Final Exam (Take Home Part)

DIRECTIONS: You must work on this exam independently but you can ask your instructor for clarifications. Read the questions carefully and follow the instructions. Write your answers/solutions legibly in the space provided. Please use a copy of this document for your submission. I will not accept answers that are written in a different document.

1. Consider the following system of linear equations:

$$\begin{cases} x_1 - 4x_2 + 2x_3 &= -1 \\ 3x_2 + 5x_3 &= -3 \\ -2x_1 + 8x_2 - 4x_3 &= 2 \end{cases}$$

matrix-vector equation of the form Ax = b.

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$
 form of improper fraction 
$$\begin{bmatrix} 1 & 0 & 16/3 & -5 \\ 0 & 1 & 5/3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (1 PT) Express the above system as a (b) (1 PT) What is the RREF of the augmented matrix of the above system? (You may use Matlab but your constants should be in the form of improper fractions.)

(c) (1 PT) Describe the solution set of the above system parametrically.

$$\begin{array}{c} x_{1} = -5 - \frac{26}{5}x_{1} \\ x_{1} = -1 - \frac{1}{3}x_{1} \\ x_{2} = 1 - \frac{1}{3}x_{2} \\ x_{3} = \frac{1}{3}x_{3} - \frac{1}{3}x_{3} - \frac{1}{3}x_{3} \\ x_{3} = \frac{1}{3}x_{3} - \frac{1}{3}x_{3} - \frac{1}{3}x_{3} \\ x_{3} = \frac{1}{3}x_{3} - \frac{1}{3}x_{3} - \frac{1}{3}x_{3} - \frac{1}{3}x_{3} - \frac{1}{3}x_{3} - \frac{1}{3}x_{3} \\ x_{3} = \frac{1}{3}x_{3} - \frac{1$$

(d) (1 PT) Describe the solution set of the associated homogeneous equation Ax = 0 parametrically.

2. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}.$$

(a) (2 PTS) Find a least-square solution to  $A\mathbf{x} = \mathbf{b}$ . That is, find  $\hat{\mathbf{x}}$  satisfying  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$ . (You may use Matlab, but write the entries of your answer in improper fraction form.) Solution:

$$A^{T}A = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} = X \qquad \left[ \begin{array}{c} X \mid Y \\ Y \end{array} \right] = \begin{bmatrix} 6 & 3 & 3 & 2 & 7 \\ 3 & 3 & 0 & 1 & 2 \\ 2 & 0 & 3 & 15 \end{array} \right] = B$$

$$A^{T}b = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix} = Y \qquad \operatorname{rref}(B) = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \stackrel{\wedge}{\gamma} = \begin{bmatrix} 5 & -\lambda_{7} \\ -1 & +\lambda_{3} \\ \lambda_{3} \end{bmatrix}$$

Answer:

(b) (2 PTS) Compute the least square error  $||A\hat{x} - b||$ . Solution:

$$A\hat{x}-b=\begin{bmatrix} -\frac{5}{2} \\ \frac{1}{1} \\ \frac{1}{0} \\ \frac{1}{1} \end{bmatrix} \qquad \sqrt{(-\frac{5}{3}^{2}+2^{2}+1^{2}+(-1)^{2}+(1)^{2}} + 4(1)^{2} = 4$$

Answer:

3. Suppose 
$$A = \begin{bmatrix} B & 0_{3\times 2} \\ D & C \end{bmatrix}$$
, where  $B = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ ,  $C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$  and  $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ 

(a) (1 PT) Find 
$$C^{-1}$$
.

Answer:

$$\begin{bmatrix} -1 & 1 \\ 1/2 & 0 \end{bmatrix}$$

(b) (3 PTS) Find  $B^{-1}$  by row reducing  $[B|I_3]$ . Show your solution, indicating the row operations in each step.

(c) (2 PTS) Compute det(A). Show your solution. Do not use Matlab.

(d) (3 PTS) Find the (4,3) entry of A<sup>-1</sup> using Cramer's Rule. Show your solution. Do not use Matlab.

## Solutions:

(Clearly indicate which item the solution is for.)

$$\begin{array}{c} \alpha ) \quad \frac{1}{0-2} \begin{bmatrix} 4-2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1/2 & 0 \end{bmatrix}$$

b) 
$$\begin{bmatrix} -1 & -1 & | & 1 & 0 & 0 \\ 1 & 6 & | & 0 & 1 & 0 \end{bmatrix} R_2 \Rightarrow R_2 + R_1 \begin{bmatrix} -1 & -1 & | & | & 1 & 0 & 0 \\ 0 & -1 & 2 & | & | & 1 & 0 & 0 \end{bmatrix} R_3 \Rightarrow R_3 + R_2 \begin{bmatrix} 1 & 1 & -1 & | & -1 & 0 & 0 \\ 0 & -1 & 2 & | & | & 1 & 0 & 0 \end{bmatrix}$$

C. 
$$det(c) = 0 - 2 = -2$$
 $det(b) = 0 + 1 + 0 - (0 - 1 - 1) = 3$ 
 $det(b) = 0 + 1 + 0 - (0 - 1 - 1) = 3$ 
 $det(b) = 0 + 1 + 0 - (0 - 1 - 1) = 3$ 
 $det(b) = 0 + 1 + 0 - (0 - 1 - 1) = 3$ 
 $det(b) = -7$ 
 $\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ 
 $\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ 
 $\begin{pmatrix} -1 & -1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ 
 $\begin{pmatrix} -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ 
 $det(a) = -2$ 
 $det(b) = -2$ 

4. (You can use Matlab for the following items.) Let  $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$  be a linear transformation such that

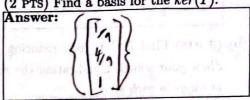
$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\2\end{bmatrix}, T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}-1\\1\\1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}1\\4\\9\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

(a) (3 PTS) Find the standard matrix rep. of T. Answer:

14,5	-16 5.5
-2.5	4 -1.5
14,5 -2.5 3.5	-16 5.5 4 -1.5 -2 .5

Solutions: (Clearly indicate which item

(b) (2 PTS) Find a basis for the ker(T).



(c) (1 PT) Is T onto? Answer:

No.

the solution is for.) 0) ty 24, 36, 3 02 - 16 0 0 14.5 0 0 15.5 tu 422 9623 0 RREF (100 4 -2.5)

$$\begin{bmatrix} L_{31} & L_{32} & L_{33} & L_{34} \\ L_{11} & L_{12} & L_{13} & L_{34} \\ L_{11} & L_{12} & L_{13} & 0 \end{bmatrix} \xrightarrow{RREP} \begin{bmatrix} 1 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & .5 \end{bmatrix}$$
enoith when the shall represent the state of the shall represent the state of the shall represent the shall represent the state of the shall represent the

 $b) A = \begin{bmatrix} 14.5 & -16.5 & .5 \\ -2.5 & 4 & -1.5 \\ 15 & 2 & 5 \end{bmatrix}$ 

$$A = \begin{bmatrix} 14.5 & -16.5 & .5 \\ 2.5 & 4 & -1.5 \\ 3.5 & 2 & .5 \end{bmatrix}$$

$$C. \text{ there is not a pivot in every row of every row of }$$

$$ref(A) = \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{4}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{4}x_3$$

$$x_2 = \frac{1}{4}x_3$$

$$x_3 = x_3$$

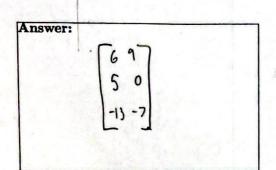
5. (You can use Matlab for the following items.) Let

$$p(t) = t^2 + t + 1,$$
  $q(t) = 3t - 2,$   $r(t) = 2 - 5t^2,$ 

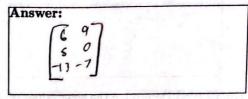
and  $T: \mathbb{R}^2 \longrightarrow \mathbb{P}_2[t]$  such that

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (2a+3b)\mathbf{p} + (a-b)\mathbf{q} + (3a+2b)\mathbf{r}$$

(a) (2 PTS) Find the matrix of T relative to the (b) (2 PTS) Find a basis for the ran(T). bases  $\{e_1, e_2\}$  and  $\{p, q, r\}$ .



Solutions: (Clearly indicate which item the solution is for.)



(c) (1 PT) Is T one-one?

Answer:	14 14 14	
Yes		

$$M = \left[ \left[ T([0]) \right]_{pq} \right]$$

$$T([0]) = 2p + q + 3r = 2t^{2} + 2t + 2 + 3t^{2} + 6 + 6t^{2} = -13t^{2} + 5t + 6$$

$$T([0]) : 3p - q + 2r = 3t^{2} + 3t + 3 + 3t + 4 + 10t^{2} = -7t^{2} + q$$

$$M = \left[ \left[ \frac{6}{9} \right]_{pq} \right]$$

$$M = \left[ \frac{6}{5} \frac{9}{0} \right]_{pq}$$

$$M = \left[ \frac{9}{5} \frac{9}{0} \right]_{pq}$$

$$M = \left[ \frac{9}{0} \frac{9}{0} \right]_{pq}$$

$$M = \left[ \frac{9}{5} \frac{9}{0}$$

b. reef(M) 2 [10] heree the colore linearly independent

6. Let 
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
.

(a) (2 PTS) Find the characteristic polynomial of A. Show your complete solution. (Do not use Matlab.)

Answer:

(b) (2 PTS) Determine the eigenvalues of A and their algebraic multiplicities.

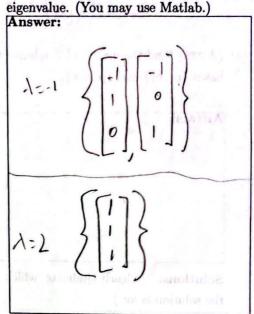
Answer:

A A M

-1 2

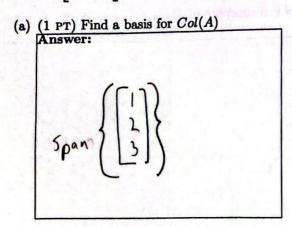
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(c) (3 PTS) Find a basis for each eigenspace and determine the geometric multiplicity of each



Solutions: (Clearly indicate which item the solution is for.)

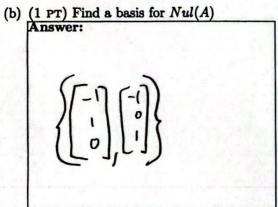
7. Let  $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ . (You may use Matlab to answer the following.)



(c) (2 PTS) Find a basis for 
$$(Col(A))^{\perp}$$

Answer:

$$\begin{bmatrix}
-2 \\ i \\ o
\end{bmatrix}, \begin{bmatrix} -3 \\ i \\ i \end{bmatrix}$$



(d) (2 PTS) Find a basis for 
$$(Nul(A))^{\perp}$$
.

Answer:

$$\begin{cases} \begin{cases} \\ \\ \\ \\ \\ \end{cases} \end{cases}$$

Solutions: (Clearly indicate which item the solution is for.)

$$rref(A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$b) \cdot Nol(A) \cdot x_{1} = -x_{2} - x_{3}$$

$$x_{2} = x_{2} \\ y_{2} = x_{3}$$

$$x_{3} = x_{3}$$

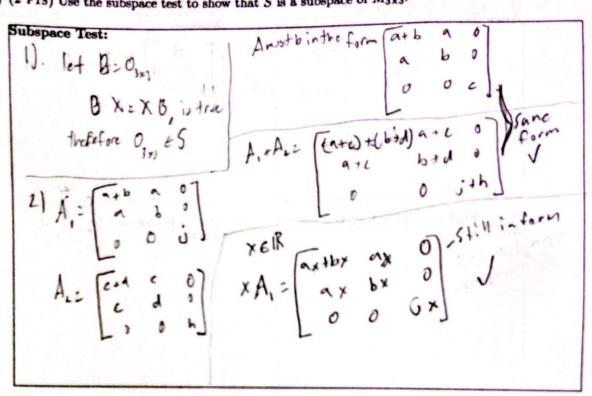
$$x_{1} = -2x_{2} - 3x_{3}$$

$$x_{2} = x_{3} \\ x_{3} = x_{3} \\ x_{4} = x_{3} \\ x_{5} = x_{5} \\ x_{$$

7

8. Let 
$$X = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$
 and set  $S = \{A \in M_{3\times 3} \mid AX = XA\}$ .

(a) (2 PTS) Use the subspace test to show that S is a subspace of  $M_{3\times3}$ .



(b) (3 PTS) Determine the dimension of S.

Solution:

9. Let 
$$S = \left\{ \begin{bmatrix} 1\\2\\3\\4 \end{bmatrix}, \begin{bmatrix} 1\\-1\\-1\\1 \end{bmatrix}, \begin{bmatrix} -1\\3\\-3\\1 \end{bmatrix} \right\}$$
 and  $W = Span(S)$ .

(a) (1 PT) Is S an orthogonal set?

Solution:

$$\langle v_1, v_2 \rangle = 1 - 2 - 3 + 4 = 0$$
  
 $\langle v_1, v_3 \rangle = -1 + 6 - 9 + 4 = 0$   
 $\langle v_2, v_3 \rangle = -1 - 3 + 3 + 1 = 0$ 

Answer: Yes