Cambridge Part III Maths

Lent 2016

Black Holes

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In this course we take $G = c = 1$ and $\Lambda = 0$.

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1 Spherical stars

Lecture 1
15/01/16

Consider a gas of cold fermions. This gas will resist compression due to *degeneracy pressure* resulting from the Pauli principle. For example, in a white dwarf star, gravity is balanced by electron degeneracy pressure. Using Newtonian gravity, we can find an upper limit on the mass of a stable white dwarf, known as the *Chandrasekhar limit*:

$$M_{\text{WD}} \lesssim 1.4 M_{\odot}$$

In a neutron star, gravity is balanced by neutron degeneracy pressure. Neutron stars are tiny; a neutron star with the mass of the sun has a radius of approximately 10 km (for comparison, the sun has a radius of $R_{\odot} \approx 7 \times 10^5$ km. At the surface of a neutron star, the gravitational potential $|\phi| \approx 0.1$. Recall that in order to be able to apply Newtonian gravity, we must have $|\phi| \ll 1$. $0.1 \not\ll 1$, so it is important to consider general relativity when reasoning about neutron stars.
In this section we will establish that $M \lesssim 3M_{\odot}$ for *any* cold star.

1.1 Spherical symmetry and time independence

Consider the round metric on S^2:

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$$

Equipped with this metric, if we exclude reflections, S^2 has $SO(3)$ as its isometry group. This motivates the following:

Definition. A spacetime is *spherically symmetric* if its isometry group has an $SO(3)$ subgroup whose orbits are 2-spheres.

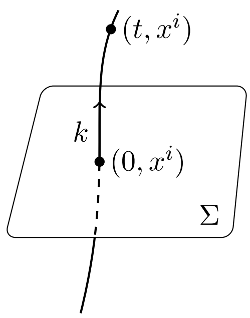
Definition. The *area-radius function* $r : \mathcal{M} \to \mathbb{R}$ is defined by:

$$A(p) = 4\pi r(p)^2, \quad r(p) \geq 0$$

where $A(p)$ is the area of the $SO(3)$ orbit through p.

A consequence of this is that the induced metric on the $SO(3)$ orbit through p is $r(p)^2\,\mathrm{d}\Omega^2$.

Definition. $(\mathcal{M}, g)$ is *stationary* if it permits a timelike Killing vector field (KVF).



Suppose we have a stationary spacetime with timelike Killing vector k. Let Σ be a spacelike 3 dimensional hypersurface, and let x^i, $i = 1, 2, 3$ be coordinates on Σ.
We define coordinates for the manifold in the following way: from each point (x^1, x^2, x^3) extend an integral curve of k; the point (t, x^i) is a parameter distance t along this curve.
In the chart (t, x^i), we can write:

$$k = \frac{\partial}{\partial t}$$

Then, using the defining property of Killing vectors, we have that the metric is independent of t. Thus, we can write:

$$\mathrm{d}s^2 = g_{00}(x^k)\,\mathrm{d}t^2 + 2g_{0i}(x^k)\,\mathrm{d}t\,\mathrm{d}x^i + g_{ij}(x^j)\,\mathrm{d}x^i\,\mathrm{d}x^j$$

and we have $g_{00} < 0$ since k is timelike.
Suppose we have a surface Σ given by $f(x) = 0$, where $f : \mathcal{M} \to \mathbb{R}$, $\mathrm{d}f\,|_\Sigma \neq 0$. Then $\mathrm{d}f$ is normal to Σ. Suppose n is another 1-form that is normal to Σ. Then we can write $n = g\,\mathrm{d}f + fn'$, where g is a function and n' is some 1-form. We have:

$$\mathrm{d}n = \mathrm{d}g \wedge \mathrm{d}f + g\,\underbrace{\mathrm{d}^2 f}_{=0} + \mathrm{d}f \wedge n' + f\,\mathrm{d}n'$$

$$\implies \mathrm{d}n\,|_\Sigma = (\mathrm{d}g - n') \wedge \mathrm{d}f \implies n \wedge \mathrm{d}n\,|_\Sigma = 0$$

In fact, the converse is true:

Theorem 1 (Frobenius)**.** *If n is a 1-form such that $n \wedge \mathrm{d}n = 0$, then there exist functions f, g such that $n = g\,\mathrm{d}f$, so that n is normal to surfaces of constant f.*

If n is a 1-form of this type, we say it is *hypersurface-orthogonal.*

> **Definition.** $(\mathcal{M}, g)$ is *static* if it contains a hypersurface-orthogonal timelike KVF.

Suppose we are in a static spacetime, and define coordinates t, x^i as before. Σ is a surface of constant t, so we have $k \propto dt$, $k_\mu \propto (1,0,0,0)$. Also note that $k_\mu = g_{\mu\nu}k^\nu = g_{\mu\nu}(\frac{\partial}{\partial t})^\nu = (g_{00}, g_{10}, g_{20}, g_{30})$. Hence we can deduce that $g_{i0} = 0$, and can write the metric as:

$$\mathrm{d}s^2 = g_{00}(x^k)\,\mathrm{d}t^2 + g_{ij}(x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j$$

where as before $g_{00} < 0$. In this metric we have a discrete isometry $(t, x^i) \to (-t, x^i)$. A static metric must be time-independent *and* invariant under time reversal. A simple case of a stationary but not static metric is that associated with a rotating star. If we reverse time the star spins in the other direction.

1.2 Static, spherically symmetric spacetimes

If we have a spacetime that is both stationary and spherically symmetric, then the isometry group must contain:

$$\underbrace{\mathbb{R}}_{\substack{\text{time}\\ \text{translation}}} \times \underbrace{SO(3)}_{S^2 \text{ orbits}}$$

It can be shown that with this condition the spacetime must also be static.
Let $\Sigma_t \perp k^a$ be a foliation of the spacetime, and use coordinates (r, θ, ϕ) on each surface, where θ, ϕ are the usual spherical coordinates and r is the area-radius function as defined earlier. Then we must have:

$$\mathrm{d}s^2\,|_{\Sigma_t} = e^{2\Psi(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega$$

for some function $\Psi(r)$. Note that we have no $\mathrm{d}r\,\mathrm{d}\theta$ or $\mathrm{d}r\,\mathrm{d}\phi$ terms because they would violate spherical symmetry. If we define t as above we can then write the entire metric as:

$$\mathrm{d}s^2 = -e^{2\Phi(r)}\,\mathrm{d}t^2 + e^{2\Psi(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega$$

for some other function $\Phi(r)$.

1.3 The TOV equations

Consider now the matter inside a stationary and spherically symmetric star. We will model the star as a perfect fluid, which means we have the following energy-momentum tensor:

$$T_{ab} = (\rho + P)u_a u_b + \rho g_{ab}$$

where ρ is the energy density, P is the pressure, and u_a is the 4-velocity of the fluid. Since the star is stationary, we can assume the fluid is at rest, so $u^a = e^{-\Phi}\left(\frac{\partial}{\partial t}\right)^a$ (since u is a unit vector pointing in the t direction). Also, since we have spherical symmetry we can assume that ρ and P are functions of r only.

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Make the following definition:

$$e^{2\Psi(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

Note that since $e^{2\Psi(r)} > 0$, we have $m(r) < \frac{r}{2}$. Using the Einstein field equations $G = 8\pi T$ it is now possible to derive the *Tolman-Oppenheimer-Volkoff equations*:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \tag{TOV1}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m + 4\pi r^3 P}{r(r-2m)} \tag{TOV2}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -(P+\rho)\frac{m + 4\pi r^3 P}{r(r-2m)} \tag{TOV3}$$

We now have three equations, but four unknowns (m,Φ,ρ and P). In order to solve this system, we will need a fourth equation, and the one most commonly chosen is an equation of state relating P and ρ. In a cold star, we can assume that the temperature $T(\rho, P) = 0$ and we can solve this to get P explicitly in terms of ρ:

$$P = P(\rho)$$

This is called a *barotropic* equation of state.
We will assume that $\rho, P > 0$. We will also assume that $\frac{\mathrm{d}P}{\mathrm{d}\rho} > 0$; this is a stability condition[1].
Let the radius of the star be R.

Outside the star ($r > R$) we can assume $\rho = P = 0$. (TOV1) then gives that $m(r) = M$ a constant. (TOV2) further provides that $\Phi = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right) + \Phi_0$, where Φ_0 is another constant. Note that since $g_t t = -e^{2\Phi} \to e^{-2\Phi_0}$ as $r \to \infty$, we can eliminate Φ_0 by making a change of coordinates $t \to e^{\Phi_0} t$, so w.l.o.g. we assume that $\Phi_0 = 0$. Hence we have the *Schwarzschild metric*:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

By taking r to be large and comparing with Newtonian gravity, we can deduce that M is in fact the mass of the star. Note that this metric has a problem. It is singular at the so-called *Schwarzschild radius* $r = 2M$. Thus a static, spherically symmetric star must have $R > 2M$ (in a normal star, $R \gg 2M$).

Inside the star ($r < R$), we now have matter to deal with. Integrating (TOV1), we have:

$$m(r) = 4\pi \int_0^r \rho(r') r'^2\,\mathrm{d}r' + m_*$$

where m_* is a constant. Consider a constant t hypersurface. The induced line element on this hypersurface is $\mathrm{d}s^2 = e^{2\Psi}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$. The proper radius (i.e. distance to $r = 0$) of a point is given by $\int_0^r e^{\Psi(r')}\,\mathrm{d}r'$. In order for our spacetime to be a manifold, we require that it is locally flat at $r = 0$, and this requires that the proper radius tends to the area-radius as $r \to 0$. Note that $\int_0^r e^{\Psi(r')}\,\mathrm{d}r' \sim e^{\Psi(0)} r$ as $r \to 0$, so we require that $e^{\Psi(0)} = 1$, or equivalently $m(0) = 0$. From this we deduce that $m_* = 0$.

If we match this expression on the boundary of the star to the Schwarzschild solution for the exterior, we see that $m(R) = M$, or:

$$M = 4\pi \int_0^R \rho(r) r^2\,\mathrm{d}r \tag{$*$}$$

[1] Consider $\frac{\mathrm{d}P}{\mathrm{d}\rho} < 0$. Then if ρ increases by a small amount in a region R, P decreases in R, but then this causes more fluid to flow into R, increasing ρ further.

The volume form on a constant t hypersurface is $e^{\Psi} r^2 \sin\theta \, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$, and so the energy of the matter in the star for t constant is:

$$E = 4\pi \int_0^R \rho e^{\Psi} r^2 \, \mathrm{d}r$$

Note that since m is increasing, so is e^{Ψ} and hence $e^{\Psi} \geq 1$ for all $0 \leq r \leq R$. Thus we have $E > M$. The reason for this is that we have a gravitational binding energy $E - M$.

If we evaluate $\frac{m(r)}{r} < \frac{1}{2}$ at $r = R$ we see that $\frac{M}{R} < \frac{1}{2}$. In fact it is possible to improve this: (TOV3) $\implies \frac{\mathrm{d}P}{\mathrm{d}r} \leq 0 \implies \frac{\mathrm{d}\rho}{\mathrm{d}r} \leq 0$, and from this we can deduce:

$$\frac{m(r)}{r} < \frac{2}{9}\left(1 - 6\pi r^2 P(r) + \left[1 + 6\pi r^2 P(r)\right]^{\frac{1}{2}}\right) \tag{$\dagger$}$$

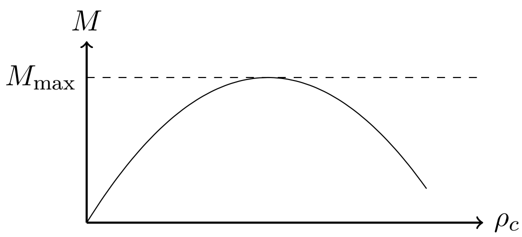
Setting $r = R$ and noting $P(R) = 0$, we obtain the so-called *Buchdahl inequality*: $\frac{M}{R} < \frac{4}{9}$.

In general, we must solve this system of equations numerically. (TOV1) and (TOV3) are a pair of coupled first order ODEs for $m(r)$ and $\rho(r)$, from which we can obtain a unique solution given $m(0) = 0$ and specifying $\rho(0) = \rho_c$, the central density. From (TOV3) we have that P is decreasing in r, so $R(\rho_c)$ is determined by fixing $P(R) = 0$. Then, using ($*$) we can obtain $M(\rho_c)$. Finally, using (TOV2) and the boundary condition that $\Phi(R) = \frac{1}{2}\log\left(1 - \frac{2M}{R}\right)$ we can deduce $\Phi(r)$.

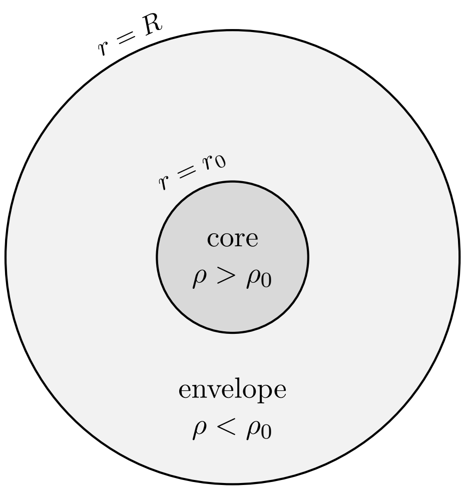
To summarise, given an equation of state, static, spherically symmetric, cold stars are a 1-parameter family labelled by ρ_c.

1.4 Maximum mass

We wish to find a limit on the maximum mass of a star.



In general, $M_{\max}$ depends on the equation of state, but here we run into a problem: we do not know the equation of state in certain conditions, namely $\rho > \rho_0$, where ρ_0 is typically on the order of the density of an atomic nucleus.



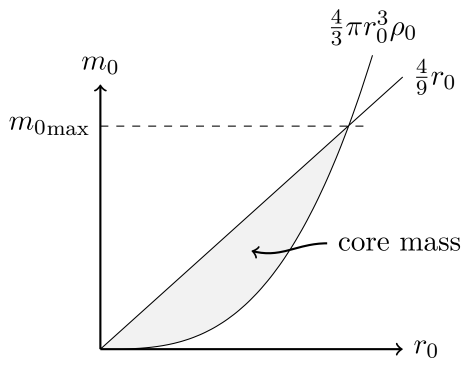
Remarkarbly, it is still possible to find an upper bound on the mass of a star. We do this by splitting the star into two regions: an *envelope*, in which we know the equation of state (so $\rho < \rho_0$), and a *core*, in which we do not ($\rho > \rho_0$). Since $\frac{\mathrm{d}\rho}{\mathrm{d}r} < 0$, the envelope does in fact envelope the core.

Let $m_0 = m(r_0)$; we call this the *core mass*. Since the minimum density in the core is ρ_0, we have $m_0 \geq \frac{4}{3}\pi r_0^3 \rho_0$. Additionally, we can apply (†) at $r = r_0$ to obtain:

$$\frac{m_0}{r_0} < \frac{2}{9}\left(1 - 6\pi r_0^2 P_0 + \left[1 + 6\pi r_0^2 P_0\right]^{\frac{1}{2}}\right)$$

where $P_0 = P(\rho_0)$. This is a decreasing function of P_0, so $\frac{m_0}{r_0} < \frac{4}{9}$.

Lets plot these two constraints:



We see that we have an upper bound on the core mass. Solving for this upper bound, we find:

$$m_0 < \sqrt{\frac{16}{23\pi\rho_0}}$$

If $\rho_0 \approx$ nuclear density, then we have $m_0 \lesssim 5M_\odot$.
Now we can extend our solution to the envelope. m_0 and r_0 together uniquely determine the envelope, as we can solve (TOV1) and (TOV3) starting at $r = r_0$ and using the known equation of state for $\rho < \rho_0$. From this we obtain M as a function of m_0 and r_0, and so can find the maximal value of M when m_0, r_0 take values in the region in the graph above.
Numerically, we can find that M is maximised when m_0 is maximised, and that the maximum mass is $M \approx m_0 \approx 5M_\odot$.
In fact, it is possible to improve this limit by imposing that the speed of sound is physical, i.e. less than the speed of light: $\sqrt{\frac{\mathrm{d}P}{\mathrm{d}\rho}} \leq 1$. Using this gives $M \lesssim 3M_\odot$.

2 The Schwarzschild solution

Lecture 3 20/01/16

We showed earlier that the only static, spherically symmetric solution of the vacuum EFEs is the Schwarzschild solution:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

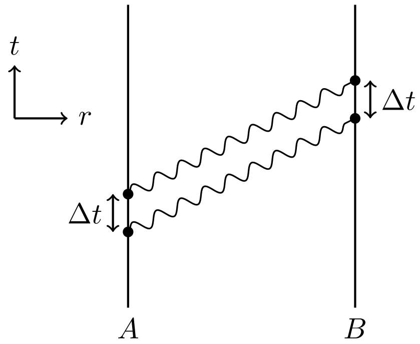
t, r, θ, ϕ are known as *Schwarzschild coordinates*. We will assume $M > 0$.
In fact:

Theorem 2 (Birkhoff). *Any spherically symmetric solution of the vacuum Einstein equations is isometric to the Schwarzschild solution.*

So in particular, spherical symmetry and a vacuum implies a static spacetime (for $r > 2M$).

2.1 Gravitational redshift

Consider two fixed observers A, B in a Schwarzschild spacetime. A sends two photons to B, seperated by a time Δt.



Because $\frac{\partial}{\partial t}$ is an isometry of the spacetime, the second photon's path is the same as that of the first, but translated by Δt. Consider the 4-velocity of a fixed observer. We have:

$$-1 = u^\mu u_\mu = g_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = -\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

Hence we have $\mathrm{d}\tau = \sqrt{1 - \frac{2M}{r}}\,\mathrm{d}t$. Therefore the proper time intervals between the photons at A and B are:

$$\Delta\tau_A = \sqrt{1 - \frac{2M}{r_A}}\Delta t, \quad \Delta\tau_B = \sqrt{1 - \frac{2M}{r_B}}\Delta t$$

So we have:

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \frac{\sqrt{1 - \frac{2M}{r_B}}}{\sqrt{1 - \frac{2M}{r_A}}}$$

If we suppose that the photons were sent at two subsequent wavecrests, then $\Delta\tau$ is the period of the waves, equal to λ, the wavelength (since $c = 1$). We define the redshift z by:

$$1 + z = \frac{\lambda_B}{\lambda_A} = \frac{\sqrt{1 - \frac{2M}{r_B}}}{\sqrt{1 - \frac{2M}{r_A}}}$$

For $r_B > r_A$, we have $z > 0$, so light is redshifted as it climbs out of the gravitational field. For $r_B \gg 2M$:

$$1 + z = \sqrt{\frac{1}{1 - \frac{2M}{r_A}}}$$

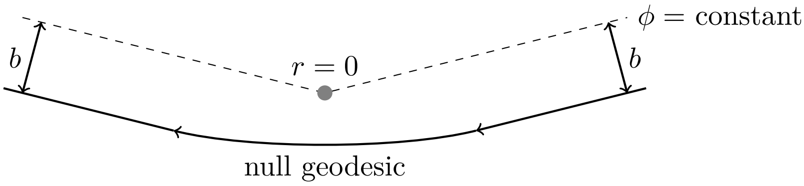
Note that this $\to \infty$ as $r_A \to 2M$.
For a star, we have the Buchdahl inequality, $R > \frac{9}{4}M$, so plugging this into the above, we find that the maximum redshift from the surface of a spherical star is $z = 2$.

2.2 Geodesics

Suppose $x^\mu(\tau)$ is an affinely parametrised geodesic, and let its 4-velocity be $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$. We have Killing fields $k = \frac{\partial}{\partial t}$ and $m = \frac{\partial}{\partial \phi}$, so along geodesics we have two conserved quantities:

$$E = -k \cdot u = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \quad \text{and} \quad h = m \cdot u = r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$$

If our geodesic is timelike and we choose τ to be proper time, we can identify E as the energy and h as the angular momentum associated with the geodesic.
In the null case, we can define the *impact parameter* $b = \left|\frac{h}{E}\right|$, and identify this as the limit of the distance between the geodesic and the star perpendicular to the geodesic as $r \to 0$.



Exercises:

1. Derive the Euler-Lagrange equation for $\theta(\tau)$. Show that one can choose coordinates such that $\theta(\tau) = \frac{\pi}{2}$, so that motion is contained in the equatorial plane.

2. Rearrange the definition of proper time:

$$g_{\mu\nu}u^\mu u^\nu = \sigma = \begin{cases} 1 & \text{timelike} \\ 0 & \text{null} \\ -1 & \text{spacelike} \end{cases}$$

   to obtain $\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V(r) = \frac{1}{2}E^2$, where $V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\sigma + \frac{h^2}{r^2}\right)$.

2.3 Eddington-Finkelstein coordinates

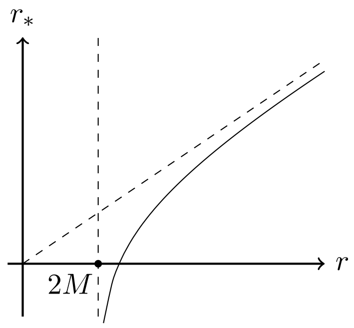
Consider radial null geodesics ($\sigma = 0$) in $r > 2M$. Since ϕ is constant, we have $h = 0$ and so $V = 0$. Since we are dealing with a null geodesic, we are free to scale τ such that $E = 1$. Hence we have:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2M}{r}\right)^{-1}, \quad \frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm 1$$

where the sign in the second equation depends on whether the geodesic is outgoing or ingoing. One thing of note is that an ingoing geodesic reaches $r = 2M$ in finite τ. The same is not true of t:

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2M}{r}\right)^{-1}$$

so $t \to \mp\infty$ as $r \to 2M$.
Define $r_* = r + 2M\log\left|\frac{r}{2M} - 1\right|$, $\mathrm{d}r_* = \frac{\mathrm{d}r}{1-\frac{2M}{r}}$ $(*)$.



We have $\frac{\mathrm{d}t}{\mathrm{d}r_*} = \pm 1$, so $t \mp r_*$ is a constant. Define $v = t + r_*$ $(\dagger)$, a constant along ingoing radial geodesics. The ingoing *Eddington-Finklestein* coordinates are v, r, θ, ϕ. In these coordinates, the line element is given by:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\Omega^2$$

This is smooth for all $r > 0$. In matrix form, the metric is:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

We have $g = \det g_{\mu\nu} = -r^4\sin^2\theta$, so $g_{\mu\nu}$ is non-degenerate for all $r > 0$ and furthermore it is Lorentzian for all $r > 0$.

In summary, spacetime can be extended through $r = 2M$ to a new region $r < 2M$.
Exercise: for $0 < r < 2M$, define r_* by $(*)$ and t by $(\dagger)$. Show that the metrix in coordinates t, r, θ, ϕ is the Schwarzschild metric with $0 < r < 2M$.
So for a ingoing radial null geodesic inside $r = 2M$ we have $\frac{\mathrm{d}r}{\mathrm{d}\tau} = -1$, so it reaches $r = 0$ in finite τ.
Consider $R_{abcd}R^{abcd}$. Some work will lead to:

$$R_{abcd}R^{abcd} \propto \frac{M^2}{r^6} \to \infty \text{ as } r \to 0$$

This quantity is a scalar, so it diverges in any coordinate system. We call $r = 0$ a *curvature singularity*. There are infinite tidal forces at $r = 0$. Note that $r = 0$ is not a part of the our spacetime, because g_{ab} is not defined there.
For $r > 2M$ we have the "static" KVF $\frac{\partial}{\partial t}$. In Eddington-Finklestein coordinates x^μ, we have:

$$k = \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial v}$$

Also, $k^2 = g_{vv} = -\left(1 - \frac{2M}{r}\right)$, so k is null at $r = 2M$, and spacelike at $r < 2M$. Only $r > 2M$ is static.

Fin