An operadic model structure on compact closed categories

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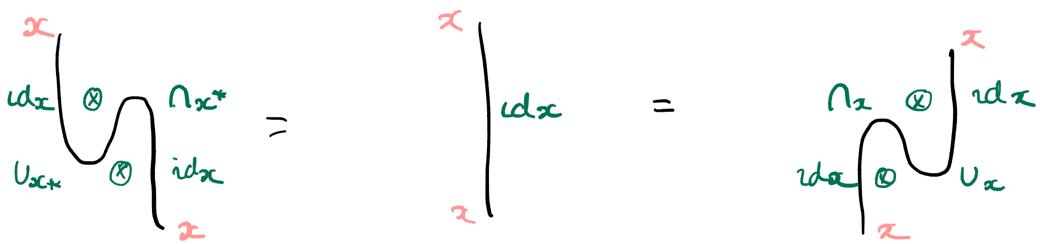
Compact Closed Categories

A compact closed category is a symm. mon. cat $(X,\otimes,I)$ such that

Let X_0 denote object monoid of X

- each object $x\in X_0$ has a dual $x^*\in X_0$:
  - there are distinguished morphisms
    - $\cap_x : I \longrightarrow x\otimes x^*$
    - $\cup^x : x^*\otimes x \longrightarrow I$ satisfying



__Trace__ for $f: x\otimes z \longrightarrow y\otimes z$ in X write

$$tr_z(f) = (id_y \otimes \cup_{z^*})(f\otimes id_z)(id_x \times \cap_z) : x \longrightarrow y$$

write $tr(x) = tr_x(id_x)$.

Note:

In this talk, we assume, for convenience that categories $X \in \mathrm{Comp}$ satisfy

- X strict monoidal

- $\forall x \in X_0$, $x^{**} = x$.

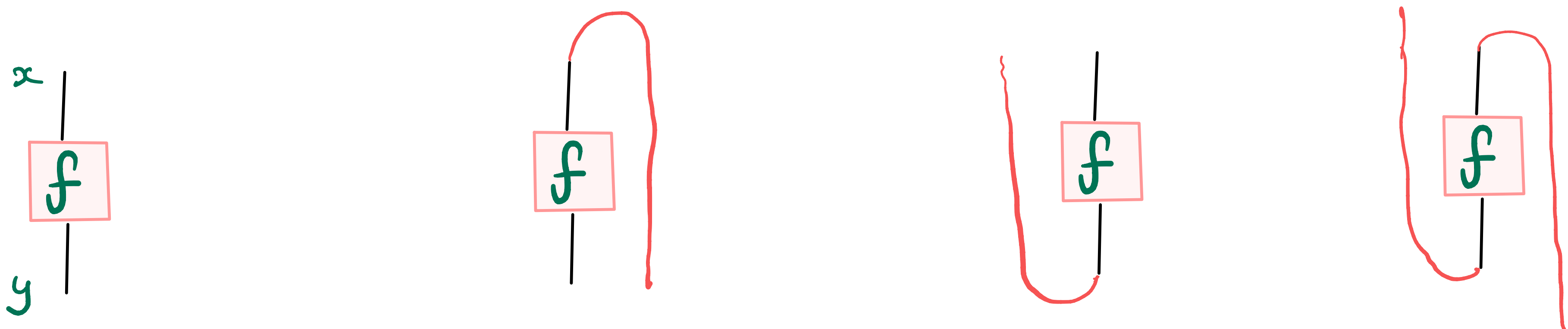
Internal homs $[x,y] = yx^*$

X compact closed with objects x, y

$$X(x,y) \cong X(I, yx^*) \cong X(y^*x, I) \cong X(y^*, x^*)$$

$$f \longmapsto \llcorner f \lrcorner \longmapsto \ulcorner f \urcorner \longmapsto f^*$$

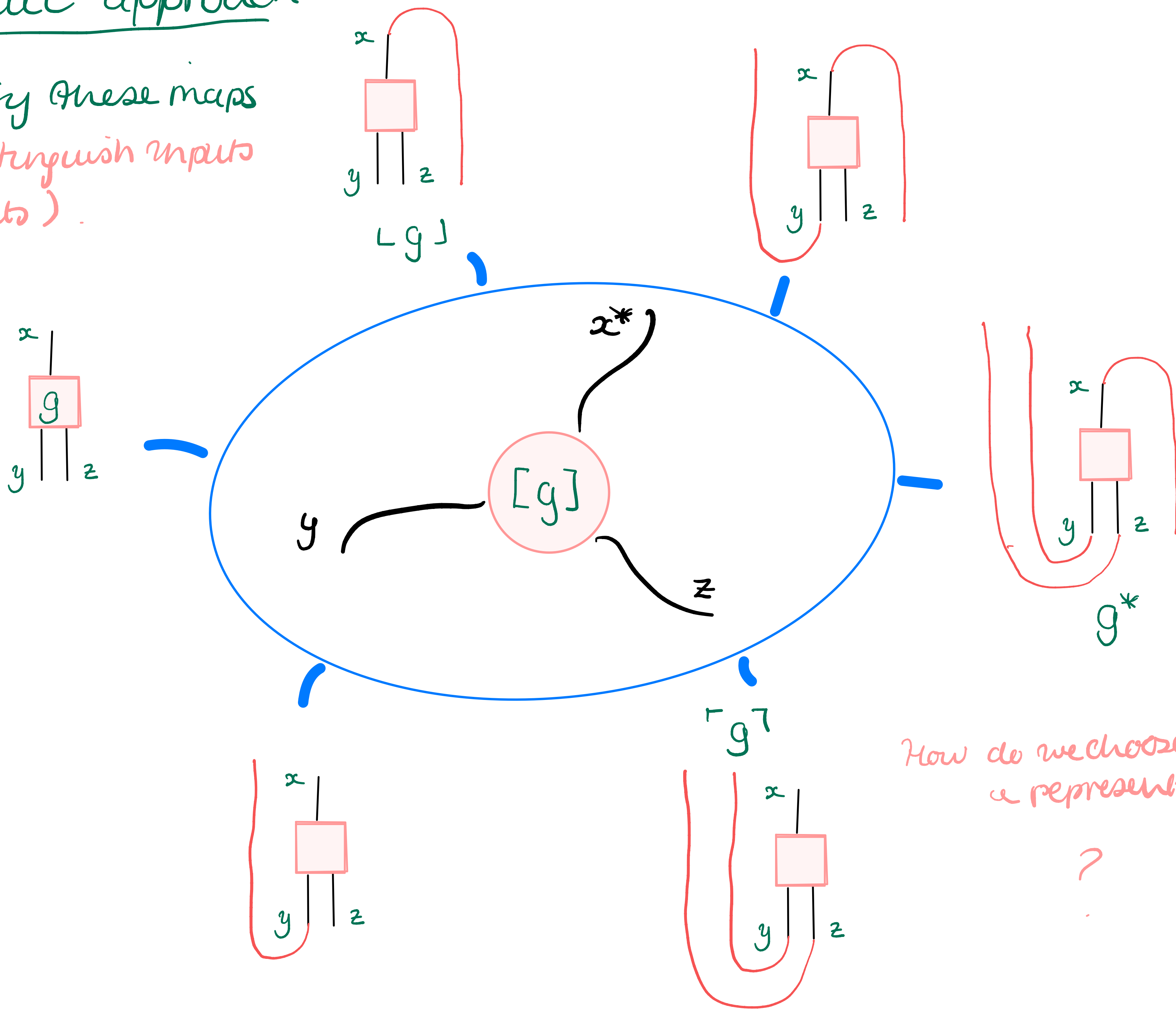
name coname



coevaluation evaluation

Operadic approach

Identify these maps
(don't distinguish inputs & outputs).

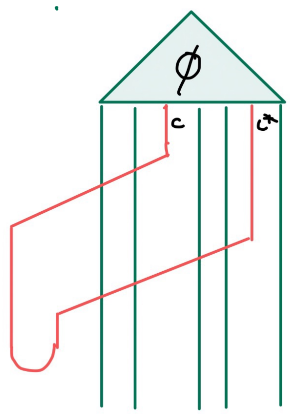


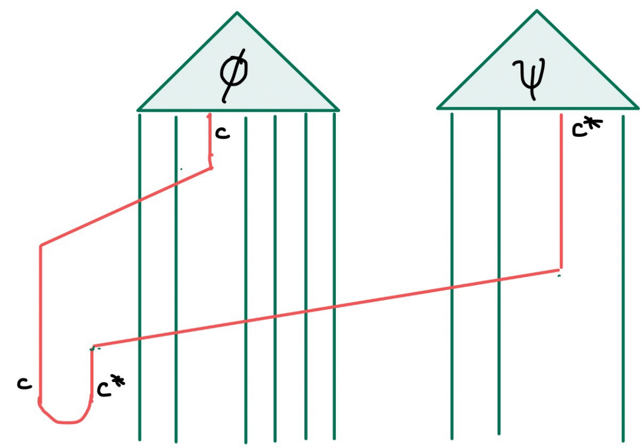
How do we choose a representative ?

Choice of Representative

We defined the monoidal functor $\mathrm{Comp} \to \mathrm{CO}$ in terms of the slice I/X of $X \in \mathrm{Comp}$.

Operations defined in terms of composition w/ cups.





$\phi, \psi \in I/X$.

- Which morphisms $F: X \to Y$ can we rebuild by looking at

$$I/_F : I/X \longrightarrow I/Y$$

- What does it mean if $I/X \simeq I/Y$?

Structure

1. Introduce 'étale' and 'torsion-free' morphisms of compact closed categories

2. Localisation: Cat. of fractions from $\cap$, $\cup$, trace morphisms + isos

3. Model Structure - factorisations + lifts.

Étale morphisms

Defn Morphism $F: X \to Y$ in Comp is étale if

- for all morphisms $g: y_1 \to y_2$ in Y,

  if $\lfloor g \rfloor : I \to y_2 y_1^*$ is in $\mathrm{Im}^{es}(F)$, then g is in $\mathrm{Im}^{es}(F)$.

Lemma: F is étale if and only if, for all $\underbrace{y \otimes z}_{\substack{\cong \\ F(a)}} \in Im^{es}(F)_0$, there are $w, x \in X_0$ s.t. $wx = a$ and $F(w) \cong y$, $F(x) \cong z$.

Proof Say $y \otimes z \in Im^{es}(F)_0$ Then so are .

$$\cap_{y \otimes z} \cong \lfloor id_y \otimes \cap_z \rfloor \cong \lfloor \cap_y \otimes id_z \rfloor .$$

Hence, if F is étale $id_y \otimes \cap_z$ and $\cap_y \otimes id_z$, and hence y and z are in $Im^{es}(F)$.

Conversely, assume $y \otimes z \in Im^{es}(F)_0 \Rightarrow y, z \in Im^{es}(F)_0$ $\forall y, z \in \mathcal{Y}_0$

Let $g \in \mathcal{Y}(y, z)$ be such that $\lfloor g \rfloor \in Im^{es}(F)$. So

By assumption $\exists\, w, x \in X$ and $f \in X(I, xw^*)$ s.t.

$$F(f) \cong \lfloor g \rfloor, \quad \text{and} \quad F(w) \cong y, \quad F(x) \cong (z).$$

But, since $X(I, xw^*) \cong X(w, x)$, there is $f \in X(w, x)$ s.t. $F(f) \cong g$.

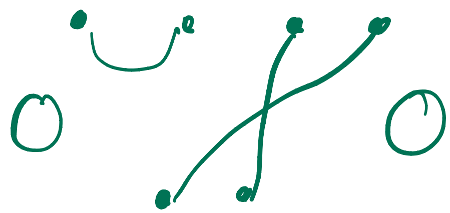
$\square$

Ex

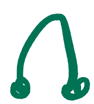
Let B have objects $n \in \mathbb{N}$

morphisms $m \to n$

isotopy classes of 1-manifolds M together w/ iso $\partial M \xrightarrow{\sim} m+n$



Let $B_2 \hookrightarrow B$ be subcategory on even objects

 $\in \mathrm{Im}(\mathrm{incl.})$ but  is not.

The inclusion $B_2 \hookrightarrow B$ is NOT étale.

Examples

- MOTIVATING EXAMPLE

Category CO of circuit operads embeds in Kleisli category CO_{IM} of monad $CO \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Comp \simeq CO^{IM}$.

Morphisms in the image are precisely étale morphisms between free compact closed categories on circuit operads.

These are the arity-preserving morphisms (called étale in ⟨Joyal-Kock, R⟩ modular operad terminology.)

- Compact closed categories of ring modules

$\varphi : R \to S$ homomorphism of finitely presented comm. rings

$$\varphi_! : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$$
$$M \mapsto M \otimes_R S \quad \text{w} \quad (m \cdot r, s) \sim (m, \varphi(r) s)$$

If $\varphi_!$ is étale on Comp then

- φ preserves prime ideals (unramified)

- To do: study other properties of these morphisms

  flatness
  infinitesimal lifting etc

Prop If $F: X \to Y$ is étale, then

i) F is isofibration $\iff$ $I/F: I/X \to I/Y$ is isofibration

other ii) " " equivalence of categories $\iff$ " " " equivalence of categories

Proof i) Immediate from definitions: $g \in \mathrm{Im}^{es}(F) = \mathrm{Im}(F) \iff gj \in \mathrm{Im}(F)$.

ii) If $F: X \to Y$ étale and $I/F: I/X \xrightarrow{\simeq} I/Y$, then for all $x_1, x_2 \in X$

f.f.:

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr] & \\ I \arrow[rr] & & x_2 x_1^* \end{tikzcd}$$

$$\begin{array}{ccc} X(I, x_2 x_1^*) & \cong & Y(I, F(x_2 x_1^*)) \\ \wr\| & & \wr\| \\ X(x_1, x_2) & & Y(F(x_1), F(x_2)) \end{array}$$

So F is fully faithful.

ess. surj. ob:

for all $y \in Y_0$ $\eta_y = \lfloor id_y \rfloor$ is in $\mathrm{Im}^{es}(I/F)_0$ $\leftarrow$ I/F is equiv.

but then id_y, and hence $y \in \mathrm{Im}^{es}(F)_0$ $\leftarrow$ F étale.

$\square$.

Trace Localisation

I) A calculus of left fractions

II) Trace localisation

I A calculus of left fractions

X compact closed

$W = W_X \subset X$ is wide subcategory with

- all isomorphisms
- all cups, caps, trace: $\forall x \in X_0$ $\cap_x, \cup_x, tr(x) \in W$.

Up to iso, a morphism in W has the form

$$w = id_t \otimes \cup_x \otimes \cap_y \otimes tr(z) : \quad t \otimes x^* \otimes x \rightsquigarrow t \otimes y \otimes y^*$$

$|_t \quad \cup_x \quad \cap_y \quad \bigcirc_z = \begin{matrix} \cap_z \\ \cup_{z^*} \end{matrix}$

$t, x, y, z \in X_0$.

NOTATION: I'll denote morphisms $w \in W_X(x,y)$ by $x \rightsquigarrow y$

PROP (X, W) defines a 'calculus of left fractions'

In other words:

F1) For all $f \in X(a,c)$ and $w \in W(a,b)$

there exists
- an object $d \in X_0$
- morphisms $f' \in X(b,d)$, $w' \in W(b,d)$

such that

$$\begin{tikzcd} a \arrow[r, squiggly, "w"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & b \arrow[d, "f'"] \\ c \arrow[r, squiggly, "w'"'] & d \end{tikzcd}$$

in X.

F2) For all $w : a \rightsquigarrow b$, $f_1, f_2 : b \rightrightarrows c$ s.t. $f_1 w = f_2 w$

there is a $v : c \rightsquigarrow d$ such that $v f_1 = v$

$$\begin{tikzcd} a \arrow[r, squiggly, "w"] & b \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & c \arrow[r, squiggly, "v"] & d \end{tikzcd}$$

PROP (X, W) defines a 'calculus of left fractions'

PROOF

F1) For all $f \in X(a,c)$ and $w \in W(a,b)$, want commuting square
$$\begin{tikzcd} a \arrow[r, squiggly, "w"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & b \arrow[d, "f'"] \\ c \arrow[r, squiggly, "w'"'] & d \end{tikzcd}$$
in X.

given such $f \in X(a,c)$ and $w \in W(a,b)$,
there are objects $t, x, y, z \in X_0$ s.t
- $a = t x^* x$
- $b = t y y^*$
- $w = id_t \otimes \cup_x \otimes \cap_y \otimes tr(z)$.

And-

$$\begin{tikzcd} a = t x^* x \arrow[r, squiggly, "id_t \otimes \cup_x \otimes \cap_y \otimes tr(z)"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & t y y^* = b \arrow[d, "f \otimes \cap_{x^*} \otimes y^*"] \\ c \arrow[r, squiggly, "tr(x \otimes y \otimes z)"'] & c \end{tikzcd}$$

F2) For all $w: a \rightsquigarrow b$, $f_1, f_2 : b \rightrightarrows c$ s.t $f_1 w = f_2 w$ }
want $v: c \rightsquigarrow d$ such that $v f_1 = v f_2$.

$$a \overset{w}{\rightsquigarrow} b \underset{f_2}{\overset{f_1}{\rightrightarrows}} c \overset{v}{\rightsquigarrow} d$$

If $w: a \rightsquigarrow b$, $f_1, f_2 : b \rightrightarrows c$ s.t, $f_1 w = f_2 w$ then $f_1 = f_2$ $\square$.

Trace Localisation

Hence, there is a category $X[W^{-1}]$:

- Objects: $X[W_X^{-1}]_0 = X_0$
- Morphisms $[f,w]: a \longrightarrow b$: represented by cospans

$$a \longrightarrow \underline{b} \leftsquigarrow b$$

up to equivalence:

$(f_1,w_1) \sim (f_2,w_2) : a \longrightarrow b$ if there is a commuting diagram

$$\begin{tikzcd}
& \underline{b}_1 \arrow[d, "h_1"] & \\
a \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \underline{c} & b \arrow[ul, squiggly, "w_1"'] \arrow[dl, squiggly, "w_2"] \\
& \underline{b}_2 \arrow[u, "h_2"'] &
\end{tikzcd}$$

where $h_1 f_1 = h_2 f_2 : a \longrightarrow \underline{c}$

$h_1 w_1 = h_2 w_2 : b \rightsquigarrow \underline{c}$

$\Rightarrow h_1, h_2 \in W$ since W satisfies 2 out of 3.

So $(f_1,w_1) \sim (f_2,w_2) : a \longrightarrow b$ if $\underline{b}_1, \underline{b}_2$ differ up to cups, caps & traces.

Write $[f,w]$ for morphism $a \longrightarrow b$ represented by (f,w).

$X[W^{-1}]$

Composition $\quad [g, w] \circ [f, v] = [\underline{g} \circ f, \underline{v} \circ w]$

$$\begin{tikzcd}
 & & \underline{c} & & \\
 & \underline{b} \arrow[ur, "\underline{g}"] & \text{by f1)} & \underline{c} \arrow[ul, squiggly, "\underline{v}"'] & \\
a \arrow[ur, "f"] & & b \arrow[ul, squiggly, "v"'] \arrow[ur, "g"] & & c \arrow[ul, squiggly, "w"']
\end{tikzcd}$$

$g \circ f$

Units

$$\begin{tikzcd}
 & a & \\
a \arrow[ur, "id"] & & a \arrow[ul, equal]
\end{tikzcd}$$

Trace localisation: $\quad Tr_X : X \longrightarrow X[W_X^{-1}], \; f \longmapsto (f, id)$

Lem For all $x \in X_0$, $\quad [\eta_x, id_{xx^*}]^{-1} = [\upsilon_{x^*}, id_I] : xx^* \longrightarrow I$

pf

$$\begin{tikzcd}
 & & I & & \\
 & I \arrow[ur, equal] \arrow[dr, squiggly, "\eta_x"'] & & I \arrow[ul, equal] & \\
I \arrow[ur, equal] & & xx^* \arrow[ur, squiggly, "\upsilon_{x^*}"'] & & I \arrow[ul, equal]
\end{tikzcd}$$

since $\eta_x, \upsilon_{x^*} \in W$.

$\square$.

$X[W^{-1}]$ is compact closed

for $[f,w] : a \to b$, construct $[f,w]^* = [f_*, w_*] : b^* \to a^*$ using F1):

$$\begin{tikzcd}
 & \underline{b} & \\
a \arrow[ur, "f"] & & b \arrow[ul, squiggly, "w"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \underline{b^*} \arrow[dl, "f^*"'] \arrow[dr, squiggly, "w^*"] & \\
a^* \arrow[dr, squiggly, "f_*"'] \arrow[rr, phantom, "(F1)"] & & b^* \arrow[dl, "w_*"] \\
 & \underline{a^*} &
\end{tikzcd}$$

$\llcorner [f,w] \lrcorner : I \to ba^*$

$$\begin{tikzcd}
 & \underline{ba^*} & \\
I \arrow[ur, "\llcorner f \lrcorner"] & & ba^* \arrow[ul, squiggly, "w \otimes id_{a^*}"']
\end{tikzcd}$$

Lem Let $F : X \to Y$ be a morphism in **Comp**. The induced morphism $F_{W^{-1}} : X[W_X^{-1}] \longrightarrow Y[W_Y^{-1}]$ is étale $\iff$ F is.

Pf • By previous lemma: If F is étale then $\forall\, y \otimes z \in \mathrm{Im}^{es}(F)$, y and z are in the essential image.

So $F : X \to Y$ étale $\Rightarrow$ $F_{W^{-1}} : X[W_X^{-1}] \to Y[W_Y^{-1}]$ is.

Conversely. Assume, $y \otimes z \in \mathrm{Im}^{es} F_{W^{-1}} \Rightarrow y, z \in \mathrm{Im}^{es}(F_{W^{-1}})$ $\forall\, y, z \in Y$.

And let $y \otimes z \in \mathrm{Im}^{es}(F)$. Then $y \otimes z \in \mathrm{Im}^{es} F_{W^{-1}}$ so $y, z \in \mathrm{Im}^{es}(F_{W^{-1}})$. But then $id_y, id_z \in \mathrm{Im}^{es}(F_{W^{-1}})$

$\rightsquigarrow$ id_y, id_z and hence $y, z \in \mathrm{Im}^{es}(F)$ by definition of localisation

$\square$

"Torsion-free" morphisms

Let X be compact closed

MX compact closed w/

objects $(MX)_0$ is the free monoid $M(X_0)$ on X_0

morphisms $(MX)((x_1,\dots,x_m),(y_1,\dots,y_n)) =$

$X(x_1 \otimes \dots \otimes x_m,\ y_1 \otimes \dots \otimes y_n)$

Defn A morphism $F : X \to Y$ of compact closed categories is torsion free

if there is a free monad Z_0 and a monoid morphism

$F_0 : MX_0 \longrightarrow Z_0$ s.t.

free

$$\begin{tikzcd} MX_0 \arrow[r, "F_0"] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & Z_0 \arrow[d] \\ X_0 \arrow[r, "F_0"'] & Y_0 \end{tikzcd}$$

is a pushout of monoid morphisms.

Operadic model structure on Comp

Cofibrations

$\Phi : X \longrightarrow Y$ injective on objects

torsion free

Fibrations

$\Psi : X \longrightarrow Y$ isofibration

étale

reflect trace morphisms:

$tr_z(g) \in Im^{es}(\Psi) \Rightarrow g \in Im^{es}(\Psi)$.

Weak equivalences

$F : X \longrightarrow Y$ induced functor

$I/_{F_{W^{-1}}} : I/_{X[W_X^{-1}]} \xrightarrow{\simeq} I/_{Y[W_Y^{-1}]}$

equivalence of categories.

Operadic model structure on Comp

Cofibrant objects:
Compact closed categories X w free object monoid
(Circuit operads!)

All objects are fibrant

Cofibrations

$\Phi : X \longrightarrow Y$ injective on objects

torsion free

Fibrations

$\Psi : X \longrightarrow Y$ isofibration

étale

reflect trace morphisms

Weak equivalences

$F : X \longrightarrow Y$ induced functor

$$I/F_{W^{-1}} : I/X[W_X^{-1}] \xrightarrow{\simeq} I/Y[W_Y^{-1}]$$

equivalence of categories.

Prop Trivial fibrations are surjective equivalences of categories.

Proof

$F : X \xrightarrow{\simeq} Y$ since F étale and reflects trace.

$F : X \twoheadrightarrow Y$ surjective since F isofibration.

Factorisations

Let $F: X \longrightarrow Y$ be any morphism.

I) There exists a factorisation

$$\begin{tikzcd} & C \arrow[dr, "\sim", "p"', two heads] \arrow[d, phantom, "\circlearrowleft"] & \\ X \arrow[ur, tail, "i"] \arrow[rr, "F"'] & & Y \end{tikzcd}$$

PROOF (up to symmetry axis).

MY_0 the free object monoid on Y_0 - object set of Y

C has objects $(x, (y_1, \dots, y_n)) \in X_0 \times MY_0$

given by pushout

$$\begin{tikzcd} MX_0 \arrow[r, "(-, I)"'] \arrow[d] & MX_0 \times MY_0 \arrow[d] \\ X_0 \arrow[r] & X_0 \times MY_0 \end{tikzcd}$$

morphisms

$$C((x, y_1 \dots, y_m), (x', y'_1, \dots, y'_n)) = Y(F(x) \otimes y_1 \otimes \dots \otimes y_m, F(x') \otimes y'_1 \otimes \dots \otimes y'_n)$$

Obvious $*$, $\otimes$, etc.

$i: X \rightarrowtail C$: $x \longmapsto (x, I_Y)$ injective on objects torsion free

$p: C \xrightarrow{\sim} Y$: projection surjective equivalence by construction

$p \circ i = F$ $\quad x \longmapsto (x, I_Y) \longmapsto F(x)$

$$(f: x \longrightarrow x') \longmapsto (F(f) : (x, I_Y) \to (x', I_Y))$$

$$\longmapsto F(f) : F(x) \longrightarrow F(x').$$

$$\begin{tikzcd} & D \arrow[dr, two heads] & \\ X \arrow[ur, hook, "\sim"] \arrow[rr, "F"'] & & Y \end{tikzcd}$$

The construction of the triv.cof − fib factorisation is more involved.

I) For $F : X \to Y$, define $\mathrm{Fib}(F) \hookrightarrow Y$ to be

smallest compact closed subcategory of Y s.t

for all morphisms $g \in Y(y_1, y_2)$ of Y,

if $\llcorner g \lrcorner \in Y(I, y_2 y_1^*)$ is in $\mathrm{Im}^{es}(F)$,

then g is in $\mathrm{Fib}(F)$, and

if $tr_2(g) \in \mathrm{Im}^{es}(F)$, then $g \in \mathrm{Fib}(F)$.

Lemma The inclusion $\mathrm{Fib}(F) \hookrightarrow Y$ is a fibration

Proof Definition of $\mathrm{Fib}(F)$ $\square$

We'll build D from $\mathrm{Fib}(F)$ in two stages.

1)

$$\begin{tikzcd} & D' \arrow[r, two heads] \arrow[dr, two heads, "p'"] & \mathrm{Fib}(F) \arrow[d, two heads] \\ X \arrow[ur, "i'", "\sim"'] \arrow[rr, "F"'] & & Y \end{tikzcd}$$

i' weak equivalence
injective on objects
But not necessarily torsion-free

II)

$$\begin{tikzcd} & D \arrow[r, two heads] \arrow[drr, two heads, "p"] & D' \arrow[r, two heads] & \mathrm{Fib}(F) \arrow[d] \\ X \arrow[ur, tail, "i", "\sim"'] \arrow[rrr, "F"'] & & & Y \end{tikzcd}$$

i trivial fibration

RMK By definition of $\mathrm{Fib}(F)$, for all $y \in \mathrm{Fib}(F)_0$ there exist $x \in X_0$, $\tilde{y} \in \mathrm{Fib}(F)_0$ s.t. $F(x) \cong y\,\tilde{y}$.

1) Construct factorisation:

$$\begin{tikzcd}
 & D' \arrow[r, two heads] \arrow[dr, "p'"] & \mathrm{Fib}(F) \arrow[d] \\
X \arrow[ur, "i'", "\sim"'] \arrow[rr, "F"'] & & Y
\end{tikzcd}$$

D' has

objects: pairs (x, y) where $x \in X_0$, $y \in \mathrm{Fib}(F)_0$, $\exists\, \tilde{y} \in \mathrm{Fib}(F)_0$, $y\tilde{y} \cong F(x)$.

morphisms $(x,y) \longrightarrow (x',y')$ are pairs $((f_i)_{i=1}^{k}, g)$ s.t.

- $g \in \mathrm{Fib}(F)(y,y')$
- $f_i \in \mathrm{Mor}(X)$ $1 \le i \le k$ and $\mathrm{dom}(f_1) = x$, $\mathrm{cod}(f_k) = x'$
- there are $w, v \in \mathrm{Mor}(W_{\mathrm{Fib}(F)}) \subset \mathrm{Mor}(W_B)$ $\mathrm{dom}(v) = y$, $\mathrm{cod}(w) = y'$ s.t.
  - $g(f) \overset{\text{def}}{=} w \circ F(\underline{f}) \circ v \in \mathrm{Fib}(F)(y,y')$

  and
  - $g \cong g(f)$ in $I/\mathrm{Fib}(F)[W^{-1}_{\mathrm{Fib}(F)}]$

  where $f \overset{\text{def}}{=} \bigotimes_{x,\, i=1}^{k} f_i \in \mathrm{Mor}(X)$
- $k \geq 1$ is minimal for these properties: None of the f_i are (vertically) composable in X s.t. properties preserved.

Sorry, what?

morphisms $(x, y) \longrightarrow (x', y')$ are pairs $((f_i)_{i=1}^{k}, g)$ s.t

- $g \in Fib(F)(y, y')$
- $f_i \in Mor(X)$ $1 \leq i \leq k$ and $dom(f_1) = x$, $cod(f_k) = x'$

- there are $w, v \in Mor(W_{Fib(F)}) \subset Mor(W_B)$ s.t $dom(v) = y$, $cod(w) = y'$
  - $g(\underline{f}) \overset{def}{=} w \circ F(\underline{f}) \circ v \in Fib(F)(y, y')$

  $\underline{f} \overset{def}{=} \bigotimes_{i=1}^{k}{}_{X} f_i \in Mor(X)$

and • $g \cong g(\underline{f})$ in $I / Fib(F)[W_{Fib(F)}^{-1}]$

- $k \geq 1$ is minimal for these properties:

  None of the f_i are (vertically) composable in X s.t properties preserved.

- there are $w, v \in Mor(W_{Fib(F)}) \subset Mor(W_B)$ $dom(v) = y$, $cod(w) = y'$ s.t.
  - $g(\underline{f}) \overset{def}{=} w \circ F(\underline{f}) \circ v \in Fib(F)(y, y')$

and
  - $g \cong g(\underline{f})$ in $\mathcal{I}/Fib(F)[W^{-1}_{Fib(F)}]$

$\underline{f} \overset{def}{=} \bigotimes_{x\, i=1}^{k} f_i \in Mor(X)$

Easier to look at $\lfloor \underline{f} \rfloor \in \mathcal{I}/X$, $\lfloor w \circ F(\underline{f}) \circ v \rfloor \in \mathcal{I}/Y$.

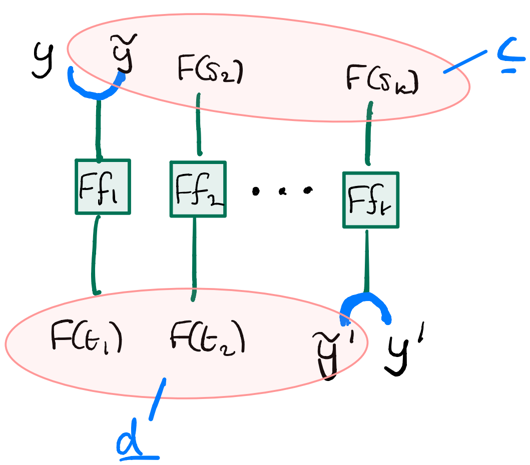
For $1 \le i \le k$ let $s_i = dom(f_i)$, $t_i = cod(f_i)$ $s_1 = x$, $t_k = x'$

Let $y\tilde{y} \cong F(x)$, $F(x') \cong \tilde{y}'y'$ in $Fib(F)_0$.

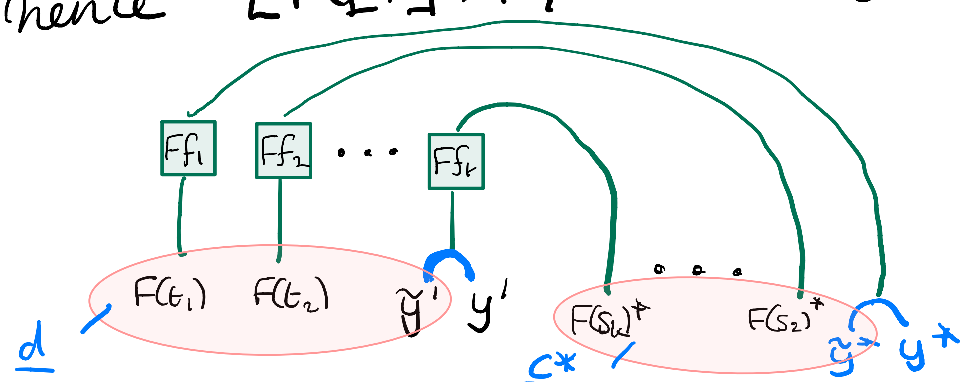
We'll assume $y\tilde{y} = F(x)$, $F(x') = \tilde{y}'y'$ for convenience.

Let $\underline{c} = \tilde{y} \otimes \left(\bigotimes_{i=2}^{k} F(s_i)\right)$, $\underline{d} = \left(\bigotimes_{j=1}^{k-1} F(t_j)\right) \otimes \tilde{y}'$ in $Fib(F)_0$.

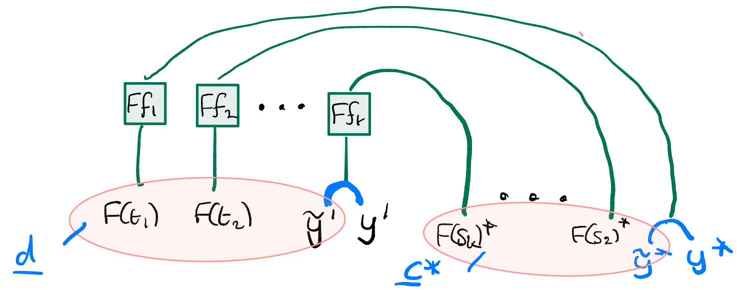
Then $F(\underline{f}) : y \otimes \underline{c} \longrightarrow \underline{d} \otimes y$ in $Fib(F)$



and hence $\lfloor F(\underline{f}) \rfloor : \mathcal{I}_Y \longrightarrow \underline{d} \otimes y \otimes \underline{c}^* \otimes y^*$



$\llcorner F(\underline{f}) \lrcorner : I_Y \longrightarrow \underline{d} \otimes y \otimes \underline{c}^* \otimes y^*$



Since $g(f) \overset{\text{def}}{=} w \circ F(\underline{f}) \circ v \in Fib(F)(y, y')$, it must be the case that

$$\llcorner g(f) \lrcorner = \left(id_{y'\tilde{y}^*} \otimes U_{\underline{b}} \otimes tr(z) \right) \circ \llcorner F(\underline{f}) \lrcorner \in Fib(F)(I, y'y^*)$$

hence • $\underline{d}\, \underline{c}^* = \underline{b}^* \underline{b}$

Can rewrite

- there are $w, v \in Mor(W_{Fib(F)}) \subset Mor(W_B)$ $dom(v) = y$, $cod(w) = y'$ s.t.
- $g(f) \overset{\text{def}}{=} w \circ F(\underline{f}) \circ v \in Fib(F)(y, y')$

and • $g \overset{\simeq}{=} g(f)$ in $I/Fib(F)[W^{-1}_{Fib(F)}]$ $\quad \underline{f} \overset{\text{def}}{=} \bigotimes_{i=1}^{k} f_i \in Mor(X)$

as

- there is $\underline{b} \in Fib(F)_0$, and $g(f) \cong g$ in $I/Fib(F)[W^{-1}_{Fib(F)}]$

such that $\llcorner g(f) \lrcorner \cong \left(id_{y'y^*} \otimes U_{\underline{b}} \right) \circ \llcorner F \underline{f} \lrcorner$.

$((f_i)_{i=1}^{k}, g) : (x, y) \longrightarrow (x', y')$ in $\mathcal{D}'$

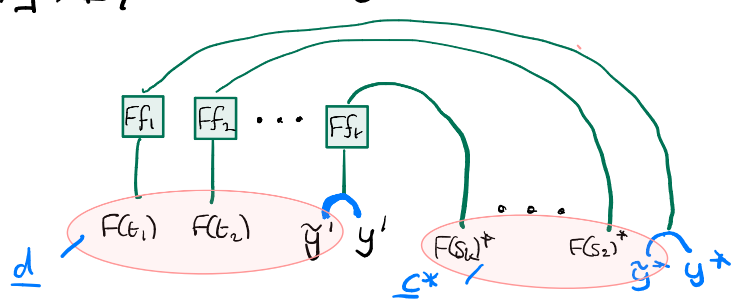
- $g \in Fib(F)(y, y')$
- $f_i \in Mor(X)$ $\quad 1 \leq i \leq k$ and $dom(f_1) = x$, $cod(f_k) = x'$
- then is $\underline{b} \in Fib(F)_0$, and $g(f) \cong g$ in $\mathcal{I}/Fib(F)[W^{-1}_{Fib(F)}]$

  such that $\lfloor g(f) \rfloor \cong (id_{y'y^*} \otimes U_{\underline{b}}) \circ \lfloor F\underline{f} \rfloor$.

- $k \geq 1$ is minimal for these properties:

  None of the f_i are (vertically) composable in X s.t properties preserved

$\lfloor F(\underline{f}) \rfloor : I_Y \longrightarrow \underline{d} \otimes y \otimes \underline{c}^* \otimes y^*$



If any $\quad * \begin{cases} F(s_i) = F(s_j)^* & 1 < i < j \leq k \\ F(s_i) = F(t_j) & 1 < i < j < k \\ F(t_i) = F(s_j) & 1 \leq i \leq j \leq k \\ F(t_i) = F(t_j)^* & 1 \leq i < j < k \end{cases}$

then can compose f_i, f_j in X.

So condition says that there are no i, j s.t $*$.

actually, it says a bit more ...

Lem Let $y, y' \in \mathrm{Im}^{es}(F)$. Then

$((f_i)_{i=1}^{k}, g) : (x, y) \longrightarrow (x', y')$ in D' has form

(f, g) with $F(f) \cong g$ in $\mathrm{Fib}(F)[W_{\mathrm{Fib}(F)}^{-1}]$.

Pf Let $y\tilde{y} \cong F(x)$, then $\tilde{y} \in \mathrm{Im}^{es}(F)$:

Namely $\mathrm{Id}_{F(x)} \cong \mathrm{id}_{y\tilde{y}}$

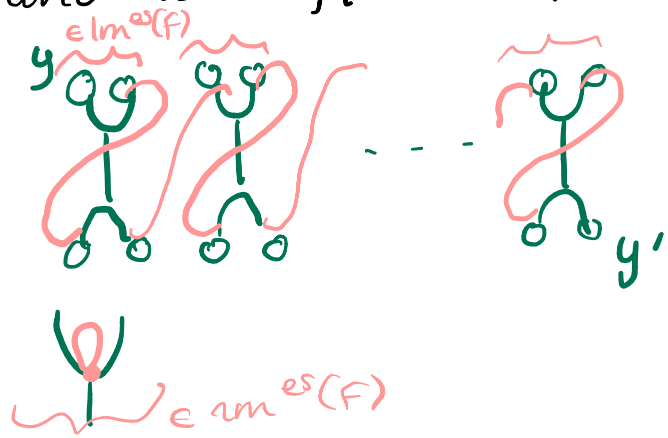
$$\underbrace{\mathrm{tr}_y \, \mathrm{id}_{F(x)}}_{\Uparrow \, \mathrm{Im}^{es}(F)} \cong \underbrace{\mathrm{tr}(y)}_{\Uparrow \, \mathrm{Im}^{es}(F)} \otimes \mathrm{id}_{\tilde{y}} .$$

Likewise $\tilde{y}'$.

Hence if $\llcorner g(f) \lrcorner \cong (\mathrm{id}_{y'y^*} \otimes U_{\underline{b}}) \circ \llcorner F f \lrcorner$

then $\underline{b} \in \mathrm{Im}^{es}(F)$ and so f_i's composable in $X_{\cdot 0}$

Some examples of morphisms in D':



Morphism $p' : D' \longrightarrow Y$ $p' : D' \twoheadrightarrow \mathrm{Fib}(F) \hookrightarrow Y$

$(x, y) \longmapsto y$, $((f_i)_i, g) \longmapsto g$

$\underbrace{\text{étale , isofibration, reflects trace}}_{\text{fibration}}$ by construction

Morphism $i' : X \longrightarrow D'$ $x \longmapsto (x, F(x))$ $f \longmapsto (f, F(f))$

clearly injective on objects.

Must show

$$\Gamma = I/_{i'_{W^{-1}}} : I/_{X[W_X^{-1}]} \longrightarrow I/_{D[W_D^{-1}]} \text{ is equivalence}$$

PROP

$$\Gamma = I/i'_{W^{-1}} : I/X[W_X^{-1}] \longrightarrow I/D[W_{D'}^{-1}]$$ is equivalence

Pf/

faithful: Let $[I \xrightarrow[\alpha_i]{} \underline{x}_i \overset{v_i}{\rightsquigarrow} x_i] \in X[W_X^{-1}](I, x_i)$ for $i = 1, 2$ has a representative of the form:

A morphism $I/X[W_X^{-1}]([\alpha_1, v_1], [\alpha_2, v_2])$ has a representation of the form:

$$\begin{tikzcd} & I \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ \underline{x}_1 \arrow[rr, "f"] & & \underline{x}_2 \\ x_1 \arrow[u, squiggly, "v_1"] & & x_2 \arrow[u, squiggly, "v_2"'] \end{tikzcd}$$

by

$$\begin{tikzcd} & & \underline{x}_{2f'} & & \\ & \underline{x}_1 \arrow[ur, "\alpha'"] & & \underline{x}_{f'} & \\ I \arrow[ur, "\alpha_1"] \arrow[uurr, bend left, "\alpha_2"] & & x_1 \arrow[ul, squiggly] \arrow[ur, "f'"] & & x_2 \arrow[ul, squiggly] \end{tikzcd}$$

In particular it follows that distinct morphisms in $I/X[W_X^{-1}]([\alpha_1, v_1], [\alpha_2, v_2])$ have distinct images in $I/D'[W_{D'}^{-1}]([(\alpha_1, F(\alpha_1)), (v_1, F(v_1))], [(\alpha_2, F(\alpha_2)), (v_2, F(v_2))])$.

fullness Let $[f, g] \in I/D'[W_{D'}^{-1}]([(\alpha_1, F(\alpha_1)), (v_1, F(v_1))], [(\alpha_2, F(\alpha_2)), (v_2, F(v_2))])$ be any morphism. Then $g = f$ in $I/D'[W_{D'}^{-1}]$ by definition

$$\begin{tikzcd} & I \arrow[dl, "(\alpha_1, F(\alpha_1))"'] \arrow[dr, "(\alpha_2, F(\alpha_2))"] & \\ (\underline{x}_1, F(\underline{x}_1)) & & (\underline{x}_2, F(\underline{x}_2)) \\ (x_1, F(x_1)) \arrow[u, squiggly, "(v_1, F(v_1))"] \arrow[urr, "g"] & & (x_2, F(x_2)) \arrow[u, squiggly, "(v_2, F(v_2))"'] \end{tikzcd}$$

PROP

$$\Gamma = I/_{i'_{W^{-1}}} : I/X[W_X^{-1}] \longrightarrow I/D'[W_{D'}^{-1}] \text{ is equivalence}$$

Pf/

ess. surj on obj.

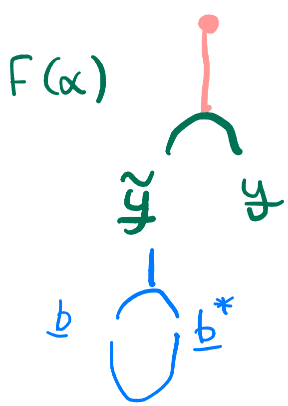
Let $I \xrightarrow[((\alpha_i)_i^k,\, \beta)]{} (\underline{x}, y) \xleftarrow[((w_j)_j,\, v)]{\rightsquigarrow} (x, y)$ represent an object of $I/D[W_D^{-1}]$

Since $\mathrm{dom}(\alpha_i) = I_X$ for all $1 \le i \le k$, $k=1$ and $(\alpha_i)_{i=1}^k = \alpha$.

Then there is a $\tilde{y} \in Fib(F)$ with $y\tilde{y} \cong F(\underline{x})$

Moreover, there is $\underline{b} \in X_0$ $\boxed{(U_{F(\underline{b})} \otimes id_y) \circ F(\alpha) \cong \beta : I \to y \text{ in } D'[W_{D'}^{-1}]}$

therefore $\tilde{y} \cong F(\underline{b}\,\underline{b}^*)$.



But then $(\alpha, \beta) \cong (\alpha, F(\alpha))$ in $I/_{D'[W_{D'}^{-1}]}$ by

□

So we have factorisation

$$\begin{tikzcd} & D' \arrow[dr, two heads] & \\ X \arrow[ur, "\sim"] \arrow[rr, "F"'] & & Y \end{tikzcd}$$

Then construct $D \simeq D'$

objects $(x, \underbrace{y_1 \cdots y_n}_{\in M(\mathrm{Fib}(F))_0})$ s.t. $F(x) \cong y_1 \otimes \cdots \otimes y_n \otimes \tilde{y}$ for some $\tilde{y} \in \mathrm{Fib}(F)_0$

morphisms $D((x, y_1, \ldots, y_m), (x', y_1', \ldots, y_n'))$

$= D'((x', y_1 \otimes \cdots \otimes y_m), (x', y_1' \otimes \cdots \otimes y_n'))$.

Then D_0 is given by pushout

$$\begin{tikzcd} MX_0 \arrow[r] \arrow[d] & MX_0 \times M\,\mathrm{Fib}(F)_0 \arrow[d] \\ X_0 \arrow[r] & D_0 \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

so have factorisation

$$\begin{tikzcd} & D \arrow[dr, two heads] & \\ X \arrow[ur, tail, "\sim"] \arrow[rr, "F"'] & & Y \end{tikzcd}$$

$\square$

Lifts

Let

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\Phi"', "\wr"] & A \arrow[d, "\Psi"] \\ Y \arrow[r, "G"'] \arrow[ur, dashed, "L"] & B \end{tikzcd}$$

commute in $\underline{\mathrm{Comp}}$

Let $g \in Y(y_1, y_2)$ $g \notin \mathrm{Im}(\Phi)$. Since Φ is w.e, there exist $\underline{x}$, $x \in X$ and $f' \in X(I, \underline{x})$, $v \in X(x, \underline{x})$ s.t there is a commuting diagram in Y:

$$\begin{tikzcd} I \arrow[r, "\Phi(f')"] \arrow[d, equal] & \Phi(\underline{x}) & \Phi(x) \arrow[l, hook', squiggly, "\Phi(v)"'] \arrow[d, equal, "\wr"] \\ I \arrow[r, "\lfloor g \rfloor"'] & y_2 y_1^* \arrow[r, equal] & y_2 y_1^* \end{tikzcd}$$

In particular, by defn of W_Y, there is a z (possibly $z = I$) in Y such that

$$\Phi(f') \cong \lfloor tr_z(g) \rfloor.$$

It follows that

$\Psi F(f') \cong G L tr_2(g) \lrcorner$ and therefore, since Ψ is a fibration, $g \in \mathrm{Im}(\Psi)$, and there are $a_1, a_2 \in A$ and $f \in A(a_1, a_2)$

s.t

$$B(\Psi(a_1), \Psi(a_2)) \ni \Psi(f) = G(g) \in B(G(y_1), G(y_2))$$

In particular, for each $y \in Y$, may take $g = id_y$.

So, we can construct a lift $Y \to A$ in the category CAT of small categories

HOWEVER. Still need to prove that there is a lift in Comp.

Since Φ is a cofibration, there is a bijection

$$\left\{ \begin{array}{l} \text{pairs of } \overset{\text{(comm)}}{\text{monoid}} \text{ morphisms} \\ (\gamma_X : X_0 \to B_0,\ \gamma_Z : Z_0 \to B_0) \\ \text{such that} \quad \begin{tikzcd} MX_0 \arrow[r, "\overline{\Phi}_0"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Z_0 \arrow[d] \\ X_0 \arrow[r] & B_0 \end{tikzcd} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{monoid morphisms} \\ \Gamma : Y_0 \to B_0 \end{array} \right\}$$

remember Z_0 is free

Let $g_X : X_0 \to B_0$, $g_Z : Z_0 \to B_0$ be the pair of morphisms

such that

$$\begin{tikzcd} MX_0 \arrow[r, "\overline{\Phi}_0"] \arrow[d] & Z_0 \arrow[d] \arrow[ddr, bend left, "g_Z"] & \\ X_0 \arrow[r] \arrow[drr, bend right, "g_X"'] \arrow[ur, phantom, "\ulcorner"] & Y_0 \arrow[dr, "g_0"] & \\ & & B_0 \end{tikzcd}$$

commutes.

We must construct a monoidal functor

$Z_0 \xrightarrow{L_Z} A_0$ such that $\Psi_0 L_Z = g_{MT} : Z_0 \to B_0$

Since Z_0 is free and Ψ reflects monoidal products, may define lift on singletons such that duals are preserved. Hence have pair of morphisms &

$\therefore$ desired nat

Let
$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, tail, "\Phi"'] & A \arrow[d, two heads, "s"', "\Psi"] \\ Y \arrow[r, "G"'] & B \end{tikzcd}$$
commute in <u>Comp</u>

Since Ψ is a trivial fibration, it is a surjective equivalence of categories.
Hence for each $g \in \mathcal{Y}(y_1, y_2)$ $g \notin Im(\Phi)$, and all $a_1, a_2 \in A_0$ s.t. $\Psi(a_i) = G(y_i)$ $i \in \{1, 2\}$, there is a unique $f \in A(a_1, a_2)$

s.t.
$$G(g) = \Psi(f).$$

Then, since Φ is a cofibration, we can construct a suitable lift $Z_0 \to A_0$ as before, and hence the desired lift

$$L : \mathcal{Y} \to A.$$

Weak equivalences of cofibrant objects

Since all objects of Comp are fibrant, fibrant-cofibrant objects for this model structure are precisely those that have free object monad.

By a previous talk, these are in the image of the free functor CO $\longrightarrow$ Comp.

circuit operads

Indeed, straightforward to show that a w.e of cofibrant objects is induced by an equivalence of circuit operads.

Example

If $X \in \mathrm{Comp}$ has a 0-object 0, then

$X \longrightarrow *$ is a weak equivalence in the

trivial compact closed category

operadic model structure

Next time

- Homotopy category Ho Comp
- Enriching the model structure..

Thank you!