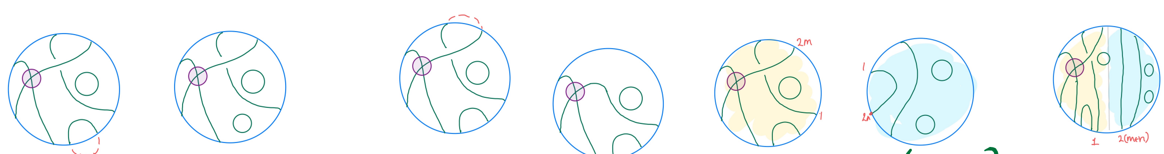
Graphs and nerves for circuit operads and compact closed categories

- **A few weeks ago**
  - Discussed categories
    - MO – of (coloured) modular operads

eg oriented surfaces

    - CO – of circuit operads

eg. virtual tuples



    - Comp – of small compact closed categories,
  - A triple of monadic adjunctions

$$\begin{tikzcd}
 & CO \arrow[dl, shift left] \arrow[dr, shift left] & \\
MO \arrow[ur, shift left] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left] \arrow[rr, phantom, "\perp"] & & Comp \arrow[ul, shift left] \arrow[ul, phantom, "\perp"] \arrow[ll, shift left]
\end{tikzcd}$$

– (in same relationship to mod ops as props to prope

- ## Much longer ago

A composite monad for modular operads (and the reasons we wanted a composite)
- using the decomposition to prove an abstract nerve theorem for modular operads.

- ## Idea for today

Do exactly the same for circuit operads

Next time

Do the same for compact closed categories

Why?

- circuit operads relevant in low dim topology
- can be used to investigate combinatorics of compact closed
- Operadic approach provides new avenues for understanding duality.

- STRUCTURE

I. Background review

- Informal review of definitions
- Outline of modular operad construction

II Circuit operads

1) - A related construction -
compact coloured props/prop-like objects
(Kock '18).
2) - an iteration of distributive laws
3) - graphical categories and the nerve
theorem

NEXT TIME

III Compact closed categories

- an iteration of distributive laws
- graphical categories and the nerve
theorem

I. • Informal review of definitions

•i) Graphical species

just as before, we'll define:

Modular and circuit operads both defined as graphical species with extra structure

- Let Σ be permutation groupoid of natural numbers $n \in \mathbb{N}_{\geq 0}$ and permutations $\underline{n} \xrightarrow{\sigma} \underline{n}$ ($\in Aut(\underline{n})$)

  where $\underline{n} = \{1, 2, \ldots, n\}$ $n \in \mathbb{N}_{\geq 0}$

  (so $\underline{0} = \emptyset$)

  Enlarge to form category $\mathbb{S}$ by adjoining distinguished element $\S \in \mathbb{S}$

$$\mathbb{S}(\S, \S) = \{id_{\S}, \tau\}$$

$$\left.\begin{matrix} ch_k^{\underline{n}} \\ ch_k^{\underline{n}} \circ \tau \end{matrix}\right\} \in \mathbb{S}(\S, \underline{n}) \quad \forall \quad 1 \leq k \leq n$$

The category of graphical species is the presheaf category $ps(\mathbb{S})$. So a graphical species

$S : \mathbb{S}^{op} \longrightarrow Set$ is described by data

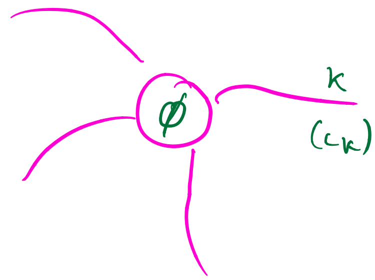
- $(\mathcal{C}, \omega) = (S_{\S}, S_{\tau})$
  $\omega : \mathcal{C} \to \mathcal{C}$ $\omega^2 = id_{\mathcal{C}}$

- $S_{\underline{n}}$ w right action of $Aut(\underline{n})$
- Equivariant maps $S(ch_k) : S_{\underline{n}} \to \mathcal{C}$
  $\omega S(ch_k) = S(ch_k \circ \tau)$

For $\underline{c} = (c_1, \dots, c_n) \in \mathcal{C}^n$

write $S_{\underline{c}} =$

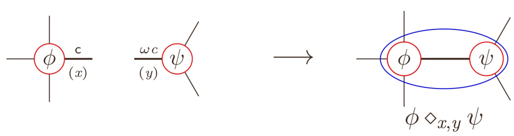
$$\left\{ \phi \in S_n \text{ w } S(ch_k)(\phi) = c_k \quad 1 \leq k \leq n \right\}$$



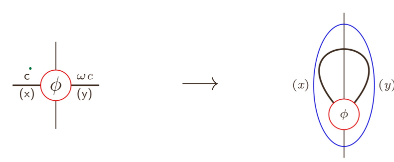
- **Modular operad:**

  Graphical species w

Multiplication



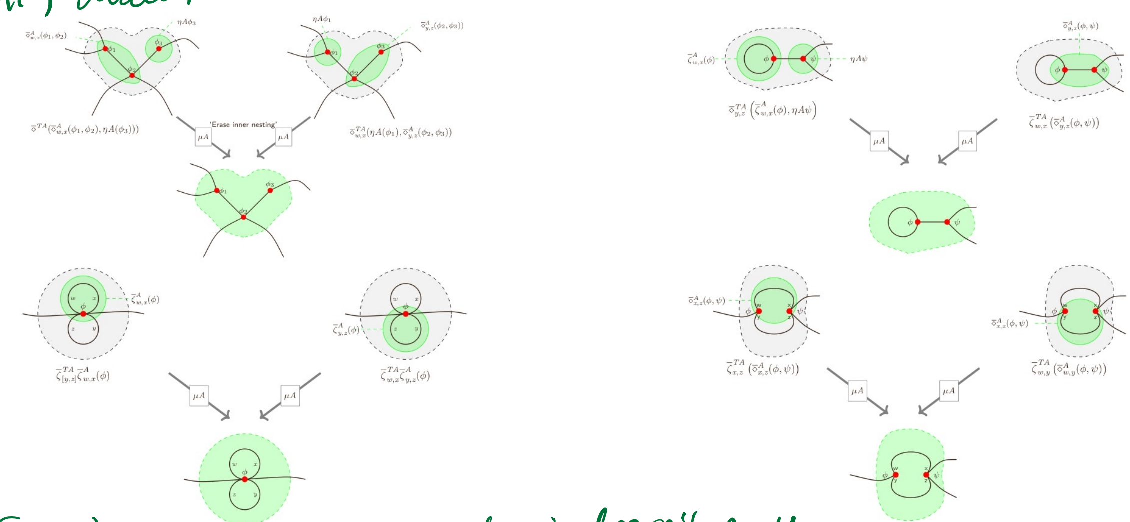
Contraction



- equivariant w.r.t Σ action etc.

S.E Axioms

(M1) Multiplication associative



(M2) order of contraction doesn't matter

- category $\mathcal{MO}^-$ of

  non-unital modular operads

  has objects $(S, \diamond, \zeta)$

  morphisms –

- Units: S is $(\mathcal{C}, \omega)$ coloured graphical species w/ multiplication $\diamond$

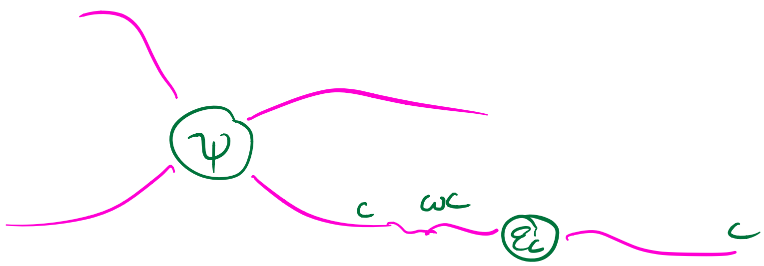
unit

$\varepsilon : \mathcal{C} \to S_2$ is injective map

$\varepsilon \circ \omega = \sigma_{\underline{2}} \cdot \varepsilon$ where $\sigma_2 \in Aut(\underline{2})$

$\neq id_{\underline{2}}$

$\forall\ \psi \quad w \quad \delta(ch_\omega)(\phi) = c$



$= \quad \phi.$

- category $\mathcal{MO}$ of modular operads $(S, \diamond, \xi, \varepsilon)$
  + morphisms in $\mathcal{GS}$ that preserve everything.

NOTE.

- If you remember if $(S, \diamond, \xi, \varepsilon)$ is a (C,ω) mod op then by definition there is a map

$$\begin{tikzcd} e \arrow[r] \arrow[dr] & S_0 \\ & e/\omega \arrow[u] \end{tikzcd}$$

given by $\xi(\varepsilon_c)$ $\forall c$

and this caused the problems for constructing a monad on graphical species

- CIRCUIT OPERADS.

graphical species $S \in GS$ +

- contraction ζ
(as w/ mod ops)

external product $\boxtimes$ making $(S_{\underline{n}})_n$ into symm graded monoid.

i.e. $\forall\ m, n \quad \boxtimes : S_{\underline{m}} \times S_{\underline{n}} \longrightarrow S_{\underline{m+n}}$

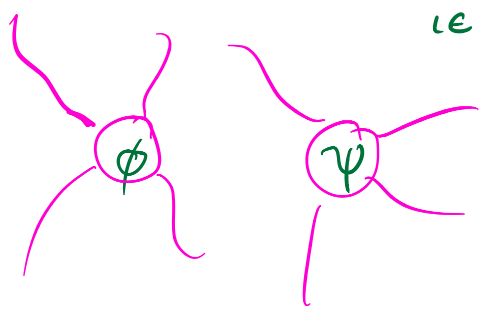
symmetries

$\sigma_{\underline{n,m}} : S_{\underline{m}} \boxtimes S_{\underline{n}} \xrightarrow{\cong} S_{\underline{n}} \boxtimes S_{\underline{m}}$

external (monoidal) unit

$\iota \in S_{\underline{0}} \qquad \phi \boxtimes \iota = \iota \boxtimes \phi = \phi$

$\forall \phi \in S_{\underline{n}} \quad \forall\ n \in \mathbb{N}$



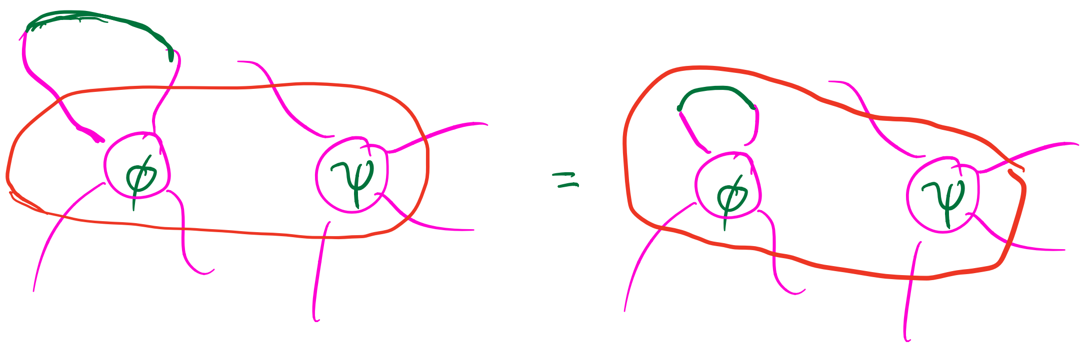
. Coherence axioms.

(C1) $\boxtimes$ is associative

up to iso

(C2) order of contraction doesn't matter (ζ2)

(C3) compatibility of $\boxtimes$, ζ



- Category CO^- of nonunital circuit operads

- <u>Units</u>

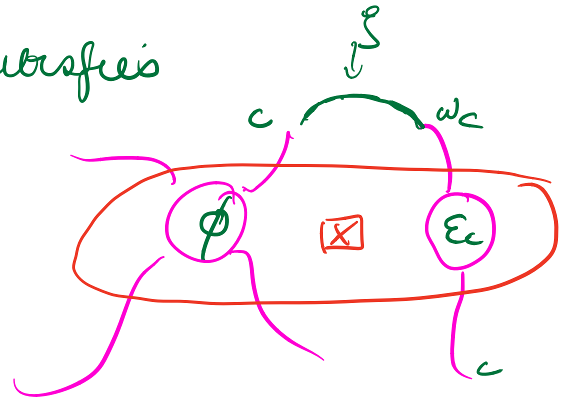
Units: S is $(\mathcal{C}, \omega)$ coloured graphical species

unit $\varepsilon : \mathcal{C} \to S_2$ is injective map

. $\varepsilon \circ \omega = \sigma_{\underline{2}} \cdot \varepsilon$ where $\sigma_2 \in Aut(\underline{2})$

$\neq id_{\underline{2}}$

satisfies

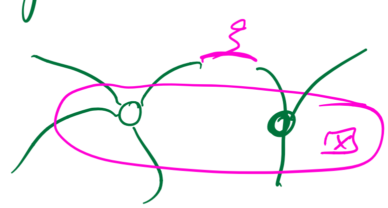
•  $= \phi \quad \forall \phi$ etc

• Category CO of circuit-operads (circuit algebras) as in Marcy

Forgetful functor
$$CO \longrightarrow MO$$

Given $(\mathcal{S}, \boxtimes, \xi, \varepsilon)$

define multiplication by



Construction of monad for modular operads

Monads $\mathbb{T}$, $\mathbb{D}$ on GS

$$\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$$

- Algebras are non unital modular operads

$\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ algebras are graphical species

S together w/ $\epsilon : S_{\S} \to S_2$

satisfying unit combinatorics

$$\begin{tikzcd} o : S_{\S} \arrow[rr] \arrow[dr] & & S_0 \\ & S_{\S}/S_{\tau} \arrow[ur] & \end{tikzcd}$$

Distributive law λ

s.t.

algebras for $\mathbb{D}\mathbb{T}$ are modular operads

In particular, there is a square of monadic adjunctions

$$\begin{tikzcd}
GS^{\mathbb{D}} \arrow[d, shift left, "U^{\mathbb{D}}"] \arrow[r, shift right, "T"'] \arrow[r, phantom, shift right=3, "U^{\mathbb{T}_*}"] & GS^{\mathbb{D}\mathbb{T}} = MO \arrow[l, shift right] \arrow[d, shift left] \\
GS \arrow[u, shift left] \arrow[u, phantom, shift left=2, "\dashv"] \arrow[r, shift right, "T"'] & GS^{\mathbb{T}} \arrow[l, shift right, "U^{\mathbb{T}}"'] \arrow[u, shift left, "U^{\mathbb{D}}"] \arrow[u, phantom, shift left=2, "\dashv"]
\end{tikzcd}$$

where $\mathbb{T}_*$ is the lift of $\mathbb{T}$ to $GS_* \overset{def}{=} GS^{\mathbb{D}}$

There is a dense full subcat $cGr \hookrightarrow GS$ (JK'11, R20) such that following diagram commutes.

$$\begin{tikzcd}
 & \Xi \arrow[r, "f.f", "dense"'] & MO \arrow[r, "N^{MO}", "f.f"'] \arrow[d, shift left, "U^{\mathbb{T}_*}"] & ps(\Xi) \arrow[d, "j^*"] \\
\mathcal{S}_* \arrow[r, "f.f", "dense"'] & cGr_* \arrow[u, "b.o", "j"'] \arrow[r, "f.f", "dense"'] & GS_* \arrow[r, "f.f"'] \arrow[u, shift left, "F^{\mathbb{T}_*}"] \arrow[d, shift left, "U^{\mathbb{D}}"] & ps(cGr_*) \arrow[d] \\
\mathcal{S} \arrow[u, "b.o"] \arrow[ur, "dense", "\Phi"'] \arrow[r, "f.f", "dense"'] & cGr \arrow[u, "b.o"] \arrow[r, "f.f", "dense"'] & GS \arrow[r, "f.f"'] \arrow[u, shift left, "F^{\mathbb{D}}"] & ps(cGr)
\end{tikzcd}$$

with Segal condition in term of $\mathcal{S}$.

i.e a Ξ-presheaf $P: \Xi^{op} \to Set$

is in the essential image of

$N^{MO}: MO \to ps(\Xi)$ if and only if

$\forall$ graphs connected $\mathcal{G}$ $\quad P(\mathcal{G}) \cong \lim_{(\mathcal{H}, f) \in \mathcal{S}/\mathcal{G}} P(j\Phi\mathcal{H})$

The monad $\mathbb{D}\mathbb{T}$ on GS does not meet the conditions outlined in (BMW '12) that guarantee that Ξ is dense in MO. "$\mathbb{D}\mathbb{T}$ doesn't have arities"

$\mathbb{T}_*$ has arities $c\mathcal{G}r_*$

• However $(\mathbb{T}_*, c\mathcal{G}r_*)$ do and since $\mathcal{S}$ is dense in $c\mathcal{G}r_*$ have same Segal conditions

• II Circuit operads

1) • A related construction - compact coloured props/prop-like objects (Kock '18).

2) • an iteration of distributive laws

3) • graphical categories and the nerve theorem

0) A little more about modular operads: $\mathbb{T}$, $\mathbb{D}$ & the distributive law.

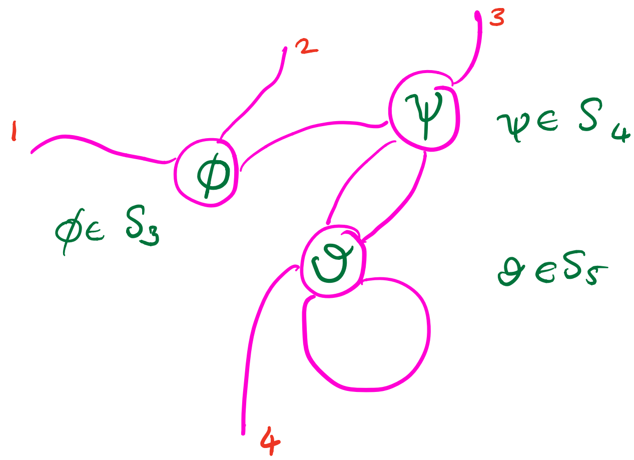
- $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ is monad for non-unital modular operads on GS.

for S w/ colours $(\mathfrak{C}, \omega)$

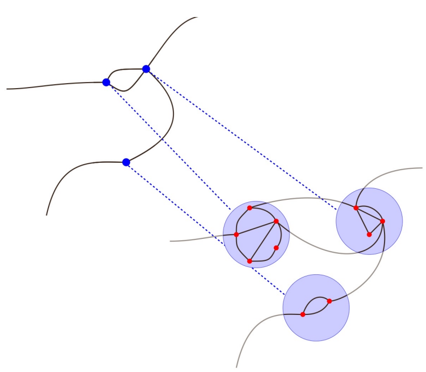
TS has " "

$TS_{\underline{n}} = \{$ Connected 'graphs' 'decorated' by S $\}/\sim$ — ω n ports

WITH AT LEAST ONE VERTEX



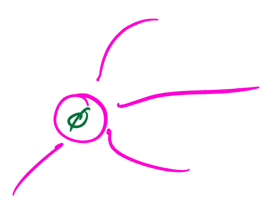
- $\mu^{\mathbb{T}}$



"graph of graphs"

forget blue bracketing

- $\eta^{\mathbb{T}}$ $\quad \phi \longmapsto$



• $\mathbb{D}$ gave combinatorics for the unit:

S is (C, ω) coloured

Adjoin elements to S.

DS is (C, ω) coloured w

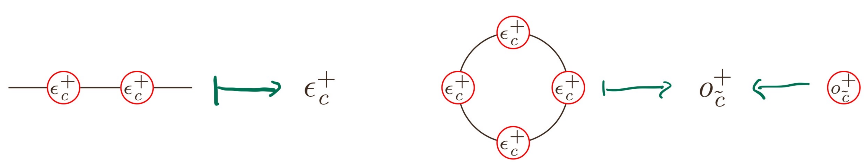
$$DS_{\underline{n}} = S_{\underline{n}} \qquad n \neq 0, 2$$

$$DS_{\underline{2}} = S_{\underline{2}} \amalg \{\amalg\, \varepsilon_c^{DS}\}_{c \in C}$$

$$DS_{\underline{0}} = S_{\underline{0}} \amalg \{\amalg\, o_{\tilde{c}}^{DS}\}_{\tilde{c} \in C/\omega}$$

Distributive law:

$$TDS_n \longrightarrow DTS_n$$



here $+$ instead of DS. in superscript.

So the monad T_* on $GS_* = GS^{\mathbb{D}}$

tells us when we can identify graphs represented by a $\mathbb{D}$-alg (S, ε, o) to "ignore" vertices decorated by the unit $\varepsilon: S_\varnothing \to S_2$ and 'contracted' unit $o: S_\varnothing \to S_0$

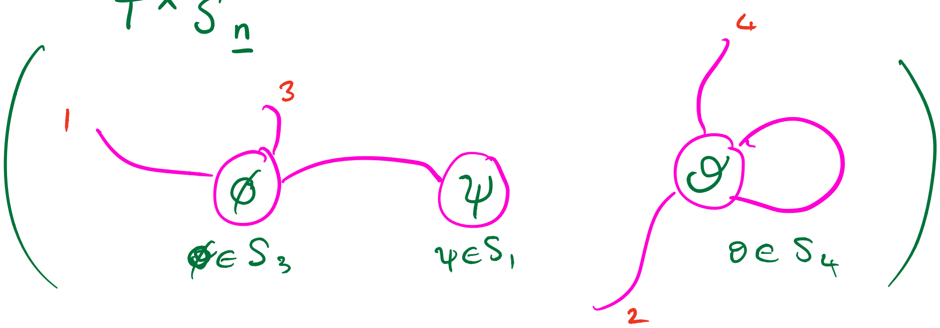
1. • Compact coloured props / prop-like objects. (Kock '18)

• Define monad $\mathbb{T}^{\times}$ on $\mathbb{D}$

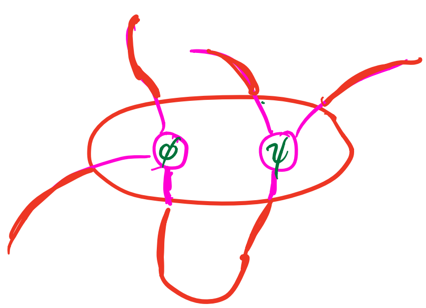
Definition identical to that of $\mathbb{T}$ but now graphs may not be connected.
(each component must have vertex.)

$T^{\times} S_{\S} = S_{\S}$

$T^{\times} S_{\underline{n}}$



$\in T^{\times} S_4$



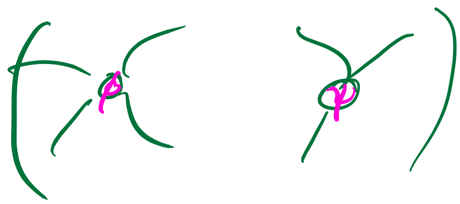
Monadic multiplication.
all fine.

Kock calls algebras for $\mathbb{T}^\times$ compact coloured props or prop like objects
BUT he's a bit vague about units

PROP: Algebras for $\mathbb{T}^\times$ are non-unital circuit operads.

PROOF: $\forall S \quad \forall n \quad TS_n \hookrightarrow T^\times S_n$
and restrictions of $\mu^{\mathbb{T}^\times}, \eta^{\mathbb{T}^\times}$ are $\mu^{\mathbb{T}}, \eta^{\mathbb{T}}$
so $\mathbb{T}^\times$ algebras have structure of an non-unital modular operads.

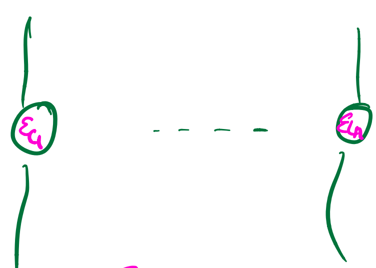
Let (A, h) be a $\mathbb{T}$ alg w induced MO structure $(A, \diamond, \zeta)$
Given any 2 elements $\phi \in A_m$, $\psi \in A_n$:

$\phi \boxtimes \psi \overset{\text{def}}{=} h$  $\in A_{m+n}$

Easy to check the coherence axioms $\square$.

So can we extend $(\lambda_{DT} : TD \Rightarrow DT$ to $T^\times D \Rightarrow DT^\times$?
~~and~~ to get unital circuit operads?

NO!! given



$\in T^\times DS_{2n}$

$\xrightarrow{?}$?? no coherent choice of image in $DT^\times S$ since $T^\times S$ has same colours as S.

HOWEVER.

Let $\mathbb{L} = (L, \mu^{\mathbb{L}}, \eta^{\mathbb{L}})$ be monad that

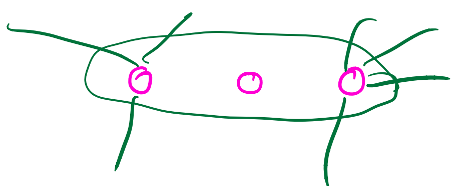
- leaves colours unchanged

$$LS_{\mathfrak{g}} = S_{\mathfrak{g}}$$

- takes graded set $(S_{\underline{n}})_{n \in \mathbb{N}}$

to free graded monoid

$$(LS_{\underline{n}})_{\underline{n} \in \mathbb{N}}$$



w obvious symmetry actions.

PROP There is a distributive law for $\mathbb{L}$ and $\mathbb{T}$

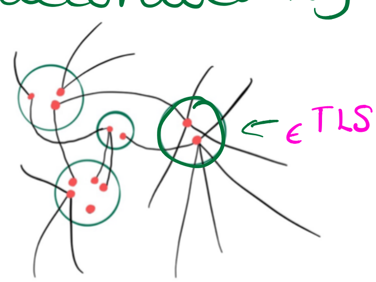
$$\lambda_{\mathbb{L}\mathbb{T}} : TL \Rightarrow LT \qquad \text{on } gS$$

And $\mathbb{L}\mathbb{T} \simeq \mathbb{T}^{\times}$.

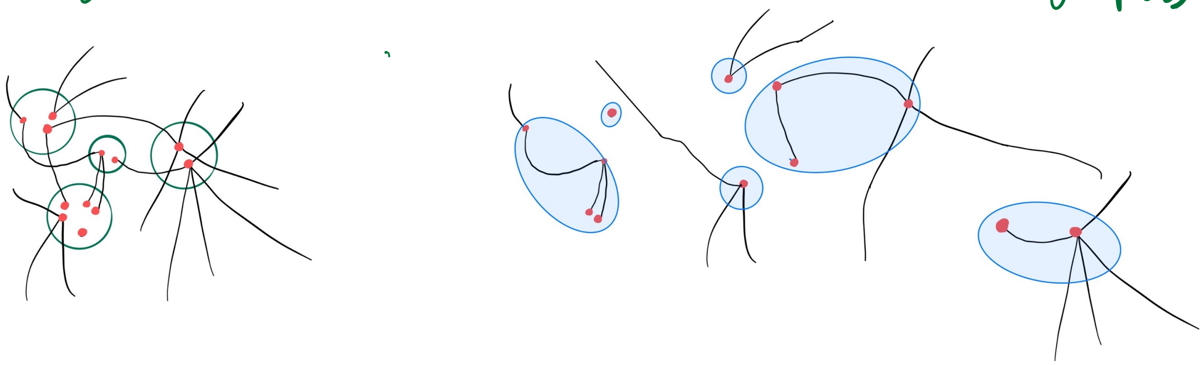
PROOF Define law.

View elements of LS as list of decorated corollas.

Connect graph decorated by LS



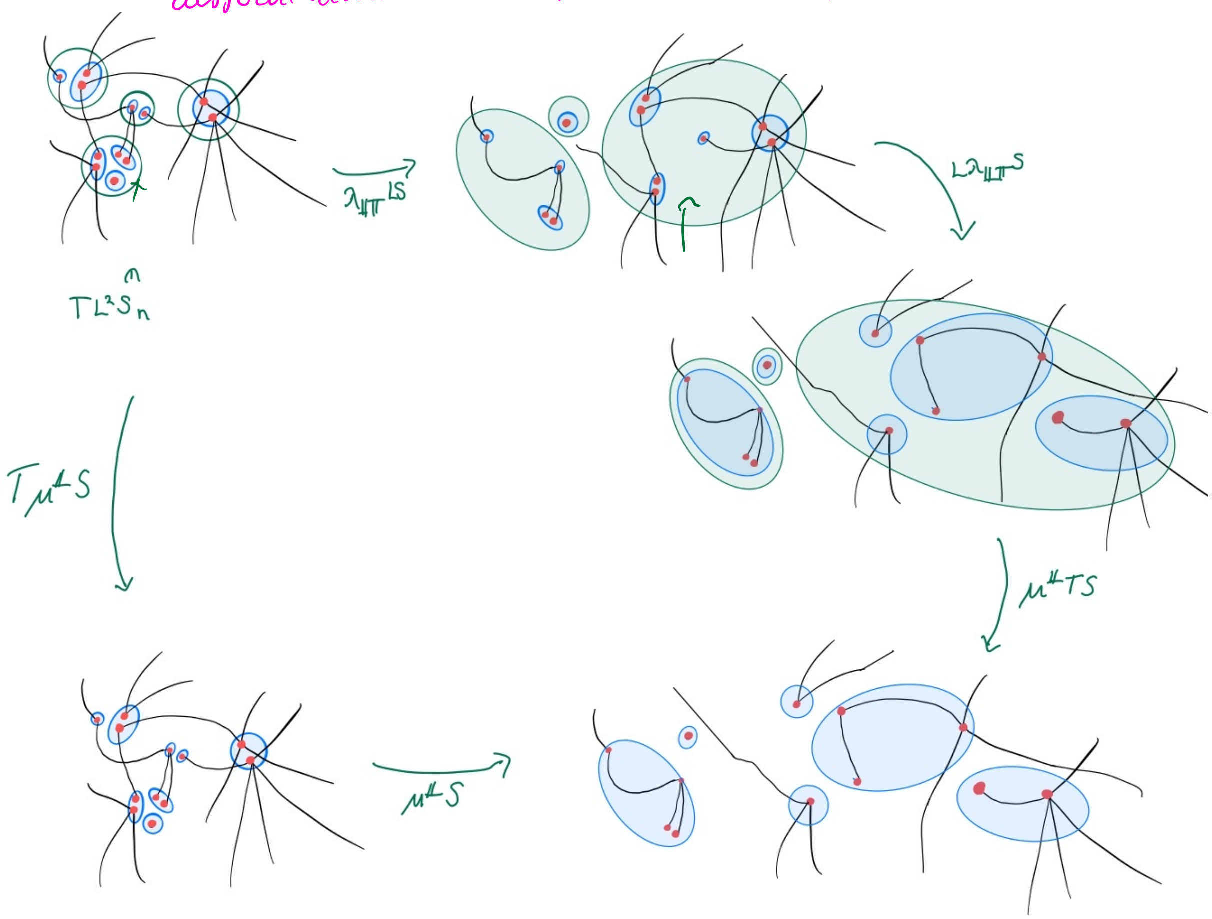
Forget the nodes to get disjoint union of connected graphs

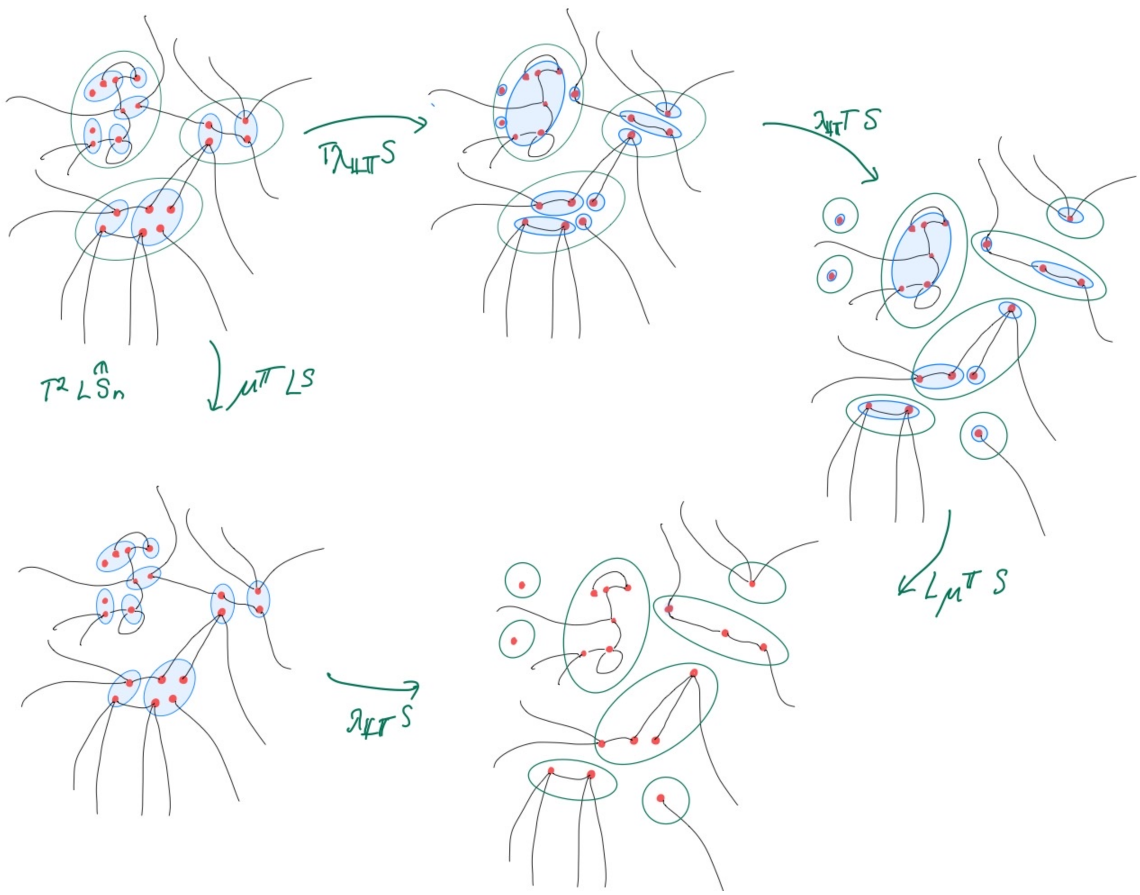


CHOICE OF ORDER doesn't matter since there is symmetric action

- Distributive law axioms

disjoint unions of disjoint unions of





$\Pi^\times \simeq \amalg\Pi$: Show that algebras agree:

Straightforward:

A triple of connected graphs is, up to permutation of components, a general graph. □

So now strategy is to show that

$\mathbb{L}$, $\mathbb{DT}$ compose:

<u>LEMMA</u>

Distributive law $\lambda_{\mathbb{LD}} : DL \Rightarrow LD$

"Pf" pretty obvious:

eg $\varepsilon_c \longmapsto (\varepsilon_c)$

Axioms are straightforward $\blacksquare$

By Cheng '11 since we have

$\mathbb{T}, \mathbb{D}, \mathbb{L}$ and $\lambda_{\mathbb{DT}} : TD \Rightarrow DT$

$\lambda_{\mathbb{LD}} : DL \Rightarrow LD$

$\lambda_{\mathbb{LT}} : TL \Rightarrow LT$

then have $\mathbb{LDT}$ on GS if "Yang-Baxter"

diagram commutes.

$$\begin{tikzcd}
 & DTL \arrow[r, "D\lambda_{\mathbb{LT}}"] & DLT \arrow[dr, "\lambda_{\mathbb{LD}}T"] & \\
TDL \arrow[ur, "\lambda_{\mathbb{DT}}L"] \arrow[dr, "T\lambda_{\mathbb{LD}}"'] & & & LDT \\
 & TLD \arrow[r, "\lambda_{\mathbb{LT}}D"'] & LTD \arrow[ur, "L\lambda_{\mathbb{DT}}"'] &
\end{tikzcd}$$

element of $TDLS_n$ is represented by

w/ colours (C, ω)

- connected graph w/ vertices decorated by
    - adjoined "units"

    OR
    - lists of elements of S.
- edges coloured by $c \in C$

OR

- isolated vertex decorated by
    - $O^{DLS}_{\langle c_1, \dots, c_k \rangle}$ $\quad c_i \in C$

this is easy to check.

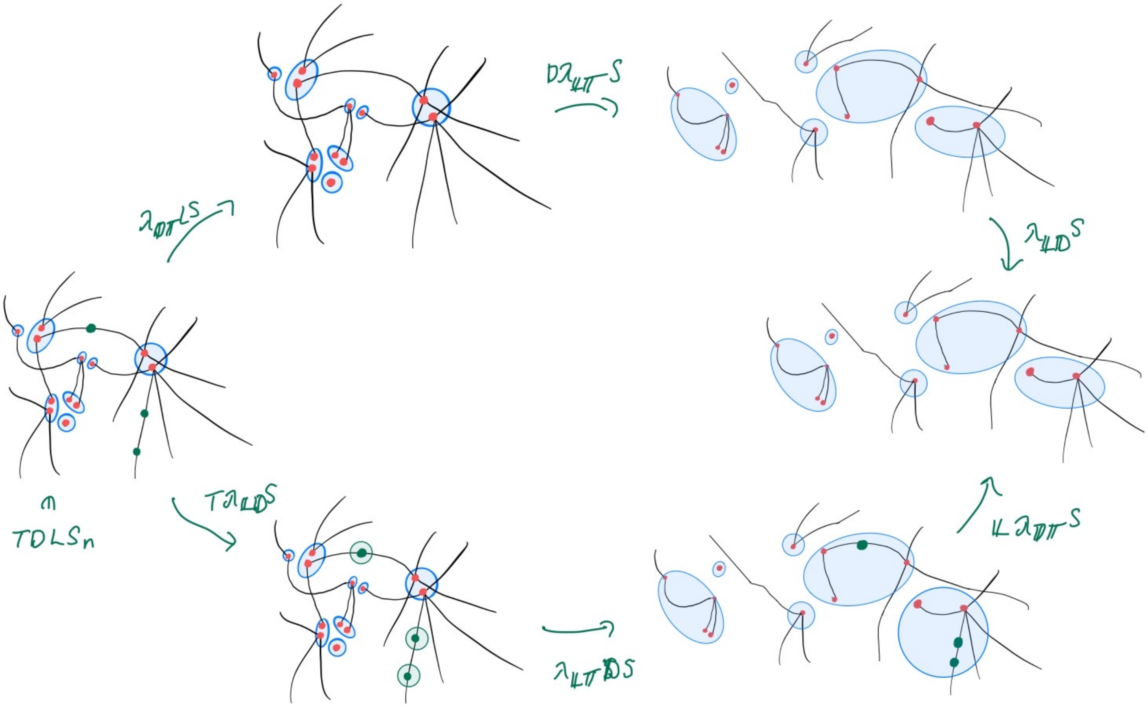
C_{nl}.

$$\begin{tikzcd}
& O^{DTLS}_{\langle c_1 \dots c_k \rangle} \arrow[r] & O^{DLTS}_{\langle c_1 \dots c_k \rangle} \arrow[dr] & \\
\bullet\ O^{DCS}_{\langle c_1 \dots c_k \rangle} \arrow[ur] \arrow[dr] & & & \left(O^{DTS}_{\langle c_1 \rangle}, \dots, O^{DTS}_{\langle c_k \rangle}\right) \\
& \bullet \left(O^{DS}_{\langle c_1 \rangle}, \dots, O^{DS}_{\langle c_k \rangle}\right) \arrow[r] & \left(\bullet\ O^{DS}_{\langle c_1 \rangle}, \dots, \bullet\ O^{DS}_{\langle c_k \rangle}\right) \arrow[ur] &
\end{tikzcd}$$

Something similar w/ graphs decorated by units (slightly more complicated)

fun to draw the possibilities!!

OTHERWISE:



<u>THEOREM</u> Algebras for composite monad are circuit operads

<u>PROOF</u> Straightforward to verify

An algebra (A, h) is

- modular operad by $\mathbb{D}\Pi$ alg-structure
- nonunital circuit operad by $\mathbb{L}\Pi$-alg structure

such that underlying nonunital modular operads agree.

- unit compatible w/ $\boxed{X}$ by $\mathbb{L}\mathbb{D}$-alg structure.

□

Now recall the modular operad picture.

$$\begin{tikzcd}
 & \Xi \arrow[r, "f.f", "dense"'] & MO \arrow[r, "N^{MO}", "f.f"'] \arrow[d, shift left, "U^{T_*}"] & ps(\Xi) \arrow[d] \\
S_* \arrow[r, "f.f", "dense"'] & cGr_* \arrow[u, "b.o"] \arrow[r, "f.f", "dense"'] & GS_* \arrow[u, shift left, "F^{T_*}"] \arrow[r, "ff"'] \arrow[d, shift left, "U^{D}"] & ps(cGr_*) \arrow[d] \\
S \arrow[u, "b.o"] \arrow[ur, "dense"] \arrow[r, "f.f", "dense"'] & cGr \arrow[u, "b.o"] \arrow[r, "f.f", "dense"'] & GS \arrow[u, shift left, "F^{D}"] \arrow[r, "ff"'] & ps(cGr)
\end{tikzcd}$$

The functor $N: MO \longrightarrow ps(\Xi)$ is f-f because Gr_* is "nice" (has arities) w respect to the lifted monad T_* on $GS_* = Alg(D)$.

BUT DT doesn't have arities

and for the same reason LDT will not have arities.

By the classical theory of distributive laws there is a cube of monadic adjunctions

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & GS^{\mathbb{L}\mathbb{D}} \arrow[rr, no head] \arrow[dd, no head] & & GS^{\mathbb{L}\mathbb{D}\mathbb{T}} = CO \arrow[dd, no head] \\
GS_* = GS^{\mathbb{D}} \arrow[urr, no head] \arrow[rr, no head] \arrow[dd, no head] & & GS^{\mathbb{D}\mathbb{T}} = MO \arrow[urr, no head] & & \\
 & & GS^{\mathbb{L}} \arrow[rr, no head] & & GS^{\mathbb{L}\mathbb{T}} = CO^- \\
GS \arrow[urr, no head] \arrow[rr, no head] & & GS^{\mathbb{T}} \arrow[urr, no head] \arrow[uu, no head] & &
\end{tikzcd}$$

Look at lift $\tilde{\mathbb{L}}\mathbb{T}_*$ of $\mathbb{L}, \mathbb{T}$ to $GS_* = GS^{\mathbb{D}}$.

$$\begin{tikzcd}
 & \widetilde{\equiv} \arrow[r, "f.f"] & MO \arrow[r, "\tilde{N}^*"] \arrow[d, shift left] & ps(\widetilde{\equiv}) \arrow[d] \\
S_* \arrow[r, "f.f"', "dense"] & Gr_* \arrow[r, "f.f", "dense"'] \arrow[u, "b.o"] & GS_* \arrow[r, "ff"] \arrow[u, shift left] \arrow[d, shift left] & ps(Gr_*) \arrow[d] \\
S \arrow[r, "f.f", "dense"'] \arrow[u, "b.o"] & Gr \arrow[r, "f.f", "dense"'] \arrow[u, "b.o"] & GS \arrow[r, "f.f"] \arrow[u, shift left] & ps(Gr)
\end{tikzcd}$$

Can show that $\widetilde{\equiv}$ isn't dense in MO so $\tilde{N}$ not f.f.

BUT...

the lift $\widetilde{\mathbb{L}\mathbb{T}}_*$ on GS_* is unsurprisingly very similar to Kock's monad $\mathbb{T}^\times = \mathbb{L}\mathbb{T}$.

Namely $\mathbb{T}_*$ defined in terms of connected graphs, but can simply replace connected to general graphs in definition of $\widetilde{\mathbb{L}\mathbb{T}}_*$

as long as all connected components have non-empty vertex sets.

That suggests how to choose the right graphical category for circuit operads:

Namely, objects of the free strict symmetric monoidal category Gr on cGr are general, not nec connected graphs:

Precisely the graphs used to define $\mathbb{T}^\times$ on GS.

and to rewrite $\widetilde{\mathbb{L}\mathbb{T}}_*$ on GS_*!

· And have dense f.f embedding

$$\mathcal{S} \xrightarrow[\text{dense}]{f.f} Gr \xrightarrow[\text{dense}]{f.f} GS$$

So, consider commuting diagram of functors

$$\begin{tikzcd}
 & \Xi^{\times} \arrow[r, "f.f"', "\text{dense}"] & MO \arrow[r, "ff", "N"'] \arrow[d, shift left] & ps(\Xi) \arrow[d] & ! \\
\mathcal{S}_* \arrow[r, "f.f", "\text{dense}"'] & Gr_* \arrow[r, "f.f", "\text{dense}"'] \arrow[u, "b.o", "j^{\times}"'] & GS_* \arrow[r, "ff"] \arrow[u, shift left] \arrow[d, shift left] & ps(Gr_*) \arrow[d] & \\
\mathcal{S} \arrow[r, "f.f", "\text{dense}"'] \arrow[u, "b.o"] \arrow[ur, "\text{dense}", "\Phi"'] & Gr \arrow[r, "f.f", "\text{dense}"'] \arrow[u, "b.o"] & GS \arrow[r, "f.f"] \arrow[u, shift left] & ps(Gr) &
\end{tikzcd}$$

$\widehat{\mathbb{L}}\ \mathbb{T}_*$ has arities Gr_* !

(By exactly the same argument as for modular operads).

Hence.

<u>Thm:</u> The category $\Xi^{\times}$ obtained in the b.o – f.f factorisation of

$$Gr \longrightarrow GS \xrightarrow{LDT} CO$$ is dense in CO.

Hence the functor $N^{CO}: CO \longrightarrow Ps(\Xi^{\times})$ is f.f w. ess. im. $P: (\Xi^{\times})^{op} \to Set$ s.t $\forall\, G$ $\quad P(G) = \lim_{(H,f)\in \mathcal{S}\downarrow G} P(j^{\times}\Phi H)$ $\square$