A Review and Application of Numerical Methods in Option Pricing

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ABSTRACT: Numerical methods have long been used for approximating the results of differential equations. These techniques have multiple applications, including in mathematical biology and financial mathematics. This paper will focus on explaining three of these techniques. Then will show how they are used in examples related to financial math.

1. Introduction

Numerical techniques for estimating the solutions of ordinary differential equations were created in the late 18th century, when Euler proposed the first method in his book *Institutionum calculi integralis* [3]. Numerical methods today have multiple practical uses, for instance in biology. In this paper we are interested in their use in option pricing. We will start be analyzing the Euler-Maruyama method. The Euler-Maruyama method is a simple method and is not used in practice by itself. Instead, other methods like the binomial options pricing model and Monte Carlo simulations are used [12]. This paper will establish the necessary background to understand the three numerical methods listed above by examining their foundations and then will proceed to explain the methods themselves. Finally, the three numerical methods will be demonstrated. The Euler-Maruyama method example will be of predicting the price of the stock given its history, this is often a necessary step when pricing options for that stock. The Binomial model example will demonstrate how to price an American option, and the Monte Carlo example will be of pricing an Asian option.

2. Background

a. Wiener Process

1) Stochastic Processes

A Wiener process is a stochastic process that can be used to model a system that involves randomness. In the case of financial mathematics it can be used to model the price of a stock. A stochastic or random process is defined as "a collection of random variables usually indexed by time." Stock prices are a continuous-time stochastic process [8].

2) History of Brownian Motion

Brownian motion has a rich history in mathematics and science. Brownian motion derives its name from the botanist Robert Brown, who described it while studying pollen. Although Brown wrote about this motion he never laid the mathematical found for it. The first person who did that was Thorvald Thiele in 1880. However, both Louis Bachelier and Albert Einstein would also independently discover the mathematics behind Brownian motion [6]. A definition would be formalized by Norbert Wiener, which is why Brownian motion is often called a Wiener process [4].

3) Definition of Wiener Process

A Wiener process must follow the following properties [5]:

1. $W_0 = 0$
2. The function $t \longrightarrow W_t$ is continuous
3. $W_{s+t} - W_s \sim N(0,t)$
4. W_t has independent increments

The first two parts of that definition are rather standard, so let's focus on the last two.

The third property can also be stated as $W_{s+t} - W_s \sim N(0, t+s-s)$. To explain this property let's look at a graph showing the price of a made up stock. For the purpose of this example, we will assume this stock follows a standard Wiener process.

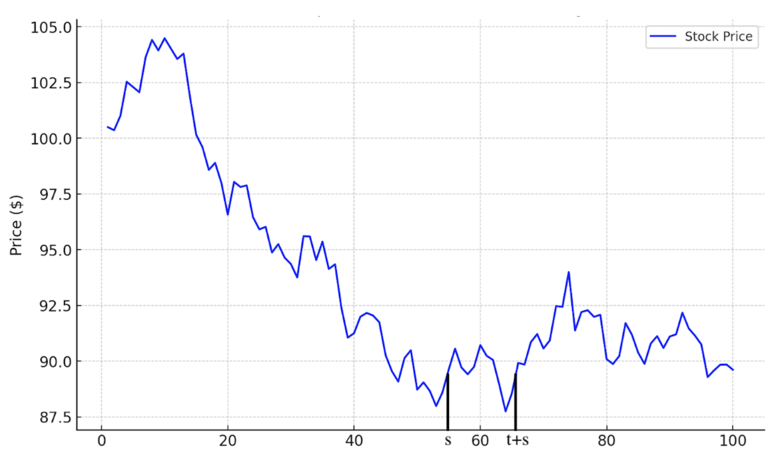

Fig. 1. Stock Price Example

We can see at time s the stock price is approximately \$89. The third property states that on average the stock price at a later time, $s+t$, will be the same (assuming there is no drift). Furthermore, There is a 50% chance the stock is above \$89 and a 50% chance the stock is below \$89, with the prices at $s+t$ being normally distributed.

The final property is important because it means Wiener process has the Markov property. This means that the future state of the process is not dependent on previous states, only the current one.

4) Geometric Brownian Motion

In practice, stocks are modeled by geometric Brownian motions. There are a couple of important differences between a geometric Brownian motion and a standard Wiener process.

1. geometric Brownian motions can be used to model exponential growth. This is much more realistic because stock prices cannot fall below 0.

2. When stocks are modeled by geometric Brownian motions, their prices follow a log-normal distribution, allowing us to find the expectation of the stock at a later time.

The fact that geometric Brownian motions can be used to model exponential growth is important because it allows for the consideration of drift. In financial mathematics, this drift term is important when modeling stock prices because it allows us to take into account the general trend of stock [7].

A stochastic process is a geometric Brownian motion if it is modeled by the following stochastic differential equation (SDE):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t$$

We will further examine this equation, often called the Black-Scholes diffusion equation later [10]. Now, let's find the solution for the Black-Scholes diffusion equation.

First we need to state Itô's lemma, which is essentialy the chain rule for SDEs. Let $X = f(t,Y)$, then:

$$dX = \frac{\partial f}{\partial t}(t,Y)dt + \frac{\partial f}{\partial y}(t,Y)dy + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t,Y)dydy$$

Then

$$g(S(t)) = ln(S(t))$$

Applying Itô's lemma results in:

$$d\ ln(S(t)) = \frac{1}{S(t)}dS - \frac{dS(t)^2}{2S(t)^2}$$

Note that $dt\ dt = dW_t\ dt = 0$ and $dW_t\ dW_t = dt$. Then

$$dS(t)^2 = \mu^2 S(t)^2 dt^2 + 2\sigma\mu S(t)^2 dtdW_t + \sigma^2 S(t)^2 dW_t dW_t = 0 + 0 + \sigma^2 S(t)^2 dt$$

Plugging $dS(t)^2$ results in

$$d\,ln(S(t)) = \frac{\mu S(t)dt + \sigma S(t)dW_t}{S_t} - \frac{\sigma^2 S(t)^2 dt}{2S(t)^2} = \mu dt + \sigma dW_t - \frac{\sigma^2}{2}dt$$

Now we integrate

$$\int_0^T d\,ln(S(t)) = \int_0^T (\mu - \frac{\sigma^2}{2})dt + \int_0^T \sigma dW_t \implies ln(\frac{S(t)}{S(0)}) = (\mu - \frac{\sigma^2}{2})T + \sigma(W_t - 0)$$

Taking the exponential results in

$$\frac{S(t)}{S(0)} = e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_t} \implies S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_t}$$

Now that we have found the solution to the Black-Scholes diffusion equation, and because we know that it follows a log-normal distribution, we can find the expected value of the stock at any time t as

$$E[S(t)] = S(0)e^{\delta t}$$

where δ is the continuously compounded rate of return [7].

b. Stochastic Differential Equations

1) History of Stochastic Differential Equations

The history of stochastic differential equations goes hand-in-hand with the history of the Wiener process, even though the theory of SDEs would not be developed until later, in that Louis Bachelier and Albert Einstein modeled Brownian motion using SDEs. At the time however, Einstein was not able to show that the process existed as the proper mathematical tools to do so did not exist at the time. It wasn't until 1923 that Norbert Wiener would formalize a definition for Brownian motion, combining ideas from Fourier series and measure theory [4]. The next influential step in creating the foundations of SDEs was made by Kolmogorov when he made strides in the theory of Markov processes. Kolmogorov proved that continuous Markov processes depend on two parameters, the speed of the drift, and the size of the random part. Kolmogorov was then able to show that the probability distributions of these continuous Markov processes to solutions of partial differential equations (PDEs). Finally, we examine the work of Kiyosi Itô, the founder of stochastic integration. In Itô's 1944 and 1951 papers, he modeled Markov processes in the form of SDEs and was able to connect his own work with Kolmogorov's [4].

2) Derivation of the Black-Scholes Equation using Stochastic Differential Equations

The solutions of a SDEs are continuous stochastic processes, in contrast to ordinary differential equations (ODEs), which have a single deterministic solution for each set of initial conditions. In mathematical finance, the Black-Scholes equation is of particular importance because it can be used to model the price of stock options [10]. The following is a derivation of the Black-Scholes equation from the Black-Scholes diffusion equation:

Remember the Black-Scholes diffusion SDE which is used to model the price of a stock

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t$$

In this case, $S(t)$ represents the stock price, μ is the drift rate, and σ is the volatility.

Note that in the following equation $V(S(t),t)$ represents the value of an option. Substituting $V(S(t),t)$ into Itô's lemma gives us:

$$dV(S(t),t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2$$

Now, let's plug dS into the above equation.

From above we know:

$$dS^2 = \sigma^2 S^2 dt$$

This gives us:

$$dV(S(t),t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\mu S dt + \frac{\partial V}{\partial S}\sigma S dW_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt$$

Next, let's assume that our position is completely hedged, meaning our portfolio should grow at the risk-free rate r. To model this we introduce a new term Δ, which we will use to eliminate randomness. We can think of Δ as the number of shares we need to completely hedge our position.

If we set $\mu = \frac{\partial V}{\partial S}$, we get:

$$d(V - \Delta S) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\mu S dt + (\frac{\partial V}{\partial S} - \Delta)\Delta S dW_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt - \mu \Delta S dt$$

$$d(V - \Delta S) = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt$$

This technique allows us to ensure there are no arbitrage opportunities, meaning growth happens at the rate r.

$$\frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt = d(V - \Delta S) = r(V - S\frac{\partial V}{\partial S})$$

The above gives us the famous Black-Scholes equation:

$$\frac{\partial V}{\partial t}dt + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt - rV = 0$$

c. Remark on the Black-Scholes Equation

The Black-Scholes equation has become one of the most important equations of all time and is often referred to as the "Trillion Dollar Equation." This is because the equation can be solved analytically in order to find the price of different European style options. However, one limitation of the Black-Scholes equation is that it cannot do the same for more exotic options, like American stock options. In order to price those options, we need to use numerical methods.

3. Numerical Methods

a. Euler-Maruyama Method

1) History of Euler-Maruyama Method

The Euler-Maruyama method, created by Gisiro Maruyama in 1955, extends Euler's original method to SDEs. The Euler-Maruyama method has many practical applications. In the field of mathematical biology, it can be used to approximate the growth of a population when there is some randomness involved [2]. In financial mathematics, the Euler-Maruyama method can be applied to the Black-Scholes differential equation, which is used to price options [10]. In the example below, we will use the Euler-Maruyama method to simulate the price of an underlying asset, a stock index in this case, which is necessary step in calculating the values of stock options.

2) Defining the Euler-Maruyama Method

As stated above, the Euler-Maruyama method is a simple method to numerically estimate the solutions of SDEs. Let us state how the Euler-Maruyama method works. Remember the Black-Scholes diffusion equation which models the price of a stock:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t$$

First we define x_0 as

$$x_0 = S(0)$$

We will use x_i at the estimate for the stock price on the ith day. Next, we define how to estimate the price on day $i+1$ based on the price from day i.

$$x_{i+1} = x_i + \mu S(t_i)\Delta t_{i+1} + \sigma S(t_i)\Delta W_{i+1}$$

Where $\Delta W_{i+1} = z_i\sqrt{\Delta t_i}$, μ is the drift which in this case is the expected return of the stock, and σ is the volatility [10].

The Euler-Maruyama method works in a very similar way to Euler's original method. The glaring difference is the addition of the $\sigma S(t_i)\Delta W_{i+1}$ term. This term helps simulate the randomness which is common in systems like the stock market.

3) Euler-Maruyama Method Example - Index Price Prediction

Both μ and σ can be calculated based on historical returns of a given stock. In the following example, the price of the S&P500 index will be estimated and compared with the actual price. To do this, we will find μ and σ using returns from the previous year. Then we will apply the Euler-Maruyama method to model the prices.

We found $\mu = .1080$ and $\sigma = .1556$. There are 250 trading days in a year, and because we simulated the index for 1 year we set $\Delta t = \frac{1}{250} = .004$. These conditions gave the following result

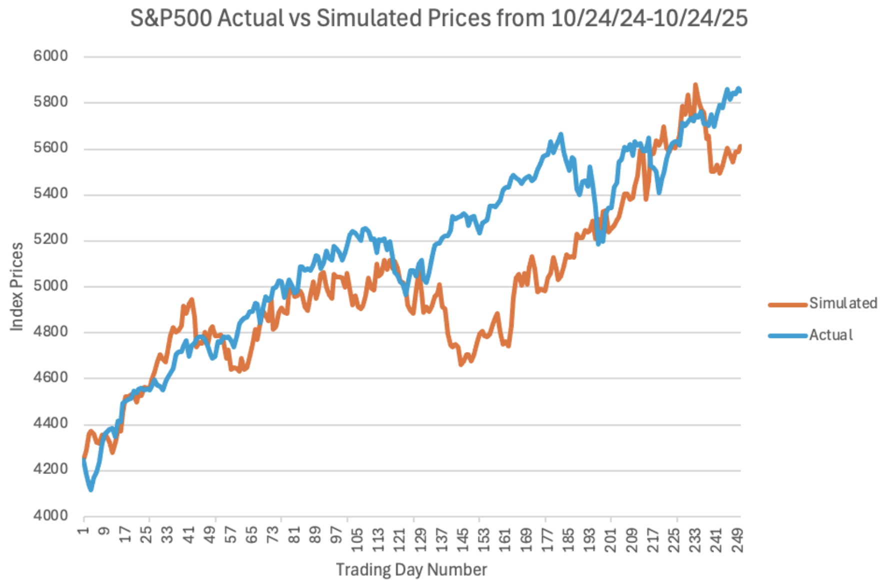

Fig. 2. S&P500 EMM example

In this case the Euler-Maruyama method was fairly accurate. However, this simulation is fairly simple. When more complex situations are encountered one of the next methods is a better choice.

b. Binomial Option Pricing Model

1) History of the Binomial Option Pricing Model

The binomial option pricing model was first introduced by William Sharpe in 1978, who also discovered the capital asset pricing model. However, it was Cox, Ross and Rubenstein in their 1979 paper who first demonstrated it. Furthermore, in 1979, Rendleman and Bartter showed that the binomial model is a discrete-time estimation of the Black-Scholes equation. This means that as the number of steps increases, the value produced by the binomial option pricing model will converge to the value provided by the Black-Scholes equation [7].

The Black-Scholes equation is great because it can be analytically solved, meaning it can be used to find exact solutions for European options. However, for more complex financial derivatives like American options, which can be exercised at any time before the expiration of the contract, the Black-Scholes equation cannot be applied. One popular method for pricing American options is the binomial option pricing model.

2) Binomial Option Pricing Model Explanation

The binomial option pricing model works by breaking the time between the start and expiration of the option contract down into discrete steps.

We can classify binomial trees into recombining trees and nonrecombining trees. At each step there is a series of nodes, for a recombining binomial tree during the nth step, there are $n+1$ nodes, for a total of $\frac{(n+1)(n+2)}{2}$ nodes in the entire tree. For a nonrecombining tree, there are a total of $2^{n+1}-1$ nodes in the tree.

At the start of the contract, there is a single node representing the current price of the underlying asset. From every node, there are two paths representing whether the price of the underlying asset went up or down, this process occurs for every step and produces the following tree structure [7][11].

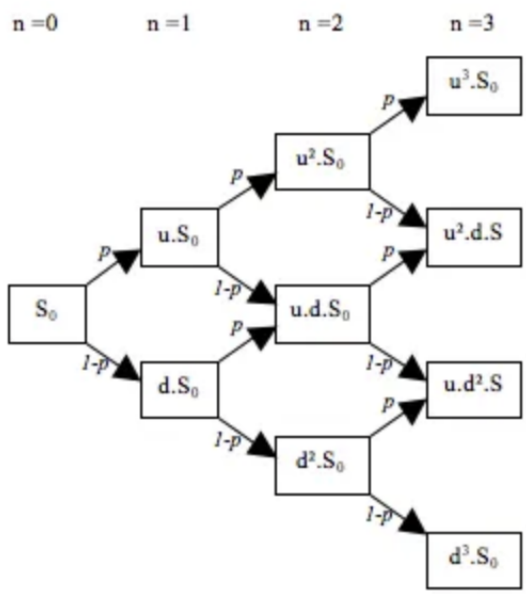

Fig. 3. Recombining Binomial Tree

Here, p is the probability the underlying asset moves up, with $1-p$ being the probability it moves down. At each path we must find the new price of the underlying asset. To do that, we multiply the previous price by either u or d, depending on the direction of the previous path. We let $u = e^{(r-\delta)h+\sigma\sqrt{h}}$ and $d = e^{(r-\delta)h-\sigma\sqrt{h}}$, where δ is the dividend yield and $h = \frac{Time\ until\ expiration\ (In\ years)}{number\ of\ steps}$ [7].

The bimonial option pricing model is able to price American options because at each node it can be calculated whether the option should be exercised or not by checking if the current payoff is greater than the expected payoff.

The binomial pricing models work by calculating the expected price of the underlying asset for each step. Once the binomial pricing model has the estimated price for each node in the nth step, it will solve for the expected payoff of the specified option. From there, we can calculate the payoffs for the $n-1$ nodes using the payoffs from the nth step. Assume that we have a node in the $n-1$ step, which will be referred to as M, whose payoff is $C(M)$. The payoff for M is[7]:

$$C(M) = e^{-rh}(pC(Mu) + (1-p)(C(Md))$$

Where $C(Mu)$ is the payoff from the node in the nth step from the upward direction of node M and $C(Md)$ is the payoff from the node in the nth step from the downward direction of node M.

3) Binomial option pricing model example

As stated above, the binomial option pricing model can prices both American and European options. The premiums for an American option should always be greater than or equal to the premiums for a European option [7]. In the following graph, the premiums for both American (blue) and European (green) call options of the S&P500 index have been calculated for various strike prices using the same data as in the example above.

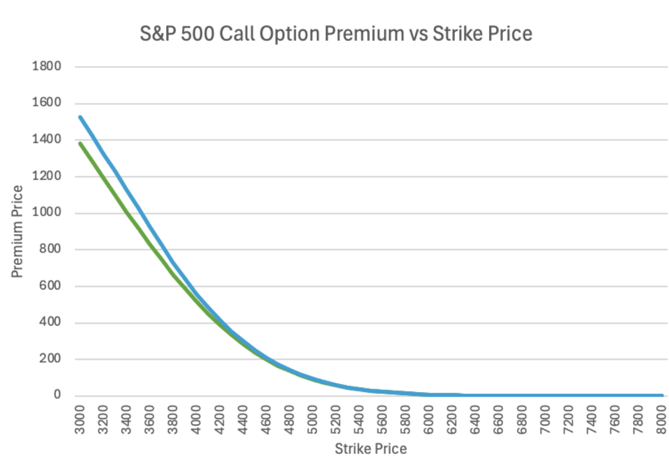

Fig. 4.

It is easy to see that the premiums for the American call option are always more expensive than those for the European call option.

c. Monte Carlo Method

1) Monte Carlo Method History

The Monte Carlo method has a rich history and is used in a wide variety of fields like physics, computer science, and finance. The first application of the modern Monte Carlo method was at the Los Alamos National Laboratory where Stanislaw Ulam, a Polish physicist, developed the method to simulate neutron diffusion in nuclear weapons [9]. The Monte Carlo method was introduced to finance by David Hertz in 1964 but was first applied to options by Phelim Boyle [1].

2) Monte Carlo Method Explanation

The Monte Carlo method is a simple but powerful idea. It essentially involves running a large number of simulations and then taking the average. The Monte Carlo method with respect to option pricing simulates the value of the underlying asset, which can then be used to price the option. Since the Monte Carlo method can simulate the entire lifetime of an option it is incredibly useful for evaluating more options that are path-dependent [7]. Now we must discuss how exactly the Monte Carlo simulates the price of the underlying asset. The first step is to generate a value of a standard random normal variable. The algorithm for the first step is [7]:

1. Using a Uniform distribution, generate a value between 0 and 1. For the purpose of this example, let us say this value is x.

2. Next, we find the value of z such that $N(z) = x$. In order to do this we apply the inverse cumulative distribution function on x, $N^{-1}(x) = z$.

Now that we have a way to randomly generate the values of a standard normal random variable, we can simulate the price of a lognormally distributed stock that does not pay dividends at the expiration date using the following equation [7]:

$$S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z}$$

Although this is a useful demonstration, in reality we use the Monte Carlo method to simulate the path of the price of a stock so we can price more complex options. To do this we break T into n steps, where $T = nh$, similar to how we did it in the binomial options pricing model. Then for each step we use the price generated from the previous step, and generate a new value for Z.

$$S_1 = S_0 e^{(r-\frac{\sigma^2}{2})h+\sigma\sqrt{h}Z_0}$$

$$S_2 = S_1 e^{(r-\frac{\sigma^2}{2})h+\sigma\sqrt{h}Z_1}$$

$$\dots$$

$$S_n = S_{n-1} e^{(r-\frac{\sigma^2}{2})h+\sigma\sqrt{h}Z_{n-1}}$$

The above method allows us to simulate the price of the stock at each step which is incredibly useful [7].

3) Monte Carlo Method Example

As stated above, the Monte Carlo method is useful for pricing more complex options, the Asian option is one such option. The payoff of an Asian option is dependent on the average price of the underlying asset over the lifetime of the option. Asian options are often used when an underlying asset is very volatile because the average price of the asset will be less volatile than the actual asset itself. Asian options vary in whether they are call or put options, are dependent on the arithmetic or geometric mean, and whether it is based upon the asset price or strike price, this results in a total of eight different types of Asian options [7]. For our example we will consider an Asian call option computed using the arithmetic mean and the underlying price of the asset. We will continue using the S&P example that we have used previously. For a long call with a strike price of $4500 the corresponding premium was computed as $84.84. This price was for a year long call option, and there were 10000 simulations run, a graph of the simulations is below.

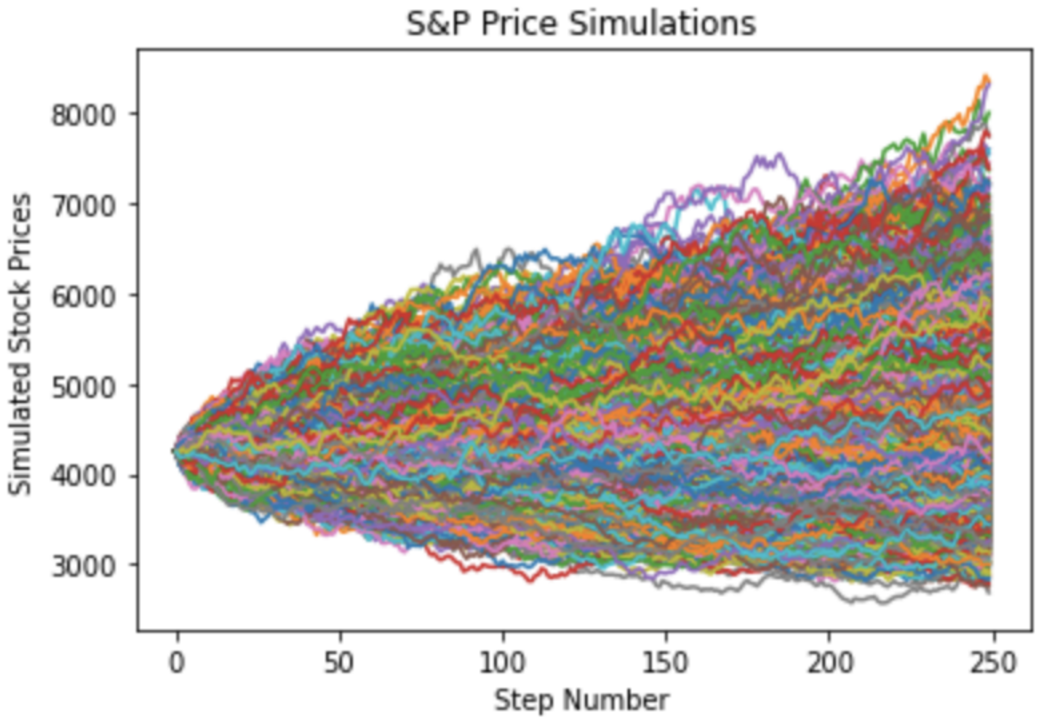

FIG. 5. Monte Carlo Simulations

This graph is fairly chaotic because there were 10000 simulations run. For all of the simulations, the price is simulated at each step. Then to price the Asian call, we take the average of the price at each step for all the simulations. After that we take the average of all of those prices and compute the payoff to get the premium price.

Acknowledgments.

Data availability statement.

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