ME 596 Spacecraft Attitude Dynamics and Control

Attitude Kinematics

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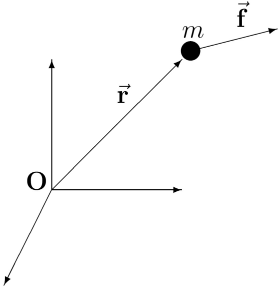
Translational dynamics (Newton's 2nd Law)

— 2nd order includes kinematics and kinetics

$$m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{f}}$$

— Rewriting as 1st order equations separates the two

kinematics
$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{p}}/m$
kinetics
$\dot{\vec{\mathbf{p}}} = \vec{\mathbf{f}}$



In applying Newton's 2nd Law, the reference origin must be inertial, meaning non-accelerating.

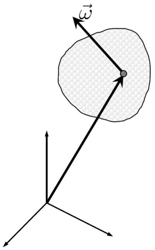
Rotational dynamics (Euler's Law)

— Includes both kinematics and kinetics

$$\dot{\vec{\mathbf{h}}} = \vec{\mathbf{g}}$$

— $\vec{\mathbf{h}}$ is the angular momentum vector

— $\vec{\mathbf{g}}$ is the applied torque vector



— Rewriting as 1st order equations separates kinematics and kinetics

kinematics
To Be Developed
kinetics
$\dot{\boldsymbol{\omega}} = -\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{I}^{-1}\mathbf{g}$

- One of the more difficult elements of modeling rotational motion is the connection between orientation of body and angular momentum
- Rotational kinematics is sufficiently important that we cover it in some detail before discussing rigid body motion

This table relates the kinetic and kinematic relationships for translational and rotational motion.

Linear momentum = mass $\times$ velocity	Angular momentum = inertia $\times$ angular velocity
d/dt (linear momentum) = applied forces	d/dt (angular momentum) = applied torques
d/dt (position) = linear momentum/mass	d/dt (angular momentum) = angular momentum "/" inertia

Back to Reference Frames

Denote frames as triads of mutually orthogonal unit vectors:

ECI	Orbital	Body-Fixed
$\left\{\hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\hat{\mathbf{i}}_3\right\}$	$\{\hat{\mathbf{o}}_1,\hat{\mathbf{o}}_2,\hat{\mathbf{o}}_3\}$	$\left\{\hat{\mathbf{b}}_1,\hat{\mathbf{b}}_2,\hat{\mathbf{b}}_3\right\}$
$\mathcal{F}_i$	$\mathcal{F}_o$	$\mathcal{F}_b$

Orthogonality means unit vectors are perpendicular to each other:

$$\begin{array}{ccc} \hat{\mathbf{i}}_1 \cdot \hat{\mathbf{i}}_1 = 1 & \hat{\mathbf{i}}_1 \cdot \hat{\mathbf{i}}_2 = 0 & \hat{\mathbf{i}}_1 \cdot \hat{\mathbf{i}}_3 = 0 \\ \hat{\mathbf{i}}_2 \cdot \hat{\mathbf{i}}_1 = 0 & \hat{\mathbf{i}}_2 \cdot \hat{\mathbf{i}}_2 = 1 & \hat{\mathbf{i}}_2 \cdot \hat{\mathbf{i}}_3 = 0 \\ \hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_1 = 0 & \hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_2 = 0 & \hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_3 = 1 \end{array}$$

may be written more concisely as

$$\hat{\mathbf{i}}_i \cdot \hat{\mathbf{i}}_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

or even more concisely as

$$\hat{\mathbf{i}}_i \cdot \hat{\mathbf{i}}_j = \delta_{ij} \qquad \text{Kronecker delta}$$

Right-Handedness

The right-handedness of the unit vectors defines the order by:

$$\begin{array}{lll}
\hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_1 = \vec{\mathbf{0}} & \hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_2 = \hat{\mathbf{i}}_3 & \hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_3 = -\hat{\mathbf{i}}_2 \\
\hat{\mathbf{i}}_2 \times \hat{\mathbf{i}}_1 = -\hat{\mathbf{i}}_3 & \hat{\mathbf{i}}_2 \times \hat{\mathbf{i}}_2 = \vec{\mathbf{0}} & \hat{\mathbf{i}}_2 \times \hat{\mathbf{i}}_3 = \hat{\mathbf{i}}_1 \\
\hat{\mathbf{i}}_3 \times \hat{\mathbf{i}}_1 = \hat{\mathbf{i}}_2 & \hat{\mathbf{i}}_3 \times \hat{\mathbf{i}}_2 = -\hat{\mathbf{i}}_1 & \hat{\mathbf{i}}_3 \times \hat{\mathbf{i}}_3 = \vec{\mathbf{0}}
\end{array}$$

This set of rules may be written more concisely as

$$\hat{\mathbf{i}}_i \times \hat{\mathbf{i}}_j = \varepsilon_{ijk} \hat{\mathbf{i}}_k$$

where ε_{ijk} is the *permutation symbol*, defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } i, j, k \text{ an even permutation of 1,2,3} \\ -1 & \text{for } i, j, k \text{ an odd permutation of 1,2,3} \\ 0 & \text{otherwise } (\textit{i.e.}, \text{ if any repetitions occur}) \end{cases}$$

The right-handedness can also be written as

$$\left\{\hat{\mathbf{i}}\right\} \times \left\{\hat{\mathbf{i}}\right\}^{\mathsf{T}} = \left\{ \begin{array}{ccc} \vec{\mathbf{0}} & \hat{\mathbf{i}}_3 & -\hat{\mathbf{i}}_2 \\ -\hat{\mathbf{i}}_3 & \vec{\mathbf{0}} & \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 & -\hat{\mathbf{i}}_1 & \vec{\mathbf{0}} \end{array} \right\} = -\left\{\hat{\mathbf{i}}\right\}^{\times}$$

Vectors

A vector, $\vec{\mathbf{v}}$, is an abstract mathematical object with two properties: length or magnitude, $||\vec{\mathbf{v}}||$, and direction

$$\vec{\mathbf{v}} = v_1\hat{\mathbf{i}}_1 + v_2\hat{\mathbf{i}}_2 + v_3\hat{\mathbf{i}}_3$$

Scalars, v_1, v_2, and v_3, are the *components* of $\vec{\mathbf{v}}$ expressed in $\mathcal{F}_i$.

The components are dot products of $\vec{\mathbf{v}}$ with the base vectors of $\mathcal{F}_i$.

Specifically, $v_1 = \vec{\mathbf{v}} \cdot \hat{\mathbf{i}}_1, \quad v_2 = \vec{\mathbf{v}} \cdot \hat{\mathbf{i}}_2, \quad v_3 = \vec{\mathbf{v}} \cdot \hat{\mathbf{i}}_3$

Since $\hat{\mathbf{i}}$ vectors are unit vectors, components may be written as

$$v_1 = v\cos\alpha_1, \quad v_2 = v\cos\alpha_2, \quad v_3 = v\cos\alpha_3$$

where $v = \|\vec{\mathbf{v}}\|$, α_j is angle between $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}_j$ for $j = 1, 2, 3$.

We frequently collect components of $\vec{\mathbf{v}}$ into a column matrix $\mathbf{v}$:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Vector Notation Summary

Throughout the course, we will work with vectors and with their representations in several reference frames.

- $\vec{\mathbf{v}}$ is a vector, $v = ||\vec{\mathbf{v}}||$ is its magnitude
- $v_1 = v \cos \alpha_1 = \vec{\mathbf{v}} \cdot \hat{\mathbf{i}}_1$ is the "1" component
- $\mathbf{v}$ is a 3×1 matrix of components in unspecified reference frame
- $\mathbf{v}_i$ is a 3×1 matrix of components in $\mathcal{F}_i$
- $\mathbf{v}_b$ is a 3×1 matrix of components in $\mathcal{F}_b$
- $\mathbf{v}_o$ is a 3×1 matrix of components in $\mathcal{F}_o$
- $\vec{\mathbf{v}} = \mathbf{v}_i^{\mathsf{T}} \left\{\hat{\mathbf{i}}\right\} = \mathbf{v}_b^{\mathsf{T}} \left\{\hat{\mathbf{b}}\right\} = \mathbf{v}_o^{\mathsf{T}} \{\hat{\mathbf{o}}\}$

The vector is an abstract mathematical object with direction and magnitude; its matrix representation is a 3×1 matrix of scalars.

These two mathematical objects are not the same thing.

Rotations

- Suppose we know components in body frame, $\mathbf{v}_b$
- We want to know components in inertial frame, $\mathbf{v}_i$

$$\vec{\mathbf{v}} = \mathbf{v}_i^\mathsf{T} \left\{\hat{\mathbf{i}}\right\} = \mathbf{v}_b^\mathsf{T} \left\{\hat{\mathbf{b}}\right\}$$

- Frames are related by a 3×3 Rotation Matrix

$$\left\{\hat{\mathbf{i}}\right\} = \mathbf{R} \left\{\hat{\mathbf{b}}\right\}$$

- We want to find $\mathbf{R}$, so substitute $\mathbf{R}\left\{\hat{\mathbf{b}}\right\}$ for $\left\{\hat{\mathbf{i}}\right\}$
- Substitution leads to $\mathbf{v}_i^\mathsf{T}\mathbf{R}\left\{\hat{\mathbf{b}}\right\} = \mathbf{v}_b^\mathsf{T}\left\{\hat{\mathbf{b}}\right\}$
- Since the $\left\{\hat{\mathbf{b}}\right\}$ vectors are mutually orthogonal,

$$\mathbf{v}_i^\mathsf{T}\mathbf{R} = \mathbf{v}_b^\mathsf{T}$$

- Transpose both sides to obtain $\mathbf{R}^\mathsf{T}\mathbf{v}_i = \mathbf{v}_b$
- So $\mathbf{R}^\mathsf{T}$ transforms vectors from $\mathcal{F}_i$ to $\mathcal{F}_b$

Rotations (continued)

- The equation $\mathbf{R}^{\mathsf{T}}\mathbf{v}_i = \mathbf{v}_b$ is a linear system of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$
- Thus, to determine the components in the inertial frame, we need to determine $\mathbf{R}$ and solve the linear system
- Can write components of $\mathbf{R}$ as

$$\begin{aligned}
\hat{\mathbf{i}}_1 &= R_{11}\hat{\mathbf{b}}_1 + R_{12}\hat{\mathbf{b}}_2 + R_{13}\hat{\mathbf{b}}_3 \\
\hat{\mathbf{i}}_2 &= R_{21}\hat{\mathbf{b}}_1 + R_{22}\hat{\mathbf{b}}_2 + R_{23}\hat{\mathbf{b}}_3 \\
\hat{\mathbf{i}}_3 &= R_{31}\hat{\mathbf{b}}_1 + R_{32}\hat{\mathbf{b}}_2 + R_{33}\hat{\mathbf{b}}_3
\end{aligned}$$

- By inspection, the elements R_{ij} are the “direction cosines” of the unit vectors of $\mathcal{F}_i$ with respect to $\mathcal{F}_b$
- That is, $R_{ij} = \hat{\mathbf{i}}_i \cdot \hat{\mathbf{b}}_j = \cos\alpha_{ij}$
- This result is why Rotation Matrices are also called Direction Cosine Matrices (DCM)

Rotations (continued)

- Can also write the rotation matrix as the dot product of two "*vectrices*":
$$\mathbf{R} = \left\{\hat{\mathbf{i}}\right\} \cdot \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}$$
- Can show that the inverse of a rotation matrix is simply its transpose
$$\mathbf{R}^{-1} = \mathbf{R}^{\mathsf{T}}$$
- Thus a rotation matrix is an *orthonormal* matrix: its rows and columns are components of *mutually orthogonal unit vectors*
- Introduce superscripts bi to denote rotation *from $\mathcal{F}_i$ to $\mathcal{F}_b$*
$$\mathbf{v}_b = \mathbf{R}^{bi}\mathbf{v}_i$$
Think of the "i" superscript on $\mathbf{R}$ as cancelling with the "i" subscript on $\mathbf{v}$
- Similarly $\mathbf{v}_i = \mathbf{R}^{ib}\mathbf{v}_b$, where $\mathbf{R}^{ib} = \mathbf{R}^{bi^{\mathsf{T}}}$

Rotation Notation

As dot product of "vectrices":

$$\mathbf{R}^{bi} = \left\{\hat{\mathbf{b}}\right\} \cdot \left\{\hat{\mathbf{i}}\right\}^{\mathsf{T}} \qquad \mathbf{R}^{ib} = \left\{\hat{\mathbf{i}}\right\} \cdot \left\{\hat{\mathbf{b}}\right\}^{\mathsf{T}}$$

As matrix transforming vectors from one frame to another

$$\mathbf{v}_b = \mathbf{R}^{bi}\mathbf{v}_i \qquad \mathbf{v}_i = \mathbf{R}^{ib}\mathbf{v}_b$$

As matrix with rows and columns being unit vectors of one frame expressed in the other

$$\mathbf{R}^{ib} = \begin{bmatrix} \hat{\mathbf{i}}_{1b}^{\mathsf{T}} \\ \hat{\mathbf{i}}_{2b}^{\mathsf{T}} \\ \hat{\mathbf{i}}_{3b}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{b}}_{1i}^{\mathsf{T}} & \hat{\mathbf{b}}_{2i}^{\mathsf{T}} & \hat{\mathbf{b}}_{3i}^{\mathsf{T}} \end{bmatrix}$$

The matrix $\mathbf{R}^{bi}$ is the rotation from $\mathcal{F}_i$ *to* $\mathcal{F}_b$

An Example

The reference frame $\mathcal{F}_b$ has the following unit vectors in $\mathcal{F}_i$. What are the rotation matrices $\mathbf{R}^{bi}$ and $\mathbf{R}^{ib}$?

$$
\begin{aligned}
\hat{\mathbf{b}}_1 &= 0.8218\,\hat{\mathbf{i}}_1 - 0.4063\,\hat{\mathbf{i}}_2 - 0.3994\,\hat{\mathbf{i}}_3 \\
\hat{\mathbf{b}}_2 &= 0.2490\,\hat{\mathbf{i}}_1 + 0.8867\,\hat{\mathbf{i}}_2 - 0.3896\,\hat{\mathbf{i}}_3 \\
\hat{\mathbf{b}}_3 &= 0.5125\,\hat{\mathbf{i}}_1 + 0.2207\,\hat{\mathbf{i}}_2 + 0.8299\,\hat{\mathbf{i}}_3
\end{aligned}
$$

The direction cosines can be identified by inspection. For $\mathbf{R}^{bi}$, the components of the $\hat{\mathbf{b}}$ vectors in $\mathcal{F}_i$ go in the rows.

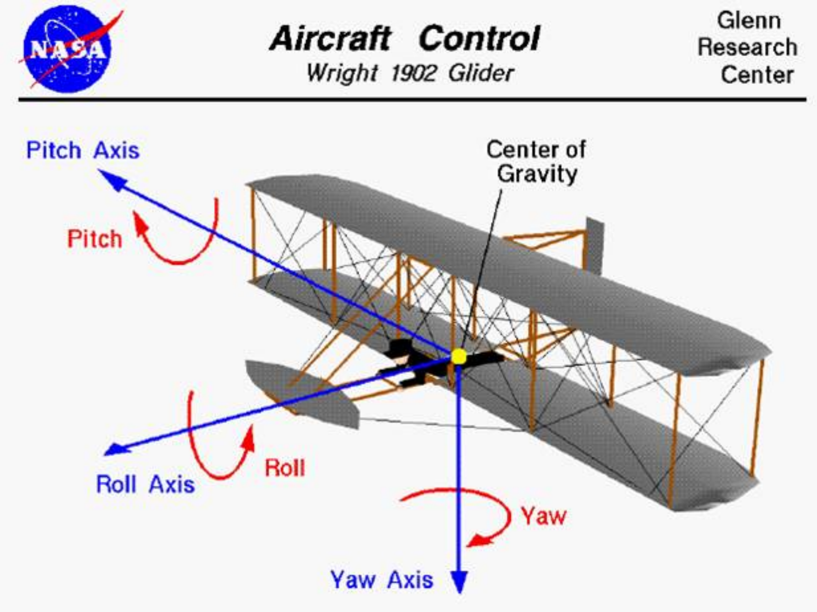
$$
\mathbf{R}^{bi} = \begin{bmatrix} 0.8218 & -0.4063 & -0.3994 \\ 0.2490 & 0.8867 & -0.3896 \\ 0.5125 & 0.2207 & 0.8299 \end{bmatrix}
$$

For $\mathbf{R}^{ib}$, the components of the $\hat{\mathbf{b}}$ vectors in $\mathcal{F}_i$ go in the columns. That is, $\mathbf{R}^{ib} = {\mathbf{R}^{bi}}^{\mathsf{T}}$

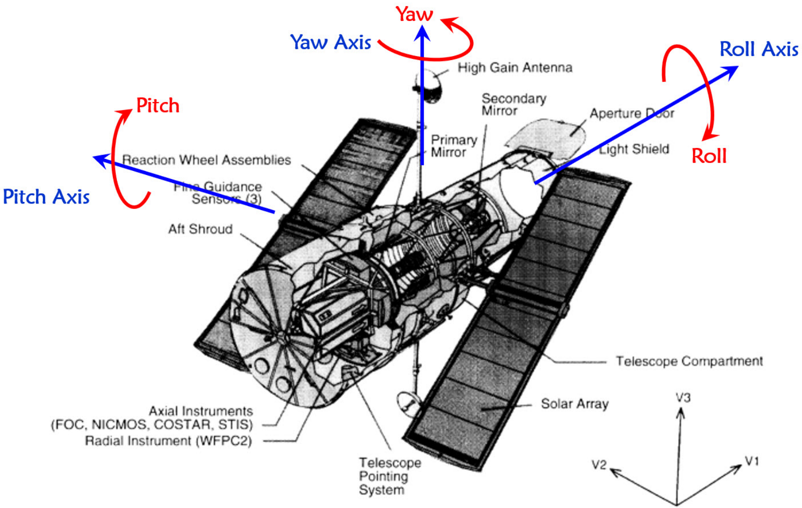
$$
\mathbf{R}^{ib} = \begin{bmatrix} 0.8218 & 0.2490 & 0.5125 \\ -0.4063 & 0.8867 & 0.2207 \\ -0.3994 & -0.3896 & 0.8299 \end{bmatrix}
$$

- The rotation matrix represents the attitude
- A rotation matrix has 9 numbers, but they are NOT independent
- There are 6 constraints on the 9 elements of a rotation matrix (what are they?)
- Thus rotation has 3 degrees of freedom
- There are many different sets of parameters that can be used to represent or parameterize rotations
- Euler angles, Euler parameters (aka quaternions), Rodrigues parameters (aka Gibbs vectors), Modified Rodrigues parameters,

Aircraft Roll, Pitch, and Yaw



Spacecraft Roll, Pitch, and Yaw



Note: RPY Axes can vary significantly from spacecraft to spacecraft, depending on specifics of a spacecraft's mission.

Ship Roll, Pitch, and Yaw

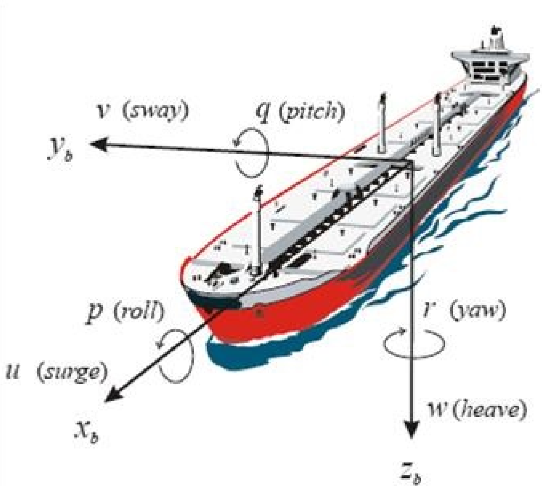


Illustration from on-line notes of T. I. Fossen, author of *Guidance and Control of Ocean Vehicles*, Wiley, 1994

Euler Angles

Leonhard Euler (1707-1783) reasoned that rotation from one frame to another can be visualized as a sequence of three simple rotations about base vectors

Each rotation is through an angle (Euler angle) about a specified axis

Thus a simple rotation is identified by the angle θ and the axis of rotation

Let's consider the rotation from $\mathcal{F}_i$ to $\mathcal{F}_b$ using three Euler angles $\theta_1, \theta_2, \theta_3$

We can make the first rotation about any of the three $\hat{\mathbf{i}}$ axes; choose $\hat{\mathbf{i}}_3$

The first rotation about the $\hat{\mathbf{i}}_3$ axis is through angle θ_1

The resulting intermediate frame is denoted $\mathcal{F}_{i'}$ or $\left\{\hat{\mathbf{i}}'\right\}$

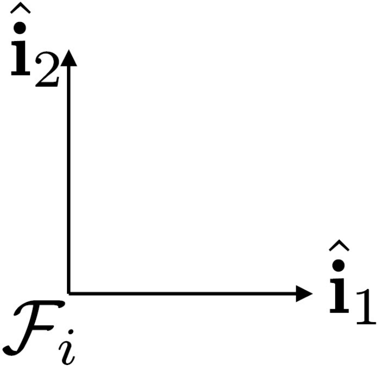
The rotation matrix from $\mathcal{F}_i$ to $\mathcal{F}_{i'}$ is denoted

$$\mathbf{R}_3(\theta_1) = \mathbf{R}^{i'i} \Rightarrow \mathbf{v}_{i'} = \mathbf{R}^{i'i}\mathbf{v}_i$$

In the next few slides, we continue the development of $\mathbf{R}^{bi}$

$\mathcal{F}_i$ looking down the $\hat{\mathbf{i}}_3$ axis

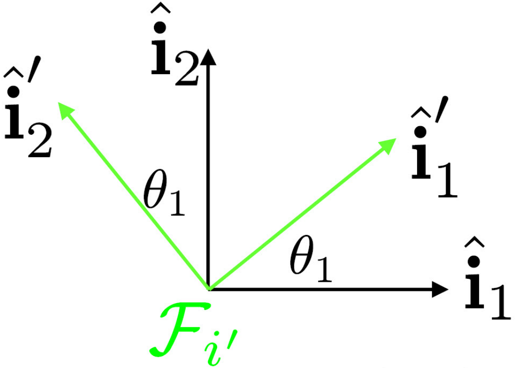
The first rotation is $\mathbf{R}_3(\theta_1)$: subscript 3 denotes rotation about the "3" axis, and subscript 1 on θ denotes that this is the first rotation in the three-Euler Angle rotation sequence.



$\hat{\mathbf{i}}_3$ is out of the page $(\hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_2 = \hat{\mathbf{i}}_3)$

Rotation from $\mathcal{F}_i$ to $\mathcal{F}_{i'}$

Remember the right-hand rule: counterclockwise rotation about $\hat{\mathbf{i}}_3$.

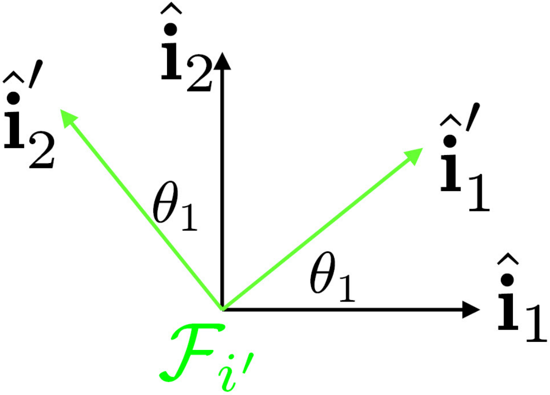


$\hat{\mathbf{i}}_3$ is out of the page ($\hat{\mathbf{i}}_1 \times \hat{\mathbf{i}}_2 = \hat{\mathbf{i}}_3$)

$\hat{\mathbf{i}}_3'$ is out of the page ($\hat{\mathbf{i}}_1' \times \hat{\mathbf{i}}_2' = \hat{\mathbf{i}}_3'$)

Rotation from $\mathcal{F}_i$ to $\mathcal{F}_{i'}$ Direction Cosines

By inspection, we can write down the $\mathcal{F}_{i'}$ vectors in terms of θ_1 and the $\mathcal{F}_i$ vectors, noting that $\hat{\mathbf{i}}_3' = \hat{\mathbf{i}}_3$



$$\hat{\mathbf{i}}_1' = \cos\theta_1 \hat{\mathbf{i}}_1 + \sin\theta_1 \hat{\mathbf{i}}_2$$
$$\hat{\mathbf{i}}_2' = -\sin\theta_1 \hat{\mathbf{i}}_1 + \cos\theta_1 \hat{\mathbf{i}}_2$$

Rotation from $\mathcal{F}_i$ to $\mathcal{F}_{i'}$ Rotation Matrix

Since $\hat{\mathbf{i}}_3$ and $\hat{\mathbf{i}}'_3$ are in the same direction, we can easily write the rotation matrix $\mathbf{R}_3(\theta_1) = \mathbf{R}^{i'i}$

$$\begin{aligned}
\hat{\mathbf{i}}'_1 &= \cos\theta_1 \hat{\mathbf{i}}_1 + \sin\theta_1 \hat{\mathbf{i}}_2 \\
\hat{\mathbf{i}}'_2 &= -\sin\theta_1 \hat{\mathbf{i}}_1 + \cos\theta_1 \hat{\mathbf{i}}_2 \\
\hat{\mathbf{i}}'_3 &= \hat{\mathbf{i}}_3
\end{aligned}$$

$$\left\{ \begin{array}{c} \hat{\mathbf{i}}'_1 \\ \hat{\mathbf{i}}'_2 \\ \hat{\mathbf{i}}'_3 \end{array} \right\} = \left[\begin{array}{ccc} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{array} \right\}$$

$$\left\{\hat{\mathbf{i}}'\right\} = \mathbf{R}_3(\theta_1) \left\{\hat{\mathbf{i}}\right\}$$

Note that the minus sign in $\mathbf{R}_3(\theta_1)$ is in the row above the row with a 1 in it.

Next we have to perform two more rotations to get from $\mathcal{F}_{i'}$ to $\mathcal{F}_b$

Second Rotation: $\mathcal{F}_{i'}$ to $\mathcal{F}_{i''}$

Second rotation can be about either "1" or "2" axis

Choose "2" axis, $\hat{\mathbf{i}}_2'$, through θ_2

Denote resulting intermediate frame as $\mathcal{F}_{i''}$ or $\left\{\hat{\mathbf{i}}''\right\}$

Rotation matrix is

$$\mathbf{R}_2(\theta_2) = \mathbf{R}^{i''i'} \Rightarrow \mathbf{v}_{i''} = \mathbf{R}^{i''i'}\mathbf{v}_{i'}$$

Note well: Rotation matrix notation for "simple"rotations is $\mathbf{R}_i(\theta_j)$, where subscript i denotes i_{th} axis and subscript j denotes j_{th} rotation in three-Euler Angle sequence

As with first rotation, we can write down $\mathbf{R}_2(\theta_2)$ by inspection

$$\mathbf{R}_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

Note that the minus sign in $\mathbf{R}_2(\theta_2)$ is in the row above the row with a 1 in it.

Now we can rotate from $\mathcal{F}_i$ to $\mathcal{F}_{i''}$

$$\mathbf{v}_{i''} = \mathbf{R}_2(\theta_2)\mathbf{v}_{i'} = \mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)\mathbf{v}_i$$

Keep in mind that we are building $\mathbf{R}^{bi}$.

Third Rotation: $\mathcal{F}_{i''}$ to $\mathcal{F}_b$

Third rotation can be about either "1" or "3" axis

Choose "1" axis, $\hat{\mathbf{i}}_1''$, through θ_3

Denote resulting "final" frame as $\mathcal{F}_b$ or $\left\{\hat{\mathbf{b}}\right\}$

Rotation matrix is

$$\mathbf{R}_1(\theta_3) = \mathbf{R}^{bi''} \Rightarrow \mathbf{v}_b = \mathbf{R}^{bi''}\mathbf{v}_{i''}$$

By inspection, the rotation matrix is

$$\mathbf{R}_1(\theta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix}$$

Note that minus sign in $\mathbf{R}_1(\theta_3)$ is in bottom row, which should be viewed as being "above" row with 1.

Note also that minus sign in all cases is on `sine`.

The pattern is:
1) 1 on diagonal corresponding to axis;
2) 0 on off-diagonal elements of row and column with 1;
3) `cosines` on remaining diagonal elements;
4) `sines` on remaining off-diagonal elements;
5) minus sign on `sine` in row above 1.

$$\boxed{\mathbf{v}_b} = \mathbf{R}_1(\theta_3)\mathbf{v}_{i''} = \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)\mathbf{v}_i = \boxed{\mathbf{R}^{bi}\mathbf{v}_i}$$

An Example

Reference frame $\mathcal{F}_a$ is transformed into reference frame $\mathcal{F}_b$ by performing a 3-1-2 Euler rotation sequence through the angles θ_1, θ_2, and θ_3.

a) Determine the direction cosine matrix $\mathbf{R}^{ba}$ (ie, from $\mathcal{F}_a$ to $\mathcal{F}_b$).

b) Derive expressions that allow you to determine θ_i for $\mathbf{R}^{ba}$.

---

a) The direction cosine matrix $\mathbf{R}^{ba}$ is

$$
\begin{aligned}
\mathbf{R}^{ba} &= \mathbf{R}_2(\theta_3)\mathbf{R}_1(\theta_2)\mathbf{R}_3(\theta_1) \\
&= \begin{bmatrix} \cos\theta_3 & 0 & -\sin\theta_3 \\ 0 & 1 & 0 \\ \sin\theta_3 & 0 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_3 c\theta_1 - s\theta_3 s\theta_2 s\theta_1 & c\theta_3 s\theta_1 + s\theta_3 s\theta_2 c\theta_1 & -s\theta_3 c\theta_2 \\ -c\theta_2 s\theta_1 & c\theta_2 c\theta_1 & s\theta_2 \\ s\theta_3 c\theta_1 + c\theta_3 s\theta_2 s\theta_1 & s\theta_3 s\theta_1 - c\theta_3 s\theta_2 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix}
\end{aligned}
$$

where we have used the shorthand $c = \cos$ and $s = \sin$.

Note that the order of simple rotation matrices in an Euler angle sequence is from left to right:

Third rotation $\times$ Second rotation $\times$ First rotation

An Example (continued): Rotation Matrix to Euler Angles

Part b) is about how we "extract" the Euler angles from a given rotation matrix

The primary reason for going through this exercise is to illustrate how tedious it is, as compared with other approaches that we cover next.

b) Denoting ij element (row i, column j) of $\mathbf{R}^{ba}$ as R_{ij}, we can determine θ_i as

$$\begin{aligned}\theta_1 &= \tan^{-1}\left(-\frac{R_{21}}{R_{22}}\right)\\ \theta_2 &= \sin^{-1}\left(R_{23}\right)\\ \theta_3 &= \tan^{-1}\left(-\frac{R_{13}}{R_{33}}\right)\end{aligned}$$

Quadrant checks are imperative. Use `atan2(y,x)`.

An Example (continued): Some numbers

For $\theta_1 = 30°$, $\theta_2 = -60°$, $\theta_3 = 45°$, we have

$$\mathbf{R}^{ba} = \mathbf{R}_2(\theta_3)\mathbf{R}_1(\theta_2)\mathbf{R}_3(\theta_1) = \begin{bmatrix} 0.9186 & -0.1768 & -0.3536 \\ -0.2500 & 0.4330 & -0.8660 \\ 0.3062 & 0.8839 & 0.3536 \end{bmatrix}$$

Using the equations for backing out θ_i, we obtain directly

$$\theta_1 = 30°, \theta_2 = -60°, \text{ and } \theta_3 = 45°$$

```
% Matlab code to do calculations
% note that R1, R2, and R3 are functions that return the
% appropriate 3\times3 rotation matrix (easy to write!)
Rba=R2(45*pi/180)*R1(-60*pi/180)*R3(30*pi/180)
th1=atan2(-Rba(2,1),Rba(2,2))*180/pi
th2=asin(Rba(2,3))*180/pi
th3=atan2(-Rba(1,3),Rba(3,3))*180/pi
```

Example of the rotation matrix functions:

```
% R = R1(angle)
% returns elementary "1" rotation matrix
function Rotmat = R1(angle)
c = cos(angle); s = sin(angle);
Rotmat = [ 1 0 0; 0 c s; 0 -s c ];
```

An Example (continued): Different numbers

For $\theta_1 = 30^\circ$, $\theta_2 = 120^\circ$, $\theta_3 = 45^\circ$, we have

$$\mathbf{R}^{ba} = \mathbf{R}_2(\theta_3)\mathbf{R}_1(\theta_2)\mathbf{R}_3(\theta_1) = \begin{bmatrix} 0.3062 & 0.8839 & 0.3536 \\ 0.2500 & -0.4330 & 0.8660 \\ 0.9186 & -0.1768 & -0.3536 \end{bmatrix}$$

Using the equations for backing out θ_i, we obtain directly

$$\theta_1 = -150^\circ, \theta_2 = 60^\circ, \text{ and } \theta_3 = -135^\circ$$

Note that these three angles are NOT the same as the three angles we used to compute $\mathbf{R}^{ba}$

Check by calculating $\mathbf{R}_2(\theta_3)\mathbf{R}_1(\theta_2)\mathbf{R}_3(\theta_1)$ using these three angles, and you will find that you obtain the same $\mathbf{R}^{ba}$

What, if anything, went wrong?

Conclusion: Euler angles extracted in this way are “correct,” but not unique.

How many Euler angle sequences are possible?

- We have considered a couple of different Euler angle sequences
- Answering the question posed above:
  — Three choices for first rotation
  — Two choices for second rotation, because we cannot rotate about the same axis
  — Two choices for third rotation, because we can rotate about the same "number" axis as for the first rotation since it is not really the same axis
  — **Thus, there are $3 \times 2 \times 2 = 12$ possible Euler angle sequences**:
    - 1-2-3, 1-3-2, 2-1-3, 2-3-1, 3-1-2, 3-2-1 ("asymmetric")
    - 1-2-1, 1-3-1, 2-1-2, 2-3-2, 3-1-3, 3-2-3 ("symmetric")
- The asymmetric set uses all three possible simple rotations
- The symmetric set uses the same rotation for the 3rd rotation as for the 1st

You need to be able to derive $\mathbf{R}$ *for any of 12 different sequences.*

Roll, Pitch, and Yaw

Roll, pitch and yaw are Euler angles and are sometimes defined as a 3-2-1 sequence and sometimes defined as a 1-2-3 sequence
What's the difference?

The 3-2-1 sequence (we did earlier) leads to

$$\mathbf{R}^{bi} = \begin{bmatrix} \mathsf{c}\theta_1\mathsf{c}\theta_2 & \mathsf{s}\theta_1\mathsf{c}\theta_2 & -\mathsf{s}\theta_2 \\ -\mathsf{c}\theta_3\mathsf{s}\theta_1 + \mathsf{c}\theta_1\mathsf{s}\theta_2\mathsf{s}\theta_3 & \mathsf{c}\theta_1\mathsf{c}\theta_3 + \mathsf{s}\theta_1\mathsf{s}\theta_2\mathsf{s}\theta_3 & \mathsf{c}\theta_2\mathsf{s}\theta_3 \\ \mathsf{c}\theta_1\mathsf{s}\theta_2\mathsf{c}\theta_3 + \mathsf{s}\theta_1\mathsf{s}\theta_3 & \mathsf{s}\theta_1\mathsf{s}\theta_2\mathsf{c}\theta_3 - \mathsf{c}\theta_1\mathsf{s}\theta_3 & \mathsf{c}\theta_2\mathsf{c}\theta_3 \end{bmatrix}$$

where θ_1 is yaw angle, θ_2 is pitch angle, and θ_3 is roll angle

The 1-2-3 sequence leads to

$$\mathbf{R}^{bi} = \begin{bmatrix} \mathsf{c}\theta_2\mathsf{c}\theta_3 & \mathsf{s}\theta_1\mathsf{s}\theta_2\mathsf{c}\theta_3 + \mathsf{c}\theta_1\mathsf{s}\theta_3 & \mathsf{s}\theta_1\mathsf{s}\theta_3 - \mathsf{c}\theta_1\mathsf{s}\theta_2\mathsf{c}\theta_3 \\ \mathsf{c}\theta_2\mathsf{s}\theta_3 & \mathsf{c}\theta_1\mathsf{c}\theta_3 - \mathsf{s}\theta_1\mathsf{s}\theta_2\mathsf{s}\theta_3 & \mathsf{s}\theta_1\mathsf{c}\theta_3 + \mathsf{c}\theta_1\mathsf{s}\theta_2\mathsf{s}\theta_3 \\ \mathsf{s}\theta_2 & -\mathsf{s}\theta_1\mathsf{c}\theta_2 & \mathsf{c}\theta_1\mathsf{c}\theta_2 \end{bmatrix}$$

where θ_1 is roll angle, θ_2 is pitch angle, and θ_3 is yaw angle

Roll, Pitch, and Yaw (continued)

Note that the two matrices differ: **Rotations do not commute**

However, if we assume that angles are small (appropriate for many vehicle dynamics problems), then approximations of the two matrices are equal

3-2-1 Sequence

$$\cos\theta \approx 1 \text{ and } \sin\theta \approx \theta$$

$$\Rightarrow \mathbf{R}^{bi} \approx \begin{bmatrix} 1 & \theta_1 & -\theta_2 \\ -\theta_1 & 1 & \theta_3 \\ \theta_2 & -\theta_3 & 1 \end{bmatrix}$$

θ_1 is yaw, θ_2 is pitch, and θ_3 is roll

1-2-3 Sequence

$$\cos\theta \approx 1 \text{ and } \sin\theta \approx \theta$$

$$\Rightarrow \mathbf{R}^{bi} \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}$$

θ_1 is roll, θ_2 is pitch, and θ_3 is yaw

$$\mathbf{R}^{bi} \approx \mathbf{1} - \begin{bmatrix} \theta_3 \\ \theta_2 \\ \theta_1 \end{bmatrix}^{\times} \Rightarrow \boxed{\mathbf{R}^{bi} \approx \mathbf{1} - \begin{bmatrix} \text{roll} \\ \text{pitch} \\ \text{yaw} \end{bmatrix}^{\times}} \Leftarrow \quad \mathbf{R}^{bi} \approx \mathbf{1} - \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}^{\times}$$

Thus, roll, pitch, and yaw are the same for 1-2-3 and 3-2-1 sequences, *if the angles are small and a linear approximation is valid.*

What You Need To Know About Euler Angles

- Given a sequence, say "3-1-2" for example, derive the Euler angle representation of $\mathbf{R}$

  — Be sure to get the order correct

- Given a sequence and some values for the angles, compute the numerical values of $\mathbf{R}$

  — Be sure to know the difference between degrees and radians

- Given the numerical values of $\mathbf{R}$, extract numerical values of the Euler angles associated with a specified sequence

  — Be sure to make appropriate quadrant checks, and to check your answer

Euler's Theorem

The complexity of using many trig and inverse trig functions leads us to **Euler's Theorem**:

The most general motion of a rigid body with a fixed point is a rotation about a fixed axis.

The axis, denoted $\mathbf{a}$, is called the **eigenaxis** or **Euler axis**.

The angle of rotation, Φ is called the **Euler angle** or the **principal Euler angle**.

Note that simple rotations used in developing Euler angles satisfy Euler's Theorem.

$\mathbf{R}^{bi}$ is a rotation matrix, so has Euler axis $\mathbf{a}$ and Euler angle Φ.

$$\mathbf{R}^{bi}\mathbf{a} = \mathbf{a} \qquad \mathbf{R}^{bi} = \cos\Phi\mathbf{1} + (1 - \cos\Phi)\mathbf{a}\mathbf{a}^{\mathsf{T}} - \sin\Phi\mathbf{a}^{\times}$$

($\mathbf{1}$ is the identity matrix)

Observations Regarding $\mathbf{a}$ and Φ

Since the rotation is about $\mathbf{a}$,

$$\mathbf{R}^{bi}\mathbf{a} = \mathbf{a}$$

from which we can deduce that $\mathbf{a}$ is an eigenvector of the rotation matrix, and that the associated eigenvalue is $+1$.

Thus every rotation matrix has an eigenvalue equal to $+1$.

This fact justifies the term **eigenaxis** for the **Euler axis**.

This parameterization requires four parameters: the three elements of $\mathbf{a}$ and the scalar Φ.

Given $(\mathbf{a}, \Phi)$, we can calculate $\mathbf{R}$:

$$\mathbf{R} = \cos\Phi\mathbf{1} + (1 - \cos\Phi)\mathbf{a}\mathbf{a}^{\mathsf{T}} - \sin\Phi\mathbf{a}^{\times}$$

Next we need to be able to calculate $(\mathbf{a}, \Phi)$ from a given $\mathbf{R}$.

Extracting $\mathbf{a}$ and Φ from $\mathbf{R}$

Just as we need to be able to compute Euler angles from a given rotation matrix, we need to be able to compute the Euler axis and Euler angle from $\mathbf{R}$.

The following relationships are fairly straightforward to derive (but you are not responsible for the derivation):

$$\begin{aligned} \Phi &= \cos^{-1}\left[\frac{1}{2}\left(\text{trace } \mathbf{R} - 1\right)\right] \\ \mathbf{a}^{\times} &= \frac{1}{2\sin\Phi}\left(\mathbf{R}^{\mathsf{T}} - \mathbf{R}\right) \end{aligned}$$

Exercise: Use Euler angles to calculate a rotation matrix, then extract $(\mathbf{a}, \Phi)$, and use $(\mathbf{a}, \Phi)$ to calculate $\mathbf{R}(\mathbf{a}, \Phi)$.

What can you say about the $\sin\Phi = 0$ case?

Another Four-Parameter Set: Quaternions

The $(\mathbf{a}, \Phi)$ representation also suffers from the complexity assoicated with its trig function terms.

Perhaps the most common rotation representation used for spacecraft attitude dynamics is the **quaternion**.

The **Euler parameter** set, or **quaternion**, is a four-parameter set with some advantages over the Euler axis/angle set.

This set of variables can be written in terms of $(\mathbf{a}, \Phi)$:

$$\begin{aligned} \mathbf{q} &= \mathbf{a} \sin \frac{\Phi}{2} \\ q_4 &= \cos \frac{\Phi}{2} \end{aligned}$$

The vector component, $\mathbf{q}$ is 3×1, and q_4 is the scalar component.

The quaternion is denoted by $\bar{\mathbf{q}} = \begin{bmatrix} \mathbf{q}^\mathsf{T} & q_4 \end{bmatrix}^\mathsf{T}$, a unit 4×1 matrix.

Quaternions: Calculating $\mathbf{R}(\bar{\mathbf{q}})$ and $\bar{\mathbf{q}}(\mathbf{R})$

We know how to calculate $\mathbf{R}$ from Euler angles and vice versa.

We know how to calculate $\mathbf{R}$ from Euler axis and angle and vice versa.

Here are the relationships between $\mathbf{R}$ and $\bar{\mathbf{q}}$

$$\mathbf{R} = \left(q_4^2 - \mathbf{q}^{\mathsf{T}}\mathbf{q}\right)\mathbf{1} + 2\mathbf{q}\mathbf{q}^{\mathsf{T}} - 2q_4\mathbf{q}^{\times}$$

$$\begin{aligned} q_4 &= \pm\frac{1}{2}\sqrt{1 + \text{trace } \mathbf{R}} \\ \mathbf{q} &= \frac{1}{4q_4}\begin{bmatrix} R_{23} - R_{32} \\ R_{31} - R_{13} \\ R_{12} - R_{21} \end{bmatrix} \end{aligned}$$

Exercise: Calculate $\mathbf{R}(\mathbf{a}, \Phi)$, extract $\bar{\mathbf{q}}$ from $\mathbf{R}$, recalculate $\mathbf{R}(\bar{\mathbf{q}})$. Check that $\bar{\mathbf{q}}$ agrees with the equations relating $\bar{\mathbf{q}}$ to $(\mathbf{a}, \Phi)$.

These calculations provide excellent methods of checking your answers.

Summary of Kinematics Notation

We have several equivalent methods of describing attitude or orientation:

- Rotation matrix = DCM = vectors of one frame expressed in the other = dot products of vectors of one frame with those of the other
- Euler angles: $3 \times 2 \times 2 = 12$ different sets
- Euler axis/angle: unit vector and angle
- Euler parameters = quaternions: unit 4×1

You must be able to compute one from the other for any given representation

Next: How does attitude vary with time?

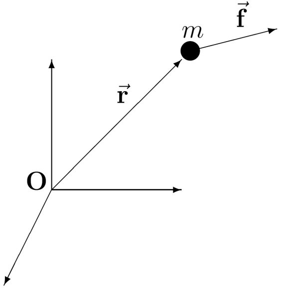
Translational dynamics (Newton's 2nd Law)

— 2nd order includes kinematics and kinetics

$$m\ddot{\vec{\mathbf{r}}} = \vec{\mathbf{f}}$$

— Rewriting as 1st order equations separates the two

kinematics
$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{p}}/m$
kinetics
$\dot{\vec{\mathbf{p}}} = \vec{\mathbf{f}}$



In applying Newton's 2nd Law, the reference origin must be inertial, meaning non-accelerating.

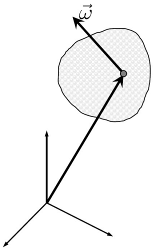
Rotational dynamics (Euler's Law)

— Includes both kinematics and kinetics

$$\dot{\vec{\mathbf{h}}} = \vec{\mathbf{g}}$$

— $\vec{\mathbf{h}}$ is the angular momentum vector

— $\vec{\mathbf{g}}$ is the applied torque vector



— Rewriting as 1st order equations separates kinematics and kinetics

kinematics
To Be Developed
kinetics
$\dot{\boldsymbol{\omega}} = -\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{I}^{-1}\mathbf{g}$

- One of the more difficult elements of modeling rotational motion is the connection between orientation of body and angular momentum
- Rotational kinematics is sufficiently important that we cover it in some detail before discussing rigid body motion

Translational vs Rotational

This table relates the kinetic and kinematic relationships for translational and rotational motion.

Linear momentum $=$ mass $\times$ velocity	Angular momentum $=$ inertia angular velocity
d/dt (linear momentum) $=$ applied forces	d/dt (angular momentum) $=$ applied torques
d/dt (position) $=$ linear momentum/mass	d/dt (angular momentum) $=$ angular momentum "/" inertia

Differential Equations of Kinematics

Given the velocity of a point mass and initial conditions for its position, we can compute its position as a function of time by integrating the differential equation:

$$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{v}}$$

We now need to develop the equivalent differential equations for the attitude when the angular velocity is known.

Sneak Preview

$$\dot{\theta} = \begin{bmatrix} 0 & \sin\theta_3/\cos\theta_2 & \cos\theta_3/\cos\theta_2 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 1 & \sin\theta_3\sin\theta_2/\cos\theta_2 & \cos\theta_3\sin\theta_2/\cos\theta_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \mathbf{S}^{-1}\boldsymbol{\omega}$$

$$\begin{aligned} \dot{\Phi} &= \mathbf{a}^{\mathsf{T}}\boldsymbol{\omega} \\ \dot{\mathbf{a}} &= \frac{1}{2}\left[\mathbf{a}^{\times} - \cot\frac{\Phi}{2}\mathbf{a}^{\times}\mathbf{a}^{\times}\right]\boldsymbol{\omega} \end{aligned}$$

$$\dot{\bar{\mathbf{q}}} = \frac{1}{2}\begin{bmatrix} \mathbf{q}^{\times} + q_4\mathbf{1} \\ -\mathbf{q}^{\mathsf{T}} \end{bmatrix}\boldsymbol{\omega} = \mathbf{Q}(\bar{\mathbf{q}})\boldsymbol{\omega}$$

Euler Angles and Angular Velocity

We develop $\dot{\boldsymbol{\theta}} = \mathbf{S}^{-1}\boldsymbol{\omega}$ one frame at a time, just as we developed rotation matrices in terms of Euler angles. (3-2-1 here)

$$\vec{\boldsymbol{\omega}}^{i'i} = \dot{\theta}_1 \hat{\mathbf{i}}_3' = \dot{\theta}_1 \hat{\mathbf{i}}_3 \qquad \boldsymbol{\omega}_{i'}^{i'i} = \boldsymbol{\omega}_{i}^{i'i} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$\vec{\boldsymbol{\omega}}^{i''i'} = \dot{\theta}_2 \hat{\mathbf{i}}_2'' = \dot{\theta}_2 \hat{\mathbf{i}}_2' \qquad \boldsymbol{\omega}_{i''}^{i''i'} = \boldsymbol{\omega}_{i'}^{i''i'} = \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix}$$

$$\vec{\boldsymbol{\omega}}^{bi''} = \dot{\theta}_3 \hat{\mathbf{b}}_1 = \dot{\theta}_3 \hat{\mathbf{i}}_1'' \qquad \boldsymbol{\omega}_{b}^{bi''} = \boldsymbol{\omega}_{i''}^{bi''} = \begin{bmatrix} \dot{\theta}_3 \\ 0 \\ 0 \end{bmatrix}$$

Add the angular velocity vectors: $\vec{\boldsymbol{\omega}}^{bi} = \vec{\boldsymbol{\omega}}^{bi''} + \vec{\boldsymbol{\omega}}^{i''i'} + \vec{\boldsymbol{\omega}}^{i'i}$

In words: $\boldsymbol{\omega}$ of $\mathcal{F}_b$ with respect to $\mathcal{F}_i = \boldsymbol{\omega}$ of $\mathcal{F}_b$ w.r.t. $\mathcal{F}_{i''}$ $+\ \boldsymbol{\omega}$ of $\mathcal{F}_{i''}$ w.r.t. $\mathcal{F}_{i'}$ $+\ \boldsymbol{\omega}$ of $\mathcal{F}_{i'}$ w.r.t. $\mathcal{F}_i$

Adding the Angular Velocities

The three angular velocities are expressed in different frames:

$$\boldsymbol{\omega}_b^{bi''} \qquad \boldsymbol{\omega}_{i''}^{i''i'} \qquad \boldsymbol{\omega}_{i'}^{i'i}$$

Remember the notation: $\boldsymbol{\omega}_b^{bi''}$ is the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_{i''}$ (superscripts), expressed in $\mathcal{F}_b$ (subscript).

Rotate all into same frame in order to add them.

Typically, we use $\mathcal{F}_b$, but we can use any reference frame.

We already have $\boldsymbol{\omega}_b^{bi''}$ in $\mathcal{F}_b$: $\begin{bmatrix}\dot{\theta}_3 & 0 & 0\end{bmatrix}^\mathsf{T}$

Rotate $\boldsymbol{\omega}_{i''}^{i''i'}$ from $\mathcal{F}_{i''}$ to $\mathcal{F}_b$, and $\boldsymbol{\omega}_{i'}^{i'i}$ from $\mathcal{F}_{i'}$ to $\mathcal{F}_{i''}$ to $\mathcal{F}_b$.

$$\begin{aligned}\boldsymbol{\omega}_b^{i''i'} &= \mathbf{R}^{bi''}\boldsymbol{\omega}_{i''}^{i''i'}\\ \boldsymbol{\omega}_b^{i'i} &= \mathbf{R}^{bi''}\mathbf{R}^{i''i'}\boldsymbol{\omega}_{i'}^{i'i}\end{aligned}$$

Carry out matrix multiplications and complete addition, keeping in mind 3-2-1 Euler angle sequence.

Completing the Operation

Carry out the matrix multiplications and add the three results:

$$\begin{aligned}
\boldsymbol{\omega}_b^{i''i'} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\theta_3\ \dot{\theta}_2 \\ -\sin\theta_3\ \dot{\theta}_2 \end{bmatrix} \\
\boldsymbol{\omega}_b^{i'i} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \\
&= \begin{bmatrix} -\sin\theta_2\ \dot{\theta}_1 \\ \cos\theta_2\ \sin\theta_3\ \dot{\theta}_1 \\ \cos\theta_2\ \cos\theta_3\ \dot{\theta}_1 \end{bmatrix}
\end{aligned}$$

Completing the addition and writing as a matrix expression, we obtain

$$\boldsymbol{\omega}_b^{bi} = \begin{bmatrix} -\sin\theta_2 & 0 & 1 \\ \cos\theta_2\ \sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2\ \cos\theta_3 & -\sin\theta_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

This result is the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$, expressed in $\mathcal{F}_b$, also written as $\boldsymbol{\omega}_b^{bi} = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$.

Completing the Operation (continued)

We have the angular velocity of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$, expressed in $\mathcal{F}_b$:

$$\boldsymbol{\omega}_b^{bi} = \left[\begin{array}{ccc} -\sin\theta_2 & 0 & 1 \\ \cos\theta_2\ \sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2\ \cos\theta_3 & -\sin\theta_3 & 0 \end{array}\right]\left[\begin{array}{c} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{array}\right] = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

We can invert $\mathbf{S}(\boldsymbol{\theta})$ to obtain

$$\begin{aligned} \dot{\boldsymbol{\theta}} &= \ = \mathbf{S}^{-1}(\boldsymbol{\theta})\boldsymbol{\omega} \\ &= \left[\begin{array}{ccc} 0 & \sin\theta_3/\cos\theta_2 & \cos\theta_3/\cos\theta_2 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 1 & \sin\theta_3\sin\theta_2/\cos\theta_2 & \cos\theta_3\sin\theta_2/\cos\theta_2 \end{array}\right]\left[\begin{array}{c} \omega_1 \\ \omega_2 \\ \omega_3 \end{array}\right] \end{aligned}$$

This result is the kinematic differential equation for rotational motion in terms of 3-2-1 Euler angles.

Note that the right-hand side of the equation goes to infinity as θ_2 approaches $\pi/2$ or $3\pi/2$.

This **kinematic singularity** is an important topic.

Kinematic Singularity in Euler Angles Differential Equation

- Note that for 3-2-1 Euler angle set, the Euler rates go to infinity when $\cos\theta_2 \to 0$
- The reason is that at $\theta_2 = \pi/2$ (or $3\pi/2$) the first and second rotations are indistinguishable
- For the "symmetric" Euler angle sequences (3-1-3, 2-1-2, 1-3-1, etc) the singularity occurs when $\theta_2 = 0$ or π
- For the "asymmetric" Euler angle sequences (3-2-1, 2-3-1, 1-3-2, etc) the singularity occurs when $\theta_2 = \pi/2$ or $3\pi/2$
- This kinematic singularity is a major disadvantage of using Euler angles for large-angle motion
- For example, for Roll-Pitch-Yaw Euler angles, the singularity is when the Pitch angle is $\pi/2$ or $3\pi/2$; for an airplane, this condition corresponds to a vertical ascent or descent

... about kinematics differential equations using Euler Angles

- Given a sequence, such as "3-1-2," derive the expression for angular velocity vector of $\mathcal{F}_b$ with respect to $\mathcal{F}_i$ expressed in $\mathcal{F}_b$; *i.e.*, derive $\mathbf{S}(\boldsymbol{\theta})$, which is different for each sequence
- Invert $\mathbf{S}(\boldsymbol{\theta})$ to obtain the Euler angle differential equations
- Given a sequence and some initial conditions for the angles, and an expression for the angular velocity as a function of time $\boldsymbol{\omega}(t)$ integrate the differential equations numerically to obtain the attitude $\boldsymbol{\theta}(t)$
- Given the numerically integrated values of $\boldsymbol{\theta}(t)$, compute the rotation matrix $\mathbf{R}$ to get the attitude of the body frame with respect to the inertial frame

Euler axis and angle differential equations

$$\begin{aligned}\dot{\Phi} &= \mathbf{a}^{\mathsf{T}}\boldsymbol{\omega} \\ \dot{\mathbf{a}} &= \frac{1}{2}\left[\mathbf{a}^{\times} - \cot\frac{\Phi}{2}\mathbf{a}^{\times}\mathbf{a}^{\times}\right]\boldsymbol{\omega}\end{aligned}$$

Kinematic singularity occurs when $\Phi = 0$ or 2π

Euler parameter differential equations

$$\dot{\bar{\mathbf{q}}} = \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix} \mathbf{q}^{\times} + q_4\mathbf{1} \\ -\mathbf{q}^{\mathsf{T}} \end{bmatrix}\boldsymbol{\omega} = \mathbf{Q}(\bar{\mathbf{q}})\boldsymbol{\omega}$$

No singularity!

Later we will learn how to get the angular velocity from the **kinetics differential equation**:

$$\dot{\boldsymbol{\omega}} = -\mathbf{I}^{-1}\boldsymbol{\omega}^{\times}\mathbf{I}\boldsymbol{\omega} + \mathbf{I}^{-1}\mathbf{g}$$

Typical Problem Involving Angular Velocity and Attitude

Given initial conditions for the attitude (in any form), and a time history of angular velocity, compute $\mathbf{R}(t)$ or any other attitude representation as a function of time

Requires integration of one of the sets of differential equations involving angular velocity

Examples:

Given Euler angle sequence, $\boldsymbol{\theta}(t_0)$, and $\boldsymbol{\omega}(t)$, integrate $\dot{\boldsymbol{\theta}} = \mathbf{S}^{-1}(\boldsymbol{\theta})$ and compute $\mathbf{R}(\boldsymbol{\theta}(t))$

Given Euler angle sequence, $\boldsymbol{\theta}(t_0)$, and $\boldsymbol{\omega}(t)$, compute $\mathbf{R}(t_0)$, $\bar{\mathbf{q}}(t_0)$, integrate $\dot{\bar{\mathbf{q}}} = \mathbf{Q}(\bar{\mathbf{q}})\boldsymbol{\omega}$, and compute $\mathbf{R}(\bar{\mathbf{q}}(t))$

Problem 1

- The orientations of two spacecraft A and B relative to an inertial frame are given through the 3-2-1 Euler angle sequence: — $\boldsymbol{\theta}_A = [30 \;\; -45 \;\; 60]^\mathsf{T}$ and $\boldsymbol{\theta}_B = [10 \;\; 25 \;\; -15]^\mathsf{T}$ degrees
- What is the relative orientation of spacecraft A relative to B in terms of the 3-2-1 Euler angles?
- In other words, what are θ_1, θ_2, and θ_3 for the rotation matrix $\mathbf{R}^{AB}$?
- The answer is (359.1, 287.7, 79.96) degrees
- Need to do the matrix multiplication to get the rotation matrix:

$$\mathbf{R}^{AB} = \mathbf{R}^{Ai}\mathbf{R}^{iB}, \quad \text{where } \mathbf{R}^{iB} = {\mathbf{R}^{Bi}}^\mathsf{T}$$

- Easy to do the calculations in Matlab ....

Problem 1 (Matlab Code)

```
% Kinematics Example Problem 1
% The orientations of two spacecraft A and B relative to an inertial fr
% are given for the 3-2-1 Euler angle sequence:
% theta_A=(30, -45, 60) and theta_B=(10, 25, -15) degrees
%
% What is the relative orientation of spacecraft A relative to B in ter
% of the 3-2-1 Euler angles?
% In other words, what are theta_1, theta_2, and theta_3
% for the rotation matrix R^{AB}?
%
% Define the 3x1 matrices of the Euler angles for spacecraft A and B
% Convert from degrees to radians
thetaA = [30; -45;  60]*pi/180;
thetaB = [10;  25; -15]*pi/180;

% Compute the two rotation matrices
RAi = R1(thetaA(3))*R2(thetaA(2))*R3(thetaA(1));
RBi = R1(thetaB(3))*R2(thetaB(2))*R3(thetaB(1));

% Compute RAB = RAi*RiB = RAi*inv(RBi) = RAi*RBi'
RAB = RAi*RBi';
```

Problem 1 (Matlab Code continued)

```
% Kinematics Example Problem 1 CONTINUED

% Extract the Euler angles from RAB
th2 = asin(-RAB(1,3))
th3 = atan2(RAB(2,3),RAB(3,3))
th1 = atan2(RAB(1,2),RAB(1,1))

% Check that these three angles give RAB
RABcheck = R1(th3)*R2(th2)*R3(th1)

% RAB*RABcheck' should be identity
identitycheck = RAB*RABcheck'

% It works, so no need to go back and do further quadrant checks.
% also nice to see the angles in degrees:
thetaABdeg = [th1; th2; th3]*180/pi

% Since some angles are negative, we could also shift them to be
% between 0 and 360 degrees by simply adding 360 to the negative ones.
% More generally, though, the function mod can do it for us

thetaABdeg = mod(thetaAB,360)
```

Problem 2

- The attitude of $\mathcal{F}_B$ relative to the inertial frame is given through the 3-2-1 Euler angle sequence
  — $\boldsymbol{\theta}_B = [10 \quad 25 \quad -15]^\mathsf{T}$ degrees
- Find the principal rotation axis $\mathbf{a}$ and angle Φ.
- Calculate the rotation matrix $\mathbf{R}^{Bi}$, then

$$\begin{aligned} \Phi &= \cos^{-1}\left[\frac{1}{2}\left(\text{trace}\,\mathbf{R} - 1\right)\right] \\ \mathbf{a}^\times &= \frac{1}{2\sin\Phi}\left(\mathbf{R}^\mathsf{T} - \mathbf{R}\right) \end{aligned}$$

  — $\Phi = 0.5546 \text{ rad} = 31.78^\circ$
  — $\mathbf{a} = [-0.5320 \;\; 0.7403 \;\; 0.4110]^\mathsf{T}$
- Easy to do in Matlab ....

Problem 2 (Matlab code)

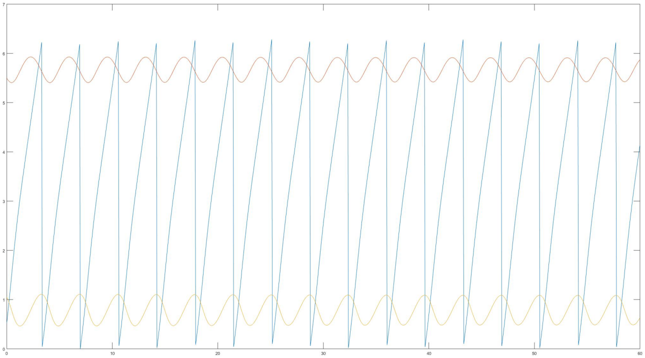
```
% Kinematics Example Problem 2
% Define the 3x1 matrix of the Euler angles for spacecraft B
% Convert from degrees to radians
thetaB = [10;  25; -15]*pi/180;

% Compute the rotation matrix
RBi = R1(thetaB(3))*R2(thetaB(2))*R3(thetaB(1));

% Extract the Euler angle and Euler axis from this matrix
cosPhi = (trace(RBi)-1)/2;
Phi    = acos(cosPhi);
sinPhi = sin(Phi);
askew  = (RBi'-RBi)/(2*sinPhi);
a = [askew(3,2); askew(1,3); askew(2,1)]

% Check that this Euler axis/angle set gives RBi
RBicheck = cosPhi*eye(3)+(1-cosPhi)*a*a'-sinPhi*askew;

% RBi*RBicheck' should be identity
identitycheck = RBi*RBicheck'

% It works, so let's show the results, with angle in rad and deg
Phirad = Phi
Phideg = Phi*180/pi
```

Problem 3

- The orientation of a spacecraft B relative to an inertial frame is given through the 3-2-1 Euler angle sequence, with initial conditions
  — $\boldsymbol{\theta}_B(0) = [30 \quad -45 \quad 60]^\mathsf{T}$ degrees
- The attitude control system causes the spacecraft to rotate with angular velocity $\boldsymbol{\omega}^{Bi} = [1 \ \ 1 \ \ 1]^\mathsf{T}$ rad/s expressed in $\mathcal{F}_B$
- Integrate the equations of motion to obtain $\boldsymbol{\theta}(t)$, for $t = 0$ to 60 seconds
- Easy to do the calculations in Matlab ....
- Matlab includes several functions that are useful for integration of ordinary differential equations, particularly `ode45`
  — Code right-hand side of differential equations in one file `rhs.m`
  — Code parameters, initial conditions, `ode45` call, and post-processing into `driver.m`
  — Name `rhs.m` and `driver.m` with descriptive names for particular problem

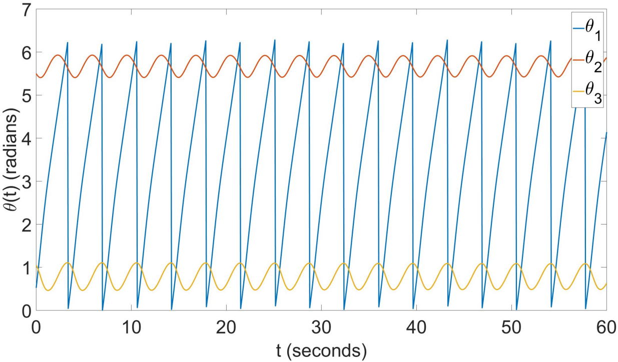
Problem 3 (Plot 1)

Matlab produces the following plot using the default settings. Clearly this plot is not as useful as it could be. Fortunately, Matlab allows the user to make modifications to the plots.



Problem 3 (Plot 2)

Changing the `linewidth` and `fontsize` parameters, adding labels and legends, produces a more satisfying plot of the Euler angles vs time.



Observations: 1) Euler angles do not approach kinematic singularity; 2) θ_1 exhibits "whirling" behavior, continuous increase from 0 to 2π; 3) $\theta_{2,3}$ exhibit oscillatory behavior

Exercise: Can you explain these observations in terms of the kinematic differential equation?

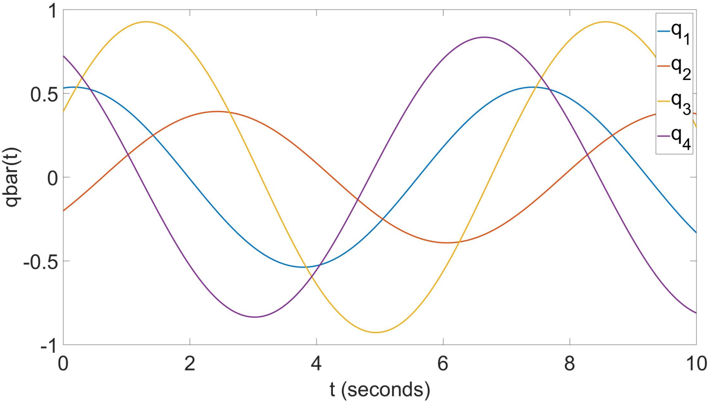
Problem 3 (Matlab Code)

```
% th321driver.m
th0=[30;-45;60]*pi/180;                          %initial conditions
tspan=linspace(0,60,1000);                       %integration timespan
options=odeset('abstol',1e-9,'reltol',1e-9);     %set tolerances
[t,th]=ode45('th321rhs',tspan,th0);              %call ode45
th(:,1:2)=mod(th(:,1:2),2*pi);                   %make angles [0,2pi]
plot(t,th);                                      %plot 1
figure;  hg=plot(t,th);                          %plot 2
set(hg,'linewidth',3)
set(gca,'fontsize',48)
hg=xlabel('t (seconds)');
set(hg,'fontsize',48);
hg=ylabel('\theta(t) (radians)');
set(hg,'fontsize',48);
hg=legend('\theta_1','\theta_2','\theta_3');
set(hg,'fontsize',48);

% th321rhs.m  right hand side function called by ode45
function thdot = th321rhs(t,th)
    s2=sin(th(2));                               %calculate trig f's
    c2=cos(th(2));
    s3=sin(th(3));
    c3=cos(th(3));
    Sinv=[0 s3/c2 c3/c2; 0 c3 -s3; 1 s3*s2/c2 c3*s2/c2]; %Sinverse
    thdot=Sinv*[1;1;1];                                  %dtheta/dt
end
```

Problem 4

- The orientation of a spacecraft B relative to an inertial frame is given through the 3-2-1 Euler angle sequence, with initial conditions:
  - $\boldsymbol{\theta}_B = [30 \quad -45 \quad 60]^\mathsf{T}$ degrees
- The attitude control system makes the spacecraft rotate with angular velocity $\boldsymbol{\omega}^{Bi} = [1 \;\; 1 \;\; 1]^\mathsf{T}$ rad/s expressed in $\mathcal{F}_B$
- Integrate the quaternion equations of motion to obtain $\bar{\mathbf{q}}(t)$, for $t = 0$ to 10 seconds
- Easy to do the calculations in Matlab

Problem 4 (Plot)



Observations: The components of the quaternion, (also called quaternions) exhibit simple oscillatory behavior, always in the interval $[-1, 1]$.

Exercise: Compare the results of Euler angle integration to quaternion integration, both of which provide equivalent results in terms of the attitude.

Summary

Learning Objective 1. Describe attitude kinematics using reference frames, rotation matrices, Euler parameters, Euler angles, and quaternions.

- Define reference frame. Define inertial origin. Define inertial reference frame.
- Derive rotation matrices using direction cosines. ... dot products. ... Euler angles. ... Euler parameters.
- Derive attitude kinematics differential equations for Euler angles.
- Identify and describe kinematic singularities.

Learning Objective 2. Accurately perform attitude kinematics calculations.

- Calculate rotation matrices using direction cosines. ... dot products. ... Euler angles. ... Euler parameters.
- Given attitude in one set of variables (e.g., Euler angles), calculate the rotation in a different set of variables (e.g., quaternion).
- Calculate attitude as function of time by integrating the attitude kinematics differential equations.

Looking Forward

- We need to develop the kinetic differential equations for rigid bodies, in order to obtain the angular velocity, $\boldsymbol{\omega}^{bi}(t)$.
- Knowing the angular velocity, we can determine the attitude as a function of time.
- Before beginning the Rigid Body Dynamics material, we will cover the topic of Attitude Determination, which requires
  — attitude sensors
  — mathematical models of subjects of attitude sensors
  — algorithms that use attitude sensor measurements and mathematical models to determine attitude
- So, the remainder of the course includes Exam 1, Attitude Determination, Rigid Body Dynamics, Exam 2, Satellite Dynamics, and Final Exam.