BRAUER DIAGRAMS, MODULAR OPERADS, AND A GRAPHICAL NERVE THEOREM FOR CIRCUIT ALGEBRAS

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Abstract. Circuit algebras, used in the study of finite-type knot invariants, are a symmetric analogue of Jones's planar algebras. They are very closely related to circuit operads, which are a variation of modular operads admitting an extra monoidal product. This paper gives a description of circuit algebras in terms categories of Brauer diagrams. An abstract nerve theorem for circuit operads – and hence circuit algebras – is proved using an iterated distributive law, and an existing nerve theorem for modular operads.

1. Introduction

Circuit algebras are symmetric (or non-planar) versions of Jones's planar algebras. They were introduced by Bar-Natan and Dansco [1] as a framework for organising virtual tangles in the study of finite type invariants of knotted objects. Recently, Dansco, Halecheva and Robertson have given an explicit proof that (directed) circuit algebras are equivalent to wheeled props [9], and they have used this to give descriptions of the Kashiwara-Vergne groups KV and KRV, and the graded Grothendieck-Teichmüller group GRT as automorphism groups of circuit algebras [10].

This work describes a general class of (coloured) circuit algebras in terms of categories of Brauer diagrams [20]. The main result is the proof of a nerve theorem for circuit algebras – closely related to the graphical nerve theorem for modular operads of [25] – which provides an explicit of the operadic combinatorics of circuit algebras.

Roughly speaking, circuit algebras are graded monoids $(A_n)_{n\in\mathbb{N}}$ that are equipped with an additional contraction operation, according to certain compatibility axioms. It will be proved in Theorem 9.8 that circuit algebras in the category Set of sets are equivalent to associative graded monoids – called *circuit operads* – in the category MO of Set-valued modular operads. Indeed, circuit algebras defined in an arbitrary symmetric monoidal category $(\mathsf{V}, \otimes, I)$ are *enriched* Set-values circuit operads, with a set (or *palette*) of objects (or colours), and a collection of V-objects of (multi-)morphisms between them. By contrast, circuit operads describe the same algebraic structures, *internal* to a category E with finite limits. Hence circuit operads are described by an *object object* $C \in \mathsf{E}$, and a filtered (morphism) object $A = (A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathsf{E}$ for all n, together with E-morphisms $A_n \to C^n$ satisfying certain axioms.

In [25], it was shown that modular operads are algebras for a monad on a certain combinatorial category GS, and – using the abstract nerve theory of [3] – a graphical category and corresponding nerve theorem for modular operads were obtained. This paper builds on that result. Circuit operads are constructed as algebras for a monad, on the same category GS.

Theorem 1.1 (Theorem 9.7)**.** *The category* CO *of circuit operads is isomorphic to the Eilenberg-Moore category of algebras for a monad* $\mathbb{O}^{\times}$ *on the category* GS *of graphical species.*

As in [25], this enables the proof of a corresponding nerve theorem. Circuit operads are characterised as functors from a full subcategory $\Xi^{\times} \subset \mathsf{CO}$ whose objects are graphs.

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Let $\mathsf{psh}(\Xi^\times)$ denote the category of presheaves (contravariant Set-valued functors) on $\Xi^\times$.

Theorem 1.2 (Theorem 10.4). *The nerve functor $N\colon \mathsf{CO} \to \mathsf{psh}(\Xi^\times)$ induced by the inclusion $\Xi^\times \hookrightarrow \mathsf{CO}$ is fully faithful.*

Moreover, there is a canonical restriction functor $R^*\colon \mathsf{psh}(\Xi^\times) \to \mathsf{GS}$, and a presheaf P on $\Xi^\times$ is equivalent to a circuit operad if and only if it satisfies the so-called 'Segal condition': P is completely described by the graphical species R^*P.

Since circuit operads have an underlying modular operad structure (see Proposition 5.23), there are the same obstructions to constructing the monad for circuit operads as for modular operads (this is the so-called 'problem of loops' [25, Section6]). Fortunately, they also admit the same resolution, via a distributive law (see [25, Section 7] for the construction of the modular operad monad). However, the modification of the results of [25] for the circuit operad case is slightly subtle, and relies on the construction of a triple of distributive laws.

Theorem 1.3 (Proposition 9.6 & Theorem 9.7). *There is a triple of monads $\mathbb{L}, \mathbb{D}$ and $\mathbb{T}$ on GS and distributive laws $\lambda_{\mathbb{DT}}, \lambda_{\mathbb{LT}}$, and $\lambda_{\mathbb{LD}}$ that describe an iterated distributive law* [7] *such that the composite $\mathbb{LDT}$ on GS is the desired monad whose EM category of algebras is isomorphic to CO.*

Theorem 1.2 is then implied by the following:

Proposition 1.4. *The lifted composite monad $\widetilde{\mathbb{L}}\mathbb{T}_*$ – whose algebras are circuit operads – on the category GS_* of $\mathbb{D}$-algebras, satisfies the conditions of* [3]*, and hence the induced nerve functor is fully faithful.*

Before discussing the contents of the paper in more detail, let us consider a few reasons why such a result may be of interest.

First, the nerve theorem gives a description of (Set-valued) circuit algebras in terms of a so-called strong 'Segal'-condition. In the case of modular operads, weakening the corresponding Segal condition gives a notion of $(\infty, 1)$ or *up-to-homotopy* modular operads, that are precisely the fibrant objects in a model structure on the relevant category of graphical presheaves (this is proved, by different means, and using slightly different graphical categories, in [14] and [25]). So, it is natural to define $(\infty, 1)$-circuit algebras as presheaves satisfying the corresponding weak Segal condition. This is relevant, for example, to the ongoing work of Dansco, Halecheva and Robertson. In [10], they have used circuit algebras to obtain results relating the *graded Grothendieck-Teichmüller* and *Kashiwara-Verne* groups GRT and KRV. However, since circuit algebra operations are not preserved strictly under passage to a pro-unipotent completion of groups, it is necessary to adopt a suitably weakened notion of the circuit algebra axioms in order to obtain a topological version of these results. (See [10, Introduction, Remark 1.1]).

Second, the description of circuit algebras in terms of categories of Brauer diagrams indicates a deep relationship between circuit algebras and (stable) representations of sequences of groups [27]. Indeed, in view of the results of [27], representations of the infinite orthogonal and symplectic groups $O(\infty)$ and $Sp(\infty)$ admit a description in terms of non-unital monochrome circuit algebras, and, similarly, representations of the infinite general linear group $GL(\infty)$ may be described in terms of non-unital two coloured circuit algebras. It is an ongoing project (with R. Street) to understand these relationships in the unital and general coloured cases.

Finally, the construction given here is based on abstract methods and may be easily modified to obtain related results. For example, in✠ [?], it forms the basis of an operadic nerve theorem for compact closed categories. Moreover, the category CA of circuit algebras in Set admits a *suspension endofunctor*, that enables the construction of *higher* or n-circuit algebras ($n \geq 1$), whereby the colours of an $(n+1)$-circuit algebra, themselves describe a n-circuit algebra. Such a construction is used in the notion of *circuit* *algebras with skeleton* described in [10], and (in a suitably weakened form) has a number of potential applications, for example to the study of extended conformal field theories.

1.1. **Overview.** The proof of Theorems 1.1 and 1.2 will rely on an iterated distributive law and the abstract nerve machinery of [3, 4]. Section 2 provides a relatively informal and accessible introduction to these notions, as well as establishing some of the categorical conventions and notations that will be used in the rest of the paper.

A detailed discussion of the categories of (coloured) Brauer diagrams and their functors is given in Section 3. Versions of these diagrams have been widely studied (see, e.g. ✠[20, 27]) since Brauer's 1937 paper [5] extending Schur-Weyl duality to describe representations of the orthogonal and symplectic groups. Section 3 emphasises the features that will be most relevant to the discussion of circuit operads in Sections 5-10.

In Section 4, circuit algebras are defined in terms of categories of Brauer diagrams, and a number of examples are discussed. Circuit operads are introduced in Section 5 as presheaves over the category GS of graphical species [15]. A canonical functor from circuit algebras in Set to circuit operads is described, as well as a monadic adjunction between circuit operads and modular operads (as defined in [25]).

The technical work of the paper starts at Section 6. This section is largely a review of basic graph constructions. As in [25], this is based on the definition of graphs introduced in [15], and also used, for example in [13, 14].

A monad $\mathbb{T}^\times$ for non-unital circuit operads is constructed in Section 7. This is very similar to the monad for non-unital modular operads in [25], and almost identical to the monad discussed in [19]. However it is not possible to obtain the monad for unital circuit operads directly from a distributive law involving $\mathbb{T}^\times$. Instead, it is necessary to supplement the construction of [25] – which is reviewed in Section 8 – by observing that $\mathbb{T}^\times$ is, itself a composite monad.

This happens in Section 9, in which an iterated distributive law is established and it is shown that the resulting composite $\mathbb{LDT}$ of three monads on GS is indeed the desired circuit operad monad. Moreover, it is proved that the forgetful functor from Set-valued circuit algebras to Set-valued circuit operads is, in fact, an equivalence of categories.

Finally, Section 10 contains a description of the graphical category $\Xi^\times$ for circuit operads and the proof of the nerve theorem (Theorem 1.2) is straightforward.

2. Categorical background

The main result of this work is the construction of a monad for circuit algebras and a graphical category and nerve theorem that give a complete description of their combinatorics. The method uses a composition of monads, via an iterated distributive law and apply this to obtain an abstract nerve theorem in the style of [3].

Since it is my hope that this work will be relevant to a wider audience than category theorists alone, the aim of this section is to provide a more intuitive explanation of the statement above.

For a more detailed review of the general theory of monads and their Eilenberg-Moore (EM) categories of algebras, see for example [22, Chapter VI]. Recall, for now, that a monad $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ on a category C is given by an endofunctor $M\colon \mathsf{C} \to \mathsf{C}$ together with natural transformations $\mu^{\mathbb{M}}\colon M^2 \Rightarrow M$ (called the *monad multiplication*) and $\eta^{\mathbb{M}}\colon id_{\mathsf{Cat}} \Rightarrow M$ (called the *monad unit*) that satisfy certain axioms (so that $\mathbb{M}$ is an associative monoid in the category of endofunctors on C). An algebra (x, θ) for $\mathbb{M}$ is an object x of C, together with a morphism $\theta\colon Mx \to x$ that is suitably compatible with $\mu^{\mathbb{M}}$ and $\eta^{\mathbb{M}}$.

2.1. **Distributive laws.** Informally, monads are gadgets that encode, via their algebras, (algebraic) structure on objects of categories. The key idea underlying this work is that, by suitably 'combining' monads that describe different simple algebraic structures, we can build up a description of more complicated structures on the objects of some category.

Let us consider how to make these notions precise: In general, monads do not compose, since given monads $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ and $\mathbb{M}' = (M', \mu^{\mathbb{M}'}, \eta^{\mathbb{M}'})$ on a category C, there is no obvious choice of natural transformation $\mu\colon (MM')^2 \Rightarrow MM'$ that defines a monadic multiplication for the endofunctor MM' on C. Nonetheless, for any natural transformation $\lambda\colon M'M \Rightarrow MM'$, there is an induced natural transformation

$$\mu_\lambda\colon\ (MM')^2 \xRightarrow{M\lambda M'} M^2M'^2. \xRightarrow{\mu^{\mathbb{M}}\mu^{\mathbb{M}'}} MM'$$

Definition 2.1. *A* distributive law [2] *for $\mathbb{M}$ and $\mathbb{M}'$ is a natural transformation $\lambda\colon M'M \Rightarrow MM'$ such that the triple $(MM', \mu_\lambda, \eta^{\mathbb{M}}\eta^{\mathbb{M}'})$ defines a monad $\mathbb{M}\mathbb{M}'$ on C.*

Distributive laws $\lambda\colon M'M \Rightarrow MM'$ are characterised by four axioms that describe coherence of λ with the monadic multiplications and units of $\mathbb{M}$ and $\mathbb{M}'$ (see [2]). So, a distributive law $\lambda\colon M'M \Rightarrow M'M$ provides a glue for composing M and M'. And, moreover, λ determines how the $\mathbb{M}$-structures and $\mathbb{M}'$-structures on C interact to form the structure encoded by the composite monad $\mathbb{M}\mathbb{M}'$.

Example 2.2. Algebras for the category monad on the category of directed graphs (described as contravariant functors from the small category $\bullet \rightrightarrows \bullet$) are small categories.

This monad may be obtained as a composite of the *semi-category monad*, which governs associative composition, and the *reflexive graph monad* that adjoins a distinguished loop at each vertex of a graph. The corresponding distributive law encodes the property that the adjoined loops provide identities for the semi-categorical composition.

Example 2.3. In [25] the monad for modular operads on the category GS of graphical species (see Definition 5.3) is obtained – as in the category case Example 2.2 – as a composite $\mathbb{D}\mathbb{T}$ where the monad $\mathbb{T}$ governs the composition structure (multiplication and contraction) and the monad $\mathbb{D}$ adjoins distinguished elements. The distributive law ensures that the distinguished elements act as (contracted) units for the multiplication (see also Section 8).

As usual, let $\mathsf{C}^{\mathbb{M}}$ denote the EM category of algebras for a monad $\mathbb{M}$ on C.

By [2, Section 3], given monads $\mathbb{M}, \mathbb{M}'$ on C, and a distributive law $\lambda\colon M'M \Rightarrow MM'$, there is a commuting square of strict monadic adjunctions:

(2.4)
$$\begin{tikzcd}
\mathsf{C}^{\mathbb{M}'} \arrow[r, shift right, "\top"'] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] & \mathsf{C}^{\mathbb{M}\mathbb{M}'} \arrow[l, shift right] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] \\
\mathsf{C} \arrow[r, shift right, "\top"'] \arrow[u, shift right] & \mathsf{C}^{\mathbb{M}}. \arrow[l, shift right] \arrow[u, shift right]
\end{tikzcd}$$

It follows that, given a n-tuple of monads $\mathbb{M}_1, \dots, \mathbb{M}_n$ on a category C, and a sequence $(\lambda_i)_{i=2}^n$ of distributive laws $\lambda_i : M_i(M_1 \dots M_{i-1}) \Rightarrow (M_1 \dots M_{i-1})M_i$, there is, by definition, a n-fold composite monad $\mathbb{M}_1 \dots \mathbb{M}_n$ on C, and hence a n-cube of adjunctions as (for the case $n = 3$) in (2.5), where only the left adjoints are marked:

$$
\begin{tikzcd}
 & \mathsf{C}^{\mathbb{M}_3\mathbb{M}_2} \arrow[rr] & & \mathsf{C}^{\mathbb{M}_3\mathbb{M}_2\mathbb{M}_1} \\
\mathsf{C}^{\mathbb{M}_3} \arrow[ur] \arrow[rr] & & \mathsf{C}^{\mathbb{M}_3\mathbb{M}_1} \arrow[ur] & \\
 & \mathsf{C}^{\mathbb{M}_2} \arrow[uu] \arrow[rr] & & \mathsf{C}^{\mathbb{M}_2\mathbb{M}_1} \arrow[uu] \\
\mathsf{C} \arrow[uu] \arrow[ur] \arrow[rr] & & \mathsf{C}^{\mathbb{M}_1} \arrow[uu] \arrow[ur] &
\end{tikzcd}
\tag{2.5}
$$

Moreover, [7], shows that the composite $\mathbb{M}_1 \dots \mathbb{M}_n$ exists if there are pairwise distributive laws $\lambda_{i,j} : M_j M_i \Rightarrow M_i M_j$, for $1 \leq i < j \leq n$ such that for each triple $1 \leq i < j < k \leq n$, the corresponding triple of monads and distributive laws, satisfies a Yang-Baxter equation (that is described in (9.1)).

2.2. **Abstract nerve theory.** Let Set be the category of sets and all set maps is denoted Set. Given any (essentially) small category C, $\mathsf{psh}(\mathsf{C})$ is the category of functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$.

Now, given a locally small category C, a functor $F \colon \mathsf{D} \to \mathsf{C}$ induces a *nerve functor* $N_{\mathsf{D}} \colon \mathsf{C} \to \mathsf{psh}(\mathsf{D})$ by $N_{\mathsf{D}}(c)(d) = \mathsf{C}(Fd, c)$ for all $c \in \mathsf{C}$ and $d \in \mathsf{D}$. If N_{D} is fully faithful, and F and D are suitably nice, then N_{D} provides a useful tool for studying C.

Example 2.6. The nerve of a small category C is described by the image of C under the nerve functor induced by the inclusion $\Delta \hookrightarrow \mathsf{Cat}$, where Cat is the category of small categories, and Δ is the (topological) *simplex category* of non-empty finite ordinals and order preserving morphisms.

Recall that every functor admits an (up to isomorphism) unique *bo-ff factorisation* as a bijective on objects functor followed by a fully faithful functor. For example, if $\mathbb{M}$ is a monad on a category C, and $\mathsf{C}^{\mathbb{M}}$ is the EM category of algebras for $\mathbb{M}$, then the free functor $\mathsf{C} \to \mathsf{C}^{\mathbb{M}}$ has bo-ff factorisation $\mathsf{C} \to \mathsf{C}_{\mathbb{M}} \to \mathsf{C}^{\mathbb{M}}$, where $\mathsf{C}_{\mathbb{M}}$ is the Kleisli category of free $\mathbb{M}$-algebras (see e.g. [22, Section VI.5]).

Hence, for any subcategory D of C, the bo-ff factorisation of the canonical functor $\mathsf{D} \hookrightarrow \mathsf{C} \xrightarrow{\text{free}} \mathsf{C}^{\mathbb{M}}$ factors through the full subcategory $\Theta_{\mathbb{M},\mathsf{D}}$ of $\mathsf{C}_{\mathbb{M}}$ with objects from D and there is an induced diagram of functors

$$
\begin{tikzcd}
\Theta_{\mathbb{M},\mathsf{D}} \arrow[r, "\text{f.f.}"] & \mathsf{C}^{\mathbb{M}} \arrow[r, "N_{\mathbb{M},\mathsf{D}}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}}) \arrow[d, "j^*"] \\
\mathsf{D} \arrow[u, "\text{b.o.}", "j"'] \arrow[r, hook] & \mathsf{C} \arrow[u, shift left, "\text{free}"] \arrow[r, "N_{\mathsf{D}}", "\text{f.f.}"'] & \mathsf{psh}(\mathsf{D}).
\end{tikzcd}
\tag{2.7}
$$

where j^* is the pullback of the bijective on objects functor $j \colon \mathsf{D} \to \Theta_{\mathbb{M},\mathsf{D}}$. The left square of (2.7) commutes by definition, and the right square commutes up to natural isomorphism.

By construction, the defining functor $\Theta_{\mathbb{M},\mathsf{D}} \to \mathsf{C}^{\mathbb{M}}$ is fully faithful. In [3] conditions on D and $\mathbb{M}$ are described that ensure that the induced nerve $N_{\mathbb{M},\mathsf{D}} \colon \mathsf{C}^{\mathbb{M}} \to \mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}})$ is also fully faithful.

Moreover, by [30, Section 4], if D and $\mathbb{M}$ satisfy these conditions, then, up to equivalence, objects in the image of $N_{\mathbb{M},\mathsf{D}}$ in $\mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}})$ are precisely those that are projected, under j^* to the image of N_{D} in $\mathsf{psh}(\mathsf{D})$. (This is called the *Segal condition* for the nerve theorem.)

Of course, it may be the case that there is no subcategory $\mathsf{D} \subset \mathsf{C}$ such that $\mathbb{M}$ and D satisfy the the conditions of [3, Sections 1 & 2]. However, it may still be possible to derive an abstract nerve theorem for $\Theta_{\mathbb{M},\mathsf{D}}$ and $\mathsf{C}^{\mathbb{M}}$.

For, if $\mathbb{M} = \mathbb{M}^n \dots \mathbb{M}^1$ is a composite monad, then any path $\mathsf{C} \to \cdots \to \mathsf{C}^{\mathbb{M}^n \dots \mathbb{M}^1} = \mathsf{C}^{\mathbb{M}}$ of left adjoints in the cube (2.5) describes a factorisation of the diagram (2.7) as

$$
\begin{tikzcd}
\Theta_{\mathbb{M},\mathsf{D}} \arrow[r, "\text{f.f.}"] \arrow[d, equal] & \mathsf{C}^{\mathbb{M}} \arrow[r, "N_{\mathbb{M},\mathsf{D}}"] \arrow[d, equal] & \mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}}) \arrow[d, equal] \\
\Theta_{\mathbb{M}_n \dots \mathbb{M}_1,\mathsf{D}} \arrow[r, "\text{f.f.}"] & \mathsf{C}^{\mathbb{M}_n \dots \mathbb{M}_1} \arrow[r, "N_{\mathbb{M}_n \dots \mathbb{M}_1,\mathsf{D}}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{psh}(\Theta_{\mathbb{M}_n \dots \mathbb{M}_1,\mathsf{D}}) \arrow[d, "j_n^*"] \\
\vdots \arrow[u, "\text{b.o.}", "j_n"'] & \vdots \arrow[u, shift left, "\text{free}"] \arrow[d, shift left] & \vdots \arrow[d] \\
\Theta_{\mathbb{M}_1,\mathsf{D}} \arrow[u] \arrow[r, "\text{f.f.}"] & \mathsf{C}^{\mathbb{M}} \arrow[u, shift left] \arrow[r, "N_{\mathbb{M}_1,\mathsf{D}}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{psh}(\Theta_{\mathbb{M}_1,\mathsf{D}}) \arrow[d, "j_1^*"] \\
\mathsf{D} \arrow[u, "\text{b.o.}", "j_1"'] \arrow[r, hook] & \mathsf{C} \arrow[u, shift left, "\text{free}"] \arrow[r, "N_{\mathsf{D}}", "\text{f.f.}"'] & \mathsf{psh}(\mathsf{D}).
\end{tikzcd}
\tag{2.8}
$$

Hence, to prove that the induced nerve $N_{\mathbb{M},\mathsf{D}} = N_{\mathbb{M}_n \dots \mathbb{M}_1,\mathsf{D}}$ on $\mathsf{C}^{\mathbb{M}}$ is fully faithful, it is sufficient to consider whether the conditions of [3] are satisfied, for any of the maps above $\mathsf{D} \to \mathsf{C}$ in (2.8) that induce a fully faithful nerve, together with their corresponding induced monad.

Example 2.9. In Section 9, the monad for circuit operads is described as a composite $\mathbb{LDT}$ of three monads defined on the category GS of graphical species. On GS, the composite $\mathbb{LDT}$ does not satisfy the conditions of [3]. However, the induced lift $\widetilde{\mathbb{L}}\mathbb{T}_*$ on the category GS_* of $\mathbb{D}$-algebras does. Hence there is a full subcategory $\Xi^\times \subset \mathsf{CO}$ of the category of circuit operads that induces a fully faithful nerve functor $\mathsf{CO} \to \mathsf{psh}(\Xi^\times)$.

2.3. **Presheaves and slice categories.** The nerve theorem for circuit operads (Theorem 10.4) is stated in terms of a composite monad $\mathbb{LDT}$ – where each monad governs a different aspect of the circuit algebra structure – on a certain category GS of coloured collections. The proof of the nerve theorem, and the corresponding Segal condition for circuit operads Theorem 10.4, rest on the fact that GS is a presheaf category. This purpose of this brief section, and the next, is to review the definitions – and establishes the notational conventions – for presheaves, and other categorical concepts that will be used throughout the paper.

Let E be a category (with finite limits), and let C be an essentially small category. A C-*presheaf* in E is a functor $S\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{E}$. I will write $\mathsf{psh}_{\mathsf{E}}(\mathsf{C})$ for the corresponding functor category.

For every $n \in \mathbb{N}$, let $\mathbf{n}$ denote both the finite ordinal $(1 < \cdots < n)$, and also the n-element set $\{1, \dots, n\}$.. In particular, $\mathbf{0}$ is the empty set $\emptyset$. For $n \in \mathbb{N}$, Σ_n is the group of permutations of $\mathbf{n}$, and Σ is the symmetric groupoid with objects $n \in \mathbb{N}$ and automorphism groups Σ_n. A presheaf $S\colon \Sigma^{\mathrm{op}} \to \mathsf{E}$ is often called a *symmetric sequence* in E, or sometimes a *species* in E (see Remark 5.1).

When $\mathsf{E} = \mathsf{Set}$, the category elements of a presheaf $P\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is defined as follows:

Definition 2.10. *Objects of the category* $\mathsf{el}_{\mathsf{C}}(P)$ *of elements of a presheaf* $P\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ *are pairs* (c, x) *– called* elements of P *– where* c *is an object of* C *and* $x \in P(c)$. *Morphisms* $(c, x) \to (d, y)$ *in* $\mathsf{el}_{\mathsf{C}}(P)$ *are given by morphisms* $f \in \mathsf{C}(c, d)$ *such that* $P(f)(y) = x$.

If a presheaf $P\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is of the form $\mathsf{C}(-,c)$, then $\mathsf{el}_{\mathsf{C}}(P)$ is the *slice category* C/c whose objects are pairs (d,f) where $f \in \mathsf{C}(d,c)$, and morphisms $(d,f) \to (d',f')$ are commuting triangles in C:

$$\begin{tikzcd} d \arrow[rr, "g"] \arrow[dr, "f"'] & & d' \arrow[dl, "f'"] \\ & c. & \end{tikzcd}$$

In general, a functor $\iota\colon \mathsf{D} \to \mathsf{C}$ induces a presheaf $\iota^*\mathsf{C}(-,c)$, $d \mapsto \mathsf{C}(\iota(d),c)$ on D. For all $c \in \mathsf{C}$, the *slice category of* D *over* c is defined by $\mathsf{D}/c \overset{\mathrm{def}}{=} \mathsf{el}_{\mathsf{D}}(\iota^*\mathsf{C}(-,c))$. (This is also – more accurately – denoted by ι/c.) The *slice category of* c *over* D, denoted by c/D (or c/ι) is the category ι^{op}/c whose objects are morphisms $f \in \mathsf{C}(c,\iota(-))$ and whose morphisms $(d,f) \to (d',f)$ are given by $g \in \mathsf{D}(d,d')$ such that $\iota(g) \circ g = f' \in \mathsf{C}(c,\iota(d'))$.

Observe, in particular, that the Yoneda embedding $\mathsf{C} \to \mathsf{psh}(\mathsf{C})$ induces a canonical isomorphism $\mathsf{el}_{\mathsf{C}}(P) \cong \mathsf{C}/P$ for all presheaves $P\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. These categories will be identified in this work.

For any category C, its *core* $\mathrm{Core}(\mathsf{C}) \subset \mathsf{C}$ is the maximal subgroupoid of C. In other words, $\mathrm{Core}(\mathsf{C})$ has the same objects as C and all its morphisms are isomorphisms.

2.4. **Symmetric monoidal categories.** For precise definitions of (symmetric) monoidal categories, see e.g. [11, Chapter 8].

Recall that a *monoidal category* is specified by a category V together with a *monoidal product* operation $\otimes\colon \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ and *monoidal unit* object $I \in \mathsf{V}$, and, for each triple x,y,z of objects of V a choice of *associator* isomorphisms $\alpha_{x,y,z}\colon (x \otimes y) \otimes z \leftrightarrows x \otimes (y \otimes z)\colon \alpha^{-}_{x,y,z}$, and *unitor* isomorphisms $u_x\colon x \otimes I \xrightarrow{\cong} x \xleftarrow{\cong} 1 \otimes x$ in V such that certain (sensible) diagrams commute. A monoidal category $(\mathsf{V},\otimes,I,\alpha,u)$ is called *strict monoidal* if the associator and unitor isomorphisms are the identity.

A monoidal structure on V is *symmetric* if, further, for all objects $x,y \in \mathsf{V}$, there are isomorphisms $\sigma_{x,y}\colon x \otimes y \leftrightarrows y \otimes x\colon \sigma_{y,x}$ that satisfy the familiar braid (Yang-Baxter) identities, and such that $\sigma_{y,x} = \sigma^{-1}_{x,y}$. To avoid excessive adjectives, the term *permutative category* is used to describe symmetric strict monoidal categories.

In what follows, I will ignore the associators, unitors and symmetry isomorphisms in the notation, and designate symmetric monoidal categories simply by $(\mathsf{V},\otimes,I)$ (or sometimes even just V). Similarly, permutative categories will usually be designated by $(\mathsf{C},\oplus,0)$.

Morphisms $\Theta\colon (\mathsf{V}_1,\otimes_1,I_1) \to (\mathsf{V}_2,\otimes_2,I_2)$ of monoidal categories are *(lax) monoidal functors*: These are functors $\Theta\colon \mathsf{V}_1 \to \mathsf{V}_2$ together with a natural transformation $\theta\colon \Theta \otimes_2 \Theta \Rightarrow \Theta(- \otimes_1 -)$ such that all the expected structure diagrams commute. Likewise, a *symmetric monoidal functor* between symmetric monoidal categories is a (lax) monoidal functor that respects the symmetry isomorphisms strictly. The category of symmetric monoidal categories and symmetric monoidal functors is denoted by Sym. A morphism Θ of (symmetric) monoidal categories is *strict* if the natural transformation it preserves the monoidal structure strictly (θ is the identity natural transformation,) and *strong* if θ is a natural isomorphism.

Example 2.11. The symmetric groupoid Σ has a permutative structure induced by addition of natural numbers. For any symmetric monoidal category $(\mathsf{V},\otimes,I)$ and any choice of object $x \in \mathsf{V}$, there is a unique strict functor $\Sigma \to \mathsf{V}$ with $0 \mapsto I$ and $1 \mapsto c$.

3. Brauer diagrams

In Section 4, circuit algebras will be defined as algebras for certain operads of wiring diagrams (Definition 4.6).

3.1. Monochrome Brauer diagrams.

Definition 3.1. *A* matching *on a finite set X is a fixed point free involution $\tau\colon X \to X$.*

In other words, τ is a partition of X into two-element subsets. In particular, a finite set X admits a matching if and only if its cardinality is even. By convention, the empty set has a unique trivial matching.

Example 3.2. The boundary $\partial\mathcal{M}$ of a compact 1-manifold $\mathcal{M}$ has a canonical matching $\tau^{\mathcal{M}}$ induced by reversing orientation on $\mathcal{M}$.

Let X, Y, Z be a triple of finite sets. Given matchings $\tau_{\widehat{Z}}$ on $X \amalg Y$ and $\tau_{\widehat{X}}$ on $Y \amalg Z$, there is a canonical matching $(\tau_{\widehat{X}} \#_Y \ \tau_{\widehat{Z}})$ on $X \amalg Y$, defined as follows: The matchings $\tau_{\widehat{Z}}$ and $\tau_{\widehat{X}}$ generate an equivalence relation R_τ on $X \amalg Y \amalg Z$ such that objects v, w are equivalent if and only if they are related by a zigzag of (alternating) applications of $\tau_{\widehat{Z}}$ and $\tau_{\widehat{X}}$ (Figure 1(b)(i)-(iv)).

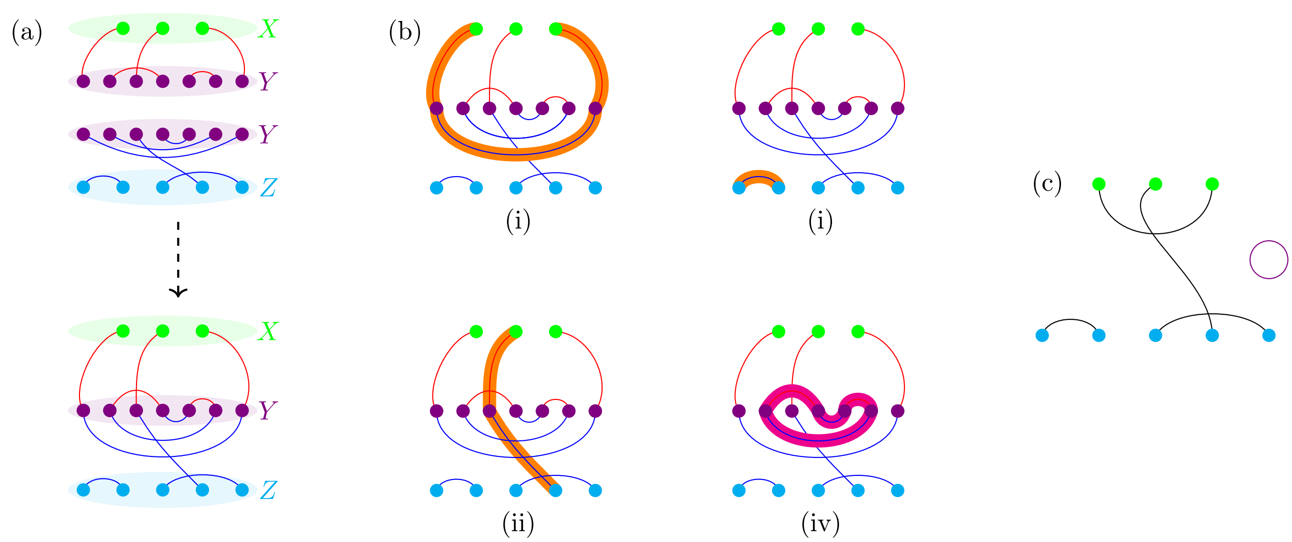


Figure 1. Composition of matchings on $X \amalg Y$ (red) and $Y \amalg Z$ (blue), (b) different equivalence classes of the relation R_τ, (c) the resulting matching on $X \amalg Z$, together with the single closed component formed in the composition.

Each $x \in X$ is directly related by involution to precisely one element, $\tau_{\widehat{Z}}(x) \in X \amalg Y$, in the chain $[x]$. Similarly, each $z \in Z$ is directly related to precisely one element, $\tau_{\widehat{X}}(y) \in Y \amalg Z$. Each $y \in Y$ is directly related to $\tau_{\widehat{Z}}(y) \in X \amalg Y$ and $\tau_{\widehat{X}}(y) \in Y \amalg Z$.

Under this relation, each equivalence class $[w]$ of R_τ (with $w \in X \amalg Y \amalg Z$) contains precisely zero, or two distinct elements of $X \amalg Z$, and is described (by e.g. the proof of [25, Proposition 4.23]) either by a closed cycle $(\dots, y_1, \dots, y_1, \dots)$ in Y (Figure 1(b)(iv)) or an open chain $(w, y_1, \dots, y_k, v)$ of distinct elements of $X \amalg Y \amalg Z$ that begins and ends in elements $w, v \in X \amalg Z$(Figure 1(b)(i)-(iii)). Hence

$$(\tau_{\widehat{X}} \#_Y \ \tau_{\widehat{Z}})(w) = v, \text{ for } w, v \in X \amalg Z \text{ such that } [w]_{R_\tau} = [v]_{R_\tau} \tag{3.3}$$

describes a fixed point-free involution on $X \amalg Z$.

The classes of the form $[w]$ with $w \in X \amalg Z$ are called *open components of the composition* $(\tau_{\widehat{X}} \#_Y \tau_{\widehat{Z}})$. The remaining classes $[y]$ describe cycles entirely contained in Y and are called *closed components created by the composition* $(\tau_{\widehat{X}} \#_Y \tau_{\widehat{Z}})$ (Figure 1(b)(iv), (c)). The number of such closed components created by the composition $(\tau_{\widehat{X}} \#_Y \tau_{\widehat{Z}})$ is denoted by $\mathfrak{k}(\tau_{\widehat{X}}, \tau_{\widehat{Z}}) \in \mathbb{N}$.

For all finite sets X, there is a canonical identity matching τ_{id_X} on $X \amalg X$ such that, for all finite sets Y and all matchings τ on $X \amalg Y$

$$\tau_{id_X} \#_X \tau = \tau = \tau \#_X \tau_{id_X}.$$

Definition 3.4. *The category* $\mathring{\mathsf{BD}}$ *of* open (monochrome) Brauer diagrams *as objects* $n \in \mathbb{N}$*, and morphisms in* $\mathring{\mathsf{BD}}(m,n)$ *are matchings on the set* $\mathbf{m} \amalg \mathbf{n}$*. Given* $\tau_f \in \mathring{\mathsf{BD}}(k,m)$ *and* $\tau_g \in \mathring{\mathsf{BD}}(m,n)$*, their composition* $\tau_{gf} \in \mathring{\mathsf{BD}}(k,n)$*, with identities* $\tau_{id_\mathbf{n}}$*, is given, as in (3.3), by* $\tau_{gf} \overset{\text{def}}{=} \tau_g \#_\mathbf{m} \tau_f$*.*

Disjoint union of finite sets and matchings, together with permutations of elements gives $\mathring{\mathsf{BD}}$ the structure of a permutative category – indeed a PROP – $(\mathring{\mathsf{BD}}, \oplus, 0)$ with $\mathring{\mathsf{BD}}(0,0) = \{*\}$.

Definition 3.5. *The category* BD *of* (monochrome) Brauer diagrams *has objects* $n \in \mathbb{N}$ *and morphisms* $f \in \mathsf{BD}(m,n)$ *are pairs* $(\tau_f, \mathfrak{k}^f)$ *where* $\tau_f \in \mathring{\mathsf{BD}}(m,n)$ *is a matching on* $\mathbf{n} \amalg \mathbf{m}$*, and* $\mathfrak{k}^f \in \mathbb{N}$*.*

The composition gf *of morphisms* $f \in \mathsf{BD}(k,m), g \in \mathsf{BD}(m,n)$ *is given by* $\tau_{gf} \overset{\text{def}}{=} (\tau_g \#_\mathbf{m} \tau_f)$ *on* $\mathbf{n} \amalg \boldsymbol{k}$ *as described in (3.3), and* $\mathfrak{k}^{gf} = \mathfrak{k}^g + \mathfrak{k}^f + \mathfrak{k}(\tau_f, \tau_g)$

The permutative structure on $\mathring{\mathsf{BD}}$ extends to one on BD. For all, m, n, $\mathsf{BD}(m,n) \cong \mathring{\mathsf{BD}}(m,n) \times \mathbb{N}$ and each $f = (\tau, \mathfrak{k}) \in \mathsf{BD}(m,n)$, decomposes as $f = \mathring{f} \oplus \kappa^f$ into an *open part* $\mathring{f} = (\tau, 0)$, with $\tau \in \mathring{\mathsf{BD}}(m,n)$, and a *closed part* $\kappa^f = (-, \mathfrak{k}) \in \mathsf{BD}(0,0) \cong \mathbb{N}$.

Given a morphism $f = (\tau, \mathfrak{k}) \in \mathsf{BD}(m,n)$, it will be convenient to use the notation $\mathfrak{S}(f) = \mathbf{m}$ and $\mathfrak{T}(f) = \mathbf{n}$ for the *source* and *target* sets of f. Motivated by Example 3.2 (and Example 3.16 below), the set $\partial f \overset{\text{def}}{=} \mathfrak{T}(f) \amalg \mathfrak{S}(f)$ is called the **boundary of** f.

Definition 3.6. *The subcategory* $\mathsf{BD}_\downarrow \subset \mathsf{BD}$ *of* downward Brauer diagrams *is the subcategory of open morphisms* $d = (\tau^\downarrow, 0) \in \mathsf{BD}(m,n)$ *such that* $\tau^\downarrow$ *is given by an injection* $\mathbf{n} \to \mathbf{m}$ *together with a matching on* $\mathbf{m} \setminus \tau^\downarrow(\mathbf{n})$*. In other words,* $d \in \mathsf{BD}_\downarrow(m,n)$ *if and only if* $\tau^\downarrow$ *satisfies* $\tau^\downarrow(y) \in \mathfrak{S}(d)$ *for all* $y \in \mathfrak{T}(d)$*.*

The category $\mathsf{BD}_\uparrow \subset \mathsf{BD}$ *of* upward Brauer diagrams *is the opposite category of* $\mathsf{BD}_\downarrow$*.*

In particular, $\mathsf{BD}_\downarrow(m,n)$, and $\mathsf{BD}_\uparrow(n,m)$ are empty, whenever $n > m$.

Example 3.7. There is a canonical inclusion $\Sigma \hookrightarrow \mathsf{BD}$: For $n \in \mathbb{N}$, and $\sigma \in \Sigma_n$, let $\sigma = (\tau_\sigma, 0) \in \mathsf{BD}(n,n)$ be the matching on $\mathbf{n} \amalg \mathbf{n}$ given by $x \mapsto \sigma x$.

Example 3.8. Since a two element set admits a unique matching, there are canonical isomorphisms

$$\mathbb{N} \cong \mathsf{BD}(2,0) \cong \mathsf{BD}(1,1) \cong \mathsf{BD}(0,2) \text{ given by } \mathfrak{k} \mapsto (*, \mathfrak{k}).$$

Then $id_1 = (*, 0) \in \mathsf{BD}(1,1)$, and, for all $n \geq 1$ $id_n = id_1 \oplus \cdots \oplus id_1$ has the form $(\tau, 0)$.

Let $\cup \overset{\text{def}}{=} (*, 0) \in \mathsf{BD}(2,0)$ and $\cap \overset{\text{def}}{=} (*, 0) \in \mathsf{BD}(0,2)$. Then $\mathfrak{k}(\cup, \cap) = 1$, and hence

$$\cup \circ \cap = (-, 1) \in \mathsf{BD}(0,0).$$

Since $(\cup \oplus id_1) \circ (id_1 \oplus \cap)$ defines a matching on $\mathbf{1} \amalg \mathbf{1}$, the *triangle identities*

$$(\cup \oplus id_1) \circ (id_1 \oplus \cap) = id_1 = (id_1 \oplus \cup) \circ (\cap \oplus id_1). \tag{3.9}$$

are satisfied (Figure 2).

Remark 3.10. In this work, the direction of morphisms is depicted as top to bottom (source above target). Hence the $\cup$ and $\cap$ conventions differ from standard (Feynman diagram) conventions.

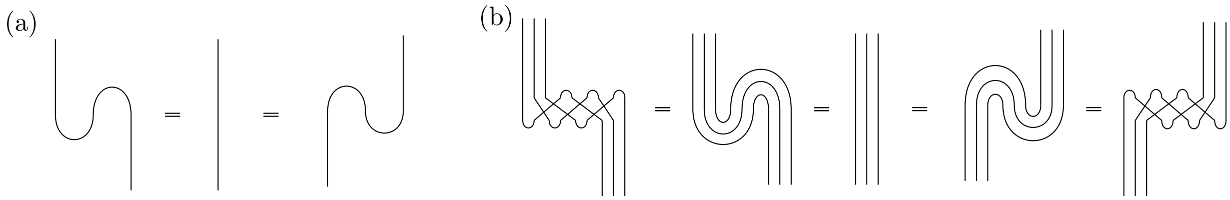


FIGURE 2. The triangle identities: (a) $n = 1$, (b) $n = 3$.

Example 3.11. For all $n \in \mathbb{N}$, $id_n = (\tau_{id_n}, 0)$ is given by the disjoint union $id_n = \bigoplus_{i=1}^{n} id_1 \in \mathsf{BD}(n,n)$. The matching τ_{id_n} on $\mathbf{n} \amalg \overleftarrow{\mathbf{n}} \cong (\mathbf{2n})$ by $(1 < \cdots < n < \overleftarrow{n} < \cdots < \overleftarrow{1}) \to (1 < \cdots < n < n+1 < \cdots < 2n)$ describes open morphisms in $\cap_n \overset{\text{def}}{=} (\tau_{id}, 0) \in \mathsf{BD}(0, 2n)$ and $\cup_n \overset{\text{def}}{=} (\tau_{id}, 0) \in \mathsf{BD}(2n, 0)$.

By construction, these satisfy the n-fold triangle identities

$$(\cup_n \oplus id_n) \circ (id_n \oplus \cap_n) = id_n = (id_n \oplus \cup_n) \circ (\cap_n \oplus id_n). \tag{3.12}$$

If $f = (\tau, \mathfrak{k}) \in \mathsf{BD}(m,n)$, then it follows immediately from Definition 3.5 that there is an *evaluation morphism* $\lfloor f \rfloor = (\tau, \mathfrak{k}) \in \mathsf{BD}(m+n, 0)$, and a *coevaluation morphism* $\lceil f \rceil = (\tau, \mathfrak{k}) \in \mathsf{BD}(0, n+m)$ corresponding to f. These satisfy $\lceil f \rceil = (f \oplus id_m) \circ \cap_m$, and $\lfloor f \rfloor = \cup_n \circ (id_n \oplus f)$.

Moreover, by definition, f defines a *dual morphism* $f^* \in \mathsf{BD}(n,m)$ described by

$$(id_m \oplus \cup_n) \circ (id_m \oplus f \oplus id_n) \circ (\cap_m \oplus id_n) = f^* = (\cup_n \oplus id_n) \circ (id_n \oplus f \oplus id_m) \circ (id_n \oplus \cap_m). \tag{3.13}$$

Remark 3.14. By e.g. [20], the category BD, is generated under horizontal (monoidal) and vertical (categorical) composition, by the morphisms id_1, $\cup$, $\cap$ and the unique non-identity permutation $\sigma_{\mathbf{2}} \in \Sigma_2 \subset \mathsf{BD}(2,2)$.

By viewing morphisms in BD as classes of tangles of 1-manifolds in 4 space, we see that the defining relations for BD are obtained from the the classical Reidemeister moves [26] by identifying the over and under crossings of undirected strands.

The category $\mathsf{BD}_{\downarrow}$ of downward Brauer diagrams is generated by $id_1, \sigma_{\mathbf{2}}$ and $\cup$ – and the category $\mathsf{BD}_{\uparrow}$ of upward Brauer diagrams by $id_1, \sigma_{\mathbf{2}}$ and $\cap$ – under horizontal and vertical composition.

In particular, the symmetric groupoid Σ is canonically isomorphic to the intersection of $\mathsf{BD}_{\downarrow}$ and $\mathsf{BD}_{\uparrow}$ in BD.

Example 3.15. Let R be a commutative ring, and R-Mod its category of modules with monoidal product given by tensor product $\otimes_R$ over R. Let Br_δ be the R-Mod-enriched *Brauer category* [20] whose objects are natural number $n \in \mathbb{N}$ and for all $m, n \in \mathbb{N}$, $\mathsf{Br}_\delta(m,n)$ is the free R-module is generated by the open Brauer diagrams $\tau \in \overset{\circ}{\mathsf{BD}}(m,n)$. Composition in Br_δ is defined by:

$$\tau_g \tau_f = \delta^{\mathfrak{k}^{gf}} \tau_{gf} \in \mathsf{Br}_\delta(k,n) \text{ for all } \tau_f \in \mathsf{Br}_\delta(k,m), \tau_g \in \mathsf{Br}_\delta(m,n).$$

The Brauer category Br_δ is the categorification of the algebras introduced by Brauer in 1937 [5] to extend Schur-Weyl duality for the orthogonal and symplectic groups

Let $\Bbbk$ be a field of characteristic 0. A strict monoidal functor $\mathcal{V}\colon \mathsf{BD} \to \mathsf{Vect}_{\Bbbk}$ satisfies $\mathcal{V}(0) = \Bbbk$, $\mathcal{V}(-,0) = 1_{\Bbbk}$, and $\mathcal{V}(-,1) = k \in \Bbbk$. By the triangle identities (3.12), that $\mathcal{V}(\cup)\colon V \otimes V \to \Bbbk$ describes a non-degenerate bilinear form on $V = \mathcal{V}(1)$, and hence an isomorphism $V \to V^*$, whereby V must be finite dimensional.

If $\mathcal{V}(\cup)$ is symmetric then $\mathcal{V}(\cup) \circ \mathcal{V}(\cap) = \mathcal{V}(-,1) = k$ is the dimension of V. And if $\mathcal{V}(\cup)$ is anti-symmetric (and hence V has even dimension $2m$), then $k = \mathcal{V}(\cup) \circ \mathcal{V}(\cap) = \mathcal{V}(-,1) = -2m$.

It is straightforward to verify that (for $R = \Bbbk$) strict monoidal functors $\mathcal{V}\colon \mathsf{BD} \to \mathsf{Vect}_{\Bbbk}$ with $k = \delta$, $\mathcal{V}(1) = V$ are equivalent to $\mathsf{Vect}_{\Bbbk}$-functors $\mathsf{Br}_{\delta} \to \mathsf{Vect}_{\Bbbk}$ such that $1 \mapsto V$. By [5, 20] (and others), $\delta = \dim(V)$ ($\delta = -\dim(V)$), these describe representations of the (finite) orthogonal groups (resp. symplectic groups).

The *downwards, and upwards Brauer categories* $\mathsf{Br}_{\downarrow}$ *and* $\mathsf{Br}_{\uparrow}$ are the subcategories of Br_{δ} (for all δ), such that elements of $\mathsf{BD}_{\downarrow}(m,n)$ ($\mathsf{BD}_{\uparrow}(m,n)$) generate $\mathsf{Br}_{\downarrow}(m,n)$ ($\mathsf{Br}_{\uparrow}(m,n)$). In particular, for all $n \in \mathbb{N}$, $\mathsf{Br}_{\downarrow}(n,n) = \mathsf{Br}_{\uparrow}(n,n)$ is a free Σ_n-module. It follows from the results of [27], that strict functors $\mathsf{BD}_{\downarrow} \to \mathsf{Vect}_{\Bbbk}$ describe representations of the infinite orthogonal group $O(\infty)$.

Example 3.16. As in Example 3.2, let $\mathcal{M}$ be a compact 1-manifold with canonical matching $\tau^{\mathcal{M}}$ on $\partial\mathcal{M}$ and n_c closed components: Then $\mathcal{M} \cong n_o(\mathcal{I}) \amalg n_c(S^1)$ where $\mathcal{I}$ is homeomorphic to the interval $[0,1]$ and $n_o = \partial\mathcal{M}/\tau^{\mathcal{M}}$ denotes the number of *open components* of $\mathcal{M}$. Then, for all $m, n \in \mathbb{N}$ such that $m+n = 2n_o$, and all order-inducing isomorphisms $\phi\colon \mathbf{n}\amalg\overleftarrow{\mathbf{m}} \to \partial\mathcal{M}$, the pair $(\mathcal{M}, \phi)$ describes a morphism $(\phi^{-1}\tau^{\mathcal{M}}\phi, n_c) \in \mathsf{BD}(m,n)$.

Conversely, for all $m, n \in \mathbb{N}$, any morphism $f = (\tau_f, \mathfrak{k}) \in \mathsf{BD}(m,n)$, is described in this way by a compact 1-manifold $\mathcal{M} \cong \frac{m+n}{2}(\mathcal{I}) \amalg \mathfrak{k}(S^1)$ and an isomorphism $\phi_f\colon \partial f \to \partial\mathcal{M}$.

Motivated by Example 3.16, define the *set of components* $\pi_0(f)$ of the morphism f to be the set of connected components $\pi_0(\mathcal{M})$ of a compact 1-manifold $\mathcal{M}$ such that $(\mathcal{M}, \phi)$ describes f as in Example 3.16.

This is well-defined and there is a canonical morphism $\partial f \to \pi_0(f)$.

So a morphism $f = (\tau, \mathfrak{k}) \in \mathsf{BD}(m,n)$ is given by a cospan of finite sets:

$$\begin{tikzcd}
\mathbf{m} \cong \mathfrak{S}(f) \arrow[r, tail, "i \mapsto \tau(i)"] \arrow[dr, dashed] & \partial f \arrow[d, dashed] & \mathfrak{T}(f) \cong \mathbf{n} \arrow[l, tail, "j \mapsto \tau(j)"'] \arrow[dl, dashed] \\
& \pi_0(f). &
\end{tikzcd} \tag{3.17}$$

Remark 3.18. (See also Remark 6.35.)

By (3.17), for composable morphisms $f \in \mathsf{BD}(k,m)$ and $g \in \mathsf{BD}(m,n)$, we may consider the pushout (pair) of cospans:

$$\begin{tikzcd}
\mathfrak{S}(f) \arrow[dr, "\tau_f"'] & & \mathfrak{T}(f) = \mathfrak{S}(g) \arrow[dl, "\tau_f"] \arrow[dr, "\tau_g"'] & & \mathfrak{T}(g) \arrow[dl, "\tau_g"] \\
& \partial f \arrow[d] \arrow[r, dashed] & P(gf) \arrow[d] & \partial g \arrow[d] \arrow[l, dashed] & \\
& \pi_0(f) \arrow[r, dashed] & \pi^P(gf). & \pi_0(g) \arrow[l, dashed] &
\end{tikzcd} \tag{3.19}$$

In general, $P(gf) \not\cong \partial(gf)$ and hence composition of morphisms in BD is not described by compositions (pushouts) of cospans as in (3.19). For example, the pushout (3.19) for the composition $\cup \circ \cap \in \mathsf{BD}(0,0)$ leads to $P(\cup \circ \cap) \cong \mathbf{2}$. But, as described in Example 3.8, $\cup \circ \cap$ is the closed map $(-,1) \in \mathsf{BD}(0,0)$ so $\partial(\cup \circ \cap) = \emptyset \not\cong \mathbf{2}$. (This is closely related to the *problem of loops* discussed in detail in [25], specifically in Section 6 thereof.)

This is, in some sense, the 'only obstruction' to defining composition in BD directly in terms of pushouts of cospans (3.17): Namely, if $f = (\tau_f, \mathfrak{k}^f) \in BD(k,m)$ and $g = (\tau_g, \mathfrak{k}^g) \in \mathsf{BD}(m,n)$ are composable morphisms such that $\mathfrak{k}(f,g) = 0$, so $\mathfrak{k}^{gf} = \mathfrak{k}^f + \mathfrak{k}^g$, then $P(gf) \cong \partial(gf)$, and $\pi^P(gf) \cong \pi_0(gf)$.

In particular, if f is not of the form $f = f_\downarrow \oplus \kappa^f$ *or* g is not of the form $g = g_\uparrow \oplus \kappa^g$ – where $f_\downarrow \in \mathsf{BD}_\downarrow(k,m)$ and $g_\uparrow \in \mathsf{BD}_\uparrow(m,n)$ and $\kappa^f, \kappa^g \in \mathsf{BD}(0,0)$ – then no closed components will be created in the composition $g \circ f$.

To represent general composition of all Brauer diagrams in BD in terms of pushouts of cospans, one can replace the cospans of the form (3.17) with cospans obtained from the manifold representation of morphisms in Example 3.16. Namely, if $(\mathcal{M}_f, \phi_f)$ represents $f = (\tau_f, \mathfrak{k}^f) \in \mathsf{BD}(k,m)$, then f is described by a cospan diagram

$$\mathfrak{S}(f) \xrightarrow{\phi_f \circ \tau_f} \partial \mathcal{M}_f \xrightarrow{\text{inc}} \mathcal{M} \xleftarrow{\text{inc}} \partial \mathcal{M}_f \xleftarrow{\phi_f \circ \tau_f} \mathfrak{T}(f). \tag{3.20}$$

And, if $(\mathcal{M}_g, \phi_g)$ represents $g = (\tau_g, \mathfrak{k}^g) \in \mathsf{BD}(m,n)$, the composition $gf \in \mathsf{BD}(k,n)$ is obtained from the pushout of cospans of spaces:

$$\begin{tikzcd}
\mathfrak{S}(f) \arrow[dr, "\text{inc} \circ \phi_f \circ \tau_f"'] & & \mathfrak{T}(f) = \mathfrak{S}(g) \arrow[dl, "\text{inc} \circ \phi_f \circ \tau_f"] \arrow[dr, "\text{inc} \circ \phi_g \circ \tau_g"'] & & \mathfrak{T}(g) \arrow[dl, "\text{inc} \circ \phi_g \circ \tau_g"] \\
& \mathcal{M}_f \arrow[dr, dashed] & & \mathcal{M}_g \arrow[dl, dashed] & \\
& & \mathcal{M}_f \amalg_{\mathfrak{T}(f)} \mathcal{M}_g. & &
\end{tikzcd} \tag{3.21}$$

Example 3.22. By definition, a (lax monoidal) functor $\mathcal{A}\colon \mathsf{BD} \to (\mathsf{V}, \otimes, I)$ induces the structure of an associative graded monoid on the graded object $(\mathcal{A}(n))_{n \in \mathbb{N}}$. The unit for the monoidal product $\boxtimes$ so obtained is induced by the structure morphism $I \to \mathcal{A}(0)$.

Observe also that, for all $n \geq 2$, and all $1 \leq i < j < n$, there is a canonical *contraction* morphism $\zeta_n^{i \ddagger j}\colon \mathcal{A}(n) \to \mathcal{A}(n-2)$ induced by the Brauer diagram $(\cup \oplus id_{n-2}) \circ \rho_n^{i,j} \in \mathsf{BD}(n, n-2)$, where $\rho_n^{i,j}$ is the shuffle permutation on $\mathbf{n} = \{1, \dots, n\}$ given by $i \mapsto 1, j \mapsto 2$, whilst leaving the relative order of the remaining elements unchanged.

If further $1 \leq k < m \leq n$ for i,j,k,m all distinct,

$$\zeta_{n-2}^{k' \ddagger m'} \circ \zeta_n^{i \ddagger j} = \zeta_{n-2}^{i' \ddagger j'} \circ \zeta_n^{k \ddagger m}\colon \mathcal{A}(n) \to \mathcal{A}(n-4). \tag{3.23}$$

Here $1 \leq k' < m' \leq n-2$ are the obvious adjusted indices ($k' = \rho_n^{i,j}(k) - 2$, $m' = \rho_n^{i,j}(m) - 2$) of k and m in $\mathbf{n}_{\widehat{i,j}} \cong \{1, \dots, n-2\}$, and the indices $1 \leq k' < m' \leq n-2$ are defined similarly.)

Example 3.24. Let $\mathcal{A}\colon \mathsf{BD} \to (\mathsf{V}, \otimes, I)$ be a functor as in Example 3.22.

The contraction ζ described above induces a 'multiplication' operation $\diamond$ (see Definition 5.19), by

$$\diamond_{m,n}^{i,j} = \zeta_{m+n}^{i,m+j} \circ \boxtimes_{m,n}\colon \mathcal{A}(m) \otimes \mathcal{A}(n) \to \mathcal{A}(m+n-2), \tag{3.25}$$

for all $m, n \geq 1$ and all $1 \leq i \leq m$, $1 \leq j \leq n$.

Moreover, since

$$(\cup \oplus id_n) \circ \rho_{2+n}^{2,i+2} \circ (\cap \oplus id_n) = id_n \in \mathsf{BD}(n,n)$$

for $1 \leq i \leq n$, it follows that

$$id_{\mathcal{A}(n)} = \diamond_{2,n}^{1,i} \circ \mathcal{A}(\cap \oplus id_n),$$

and hence the multiplication $\diamond$ on $\mathcal{A}$ has a unit $I \mapsto \mathcal{A}(2)$ given by the image of the cap morphism $\cap \in \mathsf{BD}(0,2)$.

In particular, that a symmetric functor $\mathsf{BD}_{\downarrow} \to (\mathsf{V}, \otimes, I)$ from the category of downward Brauer diagrams induces contraction, and multiplication operations on its image. However, since $\cap$ is not a morphism in $\mathsf{BD}_{\downarrow}$, there is no unit for this multiplication. This is discussed in detail in ✠[?].

3.2. **Coloured Brauer diagrams.** Let $\mathfrak{C}$ be a set. The set underlying the free associative monoid on $\mathfrak{C}$ – whose elements of $\mathsf{list}\mathfrak{C}$ are finite ordered sets $\boldsymbol{c} = (c_1, \dots, c_m) \in \mathfrak{C}^{\boldsymbol{m}}$ $(m \in \mathbb{N})$– is denoted by $\mathsf{list}\mathfrak{C}$. The symmetric groupoid Σ acts on $\mathsf{list}\mathfrak{C}$ from the right by $\sigma \colon \boldsymbol{c} = (c_1, \dots, c_m) \mapsto (\boldsymbol{c}\sigma) \overset{\text{def}}{=} (c_{\sigma 1}, \dots, c_{\sigma m})$ for all $\sigma \in \Sigma_m$.

The concatenation product on $\mathsf{list}\mathfrak{C}$ is denoted by $\oplus$. Given lists $\boldsymbol{c} = (c_1, \dots, c_m)$ and $\boldsymbol{d} = (d_1, \dots, d_n)$ in $\mathsf{list}\mathfrak{C}$,

$$\boldsymbol{c} \oplus \boldsymbol{d} \overset{\text{def}}{=} (c_1, \dots, c_m, d_1, \dots, d_n) \in \mathsf{list}\mathfrak{C},$$

and will also be denoted by $\boldsymbol{cd}$. The empty list $I_{\mathfrak{C}} \overset{\text{def}}{=} (-)$ is the unit for the concatenation product $\oplus$. In particular, $(\mathbb{N}, +, 0)$ is canonically isomorphic to the free associative monoid $(\mathsf{list}\{*\}, \oplus, I_{\{*\}})$ on a singleton.

Definition 3.26. *A pair $(\mathfrak{C}, \omega)$ of a set $\mathfrak{C}$ together with an involution $\omega \colon \mathfrak{C} \to \mathfrak{C}$ is called an (involutive) palette. Elements $c \in \mathfrak{C}$ are called* colours *in $(\mathfrak{C}, \omega)$. Objects of the category* Pal *are palettes $(\mathfrak{C}, \omega)$, and morphisms $f \colon (\mathfrak{C}, \omega) \to (\mathfrak{C}', \omega')$ given by $f \in \mathsf{Set}(\mathfrak{C}, \mathfrak{C}')$ such that $f \circ \omega = \omega' \circ f$.*

Given a palette $(\mathfrak{C}, \omega)$, and $\boldsymbol{c} = (c_1, \dots, c_m) \in \mathsf{list}\mathfrak{C}$, the operations $\boldsymbol{c} \mapsto \overleftarrow{\boldsymbol{c}} \overset{\text{def}}{=} (c_m, \dots, c_1)$ and $\boldsymbol{c} \mapsto \omega\boldsymbol{c} \overset{\text{def}}{=} (\omega c_1, \dots, \omega c_m)$ define involutions on $\mathsf{list}\mathfrak{C}$. The involution $\overleftarrow{\boldsymbol{\omega}}$ on $\mathsf{list}\mathfrak{C}$ is defined by their composite $\overleftarrow{\boldsymbol{\omega}}(\boldsymbol{c}) \overset{\text{def}}{=} \overleftarrow{\omega\boldsymbol{c}} = \omega(\overleftarrow{\boldsymbol{c}})$.

Definition 3.27. *Given a palette $(\mathfrak{C}, \omega)$, and a matching τ on a finite set X, a $(\mathfrak{C}, \omega)$*-colouring of τ *is a map $\lambda_\partial \colon X \to \mathfrak{C}$ such that $\lambda_\partial \tau = \omega \lambda_\partial$. (In other words λ_∂ is a morphism $(X, \tau) \to (\mathfrak{C}, \omega)$ in* Pal*).*

A $((\mathfrak{C}, \omega)$-) colouring λ of a Brauer diagram $f = (\tau, \mathfrak{k}) \in \mathsf{BD}(m, n)$ *is given by a pair $\lambda = (\lambda_\partial, \widetilde{\lambda})$ where λ_∂ is a colouring of τ and $\widetilde{\lambda}$ is a map $\pi_0(f) \to \widetilde{\mathfrak{C}}$ such that the following diagram of sets commutes:*

$$\begin{tikzcd}
\partial f \arrow[r, "\lambda_\partial"] \arrow[d, "\tau"', "\cong"] & \mathfrak{C} \arrow[d, "\omega", "\cong"'] \\
\partial f \arrow[r, "\lambda_\partial"] \arrow[d] & \mathfrak{C} \arrow[d, two heads] \\
\pi_0(f) \arrow[r, "\widetilde{\lambda}"] & \widetilde{\mathfrak{C}}.
\end{tikzcd} \tag{3.28}$$

The type of the colouring λ *(or just the type of λ_∂) is the pair $(\boldsymbol{c}, \boldsymbol{d}) \in (\mathsf{list}\mathfrak{C})^2$ given by*

$$\begin{aligned}
\boldsymbol{d} = (d_1, \dots, d_n) \quad &= \lambda_\partial(\mathfrak{T}(f)) && \textit{is the image in } \mathfrak{C}^{\mathbf{n}} \textit{ of } \mathfrak{T}(f) \cong \mathbf{n} \textit{ under } \lambda_\partial, \\
\boldsymbol{c} = (c_1, \dots, c_m) \quad &= \omega \circ \lambda_\partial(\mathfrak{S}(f)) && \textit{in } \mathfrak{C}^{\mathbf{m}} \textit{ is given by } \overleftarrow{(\omega\lambda_\partial(\mathbf{m}))}.
\end{aligned}$$

For each pair $\boldsymbol{c}, \boldsymbol{d} \in \mathsf{list}\mathfrak{C}$, objects of the set $\mathsf{BD}^{(\mathfrak{C}, \omega)}(\boldsymbol{c}, \boldsymbol{d})$ are pairs, (f, λ) where $f = (\tau, \mathfrak{k})$ is a morphism in $\mathsf{BD}(|\boldsymbol{c}|, |\boldsymbol{d}|)$, and λ is a colouring of f of type $(\boldsymbol{c}, \boldsymbol{d})$.

Let $(f, \lambda) \in \mathsf{BD}^{(\mathfrak{C}, \omega)}(\boldsymbol{b}, \boldsymbol{c})$ and $(g, \gamma) \in \mathsf{BD}^{(\mathfrak{C}, \omega)}(\boldsymbol{c}, \boldsymbol{d})$, and, as in the construction for composing matchings (3.3), let R_τ be the equivalence relation on $\mathfrak{T}(g) \amalg \mathfrak{T}(f) \amalg \mathfrak{S}(f)$ generated by τ_f and τ_g. Then $\gamma_\partial(y) = \omega\lambda_\partial(y)$ for each $y \in \mathfrak{T}(f) = \mathfrak{S}(f)$ by definition.

It follows that, for each closed cycle $[y] = (y, y_1 = \tau_f(y), y_2 = \tau_g(y_1), \dots) \subset \mathfrak{T}(f)$ created in the composition $g \circ f$, there is a well-defined element $\tilde{c} \in \widetilde{\mathfrak{C}}$ given by the class of $\widetilde{\lambda}(y_i)$ $(y_i \in [y])$ in $\widetilde{\mathfrak{C}}$, and therefore, the composition $g \circ f$ induces a well-defined map $\widetilde{\gamma\lambda} \colon \pi_0(gf) \to \widetilde{\mathfrak{C}}$ (see Figure 3).

And, for each $x \in \mathfrak{S}(f)$, $\lambda_\partial(\tau_{gf})(x) = \omega\lambda_\partial(x)$ and likewise, for $z \in \mathfrak{T}(g)$, $\gamma_\partial(\tau_{gf})(z) = \omega\gamma_\partial(z)$, whereby there is a well-defined colouring $(\gamma\lambda)_\partial$ of τ_{gf} that restricts to λ_∂ on $\mathfrak{S}(f)$ and to γ_∂ on $\mathfrak{T}(g)$.

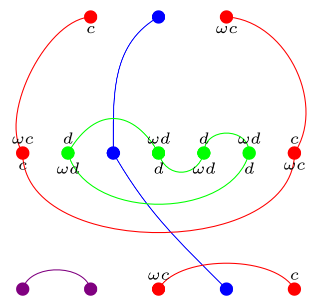


FIGURE 3. Composing coloured matchings.

Definition 3.29. *Objects of the category* $\mathsf{BD}^{(\mathfrak{C},\omega)}$ *of* $(\mathfrak{C},\omega)$-coloured Brauer diagrams *are elements of* $\mathsf{list}\mathfrak{C}$. *Morphisms in* $\mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c},\boldsymbol{d})$ *are* $(\mathfrak{C},\omega)$*-coloured Brauer diagrams of type* $(\boldsymbol{c},\boldsymbol{d})$.

Morphisms $(f,\lambda) \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{b},\boldsymbol{c})$ *and* $(g,\gamma) \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c},\boldsymbol{d})$ *compose to* $(gf,\gamma\lambda) \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{b},\boldsymbol{d})$.

The category $\mathsf{BD}^{(\mathfrak{C},\omega)}$ is permutative with monoidal structure $\oplus$ and unit $O \in \mathsf{BD}^{(\mathfrak{C},\omega)}(I_\mathfrak{C}, I_\mathfrak{C})$, induced by concatenation of object lists and disjoint union of coloured Brauer diagrams.

If $f \in \mathsf{BD}(m,n)$ is such that (f,λ) is an element of $\mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c},\boldsymbol{d})$ then there is a corresponding

- *evaluation morphism* $\lfloor (f,\lambda) \rfloor \stackrel{\text{def}}{=} (\lfloor f \rfloor, \lambda) \in \mathsf{BD}^{(\mathfrak{C},\omega)}((\overleftarrow{\omega\boldsymbol{d}}) \oplus \boldsymbol{c}, I_\mathfrak{C})$;
- *coevaluation morphism* $\lceil (f,\lambda) \rceil \stackrel{\text{def}}{=} (\lceil f \rceil, \lambda) \in \mathsf{BD}^{(\mathfrak{C},\omega)}(I_\mathfrak{C}, \boldsymbol{d} \oplus (\overleftarrow{\omega\boldsymbol{c}}))$;
- *dual morphism* $(f,\lambda)^* \stackrel{\text{def}}{=} (f^*, \lambda) \in \mathsf{BD}^{(\mathfrak{C},\omega)}((\overleftarrow{\omega\boldsymbol{d}}), (\overleftarrow{\omega\boldsymbol{c}}))$.

Example 3.30. (See Figure 4(a).) Recall that $id_1 = (*, 0) \in \mathsf{BD}(1,1)$, where $*$ denotes the unique matching on a two element set. For each $c \in \mathfrak{C}$, let $\lambda^c \colon \mathbf{2} \to \mathfrak{C}$ be the colouring $1 \mapsto \omega c$, $\mathbf{2} \mapsto c$. Then $id_c = (id_1, \lambda^c) = (*, 0, \lambda^c) \in \mathsf{BD}^{(\mathfrak{C},\omega)}$, and the triple $(*, 0, \lambda^c)$ also defines the evaluation and coevaluation morphisms $\cup_c = \lfloor id_c \rfloor \in \mathsf{BD}^{(\mathfrak{C},\omega)}((c, \omega c), I_\mathfrak{C})$ and $\cap_c = \lceil id_c \rceil \in \mathsf{BD}^{(\mathfrak{C},\omega)}(I_\mathfrak{C}, (\omega c, c))$, and these satisfy

$$(\cup_c \oplus id_c) \circ (id_c \oplus \cap_c) = id_c = (id_c \oplus \cup_{\omega c}) \circ (\cap_{\omega c} \oplus id_c). \tag{3.31}$$

More generally, if $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathsf{list}\mathfrak{C}$, then we may define $\cap_{\boldsymbol{c}} \stackrel{\text{def}}{=} \lceil id_{\boldsymbol{c}} \rceil \in \mathsf{BD}^{(\mathfrak{C},\omega)}(I_\mathfrak{C}, \boldsymbol{c} \oplus (\overleftarrow{\omega\boldsymbol{c}}))$ and $\cup_c \stackrel{\text{def}}{=} \lfloor id_{\boldsymbol{c}} \rfloor \in \mathsf{BD}^{(\mathfrak{C},\omega)}((\overleftarrow{\omega\boldsymbol{c}}) \oplus \boldsymbol{c}, I_\mathfrak{C})$ (see Figure 4), and these satisfy

$$(\cup_{\boldsymbol{c}} \oplus id_{\boldsymbol{c}}) \circ (id_{\boldsymbol{c}} \oplus \cap_{\boldsymbol{c}}) = id_{\boldsymbol{c}} = (id_{\boldsymbol{c}} \oplus \cup_{\omega\boldsymbol{c}}) \circ (\cap_{\omega\boldsymbol{c}} \oplus id_{\boldsymbol{c}}). \tag{3.32}$$

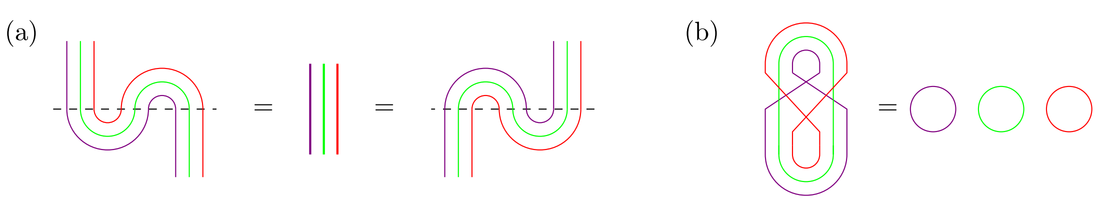


FIGURE 4. (a) The coloured triangle identities. (b) Ways of forming the trace $\mathrm{tr}(id_{\boldsymbol{c}})$.

Example 3.33. Let $\mathcal{V} \colon \mathsf{BD}^{(\mathfrak{C},\omega)} \to (\mathsf{V}, \otimes, I)$ be a strict monoidal functor. For each $\boldsymbol{c} \in \mathsf{list}\mathfrak{C}$, and each endomorphism f of $\mathcal{V}(\boldsymbol{c})$ in V, we can form the *trace* $\mathrm{tr}(f) \stackrel{\text{def}}{=} \cup_{\overleftarrow{\omega\boldsymbol{c}}} \circ (f \otimes \mathcal{V}(id_{\boldsymbol{c}})) \circ \cap_{\boldsymbol{c}} \in \mathsf{V}(I,I)$ of f. The image of $\mathcal{V}$ satisfies the axioms of a *traced monoidal category* [17]. (See also Figure 4(b).)

Example 3.34. As in the monochrome case (Example 3.22), if $(\mathfrak{C}, \omega)$ is a palette, then a functor $\mathcal{A} \colon \mathsf{BD}^{(\mathfrak{C},\omega)} \to (\mathsf{V}, \otimes, I)$ induces a graded monoid structure on the $\mathsf{list}\mathfrak{C}$-indexed object $(\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}}$. The induced product $\boxtimes$ has unit $I \mapsto \mathcal{A}(I_{\mathfrak{C}})$.

Let an n-tuple $\boldsymbol{d} = (d_1, \dots, d_n) \in \mathsf{list}\mathfrak{C}$ be such that $d_i = d = \omega d_j$ for $1 \leq i < j \leq n$. Then, as in Example 3.22, the (i,j)-contraction of $\mathcal{A}(\boldsymbol{d})$ is the V-morphism

$$\zeta^{i\ddagger j}_{\boldsymbol{d}} \overset{\text{def}}{=} \mathcal{A}\left((\cup_d \oplus id_{\boldsymbol{d}_{\widehat{i,j}}}) \circ \rho^{i,j}_n \right) \mathcal{A}(\boldsymbol{d}) \to \mathcal{A}(\boldsymbol{d}_{\widehat{i,j}}). \tag{3.35}$$

It follows from the monochrome case that contractions commute wherever defined:

$$\zeta^{i'\ddagger j'}_{\boldsymbol{d}_{\widehat{k,m}}} \circ \zeta^{k\ddagger m}_{\boldsymbol{d}} = \zeta^{k'\ddagger m'}_{\boldsymbol{d}_{\widehat{i,j}}} \circ \zeta^{i\ddagger j}_{\boldsymbol{d}} \colon \mathcal{A}(\boldsymbol{d}) \to \mathcal{A}(\boldsymbol{d}_{\widehat{i,j,k,m}}).$$

As in Example 3.24, the contraction ζ, together with the monoidal product $\boxtimes$ on the image of $\mathcal{A}$, induce a multiplication $\diamond$, with $\diamond^{i,j}_{\boldsymbol{c},\boldsymbol{d}} = \zeta^{i,m+j}_{\boldsymbol{cd}} \colon \mathcal{A}(\boldsymbol{c}) \times \mathcal{A}(\boldsymbol{d}) \to \mathcal{A}(\boldsymbol{cd}_{\widehat{i,j}})$ for all $\boldsymbol{c} = (c_1, \dots, c_m), \boldsymbol{d} = (d_1, \dots, d_n) \in \mathsf{list}\mathfrak{C}$ with $m, n \geq 1$ and $c_i = \omega d_j \in \mathfrak{C}$.

Moreover, as in Example 3.24, $\cap_c \in \mathsf{BD}^{(\mathfrak{C},\omega)}(I_{\mathfrak{C}}, (c, \omega c))$ induces a distinguished morphism $\mathcal{A}(\cap_c) \colon I \to \mathcal{A}(2)$ that acts as a two-sided unit for $\diamond$ for all $c \in \mathfrak{C}$.

Example 3.36. A monoidal category $(\mathsf{C}, \otimes, 0)$ is *compact closed* if, for every object x of C, there is a *dual object* x^* of C, and morphisms $\cap^{\mathsf{C}}_x \colon 0 \to x \otimes x^*$, $\cup^{\mathsf{C}}_x \colon x^* \otimes x \to 0$ such that

$$(\cup^{\mathsf{C}}_x \otimes id_x) \circ (id_x \otimes \cap^{\mathsf{C}}_x) = id_x = (id_x \otimes \cup^{\mathsf{C}}_{x^*}) \circ (\cap^{\mathsf{C}}_{x^*} \otimes id_x).$$

It follows that, given any small compact closed category with C with object monoid C_0 and satisfying $x^{**} = x$ for all $x \in \mathsf{C}_0$, there is a canonical faithful symmetric monoidal functor $\mathsf{BD}^{(\mathsf{C}_0,*)} \to \mathsf{C}$. This functor is an inclusion if C is permutative.

Conversely, if $F \colon \mathsf{BD}^{(\mathfrak{C},\omega)} \to (\mathsf{Set}, \times, *)$ is a symmetric strong monoidal functor (i.e. the induced maps $F(\boldsymbol{c}) \times F(\boldsymbol{d}) \to F(\boldsymbol{cd})$ are isomorphisms), then F describes a small compact closed category with object monoid indexed by $\mathsf{list}\mathfrak{C}$, and dual given by $\overleftarrow{\omega}$.

Example 3.37. (See also Example 4.12.) Let $\mathfrak{C} = \{+, -\}$ and let ω be the unique non-trivial involution on $\mathfrak{C}$. Then $\mathsf{BD}^{(\mathfrak{C},\omega)}$ is the free symmetric monoidal category with on a single object with dual (see [8, 28] ✠more?). Morphisms in $\mathsf{BD}^{(\mathfrak{C},\omega)}$ are represented, as in Figure 5, by diagrams of oriented intervals and unoriented circles.

Recall Example 3.15, and once again, let $\Bbbk$ be a field of characteristic 0. For $\delta \in \Bbbk$, the *walled Brauer category* $W\mathsf{Br}_\delta \subset \mathsf{Br}_\delta$ – studied in the representation theory of general linear groups ✠[?] – is the subcategory of Br_δ whose objects are given by pairs $(m, n) \in \mathbb{N}^2$, and $W\mathsf{Br}_\delta((m_1, n_1), (m_2, n_2)) \subset \mathsf{Br}_\delta(m_1 + n_1, m_2 + n_2))$ is subspace spanned by matchings τ that decompose as matchings on $\mathbf{m}_1 \amalg \mathbf{n}_2$ and $\mathbf{m}_2 \amalg \mathbf{n}_1$.

Every open morphism in $\mathsf{BD}^{(\mathfrak{C},\omega)}$ is isomorphic, via shuffles in $\mathsf{list}\{+, -\}$, to a generating diagram in $W\mathsf{Br}_\delta$, whereby strict functors $\mathsf{BD}^{(\mathfrak{C},\omega)} \to \mathsf{Vect}_\Bbbk$ such that $(-, 1) \mapsto \delta$, $(+) \mapsto V$ are equivalent to strict $\mathsf{Vect}_\Bbbk$-functors $W\mathsf{Br}_\delta \to \mathsf{Vect}_\Bbbk$, $(1, 0) \mapsto V$.

More generally, for any set $\mathfrak{D}$, the non-trivial involution on $\{+, -\}$ induces an involution ω on $\mathfrak{C} = \mathfrak{D} \times \{+, -\}$, and $\mathsf{BD}^{(\mathfrak{C},\omega)}$ is the free symmetric monoidal category with duals on the set $\mathfrak{D}$.

4. Circuit algebras

4.1. **Operads preliminaries.** This section summarises the basic theory of operads that will be used in the rest of the paper. See e.g. [21] for precise definitions of coloured operads (their called multicategories) and their algebras.

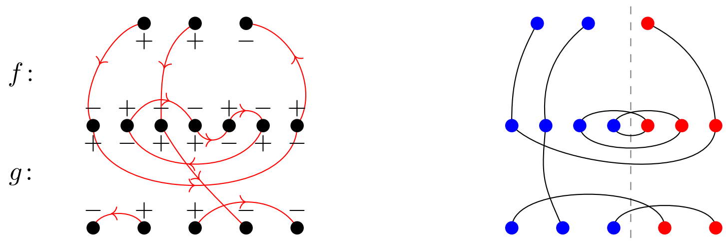


FIGURE 5. Composing directed Brauer diagrams $f\colon (-,-,+) \to (-,+,-,-,+,-,+)$ and $g\colon (-,+,-,-,+,-,+) \to (-,+,+,-,-)$. Right is an equivalent composition, viewed as a walled diagram, where horizontal arrows go from left to right (red to blue).

As in Section 3.2, let $\mathsf{list}\mathfrak{C}$ denote the set of finite lists $\boldsymbol{d} = (d_1, \dots, d_n)$ on a set $\mathfrak{D}$. Recall that a (symmetric) $\mathfrak{D}$-coloured operad of sets $\mathcal{O}$ is given by a $\mathsf{list}\mathfrak{D} \times \mathfrak{D}$-graded set $(\mathcal{O}(c, \dots, c_m; d))_{(c, \dots, c_m; d)}$ (equipped with the obvious symmetric action), and a collection of *composition morphisms*

$$\odot\colon \mathcal{O}(\boldsymbol{c}; d) \otimes \left(\bigotimes_{i=1}^{m} \mathcal{O}(\boldsymbol{b}^i; c_i)\right) \to \mathcal{O}(\boldsymbol{b}^1 \dots \boldsymbol{b}^m; d)$$

defined for all $d \in \mathfrak{D}$, $\boldsymbol{c} = (c_i)_{i=1}^m \in \mathsf{list}\mathfrak{D}$, and all $\boldsymbol{b}^i \in \mathsf{list}\mathfrak{D}$, $1 \leq i \leq m$, and such that, for all $d \in \mathfrak{D}$, there is an element $\nu_d \in \mathcal{O}(d; d)$

The composition $\odot$ is required to be associative and equivariant with respect to the Σ-action on $\mathcal{O}$. Moreover, for all $d \in D$, there is an element $\nu_d \in \mathcal{O}(d; d)$ that acts as a two-sided unit for the composition $\odot$: for all $\boldsymbol{c} = (c, \dots, c_m) \in \mathsf{list}\mathfrak{D}$, and $d \in \mathfrak{D}$, the morphisms given by the composites

$$\mathcal{O}(c, \dots, c_m; d) \xrightarrow{\cong} I \otimes \mathcal{O}(c, \dots, c_m; d) \xrightarrow{(\nu_d, id)} \mathcal{O}(d; d) \otimes \mathcal{O}(c, \dots, c_m; d) \xrightarrow{\odot} \mathcal{O}(c, \dots, c_m; d),$$

and

$$\mathcal{O}(c, \dots, c_m; d) \xrightarrow{\cong} \mathcal{O}(c, \dots, c_m; d) \otimes I \xrightarrow{(id, \bigotimes_{i=1}^m \nu_{c_i})} \mathcal{O}(c, \dots, c_m; d) \otimes \left(\bigotimes_{i=1}^{m} \mathcal{O}(c_i; c_i)\right) \xrightarrow{\odot} \mathcal{O}(c, \dots, c_m; d),$$

are the identity on $\mathcal{O}(c, \dots, c_m; d)$.

In particular, the restriction of a $\mathfrak{D}$-coloured operad $\mathcal{O}$ to (compositions of) the sets $\mathcal{O}(c; d)$ for $c, d \in \mathfrak{D}$ describes a small category.

Objects of the category Op are (coloured) operads. If $(\mathcal{O}^1, \odot^1, \nu^1)$ is a $\mathfrak{D}_1$-coloured operad, a morphism $\gamma\colon (\mathcal{O}^1, \odot^1, \nu^1) \to (\mathcal{O}^2, \odot^2, \nu^2)$ to a $\mathfrak{D}_2$-coloured operad, is given by a map of sets $\gamma_0\colon \mathfrak{D}_1 \to \mathfrak{D}_2$, and, a $(\mathsf{list}\mathfrak{D}_1 \times \mathfrak{D}_1)$-indexed collection of maps

$$\gamma_{(c, \dots, c_m k; d)}\colon \mathcal{O}^1(c, \dots, c_m; d) \to \mathcal{O}^2(\gamma_0(c), \dots, \gamma_0(c_m); \gamma_0(d))$$

that respect units and composition, and are equivariant with respect to the action of Σ.

If $\mathfrak{D}_1 = \mathfrak{D}_2 = \mathfrak{D}$, and $\gamma_0 = id_{\mathfrak{D}}$ then $\gamma\colon \mathcal{O}^1 \to \mathcal{O}^2$ is called *colour-preserving*, or *a morphism of $\mathfrak{D}$-coloured operads.*

Example 4.1. Underlying any (small) permutative category $(\mathsf{C}, \otimes, 0)$ – with object set C_0 – is a C_0-coloured (symmetric) operad $\mathcal{O}^{\mathsf{C}}$ defined by

$$\mathcal{O}^{\mathsf{C}}(x_1, \dots, x_n; y) \overset{\text{def}}{=} \mathsf{C}(x_1 \otimes \dots \otimes x_n, y)$$

and with operadic composition $\odot$ in $\mathcal{O}^{\mathsf{C}}$ induced by composition in C:

$$\odot\colon \mathcal{O}^{\mathsf{C}}(x_1, \dots, x_n; y) \times \left(\prod_{i=1}^{k} \mathcal{O}^{\mathsf{C}}(w_{i,1}, \dots, w_{i,m_i}; x_i)\right) \to \mathcal{O}^{\mathsf{C}}(w_{1,1}, \dots, w_{i,m_i}, \dots, w_{n,m_n}; y),$$

$$(\overline{g}, (\overline{f}_i)_i) \mapsto \overline{(g \circ (f_1 \otimes \cdots \otimes f_n))}$$

for all $\overline{g} \in \mathcal{O}^{\mathsf{C}}(x_1, \dots, x_n; y)$ described by $g \in \mathsf{C}(x_1 \otimes \cdots \otimes x_n, y)$, and $\overline{f}_i \in \mathcal{O}^{\mathsf{C}}(w_{i,1}, \dots, w_{i,m_i}; x_i)$ described by $f_i \in \mathsf{C}(w_{i,1} \otimes \cdots \otimes w_{i,m_i}, x_i)$ $(1 \leq i \leq n)$.

Moreover, C is precisely the category obtained from $\mathcal{O}^{\mathsf{C}}$ by restricting to (compositions of) the sets of the form $\mathcal{O}^{\mathsf{C}}(x; y)$ for all $x, y \in \mathsf{C}_0$.

Let $(\mathsf{V}, \otimes, I)$ be a symmetric monoidal category.

Example 4.2. Given a set $\mathfrak{D}$ and a $\mathfrak{D}$-indexed object $A = (A(c))_{c \in \mathfrak{D}}$ in V, the $\mathfrak{D}$-coloured endomorphism operad End^A in V has

$$End^A(c, \dots, c_k; d) = \mathsf{V}(A(c_1) \otimes \cdots \otimes A(c_k), A(d)$$

and the obvious composition and units induced by composition and identities in V.

Observe, in particular, that a $\mathfrak{D}$-coloured endomorphism operad End^A in V, underlies a small permutative category whose object monoid may be identified with $\mathsf{list}\mathfrak{D}$.

Definition 4.3. *An algebra for $\mathcal{O}$ in V is a $\mathfrak{D}$-indexed object $(A(c))_c$ in V, together with a morphism $\alpha \colon \mathcal{O} \to End^A$ of $\mathfrak{D}$-coloured operads.*

The category $Alg(\mathcal{O})$ (or $Alg_{\mathsf{V}}(\mathcal{O})$ of algebras for $\mathcal{O}$ (in V) is the full subcategory on $\mathcal{O}$ algebras (in V) of the slice category $\mathcal{O}/\mathsf{Op}$.

Remark 4.4. Definition 4.3 differs slightly from standard definitions since it allows for algebras for a Set-valued operad to be defined in arbitrary symmetric monoidal categories $(\mathsf{V}, \otimes, I)$.

Remark 4.5. If $\mathcal{O} = \mathcal{O}^{\mathsf{C}}$ is the operad underlying a small permutative category C, then an algebra for $\mathcal{O}$ in V describes a morphism $\mathsf{C} \to (\mathsf{V}, \otimes, I)$ of symmetric monoidal categories (by $\mathcal{O}(x; y) = \mathsf{C}(x, y)$ and $\overline{id}_{x \otimes y} \in \mathcal{O}(x, y; x \otimes y)$. In fact, it is straightforward to show that this describes an isomorphism between the category of $(\mathfrak{C}, \omega)$-coloured circuit algebras in $(\mathsf{V}, \otimes, I)$ and lax functors $(\mathsf{C}, \oplus, 0) \to (\mathsf{V}, \otimes, I)$. (See [21, Chapters 2-3], particularly Theorem 3.3.4(b).)

4.2. **Wiring diagrams and circuit algebras.** Operads of wiring diagrams are defined as operads underlying the categories $\mathsf{BD}^{(\mathfrak{C}, \omega)}$, for $(\mathfrak{C}, \omega) \in \mathsf{Pal}$.

Definition 4.6. *For a given palette $(\mathfrak{C}, \omega)$, the* $\mathsf{list}\mathfrak{C}$-coloured operad $WD^{(\mathfrak{C}, \omega)}$ of $(\mathfrak{C}, \omega)$-wiring diagrams *is the operad $\mathcal{O}^{\mathsf{BD}^{(\mathfrak{C}, \omega)}}$ underlying $\mathsf{BD}^{(\mathfrak{C}, \omega)}$ (Example 4.1).*

For each $(\boldsymbol{c}, \dots, \boldsymbol{c}_k; \boldsymbol{d}) \in \mathsf{list}^2\mathfrak{C}$, elements of the set

$$WD^{(\mathfrak{C}, \omega)}(\boldsymbol{c}, \dots, \boldsymbol{c}_k; \boldsymbol{d}) \stackrel{\text{def}}{=} \mathsf{BD}^{(\mathfrak{C}, \omega)}(\boldsymbol{c} \dots \boldsymbol{c}_k; \boldsymbol{d})$$

are called wiring diagrams of type $(\boldsymbol{c}, \dots, \boldsymbol{c}_k; \boldsymbol{d})$.

When $\mathfrak{C}$ is the singleton set, the $\mathbb{N}$-coloured operad $WD^{(\mathfrak{C}, \omega)} \stackrel{\text{def}}{=} \mathcal{O}^{\mathsf{BD}}$ is denoted by WD, and called the operad of (monochrome) wiring diagrams.

Let $(\mathsf{V}, \otimes, I)$ be a symmetric monoidal category.

Definition 4.7. *A* $(\mathfrak{C}, \omega)$-coloured circuit algebra in V *is a V-valued algebra (A, α) for the operad $WD^{(\mathfrak{C}, \omega)}$ of $(\mathfrak{C}, \omega)$-coloured wiring diagrams. The full subcategory of V-valued circuit algebras in $Alg(WD^{(\mathfrak{C}, \omega)})$ is denoted by $\mathsf{CA}_{\mathsf{V}}^{(\mathfrak{C}, \omega)}$. When $\mathsf{V} = \mathsf{Set}$, $\mathsf{CA}_{\mathsf{V}}^{(\mathfrak{C}, \omega)}$ is denoted simply by $\mathsf{CA}^{(\mathfrak{C}, \omega)}$.*

When $\mathfrak{C} = \{*\}$ is a singleton, write $\mathsf{CA}_{\mathsf{V}}^{\text{mono}} \stackrel{\text{def}}{=} \mathsf{CA}_{\mathsf{V}}^{(\mathfrak{C}, \omega)}$ for the category of monochrome circuit algebras in V.

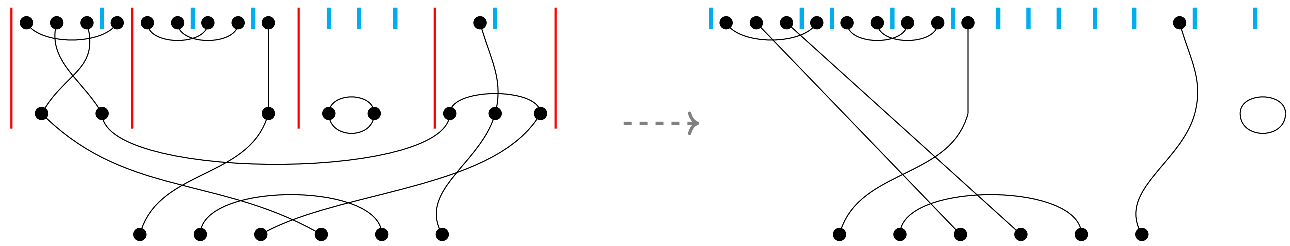


FIGURE 6. Composition in $WD^{(\mathfrak{C},\omega)}$. (See also Figure 7.)

In particular, by Remark 4.5, $(\mathfrak{C},\omega)$-coloured circuit algebras in V are defined by lax functors $\mathsf{BD}^{(\mathfrak{C},\omega)} \to (\mathsf{V},\otimes, I)$, and hence will often be denoted simply by $\mathcal{A}$.

Unpacking the definition, a $(\mathfrak{C},\omega)$-coloured circuit algebra consists of a $\mathsf{list}\mathfrak{C}$-indexed collection $(\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}\in\mathsf{list}\mathfrak{C}}$ of V-objects and, for each $(\boldsymbol{c},\dots,\boldsymbol{c}_k;\boldsymbol{d}) \in \mathsf{list}^2\mathfrak{C}\times\mathsf{list}\mathfrak{C}$, a set of V-morphisms $\mathcal{A}(f,\lambda)\colon \bigotimes_{i=1}^k \mathcal{A}(\boldsymbol{c}_i) \to \mathcal{A}(\boldsymbol{d})$ indexed by Brauer diagrams $(f,\lambda) \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c}\oplus\cdots\oplus\boldsymbol{c}_k,\boldsymbol{d})$. These satisfy:

- for all $\boldsymbol{c}\in\mathsf{list}\mathfrak{C}$, $\mathcal{A}(id_{\boldsymbol{c}}) = id_{\mathcal{A}(\boldsymbol{c})} \in \mathsf{V}(\mathcal{A}(\boldsymbol{c}),\mathcal{A}(\boldsymbol{c}))$;
- the morphisms $\mathcal{A}(f,\lambda)$ are equivariant with respect to the Σ-action on $\mathsf{list}\mathfrak{C}$ and on $WD^{(\mathfrak{C},\omega)}$;
- given composable wiring diagrams $(f,\lambda) \in WD^{(\mathfrak{C},\omega)}(\boldsymbol{c},\dots,\boldsymbol{c}_k;\boldsymbol{d})$ and, for all $1,\leq i\leq k$, $(f^i,\lambda^i) \in WD^{(\mathfrak{C},\omega)}(\boldsymbol{b}_{i,1},\dots,\boldsymbol{b}_{i,k_i};\boldsymbol{c}_i)$, the diagram

$$\begin{tikzcd}
\bigotimes_{i=1}^k \bigotimes_{j=1}^{k_i} \mathcal{A}(\boldsymbol{b}_{i,j}) \arrow[rr, "\bigotimes_{i=1}^k \mathcal{A}(f^i,\lambda^i)"] \arrow[drr, "\mathcal{A}\left((f,\lambda)\odot(f^i,\lambda^i)\right)_i"'] & & \bigotimes_{i=1}^k \mathcal{A}(\boldsymbol{c}_i) \arrow[d, "\mathcal{A}(f,\lambda)"] \\
& & \mathcal{A}(\boldsymbol{d})
\end{tikzcd} \tag{4.8}$$

  commutes in V.

Let $\mathcal{A}$ be a $(\mathfrak{C},\omega)$-coloured circuit algebra in $(\mathsf{V},\otimes,I)$.

Proposition 4.9. *The collection $(\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}\in\mathsf{list}\mathfrak{C}}$ is equipped with the structure of a $\mathsf{list}\mathfrak{C}$-graded monoid in V, contraction maps $\zeta^{i\ddagger j}_{\boldsymbol{c}}\colon \mathcal{A}(\boldsymbol{c}) \to \mathcal{A}(\boldsymbol{c}_{\widehat{i,j}})$ for all $\boldsymbol{c} = (c,\dots,c_m) \in \mathsf{list}\mathfrak{C}$ with $c_i = \omega c_j$. These satisfy:*

(c1) the graded monoidal product $\boxtimes$ on $(\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}\in\mathsf{list}\mathfrak{C}}$ is associative;
(c2) contractions commute: wherever defined,

$$\zeta^{i'\ddagger j'}_{\boldsymbol{c}_{\widehat{k,m}}} \circ \zeta^{k\ddagger m}_{\boldsymbol{c}} = \zeta^{k'\ddagger m'}_{\boldsymbol{c}_{\widehat{i,j}}} \circ \zeta^{i\ddagger j}_{\boldsymbol{c}}\colon \mathcal{A}(\boldsymbol{c}) \to \mathcal{A}(\boldsymbol{c}_{\widehat{i,j,k,m}});$$

(c3) contraction commutes with monoidal product: for all $\boldsymbol{c}$ with $c_i = \omega c_j$, and all $\boldsymbol{d}$,

$$\zeta^{i\ddagger j}_{\boldsymbol{c}\boldsymbol{d}} \boxtimes_{\boldsymbol{c},\boldsymbol{d}} = \boxtimes_{\boldsymbol{c}_{\widehat{i,j}},\boldsymbol{d}}(\zeta^{i\ddagger j}_{\boldsymbol{c}} \otimes id_{\mathcal{A}(\boldsymbol{d})})\colon \mathcal{A}(\boldsymbol{c}) \otimes \mathcal{A}(\boldsymbol{d}) \to \mathcal{A}(\boldsymbol{c}_{\widehat{i,j}}\boldsymbol{d}).$$

Moreover, for all $c \in \mathfrak{C}$, there is a distinguished morphism $\epsilon_c \in \mathsf{V}(I,\mathcal{A}(c,\omega c))$ such that

(e1) ϵ_c satisfies $\zeta^{1\ddagger i}_{(c,\omega c),\boldsymbol{c}}(\epsilon_c \boxtimes id_{\boldsymbol{c}}) = id_{\boldsymbol{c}}$ for all $\boldsymbol{c}\in\mathsf{list}\mathfrak{C}$ and all i such that $c_i = c$.

Proof. The maps $\overline{id_{\boldsymbol{c}\boldsymbol{d}}} \in WD^{(\mathfrak{C},\omega)}(\boldsymbol{c},\boldsymbol{d};\boldsymbol{c}\boldsymbol{d})$ induced by the identity morphisms $id_{\boldsymbol{c}\boldsymbol{d}} \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c}\boldsymbol{d},\boldsymbol{c}\boldsymbol{d})$ equip $(\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}\in\mathsf{list}\mathfrak{C}}$ with the structure of an associative graded monoidal product $\boxtimes$ on $(\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}\in\mathsf{list}\mathfrak{C}}$.

The contraction morphisms $\zeta^{i\ddagger j}_{\boldsymbol{d}}\colon \mathcal{A}(\boldsymbol{c}) \to \mathcal{A}(\boldsymbol{c}_{\widehat{i,j}})$ are defined as in Example 3.34 and hence satisfy (c2). Moreover, for all $\boldsymbol{c} = (c,\dots,c_m)$ with $c_i = \omega c_j$, and all $\boldsymbol{d} = (d_1,\dots,d_n)$,

$$(\cup_{c_i} \oplus id_{\boldsymbol{c}_{\widehat{i,j}}\boldsymbol{d}}) \circ \rho^{i,j}_{m+n} \circ (id_{\boldsymbol{c}} \oplus id_{\boldsymbol{d}}) = (id_{\boldsymbol{c}_{\widehat{i,j}}} \oplus id_{\boldsymbol{d}}) \circ \left(\left((\cup_{c_i} \oplus id_{\boldsymbol{c}_{\widehat{i,j}}}) \circ \rho^{i,j}_m\right) \oplus id_{\boldsymbol{d}}\right) \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c}\boldsymbol{d},\boldsymbol{c}_{\widehat{i,j}}\boldsymbol{d})$$

hence $\boxtimes$ and ζ satisfy (c3).

Since $\mathcal{A}$ is equivalently a lax functor $\mathsf{BD}^{(\mathfrak{C},\omega)} \to (\mathsf{V}, \otimes, I)$, there is a chosen morphism $I \to \mathcal{A}(I_{\mathfrak{C}})$ in V. For $c \in \mathfrak{C}$, the morphism $\epsilon_c \colon I \to \mathcal{A}(c, \omega c)$ is obtained by composing with $\mathcal{A}(\cap_c) \colon \mathcal{A}(I_{\mathfrak{C}}) \to \mathcal{A}(c, \omega c)$. This satisfies (e1) by Example 3.24. $\square$

Let $WD^{\downarrow} \subset WD$ be the suboperad of (monochrome) downward wiring diagrams that corresponds to the operad underlying the category $\mathsf{BD}_{\downarrow}$ of downward Brauer diagrams.

Corollary 4.10. *Algebras for $WD^{\downarrow}$ in V are graded monoids in V equipped with a contraction operation ζ, such that the monoid multiplication $\boxtimes$ and ζ satisfy (c1)-(c3).*

Proof. By Examples 3.22 and 3.24, the graded monoid and contraction structures of a WD-algebra depend only on morphisms in $\mathsf{BD}_{\downarrow}$, whence the corollary follows immediately. $\square$

Remark 4.11. By [27] and Example 3.15, algebras for $WD^{\downarrow}$ in $\mathsf{Vect}_{\Bbbk}$ are closely related to representations of the stable groups $O(\infty)$ and $Sp(\infty)$.

Since $\cap \in \mathsf{BD}(0,2)$ is not a morphism in $\mathsf{BD}_{\downarrow}$, $WD^{\downarrow}$-algebras do not, in general satisfy (e1), and therefore do not have a unit for the induced multiplication $\diamond$ (Example 3.24). For this reason they will sometimes be referred to as *non-unital (monchrome) circuit algebras.*

Example 4.12. As in Example 3.37, let ω be the unique non-trivial involution on the two element set $\mathfrak{C} = \{+, -\}$. In [9], it is proved that $(\mathfrak{C}, \omega)$-coloured circuit algebras in $(\mathsf{V}, \otimes, I)$ are equivalent to wheeled props in V (wheeled props are discussed in, for example [23, 24]).

The assignment $(\mathfrak{C}, \omega) \mapsto \mathsf{CA}_{\mathsf{V}}^{(\mathfrak{C},\omega)}$ defines a Cat-valued presheaf ca_{V} on the palette category Pal: A morphism $\phi \colon (\mathfrak{C}, \omega) \to (\mathfrak{C}', \omega')$ in Pal induces a strict symmetric monoidal functor $\mathsf{BD}^{(\mathfrak{C},\omega)} \to \mathsf{BD}^{(\mathfrak{C}',\omega')}$, and hence any $(\mathfrak{C}', \omega')$ coloured circuit algebra $\mathcal{A}'$ pulls back to a $(\mathfrak{C}, \omega)$-coloured circuit algebra $\phi^* \mathcal{A}'$.

Definition 4.13. *For all symmetric monoidal categories $(\mathsf{V}, \otimes, I)$, the Grothendieck construction of the functor $ca_A \colon \mathsf{Pal}^{\mathrm{op}} \to \mathsf{Cat}$ is the category CA_{V} of all circuit algebras in* V.

When $(\mathsf{V}, \otimes, I) = (\mathsf{Set}, \times *)$, the category of circuit algebras in Set is denoted by $\mathsf{CA} \overset{\text{def}}{=} \mathsf{CA}_{\mathsf{Set}}$.

So, objects of CA_{V} are pairs $((\mathfrak{C}, \omega), \mathcal{A})$ of a palette $(\mathfrak{C}, \omega)$ and a $(\mathfrak{C}, \omega)$-coloured circuit algebra $\mathcal{A}$ in V. Morphisms $((\mathfrak{C}, \omega), \mathcal{A}) \to ((\mathfrak{C}', \omega'), \mathcal{A}')$ are pairs (ϕ, γ) where $\phi \colon \mathfrak{C} \to \mathfrak{C}'$ satisfies $\phi\omega = \omega'\phi$ and $\gamma \colon \phi^* \mathcal{A}' \to \mathcal{A}$.

Example 4.14. For any collection $S = (S(\boldsymbol{c}))_{\boldsymbol{c} \in \mathsf{list}\mathfrak{C}}$ of sets equipped with a Σ-action $\sigma \colon S(\boldsymbol{c}) \xrightarrow{\cong} S(\omega \boldsymbol{c})$ for all $\boldsymbol{c} \in \mathfrak{C}^{\mathbf{n}}$ and all $\sigma \in \Sigma_n$, let $(FS(\boldsymbol{c}))_{\boldsymbol{c}}$ be the $\mathsf{list}\mathfrak{C}$-indexed set of $(\mathfrak{C}, \omega)$-coloured wiring diagrams *decorated by* S:

$$\begin{aligned} FS(\boldsymbol{d}) &= \coprod_{(\boldsymbol{c}, \dots, \boldsymbol{c}_k) \in \mathsf{list}^2 \mathfrak{C}} \left(WD^{(\mathfrak{C},\omega)}(\boldsymbol{c}, \dots, \boldsymbol{c}_k; \boldsymbol{d}) \times \textstyle\prod_{i=1}^{k} S(\boldsymbol{c}_i) \right) \\ &= \coprod_{((\boldsymbol{c} \oplus \cdots \oplus \boldsymbol{c}_k), (f, \lambda))) \in \mathsf{BD}^{(\mathfrak{C},\omega)}/\boldsymbol{d}} \left(\textstyle\prod_{i=1}^{k} S(\boldsymbol{c}_i) \right) \end{aligned} .$$

The set FS underlies the *free circuit algebra $\mathcal{F}S$ on S*:

For each $(\overline{f}, \lambda) \in WD^{(\mathfrak{C},\omega)}(\boldsymbol{c}, \dots, \boldsymbol{c}_k; \boldsymbol{d})$, the morphism $\mathcal{F}(\overline{f}, \lambda) \colon FS(\boldsymbol{c}_1) \times \cdots \times FS(\boldsymbol{c}_k) \to FS(\boldsymbol{d})$ is described by

$$\prod_{i=1}^{k} \left((\overline{f}^i, \lambda^i), (x^i_{j_i})_{j_i=1}^{m_i} \right) \mapsto \left(\left((\overline{f}, \lambda) \overline{\odot} \left((\overline{f}^i, \lambda^i)_{i=1}^{k} \right) \right), (x^i_{j_i})_{\substack{1 \le j_i \le m_i \\ 1 \le i \le k}} \right).$$

The free circuit algebra FS is *generated by* the sets $S(\boldsymbol{d})$ without any relations. A general $(\mathfrak{C}, \omega)$-coloured circuit algebra $\mathsf{CA}^{(\mathfrak{C},\omega)} \langle \mathfrak{G} | \mathfrak{R} \rangle$ in Set is formed as the quotient of a free circuit algebra $F\mathfrak{G}$ – specified by a set of generating elements $\mathfrak{G} = (Gx)_{x \in X}$ – by a set of relations $\mathfrak{R} = (Ry)_{y \in Y}$ on $\mathfrak{G}$.

Example 4.15. The circuit algebra VT of *virtual tangles* is the directed circuit algebra generated by the under and over crossings $\mathcal{G} = \{\}$, modulo the (virtual) Reidemeister moves (see [10, Definition 2.11] for the oriented, or $\{+,-\}$-coloured case).

More generally, for $1 \leq k < n$, we may consider circuit algebras of k-dimensional tangles in n-dimensional space

Example 4.16. Example 4.15 Let $\mathcal{A}$ be a $(\mathfrak{C}, \omega)$-coloured circuit algebra in Set. Then $\mathcal{A}$ defines a palette whose elements are pairs $(\boldsymbol{c}, \phi)$, and $(\overleftarrow{\omega \boldsymbol{c}}, \phi)$, with $\boldsymbol{c} \in \mathsf{list}\mathfrak{C}$, and $\phi \in \mathcal{A}(\boldsymbol{c})$, and with the obvious involution induced by $\overleftarrow{\boldsymbol{\omega}}$. (This involution is less artificial than may appear at first sight. For, the dual $(id_{\boldsymbol{c}})^*$ of the identity morphism $id_{\boldsymbol{c}} \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c}, \boldsymbol{c})$ is just $id_{\overleftarrow{\omega \boldsymbol{c}}} \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\overleftarrow{\omega \boldsymbol{c}}, \overleftarrow{\omega \boldsymbol{c}})$.)

Hence, we may define *higher circuit algebras*: circuit algebras whose palettes themselves form circuit algebras. An example of this construction is given by the *circuit algebras with skeleton* in [10]. Extended cobordisms✠ [?] provide another natural example.

Remark 4.17. Wiring diagrams are often represented pictorially (for example in [9, 10]), by immersions of compact 1-manifolds in punctured 2-discs. From this point of view, the composition in Figure 6 is represented as in Figure 7.

This representation of wiring diagrams illustrates the relationship between wiring diagrams (and hence circuit algebras) and planar diagrams and planar algebras. It also clearly exhibits the operad WD as a suboperad of the operad of wiring diagrams defined in [29].

In the context of the present work, the disc representation of wiring diagrams is highly suggestive of the graphical constructions that will follow in Section 6.

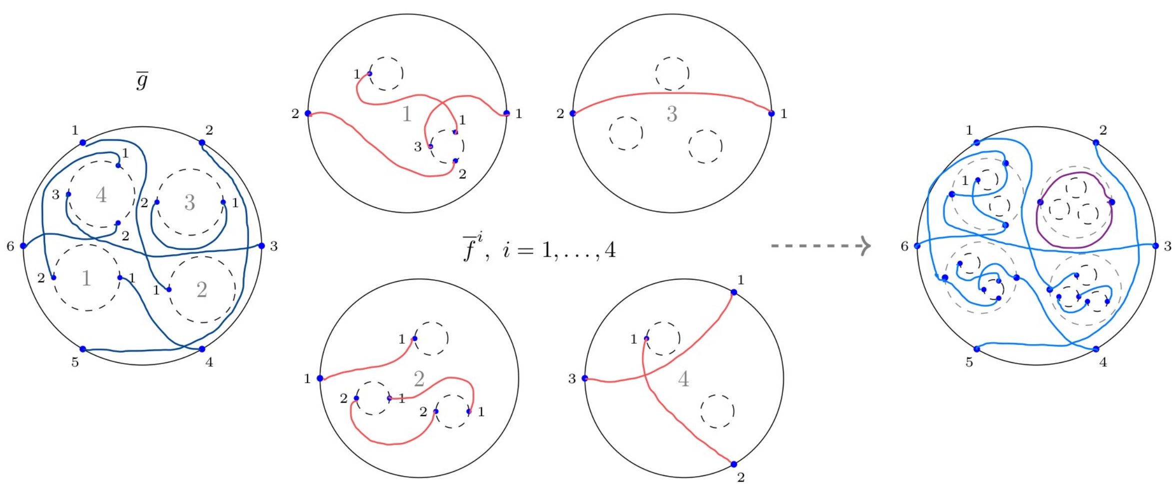


FIGURE 7. Disc representation of the composition in Figure 6.

5. Graphical species, circuit operads, and modular operads

By Proposition 4.9, a $(\mathfrak{C}, \omega)$-coloured circuit algebra $\mathcal{A}$ in a symmetric monoidal category $(\mathsf{V}, \otimes, I)$ has the structure of a $\mathsf{list}\mathfrak{C}$-graded monoid with contractions, and distinguished morphisms $\epsilon_c \in \mathsf{V}(I, \mathcal{A}(c, \omega c))$ satisfying the conditions (c1)-(c3) and (e1) stated there.

In Section 9.2, it will be shown that the converse also holds. That is, a $\mathsf{list}\mathfrak{C}$-graded object $A = (A(\boldsymbol{c}))_{\boldsymbol{c}}$ in V has the structure of a $(\mathfrak{C}, \omega)$-coloured V-valued circuit algebra if and only if it describes a graded monoid $(A, \boxtimes, I)$ in V, and is equipped with a family ζ of operations as in Examples 3.22 and 3.24, such that (c1)-(c3) and (e1) above hold.

A $\mathsf{list}\mathfrak{C}$-graded set satisfying (c1)-(c3) and (e1) of Proposition 4.9 is called a $(\mathfrak{C}, \omega)$-coloured circuit operad in Set. It is the goal of this section to define circuit operads in arbitrary categories E with finite limits.

5.1. Graphical species.

By definition monochrome circuit algebra $\mathcal{A} = (A, \alpha)$ in a symmetric monoidal category $(\mathsf{V}, \otimes, I)$ is described by a functor $A\colon \Sigma^{\mathrm{op}} \to \mathsf{V}$ together with a collection of operations

$$\alpha(\overline{f})\colon A(m_1) \otimes \cdots \otimes A(m_k) \to A(n), \ \overline{f} \in WD(\mathbf{m}_1, \dots, \mathbf{m}_k; \mathbf{n})$$

indexed by $WD(\mathbf{m}_1, \dots, \mathbf{m}_k; \mathbf{n})$ and $(m_1, \dots, m_k; n) \in \mathsf{list}\mathbb{N} \times \mathbb{N}$, and satisfying some associativity axioms. If $\mathsf{V} = \mathsf{Set}$, and $(\mathfrak{C}, \omega)$ is an arbitrary palette, then it is still possible, by enlarging Σ, to describe the $\mathsf{list}\mathfrak{C}$-coloured set $A = (A(\boldsymbol{c}))_{\boldsymbol{c}}$ underlying a $(\mathfrak{C}, \omega)$-circuit algebra of sets as a functor.

This is the idea of graphical species, that were introduced in [15], and used in the definition of coloured Set-valued modular operads (compact closed categories) [15, 25, 13, 14]. This section provides a short discussion on graphical species in an arbitrary category E with finite limits. For more details, the reader is referred to [25], where they are discussed at length.

It will be convenient to work with the groupoid $\mathbf{B}$ of finite sets and bijections, rather than its skeletal subcategory Σ. The full subcategory of finite sets in Set is denoted by $\mathsf{Set}_{\mathsf{f}}$.

Remark 5.1. (See also Section 2.3.) A presheaf $P\colon \mathbf{B}^{\mathrm{op}} \to \mathsf{E}$ on $\mathbf{B}$, also called a *(monochrome* or *single-sorted) species in* E [16], determines a presheaf on Σ by restriction.

Conversely, a Σ-presheaf Q may always be extended to a $\mathbf{B}$-presheaf $Q_{\mathbf{B}}$, by setting

$$Q_{\mathbf{B}}(X) \overset{\text{def}}{=} \lim_{(\mathbf{n}, f) \in \Sigma / X} Q(\mathbf{n}) \quad \text{for all } n \in \mathbb{N}. \tag{5.2}$$

Let the category $\mathbf{B}^{\S}$ be obtained from $\mathbf{B}$ by adjoining a distinguished object $\S$ that satisfies

- $\mathbf{B}^{\S}(\S, \S) = \{id, \tau\}$ with $\tau^2 = id$,
- for each finite set X and each element $x \in X$, there is a morphism $ch_x \in \mathbf{B}^{\S}(\S, X)$ that 'chooses' x, and $\mathbf{B}^{\S}(\S, X) = \{ch_x \ , \ ch_x \circ \tau\}_{x \in X}$,
- for all finite sets X and Y, $\mathbf{B}^{\S}(X, Y) = \mathbf{B}(X, Y)$, and morphisms are equivariant with respect to the action of $\mathbf{B}$. That is, $ch_{f(x)} = f \circ ch_x \in \mathbf{B}^{\S}(\S, Y)$ for all $x \in X$ and all bijections $f\colon X \xrightarrow{\cong} Y$.

Let E be a category with finite limits.

Definition 5.3. *A* graphical species S in E *is a presheaf* $S\colon {\mathbf{B}^{\S}}^{\mathrm{op}} \to \mathsf{E}$. *The category of graphical species by* $\mathsf{GS}_{\mathsf{E}} \overset{\text{def}}{=} \mathsf{psh}_{\mathsf{E}}(\mathbf{B}^{\S})$. *When* $\mathsf{E} = \mathsf{Set}$*, we write* $\mathsf{GS} \overset{\text{def}}{=} \mathsf{GS}_{\mathsf{Set}}$.

Hence, a graphical species S is described by an E-valued species $(S_X)_{X \in \mathbf{B}}$, and an involutive palette (see Definition 3.26) $(S_{\S}, S_{\tau})$ in E, together with, for each finite set X, and $x \in X$ a $\mathbf{B}$-equivariant projection $S(ch_x)\colon S_X \to S_{\S}$.

Let $*$ denote the terminal object of E. This exists since E is assumed to have finite limits.

Definition 5.4. *A morphism* $\gamma \in \mathsf{GS}_{\mathsf{E}}(S, S')$ *is* colour-preserving *if its component at* $\S$ *is the identity on* $S_{\S}$.

A graphical species $S \in \mathsf{GS}_{\mathsf{E}}$ *is called monochrome if* $S_{\S} = *$.

Example 5.5. The terminal graphical species K_{E} is the constant graphical species that sends $\S$ and all finite sets X to the terminal object $*$ in E.

The element category (Definition 2.10) of a graphical species S in Set is denoted by $\mathsf{el}(S) \overset{\text{def}}{=} \mathsf{el}_{\mathbf{B}^\S}(S)$.

Definition 5.6. *Elements $c \in S_\S$ are colours of S. For each element $\underline{c} = (c_x)_{x\in X} \in {S_\S}^X$, the $\underline{c}$-*(coloured) arity *$S_{\underline{c}}$ is the fibre above $\underline{c} \in {S_\S}^X$ of the map $(S(ch_x))_{x\in X} \colon S_X \to {S_\S}^X$.*

Example 5.7. (Compare Example 3.37, 4.12.) For any palette $(\mathfrak{C}, \omega)$, the terminal $(\mathfrak{C}, \omega)$-coloured graphical species $K^{(\mathfrak{C},\omega)}$ in Set, is given by ${K^{(\mathfrak{C},\omega)}}_{\underline{c}} = \{*\}$ for all $\underline{c} \in \mathfrak{C}^X$ (and all finite sets X). This is the terminal object of the category $\mathsf{GS}_{\mathsf{E}}^{(\mathfrak{C},\omega)}$ of $(\mathfrak{C}, \omega)$-coloured graphical species in Set and palette-preserving morphisms.

Let ω be the unique non-identity involution on the set $\mathfrak{C} = \{+, -\}$ as in Examples 3.37 and 4.12. For all finite sets X, ${K^{(\mathfrak{C},\omega)}}_X = \{+, -\}^X$ is the set of partitions $X = X_+ \amalg X_-$ of X into *input* and *output* sets, and morphisms ${K^{(\mathfrak{C},\omega)}}_X \to {K^{(\mathfrak{C},\omega)}}_Y$ are bijections $X \xrightarrow{\cong} Y$ that preserve the partitions. Hence, $(\mathfrak{C}, \omega)$-coloured graphical species are monochrome *'monochrome directed graphical species'*.

5.2. Circuit Operads.

The monoidal and contraction structure on circuit algebras described in Proposition 4.9 may be modified for graphical species in a category E with finite limits.

Definition 5.8. *An* external product *on a graphical species S in E is given by a collection of $\mathbf{B}$-equivariant morphisms*

$$\boxtimes_{X,Y} : S_X \times S_Y \to S_{X\amalg Y}$$

in E defined for all finite sets X and Y, and such that, for all elements $x \in X$, the following diagram commutes in E:

$$\begin{tikzcd} S_X \times S_Y \arrow[r, "S(ch_x^{X\amalg Y})"] \arrow[d] & S_\S \\ S_X. \arrow[ur, "S(ch_x^X)"'] & \end{tikzcd}$$

The external product $\boxtimes$ is unital *with* external unit *ι if there is a distinguished morphism $\iota \colon * \to S_{\mathbf{0}}$ such that, for all X, the composite*

$$S_X \xrightarrow{\cong} S_X \times * \xrightarrow{id\times\iota} S_X \times S_{\mathbf{0}} \xrightarrow{\boxtimes} S_{(X\amalg\mathbf{0})} = S_X$$

is the identity on S_X.

Remark 5.9. The monoidal unit ι for $\boxtimes$ is suppressed in the discussion (and notation) since most of the constructions in this paper carry through without the assumption that $\boxtimes$ is unital. However, most interesting/familiar examples do involve a unital external product. See also ✠REMARK and [19, REMARK].

Let S be a graphical species in E. For any finite set X with distinct elements $x \neq y$, let $(S_X)^{x\ddagger y}$ denote the equaliser

$$S_X^{x\ddagger y} \longrightarrow S_X \underset{S(ch_y\circ\tau)}{\overset{S(ch_x)}{\rightrightarrows}} S_\S. \tag{5.10}$$

Since $\tau \colon \S \to \S$ is an involution in $\mathbf{B}^\S$, $S_X^{x\ddagger y} = S_X^{y\ddagger x}$ for all finite sets X and all pairs of distinct elements x and y in X. More generally, for any finite set Z with distinct elements $x, y \in Z$, and any morphism $f \colon E \to S_Z$ in E, $E^{x\ddagger y} = E^{y\ddagger x}$ denotes the pullback of f along the universal map $S_Z^{x\ddagger y} \to S_Z$. In particular if $\boxtimes$ is an external product on S, then, for all X, Y and distinct $x, y \in X \amalg Y$, there is a well defined object $(S_X \boxtimes S_Y)^{x\ddagger y}$ in E.

In general, given distinct elements $x_1, y_1, \dots, x_k, y_k$ of S_Z, and a morphism $f \colon E \to Z$ in E, then $E^{x_1\ddagger y_1,\dots,x_k\ddagger y_k} \overset{\text{def}}{=} (E^{x_1\ddagger y_1,\dots,x_{k-1}\ddagger y_{k-1}})^{x_k\ddagger y_k}$ is the obvious limit.

Invariance of $E^{x_1\ddagger y_1,\dots,x_k\ddagger y_k}$ under permutations $(x_i,y_i)\mapsto(x_{\sigma i},y_{\sigma i})$, $\sigma\in Aut(\mathbf{k})$ follows from invariance of the defining morphisms

$$E^{x_1\ddagger y_1,\dots,x_k\ddagger y_k} \longrightarrow S_Z^{x_1\ddagger y_1,\dots,x_k\ddagger y_k} \longrightarrow S_{Z-\{x_1,y_1,\dots x_k,y_k\}}.$$

Definition 5.11. *A* (graphical species) contraction ζ *on* S *is a family of maps* $\zeta_X^{x\ddagger y}\colon S_X^{x\ddagger y}\to S_{X\setminus\{x,y\}}$ *defined for each finite set* X *and pair of distinct elements* $x,y\in X$*.*

The contraction ζ *is equivariant with respect to the* $\mathbf{B}$ *action on* S*: If* $\sigma\colon X\setminus\{x,y\}\xrightarrow{\cong}Z\setminus\{w,z\}$ *be the restriction of a bijection* $\hat\sigma\colon X\xrightarrow{\cong}Z$ *with* $\hat\sigma(x)=w$ *and* $\hat\sigma(y)=z$*. Then*

$$S(\sigma)\circ\zeta_Z^{w\ddagger z}=\zeta_X^{x\ddagger y}\circ S(\hat\sigma)\colon S_Z^{w\ddagger z}\to S_{X\setminus\{x,y\}}. \tag{5.12}$$

If ζ is a contraction on S, then by (5.12), $\zeta_X^{x\ddagger y}=\zeta_X^{y\ddagger x}$ for all finite sets X and all pairs of distinct elements $x,y\in X$.

Let $\sigma_{\mathbf{2}}$ be the unique non-identity involution on $\mathbf{2}$.

Definition 5.13. *A morphism* $\epsilon\in\mathsf{E}(S_\S,S_{\mathbf{2}})$ *is unit-like for* S *if*

$$\epsilon\circ\omega=S(\sigma_{\mathbf{2}})\circ\epsilon\colon S_\S\to S_{\mathbf{2}},\ \text{and}\ \ S(ch_1)\circ\epsilon=id_{S_\S}. \tag{5.14}$$

So a morphism ϵ that is unit-like for S is, by definition, a monomorphism.

Definition 5.15. *A* non-unital circuit operad *in* E *is a graphical species* S *in* E*, equipped with an external product* $\boxtimes$ *and a contraction* ζ*, satisfying the following three axioms, illustrated in Figures 8, 9:*

(C1) External product is associative.
For all finite sets X,Y,Z*,*

$$(S_X\boxtimes S_Y)\boxtimes S_Z=S_X\boxtimes(S_Y\boxtimes S_Z).$$

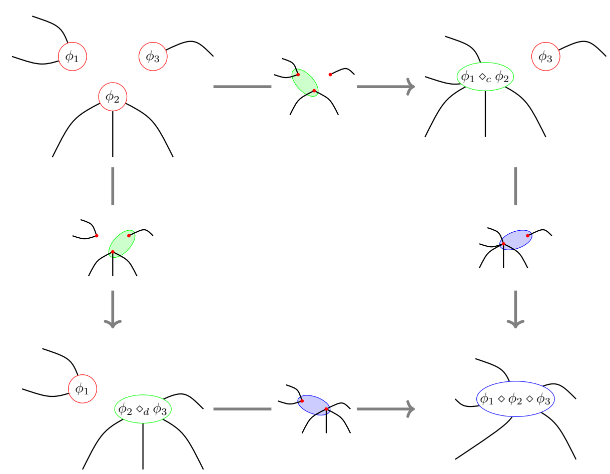


FIGURE 8. Axiom (C1).

(C2) Order of contraction does not matter.
For all finite sets X *with distinct elements* w,x,y,z*, the following square commutes:*

$$\begin{tikzcd}
S_X^{w\ddagger x,y\ddagger z} \arrow[r, "\zeta^{w\ddagger x}"] \arrow[d, "\zeta^{y\ddagger z}"'] \arrow[dr, phantom, "="] & S_{X\setminus\{w,x\}}^{y\ddagger z} \arrow[d, "\zeta^{y\ddagger z}"] \\
S_{X\setminus\{y,z\}}^{w\ddagger x} \arrow[r, "\zeta^{w\ddagger x}"'] & S_{X\setminus\{w,x,y,z\}}
\end{tikzcd}.$$

(C3) *For all finite sets X and Y and distinct elements $x_1, x_2 \in X$, the following square commutes:*

$$\begin{tikzcd} S_X^{x_1 \ddagger x_2} \times S_Y \arrow[r, "\boxtimes"] \arrow[d, "\zeta^{x_1 \ddagger x_2}"'] \arrow[dr, phantom, "="] & (S_X \boxtimes S_Y)^{x_1 \ddagger x_2} \arrow[d, "\zeta^{x_1 \ddagger x_2}"] \\ S_{X \setminus \{x_1, x_2\}} \times S_Y \arrow[r, "\boxtimes"'] & S_{X \setminus \{x_1, x_2\}} \boxtimes S_Y \end{tikzcd} .$$

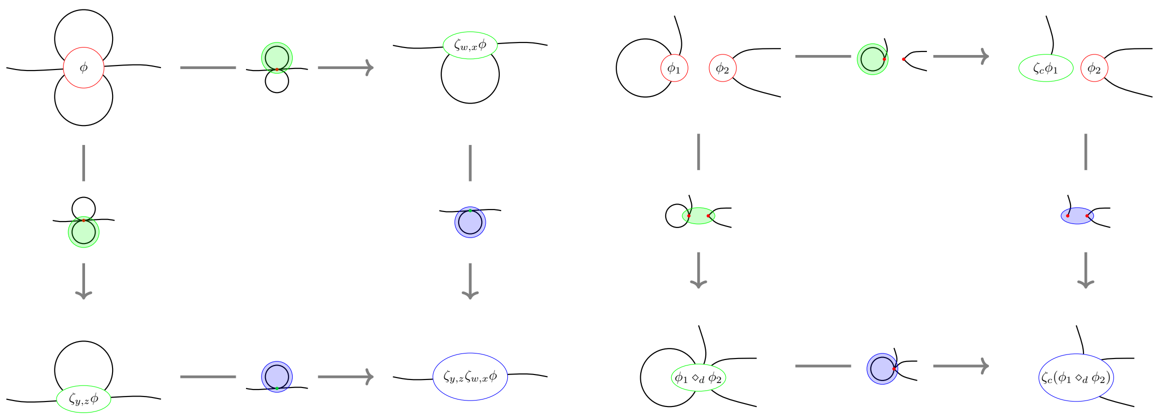


FIGURE 9. Left: axiom (C2), and right: axiom (C3).

Morphisms of non-unital circuit operads are morphisms of the underlying graphical species that preserve the external multiplication and contraction. The category of non-unital circuit operads is denoted CO^-.

A (unital) circuit operad *is given by a non-unital circuit operad $(S, \boxtimes, \zeta)$ together with a unit-like morphism $\epsilon \colon S_\S \to S_\mathbf{2}$, such that for all finite sets X and all $x \in X$, the composite*

(5.16)

$$S_X \xrightarrow{(id \times ch_x) \circ \Delta} (S_X \times S_\S)^{x \ddagger 2} \xrightarrow{id \times \epsilon} (S_X \times S_\mathbf{2})^{x \ddagger 2} \xrightarrow{\boxtimes} (S_X \boxtimes S_\mathbf{2})^{x \ddagger 2} \xrightarrow{\zeta^{x \ddagger 2}_{X \amalg \mathbf{2}}} S_X$$

is the identity on S_X. (Here $\Delta \colon S_X \to S_X \times S_X$ is the diagonal and the last map makes use of the canonical isomorphism $(X \amalg \{1\}) \setminus \{x\} \cong X$.)

Objects of the category CO are circuit operads, and morphisms are morphisms in CO^- that respect the unit.

Example 5.17. Recall Examples 3.37, 4.12 and 5.7. Monochrome directed circuit operads in Set are equivalent to wheeled PROPs. (See also [9].)

By the following proposition, that follows immediately from Definition 5.15 and Proposition 4.9, Set-valued circuit algebras describe circuit operads with the obvious product and contraction:

Proposition 5.18. *There is a canonical faithful functor $\mathsf{CA} \to \mathsf{CO}$.*

In Section 9.2, we shall see that the converse is also true.

5.3. **Modular operads.** Modular operads in Set were the subject of [25]. This section builds on that work. Modular operads in an arbitrary category E with finite limits are defined, and it is shown that circuit operads in E canonically admit a modular operad structure. This will be used in Sections 6-9.2, to construct the composite monad for circuit operads, building on a generalisation of the monad for Set-valued modular operads in [25].

Let S be a graphical species in a category E with finite limits. For finite sets X and Y with elements $x \in X$ and $y \in Y$, let $(S_X \times S_Y)^{x\ddagger y} \to S_X \times S_Y$ denote the pullback

$$\begin{tikzcd}
(S_X \times S_Y)^{x\ddagger y} \arrow[r] \arrow[d] & S_X \arrow[d, "S(ch_x)"] \\
S_Y \arrow[r, "S(ch_y \circ \tau)"'] & S_\S.
\end{tikzcd}$$

And, more generally, given distinct elements $x_1, \dots, x_k \in X$ and $y_1, \dots, y_k \in Y$, $(S_X \times S_Y)^{x_1\ddagger y_1, \dots, x_k \ddagger y_k}$ be the limit of the collection $S_X \xrightarrow{S(ch_{x_i})} S_\S \xleftarrow{S(ch_{y_i} \circ \tau)} S_Y$, $1 \leq i \leq k$ of spans.

Definition 5.19. *A* multiplication $\diamond$ *on S is a family of morphisms*

$$\diamond^{x\ddagger y}_{X,Y} \colon (S_X \times S_Y)^{x\ddagger y} \to S_{(X \amalg Y) \setminus \{x,y\}},$$

in E defined for all pairs of finite sets X and Y, with elements x and y, and such that

(1) the obvious diagram

$$\begin{tikzcd}
(S_X \times S_Y)^{x\ddagger y} \arrow[dr, "\diamond^{x\ddagger y}_{X,Y}"] \arrow[dd, "\cong"'] & \\
& S_{(X \amalg Y)\setminus\{x,y\}} \\
(S_Y \times S_X)^{y\ddagger x} \arrow[ur, "\diamond^{y\ddagger x}_{Y,X}"'] &
\end{tikzcd}$$

commutes in E,

(2) $\diamond$ is equivariant with respect to the $\mathbf{B}$-action on S: if $\sigma \colon X \setminus \{x\} \xrightarrow{\cong} W \setminus \{w\}$ and $\rho \colon Y \setminus \{y\} \xrightarrow{\cong} Z \setminus \{z\}$ restrict bijections $\hat{\sigma} \colon X \xrightarrow{\cong} W$ and $\hat{\rho} \colon Y \xrightarrow{\cong} Z$ such that $\hat{\sigma}(x) = w$ and $\hat{\rho}(y) = z$, then

$$S(\sigma \sqcup \rho) \circ \diamond^{w\ddagger z}_{W,Z} = \diamond^{x\ddagger y}_{X,Y} \circ S(\hat{\sigma} \sqcup \hat{\rho}),$$

(where $\hat{\sigma} \sqcup \hat{\rho} \colon X \amalg Y \xrightarrow{\cong} W \amalg Z$ is the block-wise permutation).

A multiplication $\diamond$ on S is unital *if S there is a unit-like morphism $\epsilon \in \mathsf{E}(S_\S, S_{\mathbf{2}})$ such that, for all finite sets X and all $x \in X$, the composite*

$$S_X \xrightarrow{(id \times ch_x)\circ\Delta} S_X \times S_\S \xrightarrow{id \times \epsilon} S_X \times S_{\mathbf{2}} \xrightarrow{\diamond^{x\ddagger 2}_{X,\mathbf{2}}} S_X \tag{5.20}$$

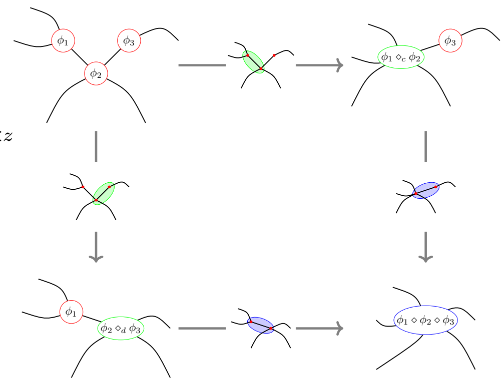
is the identity on S_X.

By (5.20) and condition (1) of Definition 5.19, if a multiplication $\diamond$ on S admits a unit ϵ, then it is unique.

Definition 5.21. *A modular operad $(S, \diamond, \zeta, \epsilon)$ is a graphical species equipped with unital multiplication $(\diamond, \epsilon)$ and a contraction ζ such that the following four coherence axioms are satisfied.*

(M1) Multiplication is associative.
For all finite set X, Y, Z and elements $x \in X$, $z \in Z$, and distinct $y_1, y_2 \in Y$,

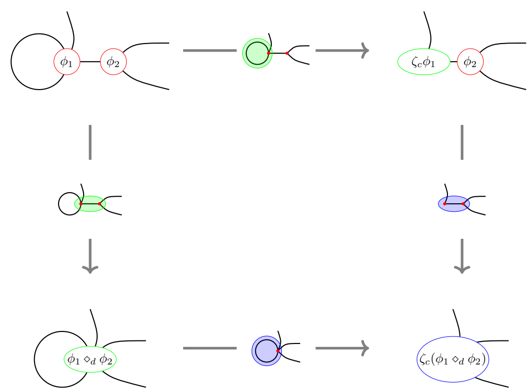
$$\begin{tikzcd}
(S_X \times S_Y \times S_Z)^{x\ddagger y_1, y_2 \ddagger z} \arrow[r, "\diamond^{x\ddagger y_1}_{X,Y} \times id_{S_Z}"] \arrow[d, "id_{S_X} \times \diamond^{y_2\ddagger z}_{Y,Z}"'] \arrow[dr, phantom, "="] & \left(S_{(X\amalg Y)\setminus\{x,y_1\}} \times S_Z\right)^{y_2\ddagger z} \arrow[d, "\diamond^{y_2\ddagger z}"] \\
\left(S_X \times S_{(Y\amalg Z)\setminus\{y_2,z\}}\right)^{x\ddagger y_1} \arrow[r, "\diamond^{x\ddagger y_1}"'] & S_{(X\amalg Y\amalg Z)\setminus\{x,y_1,y_2,z\}}.
\end{tikzcd}$$



(M2) Contraction satisfies (C2).

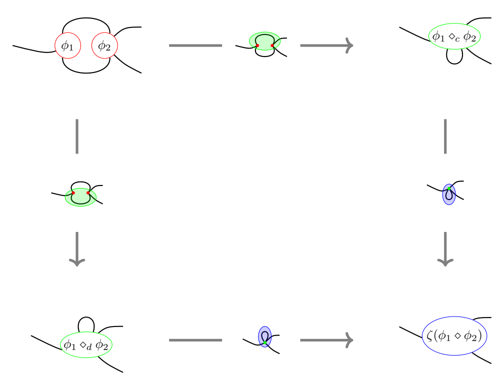
(M3) Multiplication and contraction commute.
For finite sets X and Y, mutually distinct elements x_1, x_2 and x_3 in X, and $y \in Y$, the following diagram commutes:

$$\begin{tikzcd}
(S_X^{x_1\ddagger x_2} \times S_Y)^{x_3\ddagger y} \arrow[r, "\zeta^{x_1\ddagger x_2} \times id"] \arrow[d, "\diamond^{x_3\ddagger y}"'] \arrow[dr, phantom, "="] & \left(S_{X\setminus\{x_1,x_2\}} \times S_Y\right)^{x_3\ddagger y} \arrow[d, "\diamond^{x_3\ddagger y}"] \\
S^{x_1\ddagger x_2}_{(X\amalg Y)\setminus\{x_3,y\}} \arrow[r, "\zeta^{x_1\ddagger x_2}"'] & S_{(X\amalg Y)\setminus\{x_1,x_2,x_3,y\}}.
\end{tikzcd}$$



(M4) 'Parallel multiplication' of pairs.
For finite sets X and Y, and distinct elements $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, the following digram commutes:

$$\begin{tikzcd}
(S_X \times S_Y)^{x_1\ddagger y_1, x_2\ddagger y_2} \arrow[r, "\diamond^{x_1\ddagger y_1}"] \arrow[d, "\diamond^{x_2\ddagger y_2}"'] \arrow[dr, phantom, "="] & S^{x_2\ddagger y_2}_{(X\amalg Y)\setminus\{x_1,y_1\}} \arrow[d, "\zeta^{x_2\ddagger y_2}"] \\
S^{x_1\ddagger y_1}_{(X\amalg Y)\setminus\{x_2,y_2\}} \arrow[r, "\zeta^{x_1\ddagger y_1}"'] & S_{(X\amalg Y)\setminus\{x_1,x_2,y_1,y_2\}}.
\end{tikzcd}$$



The category of (non-unital) modular operads in E *is denoted by (*$\mathsf{MO}^-_{\mathsf{E}}$*)* MO_{E}.

Circuit operads carry a canonical modular operad structure. For, given a circuit operad $(S, \boxtimes, \zeta, \epsilon)$ in E, the composition

$$\diamond^{x\ddagger y}_{X,Y} \colon\ (S_X \times S_Y)^{x\ddagger y} \xrightarrow{\ \boxtimes\ } (S_X \boxtimes S_Y)^{x\ddagger y} \xrightarrow{\ \zeta^{x\ddagger y}\ } S_{(X\amalg Y)\setminus\{x,y\}} \tag{5.22}$$

defines an equivariant multiplication defined, for all finite sets X and Y and all $x \in X$, and $y \in Y$.

It follows immediately from the definition of $\diamond$ and (C1)-(C3) for $(S, \boxtimes, \zeta)$, that $(S, \diamond, \zeta)$ satisfies the modular operad axioms (M1)-(M4). And, if ϵ is a unit for $(S, \boxtimes, \zeta)$, then ϵ satisfies (5.16) and hence is a unit for $\diamond$.

In fact, we have the following proposition:

Proposition 5.23. *There are canonical monadic adjunctions*

(5.24)
$$\begin{tikzcd} \mathsf{CO}_{\mathsf{E}}^{-} \arrow[r, shift left] \arrow[d, shift right] & \mathsf{CO}_{\mathsf{E}} \arrow[l, shift left] \arrow[l, phantom, "\top"] \arrow[d, shift right] \\ \mathsf{MO}_{\mathsf{E}}^{-} \arrow[r, shift left] \arrow[u, shift right] \arrow[u, phantom, "\vdash"] & \mathsf{MO}_{\mathsf{E}} \arrow[l, shift left] \arrow[l, phantom, "\top"] \arrow[u, shift right] \arrow[u, phantom, "\vdash"] \end{tikzcd}$$

between the categories of (non-unital) modular and circuit operads in E.

Proof. The result will follow from Proposition 7.15 and Lemma 9.4, together with the classical theory of distributive laws [2]. □

The right adjoints of the pairs $\mathsf{MO}_{\mathsf{E}}(\mathsf{MO}_{\mathsf{E}}^{-}) \rightleftarrows \mathsf{CO}_{\mathsf{E}}(\mathsf{CO}_{\mathsf{E}}^{-})$ ($\mathsf{MO}_{\mathsf{E}}^{-} \rightleftarrows \mathsf{CO}_{\mathsf{E}}^{-}$) are induced by the forgetful functor $(S, \boxtimes, \zeta) \mapsto (S, \diamond, \zeta)$ where the multiplication $\diamond$ is defined by suitable compositions $\zeta \circ \boxtimes$ as in (5.22). (See also Example 3.24.)

The left adjoint is induced by the free graded monoid monad. If S is a graphical species, then there is a graphical species LS defined by

$$LS_X = \mathrm{colim}_{(Y,f) \in \mathrm{Core}(X/\mathsf{Set_f})} \coprod_{y \in Y} S_{f^{-1}(y)}$$

where $\mathrm{Core}(X/\mathsf{Set_f})$ is the maximal subgroupoid of the slice category $X/\mathsf{Set_f}$ whose objects are morphisms of finite sets $f \colon X \to Y$, and whose morphisms $(Y, f) \to (Y', f')$ are isomorphisms $g \cong Y \to Y'$ such that $gf = f'$.

The free monoid structure induces the external product on LS, and, if $(S, \diamond, \zeta)$ is a (non-unital) modular operad, then the modular operad axioms imply that the pair of operations ζ and $\diamond$, define a contraction on LS. (See Section 9.2.)

Example 5.25. Wheeled PROPs have been described in [23, 24] and have applications in geometry and deformation theory.

By [9] and Example 4.12, wheeled PROPs (in Set) are equivalent to circuit algebras and hence, by Proposition 5.18, they describe circuit operads. The image of a wheeled PROP under the forgetful functor $\mathsf{CO} \to \mathsf{MO}$ is its underlying *wheeled properad* (see [12, 31]).

6. Background on graphs

In [25], the combinatorics of modular operads have been fully described in terms of a category Ξ of graphs [25, Section 8], and the remainder of this paper is concerned with an analogous construction for circuit operads.

This section reviews the definition of the the category etGr of graphs and étale morphisms – introduced in introduced in [15], and used in [19, 13, 14, 25] – and describes how they are related to wiring diagrams. A large number of examples are included to aid intuition. But the interested reader is referred to [25, Sections 3 & 4] for more details and explicit proofs related to the graphical formalism.

6.1. **The étale category of graphs.** A monochrome wiring diagram without closed components describes a graph as follows:

Let $(m_1, \dots, m_k; n) \in \mathsf{list}(\mathbb{N}) \times \mathbb{N}$ and let $\overline{f} \in WD(\mathbf{m}_1, \dots, \mathbf{m}_k; \mathbf{n})$ be a monochrome wiring diagram with underlying Brauer diagram $f = (\tau_f, 0) \in \mathsf{BD}(\sum_{i=1}^k m_i, n)$ where τ_f is a perfect matching on the set

$\partial f = \mathbf{n} \amalg (\coprod_{i=1}^k \mathbf{m}_i)$. Hence $\overline{f}$ is equivalently written as

$$\tau_f \circlearrowright \partial f \xleftarrow{\text{inc}} \coprod_{i=1}^k \mathbf{m}_i \xrightarrow{p} \mathbf{k} \tag{6.1}$$

where $p\colon \coprod_{i=1}^k \mathbf{m}_i \mathbf{k}$ is the canonical projection.

By forgetting the orderings on $\mathbf{k}$ and each $\mathbf{m}_i$, $1 \leq i \leq k$, we obtain the definition, originally due to Joyal and Kock [15], of a (Feynman) graph.

Definition 6.2. *A* graph $\mathcal{G}$ *is a diagram*

$$\mathcal{G} = \tau \circlearrowright E \xleftarrow{s} H \xrightarrow{t} V \tag{6.3}$$

of finite sets such that $s\colon H \to E$ is injective and $\tau\colon E \to E$ is an involution without fixed points.

Elements of V are vertices of $\mathcal{G}$ and elements of E are called *edges* of $\mathcal{G}$. In the terminology above, the set $\tilde{E}$ of τ-orbits in E, where $\tilde{e} \in \tilde{E}$ is the orbit of an edge $e \in E$, is the set of connections in $\mathcal{G}$. The set H of *half-edges* of $\mathcal{G}$ and the maps (s, t) encode the partial map $E \rightharpoonup V$ describing the incidence for the graph.

An *inner edge* of $\mathcal{G}$ is an element $e \in E$ such that $e \in im(s)$ and $\tau e \in im(s)$. The set $E_\bullet \subset E$ of inner edges of $\mathcal{G}$ is the maximal subset of $im(s) \subset E$ that is closed under τ. The set of *inner τ-orbits* $\tilde{e} \in \widetilde{E}$ with $e \in E_\bullet$ is denoted by $\widetilde{E_\bullet}$.

Elements of the set $E_0 = E \setminus im(s)$ are *ports* of $\mathcal{G}$.

A graph $\mathcal{G}$ that is obtained, as in (6.1), as the graph of a wiring diagram in WD is called *ordered.*

Remark 6.4. By the comparison with wiring diagrams and Example 3.16, a graph is a rule for gluing components of 1-dimensional manifolds together along their boundaries. So, a graph $\mathcal{G}$ may be realised geometrically by a one-dimensional space $|\mathcal{G}|$ obtained by taking the discrete space $\{*_v\}_{v \in V}$, for each $e \in E$, a copy $[0, \frac{1}{2}]_e$ of the interval $[0, \frac{1}{2}]$ subject to the identifications

- $0_{s(h)} \sim *_{t(h)}$ for $h \in H$,
- $(\frac{1}{2})_e \sim (\frac{1}{2})_{\tau e}$ for all $e \in E$.

Example 6.5. As in Example 3.8, let $*$ be the unique matching on a two element set. Recall that the pair $(*, 0)$ underlies Brauer diagrams $id_1 \in \mathsf{BD}(1,1)$, $\cap = \lceil id_1 \rceil \in \mathsf{BD}(0,2)$ and $\cup = \lfloor id_1 \rfloor \in \mathsf{BD}(2,0)$, and hence also wiring diagrams $\overline{id_1} \in WD(\mathbf{1};\mathbf{1})$, $\overline{\cap} \in WD(I_{\{*\}};\mathbf{2})$ and $\overline{\cup}_{(\mathbf{2};\mathbf{0})} \in WD\mathbf{2};\mathbf{0})$ and $\overline{\cup}_{(\mathbf{1},\mathbf{1};\mathbf{0})} \in WD(1,1;0)$. By (6.1), these describe the following graphs, each of which has one edge orbit:

- for $\overline{id_1} \in WD(\mathbf{1};\mathbf{1})$, the corresponding graph is the '1-corolla'

  $$\mathcal{C}_\mathbf{1} \overset{\text{def}}{=} \quad \circlearrowright \mathbf{2} \longleftarrow \mathbf{1} \longrightarrow \mathbf{1}$$

  with one vertex, no inner edges, and one port (see Figure 10(a));
- for $\overline{\cap} \in WD(I_{\{*\}};\mathbf{2})$, the corresponding graph is the *stick graph* $(\shortmid)$

  $$(\shortmid) \overset{\text{def}}{=} \quad \circlearrowright \mathbf{2} \longleftarrow \mathbf{0} \longrightarrow \mathbf{0}$$

  with no vertices, no inner edges and two ports (see Figure 10(b));
- for $\overline{\cup}_{(\mathbf{2};\mathbf{0})} \in WD(\mathbf{2};\mathbf{0})$, the corresponding graph is the *wheel graph* $\mathcal{W}$

  $$\mathcal{W} \overset{\text{def}}{=} \quad \circlearrowright \mathbf{2} \xleftarrow{id} \mathbf{2} \twoheadrightarrow \mathbf{1}$$

  with one vertex, one inner edge and no ports (see Figure 10(c));

- the graph corresponding to $\overline{\mho}_{(\mathbf{1},\mathbf{1};\mathbf{0})} \in WD(\mathbf{1},\mathbf{1};\mathbf{0})$ has one vertex, one inner edge and no ports (Figure 10(d)):

$$\begin{tikzcd} \mathbf{2} \arrow[loop left] & \mathbf{2} \arrow[l, "id"'] \arrow[r, "id"] & \mathbf{2} \end{tikzcd}.$$

The stick and wheel graphs $(\shortmid)$ and $\mathcal{W}$ are particularly important in what follows.

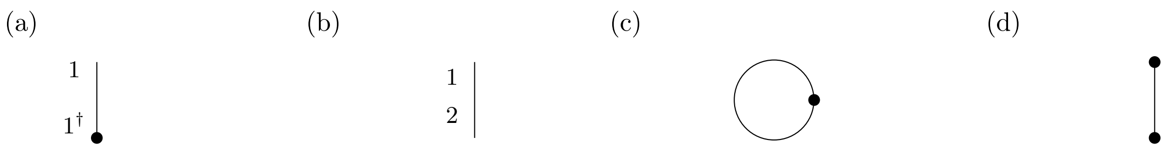


FIGURE 10. (Realisations of) graphs with one edge orbit: (a) the 1-corolla $\mathcal{C}_{\mathbf{1}}$; (b) the stick graph $(\shortmid)$; (c) the wheel graph $\mathcal{W}$ consists of a single inner edge with one end vertex; (d) a single inner edge with distinct end vertices.

Definition 6.6. *A* stick component *of a graph $\mathcal{G}$ is a pair $\{e, \tau e\}$ of edges of $\mathcal{G}$ such that e and τe are both ports.*

For any set X, let $X^\dagger \cong X$ denote its formal involution.

Example 6.7. (See also Figure 10(d).) The X-*corolla* $\mathcal{C}_X$ associated to a finite set X has the form

$$\mathcal{C}_X : \qquad \begin{tikzcd} X \amalg X^\dagger \arrow[loop left, "\dagger"] & X^\dagger \arrow[l, hook', "\text{inc}"'] \arrow[r] & \{*\}. \end{tikzcd}$$

Let $\mathcal{G}$ be a graph with vertex and edge sets V and E respectively. For each vertex v, define ${}^{H}\!/_{v} \overset{\text{def}}{=} t^{-1}(v) \subset H$ to be the fibre of t at v.

Definition 6.8. *Edges in the set* ${}^{E}\!/_{v} \overset{\text{def}}{=} s({}^{H}\!/_{v}) \subset E$ *are said to be* incident on v.

The map $|\cdot|\colon V \to \mathbb{N}$, $v \mapsto |v| \overset{\text{def}}{=} |{}^{H}\!/_{v}|$, *defines the* valency *of v and $V_n \subset V$ is the set of* n-valent *vertices of $\mathcal{G}$. A* bivalent graph *is a graph $\mathcal{G}$ with $V = V_2$.*

A vertex v is bivalent *if $|v| = 2$. An* isolated vertex *of $\mathcal{G}$ is a vertex $v \in V(\mathcal{G})$ such that $|v| = 0$.*

Bivalent and isolated vertices are particularly important in Section 8.3.

Vertex valency also induces an $\mathbb{N}$-grading on the edge set E (and half-edge set H) of $\mathcal{G}$: For $n \geq 1$, define $H_n \overset{\text{def}}{=} t^{-1}(V_n)$ and $E_n \overset{\text{def}}{=} s(H_n)$. Since $s(H) = E \setminus E_0 = \coprod_{n \geq 1} E_n$,

$$E = \coprod_{n \in \mathbb{N}} E_n.$$

Example 6.9. The stick graph $(\shortmid)$ (Example 6.5) has $E_0(\shortmid) = E(\shortmid) = \mathbf{2}$, and $E_\bullet(\shortmid) = \emptyset$. Conversely, the wheel graph $\mathcal{W}$ has $E_0(\mathcal{W}) = \emptyset$ and $E_\bullet(W) = E_2(\mathcal{W}) = E(\mathcal{W}) \cong \mathbf{2}$.

For all finite sets X, the X-corolla $\mathcal{C}_X$ (see Example 6.7) with vertex $*$ has set of ports $E_0(\mathcal{C}_X) = X$ and ${}^{E}\!/_{*} = X^\dagger$. If $|X| = k$, then $|*| = k$, so $V = V_k$, and $E = E_k \amalg E_0$ with $E_i \cong X$ for $i = 0, k$.

Definition 6.10. *An* étale morphism $f\colon \mathcal{G} \to \mathcal{G}'$ *of graphs is a commuting diagram of finite sets*

$$\text{(6.11)} \qquad \begin{tikzcd} \mathcal{G} \arrow[d] & E \arrow[r, leftrightarrow, "\tau"] \arrow[d] & E \arrow[d] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & V \arrow[d] \\ \mathcal{G}' & E' \arrow[r, leftrightarrow, "\tau'"'] & E' & H' \arrow[l, "s'"] \arrow[r, "t'"'] & V' \end{tikzcd}$$

such that the right-hand square is a pullback.

The category of graphs and étale morphisms is denoted by etGr.

It is straightforward to verify that the right hand square in (6.11) is a pullback if and only if, for all $v \in V$, the map ${}^{E}/v \to {}^{E'}/f(v)$ induced by the restriction of the map $V \to V'$ is bijective. In other words, étale morphisms describe local isomorphisms of graphs (as the name suggests).

Example 6.12. For any graph $\mathcal{G}$ with edge set E, there is a canonical (up to unique isomorphism) bijection $\mathsf{etGr}((), \mathcal{G}) \cong E$. The morphism $1 \mapsto e \in E$ in $\mathsf{etGr}((), \mathcal{G})$ that *chooses* $e \in E$ is denoted ch_e, or sometimes $ch_e^{\mathcal{G}}$.

Example 6.13. For any finite set X with $|X| = n$, and any graph $\mathcal{G}$, $\mathsf{etGr}(\mathcal{C}_X, \mathcal{G}) \cong \coprod_{v \in V_n} Aut(X)$.

Example 6.14. Observe that $\mathsf{etGr}((), ()) \cong \mathbf{B}^{\S}(\S, \S)$ canonically, so the maps $\S \mapsto (\shortmid)$ and $X \mapsto \mathcal{C}_X$ describes a full embedding $\mathbf{B}^{\S} \hookrightarrow \mathsf{etGr}$.

The restriction of the Yoneda embedding to the image of $\mathbf{B}^{\S}$ in GS induces a functor $\Upsilon \colon \mathsf{etGr} \to \mathsf{GS}$, given by

$$\Upsilon\mathcal{G} \colon \mathcal{C} \longmapsto \mathsf{etGr}(\mathcal{C}, \mathcal{G}) \text{ for } \mathcal{C} \in \mathbf{B}^{\S}.$$

Definition 6.15. *The category of elements* $\mathsf{el}(\Upsilon\mathcal{G}) \cong \mathbf{B}^{\S}/\mathcal{G}$ *is called the* category of elements of $\mathcal{G}$ *and denoted by* $\mathsf{el}(\mathcal{G})$.

Pointwise disjoint union defines a symmetric strict monoidal structure on etGr with unit given by the empty graph $\oslash \overset{\text{def}}{=} \emptyset \longleftarrow \emptyset \longrightarrow \emptyset$. Disjoint union commutes with geometric realisation: for all graphs $\mathcal{G}$ and $\mathcal{H}$, $|\mathcal{G} \amalg \mathcal{H}| = |\mathcal{G}| \amalg |\mathcal{H}|$.

Definition 6.16. *A graph is connected if it cannot be written as a disjoint union of non-empty graphs. A connected component $\mathcal{H}$ of $\mathcal{G}$ is a maximal connected subdiagram of $\mathcal{G}$.*

In particular, any graph $\mathcal{G}$ is the disjoint union of its connected components, and the inclusion $\mathcal{H} \hookrightarrow \mathcal{G}$ of a connected component $\mathcal{H}$ of $\mathcal{G}$ describes a pointwise injective étale morphism.

Example 6.17. Let X and Y be finite sets. Write $\mathcal{D}_{X,Y} \overset{\text{def}}{=} \mathcal{C}_X \amalg \mathcal{C}_Y$ for the disjoint union of X and Y-corollas:

$$\dagger \circlearrowright (X \amalg Y) \amalg (X \amalg Y)^{\dagger} \xleftarrow{\text{inc}} (X \amalg Y)^{\dagger} \xrightarrow{t_{X,Y}} \{*_X, *_Y\}$$

where the arrow $t_{X,Y}$ is the obvious projection $X^{\dagger} \to *_X$, $Y^{\dagger} \to *_Y$.

This has ports $X \amalg Y = E_0(\mathcal{C}_X) \amalg E_0(\mathcal{C}_Y)$ and no internal edges. The canonical inclusions $\iota_X : \mathcal{C}_X \to \mathcal{D}_{X,Y} \leftarrow \mathcal{C}_Y : \iota_Y$ are étale.

Example 6.18. Up to isomorphism, $(\shortmid)$ is the only connected graph with no vertices. As in [] (and following []) a *shrub* $\mathcal{S}$ is a disjoint union of stick graphs. Since $H(\mathcal{S}) = V(\mathcal{S}) = \emptyset$ for all shrubs $\mathcal{S}$, any commuting diagram of the form (6.11) where $\mathcal{G} = \mathcal{S}$ is a shrub satisfies the pullback condition and hence defines a morphism in etGr.

Example 6.19. The pair of parallel morphisms $id, \tau \colon (\shortmid) \rightrightarrows (\shortmid)$ given by the endomorphisms of $(\shortmid)$ in etGr has coequaliser $* \leftarrow \emptyset \to \emptyset$ in $\mathsf{psh}(\mathsf{D})$. Since this is not a graph, the morphisms $id_{(\shortmid)}$ and τ do not admit a coequaliser in etGr. This provided the key motivating example for [25].

The two endomorphisms $id_{\mathcal{W}}, \tau_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{W}$, viewed as morphisms of presheaves on D, have coequaliser $* \leftarrow * \to *$ which is terminal in $\mathsf{psh}(\mathsf{D})$ and is not a graph. In particular, etGr does not have a terminal object and is therefore not closed under (finite) limits.

As Example 6.19 shows, etGr does not, in general, admit finite colimits. However, if $\mathcal{G}$ is a graph and $e_1, e_2 \in E_0$ are ports of $\mathcal{G}$ with $e_1 \neq \tau e_2$, then the colimit $\mathcal{G}^{e_1 \ddagger e_2}$ of the parallel morphisms

$$ch_{e_1}, ch_{e_2} \circ \tau \colon (\shortmid) \rightrightarrows \mathcal{G} \tag{6.20}$$

is the graph described by identifying $e_1 \sim \tau e_2$ and $e_2 \sim \tau e_1$ in the diagram (6.3) defining $\mathcal{G}$.

Example 6.21. The graph $\mathcal{C}_{\mathbf{2}}^{e\ddagger\tau e}$ obtained by identifying the two ports of the **2**-corolla is isomorphic to the wheel graph $\mathcal{W}$.

Example 6.22. More generally, for X a finite set with distinct elements x and y, the graph $\mathcal{C}_X^{x\ddagger y}$ has ports in $E_0 = X \setminus \{x, y\}$, one inner τ-orbit $\{x, y\}$ (bold-face in Figure 11), and one vertex v.

$$\tau \circlearrowright (X \setminus \{y\} \amalg (X \setminus \{x\})^\dagger) \xleftarrow{\ s\ } ((X \setminus \{y\})^\dagger \amalg \{x\}) \xrightarrow{\ t\ } \{v\}.$$

Graphs of the form $\mathcal{C}_X^{x\ddagger y}$ encode formal contractions in graphical species.

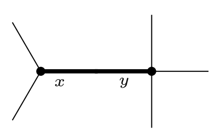
Example 6.23. For finite sets X and Y, recall that $\mathcal{D}_{X,Y}$ is the disjoint union $\mathcal{C}_X \amalg \mathcal{C}_Y$. Let $x \in X$, and $y \in Y$.

By identifying the edges $x \sim \tau y, y \sim \tau x$, we obtain a graph $\mathcal{D}_{X,Y}^{x\ddagger y} \overset{\text{def}}{=} (\mathcal{C}_X \amalg \mathcal{C}_Y)^{x\ddagger y}$, that has two vertices and one inner edge orbit $\{x, y\}$, highlighted in bold-face in Figure 11.

$$\tau \circlearrowright ((X \amalg Y) \amalg (X \amalg Y)^\dagger \amalg \{x_0, y_0\}) \xleftarrow{\ s\ } ((X \amalg Y)^\dagger \amalg \{x_0, y_0\}) \xrightarrow{\ t\ } \{v_X, v_Y\}$$

with s the obvious inclusion and the involution τ described by $z \leftrightarrow z^\dagger$ for $z \in X \amalg Y$ and $x_0 \leftrightarrow y_0$. The map t is described by $t^{-1}(v_X) = X^\dagger \amalg \{y_0\}$ and $t^{-1}(v_Y) = Y^\dagger \amalg \{x_0\}$.

In the construction of modular operads, graphs of the form $\mathcal{D}_{X,Y}^{x\ddagger y}$ are used to encode formal multiplications in graphical species.



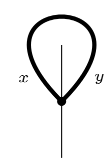


FIGURE 11. Realisations of $\mathcal{D}_{X,Y}^{x\ddagger y}$ and $\mathcal{C}_X^{x\ddagger y}$ for $X \cong \mathbf{4},\ Y \cong \mathbf{3}$.

Example 6.24. Since $H(\shortmid)$ is empty, all edges of $(\shortmid)$ are ports: $E(\shortmid) = E_0(\shortmid)$. Observe that, $ch_1^{(\shortmid)} \colon (\shortmid) \to (\shortmid)$, $1 \mapsto 1$ is the identity on $(\shortmid)$ and $ch_2^{(\shortmid)} \colon (\shortmid) \to (\shortmid)$, $1 \mapsto 2$ is precisely τ. Hence $ch_2^{(\shortmid)} \circ \tau = \tau^2 = id_{\shortmid}$, and therefore

$$(\shortmid)^{1\ddagger 2} = (\shortmid).$$

So, if ports $e_1, e_2 \in E_0(\mathcal{G})$ are such that $e_2 = \tau e_1$, then they form a stick component of $\mathcal{G}$ (Definition 6.6) and hence $\mathcal{G}^{e_1\ddagger e_2} \cong \mathcal{G}$.

It follows directly from [25, Proposition 4.8] that:

Lemma 6.25. *A morphism $f \in \mathsf{etGr}(\mathcal{G}, \mathcal{H})$ is a monomorphism if and only if it factors, up to unique isomorphism, as a colimit $\mathcal{G} \to \mathcal{G}^{e_1\ddagger e_1', \ldots, e_k\ddagger e_k'}$ of pairs of parallel maps $ch_{e_i}, ch_{e_i'} \circ \tau$ (where $e_1, e_1', \ldots, e_k, e_k'$ are distinct ports of $\mathcal{G}$), followed by an étale subgraph $\mathcal{G}^{e_1\ddagger e_1', \ldots, e_k\ddagger e_k'} \subset \mathcal{H}$.*

Example 6.26. For $k \geq 0$, the *line graph* $\mathcal{L}^k$ is the connected bivalent graph (illustrated in Figure 12) with ports in $E_0 = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\}$, and

- ordered set of edges $E(\mathcal{L}^k) = (l_j)_{j=0}^{2k+1}$ where $l_0 = 1_{\mathcal{L}^k} \in E_0$ and $l_{2k+1} = 2_{\mathcal{L}^k} \in E_0$, and the involution is given by $\tau(l_{2i}) = l_{2i+1}$, for $0 \leq i \leq k$,

- ordered set of k vertices $V(\mathcal{L}^k) = (v_i)_{i=1}^k$, such that $E/v_i = \{l_{2i-1}, l_{2_i}\}$ for $1 \leq i \leq k$.

So, $\mathcal{L}^k$ is described by a diagram of the form $\circlearrowright \mathbf{2} \amalg 2(\mathbf{k}) \longleftarrow 2(\mathbf{k}) \longrightarrow \mathbf{k}$.

The line graphs $(\mathcal{L}^k)_{k \in \mathbb{N}}$ may be defined inductively by gluing copies of $\mathcal{C}_\mathbf{2}$:

$$\begin{array}{lcl} \mathcal{L}^\mathbf{0} & = & (\shortmid) \\ \mathcal{L}^{k+1} & \cong & (\mathcal{L}^k \amalg \mathcal{C}_\mathbf{2})^{e_{\mathcal{L}^k} \ddagger e'_{\mathcal{C}_\mathbf{2}}}, \quad k \geq 0, \end{array}$$

(where $e_{\mathcal{L}^k}$ is a port of $\mathcal{L}^k$ and $e'_{\mathcal{C}_\mathbf{2}}$ is a port of $\mathcal{C}_\mathbf{2}$).

Example 6.27. For $m \geq 1$, the wheel graph $\mathcal{W}^m$ (illustrated in Figure 12) is the connected bivalent graph obtained as the coequaliser in $\mathsf{Grpsh_f(D)}$ of the morphisms $ch_{1_{\mathcal{L}^m}}, ch_{2_{\mathcal{L}^m}} \circ \tau \colon (\shortmid) \rightrightarrows \mathcal{L}^m$. So $\mathcal{W}^m$ has no ports and

- $2m$ cyclically ordered edges $E(\mathcal{W}^m) = (a_j)_{j=1}^{2m}$, such that the involution satisfies $\tau(a_{2i}) = a_{2i+1}$ for $0 \leq i \leq m$ (where $a_0 = a_{2m}$),
- m cyclically ordered vertices $V(\mathcal{W}^m) = (v_i)_{i=1}^m$, that $E/v_i = \{a_{2i-1}, a_{2i}\}$ for $1 \leq i \leq m$.

So $\mathcal{W}^m$ is described by a diagram of the form $\circlearrowright 2(\mathbf{m}) \longleftarrow 2(\mathbf{m}) \longrightarrow \mathbf{m}$.

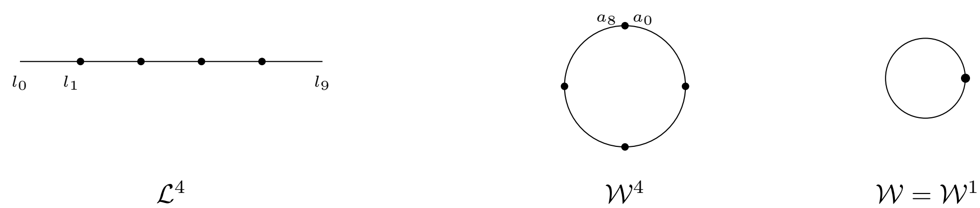


FIGURE 12. Line and wheel graphs.

By [], a connected bivalent graph is isomorphic to $\mathcal{L}^k$ or $\mathcal{W}^m$ for some $k \geq 0$ or $m \geq 1$.

Since the monoidal structure on etGr is induced by the cocartesian monoidal structure on diagrams of finite sets, colimits in etGr, when they exist, commute with $\amalg$. In particular, for any graphs $\mathcal{G}$ and $\mathcal{H}$ and distinct ports e_1, e_2 of $\mathcal{G}$, $\mathcal{G}^{e_1 \ddagger e_2} \amalg \mathcal{H} = (\mathcal{G} \amalg \mathcal{H})^{e_1 \ddagger e_2}$.

6.2. **The element category of a graph.** For every graph $\mathcal{G}$, there is a maximal shrub $\mathcal{S}(E)$ that is a strong subgraph of $\mathcal{G}$:

$$\begin{tikzcd} \mathcal{G} \arrow[d, "f"'] & E \arrow[r, leftrightarrow, "\tau"] \arrow[d, equal] & E \arrow[d, equal] & \emptyset \arrow[l] \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ \mathcal{G} & E \arrow[r, leftrightarrow, "\tau"'] & E & H' \arrow[l, "s"] \arrow[r, "t"'] & V \end{tikzcd}$$

Connected components of $\mathcal{S}(E)$ are indexed by the set of orbits $\tilde{E}$ of τ in E. For each such orbit $\tilde{e} \in \tilde{E}$, let $(\shortmid_{\tilde{e}})$ be the graph $\circlearrowright \{e, \tau e\} \longleftarrow \emptyset \longrightarrow \emptyset$. This is a strong connected subgraph of $\mathcal{G}$ under the canonical morphism $\iota_{\tilde{e}} \colon (\shortmid_{\tilde{e}}) \hookrightarrow \mathcal{G}$ at $\tilde{e}$ (for $\mathcal{G}$) induced by the inclusion $\{e, \tau e\} \hookrightarrow E$.

Recall that, for each $v \in V$, $E/v \overset{\text{def}}{=} s(t^{-1}(v))$ is the set of edges incident on v. Let $\mathbf{v} = (E/v)^\dagger$ denote its formal involution. Then the corolla $\mathcal{C}_\mathbf{v}$ is given by

$$\mathcal{C}_\mathbf{v} = \circlearrowright (E/v \amalg (E/v)^\dagger) \xleftarrow{\ s\ } H/v \xrightarrow{\ t\ } \{v\}.$$

The inclusion $E/v \hookrightarrow E$ induces a morphism $\iota_v^\mathcal{G}$ or $\iota_v \colon \mathcal{C}_\mathbf{v} \to \mathcal{G}$. Observe that, whenever there is an edge e such that e and τe are both incident on some vertex v, then ι_v is not injective on edges.

If ${}^{E}/_{v}$ is empty – so $\mathcal{C}_{\mathbf{v}}$ is an isolated vertex – then $\mathcal{C}_{\mathbf{v}} \hookrightarrow \mathcal{G}$ is a connected component of $\mathcal{G}$.

Definition 6.28. *An* (étale) neighbourhood *of a vertex $v \in V$ is an étale monomorphism $u \colon \mathcal{H} \to \mathcal{G}$ such that $\iota_v \colon \mathcal{C}_{\mathbf{v}} \to \mathcal{G}$ factors through u.*

Definition 6.29. *Let $\mathcal{G}$ be a graph. An* essential morphism *or* essential subgraph *of $\mathcal{G}$ is a morphism of the form $\iota_{\tilde{e}} \colon (\shortmid_{\tilde{e}}) \to \mathcal{G}$ $(\tilde{e} \in \widetilde{E})$ or $\iota_v \colon \mathcal{C}_{\mathbf{v}} \to \mathcal{G}$ $(v \in V)$.*

The essential category $\mathsf{es}(\mathcal{G})$ of $\mathcal{G}$ *is the full subcategory of* $\mathsf{etGr}/\mathcal{G}$ *on the essential subgraphs*

$$\iota_{\tilde{e}} \colon (\shortmid_{\tilde{e}}) \rightarrowtail \mathcal{G},\ \tilde{e} \in \tilde{E},\ \text{ and } \iota_v \colon \mathcal{C}_{\mathbf{v}} \rightarrowtail \mathcal{G},\ v \in V.$$

There is a canonical bijection between non-identity morphisms in $\mathsf{es}(\mathcal{G})$ and half-edges of $\mathcal{G}$, whereby $h = (e, v) \in H$ corresponds to a morphism $\delta_h \in \mathsf{es}(\mathcal{G})(\shortmid_{\tilde{e}}, \mathcal{C}_{\mathbf{v}})$: for if $h = (e, v) \in H$, then $s(h) = e$ is the unique element in the intersection $E(\shortmid_{\tilde{e}}) \cap E(\mathcal{C}_{\mathbf{v}}) \cap E$. So, δ_h is the unique morphism in $\mathsf{etGr}(\shortmid_{\tilde{e}}, \mathcal{C}_{\mathbf{v}})$ above $\mathcal{G}$ that fixes e. Since the objects of $\mathsf{es}(\mathcal{G})$ are indexed by $\tilde{E} \amalg V$, there are no other non-identity morphisms in $\mathsf{es}(\mathcal{G})$.

The following is [25, Lemma 4.17]:

Lemma 6.30. *Each graph $\mathcal{G}$ is canonically the colimit of the forgetful functor $\mathsf{es}(\mathcal{G}) \to \mathsf{etGr}$, $(\mathcal{C}, b) \mapsto \mathcal{C}$, and moreover $\mathsf{el}(\mathcal{G}) \simeq \mathsf{es}(\mathcal{G})$ canonically. Hence $\mathcal{G} = \mathrm{colim}_{(\mathcal{C},b)\in \mathsf{es}(\mathcal{G})} \mathcal{C} = \mathrm{colim}_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} \mathcal{C}$.*

In particular, if $\mathcal{G}$ has no stick components, then, for each $\tilde{e} \in \widetilde{E_\bullet}$, the canonical morphism $\iota_{\tilde{e}} : (\shortmid_{\tilde{e}}) \hookrightarrow \mathcal{G}$ factors in precisely two ways through the canonical morphism $\coprod_{v\in V} \mathcal{C}_{\mathbf{v}} \to \mathcal{G}$,

$$(\shortmid_{\tilde{e}}) \underset{(e,\tau e)\mapsto((\tau e)^\dagger,\tau e)}{\overset{(e,\tau e)\mapsto(e,e^\dagger)}{\rightrightarrows}} \textstyle\coprod_{v\in V} \mathcal{C}_{\mathbf{v}} \longrightarrow \mathcal{G}. \tag{6.31}$$

Hence, there exist parallel morphisms $\coprod_{\tilde{e}\in \widetilde{E_\bullet}} (\shortmid_{\tilde{e}}) \rightrightarrows \coprod_{v\in V} \mathcal{C}_{\mathbf{v}}$ such that the diagram

$$\textstyle\coprod_{\tilde{e}\in \widetilde{E_\bullet}} (\shortmid_{\tilde{e}}) \rightrightarrows \coprod_{v\in V} \mathcal{C}_{\mathbf{v}} \longrightarrow \mathcal{G} \tag{6.32}$$

describes a coequaliser in etGr. Of course the parallel morphisms in (6.32) is not unique (there are $2^{|\widetilde{E_\bullet}|}$ choices), but they are unique up to isomorphism.

6.3. **Gluing constructions and labelled graphs.** This section considers graph constructions analogous to operadic composition in WD.

Following [15]:

Definition 6.33. *Let $\mathcal{G}$ be a Feynman graph. A* $\mathcal{G}$-shaped graph of graphs *is a functor $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ (or $\mathbf{\Gamma}^{\mathcal{G}}$) such that*

$$\begin{aligned} \mathbf{\Gamma}(a) &= (\shortmid) && \text{for all } (\shortmid, a) \in \mathsf{el}(\mathcal{G}),\\ E_0(\mathbf{\Gamma}(b)) &= X && \text{for all } (\mathcal{C}_X, b) \in \mathsf{el}(\mathcal{G}), \end{aligned}$$

and, for all $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$ and all $x \in X_b$,

$$\mathbf{\Gamma}(ch_x) = ch_x^{\mathbf{\Gamma}(b)} \in \mathsf{etGr}(\shortmid, \mathbf{\Gamma}(b)).$$

A $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ is non-degenerate *if, for all $v \in V$, $\mathbf{\Gamma}(\iota_v)$ has no stick components. Otherwise, $\mathbf{\Gamma}$ is called* degenerate.

Informally, a non-degenerate $\mathcal{G}$-shaped graph of graphs is a rule for substituting graphs into vertices of $\mathcal{G}$ as in Figure 13.

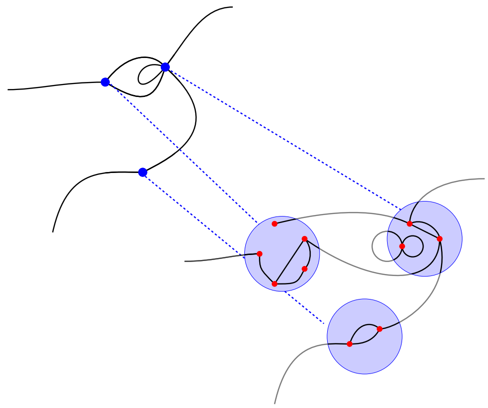


FIGURE 13. A $\mathcal{G}$ shaped graph of graphs $\mathbf{\Gamma}$ describes *graph substitution* in which each vertex v of $\mathcal{G}$ is replaced by a graph $\mathcal{G}_v$ according to a bijection $E_0(\mathcal{G}_v) \xrightarrow{\cong} (E/v)^\dagger$. When $\mathbf{\Gamma}$ is non-degenerate, taking its colimit corresponds to erasing the inner (blue) nesting.

Proposition 6.34. *A $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}\colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ admits a colimit $\mathbf{\Gamma}(\mathcal{G})$ in etGr.*

If $\mathcal{G}$ is a graph, and $\mathbf{\Gamma}$ is a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\mathbf{\Gamma}(\mathcal{G})$, then $\mathbf{\Gamma}$ induces an identity $E_0(\mathcal{G}) \xrightarrow{=} E_0(\mathbf{\Gamma}(\mathcal{G}))$, and, for each $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, the universal map $b_{\mathbf{\Gamma}}\colon \mathbf{\Gamma}(b) \to \mathbf{\Gamma}(\mathcal{G})$ is a monomorphism in etGr. In particular, $E(\mathbf{\Gamma}(\mathcal{G})) \cong E(\mathcal{G}) \amalg \coprod_{v\in V} E_\bullet(\mathbf{\Gamma}(\iota_v))$, and, there is a canonical surjective map $V(\mathbf{\Gamma}(\mathcal{G})) \twoheadrightarrow V(\mathcal{G})$ induced by the monomorphisms $\iota_{v\mathbf{\Gamma}}\colon \mathbf{\Gamma}(\iota_v) \to \mathbf{\Gamma}(\mathcal{G})$ for each vertex v of $\mathcal{G}$.

Proof. If $\mathbf{\Gamma}$ is non-degenerate, then $\mathbf{\Gamma}$ admits a colimit $\mathbf{\Gamma}(\mathcal{G})$ in etGr by [25, Proposition 5.16]. In case $\mathbf{\Gamma}$ is degenerate, the existence of a colimit in etGr follows from [25, Proposition 7.15].

The other statements follow directly from [25, Corollary 5.19]. □

Remark 6.35. Up to ordering, a graph $\mathcal{G}$ with n ports, k vertices, and such that each vertex $v_i \in V$ has $m_i = ||E/v_i||$ adjacent edges, describes a wiring diagram $\overline{g} \in WD(\mathbf{m}_1, \dots, \mathbf{m}_k; \mathbf{n})$ with no closed components.

And a $\mathcal{G}$-indexed graph of graphs $\mathbf{\Gamma}$ corresponds to a choice of k wiring diagrams $\overline{f}^i \in WD(\boldsymbol{l}_1, \dots, \boldsymbol{l}_{k_i}; \mathbf{m}_i)$ for $(l_1, \dots, l_{k_i}) \in \mathsf{list}\mathbb{N}$ and $1 \leq i \leq k$.

Non-degeneracy of $\mathbf{\Gamma}$ corresponds to the condition, discussed in Remark 3.18, that each morphism f^i of $\mathsf{BD}(l_1 + \cdots + l_{k_i}, m_i)$ does not contain caps. And, if $\mathbf{\Gamma}$ is non-degenerate, then its colimit $\mathbf{\Gamma}(\mathcal{G})$ corresponds to the composition $\overline{g} \odot (\overline{f}_i)_i$ in WD.

In both the non-degenerate and degenerate cases, the colimit of $\mathbf{\Gamma}$ agrees with the algebraic pushout of cospans (3.19), and Proposition 6.34 may also be proved using this construction.

Example 6.36. By Lemma 6.30, every graph $\mathcal{G}$ is the colimit of the (non-degenerate) *identity $\mathcal{G}$-shaped graph of graphs* $\mathbf{I}^{\mathcal{G}}$ given by the forgetful functor $\mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$, $(\mathcal{C}, b) \mapsto \mathcal{C}$. It follows from (5.9) in [25, Section 5.2] that, if $\mathcal{G}$ has no stick components, this is equivalent to the statement that $\mathcal{G}$ is the coequaliser of the canonical diagram

$$\mathcal{S}(E_\bullet) \rightrightarrows \coprod_{v\in V} \mathcal{C}_{\mathbf{v}} \xrightarrow{\coprod(\iota_v)} \mathcal{G}. \tag{6.37}$$

The identity $\mathcal{G}$-shaped graph of graphs corresponds, as in Remark 6.35 to a composition of wiring diagrams of the form $\overline{g} \circ (\overline{id}_{m_i})_{i=1}^k$.

Example 6.38. For any finite set X, a $\mathcal{C}_X$ shaped graph of graphs $\mathbf{\Gamma}$ is completely described by a pair $(\mathcal{G}, \rho)$ where $\mathcal{G}$ is a graph and $\rho : E_0(\mathcal{G}) \xrightarrow{\cong} X$ is a bijection, and the colimit $\mathbf{\Gamma}(\mathcal{C}_X)$ of $\mathbf{\Gamma}$ is $\mathcal{G}$.

In the sense of Remark 6.35, this corresponds to a composition of the form $\overline{id_n \circ g}$.

In particular, for all finite sets X with distinct elements $x, y \in X$, there is a non-degenerate $\mathcal{C}_{X \setminus \{x,y\}}$-shaped graph of graphs $\mathcal{C}_X^{x \ddagger y}$.

Example 6.39. Let X and Y be finite sets. There is a non-degenerate $\mathcal{C}_{X \amalg Y}$-shaped graph of graphs whose colimit is the disjoint union $\mathcal{C}_X \amalg \mathcal{C}_Y$.

Example 6.40. Let x be an element of a finite set X. And recall that $\mathcal{D}^{x\ddagger 2}_{X,\mathbf{2}}$ (Example 6.23) is defined as the coequaliser

$$ch_x, \tau ch_1 \colon (\shortmid) \rightrightarrows \mathcal{C}_X \amalg \mathcal{C}_{\mathbf{2}}.$$

Then, a (degenerate) $\mathcal{D}^{x\ddagger 2}_{X,\mathbf{2}}$-shaped graph of graphs of the form $\iota_{v_X} \mapsto (\mathcal{G}, \rho)$ and $\iota_{v_\mathbf{2}} \mapsto \shortmid_{\tilde{e}}$ has colimit isomorphic to $\mathcal{G}$.

Definition 6.41. *The category of* non-degenerate $\mathcal{G}$-shaped graphs of graphs $\mathsf{etGr}^{(\mathcal{G})}$ *is the full subcategory of non-degenerate $\mathcal{G}$-shaped graphs of graphs $\mathbf{\Gamma} : \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ in the category of functors $\mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ and natural transformations.*

Remark 6.42. By [25, Corollary 4.21] an étale morphism $f \colon \mathcal{G} \to \mathcal{G}'$ of connected graphs such that $f(E_0(\mathcal{G})) \subset E_0(\mathcal{G}')$ is always surjective. Moreover, if f restricts to an isomorphism of ports $E_0(\mathcal{G}) \cong E_0(\mathcal{G}')$, and $E_0(\mathcal{G})$ is non-empty, then f is an isomorphism.

It follows that, up to adjoining connected components of graphs with no ports, $\mathsf{etGr}^{(\mathcal{G})}$ is a groupoid, and, for all $\mathbf{\Gamma}^1$ and $\mathbf{\Gamma}^2$ in the same connected component of $\mathsf{etGr}^{(\mathcal{G})}$, there are graphs $\mathcal{H}^1$ and $\mathcal{H}^2$ with $E_0(\mathcal{H}^1) = E_0(\mathcal{H}^2) = \emptyset$ such that $\mathbf{\Gamma}^1(\mathcal{G}) \amalg \mathcal{H} \cong \mathbf{\Gamma}^2(\mathcal{G}) \amalg \mathcal{H}^2$.

Definition 6.43. *Let X be a finite set. A generalised* X-graph *(henceforth X-graph) is a non-degenerate $\mathcal{C}_X$-shaped graph of graphs. The groupoid of X-graphs and isomorphisms of X-graphs is denoted* $X\text{-}\mathsf{etGr}_{\mathrm{iso}} \cong \mathrm{Core}(\mathsf{etGr}^{(\mathcal{C}_X)})$.

More generally, for any graph $\mathcal{G}$ and any $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, $b\text{-}\mathsf{etGr}_{\mathrm{iso}}$ *denotes the groupoid* $\mathrm{Core}(\mathsf{etGr}^{(\mathcal{C})})$ *of non-degenerate $\mathcal{C}$-shaped graphs of graphs and isomorphisms.*

An X-graph $\mathcal{X}$ is admissible if it has no stick components.

So, an X-graph is a pair $\mathcal{X} = (\mathcal{G}, \rho)$, where $\mathcal{G}$ is a graph with no stick components and $\rho \colon E_0 \xrightarrow{\cong} X$ is a bijection of finite sets called an *X-labelling for $\mathcal{G}$*. Given X-graphs $\mathcal{X} = (\mathcal{G}, \rho)$ and $\mathcal{X}' = (\mathcal{G}', \rho')$, an *$X$-isomorphism* $\mathcal{X} \to \mathcal{X}'$ is an isomorphism $g \in \mathsf{CetGr}(\mathcal{G}, \mathcal{G}')$ that preserves the X-labelling: $\rho' \circ g_{E_0} = \rho \colon E_0 \to X$.

For any graph $\mathcal{G}$, a $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ is precisely a choice of X_b-graph $\mathcal{X}_b$ for each $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$.

7. Non-unital circuit operads

This section gives a short outline – closely following [15] and [25, Section 5] – of the construction of the non-unital circuit operad monad $\mathbb{T}^\times$ on GS_E. For $\mathsf{E} = \mathsf{Set}$, this corresponds to the monad described in [19].

From now on, it will be assumed that the category E has sufficient limits and colimits.

7.1. **Evaluating graphical species on graphs.** The fully faithful inclusion $\mathbf{B}^\S \hookrightarrow \mathsf{etGr}$ induces a fully faithful (by []) inclusion of presheaves $\mathsf{GS}_\mathsf{E} \hookrightarrow \mathsf{psh}_\mathsf{E}(\mathsf{etGr})$

$$S \longmapsto \left(\mathcal{G} \mapsto \lim_{(\mathcal{C},b) \in \mathsf{el}(\mathcal{G})} S(\mathcal{C})\right). \tag{7.1}$$

Henceforth, the same notation will be used for graphical species viewed as presheaves on $\mathbf{B}^{\S}$ and for their image as presheaves on etGr.

In fact (see [25, Section 4.4]), when $\mathsf{E} = \mathsf{Set}$, GS is equivalent to the category of sheaves for the canonical *étale topology* on etGr, and hence the embedding $\mathsf{etGr} \hookrightarrow \mathsf{GS}$, $\mathcal{G} \mapsto \Upsilon\mathcal{G}$ is fully faithful.

So, for all graphical species S in Set, and all graphs $\mathcal{G}$, there is a canonical isomorphism $S(\mathcal{G}) \cong \mathsf{GS}(\Upsilon\mathcal{G}, S)$.

Example 7.2. If S is a graphical species in Set, an element $\alpha \in S(\mathcal{G})$ is called an *S-structure on $\mathcal{G}$*. In this case, the category $\mathsf{etGr}(S)$ of *S-structured graphs*, has objects given by pairs $(\mathcal{G}, \alpha)$ with $\alpha \in S(\mathcal{G})$ and morphisms $\mathsf{etGr}(S)((\mathcal{G}, \alpha), (\mathcal{G}', \alpha'))$ are morphisms $f \in \mathsf{etGr}(\mathcal{G}, \mathcal{G}')$ such that $S(f)(\alpha') = \alpha$.

Example 7.3. Let S be a graphical species in E. Then, for any graphs $\mathcal{G}$ and $\mathcal{H}$,

$$S(\mathcal{G}) \times S(\mathcal{H}) = S(\mathcal{G} \amalg \mathcal{H})$$

by (7.1). In particular, for finite sets X and Y,

$$S_X \times S_Y = S(\mathcal{C}_X) \times S(\mathcal{C}_Y) = S(\mathcal{C}_X \amalg \mathcal{C}_Y). \tag{7.4}$$

Example 7.5. Let X be a finite set with distinct elements x and y. Recall from (5.10) that $S_X^{x \ddagger y} \in \mathsf{E}$ is the equaliser of the morphisms $S(ch_x), S(ch_y \circ \tau) \colon S_X \to S_{\S}$. Since the graph $\mathcal{C}_X^{x \ddagger y}$ is defined as the coequaliser of the morphisms $ch_x, ch_y \circ \tau \colon (\shortmid) \to \mathcal{C}_X$ (see Example 6.22), it follows from (7.1) that

$$S_X^{x \ddagger y} = S(\mathcal{C}_X^{x \ddagger y}). \tag{7.6}$$

So, a contraction ζ on a graphical species S in E is a collection of maps $S(\mathcal{C}_X^{x \ddagger y}) \to S_{X \setminus \{x,y\}}$ defined for each finite set X, and each pair x, y of distinct elements of X.

Let S be a graphical species in E, and, for some graph $\mathcal{G}$, $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ is a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\mathbf{\Gamma}(\mathcal{G})$. The composition $S \circ \mathbf{\Gamma}^{\mathrm{op}}$ defines a functor $S^{\mathbf{\Gamma}} \colon \mathsf{el}(\mathcal{G})^{\mathrm{op}} \to \mathsf{E}$ with colimit $S(\mathbf{\Gamma}(\mathcal{G}))$.

Example 7.7. If S is a graphical species in Set, then a $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$ and choice of S-structure $\alpha \in S(\mathbf{\Gamma}(\mathcal{G}))$ determines a functor $\mathsf{el}(\mathcal{G}) \to \mathsf{etGr}(S)$ by $b \mapsto S(b_{\mathbf{\Gamma}})(\alpha)$.

7.2. **A monad for non-unital circuit operads.** It is now straightforward to construct a monad on GS_{E} almost identical to the monad described in [19], whose algebras are non-unital circuit operads in E.

The *non-unital circuit operad endofunctor* $T^{\times} \colon \mathsf{GS} \to \mathsf{GS}$ is defined, on objects, by

$$\begin{array}{rcll} T^{\times} S_{\S} & = & S_{\S}, & \\ T^{\times} S_X & = & \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{X}) & \text{for all finite sets } X \end{array} \tag{7.8}$$

for all $S \in \mathsf{GS}_{\mathsf{E}}$.

The action of TS on isomorphisms in $\mathbf{B}^{\S}$ is the canonical one induced by relabellings of graph ports.

To see that TS extends to all morphisms in $\mathbf{B}^{\S}$ and therefore defines a graphical species in E, let $g \colon \mathcal{X}' \to \mathcal{X}'$ be a port-preserving isomorphism in X-$\mathsf{etGr}_{\mathrm{iso}}$, and for each $x \in X$, let $ch_x^{\mathcal{X}} \in \mathsf{CetGr}(\shortmid, \mathcal{G})$ is the map $ch_{\rho^{-1}(x)}$ defined by $1 \mapsto \rho^{-1}(x) \in E_0(\mathcal{G})$. Then,

$$S(g \circ ch_x^{\mathcal{X}}) = S(ch_x^{\mathcal{X}}) S(g) = S(ch_x^{\mathcal{X}'}),$$

and so the projections $T^{\times} S(ch_x) \colon T^{\times} S_X \to S_{\S}$ induced by $S(ch_x^{\mathcal{X}}) \colon S(\mathcal{X}) \to S(\shortmid) = S_{\S}$, are well-defined.

The assignment $S \mapsto TS$ is clearly natural on $S \in \mathsf{GS}_{\mathsf{E}}$ and hence T defines an endofunctor on GS_{E}.

Lemma 7.9. *For any graphical species S in E and any finite set $\mathcal{X}$, there is a unique isomorphism*

$$T^{\times 2} S_X \xrightarrow{\cong} \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} \lim_{\mathbf{\Gamma} \in \mathsf{etGr}^{(\mathcal{X})}} S(\mathbf{\Gamma}(\mathcal{X})),$$

where, as usual, $\mathbf{\Gamma}(\mathcal{X})$ is the colimit of $\mathbf{\Gamma}$ in $\mathsf{etGr}^{(\mathcal{X})}$.

Proof. By definition,

$$\begin{aligned} T^{\times 2} S_X &= \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} T^{\times} S(\mathcal{X}) \\ &= \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} \lim_{(\mathcal{C},b) \in \mathsf{el}(\mathcal{X})} T^{\times} S(\mathcal{C}) \\ &= \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} \lim_{(\mathcal{C},b) \in \mathsf{el}(\mathcal{X})} \mathrm{colim}_{\mathcal{Y} \in b\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{Y}). \end{aligned}$$

Now, for all $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{X})$, b-$\mathsf{etGr}_{\mathrm{iso}}$ is a groupoid, so $\mathrm{colim}_{\mathcal{Y} \in b\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{Y}) = \lim_{\mathcal{Y} \in b\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{Y})$. But, by definition of $\mathsf{etGr}^{(\mathcal{X})}$ (Definition 6.41),

$$\lim_{\mathbf{\Gamma} \in \mathsf{etGr}^{(\mathcal{X})}} S(\mathbf{\Gamma}(\mathcal{X})) \cong \lim_{(\mathcal{C},b) \in \mathsf{el}(\mathcal{X})} \lim_{\mathcal{Y} \in b\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{Y})$$

canonically. And therefore there is a unique isomorphism

$$T^{\times 2} S_X \cong \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} \lim_{\mathbf{\Gamma} \in \mathsf{etGr}^{(\mathcal{X})}} S(\mathbf{\Gamma}(\mathcal{X})). \tag{7.10}$$

□

By Proposition 6.34, each $\mathbf{\Gamma}(\mathcal{X})$ canonically inherits the structure of an X-graph from $\mathcal{X}$. Therefore, using Lemma 7.9, there is a canonical morphism $\mu^{\mathbb{T}^\times} S_X : T^{\times 2} S_X \to T^\times S_X$ in E given by

$$T^{\times 2} S_X = \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} \lim_{\mathbf{\Gamma} \in \mathsf{etGr}^{(\mathcal{X})}} S(\mathbf{\Gamma}(\mathcal{X})) \to \mathrm{colim}_{\mathcal{X}' \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{X}') = T^\times S_X. \tag{7.11}$$

This is clearly natural in X and in S and hence defines a natural transformation $\mu^{\mathbb{T}^\times} : T^{\times 2} \Rightarrow T^\times$.

Let $\eta^{\mathbb{T}^\times} : id_{\mathsf{GS}} \Rightarrow T^\times$ be the natural transformation induced by the inclusion $\mathbf{B}^\S \hookrightarrow \mathsf{etGr}$: $S_X \xrightarrow{=} S(\mathcal{C}_X) \to T^\times S_X$.

It is straightforward to verify that the triple $\mathbb{T}^\times = (T^\times, \mu^{\mathbb{T}^\times}, \eta^{\mathbb{T}^\times})$ defines a monad on GS_{E}.

Remark 7.12. In [19], empty graph missing so don't get monoid, but otherwise identical.

7.3. $\mathbb{T}^\times$-algebras are non-unital circuit operads.

Finally in this section, we prove that $\mathbb{T}^\times$-algebras are non-unital circuit operads.

Let (A, h) be an algebra for $\mathbb{T}^\times$ on GS_{E}. So, by definition of $\mathbb{T}^\times$, h is described by, for each finite set X, and X-graph $\mathcal{X}$, a morphism $A(\mathcal{X}) \to A_X$ in E, that, for convenience, will also be denoted h, (or $h_{\mathcal{X}}$).

Let X and Y be finite sets. Given an X-graph $\mathcal{X}$ and Y-graph $\mathcal{Y}$, the graph $\mathcal{X} \amalg \mathcal{Y}$ is the colimit of the identity $\mathcal{C}_X \amalg \mathcal{C}_Y$-shaped graph of graphs $\mathbf{I}^{\mathcal{D}_{X,Y}}$ (see Example 6.36). Hence, the following lemma is immediate:

Lemma 7.13. *For any graphical species S in E, disjoint union of graphs induces an external product $\boxtimes$ on $T^\times S$, that agrees with the free $\mathbb{T}^\times$-algebra structure $(T^\times S, \mu^{\mathbb{T}^\times} S)$ on its domain:*

For all X-graphs $\mathcal{X}$ and Y-graphs $\mathcal{Y}$, the diagram

$$\begin{tikzcd} S(\mathcal{X} \amalg \mathcal{Y}) \arrow[r, equal] \arrow[drr] & S(\mathcal{X}) \times S(\mathcal{Y}) \arrow[r] & T^\times S_X \times T^\times S_Y \arrow[r, equal] \arrow[d, "\boxtimes"'] & T^\times S(\mathcal{C}_X \amalg \mathcal{C}_Y) \arrow[r] & T^{\times 2} S_{X \amalg Y} \arrow[dll, "\mu^{\mathbb{T}^\times}"] \\ & & T^\times S_{X \amalg Y} & & \end{tikzcd}$$

(where the unlabelled maps are the defining morphisms for $T^\times$) commutes in E.

And similarly:

Lemma 7.14. *For any graphical species S in E, $T^{\times}S$ canonically admits a contraction ζ that agrees with the free $\mathbb{T}^{\times}$-algebra structure $(T^{\times}S, \mu^{\mathbb{T}^{\times}}S)$ on its domain:*

For all finite sets X and all X-graphs $\mathcal{X}$,

$$
\begin{tikzcd}
S(\mathcal{X}^{x\ddagger y}) \arrow[r] \arrow[dr] & T^{\times}S(\mathcal{C}_X^{x\ddagger y}) \arrow[r] \arrow[d, "\zeta"'] & {T^{\times}}^2 S_{X\setminus\{x,y\}} \arrow[dl, "\mu^{\mathbb{T}^{\times}}"] \\
& T^{\times}S_{X\setminus\{x,y\}} &
\end{tikzcd}
$$

commutes in E.

Proposition 7.15. *There is a canonical isomorphism of categories $\mathsf{GS}^{\mathbb{T}^{\times}} \cong \mathsf{CO}^{-}$.*

Proof. Let (A, h) be an algebra for $\mathbb{T}^{\times}$. For finite sets X and Y, the morphism $\boxtimes_{X,Y} \colon A_X \times A_Y \to A_{X \amalg Y}$ in E is defined as the composite

$$A_X \times A_Y = A(\mathcal{C}_X \amalg \mathcal{C}_Y) \to TA_{X\amalg Y} \xrightarrow{h} A_{X\amalg Y}.$$

And, if a finite set X has distinct elements x and y, then we may define a morphism $\zeta_X^{x\ddagger y} \colon A_X \to A_{X\setminus\{x,y\}}$ in E is defined as the composite $A_X^{x\ddagger y} = A(\mathcal{N}_X^{x\ddagger y}) \to TA_{X\setminus\{x,y\}} \xrightarrow{h} A_{X\setminus\{x,y\}}$.

To prove that $(A, \boxtimes, \zeta)$ satisfies the axioms for a circuit operad, the idea will be to represent each side of the equations (C1)-(C3) of Definition 5.15 in terms of a graph of graphs, and then show that these have the same colimit in etGr.

For example, the external product $\boxtimes_{X,Y}$ is represented by the graph $\mathcal{C}_X \amalg \mathcal{C}_Y$, and so $\boxtimes_{X\amalg Y, Z}(\boxtimes_{X,Y}, id_{A_Z})$ is represented by the $\mathcal{C}_{X\amalg Y} \amalg \mathcal{C}_Z$-shaped graphs of graphs $(\mathcal{C}_{X\amalg Y} \mapsto \mathcal{C}_X \amalg \mathcal{C}_Y,\ \mathcal{C}_Z \mapsto \mathcal{C}_Z)$ with colimit $\mathcal{C}_X \amalg \mathcal{C}_Y \amalg \mathcal{C}_Z$, which is also the colimit of the $\mathcal{C}_X \amalg \mathcal{C}_{Y\amalg Z}$-shaped graph of graphs $(\mathcal{C}_X \mapsto \mathcal{C}_X,\ \mathcal{C}_{Y\amalg Z} \mapsto \mathcal{C}_Y \amalg \mathcal{C}_Z)$.

Precisely, it follows from the monad algebra axioms that, for all finite sets X, Y and Z, the diagram

$$
\begin{tikzcd}
& & A_X \times A_Y \times A_Z \arrow[dl, "="'] \arrow[d, "="] \arrow[dr, "="] & & \\
A_{X\amalg Y} \times A_Z \arrow[d, equal] & A(\mathcal{C}_X \amalg \mathcal{C}_Y) \times A(\mathcal{C}_Z) \arrow[l, dashed, "(\boxtimes, id)"'] \arrow[r, "="] \arrow[d] & A(\mathcal{C}_X \amalg \mathcal{C}_Y \amalg \mathcal{C}_Z) \arrow[dd] & A(\mathcal{C}_X) \times A(\mathcal{C}_Y \amalg \mathcal{C}_Z) \arrow[l, "="'] \arrow[r, dashed, "(id, \boxtimes)"] \arrow[d] & A_X \times A_{Y\amalg Z} \arrow[d, equal] \\
A(\mathcal{C}_{X\amalg Y} \amalg \mathcal{C}_Z) \arrow[d] & TA(\mathcal{C}_{X\amalg Y} \amalg \mathcal{C}_Z) \arrow[l, "h"] \arrow[d] & & TA(\mathcal{C}_X \amalg \mathcal{C}_{Y\amalg Z}) \arrow[r, "h"'] \arrow[d] & A(\mathcal{C}_X \amalg \mathcal{C}_{Y\amalg Z}) \arrow[d] \\
TA_{X\amalg Y\amalg Z} \arrow[drr, "h"'] & T^2 A_{X\amalg Y\amalg Z} \arrow[l, "Th"] \arrow[r, "\mu^{\mathbb{T}^{\times}}A"'] & TA_{X\amalg Y\amalg Z} \arrow[d, "h"] & T^2 A_{X\amalg Y\amalg Z} \arrow[l, "\mu^{\mathbb{T}^{\times}}A"] \arrow[r, "Th"'] & TA_{X\amalg Y\amalg Z} \arrow[dll, "h"] \\
& & A_{X\amalg Y\amalg Z} & &
\end{tikzcd}
$$

commutes and hence $\boxtimes$ satisfies (C1) of Definition 5.15.

For (C2), if w, x, y, z are all distinct in X, then, by the monad algebra axioms, the diagram

$$\begin{tikzcd}
 & & (A_X)^{w\ddagger x, y\ddagger z} \arrow[dll, "="'] \arrow[d, "="] \arrow[drr, "="] & & \\
 (A_{X\setminus\{w,x\}})^{y\ddagger z} \arrow[d, equal] & \left(A(\mathcal{C}_X^{w\ddagger x})\right)^{y\ddagger z} \arrow[l, dashed, "\zeta^{w\ddagger x}"'] \arrow[r, "="] \arrow[d] & A\left((\mathcal{C}_X \amalg \mathcal{C}_Y)^{w\ddagger x, y\ddagger z}\right) \arrow[dd] & \left(A(\mathcal{C}_X^{y\ddagger z})\right)^{x\ddagger y} \arrow[l, "="'] \arrow[r, dashed, "\zeta^{y\ddagger z}"] \arrow[d] & (A_{X\setminus\{y,z\}})^{w\ddagger x} \arrow[d, equal] \\
 A(\mathcal{C}^{y\ddagger z}_{X\setminus\{w,x\}}) \arrow[d] & TA(\mathcal{C}^{y\ddagger z}_{X\setminus\{w,x\}}) \arrow[l, "h"] \arrow[d] & & TA(\mathcal{C}^{w\ddagger x}_{X\setminus\{y,z\}}) \arrow[r, "h"'] \arrow[d] & A(\mathcal{C}^{w\ddagger x}_{X\setminus\{y,z\}}) \arrow[d] \\
 TA_{X\setminus\{w,x,y,z\}} \arrow[drr, "h"'] & T^2A_{X\setminus\{w,x,y,z\}} \arrow[l, "Th"] \arrow[r, "\mu^{\mathbb{T}^\times} A"'] & TA_{X\setminus\{w,x,y,z\}} \arrow[d, "h"] & T^2A_{X\setminus\{w,x,y,z\}} \arrow[l, "\mu^{\mathbb{T}^\times} A"] \arrow[r, "Th"'] & TA_{X\setminus\{w,x,y,z\}} \arrow[dll, "h"] \\
 & & A_{X\setminus\{w,x,y,z\}} & &
\end{tikzcd}$$

commutes in E and hence ζ satisfies (C2).

Finally, if X and Y are finite sets and x_1 and x_2 are distinct elements of X, then the diagram

$$\begin{tikzcd}
 & & (A_X \times A_Y)^{x_1\ddagger x_2} \arrow[dl, "="'] \arrow[dr, "="] & & \\
 (A_{X\amalg Y})^{x_1\ddagger x_2} \arrow[d, equal] & (A(\mathcal{C}_X \amalg \mathcal{C}_Y))^{x_1\ddagger x_2} \arrow[l, dashed, "\boxtimes"'] \arrow[rr, equal] \arrow[d] \arrow[ddr] & & A(\mathcal{C}_X^{x_1\ddagger x_2}) \times A_Y \arrow[r, dashed, "\zeta^{x_1\ddagger x_2}"] \arrow[ddl] \arrow[d] & A_{X\setminus\{x_1,x_2\}} \times A_Y \arrow[d, equal] \\
 A(\mathcal{C}^{x_1\ddagger x_2}_{X\amalg Y}) \arrow[d] & TA(\mathcal{C}^{x_1\ddagger x_2}_{X\amalg Y}) \arrow[l, "h"] \arrow[d] & & TA(\mathcal{C}_{X\setminus\{x_1,x_2\}} \amalg \mathcal{C}_Y) \arrow[r, "h"'] \arrow[d] & A(\mathcal{C}_{X\setminus\{x_1,x_2\}} \amalg \mathcal{C}_Y) \arrow[d] \\
 TA_{(X\amalg Y)\setminus\{x_1,x_2\}} \arrow[drr, "h"'] & T^2A_{(X\amalg Y)\setminus\{x_1,x_2\}} \arrow[l, "Th"] \arrow[r, "\mu^{\mathbb{T}^\times} A"'] & TA_{(X\amalg Y)\setminus\{x_1,x_2\}} \arrow[d, "h"] & T^2A_{(X\amalg Y)\setminus\{x_1,x_2\}} \arrow[l, "\mu^{\mathbb{T}^\times} A"] \arrow[r, "Th"'] & TA_{(X\amalg Y)\setminus\{x_1,x_2\}} \arrow[dll, "h"] \\
 & & A_{(X\amalg Y)\setminus\{x_1,x_2\}} & &
\end{tikzcd}$$

commutes so (C3) is satisfied. And therefore $(A, \boxtimes, \zeta)$ is a non-unital circuit operad.

Conversely, let $(S, \boxtimes, \zeta)$ be a non-unital circuit operad. Since $T^\times S_\S = S_\S$, a morphism $h \colon T^\times S \to S$ is described by, for each finite set X, and X-graph $\mathcal{X}$, a morphism $S(\mathcal{X}) \to S_X$ in E.

Since $\mathcal{X}$ is an X-graph, it has no stick components. Hence, by (6.32)

$$S(\mathcal{X}) = \left(\prod_{v \in V(\mathcal{X})} S_{Z_v} \right)^{y_1\ddagger z_1, \dots, y_k\ddagger z_k} \tag{7.16}$$

where, for each $v \in V$, $(\mathcal{C}_{Z_v}, b_v) \in \mathsf{el}(\mathcal{X})$ is a neighbourhood of v, and $Z_v \cong |v|$ is such that $\coprod_v Z_v = X \amalg \{y_1, \dots, z_k\}$. We can construct a map $h_{\mathcal{X}} \colon S(\mathcal{X}) \to S(\mathcal{C}_X) = S_X$ as in Figure 14 by successively applying the external product $\boxtimes$, and contraction ζ, that, by (C1)-(C3), is independent of the chosen order.

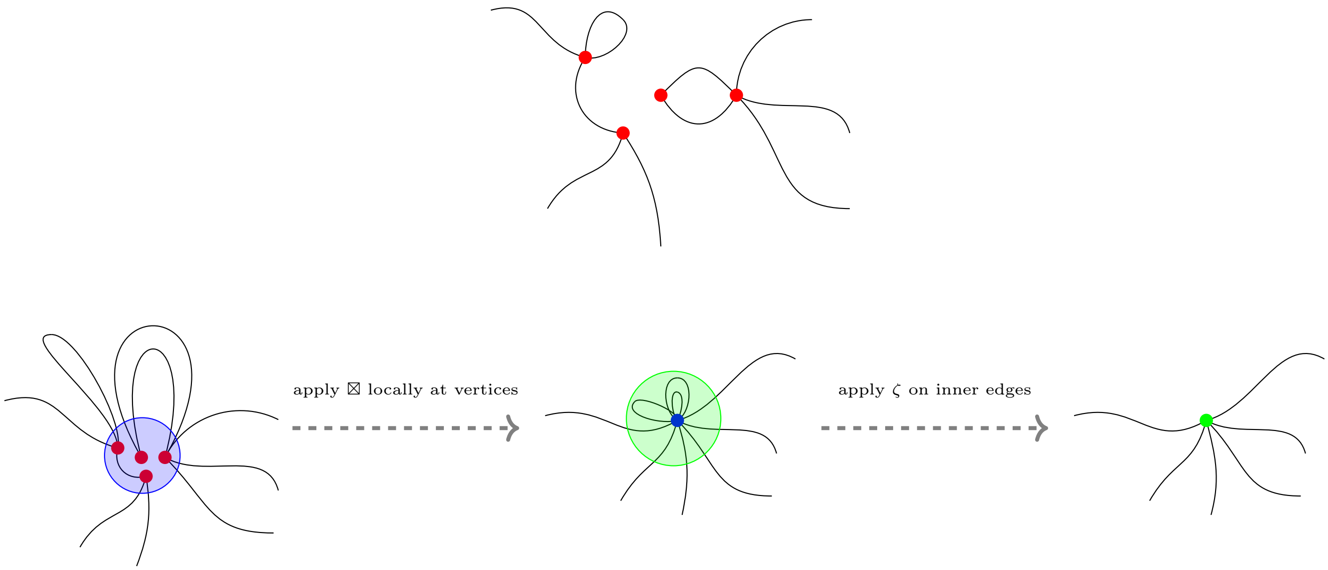


FIGURE 14. An element of $S(\mathcal{X}) \to T^\times S_X$ (top); the structure map $h_{\mathcal{X}} \colon S(\mathcal{X}) \to S_X$ (bottom).

By construction, the maps $h_{\mathcal{X}}$ so defined, extend to a morphism $h \colon T^\times S \to S$ in GS_{E}. Moreover, using the fact that, for any $\mathcal{X}$-shaped graph of graphs $\mathbf{\Gamma}$, $\mathbf{\Gamma}(b_v)$ is a Z_v-graph, we obtain morphisms $h_{\mathbf{\Gamma}(b_v)} \colon S(\mathbf{\Gamma}(b_v)) \to S_{Z_v}$, and by iterating (7.16), it is straightforward to show that $h \colon T^\times S \to S$ satisfies the axioms for a monad morphism (see Figure 15).

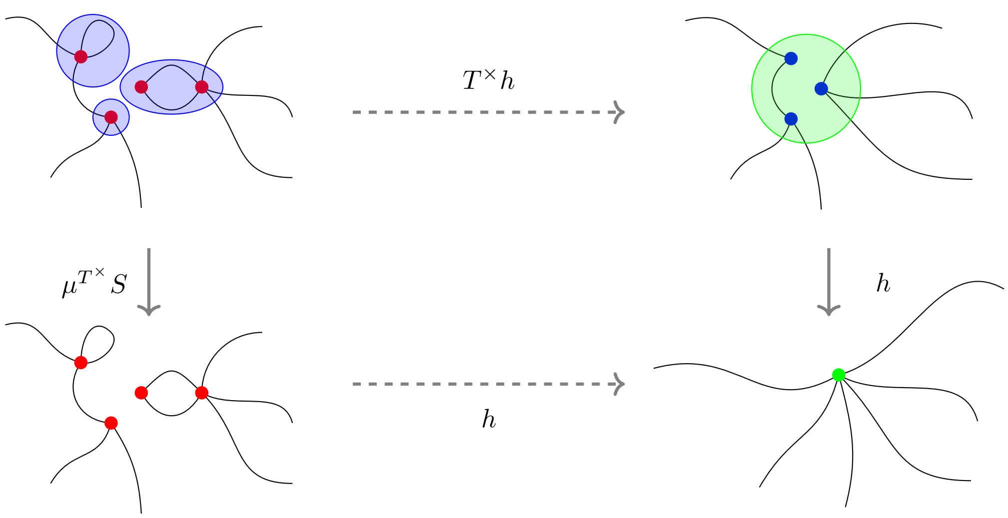


FIGURE 15. The structure map $h \colon T^\times S \to S$ is compatible with the monad multiplication $\mu^{\mathbb{T}^\times}$ on $(T^\times)^2 S$.

$\square$

Remark 7.17. It is straightforward to verify that $\mathbb{T}^\times$ and $\mathsf{etGr} \subset \mathsf{GS}$ satisfy the conditions of [3, Sections 1 & 2] and hence there is an abstract nerve theorem, analogous to Theorem 10.4, for Set-valued non-unital circuit operads. This construction is not described here since the nerve theorem for unital circuit operads is far more interesting (and challenging).

8. Modular operads and pointed graphs

This section reviews the construction of the monad for Set-valued modular operads in [25, Sections 5-7] in the more general setting of a category E – which, as usual, is assumed to have sufficient limits and colimits – and describes important features of the category etGr_* of pointed graphs, that is used in its definition.

8.1. Non-unital modular operads.

In [25], the modular operad monad on GS was defined in terms of monads $\mathbb{T}$ and $\mathbb{D}$ on GS and a distributive law $\lambda_{\mathbb{DT}} : TD \Rightarrow DT$ for composing them.

The non-unital modular operad monad $\mathbb{T}$ on GS was described In [25, Section 5], the notation $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ was used for the non-unital modular operad monad $\mathbb{T}$ on GS. And the same notation will be used here for the non-unital E-valued modular operad monad on GS_E where E is an arbitrary enriching category with all finite limits and sufficient colimits.

For all graphical species S in E, and all finite sets X,

$$TS_X \overset{\text{def}}{=} \operatorname{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} S(\mathcal{X}) \tag{8.1}$$

where $X\text{-}\mathsf{CGr}_{\mathrm{iso}} \subset X\text{-}\mathsf{etGr}_{\mathrm{iso}}$ is the full subgroupoid of connected X-graphs. In particular, there is a canonical monomorphism $TS_X \to T^\times S_X$.

By [25, Corollary 5.19] (which follows [18, Lemma 1.5.12]), if $\mathcal{G}$ is a connected graph, and $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{CetGr}$ is a non-degenerate $\mathcal{G}$-shaped graph of graphs such that $\mathbf{\Gamma}(b)$ is connected for all $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, then the colimit $\mathbf{\Gamma}(\mathcal{G})$ is also connected. Hence, the triple $(T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$, where $\mu^{\mathbb{T}}, \eta^{\mathbb{T}}$ are the obvious (co)restrictions) defines a monad on GS_E. This is the monad $\mathbb{T}$.

Proposition 8.2. *The EM category* ${\mathsf{GS}_\mathsf{E}}^{\mathbb{T}}$ *of algebras for* $\mathbb{T}$ *on* GS_E *is canonically equivalent to* MO^-_E*.*

Proof. The proof for general E closely resembles the proof of [25, Proposition 5.29] for the case $\mathsf{E} = \mathsf{Set}$. In particular, since $\mathcal{X}^{x \ddagger y}$ is connected for all connected X-graphs $\mathcal{X}$ and all pairs x, y of distinct elements of X, if (A, h) is an algebra for $\mathbb{T}$, then A admits a contraction ζ

$$\zeta^{x \ddagger y}_X \colon A^{x \ddagger y}_X = A(\mathcal{C}^{x \ddagger y}_X) \xrightarrow{h} TA_{X \setminus \{x, y\}},$$

that satisfies (M2) by the proof of Proposition 7.15.

Moreover, if (A, h) is an algebra for $\mathbb{T}$ on GS_E. Then we may define a multiplication $\diamond$ on A by the composite

$$A(\mathcal{D}^{x \ddagger y}_{X,Y}) \to TA_{(X \amalg Y) \setminus \{x, y\}} \xrightarrow{h} A_{(X \amalg Y) \setminus \{x, y\}}.$$

As with the proof of Proposition 7.15 and [25, Proposition 5.29], to show that $(A, \diamond, \zeta)$ satisfy the axioms (M1)-(M4), we construct, for each axiom, a pair of graphs of graphs that have the same colimit in CetGr.

For example, for axiom (M4), let X and Y be finite sets with distinct elements $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then the diagram

$$\begin{tikzcd}
 & & (A_X \times A_Y)^{x\ddagger y_1, x_2\ddagger y_2} \arrow[dl, "="'] \arrow[d, "="] \arrow[dr, "="] & & \\
A^{x_2\ddagger y_2}_{X\amalg Y\setminus\{x_1,y_1\}} \arrow[d, equal] & A(\mathcal{D}^{x_1\ddagger y_1}_{X,Y})^{x_2\ddagger y_2} \arrow[l, dashed, "\diamond"'] \arrow[r, "="] \arrow[d] & A(\mathcal{C}_X \amalg \mathcal{C}_Y)^{x_1\ddagger y_1, x_2\ddagger y_2} \arrow[dd] & A(\mathcal{D}^{x_2\ddagger y_2}_{X,Y})^{x_1\ddagger y_1} \arrow[l, "="'] \arrow[r, dashed, "\diamond"] \arrow[d] & A_{X\amalg Y\setminus\{x_2,y_2\}})^{x_1\ddagger y_1} \arrow[d, equal] \\
A(\mathcal{C}^{x_2\ddagger y_2}_{X\amalg Y\setminus\{x_1,y_1\}}) \arrow[d] & TA(\mathcal{C}^{x_2\ddagger y_2}_{X\amalg Y\setminus\{x_1,y_1\}}) \arrow[l, "h"] \arrow[d] & & TA(\mathcal{C}^{x_1\ddagger y_1}_{X\amalg Y\setminus\{x_2,y_2\}}) \arrow[r, "h"'] \arrow[d] & A(\mathcal{C}^{x_1\ddagger y_1}_{X\amalg Y\setminus\{x_2,y_2\}}) \arrow[d] \\
TA_{X\amalg Y\setminus\{x_1,x_2,y_1,y_2\}} \arrow[drr, "h"'] & T^2A_{X\amalg Y\setminus\{x_1,x_2,y_1,y_2\}} \arrow[l, "Th"] \arrow[r, "\mu^{\mathbb{T}^\times}A"'] & TA_{X\amalg Y\setminus\{x_1,x_2,y_1,y_2\}} \arrow[d, "h"] & T^2A_{X\amalg Y\setminus\{x_1,x_2,y_1,y_2\}} \arrow[l, "\mu^{\mathbb{T}^\times}A"] \arrow[r, "Th"'] & TA_{X\amalg Y\setminus\{x_1,x_2,y_1,y_2\}} \arrow[dll, "h"] \\
 & & A_{X\amalg Y\setminus\{x_1,x_2,y_1,y_2\}} & &
\end{tikzcd}$$

commutes, whereby it follows that $\diamond$ and ζ satisfy (M4).

Axioms (M1) and (M4) follow by the same method (when $\mathsf{E} = \mathsf{Set}$, the proof that (A, h) satisfies (M1) is described in detail in [25]).

The converse follows exactly the same method as the proof of [25, Proposition 5.29] and Proposition 7.15. $\square$

8.2. **The monad $\mathbb{D}$.** We generalise the pointed graphical species monad $\mathbb{D}$, defined on graphical species in Set in [25], to a monad $\mathbb{D}$ on GS_{E}.

The *pointed graphical species* endofunctor D on GS adjoins formal (contracted) units to a graphical species S in E according to $DS_\S = S_\S$ with $DS_\tau = S_\tau$ and, for finite sets X,

$$DS_X = \begin{cases} S_{\mathbf{2}} \amalg S_\S & X = \mathbf{2}, \\ S_{\mathbf{0}} \amalg \widetilde{S_\S} & X = \mathbf{0}, \\ S_X & X \ncong \mathbf{2}, \text{ and } X \ncong \mathbf{0}. \end{cases}$$

There are canonical natural transformations $\eta^{\mathbb{D}} \colon 1_{\mathsf{GS}_{\mathsf{E}}} \Rightarrow D$, provided by the induced morphism $S \hookrightarrow DS$, and $\mu^{\mathbb{D}} \colon D^2 \Rightarrow D$ induced by the canonical projections $D^2 S_{\mathbf{2}} \to DS_{\mathbf{2}}$, so that $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ defines the *pointed graphical species monad* on GS_{E}.

Algebras for $\mathbb{D}$ – called *pointed graphical species in* E – are graphical species S in E equipped with a distinguished unit-like morphism (Definition 5.13)

$$\epsilon \colon S_\S \to S_{\mathbf{2}},$$

and a distinguished morphism $o \colon S_\S \to S_{\mathbf{0}}$ that factors through the quotient $\widetilde{S_\S} \overset{\text{def}}{=} S_\S/_{S_\tau}$.

The EM category $\mathsf{GS}_{\mathsf{E}*}$ of algebras for $\mathbb{D}$ is precisely the category of E-presheaves on a category $\mathbf{B}^\S_*$ obtained from $\mathbf{B}^\S$ by formally adjoining morphisms $u \colon \mathbf{2} \to \S$ and $z \colon \mathbf{0} \to \S$, subject to the relations:

(i) $u \circ ch_1 = id \in \mathbf{B}^\S(\S, \S)$ and $u \circ ch_2 = \tau \in \mathbf{B}^\S(\S, \S)$;
(ii) $\tau \circ u = u \circ \sigma_{\mathbf{2}} \in \mathbf{B}^\S(\mathbf{2}, \S)$;
(iii) $z = \tau \circ z \in \mathbf{B}^\S(\mathbf{0}, \S)$.

It follows that $\mathbf{B}^\S_*$ is completely described by:

$\mathbf{B}^{\S}_*(\S, \S) = \mathbf{B}^{\S}(\S, \S)$ and $\mathbf{B}^{\S}_*(Y, X) = \mathbf{B}^{\S}(Y, X)$ whenever $Y \ncong \mathbf{0}$ and $Y \ncong \mathbf{2}$;
$\mathbf{B}^{\S}_*(\mathbf{0}, \S) = \{z\}$, and $\mathbf{B}^{\S}_*(\mathbf{0}, X) = \mathbf{B}^{\S}(\mathbf{0}, X) \amalg \{ch_x \circ z\}_{x \in X}$;
$\mathbf{B}^{\S}_*(\mathbf{2}, \S) = \{u, \tau \circ u\}$, and $\mathbf{B}^{\S}_*(\mathbf{2}, X) = \mathbf{B}^{\S}(\mathbf{0}, X) \amalg \{ch_x \circ u, ch_x \circ \tau \circ u\}_{x \in X}$.

and hence the following are equivalent:

(1) S_* is a presheaf on $\mathbf{B}^{\S}_*$ that restricts to a graphical species S on $\mathbf{B}^{\S}$;
(2) (S, ϵ, o), with $\epsilon = S_*(u)$ and $o = S_*(z)$ is a pointed graphical species.

As a consequence, the notation S_* and (S, ϵ, o) will be used interchangeably to denote the same pointed graphical species.

8.3. **Pointed graphs.** The category CetGr_* of connected graphs and *pointed étale morphisms*, obtained in the bo-ff factorisation of the functor $\mathsf{CetGr} \to \mathsf{GS}_*$ is described in detail in [25,]. Here, CetGr_* is extended to the category etGr_* of *all* graphs and pointed étale morphisms. So etGr_* is the category defined via bo-ff factorisations as in the following commuting diagram of functors,

$$
\begin{tikzcd}
\mathbf{B}^{\S}_* \arrow[r, "\text{dense}", "\text{f.f.}"'] \arrow[rr, bend left, "\text{f.f.}"] & \mathsf{etGr}_* \arrow[r, "Y_*", "\text{f.f.}"'] & \mathsf{GS}_* \arrow[d, shift left, "\text{forget}^{\mathbb{D}}"] \\
\mathbf{B}^{\S} \arrow[u, "\text{b.o.}"] \arrow[r, hook] & \mathsf{etGr} \arrow[u, "\text{b.o.}"] \arrow[r, hook, "Y"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}^{\mathbb{D}}"]
\end{tikzcd}
\tag{8.3}
$$

The inclusion $\mathbf{B}^{\S}_* \to \mathsf{CetGr}_*$ is fully faithful (by uniqueness of bo-ff factorisation), and also dense, since the induced nerve $\Upsilon_* \colon \mathsf{CetGr}_* \to \mathsf{GS}_*$ is fully faithful by construction.

Let $\mathcal{G} \in \mathsf{CetGr}$ be a graph. By Section 8.2, for each edge $e \in E$, the ch_e-coloured unit for $\Upsilon_*\mathcal{G}$ is given by $\epsilon^{\mathcal{G}}_e = ch_e \circ u \in \mathsf{CetGr}_*(\mathcal{C}_{\mathbf{2}}, \mathcal{G})$, and the corresponding contracted unit is given by $o^{\mathcal{G}}_{\tilde{e}} = ch_e \circ z \in \mathsf{CetGr}_*(\mathcal{C}_{\mathbf{0}}, \mathcal{G})$.

Since the functor Υ_* embeds CetGr_* as a full subcategory of GS_*, I will write $\mathcal{G}$, rather than $\Upsilon_*\mathcal{G}$ where there is no risk of confusion. In particular, the element category $\mathsf{el}_*(\Upsilon_*\mathcal{G})$ is denoted simply by $\mathsf{el}_*(\mathcal{G})$ and called the *category of pointed elements of a graph* $\mathcal{G}$.

The forgetful functor $\mathsf{GS}_* \to \mathsf{GS}$ induces injective-on-objects inclusions $\mathsf{el}(S) \to \mathsf{el}_*(S_*)$ for all pointed graphical species S_*. In particular, there is a canonical inclusion $\mathsf{el}(\mathcal{G}) \hookrightarrow \mathsf{el}_*(\mathcal{G})$, and, by [25, Section 4],

$$
\lim_{(\mathcal{C}, b) \in \mathsf{el}_*(\mathcal{G})} S(\mathcal{C}) = \lim_{(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})} S(\mathcal{C}) = S(\mathcal{G})
\tag{8.4}
$$

for all pointed graphical species $S_* = (S, \epsilon, o) \in \mathsf{GS}_{\mathsf{E}*}$.

Moreover, by [25, Section 7.3], the induced inclusion $\mathsf{GS}_{\mathsf{E}*} \to \mathsf{psh}_{\mathsf{E}}(\mathsf{etGr}_*)$ is fully faithful. (In fact, GS_{E} is equivalent to the category of E-sheaves for the obvious extension of the canonical étale topology on etGr to etGr_*.)

To better understand morphisms in etGr_*, assume first that $\mathcal{G}$ has no isolated vertices and let $W \subset V_0 \amalg V_2$ a subset of isolated and bivalent vertices of $\mathcal{G}$.

The *vertex deletion functor (for W)* is the $\mathcal{G}$-shaped graph of graphs $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ given by

$$
\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}(\mathcal{C}_X, b) = \begin{cases} (\mid) & \text{if } (\mathcal{C}_X, b) \text{ is a neighbourhood of } v \in W \text{ (so } X \cong \mathbf{0} \text{ or } X \cong \mathbf{2}), \\ \mathcal{C}_X & \text{otherwise.} \end{cases}
$$

By Proposition 6.34, this has a colimit $\mathcal{G}_{\setminus W}$ in etGr. Therefore $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ induces a morphism $\mathsf{del}_{\setminus W} \in \mathsf{etGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$.

Definition 8.5. *The morphism* $\mathsf{del}_{\setminus W} \in \mathsf{etGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ *induced by* $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}\colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ *is called the* vertex deletion morphism corresponding to $W \subset V_0 \cong V_2$.

Example 8.6. If $\mathcal{G} = \mathcal{C}_{\mathbf{0}}$ is an isolated vertex $*$ and $W = \{*\}$, then $\mathsf{del}_{\setminus W}$ is precisely $z\colon \mathcal{C}_{\mathbf{0}} \to (\mid)$.

Example 8.7. For $\mathcal{G} = \mathcal{C}_{\mathbf{2}}$ and $W = V = \{*\}$, $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ is the constant functor $\mathsf{el}(\mathcal{G}) \to (\mid)$:

$$\begin{tikzcd}
(\mid) \arrow[r, "ch_1"] \arrow[dr, "id_{(\mid)}"'] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & (\mid) \arrow[l, "ch_2 \circ \tau"'] \arrow[dl, "id_{(\mid)}"] \\
& (\mid) &
\end{tikzcd} \tag{8.8}$$

and hence $\mathcal{G}_{\setminus W} = (\mid)$ and $\mathsf{del}_{\setminus W} = u \in \mathsf{etGr}_*(\mathcal{C}_{\mathbf{2}}, \mid)$.

More generally, if $\mathcal{G} = \mathcal{L}^k$, and $W = V$, then $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ is also the constant functor to $(\mid)$ and hence $\mathcal{G}_{\setminus W} = (\mid)$. The induced morphism is denoted by $u^k \stackrel{\text{def}}{=} \mathsf{del}_{\setminus W}\colon \mathcal{L}^k \to \mathsf{etGr}_*$:

$$\begin{tikzcd}
(\mid) \arrow[r, "ch_1"] \arrow[dr, "id_{(\mid)}"'] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & \dots \arrow[l, "ch_2 \circ \tau"'] \arrow[r, "ch_1"] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & (\mid) \arrow[l, "ch_2 \circ \tau"'] \arrow[dl, "id_{(\mid)}"] \\
& (\mid) \arrow[r, equal] & \dots \arrow[r, equal] & (\mid). &
\end{tikzcd}$$

In particular, $u^1 = u\colon \mathcal{C}_{\mathbf{2}} \to (\mid)$ and u^0 is just the identity on $(\mid)$.

Example 8.9. Let $\mathcal{G} = \mathcal{W}$ be the wheel graph with one vertex v. Then $\mathcal{W}_{/\{v\}} = \operatorname{colim}_{\mathsf{el}(\mathcal{W})} \mathbf{\Lambda}^{\mathcal{W}}_{/\{v\}}$ exists and is isomorphic to $(\mid)$ in etGr. The induced morphism $\kappa \stackrel{\text{def}}{=} \mathsf{del}_{\setminus \{v\}}\colon \mathcal{W} \to (\mid)$ is described in (8.10).

Hence, there are precisely two morphisms κ and $\tau \circ \kappa$ in $\mathsf{etGr}_*(\mathcal{W}, \mid)$:

$$\begin{tikzcd}
& \mathcal{W} & \\
(\mid) \arrow[ur, "ch_a"] \arrow[rr, shift left, "ch_1"] \arrow[rr, shift right, "ch_2 \circ \tau"'] \arrow[dr, equal] & & \mathcal{C}_{\mathbf{2}} \arrow[ul, "1_{\mathcal{C}_{\mathbf{2}}} \mapsto a"'] \arrow[dl, "u"] \\
& (\mid) &
\end{tikzcd} \tag{8.10}$$

$$\begin{tikzcd}
& \mathcal{W} & \\
(\mid) \arrow[ur, "ch_a"] \arrow[rr, shift left, "ch_1"] \arrow[rr, shift right, "ch_2 \circ \tau"'] \arrow[d, equal] & & \mathcal{C}_{\mathbf{2}} \arrow[ul, "1_{\mathcal{C}_{\mathbf{2}}} \mapsto a"'] \arrow[d, "\sigma_{\mathbf{2}}"] \\
(\mid) \arrow[rr, shift left, "ch_2"] \arrow[rr, shift right, "ch_1 \circ \tau"'] \arrow[dr, equal] & & \mathcal{C}_{\mathbf{2}} \arrow[dl, "u"] \\
& (\mid). &
\end{tikzcd} \tag{8.11}$$

More generally, let $\mathcal{W}^m$ be the wheel graph with m cyclically ordered vertices $(v_i)_{i=1}^m$, and let $\iota \in \mathsf{CetGr}(\mathcal{L}^{m-1}, \mathcal{W}^m)$ be a pointwise étale inclusion that respects the (cyclic) ordering of vertices. So, if W is the image of $V(\mathcal{L}^{m-1})$ in $V(\mathcal{W}^m)$, then $V(\mathcal{W}^m) \cong W \amalg \{*\}$, and by (8.10) and the Example 8.7, there are two distinct pointed morphisms, κ^m and $\tau \circ \kappa^m$, in $\mathsf{etGr}_*(\mathcal{W}^m, \mid)$. Hence, for all $\mathcal{G}$,

$$\mathsf{etGr}_*(\mathcal{W}^m, \mathcal{G}) = \mathsf{etGr}(\mathcal{W}^m, \mathcal{G}) \amalg \{ch_e \circ \kappa^m\}_{e \in E(\mathcal{G})} \cong \mathsf{etGr}_*(\mathcal{W}, \mathcal{G}).$$

Remark 8.12. Recall from Remark 3.18 that the composition $\cup \circ \cap$ of Brauer diagrams is not described by a pushout of cospans as in (3.19), and instead, the pushout (3.19) described by $\cup \circ \cap$ is induced by the trivial diagram $(\mid) \xrightarrow{id} (\mid) \xleftarrow{id} (\mid)$, and therefore agrees with $\kappa\colon \mathcal{W} \to (\mid)$ described in Example 8.9.

(See [25, Section 6] for more details on the tricky combinatorics of contractions.)

For convenience, let $\mathcal{W}^0 \stackrel{\text{def}}{=} (\mid_{\tilde{e}}) = \mathcal{W}^m_{/V(\mathcal{W}^m)}$ for all m. Then for all $k \geq 0$, $u^k\colon \mathcal{L}^k \to (\mid)$ factors canonically as $\mathcal{L}^k \to \mathcal{W}^k \xrightarrow{\kappa^k} (\mid)$.

Example 8.13. Assume that $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$ is connected (so $V_0 = \emptyset$). Let $W \subset V_2$ be a subset of bivalent vertices of $\mathcal{G}$. Unless $W = V(\mathcal{G})$ and $\mathcal{G} = \mathcal{W}^m$ for some $m \geq 1$, the graph $\mathcal{G}_{\setminus W}$ may be intuitively described as '$\mathcal{G}$ with a subset of bivalent vertices deleted' as in Figure 16.

In this case, the graph $\mathcal{G}_{\setminus W}$ is described explicitly by:

$$\mathcal{G}_{\setminus W} = \quad \tau_{\setminus W} \circlearrowright E_{\setminus W} \xleftarrow{s_{\setminus W}} H_{\setminus W} \xrightarrow{t_{\setminus W}} V_{\setminus W},$$

where

$$\begin{aligned} V_{\setminus W} &= V \setminus W, \\ H_{\setminus W} &= H \setminus \left(\textstyle\coprod_{v \in W} {}^{H}\!/v\right), \\ E_{\setminus W} &= E \setminus \left(\textstyle\coprod_{v \in W} {}^{E}\!/v\right), \end{aligned}$$

$s_{\setminus W}, t_{\setminus W}$ are just the restrictions of s and t. The precise description of the involution $\tau_{\setminus W}$ is more complicated than needed here. (The interested reader is referred to [25, Section 7.2] which contains full descriptions of the vertex deletion morphisms).

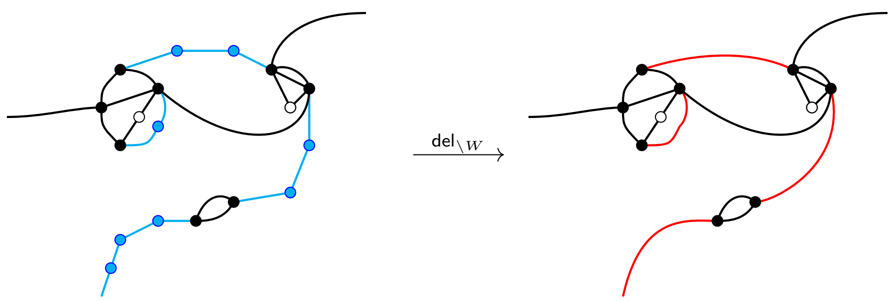


FIGURE 16. Vertex deletion $\mathsf{del}_{\setminus W} \colon \mathcal{G} \to \mathcal{G}_{\setminus W}$, with $W \subset V_2$.

Definition 8.14. *The* similarity category $\mathsf{CGr_{sim}} \hookrightarrow \mathsf{CetGr}_*$ *is the identity-on-objects subcategory of* CetGr_* *whose morphisms are generated under composition by* $z \colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$*, the vertex deletion morphisms, and graph isomorphisms. Morphisms in* $\mathsf{CGr_{sim}}$ *are called* similarity morphisms, *and connected components of* $\mathsf{CGr_{sim}}$ *are* similarity classes. *Graphs in the same connected component of* $\mathsf{CGr_{sim}}$ *are* similar.

The following proposition compiles basic important properties of vertex deletion morphisms. Detailed proofs may be found in [25, Section 7.2].

Proposition 8.15. *For all graphs $\mathcal{G}$ and all subsets $W \subset V_2$ of bivalent vertices of $\mathcal{G}$,*

(1) $\mathsf{del}_{\setminus W} \colon \mathcal{G} \to \mathcal{G}_{\setminus W}$ *preserves connected components, and may be defined componentwise;*
(2) if $\mathcal{G}$ is connected and $W = V$, then $\mathcal{G} \cong \mathcal{L}^k$ or $\mathcal{G} \cong \mathcal{W}^m$ for $k, m \geq 0$ and $\mathcal{G}_{\setminus W} \cong (\shortmid)$;
(3) unless $\mathcal{G} \cong \mathcal{W}^m$ and $W = V$ for $m \geq 1$, if $\mathcal{G}$ is connected, then $\mathsf{del}_{\setminus W}$ *induces an identity on ports:* $\mathsf{del}_{\setminus W} \colon E_0(\mathcal{G}) \xrightarrow{=} E_0(\mathcal{G}_{\setminus W})$;
(4) the pair $(\mathsf{CGr_{sim}}, \mathsf{CetGr})$ *of subcategories of* CetGr_* *defines an orthogonal factorisation system on* CetGr_*.

Example 8.16. For all graphical species S in E, and all graphs $\mathcal{G}$ with no isolated vertices,

$$DS(\mathcal{G}) = \coprod_{W \subset V_2} S(\mathcal{G}_{\setminus W}). \tag{8.17}$$

More generally, since $DS_{\mathbf{0}} = DS(\mathcal{C}_{\mathbf{0}}) = S_{\mathbf{0}} \amalg \widetilde{S_\S}$ where $\widetilde{S_\S}$ is the coequaliser of the identity and S_τ on $S_\S$,

$$DS(\mathcal{G}) \cong \mathrm{colim}_{(\mathcal{H}, f) \in \mathcal{G}/\mathsf{Gr_{sim}}} S(\mathcal{H}) \tag{8.18}$$

for all graphs $\mathcal{G} \in \mathsf{etGr}$.

Example 8.19. Let $\mathsf{E} = \mathsf{Set}$ and, as usual, let Y be the Yoneda embedding $\mathsf{etGr} \hookrightarrow \mathsf{GS}$, $\mathcal{H} \mapsto \mathsf{etGr}(-, \mathcal{H})$. Let $\mathcal{G}$ be any graph and $\mathcal{H}$ a graph without isolated vertices. Then the factorisation of morphisms in $\mathsf{etGr}_*(\mathcal{G}, \mathcal{H})$ also follows from the identity

$$\begin{array}{rcll} \mathsf{etGr}_*(\mathcal{H}, \mathcal{G}) & = & DY(\mathcal{H})(\mathcal{G}) & \text{by (8.3)} \\ & = & \coprod_{W \in \mathcal{P}(V_2)} \mathsf{etGr}(\mathcal{G}_{\setminus W}, \mathcal{H}) & \text{by (8.17).} \end{array}$$

Example 8.20. Let $S_* = (S, \epsilon, o)$ be a pointed graphical species in E.

For $k, m \geq 0$ and $m \geq 1$, the vertex deletion morphisms $u^k \colon \mathcal{L}^k \to (\shortmid)$ and $\kappa^m \colon \mathcal{W}^m \to (\shortmid)$ in CetGr_* induce monomorphisms in E:

$$\epsilon^k \overset{\text{def}}{=} S_*(u^k) \colon S_\S \to S(\mathcal{L}^k), \quad \text{and} \quad o^m \overset{\text{def}}{=} S_*(\kappa^m) \colon S_\S \to S(\mathcal{W}^m), \tag{8.21}$$

where $\epsilon^0 = o^0 = id_{S_\S}$, $\epsilon^1 = \epsilon$, and ϵ^k, o^m factor through $\epsilon \colon S_\S \to S_{\mathbf{2}}$ for all $k \geq 0, m \geq 1$.

8.4. **The distributive law $TD \Rightarrow DT$.** Similar connected X-graphs

In the remainder of this chapter, we work with connected graphs and their pointed étale morphisms. Recall (Definition 6.43) that an X-graph $\mathcal{X}$ is said to be admissible if it has no stick components.

For any finite set X, we may enlarge the category $X\text{-}\mathsf{CGr}_{\text{iso}}$ of connected X-graphs and port-preserving isomorphisms, to form a category $X\text{-}\mathsf{CGr}_{\mathsf{sim}}$ that includes vertex deletion morphisms $\mathsf{del}_{\setminus W}$ and $z \colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$ out of X-graphs together with any labelling of ports from the domain:

By **??**, any similarity morphism that is not of the form $z \colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$ or $\kappa^m \colon \mathcal{W}^m \to (\shortmid)$ is boundary preserving. So, let $\mathcal{G}$ be a connected graph with non-empty vertex set and non-empty boundary $E_0 \cong X$, and let $f \in \mathsf{CGr}_{\mathsf{sim}}(\mathcal{G}, \mathcal{G}')$ be a similarity morphism. Then f, together with an X-labelling $\rho \colon E_0 \to X$ of $\mathcal{G}$, induces an X-labelling on $\mathcal{G}'$. The category $X\text{-}\mathsf{CGr}_{\mathsf{sim}}$ is obtained by adjoining to the category $X\text{-}\mathsf{CGr}_{\text{iso}}$ of (admissible) X-graphs and port-preserving isomorphisms, all similarity morphisms from admissible X-graphs, and their codomains, equipped with the induced labelling.

- If $X \not\cong \mathbf{0}$, $X \not\cong \mathbf{2}$, then $X\text{-}\mathsf{CGr}_{\mathsf{sim}}$ is the category whose objects are connected X-graphs and whose morphisms are similarity morphisms that preserve the labelling of the ports.
- For $X = \mathbf{2}$, $\mathbf{2}\text{-}\mathsf{CGr}_{\mathsf{sim}}$ contains the morphisms $\mathsf{del}_{\setminus V} \colon \mathcal{L}^k \to (\shortmid)$, and hence the labelled stick graphs $(\shortmid, id)$ and $(\shortmid, \tau)$. There are no non-trivial morphisms out of these special graphs, and $\mathcal{X}$ is in the same connected component as $(\shortmid, id)$ if and only if $\mathcal{X} = \mathcal{L}^k$ (with the identity labelling) for some $k \in \mathbb{N}$. In particular, $\tau \colon (\shortmid) \to (\shortmid)$ does not induce a morphism in $\mathbf{2}\text{-}\mathsf{CGr}_{\mathsf{sim}}$.
- Finally, when $X = \mathbf{0}$, the morphisms $\mathsf{del}_{\setminus V} \colon \mathcal{W}^m \to (\shortmid)$, and $z \colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$ are not boundary-preserving, and, in particular, do not equip $(\shortmid)$ with any labelling of its ports. So, the objects of $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}$ are the admissible $\mathbf{0}$-graphs, and $(\shortmid)$.

  In particular, $\mathcal{W}^m, \mathcal{C}_{\mathbf{0}}$ and $(\shortmid)$ are in the same connected component of $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}$. Since $(\shortmid)$ is not admissible, there are no non-trivial morphisms in $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}$ with $(\shortmid)$ as domain.

  In what follows, to simplify notation, we'll write $\mathcal{C}_{\mathbf{0}\setminus V} \overset{\text{def}}{=} (\shortmid)$ and $\mathsf{del}_{\setminus V} = z \colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$.

The following lemma follows immediately from the definitions

Lemma 8.22. *For all finite sets X and all connected X-graphs $\mathcal{X}$, the connected component of $X\text{-}\mathsf{CGr}_{\mathsf{sim}}$ containing $\mathcal{X}$ has a terminal object (without bivalent or isolated vertices) given by $\mathcal{X}_{/(V_2 \amalg V_0)}$.*

This terminal object is an X-graph, and hence has non-trivial vertex set, unless $\mathcal{X} \cong \mathcal{L}^k$ or $\mathcal{X} \cong \mathcal{W}^m$, $\mathcal{X} \cong \mathcal{C}_{\mathbf{0}}$ for some $k, m \geq 0$.

By (8.18), if $\mathcal{X}$ is a connected X-graph and $X\text{-}\mathsf{CGr}^{\mathcal{X}}_{\mathsf{sim}}$ is the connected component of $\mathcal{X}$ in $X\mathsf{CGr}_{\mathsf{sim}}$, then there is a canonical isomorphism

$$DS(\mathcal{X}) \cong \operatorname{colim}_{\mathcal{X}' \in X\text{-}\mathsf{CGr}^{\mathcal{X}}_{\mathsf{sim}}} S(\mathcal{X}').$$

Let S be a graphical species in E. Hence, by (8.17), TDS_X is described by a colimit – indexed by pairs $(\mathcal{X}, W)$, of a connected X-graph $\mathcal{X}$ and a subset $W \subset V_0 \amalg V_2$ of vertices of $(\mathcal{X})$, and isomorphisms $\mathcal{X} \xrightarrow{\mathcal{X}} {}'$ in $X\text{-}\mathsf{CGr}_{\mathrm{iso}}$ – of objects $S(\mathcal{X}_{\setminus W})$ in E. The distributive law $\lambda_{\mathbb{D},\mathbb{T}} \colon TD \Rightarrow DT$ is induced by forgetting the pair $(\mathcal{X}, W)$, and replacing it with $\mathcal{X}_{\setminus W}$, with canonical map $S(\mathcal{X}_{\setminus W}) \to DTS_X$.

Precisely, for $X \ncong \mathbf{0}$,

$$TDS_X = \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} DS(\mathcal{X}) = \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} \coprod_{W \subset V_2(\mathcal{X})} S(\mathcal{X}_{\setminus W}).$$

Since vertex deletion morphisms preserve ports when $X \neq \mathbf{0}$, there are canonical morphisms

$$TDS_X \to \begin{cases} \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} S(\mathcal{X}) & = TS_X \quad = DTS_X \quad \text{when } X \ncong \mathbf{0}, X \ncong \mathbf{2}, \\ (\mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} S(\mathcal{X})) \amalg S_{\S} & = TS_{\mathbf{2}} \amalg S_{\S} \quad = DTS_{\mathbf{2}} \quad \text{when } X = \mathbf{2} \end{cases}$$

(since $(\shortmid)$ is not a $\mathbf{2}$-graph by convention), and these describe the morphisms $\lambda_{\mathbb{DT}} S_X \colon TDS_X \to DTS_X$ in E for $X \ncong \mathbf{0}$.

When $X = \mathbf{0}$,

$$\begin{aligned} TDS_{\mathbf{0}} &= \mathrm{colim}_{\mathcal{X} \in \mathbf{0}\text{-}\mathsf{CGr}_{\mathrm{iso}}} DS(\mathcal{X}) \\ &= \Big(\mathrm{colim}_{\substack{\mathcal{X} \in \mathbf{0}\text{-}\mathsf{CGr}_{\mathrm{iso}} \\ \mathcal{X} \ncong \mathcal{C}_{\mathbf{0}}}} \textstyle\coprod_{W \subset V_2(\mathcal{X})} S(\mathcal{X}_{\setminus W}) \Big) \amalg DS(\mathcal{C}_{\mathbf{0}}) \\ &= \Big(\mathrm{colim}_{\substack{\mathcal{X} \in \mathbf{0}\text{-}\mathsf{CGr}_{\mathrm{iso}} \\ \mathcal{X} \ncong \mathcal{C}_{\mathbf{0}}}} \textstyle\coprod_{W \subset V_2(\mathcal{X})} S(\mathcal{X}_{\setminus W}) \Big) \amalg S_{\mathbf{0}} \amalg \widetilde{S_{\S}}. \end{aligned}$$

Then $\lambda_{\mathbb{DT}} S_{\mathbf{0}} \colon TDS_{\mathbf{0}} \to DTS_{\mathbf{0}}$ is given by the canonical morphism

$$TDS_{\mathbf{0}} = \Bigg(\mathrm{colim}_{\substack{\mathcal{X} \in \mathbf{0}\text{-}\mathsf{CGr}_{\mathrm{iso}} \\ \mathcal{X} \ncong \mathcal{C}_{\mathbf{0}}}} \coprod_{W \subset V_2(\mathcal{X})} S(\mathcal{X}_{\setminus W}) \Bigg) \amalg S_{\mathbf{0}} \amalg \widetilde{S_{\S}} \to \mathrm{colim}_{\mathcal{X} \in \mathbf{0}\text{-}\mathsf{CGr}_{\mathrm{iso}}} S(\mathcal{X}) \amalg \widetilde{S_{\S}} = DTS_{\mathbf{0}}.$$

The verification that $\lambda_{\mathbb{DT}}$ satisfies the four distributive law axioms [2] follows by a straightforward application of the definitions. To help the reader gain familiarity with the constructions, I describe just one here, namely that the following diagram of endofunctors on GS_{E} commutes:

(8.23)
$$\begin{tikzcd} T^2 D \arrow[r, Rightarrow, "T\lambda_{\mathbb{DT}}"] \arrow[d, Rightarrow, "\mu^{\mathbb{T}} D"'] & TDT \arrow[r, Rightarrow, "\lambda_{\mathbb{DT}} T"] & DT^2 \arrow[d, Rightarrow, "D\mu^{\mathbb{T}}"] \\ TD \arrow[rr, Rightarrow, "\lambda_{\mathbb{DT}}"] & & DT. \end{tikzcd}$$

Recall, from (7.11) and (8.17), that, for all graphical species S in E and all finite sets X,

$$\begin{aligned} T^2 DS_X &= \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} \lim_{(\mathcal{C}, b) \in \mathsf{el}(\mathcal{X})} \mathrm{colim}_{\mathcal{Y}^b \in b\text{-}\mathsf{CGr}_{\mathrm{iso}}} \textstyle\coprod_{W^b \subset V_2} S(\mathcal{Y}^b_{\setminus W^b}) \\ &= \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} \lim_{\mathbf{\Gamma} \in \mathsf{CetGr}(\mathcal{X})} \textstyle\coprod_{W^{\mathbf{\Gamma}} \subset V_2(\mathbf{\Gamma}(\mathcal{X}))} S(\mathbf{\Gamma}(\mathcal{X})_{\setminus W^{\mathbf{\Gamma}}}). \end{aligned}$$

So, to describe the two paths in (8.23), let $\mathcal{X}$ be a connected X-graph, $\mathbf{\Gamma}$ a non-degenerate $\mathcal{X}$-shaped graph of connected graphs, and let $W^{\mathbf{\Gamma}} \subset (V_0 \amalg V_2)(\mathbf{\Gamma}(\mathcal{X}))$ be a subset of (bivalent or isolated) vertices of the colimit $\mathbf{\Gamma}(\mathcal{X})$ of $\mathbf{\Gamma}$ in CetGr.

Unless $\mathcal{X} \cong \mathcal{C}_{\mathbf{0}}$, $\mathbf{\Gamma}$ is trivial and $W = V(\mathcal{C}_{\mathbf{0}})$), it must be the case that $W^{\mathbf{\Gamma}} \subset V_2(\mathbf{\Gamma}(\mathcal{X}))$. Then, $W^{\mathbf{\Gamma}}$ is equivalently described by subsets $W^b \subset V_2(\mathbf{\Gamma}(\mathcal{X}))$, given by

$$W^b \overset{\mathrm{def}}{=} V(\mathbf{\Gamma}(b)) \cap W^{\mathbf{\Gamma}} \text{ in } V(\mathbf{\Gamma}(\mathcal{X})) \text{ for each } (\mathcal{C}, b) \in \mathsf{el}(\mathcal{X}).$$

In particular, since $E_0(\mathbf{\Gamma}(b)) \cong \mathbf{2}$ is not empty, isomorphisms $g \colon (\mathcal{C}, b) \to (\mathcal{C}', b')$ in $\mathsf{el}(\mathcal{G})$ induce bijections $W^b \cong W^{b'}$.

Otherwise – if $\mathcal{X} \cong \mathcal{C}_{\mathbf{0}}$, $\mathbf{\Gamma}$ is trivial and $W = V(\mathcal{C}_{\mathbf{0}})$ contains the unique isolated vertex – write, let $W^b = W^{\mathbf{\Gamma}} - V(\mathbf{\Gamma}(b))$, with $(\mathcal{C}_{\mathbf{0}}, b)$ the unique element of $\mathcal{X}$.

So, in both cases, we may define $W \subset V(\mathcal{X})$ to be the set of vertices v of $\mathcal{X}$ such that $W^b = V(\mathbf{\Gamma}(b))$ for any neighbourhood $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{X})$ of v. Then, $W \subset V_2(\mathcal{X})$ or $\mathcal{X} \cong \mathcal{C}_{\mathbf{0}}$ and hence $W \subset V_0(\mathcal{X})$. In each case, we obtain a unique non-degenerate $\mathcal{X}_{\setminus W}$-shaped graph of connected graphs $\mathbf{\Gamma}_{\setminus W}$

$$\mathbf{\Gamma}_{\setminus W}(\mathsf{del}_{\setminus W} \circ b) = \mathbf{\Gamma}(b)_{\setminus W^b}, \text{ and } \mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W}) = \mathbf{\Gamma}(\mathcal{X})_{\setminus W^{\mathbf{\Gamma}}}. \tag{8.24}$$

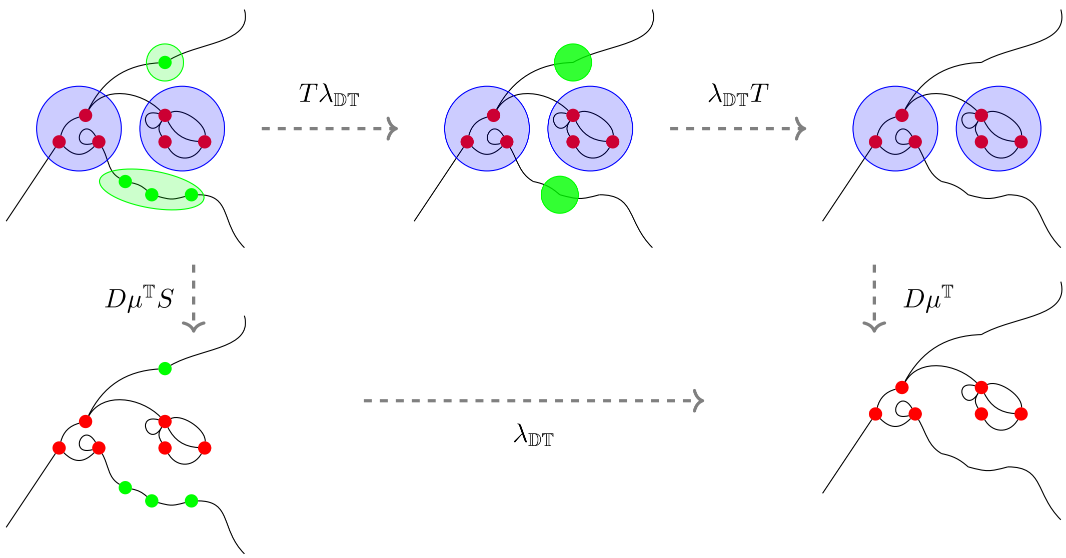


FIGURE 17. Diagram (8.23) commutes. Here the vertices in W are marked in green.

Then the two paths in (8.23), illustrated in Figure 17, are induced by the following canonical maps (where, in each case, the vertical arrows are the defining universal morphisms):

Top-right:

$$\begin{tikzcd}
(\mathcal{X}, (\mathbf{\Gamma}(b))_b, (W^b)_b) \arrow[r, dotted, "\text{delete } W^b \text{ in } \mathbf{\Gamma}(b)"] \arrow[d, dotted] & (\mathcal{X}, (\mathbf{\Gamma}(b)_{\setminus W^b})_b, W) \arrow[r, dotted, "\text{delete } W \text{ in } \mathcal{X}"] \arrow[d, dotted] & (\mathcal{X}_{\setminus W}, \mathbf{\Gamma}_{\setminus W}) \arrow[r, dotted, "\text{evaluate colimit } \mathbf{\Gamma}_{\setminus W}"] \arrow[d, dotted] & \mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W}) \arrow[d, dotted] \\
\lim_{\mathsf{el}(\mathcal{X})} S(\mathbf{\Gamma}(b)_{\setminus W^b}) \arrow[r] \arrow[d] & \lim_{\mathsf{el}(\mathcal{X})} S(\mathbf{\Gamma}(b)_{\setminus W^b}) \arrow[r] \arrow[d] & S(\mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W}) \arrow[r] \arrow[d] & S(\mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W})) \arrow[d] \\
T^2DS_X \arrow[r, "T\lambda_{\mathbb{DT}}S_X"'] & TDTS_X \arrow[r, "\lambda_{\mathbb{DT}}TS_X"'] & DT^2S_X \arrow[r, "D\mu^{\mathbb{T}}S_X"'] & DTS_X
\end{tikzcd}$$

Left-bottom:

$$\begin{tikzcd}
(\mathcal{X}, \mathbf{\Gamma}, W^{\mathbf{\Gamma}}) \arrow[r, dotted, "\text{evaluate colimit } \mathbf{\Gamma}"] \arrow[d, dotted] & (\mathbf{\Gamma}(\mathcal{X}), W^{\mathbf{\Gamma}}) \arrow[r, dotted, "\text{delete } W^{\mathbf{\Gamma}}"] \arrow[d, dotted] & \mathbf{\Gamma}(\mathcal{X})_{\setminus W^{\mathbf{\Gamma}}} \arrow[d, dotted] \\
S(\mathbf{\Gamma}(\mathcal{X})_{\setminus W^{\mathbf{\Gamma}}}) \arrow[r, equal] \arrow[d] & S(\mathbf{\Gamma}(\mathcal{X})_{\setminus W^{\mathbf{\Gamma}}}) \arrow[r, equal] \arrow[d] & S(\mathbf{\Gamma}(\mathcal{X})_{\setminus W^{\mathbf{\Gamma}}}) \arrow[d] \\
T^2DS_X \arrow[r, "\mu^{\mathbb{T}}DS_X"'] & TDS_X \arrow[r, "\lambda_{\mathbb{DT}}S_X"'] & DTS_X.
\end{tikzcd}$$

These are equal since $S(\mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W})) = S(\mathbf{\Gamma}(\mathcal{X})_{\setminus W^{\mathbf{\Gamma}}})$ by construction, and hence (8.23) commutes.

The proofs that $\lambda_{\mathbb{DT}}$ satisfies the remaining three distributive law axioms follow similarly (and are somewhat simpler than the proof that (8.23) commutes.

By [2, Section 3], there is a monad $\mathbb{T}_* = (T_*, \mu^{\mathbb{T}_*}, \eta^{\mathbb{T}_*})$ on $\mathsf{GS}_{\mathsf{E}*}$ whose EM category of algebras is equivalent to the EM category of algebras for $\mathbb{DT}$ on GS_{E}. In case $\mathsf{E} = \mathsf{Set}$, $\mathbb{T}_*$ has been described in detail in [25, Section 7], and the description of $\mathbb{T}_*$ given in [25, Corollary 7.43] for graphical species in Set, holds for graphical species in a general category E with sufficient (co)limits.

Precisely, the endofunctor T_* is the quotient of T given by

$$T_* S_{\S} = S_{\S}, \text{ and } T_* S_X = \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}} S(\mathcal{X}) \tag{8.25}$$

for all pointed graphical species $S_* = (S, \epsilon, o)$ in E. The unit $\epsilon^{T_* S} \overset{\text{def}}{=} T_*(\epsilon)\colon S_{\S} \to T_* S_{\mathbf{2}}$ is described by the obvious composite

$$S_{\S} \xrightarrow{\epsilon} S_{\mathbf{2}} \xrightarrow{\eta^{\mathbb{T}} S} TS_{\mathbf{2}} \twoheadrightarrow T_* S_{\mathbf{2}},$$

and the contracted unit $o^{T_* S} \overset{\text{def}}{=} T_*(o)\colon S_{\S} \to T_* S_{\mathbf{0}}$ is described by

$$S_{\S} \xrightarrow{o} S_{\mathbf{0}} \xrightarrow{\eta^{\mathbb{T}} S} TS_{\mathbf{0}} \twoheadrightarrow T_* S_{\mathbf{0}}.$$

In particular, $\epsilon^{T_* S}$ is defined by maps $\epsilon^k\colon S_{\S} \to S(\mathcal{L}^k)$, $k \geq 1$, and $o^{T_* S}$ by maps $o^m\colon S_{\S} \to S(\mathcal{W}^m)$, $m \geq 1$ and $o\colon S_{\S} \to S(\mathcal{C}_{\mathbf{0}})$.

The multiplication $\mu^{\mathbb{T}_*}$ for $\mathbb{T}_*$ is the one induced by $\mu^{\mathbb{T}}$, that forgets the pair $(\mathcal{X}, \mathbf{\Gamma})$ of an X-graph $\mathcal{X}$ and a non-degenerate $\mathcal{X}$-shaped graph of connected graphs $\mathbf{\Gamma}$ and replaces it with the colimit $\mathbf{\Gamma}(\mathcal{X})$, and the unit $\eta^{\mathbb{T}_*}$ for $\mathbb{T}_*$ is the obvious map

$$S_X = S(\mathcal{C}_X) \mapsto \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}} S(\mathcal{X}).$$

It is then straightforward to derive the following theorem (that was the main result, Theorem 7.46, of [25] in the case $\mathsf{E} = \mathsf{Set}$) for general E.

Theorem 8.26. *The EM category $\mathsf{GS}^{\mathbb{DT}}$ of algebras for $\mathbb{DT}$ is canonically isomorphic to* MO.

Proof. Let (A, h) be a $\mathbb{DT}$-algebra in E. Then h induces a $\mathbb{T}$-algebra structure $h^{\mathbb{T}}\colon TA \to A$ by restriction, and a $\mathbb{D}$-algebra structure $h^{\mathbb{D}}\colon DA \to A$. In particular, $h^{\mathbb{T}}$ equips A with the a multiplication $\diamond$ and contraction ζ such that $(A, \diamond, \zeta)$ is a non-unital modular operad in E, and $h^{\mathbb{D}}$, equips A with the structure of a pointed graphical species $A_* = (A, \epsilon, o)$ in E. To prove that (A, h) describes a modular operad, we must show that ϵ is a unit for $\diamond$.

This is straightforward since (A, h) is a $\mathbb{DT}$ algebra and hence (A, ϵ, o), together with $h^{\mathbb{T}}$ describe a $\mathbb{T}_*$-algebra. Whence it follows from (8.25) that ϵ is a unit for $\diamond$.

Conversely, if $(S, \diamond, \zeta, \epsilon)$ is a modular operad in E, then $(S, \diamond, \zeta)$ defines a $\mathbb{T}$-algebra $p^{\mathbb{T}}\colon TS \to S$ by Proposition 8.2, and $(S, \epsilon, \zeta\epsilon)$ defines a $\mathbb{D}$-algebra by Section 8.2. In particular, since ϵ is a unit for $\diamond$, and $p^{\mathbb{T}}$ is induced by iterations of $\diamond$ and ζ, $p^{\mathbb{T}}$ induces a $\mathbb{T}_*$-algebra structure on $(S, \epsilon, \zeta\epsilon)$ by (8.25), and hence $(S, \diamond, \zeta, \epsilon)$ describes an algebra for $\mathbb{DT}$. ☐

9. A monad for circuit operads

Despite the similarity between the monads $\mathbb{T}$ and $\mathbb{T}^{\times}$ on GS_{E}, it is not possible to modify $\lambda_{\mathbb{DT}}$ by replacing $\mathbb{T}$ with $\mathbb{T}^{\times}$ to obtain a distributive law for $\mathbb{D}$ and $\mathbb{T}^{\times}$. Namely, for all graphical species S, there is a canonical morphism

$$S(\shortmid \amalg \shortmid) = S_{\S} \times S_{\S} \to DS_{\mathbf{2}} \times DS_{\mathbf{2}} = DS(\mathcal{C}_{\mathbf{2}} \amalg \mathcal{C}_{\mathbf{2}}) \to TDS_{\mathbf{2} \amalg \mathbf{2}},$$

and $DS_{\mathbf{2}} \times DS_{\mathbf{2}} = (S_{\mathbf{2}} \amalg S_{\S}) \times (S_{\mathbf{2}} \amalg S_{\S})$ by definition. However, there is, in general, no morphism

$$(S_{\mathbf{2}} \amalg S_{\S}) \times (S_{\mathbf{2}} \amalg S_{\S}) \to TS_{\mathbf{2} \amalg \mathbf{2}} = DTS_{\mathbf{2} \amalg \mathbf{2}}.$$

The problem of finding a circuit operad monad on GS_E is easily solved when we observe that the monad $\mathbb{T}^\times$ is, itself a composite of $\mathbb{T}$ with the monad $\mathbb{L} = (L, \mu^\mathbb{L}, \eta^\mathbb{L})$ induced by the free graded monoid monad on $\mathsf{psh}(\Sigma)$. Then, we can form the composite monad $\mathbb{LDT}$ if there is a distributive law $\lambda_{\mathbb{L}(\mathbb{DT})} \colon (DT)L \Rightarrow L(DT)$.

By [7], such a distributive law $\lambda_{\mathbb{L}(\mathbb{DT})}$ exists if there are pairwise distributive laws $\lambda_{\mathbb{LT}} \colon TL \Rightarrow LT$, and $\lambda_{\mathbb{LD}} \colon DL \Rightarrow LD$ on GS_E, such that the diagram

$$\begin{tikzcd}
 & DTL \arrow[r, "D\lambda_{\mathbb{LT}}"] & DLT \arrow[dr, "\lambda_{\mathbb{LD}}T"] & \\
TDL \arrow[ur, "\lambda_{\mathbb{DT}}L"] \arrow[dr, "T\lambda_{\mathbb{LD}}"'] & & & LDT \\
 & TLD \arrow[r, "\lambda_{\mathbb{LT}}D"'] & LTD \arrow[ur, "L\lambda_{\mathbb{DT}}"'] &
\end{tikzcd} \tag{9.1}$$

commutes.

In this section, we construct the monad $\mathbb{L}$ and show that there is a distributive law $\lambda_{\mathbb{LT}} : TL \Rightarrow LT$ such that the induced composite monad is precisely $\mathbb{T}^\times$.

In the following section, we describe the distributive law $\lambda_{\mathbb{LD}} : DL \Rightarrow LD$ such that the diagram (9.1) commutes, and then show that the EM category of algebras for the composite monad $\mathbb{LDT}$ on GS_E is canonically equivalent to the category CO_E of circuit operads in E.

9.1. **The free external product monad $\mathbb{L}$.** As usual, let E be a category with sufficient (co)limits. The free graded monoid endofunctor L on GS_E is given by

$$\begin{array}{rcl} LS_\S & = & S_\S \\ LS_X & = & \mathrm{colim}_{(Y,f)\in \mathrm{Core}(X/\mathsf{Set_f})} \prod_{y\in Y} S_{f^{-1}(y)} \end{array} \tag{9.2}$$

for all finite sets X.

Then L underlies a monad $\mathbb{L} = (L, \mu^\mathbb{L}, \eta^\mathbb{L})$ where the monadic multiplication $\mu^\mathbb{L}$ is induced by concatenation of lists, and the monadic unit $\eta^T T$ is induced by inclusion of one element lists.

The following lemma is immediate:

Lemma 9.3. *The EM category of algebras for the monad $\mathbb{L}$ on GS_E is equivalent to the category of graphical species with external product in E.*

In order to describe a distributive law $\lambda_{\mathbb{LT}} \colon TL \Rightarrow$ for $\mathbb{L}$ and $\mathbb{T}$, we will first rewrite (9.2) as follows: Let S be a graphical species in E, and X a finite set, then

$$\begin{array}{rcl} LS_X & = & \mathrm{colim}_{(Y,f)\in \mathrm{Core}(X/\mathsf{Set_f})} \prod_{y\in Y} S_{f^{-1}(y)} \\ & = & \mathrm{colim}_{\mathcal{X}\in X\text{-}\mathsf{Cor}_{\boxtimes\mathrm{iso}}} S(\mathcal{X}) \end{array}$$

where $X\text{-}\mathsf{Cor}_{\boxtimes\mathrm{iso}} \subset X\text{-}\mathsf{etGr}_{\mathrm{iso}}$ is the groupoid of X-graphs $(\mathcal{G}, \rho)$ such that $\mathcal{G}$ is a disjoint union of corollas. In particular, from this point of view, the multiplication $\mu^\mathbb{L}$ for $\mathbb{L}$ is just the restriction of the multiplication $\mu^{\mathbb{T}^\times}$ for $\mathbb{T}^\times$ (Proposition 7.15) to $\mathcal{X}$-shaped graphs of disjoint unions of corollas, where $\mathcal{X}$ itself is a disjoint union of corollas. And the unit $\eta^\mathbb{L}$ for $\mathbb{L}$, is just the obvious corestriction of the unit $\eta^{\mathbb{T}^\times}$.

It follows that, for all finite sets X,

$$\begin{array}{rcl} TLS_X & = & \mathrm{colim}_{\mathcal{X}\in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{X})} LS(\mathcal{C}) \\ & = & \mathrm{colim}_{\mathcal{X}\in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} \lim_{\mathbf{\Gamma}\in \mathsf{Cor}_\boxtimes^{(\mathcal{X})}} S(\mathbf{\Gamma}(\mathcal{X})) \end{array}$$

where $\mathsf{Cor}_{\boxtimes}^{(\mathcal{X})}$ is the full subgroupoid of $\mathsf{etGr}^{(\mathcal{X})}$ whose objects are non-degenerate $\mathcal{X}$-shaped graphs of graphs $\mathbf{\Gamma}$ such that $\mathbf{\Gamma}(\mathcal{C}_{X_b}, b) \in X\text{-}\mathsf{Cor}_{\boxtimes\mathrm{iso}}$ is a disjoint union of corollas for each $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{X})$.

Now, each $\mathbf{\Gamma}(\mathcal{X})$ canonically has the structure of an X-graph, so there is a canonical morphism $TLS_X \to \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{X})$, where now the colimit is over all X-graphs and their isomorphisms. Then, since a graph $\mathcal{X}$ is a finite disjoint union of its connected components $\mathcal{X} = \coprod_{k \in K} \mathcal{X}_k$ and therefore $S(\mathcal{X}) = \prod_{k \in K} S(\mathcal{X}_k)$ describes an element of LTS_X. Let $\lambda_{\mathbb{LT}} \colon TL \Rightarrow LT$ be the natural transformation so obtained. So, $\lambda_{\mathbb{LT}}$ takes a colimit – evaluated over pairs $(\mathcal{X}, \mathbf{\Gamma})$ where $\mathcal{X}$ is an X-graph and $\mathbf{\Gamma} \in \mathsf{Cor}_{\boxtimes}^{(\mathcal{X})}$ with colimit $\mathbf{\Gamma}(\mathcal{X}) = \coprod_{k \in K} \mathbf{\Gamma}(\mathcal{X})_k$, and isomorphisms of $\mathcal{X}$ – of objects $S(\mathbf{\Gamma}(\mathcal{X}))$, and forgets the pair $(\mathcal{X}, \mathbf{\Gamma})$, to obtain a colimit of elements $S(\mathbf{\Gamma}(\mathcal{X}))$, evaluated over $X\text{-}\mathsf{etGr}_{\mathrm{iso}}$.

Lemma 9.4. *The natural transformation $\lambda_{\mathbb{LT}} \colon TL \Rightarrow LT$ satisfies the four axioms of a distributive law, and $\mathbb{LT} = \mathbb{T}^{\times}$ on GS_{E}. Hence, algebras for $\mathbb{LT}$ are non-unital circuit operads.*

Proof. Since a graph is the disjoint union of its connected components, $LT = T^{\times} \colon \mathsf{GS}_{\mathsf{E}} \to \mathsf{GS}_{\mathsf{E}}$.

Moreover, by the description of $\mathbb{L}$ in terms of graphs of graphs, TL is a subfunctor of $T^{\times 2}$, and $\lambda_{\mathbb{LT}}$ is just the restriction to TL of $\mu^{\mathbb{T}^{\times}} \colon T^{\times 2} \Rightarrow T^{\times} = LT$ on GS_{E}. The four distributive law axioms then follow immediately from the monad axioms. □

There is an obvious choice of natural transformation $\lambda_{\mathbb{LD}} \colon DL \Rightarrow LD$ to define the distributive law for $\mathbb{L}$ and $\mathbb{D}$: By definition of D, we may define $\lambda_{\mathbb{LD}}$ in terms of the maps $L\eta^{\mathbb{D}} \colon L \Rightarrow LD$ on $LS \hookrightarrow DLS$, $S \in \mathsf{GS}_{\mathsf{E}}$, and extend this to a morphism on DLS by

$$S_{\S} \to DS_{\mathbf{2}} \xrightarrow{\eta^{\mathbb{L}} DS} LDS_{\mathbf{2}}, \quad \text{and } S_{\S} \to DS_{\mathbf{0}} \xrightarrow{\eta^{\mathbb{L}} DS} LDS_{\mathbf{0}}.$$

Informally, $\lambda_{\mathbb{LD}}$ takes a tuple $(S_{X_i})_{i=1}^n$ of objects in the image of S, and regards it as a tuple in the image of DS, and takes the image of $S_{\S}$ in $DS_{\mathbf{0}}$ or $DS_{\mathbf{2}}$ and regards it as a 1-tuple in the image of DS.

This construction is natural in S, and it is simple to show that $\lambda_{\mathbb{LD}}$ so obtained describes a distributive law for $\mathbb{L}$ and $\mathbb{D}$.

9.2. **An iterated distributive law for circuit operads.** First observe that, given a graph $\mathcal{G}$, $W \subset V_2$ a subset of bivalent vertices of $\mathcal{G}$, and $\mathbf{\Gamma}_{\setminus W}$ a non-degenerate $\mathcal{G}_{\setminus W}$-shaped graph of graphs, we can always define a non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}^W$ by

$$(\mathcal{C}, b) \mapsto \begin{cases} \mathcal{C}_{\mathbf{2}} & (\mathcal{C}, b) \text{ is a neighbourhood of } v \in W \\ \mathbf{\Gamma}_{\mathcal{W}}(\mathsf{del}_{\setminus W} \circ b) & \text{otherwise.} \end{cases} \tag{9.5}$$

(See Figure 18.) Then $\mathbf{\Gamma}^W$ induces an inclusion $W \hookrightarrow V_2(\mathbf{\Gamma}^W(\mathcal{X}))$, and $\mathbf{\Gamma}(\mathcal{X})_{\setminus W} = \mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W})$. Hence there is a canonical full embedding $\mathsf{etGr}^{(\mathcal{G}_{\setminus W})} \hookrightarrow \mathsf{etGr}^{(\mathcal{G})}$, that restricts to an embedding $\mathsf{Cor}_{\boxtimes}^{(\mathcal{G}_{\setminus W})} \hookrightarrow \mathsf{Cor}_{\boxtimes}^{(\mathcal{G})}$ and preserves similarity classes of colimits.

Proposition 9.6. *The triple of distributive laws $\lambda_{\mathbb{DT}} \colon TD \Rightarrow DT$, $\lambda_{\mathbb{LT}} \colon TL \Rightarrow LT$, and $\lambda_{\mathbb{LDT}} \colon DL \Rightarrow LD$ on GS satisfy the Yang-Baxter equation (9.1), and hence there exists a composite monad $\mathbb{LDT}$ on GS_{E}.*

Proof. By [7, Theorem 1.6], we must check that the diagram (9.1) commutes.

For all graphical species S in E and all finite sets X,

$$TDLS_X = \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathrm{iso}}} \coprod_{W \subset (V_0 \amalg V_2)(\mathcal{X})} \lim_{\mathbf{\Gamma}_{\setminus W} \in \mathsf{Cor}_{\boxtimes}^{(\mathcal{X}_{\setminus W})}} S(\mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W}))$$

by the previous discussion. So, let be given a triple $(\mathcal{X}, W, \mathbf{\Gamma}_{\setminus W})$ where $\mathcal{X}$ is a connected X-graph, $W \subset V_2(\mathcal{X})$ (or $\mathcal{X} \cong \mathcal{C}_{\mathbf{0}}$ and $W = V(\mathcal{C}_{\mathbf{0}})$) and $\mathbf{\Gamma}_{\setminus W}$ is a non-degenerate $\mathcal{X}_{\setminus W}$-shaped graph of graphs such that $\mathbf{\Gamma}_{\setminus W}(b)$ is a disjoint union of corollas for each $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{X}_{\setminus W})$.

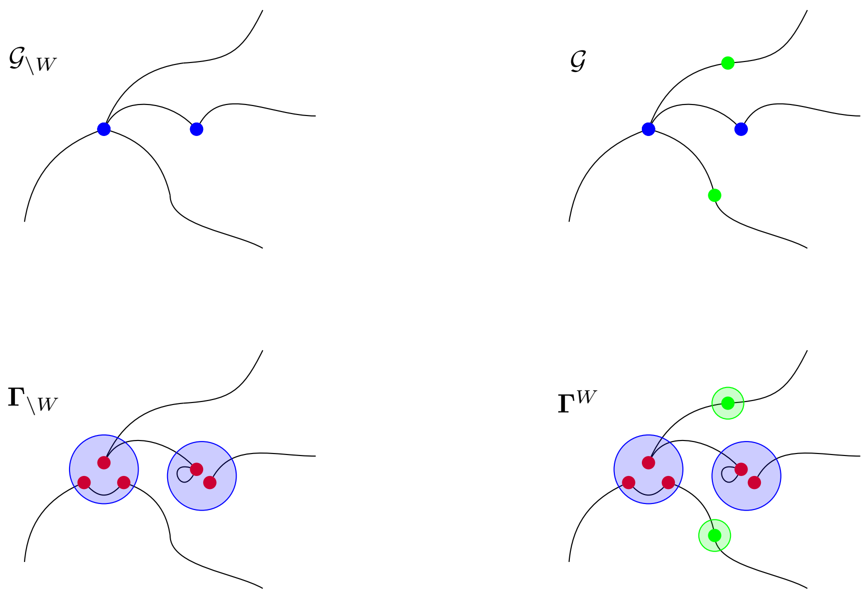


FIGURE 18. The $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}^W$ described in (9.5), and the $\mathcal{G}_{\setminus W}$-shaped graph of graphs $\mathbf{\Gamma}_{\setminus W}$ have similar colimits.

First observe that if $W = V(\mathcal{X})$, and then either $\mathcal{X} \cong \mathcal{C}_{\mathbf{0}}$ or $\mathcal{X} \cong \mathcal{L}^k \in \mathbf{2}\text{-}\mathsf{CGr}_{\mathrm{iso}}$ or $\mathcal{X} \cong \mathcal{W}^m \in \mathbf{0}\text{-}\mathsf{CGr}_{\mathrm{iso}}$. In each case $\mathcal{X}_{\setminus W} \cong (\shortmid)$ and hence $\mathbf{\Gamma}_{\setminus W}$ is the trivial graph of graphs $id \mapsto (\shortmid)$. So the canonical map $S(\mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W})) \to FS_X$ where $F \in \{DTL, DLT, LDT, TLD, LTD\}$ factors through $S_{\S}$, and it is immediate that the top and bottom paths of (9.1) coincide for this representative.

Assume therefore that $W \neq V(\mathcal{X})$. The two paths in (9.1) are then described by the following data:

Top:

$$\begin{tikzcd}
 & (\mathcal{X}_{\setminus W}, \mathbf{\Gamma}_{\mathcal{W}}) \arrow[r, dotted, "\text{take colimit } \mathbf{\Gamma}_{\setminus W}"] \arrow[d, dotted] & \mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W}) \arrow[d, dotted] \arrow[dr, dotted, "L\eta^{\mathbb{D}}T"] & \\
(\mathcal{X}, W, \mathbf{\Gamma}_{\setminus W}) \arrow[ur, dotted, "\text{delete } W \text{ in } \mathcal{X}"] \arrow[d, dotted] & S(\mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W})) \arrow[r] \arrow[d] & \operatorname{colim}_{\mathcal{X}' \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{X}') \arrow[d] \arrow[dr] & \mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W}) \arrow[d, dotted] \\
S(\mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W})) \arrow[ur] \arrow[d] & DTLS_X \arrow[r, "D\lambda_{\mathbb{LT}}S"'] & DLTS_X \arrow[dr, "\lambda_{\mathbb{LD}}TS"] & \operatorname{colim}_{\mathcal{X}' \in X\text{-}\mathsf{etGr}_{\mathrm{iso}}} S(\mathcal{X}') \arrow[d] \\
TDLS_X \arrow[ur, "\lambda_{\mathbb{DT}}LS"] & & & LDTS_X
\end{tikzcd}$$

where, in each case, we evaluate $S(\mathbf{\Gamma}_{\setminus W}(\mathcal{X}_{\setminus W}))$ and take the appropriate colimit of such objects.

Bottom: The bottom path of (9.1) gives

$$\begin{tikzcd}
(\mathcal{X}, W, \mathbf{\Gamma}_{\backslash W}) \arrow[d, dotted] \arrow[dr, dotted, "\text{forget } W"] & & & \mathbf{\Gamma}_{\backslash W}(\mathcal{X}_{\backslash W}) \arrow[d, dotted] \\
S(\mathbf{\Gamma}_{\backslash W}(\mathcal{X}_{\backslash W})) \arrow[d] \arrow[dr] & (\mathcal{X}, \mathbf{\Gamma}^W, W) \arrow[r, dotted, "\text{take colimit } \mathbf{\Gamma}^W"] \arrow[d, dotted] & (\mathbf{\Gamma}^W(\mathcal{X}), W) \arrow[ur, dotted, "\text{delete } W"] \arrow[d, dotted] & S(\mathbf{\Gamma}(\mathcal{X})_{\backslash W^{\mathbf{\Gamma}}}) \arrow[d] \\
TDLS_X \arrow[dr, "T\lambda_{\mathbb{DL}}S"'] & S(\mathbf{\Gamma}^W(\mathcal{X})_{\backslash W}) \arrow[r] \arrow[d] & S(\mathbf{\Gamma}^W(\mathcal{X})_{\backslash W}) \arrow[ur] \arrow[d] & LDTS_X \\
& TLDS_X \arrow[r, "\lambda_{\mathbb{LT}}DS"'] & LTDS_X \arrow[ur, "L\lambda_{\mathbb{DT}}S"'] &
\end{tikzcd}$$

where $\mathbf{\Gamma}^W$ is the non-degenerate $\mathcal{X}$-shaped graph of graphs in $\mathsf{Cor}_{\boxtimes}^{(\mathcal{X})}$ obtained from $\mathbf{\Gamma}_{/W}$ as in (9.5).

Hence the diagram (9.1) commutes, and so $(\lambda_{\mathbb{LD}}, \lambda_{\mathbb{LT}}, \lambda_{\mathbb{DT}})$ describe an iterated distributive law for $\mathbb{L}, \mathbb{D}$ and $\mathbb{T}$ on GS_{E}. $\square$

It therefore remains to prove:

Theorem 9.7. *The EM category of algebras for the composite monad $\mathbb{LDT}$ on GS_{E} is isomorphic to the category CO_{E} of circuit operads in E.*

Proof. By [2], and Lemma 9.4, an algebra (A, h) for the composite monad $\mathbb{LDT}$ on GS_{E} has an $\mathbb{LT}$-algebra structure $h^{\mathbb{LT}} \colon LTA \to A$, and also a $\mathbb{DT}$ algebra structure $h^{\mathbb{DT}} \colon DTA \to A$ with each structure given by the corresponding restriction $LTA \to LDTA \xrightarrow{h} A$, respectively $DTA \to LDTA \xrightarrow{h} A$. Hence, by Lemma 9.4, h induces an external product $\boxtimes$ on A, and contraction ζ on A such that $(A, \boxtimes, \zeta)$ satisfies (B1)-(B3). Moreover, by Theorem 8.26, h induces a unital multiplication $(\diamond, \epsilon)$ on A where $\diamond_{X,Y}^{x \ddagger y} = \zeta_{X \amalg Y}^{x \ddagger y} \boxtimes_{X,Y} \colon (A_X \times A_Y)^{x \ddagger y} \to A_{X \amalg Y \setminus \{x,y\}}$ and, hence

$$S_X \xrightarrow{(id \times ch_x) \circ \Delta} (S_X \times S_{\S})^{x \ddagger 2} \xrightarrow{id \times \epsilon} (S_X \times S_{\mathbf{2}})^{x \ddagger 2} \xrightarrow{\zeta_{X \amalg \mathbf{2}}^{x \ddagger 2} \circ \boxtimes} S_X$$

is the identity on S_X. So, $(A, h))$ has a canonical circuit operad structure.

Conversely, a circuit operad $(S, \boxtimes, \zeta, \epsilon)$ in E, has an underlying modular operad structure $(S, \diamond, \zeta, \epsilon)$, where $\diamond_{X,Y}^{x \ddagger y} = \zeta_{X \amalg Y}^{x \ddagger y} \boxtimes_{X,Y}$ and hence $(S, \boxtimes, \zeta, \epsilon)$ defines a $\mathbb{DT}$-algebra structure $h^{\mathbb{DT}} \colon DTS \to S$ by Theorem 8.26. By Lemma 9.3, $(S, \boxtimes)$ has an $\mathbb{L}$-algebra structure where $h^{\mathbb{L}} \colon LS \to S$ is given by $\boxtimes_{X,Y} \colon S_X \times S_Y$ on $S_X \times S_Y$. It follows immediately from the circuit operad axioms that $h^{\mathbb{L}} \circ Lh^{\mathbb{DT}} \colon LDTS \to S$ gives S the structure of an $\mathbb{LDT}$-algebra.

These maps are clearly functorial and extend to an equivalence $\mathsf{CO}_{\mathsf{E}} \simeq \mathsf{GS}_{\mathsf{E}}{}^{\mathbb{LDT}}$. $\square$

Theorem 9.8. *For each palette $(\mathfrak{C}, \omega)$, there is an equivalence of the categories $\mathsf{CA}^{(\mathfrak{C}, \omega)} \simeq {}^{(\mathfrak{C}, \omega)}\mathsf{CO}$ of $(\mathfrak{C}, \omega)$-coloured circuit algebras and circuit operads in Set. This extends to an equivalence $\mathsf{CA} \simeq \mathsf{CO}$ of the categories of all circuit algebras and all circuit operads in Set.*

Proof. By Proposition 5.18, there is a faithful functor $\mathsf{CA} \to \mathsf{CO}$ described in Section 5.

For the converse, let (A, h) be an algebra for $\mathbb{LDT}$, with $(A_{\S}, A_{\tau}) = (\mathfrak{C}, \omega)$, and let $(\overline{f}, \lambda)$ in $WD^{(\mathfrak{C}, \omega)}(\boldsymbol{c}, \dots, \boldsymbol{c}_k; \boldsymbol{d})$ be a wiring diagram with underlying uncoloured Brauer diagram $f = (\tau, \mathfrak{k}) \in \mathsf{BD}(\sum_{i=1}^k m_i, n)$, where n is the length of $\boldsymbol{d}$ and m_i is the length of $\boldsymbol{c}_i$ for each $1 \leq i \leq k$.

By (6.1), the open part $(\tau, 0)$ of $\overline{f}$ describes an $\mathbf{n}$-graph $\mathring{\mathcal{X}}$ with k ordered vertices $(v_i)_{i=1}^k$ such that $v_i \in V_{m_i}$ for all i, such that there is a given choice of essential morphism $\iota_i \colon \mathcal{C}_{\mathbf{v_i}} \to \mathring{\mathcal{X}}$. Moreover, the colouring λ of $\overline{f}$ describes an involution preserving map $\lambda_\partial \colon E(\mathring{\mathcal{X}}) \to \mathfrak{C}$, such that the $\mathbf{n}$ ports of $\mathring{\mathcal{X}}$ are coloured by $\boldsymbol{d}$.

So, for any k-tuple $(\phi_1, \dots, \phi_k) \in A_{\boldsymbol{c}_1} \times \cdots \times A_{\boldsymbol{c}_k}$ there is an element $\alpha \in A(\mathring{\mathcal{X}})$ defined by $A(\iota_i)(\alpha) = \phi_i \in A(\mathcal{C}_{\mathbf{v_i}})$ for each vertex v_i, and $A(ch_e)(\alpha) = \lambda_\partial(e)$ for each edge $e \in E(\mathring{\mathcal{X}})$. Since there is a canonical inclusion $LT(A) \subset LDT(A)$, the assignment $(\phi_1, \dots, \phi_k) \mapsto \alpha \in A(\mathring{\mathcal{X}})$ extends to a map

$$A(\mathring{\overline{f}}) \colon A_{\boldsymbol{c}_1} \times \cdots \times A_{\boldsymbol{c}_k} \to A(\mathring{\mathcal{X}}) \to LT(A)_{\boldsymbol{d}} \to LDT(A)_{\boldsymbol{d}} \xrightarrow{h} A_{\boldsymbol{d}}$$

induced by the open part of $\overline{f}$.

Moreover, the colouring λ of $\overline{f}$ determines a $\mathfrak{k}$-tuple $(\tilde{c}^1, \dots, \tilde{c}^{\mathfrak{k}}) \in \widetilde{\mathfrak{C}}^{\mathfrak{k}}$, and hence, there is a morphism of sets

$$A(\overline{f}, \lambda) \colon A_{\boldsymbol{c}_1} \times \cdots \times A_{\boldsymbol{c}_k} \to A_{\boldsymbol{d}}$$

determined by

$$(\phi_1, \dots, \phi_k) \mapsto \Big(A(\mathring{\overline{f}})(\phi_1, \dots, \phi_k)\Big) \boxtimes o_{\tilde{c}^1} \boxtimes \cdots \boxtimes o_{\tilde{c}^{\mathfrak{k}}} \in A(\boldsymbol{d}).$$

Since composition in $WD^{(\mathfrak{C},\omega)}$ agrees (up to closed components of wiring diagrams) with the multiplication $\mu^{\mathbb{LDT}}$ for the monad $\mathbb{LDT}$, it follows from Theorem 9.7 – and in particular, the definition of the distributive law $\lambda_{\mathbb{DT}}$ – that the collection of morphisms $\big(A(\overline{f}, \lambda)\big)_{(\overline{f},\lambda) \in WD^{(\mathfrak{C},\omega)}}$ so defined endows the $\mathsf{list}\mathfrak{C}$-indexed collection $(A_{\boldsymbol{c}})_{\mathsf{list}\mathfrak{C}}$ with the structure of a Set-valued algebra for $WD^{(\mathfrak{C},\omega)}$.

It is straightforward to verify that this assignment is natural in (A, h), and hence extends to a functor $\mathsf{CO}_{\mathsf{Set}} \to \mathsf{CA}_{\mathsf{Set}}$ such that, moreover, the pair of functors $\mathsf{CA}_{\mathsf{Set}} \leftrightarrows \mathsf{CO}_{\mathsf{Set}}$ in fact define an (adjoint) equivalence of categories. □

In other words, for any palette $(\mathfrak{C}, \omega)$, the operad $WD^{(\mathfrak{C},\omega)}$ of $(\mathfrak{C}, \omega)$-coloured wiring diagrams is the *operad of* $(\mathfrak{C}, \omega)$*-coloured circuit operads in* Set. This is important for the description of weak circuit operads in [?].

10. A nerve theorem for circuit operads

This section recalls the nerve theorem for Set-valued modular operads [25, THM], and gives a proof – by the same methods – of an analogous result for circuit operads in Set.

10.1. Extending the nerve theorem for modular operads. The nerve theorem for modular operads, in the style of [30, 3], was proved in [25, Theorem 8.2]. There, it was shown that, though the modular operad monad $\mathbb{DT}$ on GS does not satisfy the conditions of [3, Sections 1 & 2] ($\mathbb{DT}$ does not have arities CetGr), its lift $\mathbb{T}_*$ to the category GS_* of $\mathbb{D}$-algebras, together with the category CetGr_* does.

Once again, let GS_* be the EM category of algebras for the monad $\mathbb{D}$ on GS and let $\mathsf{CetGr}_* \hookrightarrow \mathsf{GS}_*$ be the dense category obtained in the bo-ff factorisation of the functor $\mathsf{CetGr} \to \mathsf{GS} \to \mathsf{GS}_*$. By [2, Section 3], there is a monad $\mathbb{T}_*$ on GS_* whose EM category of algebras is equivalent to the category MO of modular operads, and hence a commuting diagram of functors, where Ξ is obtained in bo-ff factorisation of the functor $\mathsf{CetGr} \to \mathsf{GS} \to \mathsf{MO}$:

(10.1)
$$\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}^{\mathbb{T}_*}"] & \mathsf{psh}(\Xi) \arrow[d, "j^*"] \\
\mathbf{B}^{\S}_* \arrow[r, hook, "\text{f.f.}"'] & \mathsf{CetGr}_* \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS}_* \arrow[u, shift left, "\text{free}^{\mathbb{T}_*}"] \arrow[r, hook, "\text{f.f.}"'] \arrow[d, shift left, "\text{forget}^{\mathbb{D}}"] & \mathsf{psh}(\mathsf{CetGr}_*) \arrow[d] \\
\mathbf{B}^{\S} \arrow[u, "\text{b.o.}"] \arrow[ur, "\text{dense}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{CetGr} \arrow[u, "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}^{\mathbb{D}}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{psh}(\mathsf{CetGr}).
\end{tikzcd}$$

Using the methods of [3, Sections 1 & 2], the following was proved in [25, Theorem ??].

Theorem 10.2. *The monad $\mathbb{T}_*$ has arities CetGr_*. Moreover, $\mathbf{B}^\S$ is dense in CetGr_* and therefore, the nerve $N\colon \mathsf{MO} \to \mathsf{psh}(\Xi)$ is fully faithful and its essential image consists of those presheaves P on Ξ such that*

$$P(\mathcal{G}) = \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \textit{ for all graphs } \mathcal{G}.$$

Similarly, the circuit operad monad $\mathbb{LDT}$ on GS does not have arities. However, by (2.5) there exists a lift $\tilde{\mathbb{L}}$ of $\mathbb{L}$ to GS_* whose EM category $\mathsf{GS}_*^{\tilde{\mathbb{L}}}$ of algebras is equivalent to the EM category of algebras $\mathsf{GS}^{\mathbb{DL}}$ for $\mathbb{LD}$ on GS.

In particular, there is a distributive law $\tilde{\lambda}_{\mathbb{LT}}\colon T_*\tilde{L} \Rightarrow \tilde{L}T_*$ on GS_* induced by $\lambda_{\mathbb{LT}}$ on GS such that algebras for the composite monad $\tilde{\mathbb{L}}\mathbb{T}_*$ on GS_* are canonically equivalent to algebras for $\mathbb{LDT}$ on GS, and hence to circuit operads.

As will be shown in Remark 10.15, the monad $\tilde{\mathbb{L}}\mathbb{T}_*$ does not have arities CetGr_*. However, the category etGr_* is also dense in GS_*, and so we consider the commuting diagram of functors

(10.3)
$$\begin{tikzcd}
 & \Xi^{\times} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N_{\Xi^\times}"] \arrow[d, shift left, "\mathrm{forget}^{\tilde{\mathbb{L}}\mathbb{T}_*}"] & \mathsf{psh}(\Xi^\times) \arrow[d, "j^*"] \\
\mathbf{B}^\S_* \arrow[r, hook, "\text{f.f.}"'] & \mathsf{etGr}_* \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS}_* \arrow[u, shift left, "\mathrm{free}^{\tilde{\mathbb{L}}\mathbb{T}_*}"] \arrow[r, hook, "\text{f.f.}"'] \arrow[d, shift left, "\mathrm{forget}^{\mathbb{D}}"] & \mathsf{psh}(\mathsf{etGr}_*) \arrow[d] \\
\mathbf{B}^\S \arrow[u, "\text{b.o.}"] \arrow[ur, "\text{dense}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{etGr} \arrow[u, "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\mathrm{free}^{\mathbb{D}}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{psh}(\mathsf{CetGr}).
\end{tikzcd}$$

where $\Xi^\times$ is the category obtained in the bo-ff factorisation of $\mathsf{etGr} \to \mathsf{GS} \to \mathsf{CO}$.

Since the induced nerve $\mathsf{GS}_* \to \mathsf{psh}(\mathsf{etGr}_*)$ is fully faithful, it follows from [3] that the nerve functor $N_{\Xi^\times}$ is fully faithful if $\tilde{\mathbb{L}}\mathbb{T}_*$ on GS_* has arities etGr_*. In this case, since $\mathbf{B}^\S$ is dense in etGr_*:

Theorem 10.4. *The functor $N\colon \mathsf{CO} \to \mathsf{psh}(\Xi^\times)$ is full and faithful. Its essential image consists of precisely those presheaves P on $\Xi^\times$ whose restriction to $\mathsf{psh}(\Xi)$ are graphical species. In other words,*

(10.5)
$$P(\mathcal{G}) = \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \textit{ for all graphs } \mathcal{G}.$$

10.2. **The monad $\tilde{\mathbb{L}}$ on GS_*.** The lift $\tilde{\mathbb{L}} = (\tilde{L}, \mu^{\tilde{\mathbb{L}}}, \eta^{\tilde{\mathbb{L}}})$ of $\mathbb{L}$ to $\mathsf{GS}_{\mathsf{E}*}$ is easy to describe: Given a pointed graphical species $S_* = (S, \epsilon, o)$ in E, $\tilde{L}(S_*) = (LS, \eta^{\mathbb{L}}\epsilon, \eta^{\mathbb{L}}o)$, and $\mu^{\tilde{\mathbb{L}}}, \eta^{\tilde{\mathbb{L}}}$ are induced by $\mu^{\mathbb{L}}$ and $\eta^{\mathbb{L}}$ in the obvious way.

Similarity morphisms in etGr_* preserve connected components of graphs, and therefore for all finite sets X, we may define the category $X\text{-}\mathsf{Gr}_{\mathsf{sim}}$ of all X-graphs and similarity morphisms in terms of the categories $f^{-1}(y)\text{-}\mathsf{CGr}_{\mathsf{sim}}$ of connected $f^{-1}(y)$-graphs and similarity morphisms by

$$X\text{-}\mathsf{Gr}_{\mathsf{sim}} \overset{\text{def}}{=} \operatorname{colim}_{(Y,f)\in \mathrm{Core}(X/\mathsf{Set}_{\mathsf{f}})} \coprod_{y\in Y} f^{-1}(y)\text{-}\mathsf{CGr}_{\mathsf{sim}}.$$

Remark 10.6. Since $X\text{-}\mathsf{Gr}_{\mathsf{sim}}$ includes disjoint unions of morphisms of the form $\mathsf{del}_{\backslash V}$ and z with codomain $(\shortmid)$, objects of $X\text{-}\mathsf{Gr}_{\mathsf{sim}}$ are pairs of graphs $\mathcal{G} \amalg \mathcal{S}$ where $\mathcal{S}$ is a (possibly) empty shrub, together with an X-graph structure $\rho\colon E_0(\mathcal{G}) \xrightarrow{\cong} X$. In particular, each connected component of $X\text{-}\mathsf{CGr}_{\mathsf{sim}}$ contains an X-graph.

Since morphisms in $X\text{-}\mathsf{Gr}_{\mathsf{sim}}$ preserve connected components of graphs, for any X-graph $\mathcal{X}$, the connected component of $X\text{-}\mathsf{Gr}_{\mathsf{sim}}$ containing $\mathcal{X}$ has a terminal object $\mathcal{X}^\perp$ without isolated of bivalent vertices.

So, the endofunctor $\tilde{L}T_*$ admits a description as the obvious quotient of $T^{\times} = LT$:

$$\begin{array}{rcl}\tilde{L}T_*S_X & = & \mathrm{colim}_{(Y,f)\in \mathrm{Core}(X/\mathsf{Set_f})} \prod_{y\in Y} T_* S_{f^{-1}(y)} \\ & = & \mathrm{colim}_{(Y,f)\in \mathrm{Core}(X/\mathsf{Set_f})} \prod_{y\in Y} \left(\mathrm{colim}_{(\mathcal{G},\rho)\in f^{-1}(y)\text{-}\mathsf{CGr_{sim}}} S(\mathcal{G})\right) \\ & = & \mathrm{colim}_{\mathcal{X}\in X\text{-}\mathsf{Gr_{sim}}} S(\mathcal{X}).\end{array} \tag{10.7}$$

10.3. **A graphical category for circuit operads.** The category etGr_* obtained in the bo-ff factorisation of $\mathsf{etGr} \to \mathsf{GS} \to \mathsf{GS}_*$ has been discussed in Section 8.3 Now let $\Xi^{\times}$ be the category obtained in the bo-ff factorisation of $\mathsf{etGr} \to \mathsf{GS} \to \mathsf{CO}$. This is the full subcategory of the Kleisli category $\mathsf{GS}_{*\tilde{\mathbb{L}}\mathbb{T}_*}$ of $\tilde{\mathbb{L}}\mathbb{T}_*$ on free circuit operads of the form $\tilde{L}T_*Y_*\mathcal{G} = LDTY\mathcal{G}$ for graphs $\mathcal{G} \in \mathsf{etGr}$. So, we may regard objects of $\Xi^{\times}$ as graphs, and morphisms $\beta \in \Xi^{\times}(\mathcal{G},\mathcal{H})$ are given by morphisms in $\mathsf{GS}_*(Y_*\mathcal{G}, \tilde{L}T_*Y_*\mathcal{H})$.

The first step in describing $\Xi^{\times}$, is therefore to describe $\tilde{L}T_*Y_*\mathcal{H}$ for all graphs $\mathcal{H}$.

Since each connected component of X-$\mathsf{Gr_{sim}}$ contains an X-graph $\mathcal{X}$ it follows from (10.7) that, for all finite sets X, elements of $(\tilde{L}T_*Y_*\mathcal{H})_X \cong \Xi^{\times}(\mathcal{C}_X,\mathcal{H})$ are represented by pairs $(\mathcal{X}, f)$ where $\mathcal{X}$ is an X-graph and $f \in \mathsf{etGr}_*(\mathcal{X},\mathcal{H})$. In particular, f factors uniquely as a morphism $\mathcal{X} \to \mathcal{X}_{\setminus W_f}$ in $\mathsf{Gr_{sim}}$ followed by a morphism $f^{\perp} \in \mathsf{etGr}(\mathcal{X}_{\setminus W_f},\mathcal{H})$. It follows that pairs $(\mathcal{X}^1, f^1)$ and $(\mathcal{X}^1, f^2)$ represent the same element $[\mathcal{X}, f]_* \in (\tilde{L}T_*Y_*\mathcal{H})_X$ if and only if there is an object $\mathcal{X}_f \in X$-$\mathsf{Gr_{sim}}$ and a commuting diagram

$$\begin{tikzcd} \mathcal{X}^1 \arrow[r, "g^1"] \arrow[dr, "f^1"'] & \mathcal{X}^f \arrow[d, "f^{\perp}"] & \mathcal{X}^2 \arrow[l, "g^2"'] \arrow[dl, "f^2"] \\ & \mathcal{H} & \end{tikzcd} \tag{10.8}$$

in etGr_* such that g^j are morphisms in X-$\mathsf{Gr_{sim}}$ for $j = 1, 2$ and $f^{\perp} \in \mathsf{etGr}(\mathcal{X}^f,\mathcal{H})$ is an unpointed morphism.

In particular, for all $\mathcal{H}$, and all $m \geq 1$, the following special case of (10.8) commutes in etGr_* for all $e \in E$:

$$\begin{tikzcd} \mathcal{C}_{\mathbf{0}} \arrow[r, "z"] \arrow[dr, "ch_e \circ z"'] & (\mid) \arrow[d, "ch_e"] & \mathcal{W}^m \arrow[l, "\kappa"'] \arrow[dl, "ch_e \circ \kappa"] \\ & \mathcal{H}. & \end{tikzcd} \tag{10.9}$$

Since $\Xi^{\times}(\mathcal{G},\mathcal{H}) \cong \tilde{L}T_*Y_*\mathcal{H}(\mathcal{G}) = \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} \tilde{L}T_*Y_*\mathcal{H}(\mathcal{C})$, morphisms $\gamma \in \Xi^{\times}(\mathcal{G},\mathcal{H})$ are represented by pairs $(\mathbf{\Gamma}, f)$ – where $\mathbf{\Gamma} : \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ is a non-degenerate $\mathcal{G}$-shaped graph of graphs, and $f \in \mathsf{etGr}_*(\mathbf{\Gamma}(\mathcal{G}),\mathcal{H})$ is a morphism in etGr_* (see [25, ??]).

The following lemma generalises [25, Lemma 8.9], and is proved in exactly the same manner.

Lemma 10.10. *Let $\mathcal{G}$ and $\mathcal{H}$ be graphs and, for $i = 1, 2$, let $\mathbf{\Gamma}^i$ be a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\mathbf{\Gamma}^i(\mathcal{G})$, and $f^i \in \mathsf{etGr}_*(\mathbf{\Gamma}(\mathcal{G}),\mathcal{H})$. Then $(\mathbf{\Gamma}^1, f^1), (\mathbf{\Gamma}^2, f^2)$ represent the same element γ of $\Xi^{\times}(\mathcal{G},\mathcal{H})$ if and only if there is a representative $(\mathbf{\Gamma}, f)$ of γ, and a commuting diagram in etGr_* of the following form:*

$$\begin{tikzcd} \mathbf{\Gamma}^1(\mathcal{G}) \arrow[r] \arrow[dr, "f^1"'] & \mathbf{\Gamma}(\mathcal{G}) \arrow[d, "f"] & \mathbf{\Gamma}^2(\mathcal{G}) \arrow[l] \arrow[dl, "f^2"] \\ & \mathcal{H}, & \end{tikzcd} \tag{10.11}$$

where the morphisms in the top row are component preserving vertex deletion morphisms (including $z\colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$*) and* $f \in \mathsf{etGr}(\mathbf{\Gamma}(\mathcal{G}), \mathcal{H})$ *is an unpointed étale morphism of graphs.*

The following terminology is from [18].

Definition 10.12. *A* (pointed) free map in $\Xi^\times$ *is a morphism in the image of the inclusion* $\mathsf{CetGr}_* \hookrightarrow \Xi^\times$*. An* unpointed free map *in* $\Xi^\times$ *is a morphism in the image of* $\mathsf{CetGr} \hookrightarrow \mathsf{CetGr}_* \hookrightarrow \Xi^\times$*. A* generic morphism in $\Xi^\times$ *is a morphism in* $\Xi^\times(\mathcal{G}, \mathcal{H})$ *with a representative of the form* $(\mathbf{\Gamma}, id_{\mathcal{H}})$*.*

So, a free map in $\Xi^\times(\mathcal{G}, \mathcal{H})$ has a representative of the form (id, f) where $id\colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ is the trivial identity $\mathcal{G}$-shaped graph of graphs with colimit $\mathcal{G}$ and $f \in \mathsf{etGr}_*(\mathcal{G}, \mathcal{H})$, and a generic map in $\Xi^\times(\mathcal{G}, \mathcal{H})$ is described by a non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$ with colimit $\mathcal{H}$, and hence $\mathbf{\Gamma}$ induces an identity on boundaries.

In particular, morphisms in $\Xi^\times(\mathcal{G}, \mathcal{H})$ factor as $\mathcal{G} \xrightarrow{\mathbf{\Gamma}} (\mathcal{G}) \xrightarrow{\mathsf{del}_{\setminus W}} \mathbf{\Gamma}(\mathcal{G})_{\setminus W} \xrightarrow{f} \mathcal{H}$, where $\mathbf{\Gamma}\colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ is a nondegenerate $\mathcal{G}$-shaped graph of graphs (that do not have to be connected), and $f \circ \mathsf{del}_{\setminus W} \in \mathsf{etGr}_*(\mathbf{\Gamma}(\mathcal{G}), \mathcal{H})$ where $\mathbf{\Gamma}(\mathcal{G})$ is the colimit of $\mathbf{\Gamma}$ and f is a morphism in etGr.

10.4. **Factorisation categories.** Let $\mathsf{GS}_{*\tilde{\mathbb{L}}\mathbb{T}_*}$ be the Kleisli category of the monad $\tilde{\mathbb{L}}\mathbb{T}_*$ on GS_*. Following [3, Section 2.4], we may associate to each morphism $\beta \in \mathsf{GS}_{*\tilde{\mathbb{L}}\mathbb{T}_*}(\mathcal{G}, S_*)$ where $\mathcal{G}$ is a graph and S_* is a pointed graphical species in Set, a category $\mathsf{fact}_*(\beta)$ called the *factorisation category* $\mathsf{fact}_*(\beta)$ *of* β. If $\mathsf{fact}_*(\beta)$ is connected for all choices of $\mathcal{G}$, S_* and β, then, by [3, Proposition 2.5], the monad $\tilde{\mathbb{L}}\mathbb{T}_*$ together with the category etGr_* satisfy the conditions of the nerve theorem [3, Theorem 1.10] (that is, $\tilde{\mathbb{L}}\mathbb{T}_*$ has arities etGr_*), and hence the conclusions of Theorem 10.4 hold.

By Yoneda, $\mathsf{GS}_{*\tilde{\mathbb{L}}\mathbb{T}_*}(\mathcal{G}, S_*) \cong \mathsf{GS}_*(\mathcal{G}, \tilde{L}T_*S_*) \cong \tilde{L}T_*S_*(\mathcal{G})$ canonically, and

$$\tilde{L}T_*S_*(\mathcal{G}) = \mathrm{colim}_{(\mathcal{C},\mathrm{b}) \in \mathsf{el}(\mathcal{G})} \tilde{L}T_*S_*(\mathcal{C})$$

where

$$\tilde{L}T_*S_*(\mathcal{C}) \cong \begin{cases} S_\S & \mathcal{C} \cong (\shortmid), \\ \mathrm{colim}_{\mathcal{X} \in X\text{-}\mathsf{Gr}_{\mathsf{sim}}} S(\mathcal{X}) & \mathcal{C} \cong \mathcal{C}_X. \end{cases}$$

In other words, a morphism $\beta \in \mathsf{GS}_{*\mathbb{T}_*}(\mathcal{G}, S_*)$ is represented by a pair $(\mathbf{\Gamma}, \alpha)$ of

- a non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$ with colimit $\mathbf{\Gamma}(\mathcal{G})$, viewed as a morphism generic morphism $\mathbf{\Gamma} \in \Xi^\times(\mathcal{G}, \mathbf{\Gamma}(\mathcal{G})) = \mathsf{GS}_{*\mathbb{T}_*}(\mathcal{G}, \mathbf{\Gamma}(\mathcal{G}))$,
- an element $\alpha \in S(\mathbf{\Gamma}(\mathcal{G}))$ viewed, by Yoneda, as a morphism $\alpha \in \mathsf{GS}_{*\mathbb{T}_*}(Y_*\mathbf{\Gamma}(\mathcal{G}), S_*)$

such that the induced composition $\mathcal{G} \xrightarrow{\mathbf{\Gamma}} \mathbf{\Gamma}(\mathcal{G}) \xrightarrow{\alpha} S(\mathbf{\Gamma}(\mathcal{G})) \twoheadrightarrow (\tilde{L}T_*S_*)(\mathcal{G})$ is precisely β.

In particular, by (10.7), if $(\mathbf{\Gamma}^1, \alpha^1)$ and $(\mathbf{\Gamma}^2, \alpha^2)$ represent the same element $\beta \in \beta \in \mathsf{GS}_{*\mathbb{T}_*}(\mathcal{G}, S_*)$, then $\mathbf{\Gamma}^1(\mathcal{G})$ and $\mathbf{\Gamma}^2(\mathcal{G})$ are similar in etGr_*.

For each $\beta \in \mathsf{GS}_{*\mathbb{T}_*}(\mathcal{G}, \mathbf{\Gamma}(\mathcal{G}))$, objects of $\mathsf{fact}_*(\beta)$ are given by pairs $(\mathbf{\Gamma}, \alpha)$ as above, and morphisms in $\mathsf{fact}_*(\beta)((\mathbf{\Gamma}^1, \alpha^1), (\mathbf{\Gamma}^2, \alpha^2))$ are commuting diagrams in $\mathsf{GS}_{*\tilde{\mathbb{L}}\mathbb{T}_*}$

(10.13)
$$\begin{tikzcd}
& \mathbf{\Gamma}^1(\mathcal{G}) \arrow[dd, "g"] \arrow[dr, "\alpha^1"] & \\
\mathcal{G} \arrow[ur] \arrow[dr] & & S_* \\
& \mathbf{\Gamma}^2(\mathcal{G}) \arrow[ur, "\alpha^2"'] &
\end{tikzcd}$$

such that g is a morphism in $\mathsf{etGr}_* \hookrightarrow \mathsf{GS}_{*\tilde{\mathbb{L}}\mathbb{T}_*}$.

Lemma 10.14. *For all graphs $\mathcal{G}$ and all $\beta \in \mathsf{GS}_*(\mathcal{G}, \tilde{L}T_*S)$, the category $\mathsf{fact}_*(\beta)$ is connected.*

Proof. If $\mathcal{G} \cong \mathcal{C}_X$ for some finite set X, then objects of $\mathsf{fact}_*(\beta)$ have the form $(\mathcal{X}, \alpha)$ where $\mathcal{X}$ is an X-graph, $\alpha \in S_*(\mathcal{X})$ and $\alpha \mapsto \beta$ under the universal morphism $S_*(\mathcal{X}) \to (\tilde{L}T_*S_*)_X$. Since $(\tilde{L}T_*S_*)_X = \mathrm{colim}_{\mathcal{X}' \in X\text{-}\mathsf{Gr}_{\mathsf{sim}}} S(\mathcal{X}')$, it follows that $\mathcal{X}$ and $\mathcal{X}'$ are in the same connected component of X-$\mathsf{CGr}_{\mathsf{sim}}$ and therefore, they are connected in etGr_*,whence the lemma holds when $\mathcal{G}$ is a corolla.

Since colimits of graphs of graphs are computed componentwise, we may assume that $\mathcal{G}$ is connected. If $(\mathbf{\Gamma}^i, \alpha^i)$, $i = 1, 2$ are objects of $\mathsf{fact}_*(\beta)$ then, for each $(\mathcal{C}_X, b) \in \mathsf{el}(\mathcal{G})$, $\mathbf{\Gamma}^1(b)$ and $\mathbf{\Gamma}^2(b)$ are in the same connected component of X-$\mathsf{Gr}_{\mathsf{sim}}$, and hence $\mathbf{\Gamma}^1(\mathcal{G})$ and $\mathbf{\Gamma}^2(\mathcal{G})$ are in the same connected component of etGr_*. □

Theorem 10.4 now follows from [3, Sections 1 & 2].

Proof of Theorem 10.4. The category etGr_* is dense in GS_*. By [3, Proposition 2.5], the statement of Lemma 10.14 is equivalent to the statement that the monad $\tilde{\mathbb{L}}\mathbb{T}_*$ has arities etGr_*. Hence the induced nerve functor $N \colon \mathsf{CO} \to \mathsf{psh}(\Xi^\times)$ is fully faithful by [3, Propositions 1.5 & 1.9]. Moreover, by [3, Theorem 1.10] its essential image is the subcategory of those presheaves on $\Xi^\times$ whose restriction to etGr_* are in the image of the fully faithful embedding $\mathsf{GS}_* \hookrightarrow \mathsf{psh}(\mathsf{etGr}_*)$.

So a presheaf P on $\Xi^\times$ is in the image of N if and only if

$$P(\mathcal{G}) = \lim_{(\mathcal{C},b) \in \mathsf{el}_*(\mathcal{G})} P(\mathcal{C}),$$

and, by finality of $\mathsf{el}(\mathcal{G}) \subset \mathsf{el}_*(\mathcal{G})$, this is the case precisely if

$$P(\mathcal{G}) = \lim_{(\mathcal{C},b) \in \mathsf{el}(\mathcal{G})} P(\mathcal{C}).$$

That is, the Segal condition (10.5) is satisfied. □

Remark 10.15. Let ${\Xi^\times}_c \subset \Xi^\times$ be the full subcategory whose objects are connected graphs. So, ${\Xi^\times}_c$ is the category obtained in the bo-ff factorisation of $\mathsf{CetGr} \overset{\text{dense}}{\hookrightarrow} \mathsf{GS} \to \mathsf{CO}$.

Then, morphisms $\beta \in {\Xi^\times}_c(\mathcal{G}, \mathcal{H})$ are described by a $\tilde{L}T_*Y_*\mathcal{H}$-structure on $\mathcal{G}$, and hence represented by pairs $(\mathbf{\Gamma}, \underline{f})$ where

- $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{CetGr}$ is a non-degenerate $\mathcal{G}$-shaped graph of *connected* graphs with colimit $\mathbf{\Gamma}(\mathcal{G})$,
- and $\underline{f} \in LY_*\mathcal{H}(\mathbf{\Gamma}(\mathcal{G}))$ is represented by a pair $(\mathbf{\Gamma}_f, f)$ where
  - $\mathbf{\Gamma}_f \colon \mathsf{el}(\mathbf{\Gamma}(\mathcal{G})) \to \mathsf{etGr}$ is a non-degenerate $\mathbf{\Gamma}(\mathcal{G})$-shaped graph of graphs in $\mathsf{Cor}_{\boxtimes}^{(\mathbf{\Gamma}(\mathcal{G}))}$: in other words $\mathbf{\Gamma}_f(\mathcal{C}_X, b)$ is a disjoint union of corollas $\coprod_{i=1}^k \mathcal{C}_{X_i}$ for all $(\mathcal{C}_X, b) \in \mathsf{el}(\mathbf{\Gamma}(\mathcal{G}))$,
  - and $f \in \mathsf{etGr}_*(\mathbf{\Gamma}_f(\mathbf{\Gamma}(\mathcal{G})), \mathcal{H})$ (where $\mathbf{\Gamma}_f(\mathbf{\Gamma}(\mathcal{G}))$ is the colimit of $\mathbf{\Gamma}_f$ in etGr).

More generally, if $\mathcal{G}$ is connected, a morphism $\beta \in \mathsf{GS}_{*\tilde{\mathbb{L}}\mathbb{T}_*}(\mathcal{G}, S_*)$ in the Kleisli category of $\tilde{\mathbb{L}}\mathbb{T}_*$ is represented by a pair $(\mathbf{\Gamma}, \underline{\alpha})$, of a non-degenerate $\mathcal{G}$-shaped graph of connected graphs $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{CetGr}$ with connected colimit $\mathbf{\Gamma}(\mathcal{G})$, and an $\underline{\alpha} \in \tilde{L}S_*(\mathbf{\Gamma}(\mathcal{G}))$ that assigns to each $(\mathcal{C}_X, b) \in \mathsf{el}(\mathbf{\Gamma}(\mathcal{G}))$, a tuple $(\phi_i)_I = 1^k$ – with $\phi_i \in S_{X_i}$, and $\coprod_{i=1}^k X_i = X$ – of elements of S.

Such pairs $(\mathbf{\Gamma}, \underline{\alpha})$ are the objects of the *connected factorisation category* $\mathsf{fact}_*(\beta)_c$ of β. Morphisms in $\mathsf{fact}_*(\beta)_c((\mathbf{\Gamma}, \underline{\alpha}), (\mathbf{\Gamma}', \underline{\alpha}'))$ are morphisms $g \in \mathsf{CetGr}_*(\mathbf{\Gamma}(\mathcal{G}), \mathbf{\Gamma}'(\mathcal{G}))$ between connected graphs, such that the diagram

(10.16)
$$\begin{tikzcd} & \mathbf{\Gamma}(\mathcal{G}) \arrow[dd, "g"] \arrow[dr, "\underline{\alpha}"] & \\ \mathcal{G} \arrow[ur] \arrow[dr] & & S_* \\ & \mathbf{\Gamma}'(\mathcal{G}) \arrow[ur, "\underline{\alpha}"'] & \end{tikzcd}$$

commutes. By [3, Proposition 2.5], the full subcategory of $\Xi^\times$ on the connected graphs induces a fully faithful nerve functor from CO (i.e. CetGr_* provides arities for $\tilde{\mathbb{L}}\mathbb{T}_*$) if, for all connected graphs $\mathcal{G}$, all pointed graphical species S_*, and all $\beta \in \mathsf{GS}_{*\tilde{\mathbb{L}}\mathbb{T}_*}(\mathcal{G}, S_*)$, the category $\mathsf{fact}_*(\beta)_c$ is connected.

To see that this is not the case, consider the case that $S_* = Y_*(\shortmid \amalg \shortmid)$, $\mathcal{G} = \mathcal{C}_{\mathbf{0}}$. A $\mathcal{C}_{\mathbf{0}}$-shaped graph of connected graphs is just a connected graph $\mathcal{H}$ with empty boundary, and an object $\underline{\alpha} \in Y_*(\shortmid \amalg \shortmid)(\mathcal{H})$ is just an $\mathcal{H}$-shaped graph of graphs in $\mathbf{\Gamma}_{\boxtimes} \in \mathsf{Cor}_{\boxtimes}^{(\mathcal{H})}$ (so $\mathbf{\Gamma}_{\boxtimes}(b)$ is a disjoint union of corollas for each $(\mathcal{C}_X, b) \in \mathsf{el}(\mathcal{H})$) followed by a morphism $f \in \mathsf{etGr}_*(\mathbf{\Gamma}_{\boxtimes}(\mathcal{H}), (\shortmid \amalg \shortmid))$ from the colimit $\mathbf{\Gamma}_{\boxtimes}(\mathcal{H})$ of $\mathbf{\Gamma}_{\boxtimes}$. In particular, each component of $\mathbf{\Gamma}_{\boxtimes}(\mathcal{H})$ must either be an isolated vertex, or of the form $\mathcal{L}^k$ or $\mathcal{W}^m$ $(k, m \geq 0)$.

Let $\beta \in \tilde{L}T_*Y_*(\shortmid \amalg \shortmid)$ be defined by the $\mathcal{C}_{\mathbf{0}}$-shaped graph of graphs $\mathbf{\Gamma}_{\boxtimes}^{\beta}$ with colimit $\mathcal{C}_{\mathbf{0}} \amalg \mathcal{C}_{\mathbf{0}}$, and the componentwise morphism $(z, z) \in \mathsf{etGr}_*(\mathcal{C}_{\mathbf{0}} \amalg \mathcal{C}_{\mathbf{0}}, \shortmid \amalg \shortmid)$.

A triple $(\mathcal{H}, \mathbf{\Gamma}_{\boxtimes}, f)$ as above, represents an object of $\mathsf{fact}_*(\beta)$ if $\mathbf{\Gamma}_{\boxtimes}(\mathcal{H})$ is in the same connected component as $\mathcal{C}_{\mathbf{0}} \amalg \mathcal{C}_{\mathbf{0}}$ in $\mathbf{0}\text{-}\mathsf{Gr}_{\mathsf{sim}}$ and hence there are morphisms

$$\mathbf{\Gamma}_{\boxtimes}(\mathcal{H}) \longrightarrow (\shortmid \amalg \shortmid) \xleftarrow{z \amalg z} \mathcal{C}_{\mathbf{0}} \amalg \mathcal{C}_{\mathbf{0}}$$

in $\mathbf{0}\text{-}\mathsf{Gr}_{\mathsf{sim}}$.

So, consider the object $(\mathcal{H}, \mathbf{\Gamma}_{\boxtimes}, f)$ of $\mathsf{fact}_*(\beta)$, illustrated in Figure 19, where $\mathcal{H} = \mathcal{C}_{\mathbf{4}}^{1\ddagger 2, 3\ddagger 4}$ is the graph with two loops at one vertex, $\mathbf{\Gamma}_{\boxtimes}$ is given by

$$(\mathcal{C}_{\mathbf{4}}, b) \mapsto \mathcal{C}_{\mathbf{2}} \amalg \mathcal{C}_{\mathbf{2}}$$

with colimit $\mathbf{\Gamma}_{\boxtimes}(\mathcal{H}) = \mathcal{W} \amalg \mathcal{W}$, and $f = (\kappa, \kappa) \in \mathsf{etGr}_*(\mathcal{W} \amalg \mathcal{W}, \shortmid \amalg \shortmid)$.

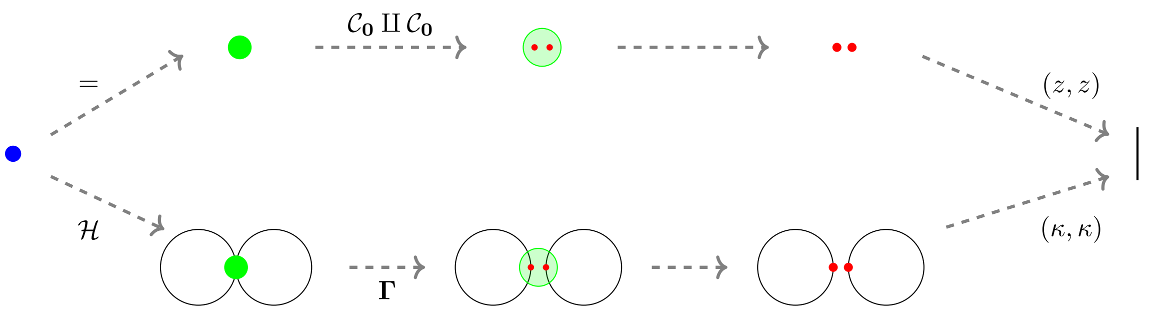


FIGURE 19. There is no zigzag of morphisms $\mathcal{C}_{\mathbf{0}} \rightsquigarrow \mathcal{H}$ in CetGr_* that connect $(\mathcal{C}_{\mathbf{0}}, \mathcal{C}_{\mathbf{0}} \amalg \mathcal{C}_{\mathbf{0}})$ and $(\mathcal{H}, \mathbf{\Gamma}_{\boxtimes})$ in $\mathsf{fact}_*(\beta)_c$.

Since morphisms in CetGr_* factor as vertex deletion morphisms followed by morphisms in CetGr, and the single vertex of $\mathcal{H}$ has valency four, a zigzag of morphisms connecting $\mathcal{C}_{\mathbf{0}}$ and $\mathcal{H}$ in CetGr_* may be replaced by a morphism of the form $ch_e \circ z \in \mathsf{CetGr}(\mathcal{C}_{\mathbf{0}}, \mathcal{H})$. But then the diagram (10.16) does not commute.

Remark 10.17. Both the nerve theorem [25, Theorem 8.2] for modular operads on which Theorem 10.4 is built, as well as Hackney, Robertson and Yau's nerve theorem [14, 3.6], led to well-defined models of $(\infty, 1)$-modular operads in which the fibrant objects in the appropriate category of presheaves are precisely those that satisfy the weak Segal condition: in the case of [25] it followed from the results of [6] that these were given by $P\colon \Xi^P\mathrm{op} \to \mathsf{sSet}$ such that, for all (connected) graphs $\mathcal{G} \in \Xi$,

$$P(\mathcal{G}) \simeq \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}).$$

However, describing a similar model for circuit operads (algebras) is more challenging. The presence of unconnected graphs means that, neither the method of [14], nor that of [25, Corollary 8.14] may be directly extended to a model structure on functors from $\Xi^{\times\mathrm{op}}$ (or some subcategory thereof) to sSet that gives a good description of weak circuit operads. For this reason, the description of a model for $(\infty, 1)$-circuit operads is deferred to a separate paper✠ [?] (in progress).

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