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Cambridge Part III Maths

Lent 2016

String Theory

based on a course given by   written up by

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In these notes, variables with arrows above them $\vec{x}$ are ordinary spatial vectors in the temporal gauge, whereas boldface variables $\mathbf{x}$ are transverse vectors in the lightcone gauge.
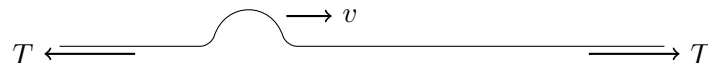
Contents

1 Introduction

Lecture 1
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An *ideal string* is one with negligible cross-sectional area and uniform mass density ρ. Suppose an ideal string is placed under a tension T. It is a simple exercise to deduce that the speed of propagation v of a disturbance in the displacement of the string is given by $v = \sqrt{T/\rho}$.


Special relativity tells us that v must be less than or equal to c ಠ_ಠ, so we can immediately deduce the inequality $T \leq \rho c^2$. For a non-relativistic string (e.g. a violin string), $T \ll \rho c^2$. However the strings we are interested in are said to be *ultra-relativistic*: they have $T = \rho c^2$. In this case, the only relevant quantity is T.

In a quantum theory of strings, we can take advantage of Planck's constant to define a length scale for a string under tension T: the *string length* $l_s \sim \sqrt{\hbar c^2/T}$. From now on, unless otherwise indicated, we will use natural units where $\hbar = c = 1$. Sometimes we will write $T = 1/2\pi\alpha'$, where α' is the *Regge slope parameter*, and choose $l_s = \sqrt{\alpha'}$.

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Initially, it was supposed that strings operated on an atomic scale, so that $l_s \sim 1$ fermi $= 10^{-15}$ m, but this failed to agree with experimental results. In electron-proton interactions, this string model predicted weak scattering, but in fact strong scattering was observed, suggesting pointlike constituent particles (eventually explained as the quarks of QCD).

A persistent prediction of all string theories is the existence of a massless spin 2 particle. This is what you would expect from a theory of quantum gravity because if you linearise general relativity about a Minkowski background you in fact obtain a massless spin 2 particle, known as the graviton. So in 1974, string theory was reinterpreted as a theory of quantum gravity. Since we are dealing with gravity we have access to Newton's constant G, and so can define another lengthscale, the Planck length $l_P = \sqrt{\hbar G/c^3} \approx 10^{-35}$ m. Under this reinterpretation we say that the string length is on the order of magnitude of a Planck length. In fact, for string theory in a Minkowski background to be consistent, we require $l_s \ll l_P$.

Strings can be one of two types:

open   closed
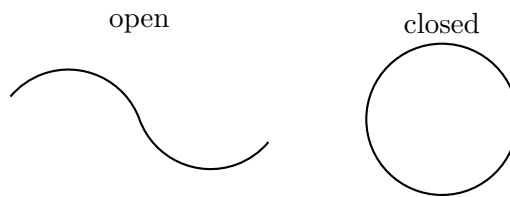
As it evolves, a string sweeps out a *worldsheet* in spacetime:
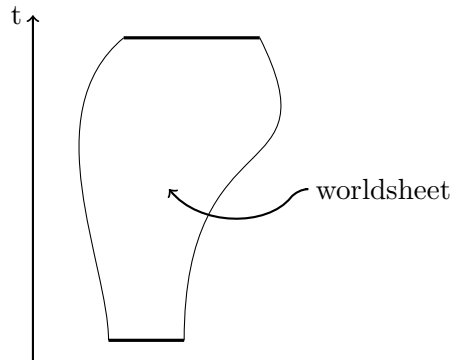
They interact by splitting and joining. One of the main faults of QFT is that interactions occur at distinct points, leading to large ultraviolet divergences, but in string theory we can avoid this by imposing that the worldsheets of interacting strings must be smooth manifolds embedded in spacetime. So for example, two strings scattering off one another might be represented in the following way:

Confer with Feynman diagrams:
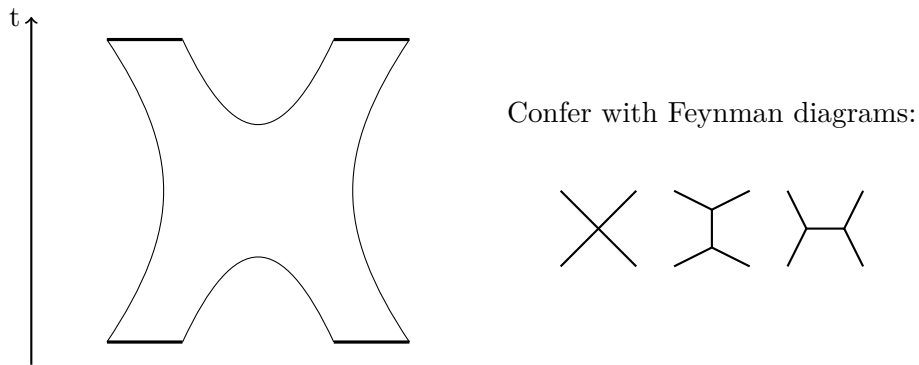
Note that it is always possible to produce a closed string from an open one:
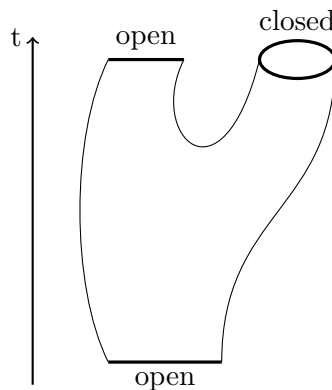
Since the massless spin 2 particle appears in the spectrum of closed strings, this means that a theory of strings is always a theory of quantum gravity.

Note that strings can only interact with themselves. String theory is often popularly described as a "Theory of Everything". In fact, string theory must either be a "Theory of Everything" or a "Theory of Nothing". This is because if it doesn't describe everything, then certain things will not interact with gravity, but we observe that everything must interact with gravity.

There are several different types of string theory. The simplest is bosonic string theory, but it has several problems:

- It allows tachyons to exist. These are particles with $m^2 < 0$ and which lead to instabilities.

- As the name indicates, it does not predict fermions.

- Self consistency requires that $D = 26$, where D is the number of spacetime dimensions.

The only fully consistent string theories are the *superstring* theories, which make use of supersymmetry. These do not allow tachyons, and have $D = 10$. In the 1980s, 5 types of superstring theory were found: Type I, Type IIA, Type IIB, Heterotic E, and Heterotic B. We will discuss Type IIA and Type IIB. All of these theories appear to be different because the coupling is weak, but in the 1990s they were unified into one (arbitrarily coupled) theory known as *M-theory*, with $D = 11$.

2 Relativistic point particle

Lecture 2
16/01/16

What is an *elementary* particle? A very mathematical definition due to Wigner is: "a unitary irrep of the Poincaré group classified by mass and spin". We however will focus on a more physical

definition: "a particle without structure". This definition requires that the classical action of a particle depends only on the geometry of its worldline. The simplest possibility is that the action is proportional to the length of the worldline:

$$I = -mc^2 \int_A^B d\tau \qquad \text{where } c^2\,d\tau^2 = -\,ds^2 = c^2\,dx^{0^2} - \sum_{i=1}^{D-1} dx^{i^2}$$

$$\implies I = -mc^2 \int_{t_A}^{t_B} dt\,\sqrt{-\dot{x}^2} \qquad \dot{x}^m = \frac{\partial x^m(t)}{\partial t},\ t = \text{arbitrary monotonic worldline time}$$

Suppose we tried to include terms involving the extrinsic curvature K:

$$I = -mc^2 \int_{t_A}^{t_B} dt\,\sqrt{-\dot{x}^2}\left(1 + (lK)^2 + \dots\right)$$

where l is some characteristic length scale. If $K^{-1} \gg l$, then such extra terms are irrelevant, but this would suggest some kind of structure. So for elementary particles, we must have $l = 0$.

We have a gauge invariance in the above description: t is arbitrary. Thus we can reparametrise without consequence:

$$t \to t^*(t) \quad x(t) \to x^*(t^*) = x(t)$$

Lets check this for infinitesimal reparametrisations $t^*(t) = t - \xi(t)$. We have:

$$x^*(t - \xi(t)) = x(t) \implies \delta_\xi x(t) = x^*(t) - x(t) = \xi\dot{x}$$

Also:

$$\begin{aligned}
\delta_\xi \sqrt{-\dot{x}^2} &= -\frac{\dot{x}}{\sqrt{-\dot{x}^2}} \cdot \delta_\xi \dot{x} \\
&= -\frac{\dot{x}}{\sqrt{-\dot{x}^2}} \cdot \frac{d\delta_\xi x}{dt} \\
&= -\frac{\dot{x}}{\sqrt{-\dot{x}^2}} \cdot \left(\dot{\xi}\dot{x} + \xi\ddot{x}\right) \\
&= -\dot{\xi}\sqrt{-\dot{x}^2} - \frac{\xi\dot{x} \cdot \ddot{x}}{\sqrt{-\dot{x}^2}} \\
&= \frac{d}{dt}\left(\xi\sqrt{-\dot{x}^2}\right)
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
I(x) = -mc \int_{t_A}^{t_B} dt\,\sqrt{-\dot{x}^2} &\to -mc \int_{t_A^*}^{t_B^*} dt^*\,\sqrt{-\left(\frac{dx^*}{dt^*}\right)^2} \\
&= -mc \int_{t_A - \xi(t_A)}^{t_B - \xi(t_B)} dt^*\,\sqrt{-\left(\frac{dx^*}{dt^*}\right)^2}
\end{aligned}$$

Relabel t^* to t:

$$\begin{aligned}
&= -mc \int_{t_A - \xi(t_A)}^{t_B - \xi(t_B)} dt\,\sqrt{-(\dot{x}^*)^2} \\
&= -mc \int_{t_A - \xi(t_A)}^{t_B - \xi(t_B)} dt\,\left[\sqrt{-\dot{x}^2} + \frac{d}{dt}\left(\xi\sqrt{-\dot{x}^2}\right)\right]
\end{aligned}$$

$$\implies \delta I(x) = -mc\left(\int_{t_A-\xi(t_A)}^{t_B-\xi(t_B)} - \int_{t_A}^{t_B}\right)\mathrm{d}t\,\sqrt{-\dot{x}^2} - mc\int_{t_A-\xi(t_A)}^{t_B-\xi(t_B)}\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\xi\sqrt{-\dot{x}^2}\right)$$

$$= mc\left(\int_{t_B-\xi(t_B)}^{t_B} - \int_{t_A-\xi(t_A)}^{t_A}\right)\mathrm{d}t\,\sqrt{-\dot{x}^2} - mc\left[\xi\sqrt{-\dot{x}^2}\right]_{t_A-\xi(t_A)}^{t_B-\xi(t_B)}$$

$$= mc\left(\xi(t_B-\xi(t_B))\sqrt{-\dot{x}^2}|_{t_B-\xi(t_B)} - \xi(t_A-\xi(t_A))\sqrt{-\dot{x}^2}|_{t_A-\xi(t_A)}\right) - mc\left[\xi\sqrt{-\dot{x}^2}\right]_{t_A-\xi(t_A)}^{t_B-\xi(t_B)}$$

$$= 0$$

as expected.

Note that gauge invariance is *not* a symmetry, but instead implies redundancy. We can eliminate this by imposing a *gauge condition*. For example, the *temporal gauge* is $x^0(t) = t$. In this gauge we have $I = -mc^2 \int \mathrm{d}t\,\sqrt{1 - \frac{v^2}{c^2}}$ where $v = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right|$. If we subtract the negative rest energy $-mc^2$ and take $c \to \infty$, we recover the normal non-relativistic action:

$$I \to I_{\mathrm{NR}} = \int \mathrm{d}t\,\frac{1}{2}mv^2$$

From now on we will take $c = 1$.

2.1 Hamiltonian formulation

We have a Lagrangian $\mathcal{L} = -m\sqrt{-\dot{x}^2}$, from which we can obtain a conjugate momentum:

$$p_m = \frac{\partial \mathcal{L}}{\partial \dot{x}^m} = \frac{m\dot{x}_m}{\sqrt{-\dot{x}^2}}$$

In particular, we have $p^2 + m^2 = 0$. Not all of the components of p are independent, and we can't solve for $\dot{x}$ in terms of p. Also, the Hamiltonian is identically zero: $H = p \cdot \dot{x} - \mathcal{L} = \frac{m\dot{x}^2}{\sqrt{-\dot{x}^2}} + m\sqrt{-\dot{x}^2} = 0$. Dirac's formalism (developed to deal with these cases) tells us that we should instead promote p to a free variable, and instead use a Lagrange multiplier to impose the $p^2 + m^2 = 0$ constraint. We obtain the following action:

$$I[x, p; e] = \int \mathrm{d}t\left(\dot{x}^m p_m - \frac{1}{2}e(p^2 + m^2)\right) \tag{$*$}$$

We can check this is valid by eliminating p and e. We have $\frac{\delta I}{\delta p} = 0 \implies p = e^{-1}\dot{x}$, and we can use this to eliminate p:

$$I[x, p; e] \to I[x; e] = \frac{1}{2}\int \mathrm{d}t\left(e^{-1}\dot{x}^2 - em^2\right)$$

We could interpret this as a sort of 1D gravity with metric $g = -e^2$ and cosmological constant m^2, coupled to D 1D scalar fields.

Now, $\frac{\delta I[x;e]}{\delta e} = 0 \implies e = \frac{\sqrt{-\dot{x}^2}}{m}$ and we can use this to eliminate e:

$$I[x; e] \to I[x] = -m\int \mathrm{d}t\,\sqrt{-\dot{x}^2}$$

So we regain the original action.

Note that $(*)$ is still invariant under time reparametrisations (which we will refer to as Diff_1). Under such transformations we have:

$$\delta_\xi x = \xi\dot{x}, \quad \delta_\xi p = \xi\dot{p}, \quad \delta_\xi e = \frac{\partial}{\partial t}(e\xi)$$

In other words, x and p are scalars, while e is a scalar density. Combining these gives that $\delta I =$ a boundary term, which we choose to ignore. $(*)$ is also invariant under the *canonical* gauge transformation:

$$\delta_\alpha x = \alpha p, \quad \delta_\alpha p = 0, \quad \delta e = \dot{\alpha}$$

> **Definition.** Given an action $I[\phi, \psi]$, a *trivial gauge transformation* is one that can be written as follows:
> $$\delta_f \phi = f \frac{\delta I}{\delta \psi}, \quad \delta_f \psi = -f \frac{\delta I}{\delta \phi}$$
> for some function f.

Note that under a trivial gauge transformation we have $\delta I = \delta_f \phi \frac{\delta I}{\delta \phi} + \delta_f \psi \frac{\delta I}{\delta \psi} = 0$, so it is indeed a gauge transformation.

The Diff_1 and canonical gauge invariances differ by a trivial gauge transformation; we say they are equivalent.

2.2 Lightcone gauge

If we choose the temporal gauge $x^0(t) = t$, we have $\dot{x}_m p^m = \dot{x}^0 p_0 + \dot{\vec{x}} \cdot \vec{p} = \dot{\vec{x}} \cdot \vec{p} - p^0$. If we use $(*)$ and compare this to $L = \dot{x}^m p_m - H$, we have $H = p^0 = \pm\sqrt{|\vec{p}|^2 + m^2}$, where we used the constraint $p^2 + m^2 = 0$ to find the second inequality.

The Hamiltonian depends on the choice of gauge. Another important possible choice is the *lightcone gauge*, which we will now describe. We define lightcone coordinates as follows:

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1}), \quad \mathbf{x} = (x^1, \ldots, x^{D-2})$$

x^I, $I = 1, \ldots, D - 2$, are said to be coordinates in *transverse space*. Similarly, we have:

$$p_\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_{D-1}), \quad \mathbf{p} = (p_1, \ldots, p_{D-2})$$

Using these coordinates we have:

$$\dot{x}^m p_m = \dot{x}^- p_- + \dot{x}^+ p_+ + \dot{\mathbf{x}} \cdot \mathbf{p} \quad \text{and} \quad p^2 = 2p_+ p_- + |\mathbf{p}|^2$$

Lightcone gauge is the choice $x^+(t) = t$. This choice is ok provided that $p_- \neq 0$, but note that $p_- = 0$ implies via $p^2 + m^2 = 0$ that both $\mathbf{p}$ and m are equal to 0. In the lightcone gauge, we have $\dot{x}^m p_m = \dot{\mathbf{x}} \cdot \mathbf{p} + \dot{x}^- p_- + p_+$, so the Hamiltonian is now $-p_+$. Using $p^2 + m^2 = 0$, we have then that:

$$I = \int \mathrm{d}t \, (\dot{\mathbf{x}} \cdot \mathbf{p} + \dot{x}^- p_- - H) \quad \text{where} \quad H = \frac{|\mathbf{p}|^2 + m^2}{2p_-}$$

The action is still Lorentz invariant, despite the fact that $x^+ = t$ is not Lorentz invariant (neither for that matter is $x^0 = t$). In fact, it is not possible to break a symmetry by choosing a gauge.

2.3 Poisson brackets

For a mechanical model with action

$$I[q, p] = \int \mathrm{d}t \, \dot{q}^I p_I - H(q, p)$$

q and p are here coordinates on phase space. A *Possion bracket* is a way of taking two functions f and g on phase space and producing a third function on phase space:

$$\{f,g\}_{\mathrm{PB}} = f\left[\frac{\overleftarrow{\partial}}{\partial q^I}\frac{\partial}{\partial p_I} - \frac{\overleftarrow{\partial}}{\partial p^I}\frac{\partial}{\partial q_I}\right]g = -\{g,f\}_{\mathrm{PB}}$$

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For example, we have $\{q^I, p_J\}_{\mathrm{PB}} = \delta^I_J$.

More generally, we could use any coordinates z^A, $A = 1, \ldots, 2N$ on phase space. Phase space is a symplectic manifold[1]. Now we define the Poisson bracket in a slightly different way:

$$\{f,g\}_{\mathrm{PB}} = f\left[\frac{\overleftarrow{\partial}}{\partial z^A}(\Omega^{-1})^{AB}\frac{\partial}{\partial z^B}\right]g$$

This definition is also antisymmetric and obeys the Jacobi identity, so it is a valid Lie bracket.

𝕿heorem 1 (Darboux)**.** *There exist local coordinates p_I, q^I such that $\Omega = \mathrm{d}p_I \wedge \mathrm{d}q^I$, where $I = 1, \ldots, N$.*

p_I and q^I are known as *Darboux* coordinates. When Ω is written in terms of Darboux coordinates, the two definitions for the Poisson bracket above coincide.

> **Definition.** A *canonical transformation* (or *symplectic diffeomorphism*) is a diffeomorphism that preserves the symplectic 2-form.

Any function Q on phase space is the generator of a 1-parameter group of diffeomorphisms. Infinitesimally, functions transform in the following way:

$$f(z) \to f(z) + \{f, \epsilon Q\}_{\mathrm{PB}} + O(\epsilon^2)$$

where we take $\epsilon \ll 1$. For example, Darboux coordinates have:

$$\delta_\epsilon q^I = \epsilon\{q^I, Q\}_{\mathrm{PB}} = \epsilon\frac{\partial Q}{\partial p_I}$$

$$\delta_\epsilon p_I = \epsilon\{p_I, Q\}_{\mathrm{PB}} = -\epsilon\frac{\partial Q}{\partial q^I}$$

Notice that:

$$\begin{aligned}
\delta_\epsilon \Omega &= \delta_\epsilon\left(\mathrm{d}p_I \wedge \mathrm{d}q^I\right)\\
&= \epsilon\left(\mathrm{d}p_I \wedge \mathrm{d}\frac{\partial Q}{\partial p_I} - \mathrm{d}\frac{\partial Q}{\partial q^J} \wedge \mathrm{d}q^J\right)\\
&= \epsilon\frac{\partial^2 Q}{\partial p_I \partial q^J}\underbrace{\left(\mathrm{d}p_I \wedge \mathrm{d}q^J - \mathrm{d}p_I \wedge \mathrm{d}q^J\right)}_{=0} + \epsilon\underbrace{\frac{\partial^2 Q}{\partial p_I \partial p_J}}_{\text{symmetric}}\underbrace{\mathrm{d}p_I \wedge \mathrm{d}p_J}_{\text{antisymmetric}} - \epsilon\underbrace{\frac{\partial^2 Q}{\partial q^I \partial q^J}}_{\text{symmetric}}\underbrace{\mathrm{d}q^I \wedge \mathrm{d}q^J}_{\text{antisymmetric}}\\
&= 0
\end{aligned}$$

So this is a canonical transformation.

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[1]That is, it permits a closed invertible 2-form Ω, called a *symplectic* 2-form.

2.4 Gauge invariance and first-class constraints

Consider the following action:

$$I = \int \mathrm{d}t \left(\dot{q}^I p_I - \lambda^i \varphi_i(q, p) \right) \tag{†}$$

where $I = 1, \ldots, N$ and $i = 1, \ldots, n < N$. Note that we could instead consider:

$$I = \int \mathrm{d}t \left(\dot{q}^I p_I - H(q, p) - \lambda^i \varphi_i \right)$$

but this is no more general as it is always possible to rewrite this in the form (†) by enlarging the phase space $(q^I, p_I) \to (q^I, q^0, p_I, p_0)$.
We have n phase-space contraints $\varphi_i = 0$ with Lagrange multipliers λ^i.

> **Definition.** If $\{\varphi_i, \varphi_j\}_{\mathrm{PB}} = f_{ij}^k(z)\varphi_k$ for some functions of phase space f_{ij}^k, then we say that the set of constraints is *first-class*.

If the f_{ij}^k are constants, then the Jacobi identity holds, so the f_{ij}^k are structure constants of a Lie algebra (we require in general f_{ij}^k to be constant because the Poisson bracket contains derivatives). Let's calculate $\delta_\epsilon \dot{q}^I p_I$:

$$
\begin{aligned}
\delta_\epsilon \dot{q}^I p_I &= p_I \frac{\partial \delta_\epsilon q^I}{\partial t} + \dot{q}^I \delta_\epsilon p_I \\
&= p_I \frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon \frac{\partial Q}{\partial p_I} \right) - \dot{q}^I \epsilon \frac{\partial Q}{\partial q_I} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon p_I \frac{\partial Q}{\partial p_I} \right) - \epsilon \left(\dot{p}_I \frac{\partial Q}{\partial p_I} + \dot{q}^I \frac{\partial Q}{\partial q^I} \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon p_I \frac{\partial Q}{\partial p_I} \right) - \epsilon \dot{Q} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\epsilon \left(p_I \frac{\partial Q}{\partial p_I} - Q \right) \right] + \underbrace{\dot{\epsilon} Q}_{\substack{\text{if we allow} \\ \epsilon = \epsilon(t)}}
\end{aligned}
$$

If we let $\epsilon Q = \epsilon^i \varphi_i$, we get $\delta_\epsilon \dot{q}^I p_I = \dot{\epsilon}^i \varphi_i + \frac{\mathrm{d}}{\mathrm{d}t}(\ldots)$. We also have:

$$\delta_\epsilon \lambda^i \varphi_i = \varphi_i \delta_\epsilon \lambda^i + \lambda^i \epsilon^j \{\varphi_i, \varphi_j\}_{\mathrm{PB}} = \left(\delta_\epsilon \lambda^k + \lambda^i \epsilon^j f_{ij}^k \right) \varphi_k$$

where we can make the last equality if the constraints $\{\varphi_i\}$ are first class. Therefore we have:

$$\delta_\epsilon L = (\dot{\epsilon}^k - \delta_\epsilon \lambda^k - \lambda^i \epsilon^k f_{ij}^k)\varphi_k + \frac{\mathrm{d}}{\mathrm{d}t}(\ldots)$$

So we see that δI is just a boundary term if we choose the folowing:

$$\delta \lambda^k = \dot{\epsilon}^k + \epsilon^i \lambda^j f_{ij}^k \tag{∗}$$

For constant f_{ij}^k this is in fact a Yang-Mills gauge transformation for a group whose Lie algebra has structure constants f_{ij}^k. To see this, compare to the expression for the infinitesimal difference in a Yang-Mills field after a gauge transformation:

$$\delta A_\mu^C = \partial_\mu \epsilon^C + \epsilon^A A_\mu^B f_{AB}^C$$

In our case, the Lagrange multipliers are 1D gauge potentials for a gauge group.

> **Example.** For the point particle, we have a single constraint $\phi = \frac{1}{2}(p^2 + m^2)$. Since there is only one constraint, it must be first-class, and the single structure constant is a constant (zero). We have:
> $$\delta_\alpha x = \alpha \left\{ x, \frac{1}{2}(p^2 + m^2) \right\}_{\text{PB}} = \alpha p \quad \delta_\alpha p = \alpha \left\{ p, \frac{1}{2}(p^2 + m^2) \right\} = 0$$
> and using ($*$) we see that $\delta_\alpha \epsilon = \dot{\alpha}$. This is the canonical gauge transformation mentioned above.

This was an abelian example. When we get to strings, we will see that the gauge group is non-abelian.

2.5 Gauge fixing

We can fix gauge invariance (generated by the constraints φ_i) by choosing n *gauge fixing functions* $\chi^i(q, p)$. A general gauge transformation acts on χ^i as follows:

$$\delta_\epsilon \chi^i = \{\chi^i, \epsilon^j \varphi_j\}_{\text{PB}} = \{\chi_i, \varphi_j\}_{\text{PB}} \epsilon^j$$

In order for χ^i to actually fix the gauge, we want for $\delta_\epsilon \chi^i = 0$ only if $\epsilon^i = 0$ for all i. This is true if and only if $\det\{\chi^i, \varphi_j\}_{\text{PB}} \neq 0$.

If $\chi^i \ll 1$ and is initially non-zero, we want to be able to make a gauge transformation to bring it to zero, and again we can do this if and only if $\det\{\chi^i, \varphi_j\}_{\text{PB}} \neq 0$.

Lecture 5
23/01/16

2.6 Continuous symmetries and Noether's Theorem

Many actions have continuous symmetries. For example, the relativistic particle

$$\mathscr{F}\!in$$