Approximating the Gravitational Potential Using Multipole Expansions

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Introduction

A galaxy is a system of stars, gas, dust and dark matter held together by gravitational attraction (Binney & Tremaine 1987). One of the main techniques used to study gravitational systems is by direct N-body simulation. N-body simulations require the gravitational force in order to calculate the dynamics of the system. The gravitational force can be calculated by direct summation of the Newtonian gravitational force equation, requiring N^2 operations per time step. Another method is to implement a tree algorithm, which makes use of neglecting individual bodies' that are far away from the body of interest, then calculating the net force from the collection of bodies center of mass. A tree algorithm requires $\sim N \log N$ operations per time step. To contrast with direct calculation, the gravitational potential, which can be used to find forces, can be approximated by a Self-Consistent Field (SCF) method or Multipole Expansion (MEX). SCF involves an expansion in the angular and radial parts while MEX is just an expansion in the angular part (Meiron et al. 2014). MEX has been advocated for as it is able to easily capture radial structure as there is only an expansion in the angular part (Meiron et al. 2014). It also has the benefit of being able to capture the potentials of arbitrary distributions of matter (Binney & Tremaine 1987).

MEX seeks the solutions of Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$, for a thin spherical shell of variable surface density. If the shell is sufficiently thin, the problem reduces to solving Laplace's equation, $\nabla^2\Phi = 0$, on both sides of the shell, with suitable boundary conditions at the origin, on the shell, and at infinity. Laplace's equation can be solved by separation of variables. The potential for the system is then found by the contribution of all the spherical shells. The specific potential (gravitational potential per unit mass) is described by equation (2-122) of Binney & Tremaine 1987,

$$\Phi(r,\theta,\phi) = -4\pi G \times \sum_{l=0}^{l_{max}} \sum_{m=-l}^{l} \frac{Y_l^m(\theta,\phi)}{2l+1} \left[\frac{1}{r^{(l+1)}} \int_0^r \rho_{lm}(a) a^{(l+2)}\, da + r^l \int_r^\infty \rho_{lm}(a) \frac{1}{a^{(l-1)}}\, da \right] \quad (1)$$

where ρ_{lm} are radial basis functions and l_{max} is the highest order expansion term.

From examining equation (1) above, it is clear that the potential per unit mass for MEX can be found in a very algorithmic way. Since the potential using MEX can be found in a very algorithmic way, by taking the negative gradient of (1) the forces per unit mass can be found (Thornton S., Marion J., 2003).

This thesis aims to develop a code successful in calculating the gravitational potential and successfully calculating the forces per unit mass using MEX.

Methodology

The code that was developed to compute the MEX potential was written in Python using two pre-existing Python packages; NumPy and SciPy. NumPy was used for its N-dimensional array and its math functions, such as sine and cosine, while SciPy was used for numerical integration and interpolation.

From examining equation (1), it is clear that the approximation should be calculated in separate steps. The steps in which the MEX approximation is calculated are as follows:

Step 1: Radially bin the particles

Step 2: Calculate the radial basis functions, ρ_{lm}.

Step 3: Calculate the integral over each combination of lm.

Step 4: Compute the sum over each particle position for each combination of lm.

$$\rho_{lm}(r) = \sum_i \frac{m_i Y_l^{m^*}(\theta_i,\phi_i)}{r_i^2 \Delta r} \quad (3)$$

In order to calculate the potential, the particles of the system need to be binned radially. Choosing to use bins spaced linearly allows sub structure and sharp edges in halos to be captured.

After calculating the basis functions and the integrals from (1), the potential at the particle points can be found. By specifying the position of the particle in spherical coordinates, the potential is then returned at that position. The sum to find the potential at any given specified point is mass independent and does not need to be an actual particle position.

As the force equations resemble the potential equation, the forces are calculated using a similar algorithm.

Results

As the forces are directly related to the potential, a uniform density toy model will be used to compare the forces of that produced by the MEX approximation. The analytic force for the toy model is,

$$f_{r,\ \text{inside}} = -\frac{8\pi G\rho r}{6} \qquad f_{r,\ \text{outside}} = \frac{8\pi G\rho {r_{halo}}^3}{6r^2} \quad (3)$$

By centering the halo on the origin and at a position of z = 400 kpc, MEX will be able to be tested for a spherically symmetric and axisymmetric system.

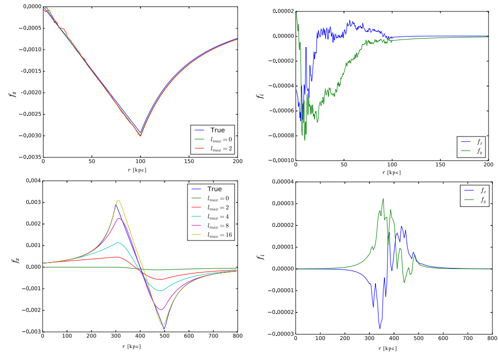

Figure 1: Force produced by MEX compared to that of the toy model. Left side is the z component of force, the right side is the off axis forces for highest l_{max} in corresponding left hand side figure. The top two panels are origin centered, the bottom two are centered on z = 400kpc.

Looking at Fig 1. it is clear that MEX is capturing the force of the toy model. With the spherically symmetric case there isn't improvement increasing l_{max}, but with the axisymmetric case there is improvement. Looking at the off axis force we see that they are noisy inside the halo and converge to zero outside. This is expected as the particles are randomly distributed inside the halo and are not perfectly uniform.

As MEX is able to capture the forces of a uniform density toy model, the next step is to compare the forces produced by MEX to that of a full N-body code. The N-body code used was Gadget (Springel 2005). Gadget evolves self-gravitating collision-less systems using a tree code to compute the gravitational force. A realistic cosmological halo with visible sub-structure will be used for the comparison.

To compare how varying l_{max} and number of bins m_r impact the accuracy of MEX, the root mean square (RMS) values will be calculated between the forces produced by Gadget and MEX for each component of force, that is,

$$\text{RMS}_i = \frac{1}{N} \sum_j^N \sqrt{(f_{i,j\ \text{MEX}} - f_{i,j\ \text{Gadget}})^2} \quad (4)$$

The RMS values will be calculated for each combination of l_{max}= 0,2,4,6 and m_r = 200,250,300,350,400.

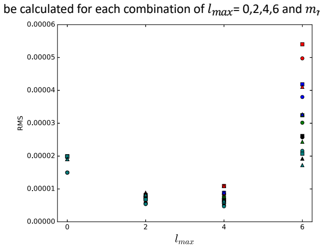

Figure 2: RMS vs. l_{max} for the cosmological halo, circle points are RMS_x, square points are RMS_y and triangular points are RMS_z. The red points are m_r = 200, blue points are m_r = 250, green points are m_r = 300, black points are m_r = 350, teal points are m_r = 400.

We see that the RMS decreases with the number of bins. To a certain point, the RMS decreases with l_{max} and then blows up past an l_{max} of 4. The RMS increases for higher l_{max} because MEX begins to over approximate high density regions. This can be shown by plotting the absolute fractional difference of the two force schemes versus the radius.

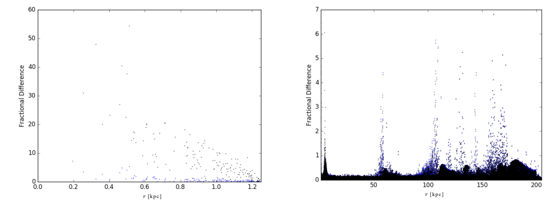

Figure 3: Fractional difference versus radius. Blue points l_{max} = 4, black l_{max} = 6, left figure is for 0 < r < 1.25, right figure is r > 1.25.

Looking at Fig. 3, we see that overall the difference decreases with increase in l_{max}, but in high density regions the fractional difference begins to blow up. This suggests that MEX may need the introduction of some softening parameter to prevent the forces from blowing up in high density regions.

To visually see the improvement between l_{max} = 0 and l_{max} = 4, the forces produced by Gadget will be plotted against those of MEX for each component of force.

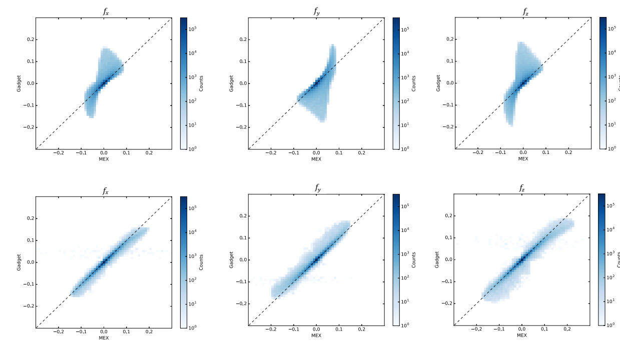

Figure 4: Forces produced by Gadget Vs. forces produced by MEX. Top plots are l_{max} = 0 , bottom l_{max} = 4. Plots from left to right are, x-component, y-component, z-component of force.

Looking at Fig. 4, it is clear that the forces are converging as l_{max} is increased. Looking closely at the lower half of Fig. 4, the presence of particles that are wildly off the line can be seen, this again suggests the need for some softening parameter to prevent the blow up of forces in high density regions.

Conclusion

In conclusion, the MEX approximation is able to produce the forces for N-body distributions. In order to improve accuracy a softening parameter could be introduced to eliminate error in high density regions. Once the softening parameter is successfully introduced, forces as functions of time could be created to evolve N-body systems.

References

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