Chapter 1

Week 1

1.1 Q 1.2.9

(a) I hate this question.

(b) And this one too.

1.2 Q 1.3.1

(a) By definition an edge is incident to 2 and only 2 vertices, thus the sum of objects incident to any edge is 2.

(b) Without any restriction, the column sums of the adjancy matrix could take any value. For a simple graph the column sum has a maximum value of $v - 1$ since there is a maximum of 1 edge to any vertex pair.

1.3 Q 1.6.8

(a) Consider a single component graph. If one removes an edge from this graph there are two possible scenarios:
1. The edge is removed, but there is at least one other edge connecting any two disjoint subgraphs and so $\omega(G)$ does not change.
2. The edge was the only connection between two disjoint subgraphs and so the subgraphs become disconected and $\omega(G)$ increases by one. Since any edge can only connect two vertices, and thus only two subgraphs, it is not possible for the removal of an edge to increase $\omega(G)$ by more than one.

Any multi-component graph will act the same, as we can only take a single edge away from a single component at a time.

(b) Removing a vertex can have a considerably larger effect than removing an edge. Consider the following two cases that break the equality:
1. We remove a vertex that is incident with no edges. In this case, the vertex makes up an entire disconnected component of its own, and so removing it removes a component; i.e. $\omega(G - v) \leq \omega(G)$
2. We remove a vertex that is the only vertex shared by 3 different subgraphs: Let v be the vertex as described and G_x be a subgraph containing v, $x = 0, 1, 2$. Then we have $G_0 \cap G_1 \cap G_2 = \{v\}$
If we remove v then we have $G_0 \cap G_1 \cap G_2 = \emptyset$
So we went from $\omega(G) = 1$ to $\omega(G) = 3$. Clearly $\omega(G - v) \geq \omega(G) + 1$

1.4 Q 1.6.14

(a) Let G be simple, connected and incomplete. Consider 3 vertices u, v, and w with edges $uv, uw, vw \in E \ \forall u, v, w \in G$. Clearly each vertex in G is connected to every other vertex in G and so G must be complete. This is a contradiction, to avoid the contradiction at least 1 of the possible set of three vertices must have the condition that at least 1 of $uv, uw, vw \notin E$

1.5 Q 1.7.2

(a) Assume that G is connected and acyclic. Therefore G is a tree, which means it must have $\delta = 1$. We have a contradiction and so G must contain a cycle.

Chapter 2

Week 2

2.1 Q 2.1.5

(a) Since the number of edges of G is constrained to $v - 1$ we can construct any arbitrary connected graph by sequentially adding a vertex and edge pair to a kernel vertex. They must be added as a pair otherwise the # edges constraint would be violated. Creating a cycle requires adding an edge between two vertices that already exist. Since we cannot at any stage do this, G must be acyclic.
Since G is connected and acyclic, it is a tree.

(b) G is acyclic and so it is a forest. Because it is a forest, we know that $\varepsilon(G) = v(G) - \omega(G)$ and we also have the condition that $\varepsilon(G) = v(G) - 1$. It is clear then that $\omega(G) = 1$ and therefore we have only 1 component which means that G is connected.
Since G is acyclic and connected, it is a tree.

(c) G is a tree which means it is acyclic and connected by definition.

2.2 Q 2.2.4

(a) Since the textbook does not define a Maximal Forest, let's define it as follows: F is a maximal forest of G when for every component H of G, $F \cap H$ is a spanning tree of H.
Thus the solution is: by the definition above.

(b) Each component of the forest is a tree with $v_{sub} - 1$ edges where $\sum_{\omega}(v_{sub}) = v(G)$, so the total number of edges of the forest epsilon(F) is simply $\sum_{\omega}(v_{sub}) - \omega * 1$, i.e. $\varepsilon(F) = v(G) - \omega(G)$.

2.3 Q 2.2.6

(a) Assuming that there IS a cut edge, imagine two disjoint subgraphs A and B of G s.t. there is only one edge e, not in A or B, connecting the two subgraphs, i.e. $A + B + e = G$. Now consider the vertex of A that is the vertex in G incident with e, since e is not included in A, this vertex must have an odd degree ($evennumber - 1 = oddnumber$). However, since every other vertex in A must have an even degree, this is clearly (explain!) impossible so we have a contradiction.

(b) Yuck. To be done.

2.4 Q 2.3.1

(a) Clearly the cut edge must incident to two vertices, if we were to remove either of those vertices it would result in a loss of the edge as well. At a minimum such a cut vertex must increase the number of components by 1, but if it is also incident with other cut edges (which would also disappear) then correspondingly more components are created.

(b) Consider a graph with a central node that is connected to all other vertices, and each pair of outer vertices is also connected. Such a graph has no cut edge but the central node is a cut vertex - removing it would increase the number of components drastically.