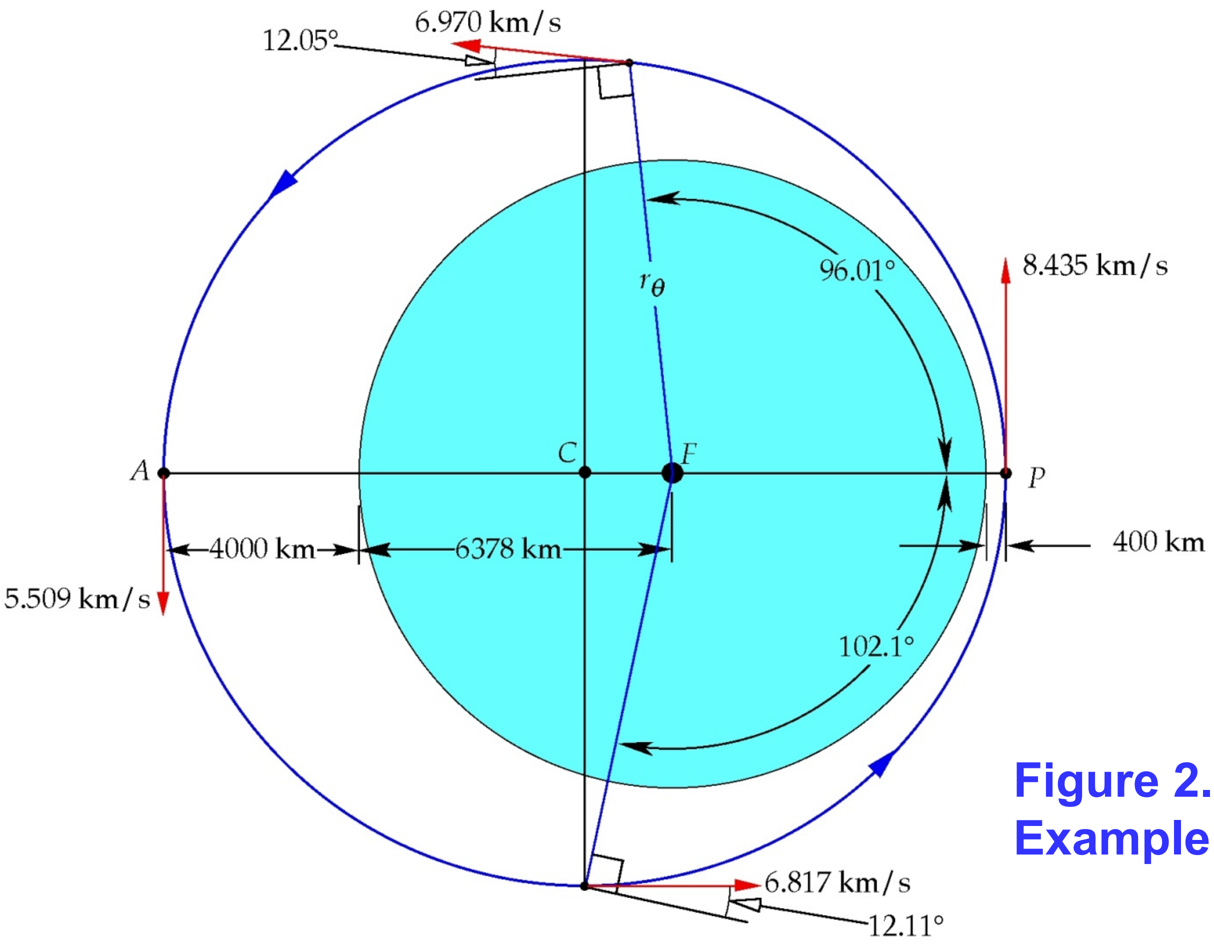
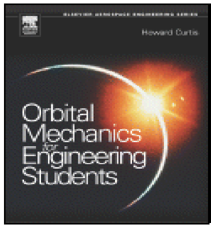
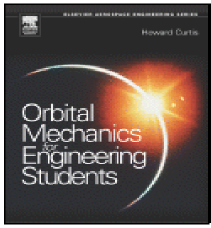


Figure 2.21: The orbit of Example 2.7.



2.8 Parabolic Trajectories (e=1)

- **A parabolic trajectory has the shape of a parabola & is OPEN (i.e. doesn't intersect itself)**
- **A parabolic path has ZERO total energy; *h & p* are finite, but *a & T* are infinite**
- **The speed of an object on a parabolic path** $v = \sqrt{\dfrac{2\mu}{r}}$
    - **Note as r →∞, v →0**
- **A parabolic path is called an escape trajectory; if an object on a closed orbit (circular or elliptical) can increase its velocity to this value, it will escape the gravitational attraction of the central body**
- **Flight path angle is half the true anomaly (Eqn 2.93)**
    - **As r →∞, θ →180º, γ →90º, v →v_r**
    - **Object's motion becomes rectilinear (no longer a curved path)**



2.8 Parabolic Trajectories (e=1)

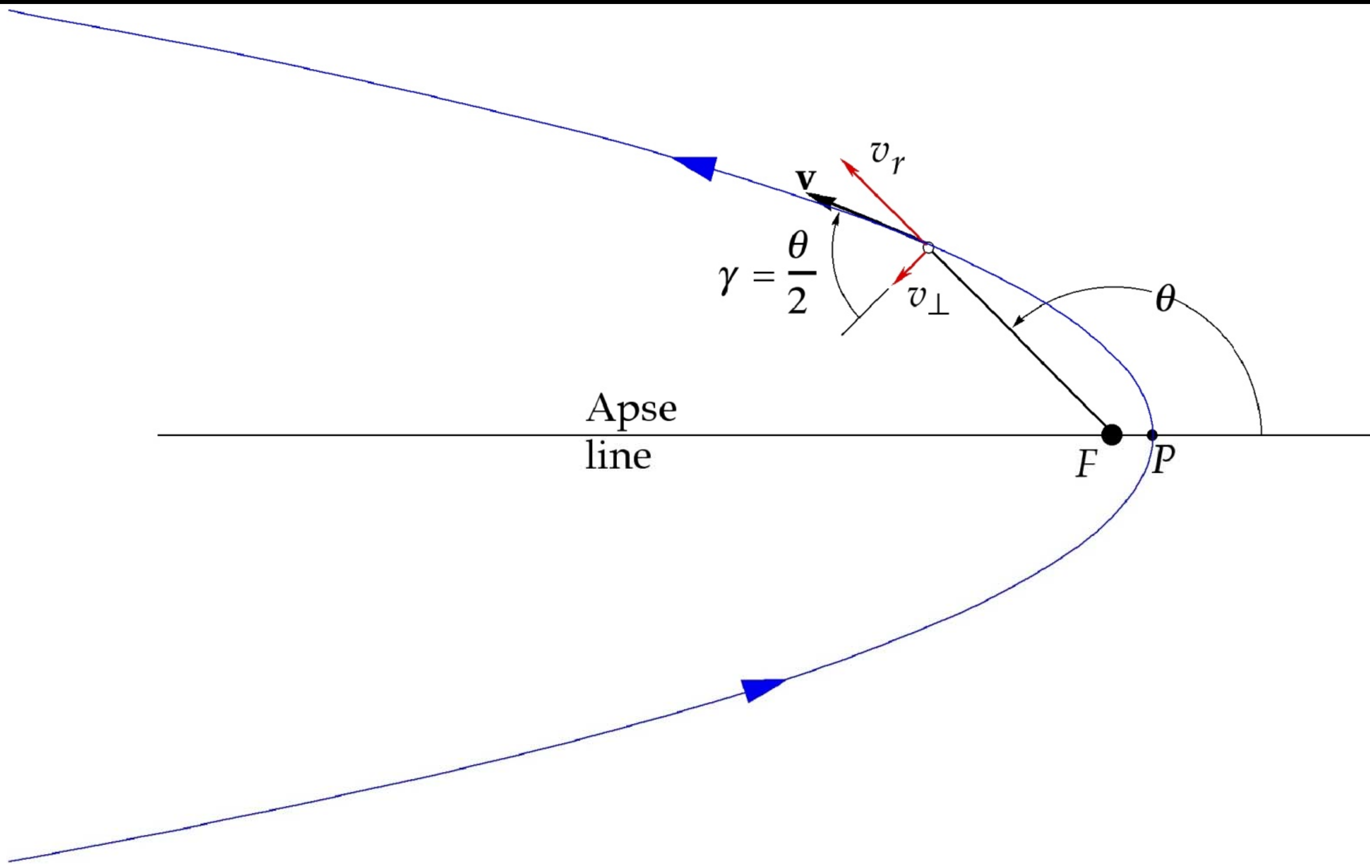
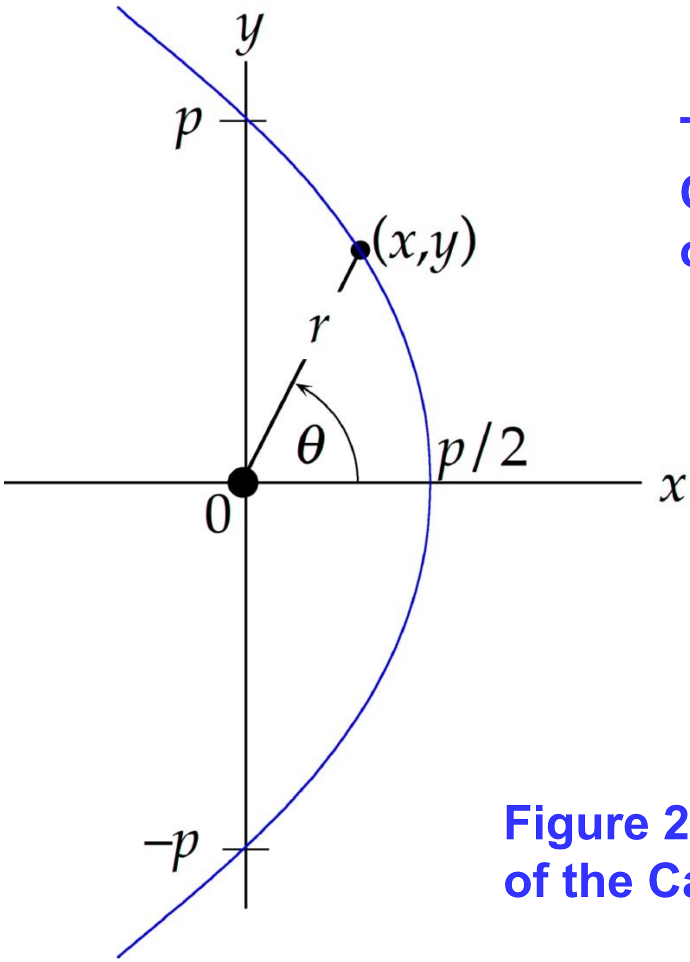


Figure 2.22: Parabolic trajectory around the center of attraction ***F***.

2.8 Parabolic Trajectories (e=1)



The equation of a parabola in a Cartesian coordinate system whose origin serves as the focus is

$$x = \frac{p}{2} - \frac{y^2}{2p}$$

Figure 2.23: Parabola with focus at the origin of the Cartesian coordinate system.

Figure for Example 2.9

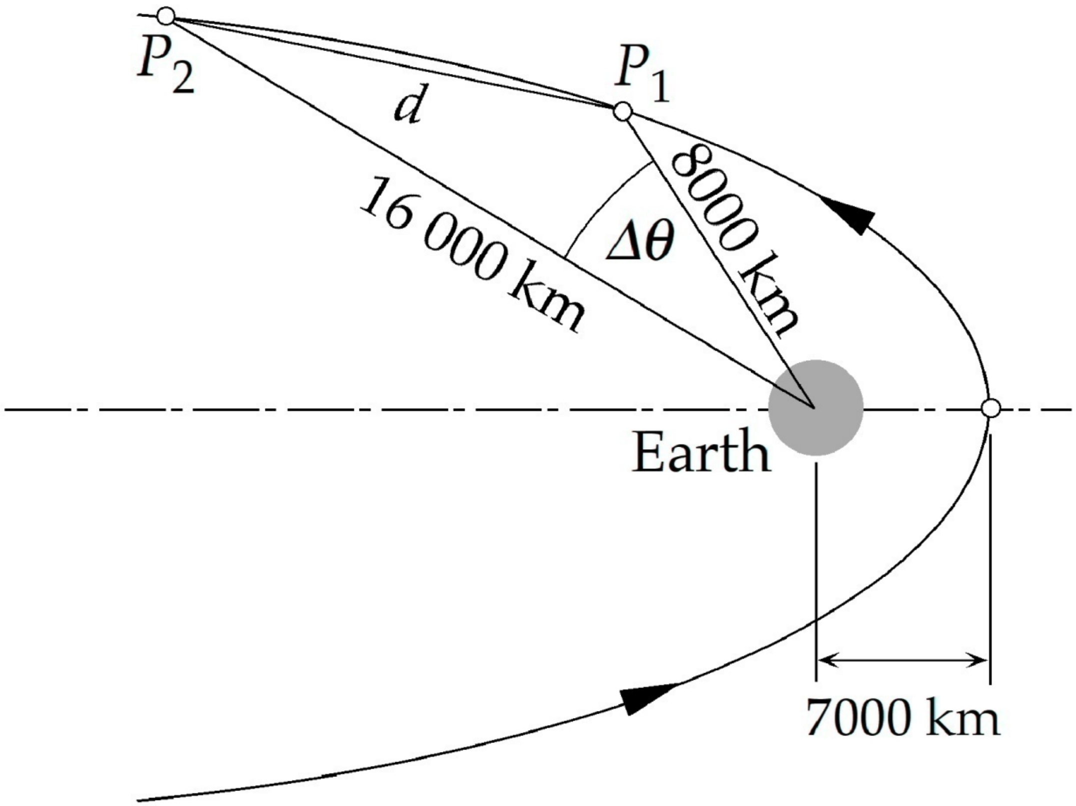
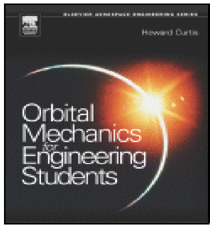


Figure 2.24: Parabolic geocentric trajectory.



2.8 Parabolic Trajectories (e=1)

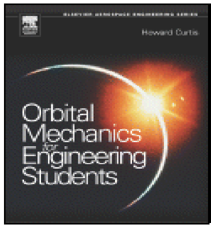
- A parabolic trajectory has the shape of a parabola & is OPEN (i.e. doesn't intersect itself)

- Inserting *e* = 1 into Eqn 2.60, we see that a parabolic path has ZERO total energy; *h* & *p* are finite, but *a* & *T* are infinite

- Some of the formulas from the previous section apply to parabolic trajectories & some do not

- Because a parabolic path has eccentricity of 1, the formulas from the previous section will not contain "*e*" when applied here ("1" has already been inserted for "e"); an example is Eqn 2.45:

$$r = \frac{h^2}{\mu}\left(\frac{1}{1+\cos\theta}\right)$$



2.8 Parabolic Trajectories (e=1)

- Inserting $\varepsilon = 0$ into Eqn 2.60, we see that the speed of an object on a parabolic path is $v = \sqrt{\frac{2\mu}{r}}$; as r → ∞, v → 0

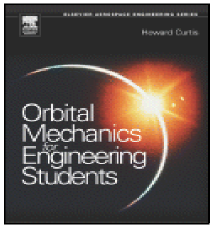
- A parabolic path is called an **escape trajectory;** if an object on a closed orbit (circular or elliptical) can increase its velocity to this value, it will escape the gravitational attraction of the central body; thus we often call the velocity at any point on a parabolic path the **escape velocity** at that point, or "v_{esc}"; comparing with Eqn 2.63, this yields $v_{esc} = \sqrt{2}v_{circ}$

    - 41.4% boost in velocity from a circular orbit

- Inserting $e = 1$ into Eqn 2.52, we see that flight path angle is half the true anomaly (Eqn 2.93)

    - As $r \to \infty$, $\theta \to 180°$, $\gamma \to 90°$, $v \to v_r$

    - Object's motion becomes rectilinear (no longer a curved path)



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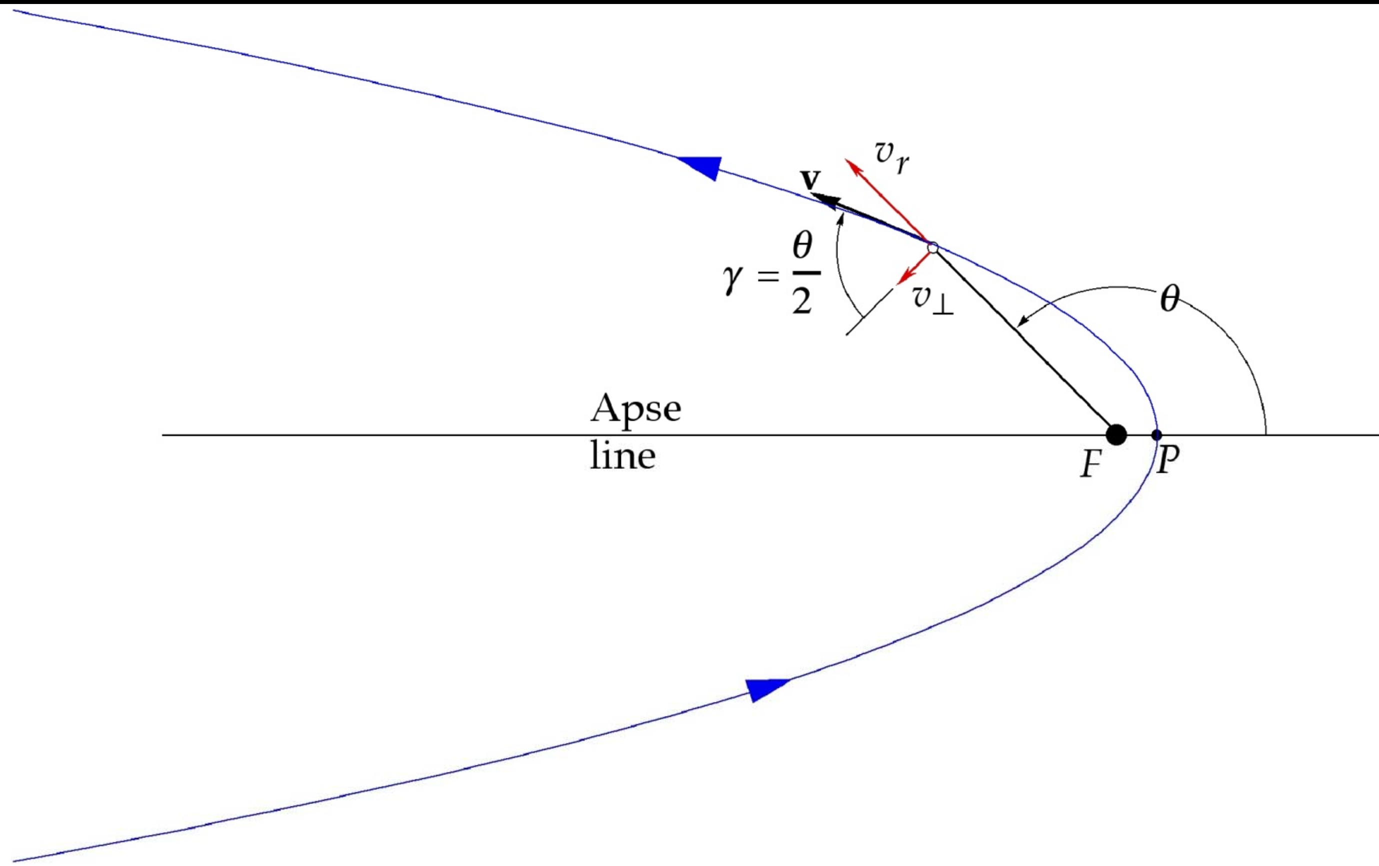

Figure 2.22: Parabolic trajectory around the center of attraction *F*.

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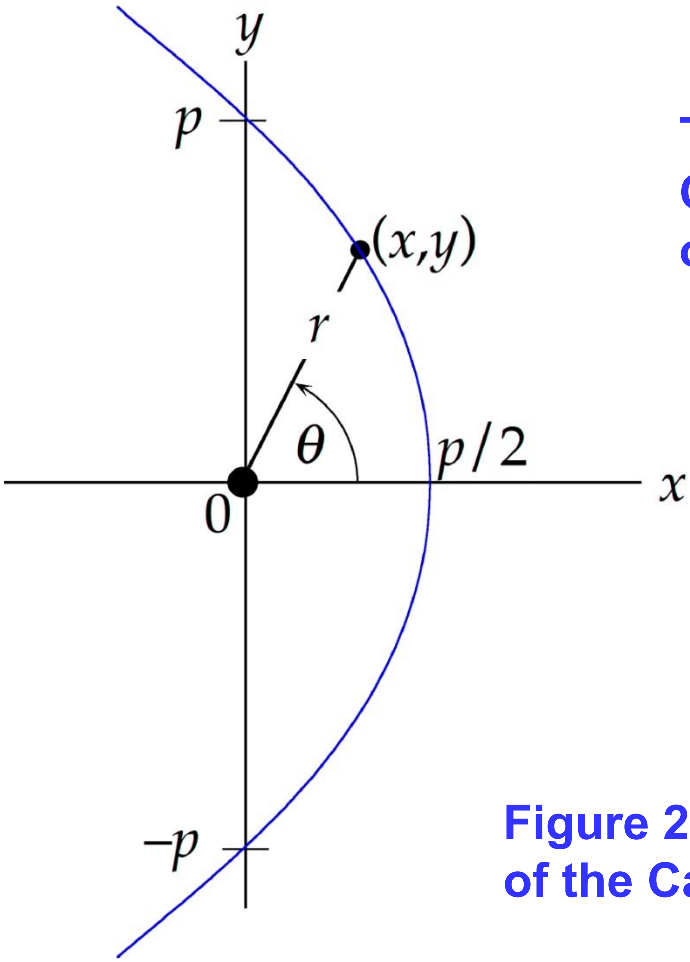

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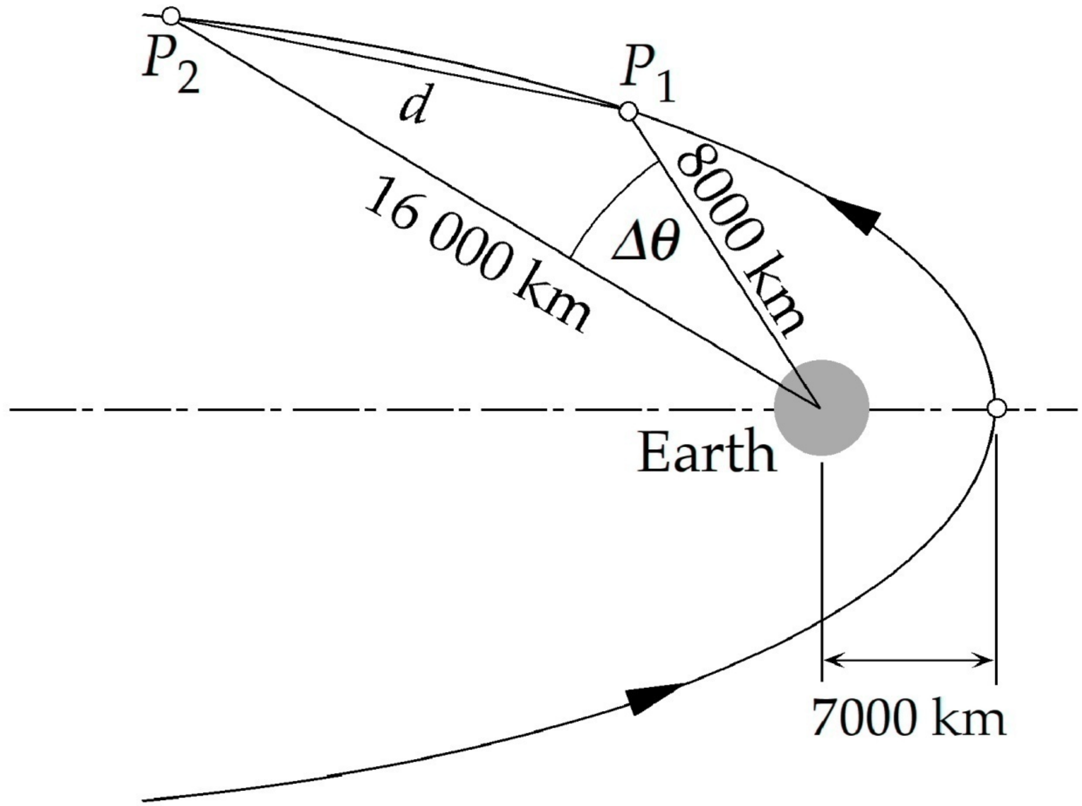

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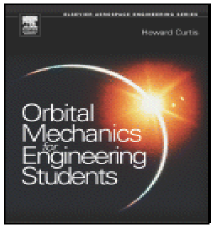
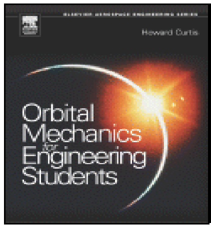


2.9 Hyperbolic Trajectories ($e>1$)

- A hyperbolic trajectory has the shape of a hyperbola & is OPEN; however, hyperbolic trajectories have more "structure" to them than parabolic
- (A parabola can be thought of as a special case of a hyperbola, just as a circle can be thought of as a special case of an ellipse)
- Most of the formulas from the elliptical orbit section apply to hyperbolic trajectories, but many are slightly different in form (sign change, etc)
- A hyperbolic path has POSITIVE total energy; *T* is infinite, most every other qty is finite



2.9 Hyperbolic Trajectories (*e*>1)

Aiming radius: $\Delta = a\sqrt{e^2 - 1}$

Turn angle: $\delta = 2\sin^{-1}(1/e)$

Semiminor axis *b* = aiming radius *Δ*.

True anomaly of the asymptote: $\theta_\infty = \cos^{-1}(-1/e)$

Semimajor axis: $a = \frac{h^2}{\mu}\frac{1}{e^2 - 1}$

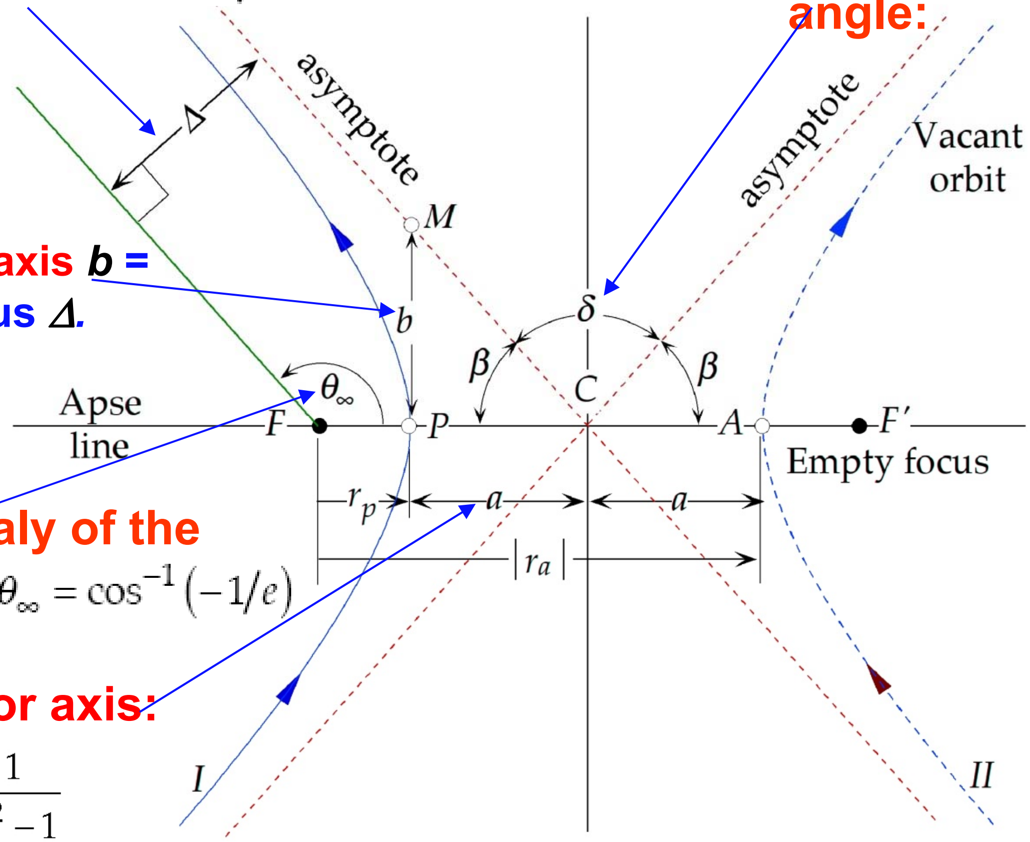
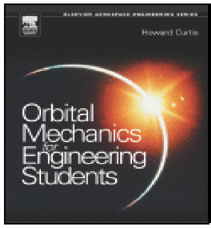


Figure 2.25: Hyperbolic trajectory (on the left)



2.9 Hyperbolic Trajectories ($e > 1$)

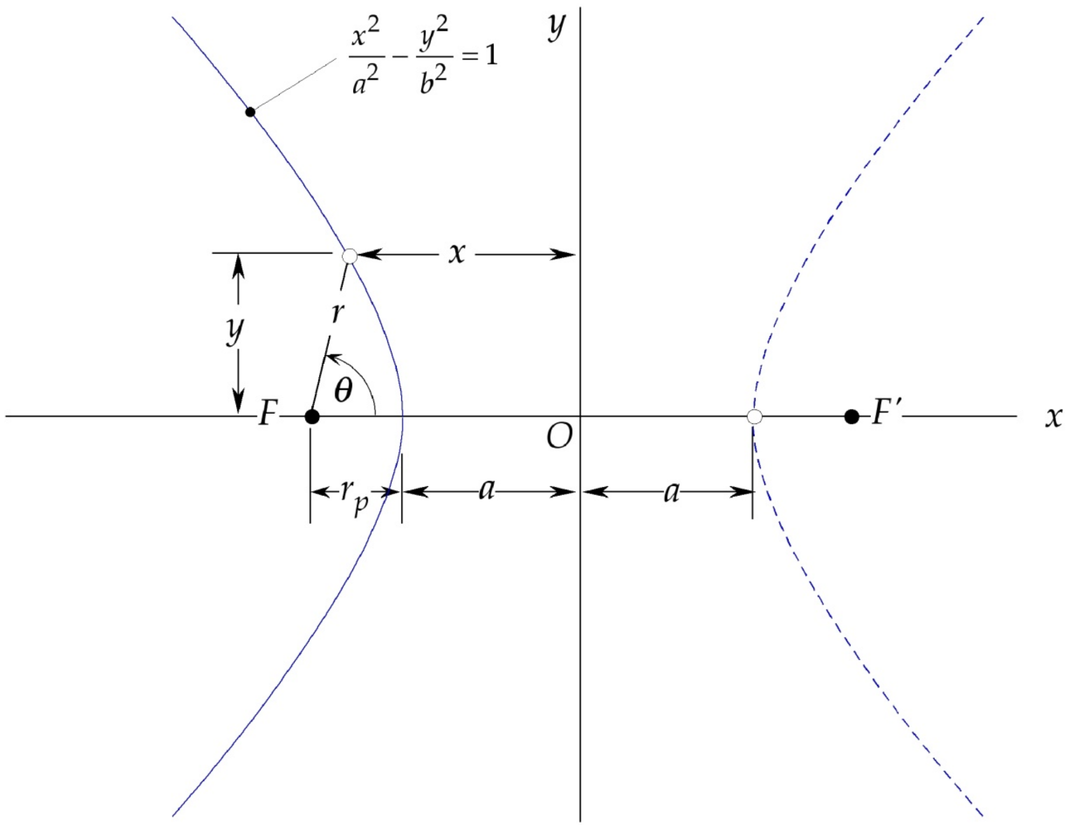
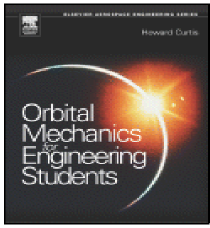


Figure 2.26: Cartesian coordinate description of the hyperbolic trajectory.



2.9 Hyperbolic Trajectories ($e>1$)

The energy equation for a hyperbolic trajectory is

$$\frac{v^2}{2}-\frac{\mu}{r}=\frac{\mu}{2a} \quad \textbf{(positive)}$$

Thus, the speed of an object on a hyperbolic path is $v=\sqrt{\dfrac{2\mu}{r}+\dfrac{\mu}{a}}$

Hyperbolic excess speed is the speed as *r* approaches infinity: $v_\infty=\sqrt{\dfrac{\mu}{a}}$

Therefore, the speed of an object on a hyperbolic path can be stated as:

$$v=\sqrt{v_{esc}{}^2+v_\infty{}^2}$$

Characteristic energy $C_3=v_\infty{}^2$

2.9 Hyperbolic Trajectories ($e > 1$)

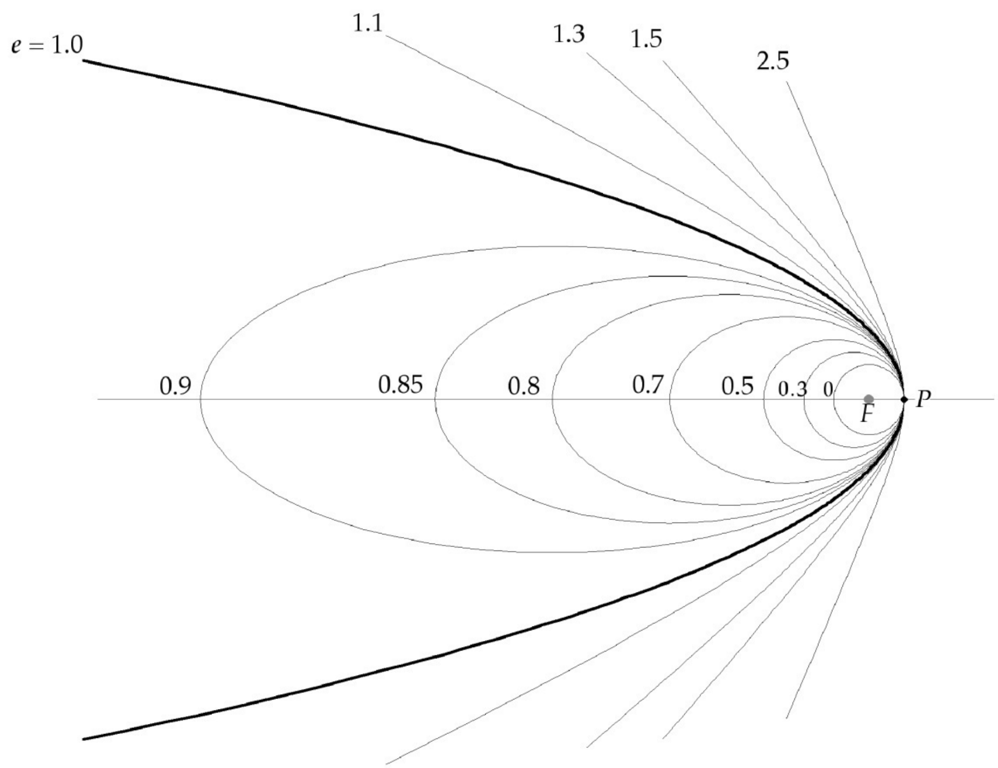
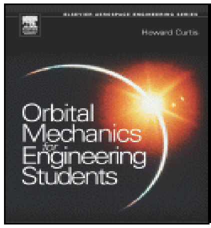


Figure 2.27: Orbits of various eccentricities, having a common focus *F* and periapsis *P*.



Tool Box of Equations Necessary for Solving Two-dimensional Curvilinear Orbital Problems

All orbits

$$h = r v_{\perp}$$

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$$

$$v_r = \frac{\mu}{h} e \sin\theta$$

$$\tan\gamma = \frac{v_r}{v_{\perp}}$$

$$v = \sqrt{{v_r}^2 + {v_{\perp}}^2}$$

Ellipses ($0 \le e < 1$)

$$a = \frac{r_p + r_a}{2} = \frac{h^2}{\mu}\frac{1}{1 - e^2}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$e = \frac{r_a - r_p}{r_a + r_p}$$

Hyperbolas ($e > 1$)

$$\theta_{\infty} = \cos^{-1}\left(-\frac{1}{e}\right)$$

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right)$$

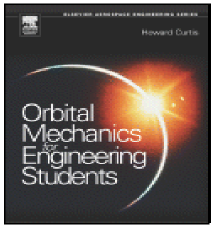
$$a = \frac{h^2}{\mu}\frac{1}{e^2 - 1}$$

$$\Delta = a\sqrt{e^2 - 1}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$$

Parabolas ($e = 1$)

$$\frac{v^2}{2} - \frac{\mu}{r} = 0$$



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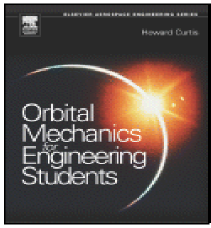

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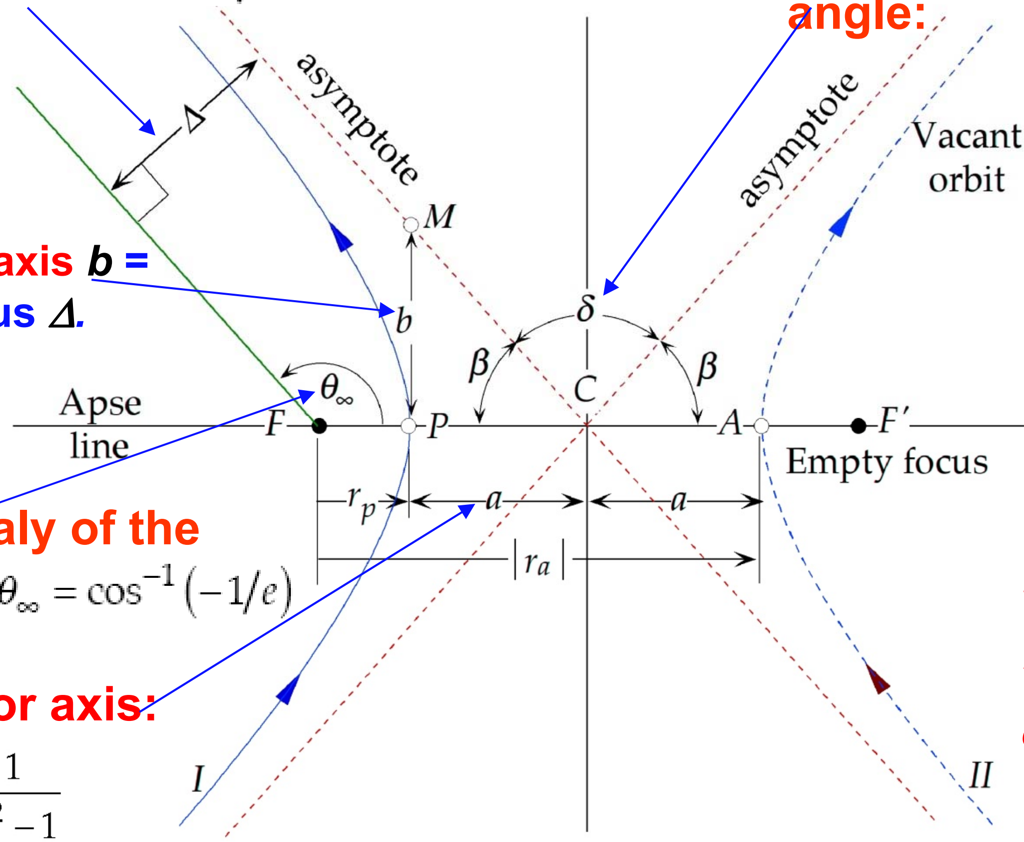


For hyperbolic paths:

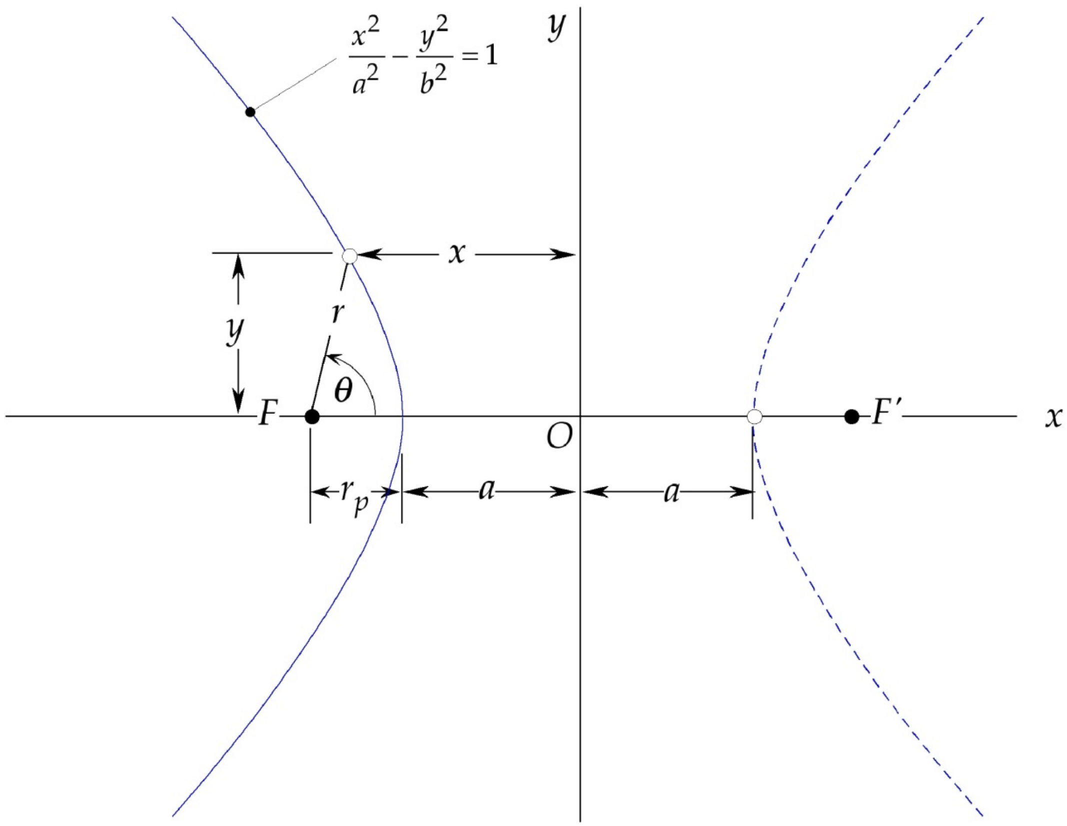
$r_p > 0$

$r_a < 0$

$a = (-r_a - r_p)/2$

Figure 2.25: Hyperbolic trajectory (on the left)

2.9 Hyperbolic Trajectories ($e > 1$)



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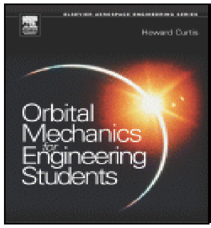

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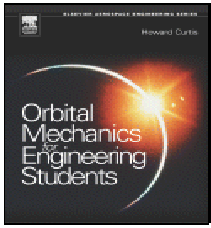
Thus, the speed of an object on a hyperbolic path is $v = \sqrt{\dfrac{2\mu}{r} + \dfrac{\mu}{a}}$

Hyperbolic excess speed is the speed as *r* approaches infinity: $v_\infty = \sqrt{\dfrac{\mu}{a}}$

Therefore, the speed of an object on a hyperbolic path can be stated as:

$$v = \sqrt{{v_{esc}}^2 + {v_\infty}^2}$$

Recall that an object on a parabolic path slows toward zero speed as *r* increases → $v_\infty = 0$



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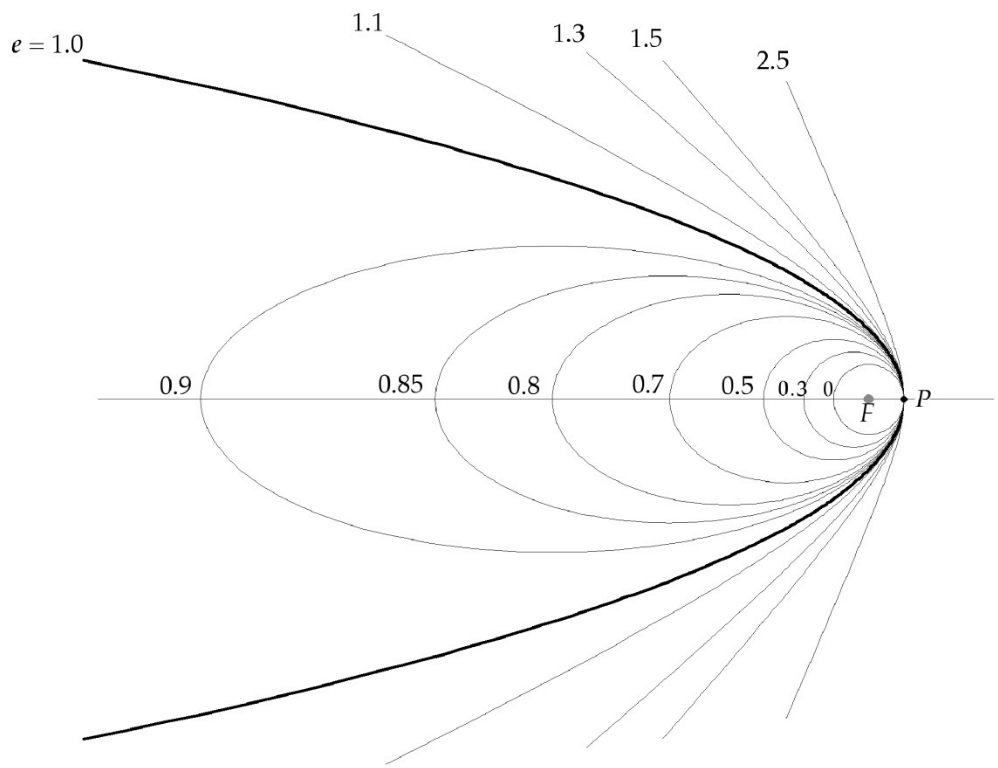
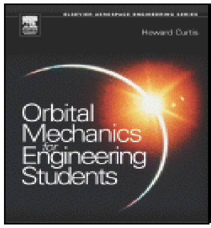


Figure 2.27: Orbits of various eccentricities, having a common focus *F* and periapsis *P*.



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$$v_r = \frac{\mu}{h} e \sin\theta$$

$$\tan\gamma = \frac{v_r}{v_\perp}$$

$$v = \sqrt{{v_r}^2 + {v_\perp}^2}$$

Ellipses ($0 \le e < 1$)

$$a = \frac{r_p + r_a}{2} = \frac{h^2}{\mu}\frac{1}{1 - e^2}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$e = \frac{r_a - r_p}{r_a + r_p}$$

Hyperbolas ($e > 1$)

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right)$$

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right)$$

$$a = \frac{h^2}{\mu}\frac{1}{e^2 - 1}$$

$$\Delta = a\sqrt{e^2 - 1}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$$

Parabolas ($e = 1$)

$$\frac{v^2}{2} - \frac{\mu}{r} = 0$$

2.10 Perifocal Frame

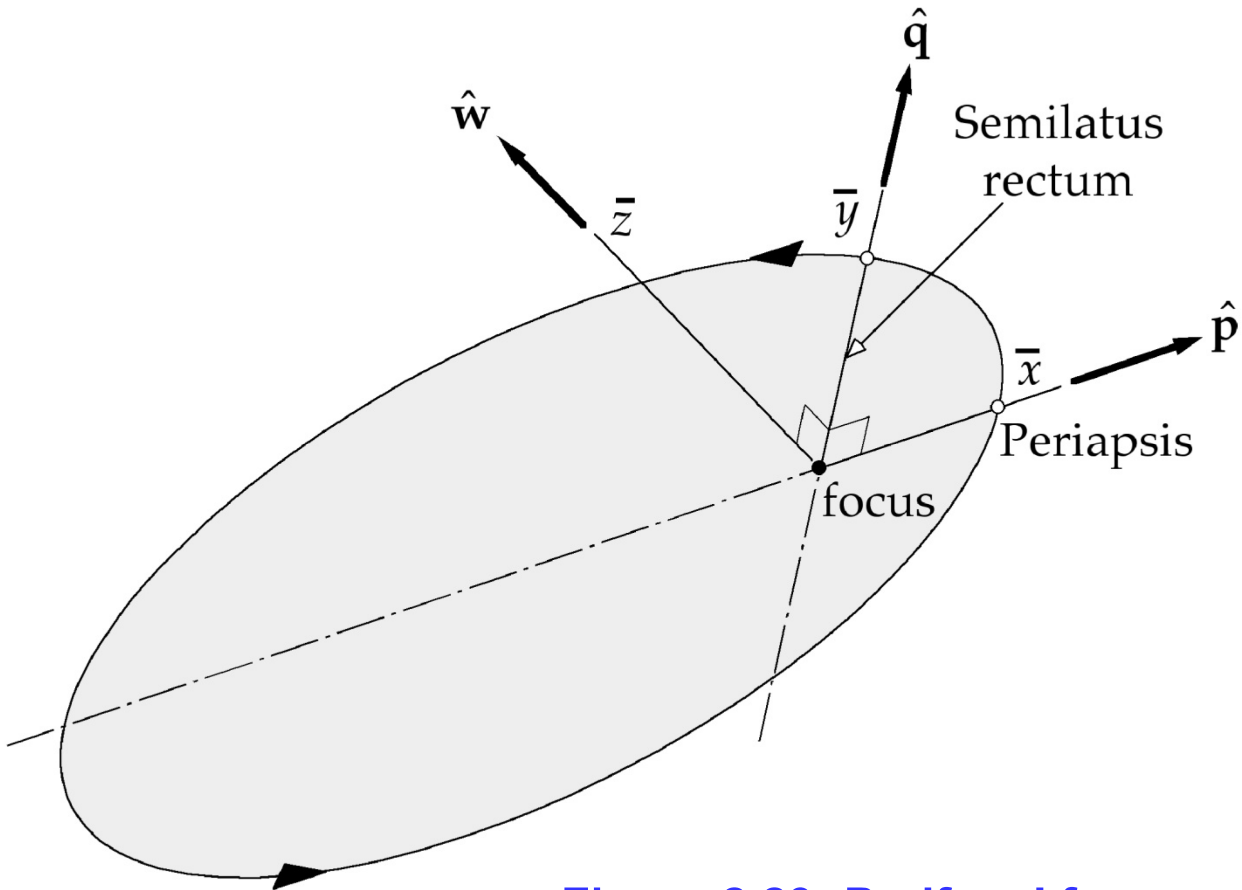
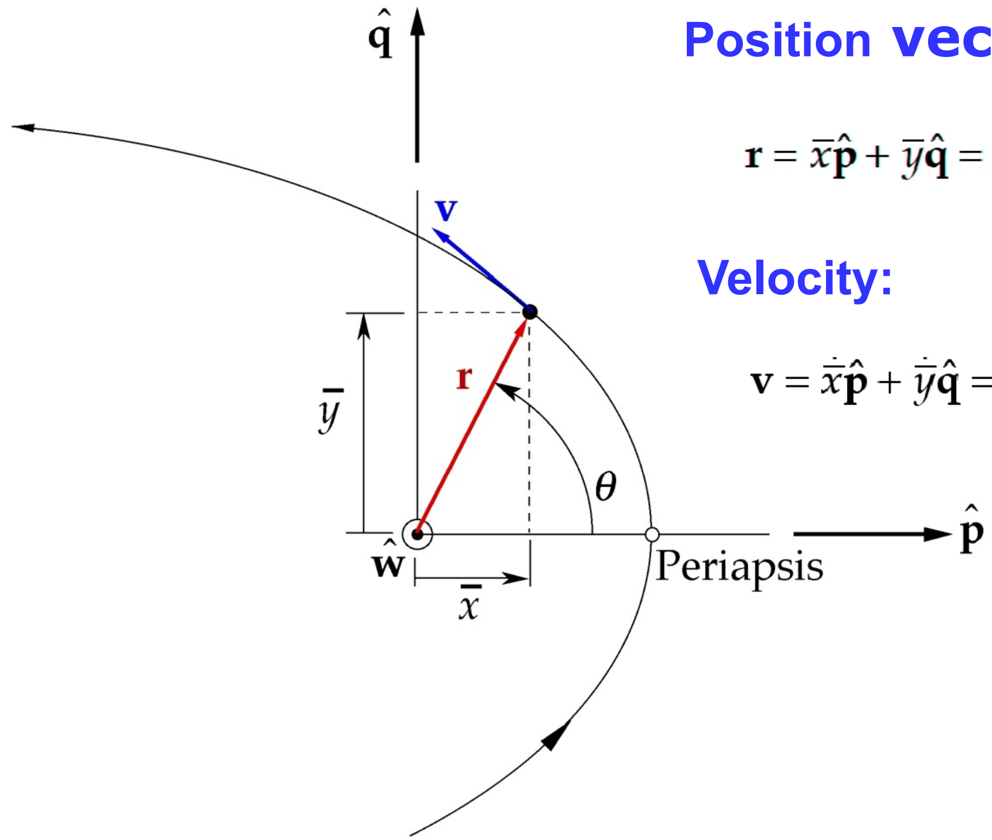


Figure 2.29: Perifocal frame.

2.10 Perifocal Frame



Position vector:

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\left(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}\right)$$

Velocity:

$$\mathbf{v} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} = \frac{\mu}{h}\left[-\sin\theta\hat{\mathbf{p}} + \left(e+\cos\theta\right)\hat{\mathbf{q}}\right]$$

Figure 2.30: Position and velocity relative to the perifocal frame.