

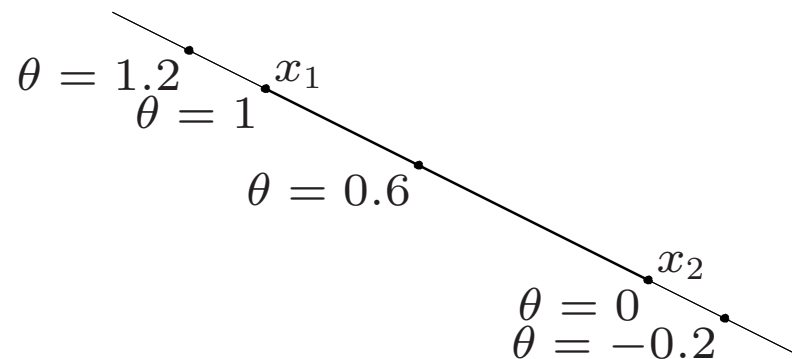
## 2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

# Affine set

**line** through  $x_1, x_2$ : all points

$$x = \theta x_1 + (1 - \theta)x_2 \quad (\theta \in \mathbf{R})$$



**affine set:** contains the line through any two distinct points in the set

**example:** solution set of linear equations  $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)

# Convex set

**line segment** between  $x_1$  and  $x_2$ : all points

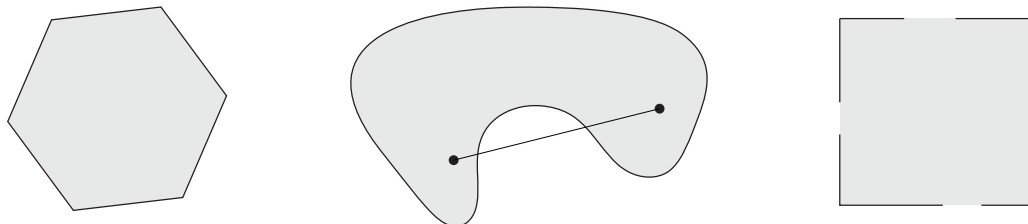
$$x = \theta x_1 + (1 - \theta)x_2$$

with  $0 \leq \theta \leq 1$

**convex set**: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

**examples** (one convex, two nonconvex sets)



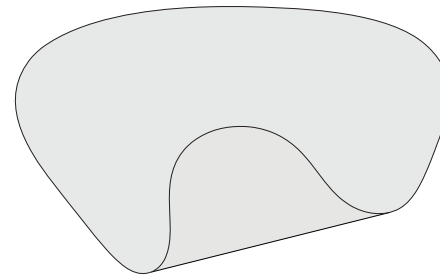
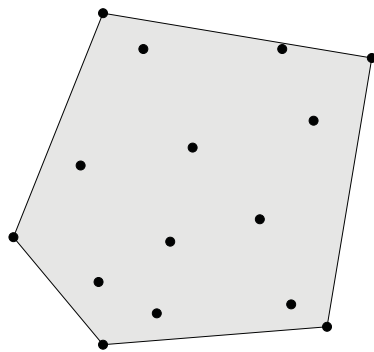
# Convex combination and convex hull

**convex combination** of  $x_1, \dots, x_k$ : any point  $x$  of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$$

with  $\theta_1 + \dots + \theta_k = 1$ ,  $\theta_i \geq 0$

**convex hull**  $\text{conv } S$ : set of all convex combinations of points in  $S$

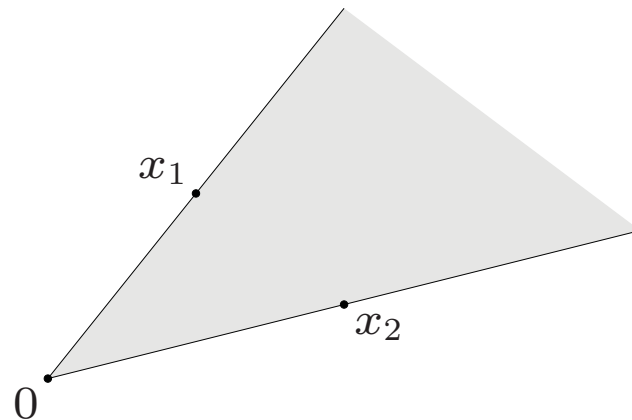


# Convex cone

**conic (nonnegative) combination** of  $x_1$  and  $x_2$ : any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

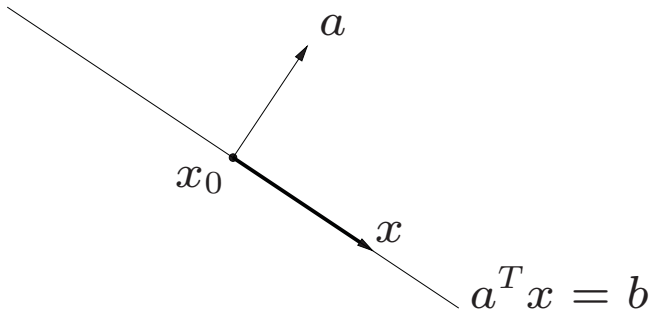
with  $\theta_1 \geq 0$ ,  $\theta_2 \geq 0$



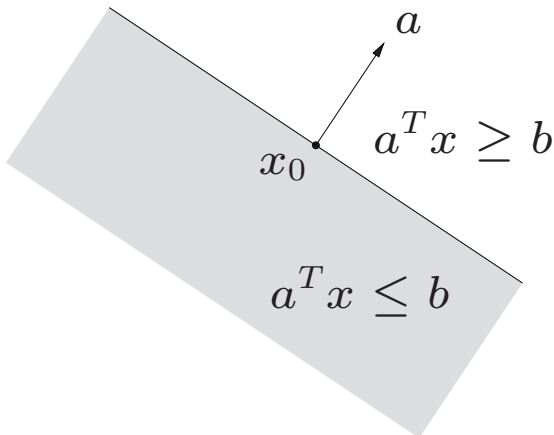
**convex cone:** set that contains all conic combinations of points in the set

# Hyperplanes and halfspaces

**hyperplane:** set of the form  $\{x \mid a^T x = b\}$  ( $a \neq 0$ )



**halfspace:** set of the form  $\{x \mid a^T x \leq b\}$  ( $a \neq 0$ )



- $a$  is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

# Euclidean balls and ellipsoids

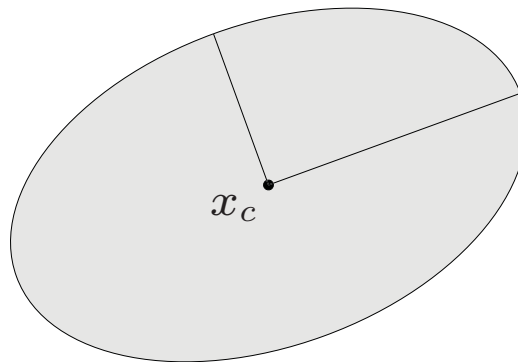
**(Euclidean) ball** with center  $x_c$  and radius  $r$ :

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

**ellipsoid:** set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with  $P \in \mathbf{S}_{++}^n$  (*i.e.*,  $P$  symmetric positive definite)



other representation:  $\{x_c + Au \mid \|u\|_2 \leq 1\}$  with  $A$  square and nonsingular

# Norm balls and norm cones

**norm:** a function  $\|\cdot\|$  that satisfies

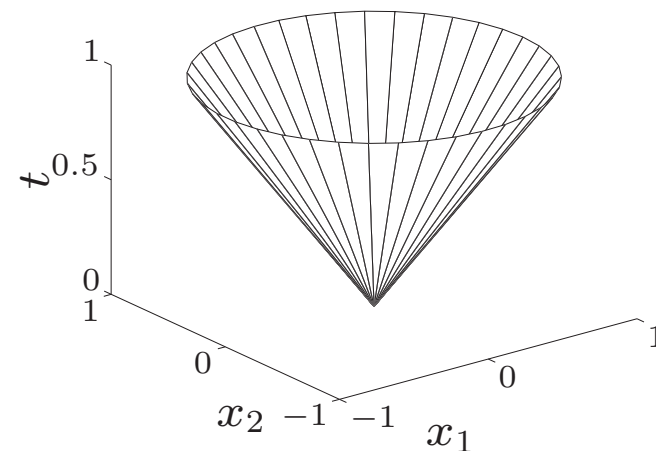
- $\|x\| \geq 0$ ;  $\|x\| = 0$  if and only if  $x = 0$
- $\|tx\| = |t| \|x\|$  for  $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation:  $\|\cdot\|$  is general (unspecified) norm;  $\|\cdot\|_{\text{symb}}$  is particular norm

**norm ball** with center  $x_c$  and radius  $r$ :  $\{x \mid \|x - x_c\| \leq r\}$

**norm cone:**  $\{(x, t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone



norm balls and cones are convex

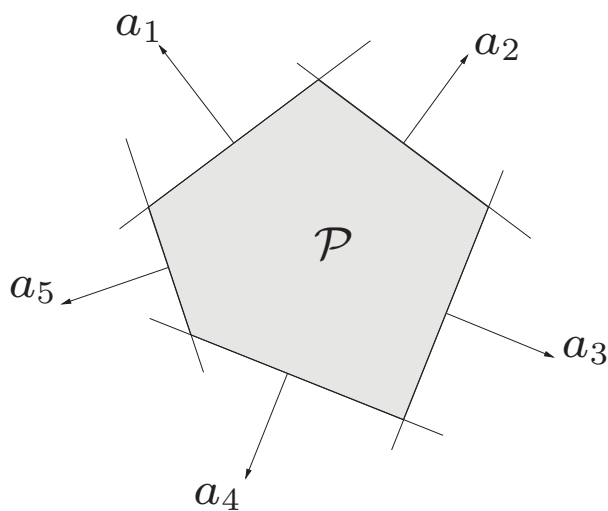


# Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

( $A \in \mathbf{R}^{m \times n}$ ,  $C \in \mathbf{R}^{p \times n}$ ,  $\preceq$  is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

# Positive semidefinite cone

notation:

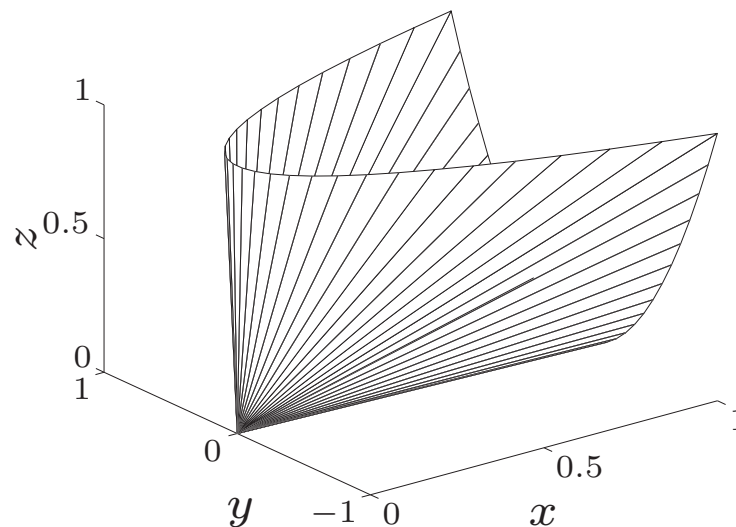
- $\mathbf{S}^n$  is set of symmetric  $n \times n$  matrices
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$ : positive semidefinite  $n \times n$  matrices

$$X \in \mathbf{S}_+^n \iff z^T X z \geq 0 \text{ for all } z$$

$\mathbf{S}_+^n$  is a convex cone

- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$ : positive definite  $n \times n$  matrices

example:  $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$



# Operations that preserve convexity

practical methods for establishing convexity of a set  $C$

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

2. show that  $C$  is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . . ) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions