Cambridge Part III Maths

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Advanced Quantum Field Theory

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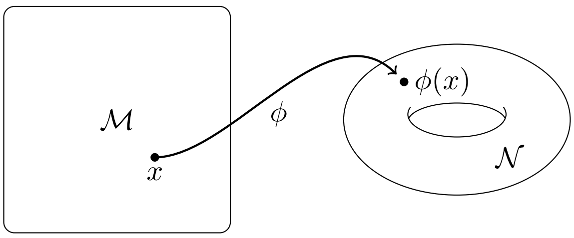
Contents

1 Introduction

Lecture 1
15/01/16

We must do three things to construct a QFT:

1. Pick a smooth manifold $\mathcal{M}$ (which we call *space*, or *spacetime* if it is Lorentzian) in which our QFT is going to live, and choose a metric g for $\mathcal{M}$. For example:
   - In particle physics, $\mathcal{M} = \mathbb{R}^4$ and $g =$ the Minkowski metric η.
   - In statistical physics, $\mathcal{M} = \mathbb{R}^4$ and $g =$ the Euclidean metric δ.
   - In string theory, $\mathcal{M} = \Sigma$, some Riemann surface.
2. Pick some *fields*. These might be:
   - Scalar fields $\phi^a : \mathcal{M} \to \mathbb{R}$, $a = 1, \ldots, m$. This is known as a *linear sigma model.*
   - $\phi : \mathcal{M} \to \mathcal{N}$ where $(\mathcal{N}, G)$ is a general Riemannian manifold. If $\mathcal{N} \neq \mathbb{R}^m$, this is called a *non-linear sigma model.*



- ψ a (Dirac) spinor on $\mathcal{M}$.
- Gauge fields $A_\mu(x)$ (taking values in $\Omega^1(\mathcal{M}, \mathcal{L}(G))$, i.e. the set of all 1-forms on $\mathcal{M}$ with coefficients in the Lie algebra of some Lie group G).

3. We must pick an action $S : \mathcal{C} \to \mathbb{R}$, where $\mathcal{C}$ is the space of fields. For example:
   - For a scalar field ϕ:
   $$S[\phi] = \int_{\mathcal{M}} \mathrm{d}^d x \, \sqrt{g} \Big[\underbrace{\frac{g^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi}_{\text{kinetic term}} - \underbrace{V(\phi)}_{\substack{\text{potential} \\ \text{term}}} \Big]$$
   - For a gauge field A:
   $$S[A] = \frac{1}{4e^2} \int \mathrm{d}^d x \, \sqrt{g} g^{\mu\nu} g^{\rho\sigma} \underbrace{(F_{\mu\rho}, F_{\nu\sigma})}_{\text{Killing form}}$$

   In general, the action, even for a scalar, can be the integral of an arbitrary differential polynomial in the fields:
   $$S[\phi] = \int \underbrace{\mathcal{L}(\phi, \partial^p \phi)}_{\substack{\text{polynomial in } \phi \\ \text{and all derivatives}}} \sqrt{g} \, \mathrm{d}^d x$$

2 QFT in zero dimensions

If $\dim \mathcal{M} = 0$ and $\mathcal{M}$ is connected, then $\mathcal{M} = \{\mathrm{pt}\}$. We'll choose our fields to be $\phi : \{\mathrm{pt}\} \to \mathbb{R}$ (i.e. $\phi \in \mathbb{R}$).
The basic object we want to compute (in any QFT) is the partition function:

$$Z = \int_{\mathcal{C}} \mathrm{D}\phi \, e^{-S[\phi]}$$

which in this case is just:

$$Z = \int_{\mathbb{R}} \mathrm{d}\phi \, e^{-S(\phi)}$$

We'll assume that $e^{-S(\phi)}$ decays sufficiently rapidly as $|\phi| \to \infty$ that this integral converges. In practice, we take $S(\phi)$ to be an even degree polynomial in ϕ, for example:

$$S(\phi) = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4$$

In this case $Z = Z(m, \lambda, \dots)$ is a function of all the coupling constants in $S(\phi)$. For example:

$$Z(m^2) = \int_{\mathbb{R}} \mathrm{d}\phi \, e^{-m^2 \phi^2 / 2} = \frac{\sqrt{2\pi}}{m}$$

If we include more terms, integrals become hard to solve:

$$Z(m^2, \phi) = \int_{\mathbb{R}} e^{-m^2 \phi^2/2 - \lambda \phi^4 / 4!}$$

We can try to evaluate this perturbatively in λ:

$$Z(m^2,\phi) = \int \mathrm{d}\phi \left(e^{-m^2\phi^2/2} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \frac{\phi^{4n}}{n!} \right)$$
$$\overset{?}{\sim} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \frac{1}{n!} \int \mathrm{d}\phi\, \phi^{4n} e^{-m^2\phi^2/2}$$

This integral can't possibly converge as a power series in λ. In fact this is an asymptotic series[1] for $Z(m^2,\lambda)$. Feynman diagram expansions in QFT are (almost) always asymptotic expansions. It can be shown :

$$Z(m^2\lambda) \sim \frac{\sqrt{2\pi}}{m}\left[1 - \frac{\lambda}{8m^4} + \frac{35}{384}\frac{\lambda^2}{m^4} + \dots\right]$$

Inductively, the coefficient of λ^n is $\frac{1}{(4!)^n n!} \times \frac{(4n)!}{4^n(2n)!}$. We expect $\frac{1}{(4!)^n n!}$ as the coefficient in a series expansion. $\frac{(4n)!}{4^n(2n)!}$ is the number of ways of joining $4n$ objects into pairs.

2.1 Feynman rules

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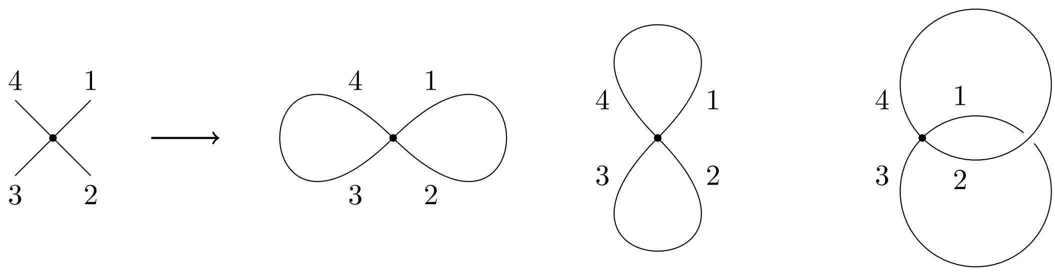
We represent each term in the asymptotic series as a graph. Each propagator —— contributes a factor of $\frac{1}{m^2}$, and each vertex $\times$ contributes a factor of $-\lambda$. For the partition function, we just want vacuum graphs, so we have no external legs. Feynman rules tell us to draw all such graphs, imagining that the lines emanating from each vertex are labelled. For example, at first order (one vertex), we can make the following graphs:



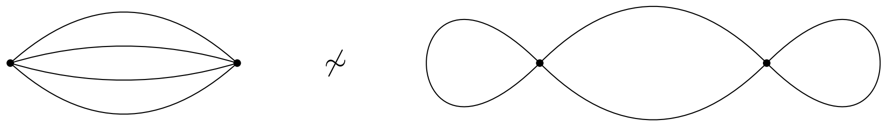
Hence, the first order term in the expansion is given by:

$$\begin{array}{c} \text{counting graphs} \\ \downarrow \end{array}$$
$$\begin{array}{r} \text{one vertex} \longrightarrow \\ \text{two propagators} \longrightarrow \end{array} \frac{-\lambda}{m^4}\frac{3}{4!} = -\frac{\lambda}{8m^4}$$
$$\begin{array}{c} \uparrow \\ \text{expanding exponential} \end{array}$$

[1] Recall: $\sum_{n=0}^{\infty} a_n\lambda^n$ is an asymptotic series for $Z(\lambda)$ if:

$$\lim_{\lambda\to 0} \frac{|Z(\lambda) - \sum_{n=0}^{N} a_n\lambda^n|}{\lambda^N} = 0 \quad \text{for all } N \in \mathbb{N}$$

Drawing *all* of these graphs can be very laborious. There is a shortcut.
Let D_n be the set of all labelled graphs with n vertices. Each will come with a factor of $-\lambda^n$, but as topological (i.e. unlabelled) graphs, we've overcounted, and we should account for this. D_n is naturally acted on by the group $(S_4)^n \times S_n$, where the S_4s act by permuting the labels at each vertex, and the S_n acts be permuting the vertices themselves. This group has order $(4!)^n n!$ and so the weight of Feynman graphs with n vertices is $|D_n|/(4!)^n n!$.
It's useful to rewrite this in the following way. Let Γ be an orbit of $G_n = (S_4)^n \times S_n$ in D_n, i.e. Γ is a topologically distinct unlabelled graph. As an example, the following two graphs are distinct:



Also, let $\mathcal{O}_n$ be the set of these orbits (i.e. the set of topologically distinct graphs with n vertices). Then the orbit-stabilizer theorem states that:

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma \in \mathcal{O}_n} \frac{1}{|\mathrm{Aut}\,\Gamma|}$$

where $\mathrm{Aut}\,\Gamma$ is the set of elements in G_n that fix Γ. $|\mathrm{Aut}\,\Gamma|$ is sometimes called the *symmetry factor*. Thus the weight of the n vertex contribution to the asymptotic series for $Z(m, \lambda)$ is:

$$\frac{(-\lambda)^n}{m^{4n}} \sum_{\Gamma \in \mathcal{O}_n} \frac{1}{|\mathrm{Aut}\,\Gamma|}$$

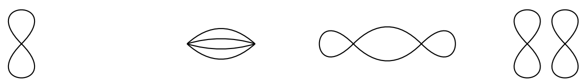
and therefore we have:

$$\begin{aligned} \frac{Z(m,\lambda)}{Z(m,0)} &= \sum_{n=0}^{\infty} \left(\frac{(-\lambda)^n}{m^{4n}} \sum_{\Gamma \in \mathcal{O}_n} \frac{1}{|\mathrm{Aut}\,\Gamma|} \right) \\ &= \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} \frac{(-\lambda)^{|v(\Gamma)|}}{(m^2)^{|e(\Gamma)|}} \end{aligned}$$

where $|e(\Gamma)|$ is the number of edges, and $|v(\Gamma)|$ is the number of vertices.
As a sanity check, lets use this to calculate the first few terms:

$$\frac{Z(m,\lambda)}{Z(m,0)} = 1 + \frac{-\lambda}{m^4} \cdot \frac{1}{8} + \frac{\lambda^2}{m^8} \left[\frac{1}{2} \cdot \frac{1}{4!} + \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{2} \cdot \left(\frac{1}{8} \right)^2 \right] + \cdots$$

$\emptyset$



which if we do the arithmetic will lead to agreement with the above.

More generally we may have several types of field ϕ_i, each with some propagator $\frac{1}{P_i}$. These may interact through many types of vertices α, each with coupling constant λ_α. Then the partition function has an asymptotic series expansion:

$$\begin{aligned}\frac{Z}{Z|_{\lambda_\alpha=0}} &= \sum_\Gamma \frac{1}{|\mathrm{Aut}\,\Gamma|}\frac{\prod_\alpha(-\lambda_\alpha)^{|v_\alpha(\Gamma)|}}{\prod_i P_i^{|e_i(\Gamma)|}} \\ &= \exp\left(\sum_{\Gamma \text{ connected}} \frac{1}{|\mathrm{Aut}\,\Gamma|}\frac{\prod_\alpha(-\lambda_\alpha)^{|v_\alpha(\Gamma)|}}{\prod_i P_i^{|e_i(\Gamma)|}}\right) \\ &= \exp(W)\end{aligned}$$

where we use the last line to define W, the (Helmholtz) free energy.

2.2 Schwinger-Dyson equations

Trivially, $Z = \int_{\mathbb{R}} \mathrm{d}\phi\, e^{-S(\phi)} = \int_{\mathbb{R}} \mathrm{d}(\phi+\epsilon)\, e^{-S(\phi+\epsilon)}$ for any constant ϵ. By translational invariance of the measure, $\mathrm{d}(\phi+\epsilon) = \mathrm{d}\phi$, so we have:

$$Z = \int_{\mathbb{R}} \mathrm{d}\phi\, e^{-S(\phi+\epsilon)} = \int_{\mathbb{R}} \mathrm{d}\phi\, e^{-S(\phi)}\left[1 - \epsilon\frac{\partial S}{\partial\phi}\right] + O(\epsilon^2)$$

If we compare this to our original expression for Z, we see $Z = Z - \epsilon\int \mathrm{d}\phi\, e^{-S}\frac{\partial S}{\partial\phi} + O(\epsilon^2)$, which implies:

$$\left\langle \frac{\partial S}{\partial \phi}\right\rangle = 0$$

where we take the normal definition of the expectation of a function $f(\phi)$ for a given partition function:

$$\langle f(\phi)\rangle = \frac{1}{Z}\int \mathrm{d}\phi\, e^{-S(\phi)} f(\phi)$$

This is essentially Ehrenfest's theorem: the expectation value of the classical equation of motion holds. For example, if $S = \frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!}$, then we have $\left\langle\frac{\partial S}{\partial\phi}\right\rangle = \left\langle m^2\phi + \frac{\lambda\phi^3}{3!}\right\rangle = 0$.

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However, to probe how the fluctuations take us away from the classical equations of motion, we should compute more general correlation functions. We let $f : \mathcal{C} \to \mathbb{R}$ and compute $\langle f\rangle$:

$$\begin{aligned}\langle f\rangle &= \frac{1}{Z}\int_{\mathbb{R}} \mathrm{d}\phi\, e^{-S(\phi)} f(\phi) \\ &= \frac{1}{Z}\int_{\mathbb{R}} \mathrm{d}(\phi+\epsilon)\, e^{-S(\phi+\epsilon)} f(\phi+\epsilon) \\ &= \frac{1}{Z}\int_{\mathbb{R}} \mathrm{d}\phi\, e^{-S(\phi)}\left(f - \epsilon\frac{\partial S}{\partial\phi}f + \epsilon\frac{\partial f}{\partial\phi}\right) + O(\epsilon^2)\end{aligned}$$

The 0^{th} order term is just $\langle f\rangle$, so comparing to the left hand side we have:

$$\left\langle \frac{\partial S}{\partial\phi} f(\phi)\right\rangle = \left\langle\frac{\partial f}{\partial\phi}\right\rangle$$

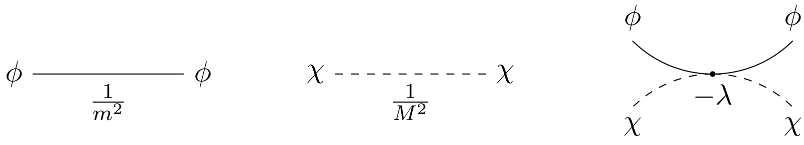
So the classical equation of motion $\frac{\partial S}{\partial\phi} = 0$ does not hold in the presence of so-called *operator insertions*.

2.3 Effective Theories

Let's consider a $d = 0$ QFT involving two fields $(\phi, \chi) \in \mathbb{R}^2$ with the following action (chosen for its simplicity):

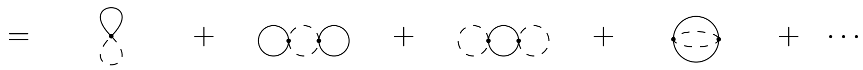
$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2$$

With this action we have a different set of Feynman rules, contributing the following factors to each diagram:



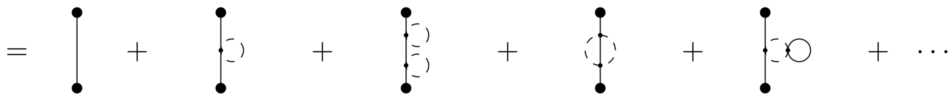
We can use these to compute the partition function. Let $Z_0 = Z|_{\lambda=0}$. We have:

$$\log\frac{Z}{Z_0} = \sum \text{connected vacuum diagrams}$$



$$= \frac{1}{4}\frac{-\lambda}{m^2M^2} + \frac{1}{16}\frac{\lambda^2}{m^4M^4} + \frac{1}{16}\frac{\lambda^2}{m^4M^4} + \frac{1}{8}\frac{\lambda^2}{m^4M^4} + \cdots$$

We can also compute expectation values, for example:

$\left\langle\phi^2\right\rangle =$ 

$$= \frac{1}{m^2} + \frac{1}{2}\frac{-\lambda}{m^4M^2} + \frac{1}{4}\frac{\lambda^2}{m^6M^4} + \frac{1}{2}\frac{\lambda^2}{m^6M^4} + \frac{1}{4}\frac{\lambda^2}{m^6M^4} + \cdots$$

Note that we have no disconnected vacuum diagrams here because we divided by Z in $\langle\cdot\rangle$.
Let's compute these results in a different way. We will briefly include $\hbar$ in our calculations. We define the *effective action* S_{eff} for ϕ by the result of doing just the χ path integral:

$$e^{-S_{\text{eff}}/\hbar} = \int_{\mathbb{R}} \mathrm{d}\chi\, e^{-S(\phi,\chi)/\hbar}$$

We can actually calculate this integral exactly for our choice of $S(\phi, \chi)$:

$$= e^{-m^2\phi^2/2}\sqrt{\frac{2\pi\hbar}{M^2 + \lambda\phi^2/2}}$$

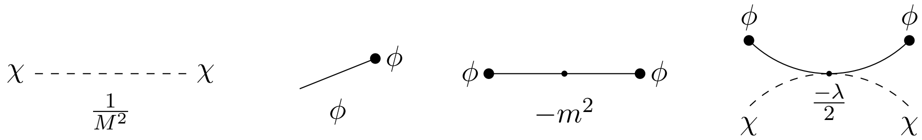
So we have:

$$\begin{aligned}
S_{\text{eff}} &= -\hbar\log\left[e^{-m^2\phi^2/2}\sqrt{\frac{2\pi\hbar}{M^2 + \lambda\phi^2/2}}\right] \\
&= \frac{m^2\phi^2}{2} + \frac{\hbar}{2}\log\left(1 + \frac{\lambda\phi^2}{2M^2}\right) + \underbrace{\frac{\hbar}{2}\log\left(\frac{M^2}{2\pi\hbar}\right)}_{\text{independent of } \phi \text{ so drop it}} \\
&= \frac{m^2\phi^2}{2} + \frac{\hbar\lambda}{4M^2}\phi^2 - \frac{\hbar^2\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots \\
&= \frac{m_{\text{eff}}^2\phi^2}{2} + \frac{\lambda_4\phi^4}{4!} + \frac{\lambda_6\phi^6}{6!} + \dots
\end{aligned}$$

where $m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{4M^2}$, $\lambda_4 = -\frac{\hbar\lambda^2 4!}{16M^4}$, and so on.
Integrating out χ has shifted the effective mass of ϕ. It has also produced an infinite series of new ϕ interactions. These effects are quantum in the sense that they vanish as $\hbar = 0$. More generically, we'd start with an action containing arbitrarily many vertices for ϕ and χ:

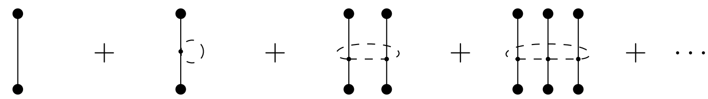
$$S(\phi, \chi) = \sum_{i,j=1}^{\infty} \frac{\lambda_{i,j}\phi^{2i}\chi^{2j}}{i!j!}$$

Then integrating out χ will shift the values of all of the ϕ vertices, $\lambda_{i,0} \to \lambda_{i,0}^{\text{eff}}$. These shifts will be proportional to polynomial powers of $\hbar$. We will usually only be able to carry out the χ integral perturbatively. Lets return to the simple action and do this.
If we just do a χ integral, the Feynman rules are like before, except that we can have external ϕs, which have to be counted explicitly, but can't have any ϕ propagators (unless they are between two external ϕs):



With these rules we obtain:

$-S_{\text{eff}}(\phi) =$



$$= \frac{-m^2\phi^2}{2} + \frac{-\lambda}{4M^2}\phi^2 + \frac{\lambda^2}{16M^4}\phi^4 + \cdots$$

Note that only connected diagrams because $S_{\text{eff}}(\phi)$ is the logarithm of the path integral. We see that the new vertices in $S_{\text{eff}}(\phi)$ and the shift $m^2 \to m_{\text{eff}}^2$ have been generated by loops of the χ field (in this case, just one loop).
We can use our *effective field theory* to compute Z and all correlation functions where the operator insertions depend only on ϕ:

$$\langle \mathcal{O}_1(\phi) \dots \mathcal{O}_n(\phi) \rangle_{S(\phi,\chi)} = \langle \mathcal{O}_1(\phi) \dots \mathcal{O}_n(\phi) \rangle_{S_{\text{eff}}(\phi)}$$

It's better to use S_{eff} if we are interested in many different correlation functions. A general principle to adhere to is: "We should describe physics using just those degrees of freedom that are relevant to the experiments that we are conducting."
If correlation functions involve both ϕ and χ, we require the full interacting theory.

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