GRAPHICAL COMBINATORICS AND A DISTRIBUTIVE LAW FOR MODULAR OPERADS

SOPHIE RAYNOR

Abstract. This work presents a detailed analysis of the combinatorics of modular operads. These are operad-like structures that admit a contraction operation as well as an operadic multiplication. Their combinatorics are governed by graphs that admit cycles, and are known for their complexity. In 2011, Joyal and Kock introduced a powerful graphical formalism for modular operads. This paper extends that work. A monad for modular operads is constructed and a corresponding nerve theorem is proved, using Weber's abstract nerve theory, in the terms originally stated by Joyal and Kock. This is achieved using a distributive law that sheds new light on the combinatorics of modular operads.

Introduction

Modular operads, introduced in [16] to study moduli spaces of Riemann surfaces, are a "'higher genus' analogue of operads . . . in which graphs replace trees in the definition." [16, Abstract].

Roughly speaking, modular operads are $\mathbb{N}$-graded objects $P = \{P(n)\}_{n\in\mathbb{N}}$ that, alongside an operadic multiplication (or composition) $\circ\colon P(n) \times P(m) \to P(m+n-2)$ for $m,n \geq 1$, admit a contraction operation $\zeta\colon P(n) \to P(n-2)$, $n \geq 2$. For example, as in Figure 1, we may multiply two oriented surfaces by gluing them along chosen boundary components, or contract a single surface by gluing together two distinct boundary components.

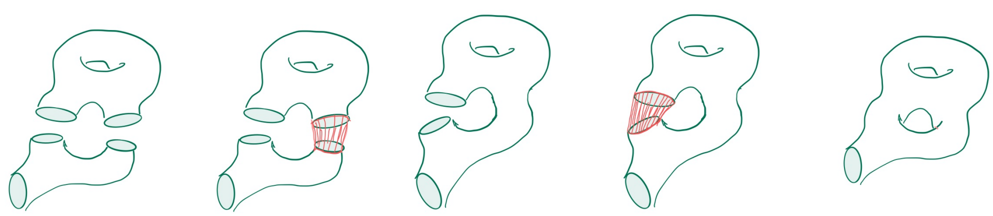

Figure 1. Gluing (multiplication) and self-gluing (contraction) of surfaces along boundary components. Moduli of geometric structures – such as Riemann surfaces – provide many examples of modular operads.

This work considers a notion of modular operads due to Joyal and Kock [22],[1] that incorporates a broad compass of related structures, including modular operads in the original sense of [16] (see Example 1.26) and their coloured counterparts [17], but also wheeled properads [18, 42] (see Example 1.29). More generally, compact closed categories [26] provide examples of modular operads [37, 38] (see Example 1.27). These are closely related to circuit algebras that are used in the study of finite-type knot invariants [1, 13] (see Example 1.28). As such, modular operads have applications across a range of disciplines.

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[1] Joyal and Kock used the term 'compact symmetric multicategories (CSMs)' in [22] to refer to what are here called 'modular operads'. Indeed, I adopted this terminology in a previous version of this paper and in [35].

However, the combinatorics of modular operads are complex. In modular operads equipped with a multiplicative unit, contracting this unit leads to an exceptional ‘loop’, that can obstruct the proof of general results. This paper undertakes a detailed investigation into the graphical combinatorics of modular operads, and provides a new understanding of these loops.

In [22], which forms the inspiration for this work, Joyal and Kock construct modular operads as algebras for an endofunctor on a category GS of coloured collections called ‘graphical species’. Their machinery is significant in its simplicity. It relies only on minimal data and basic categorical constructions, that lend it considerable formal and expressive power.

However, the presence of exceptional loops means that their modular operad endofunctor does not extend to a monad on GS. As a consequence, it does not lead to a precise description of the relationship between modular operads and their graphical combinatorics. (See Section 6 for details.)

This paper contains proofs of the following statements that first appeared in [22] (and were proved – by similar, though slightly less general methods than those presented here – in my PhD thesis [35]):

Theorem 0.1 (Monad existence Theorem 7.47)**.** *The category* MO *of modular operads is isomorphic to the Eilenberg-Moore category of algebras for a monad* $\mathbb{O}$ *on the category* GS *of graphical species.*

In particular, $\mathbb{O}$ is the *algebraically free monad* [25] on the endofunctor of [22]. Theorem 0.2 – the ‘nerve theorem’ – characterises modular operads in terms of presheaves on a category Ξ of graphs.

Theorem 0.2 (Nerve Theorem 8.2)**.** *The category* MO *has a full subcategory* Ξ *whose objects are graphs. The induced (nerve) functor* N *from* MO *to the category* $\mathsf{psh}(\Xi)$ *of presheaves on* Ξ *is fully faithful.*

There is a canonical (restriction) functor $R^*\colon \mathsf{psh}(\Xi) \to \mathsf{GS}$*, and the essential image of* N *consists of precisely those presheaves* $P\colon \Xi^{\mathrm{op}} \to \mathsf{Set}$ *that satisfy the so-called ‘Segal condition’:*
P *is in the essential image of* N *if and only if it is completely determined by the graphical species* R^*P*.*

An obvious motivation for establishing such results is provided by the study of weak, or $(\infty,1)$-modular operads, by weakening the Segal condition of Theorem 0.2. A number of potential applications of such structures are discussed in the introduction to [20]. To this end, Hackney, Robertson and Yau have also recently proved versions of Theorems 0.1 and 0.2, by different methods, and used them to obtain a model of $(\infty,1)$-modular operads that are characterised in terms of a weak Segal condition [20, 21].

The aim of this work is to prove Theorems 0.1 and 0.2 in the manner originally proposed by [22] – using the abstract nerve machinery introduced by Weber [41, 5] (see Section 2) – and to use these proofs as a route to a full understanding of the underlying combinatorics, and the contraction of multiplicative units in particular. This places strict requirements on the relationship between the modular operad monad $\mathbb{O}$ and the graphical category Ξ. In fact, to apply the results of [5], the category Ξ must – in a sense that will be made precise in Section 2 – arise naturally from the definition of $\mathbb{O}$.

Neither the construction of the monad $\mathbb{O}$ for modular operads, nor the proof of Theorem 0.2 is entirely straightforward. First, the method of [22], which is closely related to analogous constructions for operads (Examples 5.1, 6.1, c.f. [18, 27, 32, 34]) does not lead to a well-defined monad. Second, as a consequence of the contracted units, the desired monad, once obtained, does not satisfy the conditions for applying the machinery of [5]. To prove Theorems 0.1 and 0.2, it is therefore necessary to break the problem into smaller pieces, thereby rendering the graphical combinatorics of modular operads completely explicit.

Since the obstruction to obtaining a monad in [22] arises from the combination of the modular operadic contraction operation and the multiplicative units (see Section 6), the approach of this work is to first treat these structures separately – via a monad $\mathbb{T}$ on GS whose algebras are non-unital modular operads, and a monad $\mathbb{D}$ on GS that adjoins distinguished ‘unit’ elements – and then combine them, using the theory of distributive laws [3].

Theorem 0.1 is then a corollary of:

Theorem 0.3 (Proposition 7.40 & Theorem 7.47)**.** *There is a distributive law λ for $\mathbb{T}$ over $\mathbb{D}$ such that the resulting composite monad $\mathbb{DT}$ on GS is precisely the modular operad monad $\mathbb{O}$ of Theorem 0.1.*

The graphical category Ξ, used to define the modular operadic nerve, arises canonically via the unique fully faithful–bijective on objects factorisation of a functor used in the construction of $\mathbb{O}$. Therefore, if the monad $\mathbb{O}$ satisfies certain formal conditions – if it 'has arities' (see [5]) – then Theorem 0.2 follows from [5, Section 1].

Though the monad $\mathbb{O}$ on GS does not have arities, the distributive law in Theorem 0.3 implies that there is a monad $\mathbb{T}_*$, on the category GS_* of $\mathbb{D}$-algebras, whose algebras are modular operads. Moreover, Theorem 0.2 follows from:

Lemma 0.4 (Lemma 8.11)**.** *The monad $\mathbb{T}_*$ on GS_* has arities, and hence satisfies the conditions of* [5, Theorem 1.10].

I conclude this introduction by briefly mentioning three (related) benefits of this abstract approach.

In the first place, the results obtained by this method provide a clear overview of how modular operads fit into the wider framework of operadic structures, and how other general results may be modified to this setting. For example, by Lemma 0.4, $\mathbb{T}_*$ and Ξ satisfy the Assumptions 7.9 of [9], which leads to a suitable notion of weak modular operad via the following corollary:

Corollary 0.5 (Corollary 8.14)**.** *There is a model structure on the category of presheaves in simplicial sets on Ξ. The fibrant objects are precisely those presheaves that satisfy a weak Segal condition.*

Second, since this work makes the combinatorics of modular operads – including the tricky bits – completely explicit, it provides a clear road map for working with and extending the theory.

One fruitful direction for extending this work is to use iterated distributive laws [10] to generalise constructions presented here. In [36], an iterated distributive law is used to construct circuit operads – modular operads with an extra product operation, closely related to small compact closed categories – as algebras for a composite monad on GS (Example 1.28). Once again, the distributive laws play an important role in describing the corresponding nerve. The approach of [10, Section 3] may also be used to construct higher (or (n,k)-) modular operads. This can be used to give a modular operadic description of extended cobordism categories.

Finally, the complexities of the combinatorics of contractions can provide new insights into the structures they are intended to model. In current work, also together with L. Bonatto, S. Chettih, A. Linton, M. Robertson, N. Wahl, I am using these ideas to explore singular curves in the compactification of moduli spaces of algebraic curves. (See also Example 1.26, and consider Figure 25.)

This work owes its existence to the ideas of A. Joyal and J. Kock and I thank Joachim for taking time to speak with me about it. P. Hackney, M. Robertson and D. Yau's work has been an invaluable resource. Conversations with Marcy have been particularly helpful. I gratefully acknowledge the anonymous reviewer whose insights have not only improved the paper, but also increased my appreciation of the mathematics.

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Overview and key points. The opening two sections provide context and background for the rest of the work. An axiomatic definition of modular operads is given in Section 1. Section 2 gives a brief review of Weber's abstract nerve theory, that provides a framework for the later sections. Both these introductory sections include a number of examples to motivate the constructions that follow.

Section 3 is a detailed introduction to the (Feynman) graphs of [22], and Section 4 focuses on their étale morphisms. The monad $\mathbb{T}$ for non-unital modular operads is constructed in Section 5.

Section 6 acts as a short intermezzo in which the appearance of exceptional loops in the theory, and why they are problematic in the construction of [22], is explained.

The construction of the monad $\mathbb{O}$ for modular operads happens in Section 7. This is the longest and most important section of the work, and contains most of the new contributions. Finally, Section 8 contains the proof of the Nerve Theorem 0.2, as well as a short discussion on weak modular operads.

There have been many other approaches to the issue of loops, some of which are mentioned in Remarks 6.7 and 6.8. But the graphical construction presented in this paper is unique, as far as I am aware, in that it *does not* incorporate some version of the exceptional loop into the graphical calculus, in order to model contractions of units. (Remark 6.9.)

In other approaches, the contraction of units is described by adjoining a formal colimit of a diagram of graphs, resulting in the exceptional loop object (see Example 3.16). By contrast, we will see in Section 7 that the definition of modular operads (Definition 1.24) implies that the contracted units are, in fact, described in terms of a formal limit of the very same diagram. This is illustrated in Figure 2.

Moreover, this construction leads to a graphical description of the unit contraction, not by an exceptional loop, but as the singularity of a 'double cone' of wheel-shaped graphs (see Section 7.4 and Figure 25).

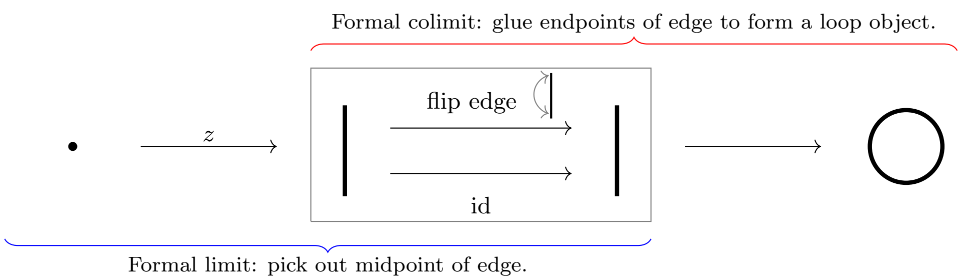


FIGURE 2. An edge graph with no vertices may be flipped or left unchanged. The exceptional loop that 'glues the edge ends together' arises as the formal colimit of these endomorphisms. In Section 7, the graph category of Sections 3-5 (and [22]) is enlarged to include the morphism z that 'picks out the midpoint' of the edge graph with no vertices.

1. Definitions and examples

The goal of this section is to give an axiomatic definition of modular operads (Definition 1.24), and to provide some motivating examples. As mentioned in the introduction, the term 'modular operad' refers here to what are called 'compact symmetric multicategories (CSMs)' in [22].

1.1. **Graphical species.**

After establishing some basic notional conventions, we discuss Joyal and Kock's graphical species [22] that generalise various notions of coloured collection used in the study of operads.

A *presheaf* on a category C is a functor $P\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. The corresponding functor category is denoted $\mathsf{psh}(\mathsf{C})$. The notion of elements of a presheaf will be used extensively in what follows:

Definition 1.1. *Objects of the category* $\mathsf{el}_{\mathsf{C}}(P)$ *of elements of a presheaf* $P\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ *are pairs* (c, x) – *called* elements of P – *where* c *is an object of* C *and* $x \in P(c)$. *Morphisms* $(c, x) \to (d, y)$ *in* $\mathsf{el}_{\mathsf{C}}(P)$ *are given by morphisms* $f \in \mathsf{C}(c, d)$ *such that* $P(f)(y) = x$.

If a presheaf P on C is of the form $\mathsf{C}(-, c)$, then $\mathsf{el}_{\mathsf{C}}(P)$ is the *slice category* C/c whose objects are pairs (d, f) where $f \in \mathsf{C}(d, c)$, and morphisms $(d, f) \to (d', f')$ are given by by commuting triangles in C:

$$\begin{tikzcd} d \arrow[rr, "g"] \arrow[dr, "f"'] & & d' \arrow[dl, "f'"] \\ & c. & \end{tikzcd}$$

In general, a functor $\iota\colon \mathsf{D} \to \mathsf{C}$ induces a presheaf $\iota^*\mathsf{C}(-, c)$, $d \mapsto \mathsf{C}(\iota(d), c)$ on D. For all $c \in \mathsf{C}$, the *slice category of* D *over* c is defined by $\mathsf{D}/c \overset{\text{def}}{=} \mathsf{el}_{\mathsf{D}}(\iota^*\mathsf{C}(-, c))$. (This is also – more accurately – denoted by ι/c.)

In particular, the Yoneda embedding $\mathsf{C} \to \mathsf{psh}(\mathsf{C})$ induces a canonical isomorphism $\mathsf{el}_{\mathsf{C}}(P) \cong \mathsf{C}/P$ for all presheaves P on C, and these categories will be identified in this work.

The groupoid of finite sets and bijections is denoted by $\mathbf{B}$. For $n \in \mathbb{N}$, the set $\{1, \ldots, n\}$ is denoted by $\mathbf{n}$. So $\mathbf{0} = \emptyset$ is the empty set.

Remark 1.2. Let $\Sigma \subset \mathbf{B}$ denote the skeletal subgroupoid on the objects $\mathbf{n}$, for $n \in \mathbb{N}$. A presheaf $P\colon \mathbf{B}^{\mathrm{op}} \to \mathsf{Set}$ on $\mathbf{B}$, also called a *(monochrome* or *single-sorted) species* [23], determines a presheaf on Σ by restriction. Conversely, a Σ-presheaf Q may always be extended to a $\mathbf{B}$-presheaf $Q_{\mathbf{B}}$, by setting

$$Q_{\mathbf{B}}(X) \overset{\text{def}}{=} \lim_{(\mathbf{n}, f) \in \Sigma/X} Q(\mathbf{n}) \ \text{ for all } n \in \mathbb{N}.$$

Graphical species, defined in [22, Section 4], are a *coloured* or *multi-sorted* version of species.

Let the category $\mathbf{B}^{\S}$ be obtained from $\mathbf{B}$ by adjoining a distinguished object $\S$ that satisfies

- $\mathbf{B}^{\S}(\S, \S) = \{id, \tau\}$ with $\tau^2 = id$,
- for each finite set X and each element $x \in X$, there is a morphism $ch_x \in \mathbf{B}^{\S}(\S, X)$ that 'chooses' x, and $\mathbf{B}^{\S}(\S, X) = \{ch_x\ ,\ ch_x \circ \tau\}_{x \in X}$,
- for all finite sets X and Y, $\mathbf{B}^{\S}(X, Y) = \mathbf{B}(X, Y)$, and morphisms are equivariant with respect to the action of $\mathbf{B}$. That is, $ch_{f(x)} = f \circ ch_x \in \mathbf{B}^{\S}(\S, Y)$ for all $x \in X$ and all bijections $f\colon X \xrightarrow{\cong} Y$.

Definition 1.3. *A* graphical species *is a presheaf* $S\colon \mathbf{B}^{\S\,\mathrm{op}} \to \mathsf{Set}$.

The element category of a graphical species S *is denoted by* $\mathsf{el}(S) \overset{\text{def}}{=} \mathsf{el}_{\mathbf{B}^{\S}}(S)$, *and the category of graphical species by* $\mathsf{GS} \overset{\text{def}}{=} \mathsf{psh}(\mathbf{B}^{\S})$.

Hence, a graphical species S is described by a species $(S_X)_{X \in \mathbf{B}}$, and a set $S_{\S}$ with involution $S_\tau\colon S_{\S} \to S_{\S}$, together with, for each finite set X, and $x \in X$ a $\mathbf{B}$-equivariant projection $S(ch_x)\colon S_X \to S_{\S}$.

Definition 1.4. *Given a graphical species* S, *the pair* $(S_{\S}, S_\tau)$ *is called the* (involutive) palette *of* S *and elements* $c \in S_{\S}$ *are colours of* S. *If* $S_{\S}$ *is trivial then* S *is a* monochrome graphical species.

For each element $\underline{c} = (c_x)_{x \in X} \in {S_{\S}}^X$, *the* $\underline{c}$-(coloured) arity $S_{\underline{c}}$ *is the fibre above* $\underline{c} \in {S_{\S}}^X$ *of the map* $(S(ch_x))_{x \in X}\colon S_X \to {S_{\S}}^X$.

Remark 1.5. The involution τ on $\S$ is responsible for much of the heavy lifting in the constructions that follow. Initially however, its role may seem obscure. I mention two key features here. First, the involution provides the expressive power necessary to describe composition rules involving colours, such as particle spin, that may have an orientation, or direction. (Directed graphical species are discussed in Example 1.10.)

The second is more fundamental. As will be explained in Example 3.20, $\mathbf{B}^\S$ embeds in a certain category of graphs. Under this embedding, the distinguished object $\S$ is represented as the exceptional edge with no vertices, and the involution τ as the 'flip' map that swaps its ends (see Figure 2). This enables us to encode formal compositions in graphical species – described in terms of graphs – as categorical limits, and thereby derive the results of this paper by purely abstract methods. For example, the involution underlies a well-defined notion of graph nesting, or substitution, in terms of diagram colimits, without the need to specify extra data (see Sections 5 and 6, and compare with, e.g. [42, 18]).

Example 1.6. The terminal graphical species K has trivial palette and $K_X = \{*\}$ for all finite sets X.

Definition 1.7. *A morphism $\gamma \in \mathsf{GS}(S, S')$ is* palette-preserving *if its component $\gamma_\S$ at $\S$ is the identity on $S_\S$. For a fixed palette $(\mathfrak{C}, \omega)$, $\mathsf{GS}^{(\mathfrak{C},\omega)}$ is the subcategory of GS on the $(\mathfrak{C}, \omega)$-coloured graphical species and palette-preserving morphisms.*

Example 1.8. For any palette $(\mathfrak{C}, \omega)$, the terminal $(\mathfrak{C}, \omega)$-coloured graphical species $K^{(\mathfrak{C},\omega)}$ in $\mathsf{GS}^{(\mathfrak{C},\omega)}$ is described by $K_X^{(\mathfrak{C},\omega)} = \mathfrak{C}^X$ with $K_{\underline{c}}^{(\mathfrak{C},\omega)} = \{*\}$ for all finite sets X and all $\underline{c} \in \mathfrak{C}^X$.

In particular, let $\sigma_{\mathfrak{Di}}$ be the unique non-identity involution on the set $\mathfrak{Di} \overset{\text{def}}{=} \{\text{in}, \text{out}\}$. A *monochrome directed graphical species* is a graphical species with palette $(\mathfrak{Di}, \sigma_{\mathfrak{Di}})$. The terminal monochrome directed graphical species is denoted by $Di \overset{\text{def}}{=} K^{(\mathfrak{Di},\sigma_{\mathfrak{Di}})}$. See also Example 1.10.

Remark 1.9. In the graphical representation of the category $\mathbf{B}^\S$, mentioned in Remark 1.5, a finite set X is represented by a *corolla* or *star graph* $\mathcal{C}_X$ with legs in bijection with X (Figure 3 left side).

An element $\phi \in S_X$ of a graphical species S is represented as a labelling or *decoration* of the unique vertex of $\mathcal{C}_X$, and a *colouring* of the legs of $\mathcal{C}_X$ by $S_\S$ according to $S(ch_x)$ for $x \in X$ (Figure 3 right side).

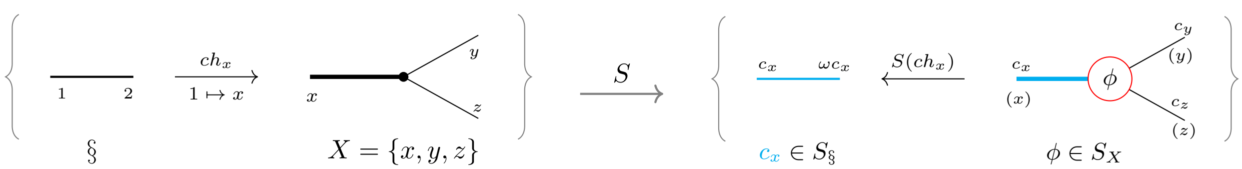


FIGURE 3. Graphical species may be represented graphically: $\phi \in S_X$ is represented as a X-corolla $\mathcal{C}_X$ with vertex decorated by ϕ and x-leg coloured by $c_x = S(ch_x)$.

Example 1.10. Recall the graphical species Di defined in Example 1.8. For each finite set X, $Di_X = \{\text{in}, \text{out}\}^X$ is the set of partitions $X = X_{\text{in}} \amalg X_{\text{out}}$ of X into *input* and *output* sets, with blockwise action of the partition-preserving isomorphisms in $\mathsf{el}(Di)$.

In other words, $\mathsf{el}(Di)$ is equivalent to the category $(\mathbf{B} \times \mathbf{B}^{\text{op}})^{\downarrow}$, obtained from $\mathbf{B} \times \mathbf{B}^{\text{op}}$ by adjoining a distinguished object $(\downarrow)$ (see Figure 4(a)) with trivial endomorphism group, and, for all pairs (X, Y) of finite sets, and all elements $x \in X$ and $y \in Y$, *input* morphisms $i_x : (\downarrow) \to (X, Y)$, and *output* morphisms $o_y : (\downarrow) \to (X, Y)$ that are compatible with the action of $\mathbf{B} \times \mathbf{B}^{\text{op}}$ (see Figure 4(d)).

The objects (X, Y) of $(\mathbf{B} \times \mathbf{B}^{\text{op}})^{\downarrow}$ may be represented, as in Figure 4(b), as *directed corollas* and the distinguished object $(\downarrow)$ as a *directed exceptional edge* (Figure 4(a)). If $Y = \{*\}$ is a singleton, then $(X, \{*\})$ describes a *rooted corolla* as in Figure 4(c).

Hence $\mathsf{GS}/\mathfrak{Di}$ is equivalent to the category $\mathsf{psh}((\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow})$ of *directed graphical species*. The subcategory $\mathsf{GS}^{(\mathfrak{Di},\sigma_{\mathfrak{Di}})}/Di$ of *monochrome directed graphical species* is equivalent to $\mathsf{psh}(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})$.

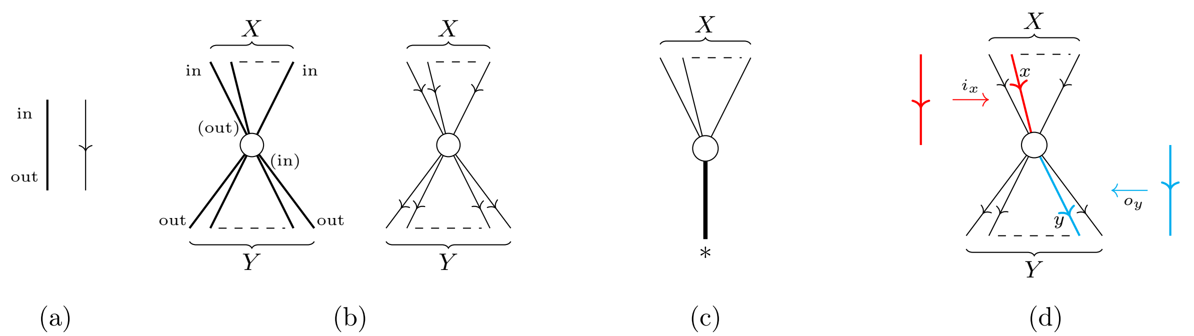


FIGURE 4. (a) the directed exceptional edge ($\downarrow$), (b) the pair $(X,Y)\in(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow}$ describes a directed corolla, (c) $(X,\{*\})$ describes a rooted corolla, (d) input and output morphisms in $(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow}$.

A *PROP* [29] is a strict symmetric monoidal category $(\mathsf{E},+,0)$ whose objects are natural numbers and whose monoidal product $+$ is addition on objects. More generally, for any set $\mathfrak{D}$, a $\mathfrak{D}$-*coloured PROP* $(\mathsf{E}^{\mathfrak{D}},\oplus,\emptyset)$ is a strict symmetric monoidal category whose monoid of objects is freely generated by $\mathfrak{D}$. This is equivalently a presheaf $P^{\mathfrak{D}}$ on $(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow}$ with $P^{\mathfrak{D}}(\downarrow)=\mathfrak{D}$ and, for all pairs (X,Y) of finite sets,

$$P^{\mathfrak{D}}(X;Y)=\coprod_{(\underline{c},\underline{d})\in\mathfrak{D}^{|X|}\times\mathfrak{D}^{|Y|}}\mathsf{E}^{\mathfrak{D}}(\underline{c},\underline{d}),$$

together with composition and monoidal product maps, and an injection $P^{\mathfrak{D}}(\downarrow)\hookrightarrow P^{\mathfrak{D}}(\mathbf{1};\mathbf{1})$ that induces the identities for composition. In particular, PROPs may be described in terms of graphical species.

1.2. **Multiplication and contraction on graphical species.** Intuitively, a multiplication $\diamond$ on a graphical species S is a rule for combining (gluing) distinct elements of S along pairs of legs (called 'ports') with dual colouring as in Figure 5:

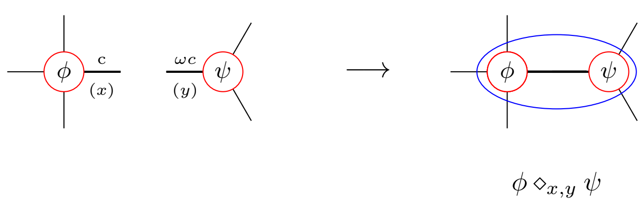


FIGURE 5. Multiplication

The notation '$\rightharpoonup$' denotes a partial map of sets. So $f\colon A\rightharpoonup B$ is given by a subset $A'\subset A$ and a function $A'\to B$.

Definition 1.11. *A* multiplication $\diamond$ *on a graphical species S is given by a family of partial maps*

$$-\diamond_{x,y}^{X,Y}-\colon S_{X\amalg\{x\}}\times S_{Y\amalg\{y\}}\rightharpoonup S_{X\amalg Y}, \tag{1.12}$$

defined (for all X,Y) when $\phi\in S_{X\amalg\{x\}},\psi\in S_{Y\amalg\{y\}}$ satisfy $S(ch_x)(\phi)=S(ch_y\circ\tau)(\psi)$.

The multiplication $\diamond$ satisfies the following conditions:

(m1) (Commutativity axiom.)
Wherever $\diamond^{X,Y}_{x,y}$ *is defined,*

$$\psi \diamond^{Y,X}_{y,x} \phi = \phi \diamond^{X,Y}_{x,y} \psi$$

(m2) (Equivariance axiom.)
For all bijections $\hat{\sigma}\colon X \xrightarrow{\cong} W$ *and* $\hat{\rho}\colon Y \xrightarrow{\cong} Z$ *that extend to bijections* $\sigma\colon X \amalg \{x\} \xrightarrow{\cong} W \amalg \{w\}$ *and* $\rho\colon Y \amalg \{y\} \xrightarrow{\cong} Z \amalg \{z\}$,

$$S(\hat{\sigma} \sqcup \hat{\rho})(\phi \diamond^{W,Z}_{w,z} \psi) = S(\sigma)(\phi) \diamond^{X,Y}_{x,y} S(\rho)(\psi),$$

(where $\hat{\sigma} \sqcup \hat{\rho}\colon X \amalg Y \xrightarrow{\cong} W \amalg Z$ *is the block permutation).*

A unit for the multiplication $\diamond$ *is a map* $\epsilon\colon S_\S \to S_{\mathbf{2}}$, $c \mapsto \epsilon_c$ *such that, for all* X *and all* $\phi \in S_{X \amalg \{x\}}$ *with* $S(ch_x) = c \in S_\S$,

$$\phi \diamond^{X,\{1\}}_{x,2} \epsilon_c = \epsilon_c \diamond^{\{1\},X}_{2,x} \phi = \phi.$$

A multiplication $\diamond$ *is called* unital *if it has a unit* ϵ*. In this case* ϵ_c *is a* c-coloured unit for $\diamond$.

If $(\diamond, \epsilon\colon \mathfrak{C} \to S_{\mathbf{2}})$ is a unital multiplication on a $(\mathfrak{C}, \omega)$-coloured graphical species S, then $\epsilon_c \in S_{(c,\omega c)}$ for all $c \in \mathfrak{C}$. So a unit map $\epsilon\colon \mathfrak{C} \to S_{\mathbf{2}}$ is injective. Let $\sigma_{\mathbf{2}} \in \mathbf{B}^\S(\mathbf{2}, \mathbf{2})$ be the unique non-identity endomorphism.

Lemma 1.13. *If* $\diamond$ *admits a unit* $\epsilon : \mathfrak{C} \to S_{\mathbf{2}}$*, it is unique. Moreover,* ϵ *is compatible with the involutions* $\omega = S_\tau$ *on* $\mathfrak{C}$ *and* $S(\sigma_{\mathbf{2}})$ *in that*

$$\epsilon \circ \omega = \epsilon \circ S_\tau = S(\sigma_{\mathbf{2}}) \circ \epsilon\colon\ \mathfrak{C} \to S_{\mathbf{2}}. \tag{1.14}$$

Proof. Let $\lambda\colon \mathfrak{C} \to S_{\mathbf{2}}$, $c \mapsto \lambda_c$ be another unit for $\diamond$. Then, for all $c \in \mathfrak{C}$,

$$\epsilon_c = \epsilon_c \diamond^{\{1\},\{1\}}_{2,2} \lambda_c = \lambda_c \diamond^{\{1\},\{1\}}_{2,2} \epsilon_c = \lambda_c,$$

whence ϵ is unique. The second statement follows from the defining properties of multiplication. Namely,

$$S(\sigma_{\mathbf{2}})(\epsilon_c) = S(\sigma_{\mathbf{2}})(\epsilon_c) \diamond^{\{1\},\{1\}}_{2,2} \epsilon_{\omega c} = S(\hat{\sigma}_{\mathbf{2}} \sqcup id_{\{1\}})(\epsilon_c \diamond^{\{2\},\{1\}}_{1,2} \epsilon_{\omega c}) = \epsilon_c \diamond^{\{2\},\{1\}}_{1,2} \epsilon_{\omega c} = \epsilon_{\omega c} \diamond^{\{2\},\{1\}}_{1,2} \epsilon_c = \epsilon_{\omega c}.$$

□

Remark 1.15. Equivalently, a multiplication $\diamond$ on $(\mathfrak{C}, \omega)$-coloured graphical species S is a family of maps

$$- \diamond^{\underline{c},\underline{d}}_{c} -\colon S_{(\underline{c},c)} \times S_{(\underline{d},\omega c)} \to S_{(\underline{cd})}, \quad \text{for } c \in \mathfrak{C},\ \underline{c} \in \mathfrak{C}^X,\ \underline{d} \in \mathfrak{C}^Y, \tag{1.16}$$

Both (1.12) and (1.16) are used in what follows. Where the context is clear, the superscripts may be dropped altogether.

As one would expect, a multiplication $\diamond$ on a graphical species S is called 'associative' if the result of several consecutive multiplications does not depend on their order. This is stated precisely in condition (M1) of Definition 1.24, and visualised in the figure therein.

Example 1.17. A graphical species O equipped with a unital, associative multiplication $(\diamond, \epsilon)$ is a cyclic operad in the sense of [14]. When the involution is trivial, these are the *entries-only* cyclic operads of [12] (see there for a comparison with cyclic operads as introduced in [15]).

Some advantages of the involutive, graphical species approach to cyclic operads are discussed in [14] and [21, Introduction].

Example 1.18. Recall, from Example 1.10, the category $(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow}$ – whose objects of $(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow}$ are either the exceptional directed edge $(\downarrow)$, or pairs (X,Y) of finite sets – and the graphical species Di (Example 1.8), described by the equivalence $\mathsf{el}(Di)\simeq(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow}$.

If $Y\cong\{*\}$ is a singleton, then $(X,\{*\})$ is called a rooted corolla, and denoted by t_X (Figure 4(c)). Let $\mathbf{B}^{\downarrow}\subset(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow}$ be the full subcategory on $(\downarrow)$ and all rooted corollas t_X.

Presheaves $O\colon\mathbf{B}^{\downarrow\mathrm{op}}\to\mathsf{Set}$ are described by a set $\mathfrak{D}=O(\downarrow)$ and sets $O(\underline{c};d)$, defined for all $d\in\mathfrak{D}$ and $\underline{c}\in\mathfrak{D}^X$ (for all X), and such that the action of $\mathbf{B}$ on O induces isomorphisms $O((c_x)_x;d)\cong O((c_{f(x)})_x;d)$ for all $f\colon X\xrightarrow{\cong}Y$. Hence, a $\mathfrak{D}$- coloured operad is a $\mathbf{B}^{\downarrow}$ presheaf O, together with an operadic composition, and, for each $d\in\mathfrak{D}$, a unit element $1_d\in O(d;d)$.

The graphical species $RC\subset Di$ is given by $\mathsf{el}(RC)\xrightarrow{\simeq}\mathbf{B}^{\downarrow}$ under the equivalence $\mathsf{el}(Di)\xrightarrow{\simeq}(\mathbf{B}\times\mathbf{B}^{\mathrm{op}})^{\downarrow}$. So, $RC_{\S}=Di_{\S}=\{\mathrm{in},\mathrm{out}\}$, $RC_{\mathbf{0}}=\emptyset$, and $RC_{X\amalg\{*\}}$ consists of those $\phi\in Di_{X\amalg\{*\}}$ such that

$$Di(ch_x)(\phi)=(\mathrm{in})\text{ for all }x\in X,\text{ and }Di(ch_*)(\phi)=(\mathrm{out}).$$

Clearly, RC inherits the trivial unital multiplication from Di. Moreover, a presheaf $O\colon\mathbf{B}^{\downarrow\mathrm{op}}\to\mathsf{Set}$ has the structure of an operad precisely when the corresponding graphical species $O^{\mathsf{GS}}\in\mathsf{GS}/RC$ is equipped with an associative unital multiplication. Hence, the category Op of (symmetric) operads is equivalent to the category whose objects are objects of GS/RC with an associative unital multiplication, and whose morphisms are morphisms in GS/RC that preserve the multiplication and units.

Remark 1.19. Examples 1.17 and 1.18 highlight the expressive power of graphical species. The involution τ on $\S$ means that (undirected) cyclic operads and (directed) operads may be expressed in terms of presheaves on the same underlying category. (See also Examples 1.10 and 1.29.)

Intuitively, a contraction ζ on a graphical species S may be thought of as a rule for 'self-gluing' single elements of S along pairs of ports with dual colouring (Figure 6). The presence of a contraction operation enables modular operads to encode algebraic structures – such as those involving trace – that ordinary operads cannot [32, 33].

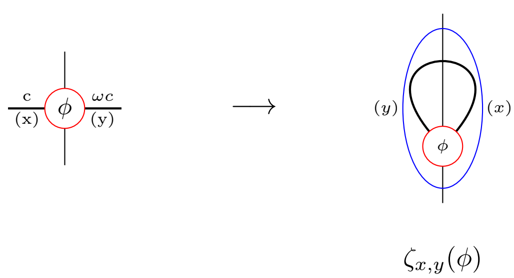


FIGURE 6. Contraction

Definition 1.20. *A* contraction ζ *on* S *is given by a family of partial maps*

$$\zeta^X_{x,y}\colon S_{X\amalg\{x,y\}}\rightharpoonup S_X \tag{1.21}$$

defined for all finite sets X *and all* $\phi\in S_{X\amalg\{x,y\}}$, *such that* $S(ch_x)(\phi)=S(ch_y\circ\tau)(\phi)$, *and equivariant with respect to the action of* $\mathbf{B}$ *on* S*: If* $\sigma\colon X\amalg\{x,y\}\xrightarrow{\cong}Z\amalg\{w,z\}$ *extends the bijection* $\hat\sigma\colon X\xrightarrow{\cong}Z$ *by* $\sigma(x)=w,\sigma(y)=z$, *then for any* $\phi\in S_{Z\amalg\{w,z\}}$, *we have*

$$S(\hat\sigma)\left(\zeta^Z_{w,z}(\phi)\right)=\zeta^X_{x,y}\left(S(\sigma)(\phi)\right).$$

If ζ is a contraction on S, then by, equivariance, $\zeta^X_{x,y}(\phi)=\zeta^X_{y,x}(\phi)$ wherever defined.

Remark 1.22. A contraction ζ on a $(\mathfrak{C},\omega)$-coloured graphical species S is equivalently a family of maps

$$\zeta_c^{\underline{c}}\colon S_{(\underline{c},c,\omega c)} \to S_{\underline{c}}$$

for $c \in \mathfrak{C}$, and $\underline{c} \in \mathfrak{C}^X$. Depending on context, both $\zeta_c^{\underline{c}}$ (and even ζ_c) and (1.21) will be used.

Let S be a $(\mathfrak{C},\omega)$-coloured graphical species equipped with a unital multiplication $(\diamond,\epsilon)$ and contraction ζ. By Lemma 1.13, there is a *contracted unit* map

$$o \overset{\text{def}}{=} \zeta\epsilon : \mathfrak{C} \to S_{\mathbf{0}}, \text{ satisfying } \zeta_c(\epsilon_c) = \zeta_{\omega c}(\epsilon_{\omega c}) \text{ for all } c \in \mathfrak{C}. \tag{1.23}$$

As will be explained in Sections 6 and 7, the contracted units $o\colon S_{\S} \to S_{\mathbf{0}}$ present the main challenge for describing the combinatorics of modular operads.

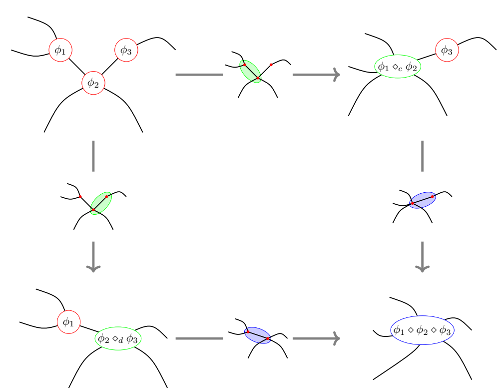
1.3. **Modular operads: definition and examples.**

Modular operads are graphical species with multiplication and contraction operations that satisfy the nicest possible (mutual) coherence axioms.

Definition 1.24. *A* modular operad *is a graphical species S, with palette $(\mathfrak{C},\omega)$, say, together with a unital multiplication $(\diamond,\epsilon)$, and a contraction ζ, satisfying the following four* coherence axioms *governing their composition:*

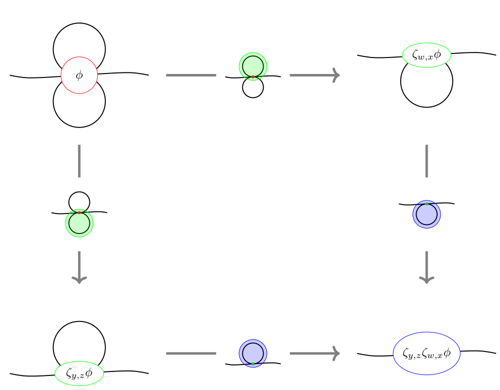
(M1) Multiplication is associative.
For all $\underline{b} \in \mathfrak{C}^{X_1}, \underline{c} \in \mathfrak{C}^{X_2}, \underline{d} \in \mathfrak{C}^{X_3}$ and all $c,d \in \mathfrak{C}$, the following square commutes:

$$\begin{tikzcd} S_{(\underline{b},c)} \times S_{(\underline{c},\omega c,d)} \times S_{(\underline{d},\omega d)} \arrow[r, "\diamond_c \times id"] \arrow[d, "id\times\diamond_d"'] & S_{(\underline{bc},d)} \times S_{(\underline{d},\omega d)} \arrow[d, "\diamond_d"] \\ S_{(\underline{b},c)} \times S_{(\underline{c},\omega c,\underline{d})} \arrow[r, "\diamond_c"'] & S_{\underline{bcd}}. \end{tikzcd}$$



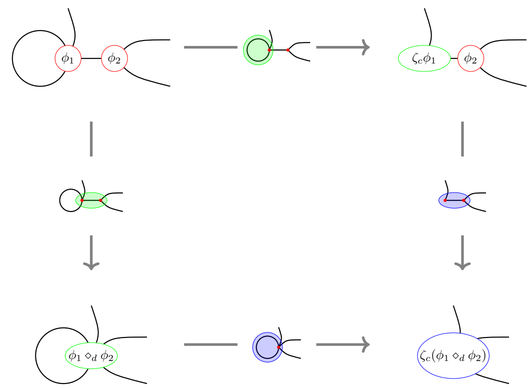
(M2) Order of contraction does not matter.
For all $\underline{c} \in \mathfrak{C}^X$ and $c,d \in \mathfrak{C}$, the following square commutes:

$$\begin{tikzcd} S_{(\underline{c},c,\omega c,d,\omega d)} \arrow[r, "\zeta_c"] \arrow[d, "\zeta_d"'] & S_{(\underline{c},d,\omega d)} \arrow[d, "\zeta_d"] \\ S_{(\underline{c},c,\omega c)} \arrow[r, "\zeta_c"'] & S_{\underline{c}}. \end{tikzcd}$$



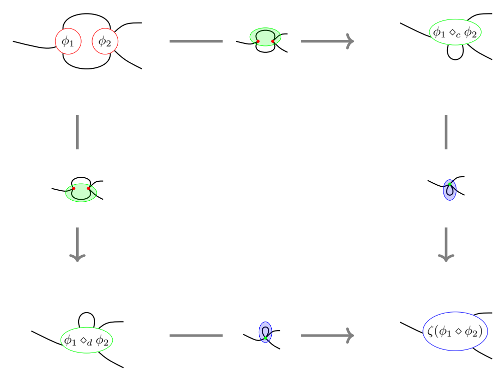
(M3) Multiplication and contraction commute.
For all $\underline{c} \in \mathfrak{C}^{X_1}$, $\underline{d} \in \mathfrak{C}^{X_2}$ *and* $c, d \in \mathfrak{C}$, *the following square commutes.*

$$\begin{tikzcd}
S_{(\underline{c},c,\omega c,d)} \times S_{(\underline{d},\omega d)} \arrow[r, "\zeta_c \times id"] \arrow[d, "\diamond_d"'] & S_{(\underline{c},d)} \times S_{(\underline{d},\omega d)} \arrow[d, "\diamond_d"] \\
S_{(\underline{c},c,\omega c,\underline{d})} \arrow[r, "\zeta_c"'] & S_{\underline{cd}}
\end{tikzcd}$$



(M4) ‘Parallel multiplication’ of pairs.
For all $\underline{c} \in \mathfrak{C}^{X_1}$, $\underline{d} \in \mathfrak{C}^{X_2}$, *and* $c, d \in \mathfrak{C}$, *the following square commutes:*

$$\begin{tikzcd}
S_{(\underline{c},c,d)} \times S_{(\underline{d},\omega c,\omega d)} \arrow[r, "\diamond_c"] \arrow[d, "\diamond_d"'] & S_{(\underline{c},d,\underline{d},\omega d)} \arrow[d, "\zeta_d"] \\
S_{(\underline{c},c,\underline{d},\omega c)} \arrow[r, "\zeta_c"'] & S_{\underline{cd}}
\end{tikzcd}$$



Modular operads form a category MO *whose morphisms are morphisms of the underlying graphical species that preserve multiplication, contraction and multiplicative units.*

Informally, the multiplication and contraction operations describe rules for *collapsing* edges of graphs that represent formal compositions of elements. The coherence axioms (M1)-(M4) say that this is independent of the order in which the edges are collapsed.

Remark 1.25. A *non-unital modular operad* $(S, \diamond, \zeta)$ is a graphical species S equipped with a multiplication $\diamond$ and contraction ζ satisfying (M1)-(M4), but without the requirement of a multiplicative unit. These form a category MO^- whose morphisms are morphisms in GS that preserve the multiplication and contraction operations. Non-unital modular operads are the subject of Section 5.

To provide context and motivation for the constructions that follow, the remainder of this section is devoted to examples.

Example 1.26. **Getzler-Kapranov modular operads.** We equip the monochrome graphical species $M : (M_X = \mathbb{N})_{X \in \mathbf{B}}$, with a unital multiplication $(+, 0 \in M_{\mathbf{2}})$ induced by addition in $\mathbb{N}$:

$$+\colon M_{\mathbf{m}} \times M_{\mathbf{n}} \to M_{\mathbf{m+n-2}},\ (g_m, g_n) \mapsto g_m + g_n \text{ for } m, n \geq 1,$$

and a contraction t induced by the successor operation:

$$t\colon M_{\mathbf{n}} \to M_{\mathbf{n-2}},\ g_n \mapsto g_n + 1 \text{ for } n \geq 2.$$

Since a compact oriented surface with boundary is determined, up to homeomorphism, by its genus and number of boundary components, $(M, +, s)$ models gluing of topological surfaces along boundary components (see Figure 1). Monochrome objects (S,γ) of the slice category MO/M describe a bigraded set $(S^\gamma(g,n))_{g,n}$ with operations

$$+^S\colon S^\gamma(g_1,n_1)\times S^\gamma(g_2,n_2)\to S^\gamma(g_1+g_2,n_1+n_2-2) \text{ for } n_1,n_2\geq 1,$$
$$t^S\colon S^\gamma(g,n)\to S^\gamma(g+1,n-2) \text{ for } n\geq 2,$$

and may encode (moduli spaces) of geometric structures on surfaces. For example, the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of genus g smooth algebraic curves with n marked points, may be described, via Belyi's Theorem, in terms of the space of genus g Riemann surfaces with n nodes, and the spaces $\overline{\mathcal{M}}_n \overset{\text{def}}{=} \coprod_{g\in\mathbb{N}}\overline{\mathcal{M}}_{g,n}$ form a monochrome modular operad (c.f. [16, Example 6.2]).

Getzler and Kapranov [16] originally defined modular operads in terms of the restriction to the *stable part* $M^{st}\subset M$ of the graphical species M. An oriented surface Γ is stable if and only if $2g+n-2>0$. So $M^{st}_{\mathbf{n}} = M_{\mathbf{n}}$ for $n>2$ but $M^{st}_{\mathbf{0}} = \{2,3,4,\dots\}$ and $M^{st}_{\mathbf{1}} = M^{st}_{\mathbf{2}} = \{1,2,3,4,\dots\}$.

In particular, since $0\notin M^{st}_{\mathbf{2}}$, modular operads in the original sense of [16] are non-unital.

These ideas may be extended to many-coloured cases: for example, one can describe a 2-coloured modular operad for gluing surfaces along open and closed subsets of their boundaries. (See, e.g. [17].)

Example 1.27. **Compact closed categories**, introduced in [24], are symmetric monoidal categories $(\mathsf{C},\otimes,e)$ for which every object $c\in\mathsf{C}$ has a symmetric *categorical dual* (see [4, 26]): there is an object $c^*\in\mathsf{C}$, and morphisms $\cap_c\colon e\to c^*\otimes c$ and $\cup_c\colon c\otimes c^*\to e$ such that

$$(\cup_c\otimes id_c)\circ(id_c\otimes\cap_c) = id_c = (\cap_{c^*}\otimes id_c)\circ(id_c\otimes\cup_{c^*}).$$

Examples of compact closed categories include categories of finite dimensional vector spaces over a given field, or, more generally, finite dimensional projective modules over a commutative ring. Cobordism categories provide other important examples.

The category of small compact closed categories contains an equivalent full subcategory Comp of those categories C such that $c=c^{**}$ for all objects. Moreover, there is a canonical monadic adjunction $\mathsf{MO}\leftrightarrows\mathsf{Comp}$. The right adjoint takes a small compact closed category $(\mathsf{C},\otimes,e,*)$ with object set C_0 to a $(\mathsf{C}_0,*)$-coloured modular operad S^{C} with coloured arities

$$S^{\mathsf{C}}_{(d_1,\dots,d_n,c^*_m,\dots,c^*_1)} = \mathsf{C}(c_1\otimes\cdots\otimes c_m, d_1\otimes\cdots\otimes d_n)$$

and the modular operad structure on S^{C} is induced by composition in C together with $\cup$ and $\cap$. The left adjoint $\mathsf{MO}\to\mathsf{Comp}$ is induced by the free monoid functor on palettes and arities.

These observations underly the proof, in [37], of an 'operadic' nerve theorem for compact closed categories in the style of Section 8.

Example 1.28. **Circuit algebras** – so named because of their resemblance to electronic circuits – are a symmetric version of Jones's planar algebras, introduced to study finite-type invariants in low-dimensional topology [1, 13].

Objects of the category CO of *circuit operads* [36] are modular operads equipped, via a monadic adjunction $\mathsf{MO}\leftrightarrows\mathsf{CO}$, with an extra 'external product' operation. Circuit operads in a Set (or a linear category $\mathcal{E}$) are precisely the circuit algebras in Set (or $\mathcal{E}$).

The adjunction between modular operads and compact closed categories in Example 1.27 factors though the adjunction $\mathsf{MO}\leftrightarrows\mathsf{CO}$. Moreover, every compact closed category in Comp is equivalent to the free compact closed category on a circuit operad.

This formal perspective on modular operads, circuit algebras, and compact closed categories leads to interesting questions in a number of directions. For example, we can study the analogous relationships if the definition of modular operads is relaxed by replacing the symmetric action with a braiding, or by considering higher dimensional versions. Related ideas are being explored by Dansco, Halacheva and Robertson in their work on algebraic and categorical structures in low-dimensional topology.

Example 1.29. **Wheeled properads.** Wheeled properads have been studied extensively in [18] and [42]. They describe the *connected part* (c.f. [40, Introduction]) of wheeled PROPs – coloured PROPs with a contraction – that have applications in geometry, deformation theory, and other areas [33, 32].

The category WP of (Set-valued) wheeled properads is canonically equivalent to the slice category MO/Di of *directed modular operads.* This is well-defined since the terminal directed graphical species Di trivially admits the structure of a modular operad (see Example 1.10).

An equivalence between wheeled PROPs in linear categories and directed circuit algebras is established in [13]. The equivalence between MO/Di and WP is extended to an equivalence between wheeled PROPs and circuit operads in [36].

2. Abstract nerve theorems and distributive laws

The purpose of this largely formal section is to review some basic theory of distributive laws, and provide an overview of Weber's abstract nerve theory. The simplicial nerve for categories, and the dendroidal nerve for operads provide motivating examples for the latter.

For an overview of monads and their Eilenberg–Moore (EM) categories of algebras, see for example [30, Chapter VI].

2.1. Monads with arities and abstract nerve theory.

Given a locally small category C, a functor $F\colon \mathsf{D} \to \mathsf{C}$ induces a *nerve functor* $N_{\mathsf{D}}\colon \mathsf{C} \to \mathsf{psh}(\mathsf{D})$ by $N_{\mathsf{D}}(c)(d) = \mathsf{C}(Fd, c)$ for all $c \in \mathsf{C}$ and $d \in \mathsf{D}$. If N_{D} is fully faithful, and F and D are suitably nice, then N_{D} provides a useful tool for studying C.

In the crudest sense, monads with arities are monads whose EM category of algebras may be characterised in terms of a purely abstract fully faithful nerve. The aim of this section is to explain, without proofs, the key points of this abstract nerve theory. This motivates the framework of this paper, and underlies the proof of the nerve theorem for modular operads, Theorem 8.2, in Section 8. Details may be found in [5, Sections 1-3].

Recall that every functor admits an (up to isomorphism) unique *bo-ff factorisation* as a bijective on objects functor followed by a fully faithful functor. For example, if $\mathbb{M}$ is a monad on a category C, and $\mathsf{C}^{\mathbb{M}}$ is the EM category of algebras for $\mathbb{M}$, then the free functor $\mathsf{C} \to \mathsf{C}^{\mathbb{M}}$ has bo-ff factorisation $\mathsf{C} \to \mathsf{C}_{\mathbb{M}} \to \mathsf{C}^{\mathbb{M}}$, where $\mathsf{C}_{\mathbb{M}}$ is the Kleisli category of free $\mathbb{M}$-algebras (see e.g. [30, Section VI.5]).

Hence, for any subcategory D of C, the bo-ff factorisation of the canonical functor $\mathsf{D} \hookrightarrow \mathsf{C} \xrightarrow{\text{free}} \mathsf{C}^{\mathbb{M}}$ factors through the full subcategory $\Theta_{\mathbb{M},\mathsf{D}}$ of $\mathsf{C}_{\mathbb{M}}$ with objects from D.

By construction, the defining functor $\Theta_{\mathbb{M},\mathsf{D}} \to \mathsf{C}^{\mathbb{M}}$ is fully faithful. So, it is natural to ask if there are conditions on D and $\mathbb{M}$ that ensure that the induced nerve $N_{\mathbb{M},\mathsf{D}}\colon \mathsf{C}^{\mathbb{M}} \to \mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}})$ is also fully faithful. This is the motivation for describing monads with arities.

Definition 2.1. *As usual, a subcategory* $\mathsf{D} \hookrightarrow \mathsf{C}$ *is called* replete *if any object* $c \in \mathsf{C}$ *such that* $c \cong d$ *for some* $d \in \mathsf{D}$*, is, itself, an object of* D*. The* essential image $im^{es}(F)$ *of a functor* $F\colon \mathsf{E} \to \mathsf{C}$ *is the smallest replete subcategory* $im^{es}(F) \hookrightarrow \mathsf{C}$ *containing the image* $im(F)$ *of* F *in* C*.*

A subcategory $\iota\colon \mathsf{D} \hookrightarrow \mathsf{C}$ *is a* dense subcategory *(and* ι *is a* dense functor*) if the induced nerve* $N_{\mathsf{D}}\colon \mathsf{C} \to \mathsf{psh}(\mathsf{D})$ *is full and faithful.*

Once again, let $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ be a monad on C. Let $\iota\colon \mathsf{D} \to \mathsf{C}$ be a dense subcategory, and let $\Theta_{\mathbb{M},\mathsf{D}}$ be obtained in the bo-ff factorisation of $\mathsf{D} \to \mathsf{C}^{\mathbb{M}}$. There is an induced diagram of functors

$$\begin{tikzcd} \Theta_{\mathbb{M},\mathsf{D}} \arrow[r, "\text{f.f.}"] & \mathsf{C}^{\mathbb{M}} \arrow[r, "N_{\mathbb{M},\mathsf{D}}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}}) \arrow[d, "j^*"] \\ \mathsf{D} \arrow[u, "\text{b.o.}", "j"'] \arrow[r, hook, "\text{dense}"] & \mathsf{C} \arrow[u, shift left, "\text{free}"] \arrow[r, "N_{\mathsf{D}}", "\text{f.f.}"'] & \mathsf{psh}(\mathsf{D}). \end{tikzcd} \tag{2.2}$$

where j^* is the pullback of the bijective on objects functor $j\colon \mathsf{D} \to \Theta_{\mathbb{M},\mathsf{D}}$. The left square of (2.2) commutes by definition, and the right square commutes up to natural isomorphism.

By [28, Proposition 5.1], the inclusion $\iota\colon \mathsf{D} \to \mathsf{C}$ is dense if and only if every object c of C is given canonically by the colimit of the functor $\mathsf{D}/c \to \mathsf{C}$, $(d, f) \mapsto \iota(d)$.

The monad $\mathbb{M}$ *has arities* D if $N_{\mathsf{D}} \circ M$ takes the canonical colimit cocones D/c in C to colimit cocones in $\mathsf{psh}(\mathsf{D})$. In this case, by [41, Section 4], the full inclusion $\Theta_{\mathbb{M},\mathsf{D}} \to \mathsf{C}^{\mathbb{M}}$ is dense, and the essential image of the induced fully faithful nerve $N_{\mathbb{M},\mathsf{D}}\colon \mathsf{C}^{\mathbb{M}} \to \mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}})$ is the full subcategory of $\mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}})$ on those presheaves P whose restriction j^*P to D is in the essential image of $N_{\mathsf{D}}\colon \mathsf{C} \to \mathsf{psh}(\mathsf{D})$.

Remark 2.3. The condition that $\mathbb{M}$ has arities $\mathsf{D} \hookrightarrow \mathsf{C}$ is sufficient, but not necessary, for the induced nerve $\mathsf{C}^{\mathbb{M}} \to \mathsf{psh}(\Theta_{\mathbb{M},\mathsf{D}})$ to be fully faithful.

In fact, by Theorem 8.2 and Remark 8.12, the modular operad monad $\mathbb{O}$ on the category of graphical species, together with the full dense subcategory $\mathsf{CetGr} \hookrightarrow \mathsf{GS}$ of connected graphs and étale morphisms (described in Section 4), provides an example of a monad that does not have arities, but for which the nerve theorem holds.

Necessary conditions on $\mathbb{M}$ and $\mathsf{D} \hookrightarrow \mathsf{C}$, for the induced nerve to be fully faithful are described in [8].

Example 2.4. Recall that directed graphs $\mathbf{G} = (\mathfrak{s}, \mathfrak{t}\colon E \rightrightarrows V)$ are presheaves over the small diagram category $\mathcal{E} \overset{\text{def}}{=} \bullet \rightrightarrows \bullet$, and that the canonical forgetful functor from Cat to $\mathsf{psh}(\mathcal{E})$ – that takes a small category C to the directed graph $\mathbf{G}_{\mathsf{C}}$ with vertex set V_{C} indexed by objects of C, and edge set E_{C} indexed by morphisms of C – is monadic. So, every directed graph freely generates a small category.

For all $n \in \mathbb{N}$, the finite ordinal $[n]$ may be viewed as a directed linear graph:

$$[n] = \overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n}{\bullet}. \tag{2.5}$$

The free category on $[n]$ is the n-simplex $\Delta(n)$, and Δ is the *simplex category* of *simplices* $\Delta(n)$, $n \in \mathbb{N}$, and functors between them. The category of Δ-presheaves, or *simplicial sets*, is denoted by sSet.

The *classical nerve theorem* states that the induced nerve functor $N_{\Delta}\colon \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful. Moreover, its essential image consists of precisely those $P \in \mathsf{sSet}$ that satisfy the classical Segal condition (originally formulated in [39]): a simplicial set P is the nerve of a small category if and only if, for $n > 1$, the set P_n of n-simplices is isomorphic to the n-fold fibred product

$$P_n \cong \underbrace{P_1 \times_{P_0} \cdots \times_{P_0} P_1}_{n \text{ times}}. \tag{2.6}$$

The nerve theorem and Segal condition (2.6) may be derived using the abstract nerve theory of [5]:

Let $\Delta_0 \subset \mathsf{psh}(\mathcal{E})$ be the full subcategory on the directed linear graphs $[n]$ whose morphisms are *successor-preserving*: $f\colon [m] \to [n]$ satisfies $f(i+1) = f(i) + 1$ for all $0 \leq i < m$. In particular, $\mathcal{E}$ embeds in Δ_0 as the full subcategory on the objects $[0]$ and $[1]$, and the full inclusion $\Delta_0 \hookrightarrow \mathsf{psh}(\mathcal{E})$ is precisely the nerve induced by the inclusion $\mathcal{E} \hookrightarrow \Delta_0$. Hence $\mathcal{E}$ is dense in Δ_0. Since $\mathcal{E} \hookrightarrow \Delta_0$ is fully faithful, so is N_{Δ_0} (by [31, Section VII.2]), so Δ_0 is also dense in $\mathsf{psh}(\mathcal{E})$.

Since Δ is the category obtained in the bo-ff factorisation of $\Delta_0 \to \mathsf{psh}(\mathcal{E}) \to \mathsf{Cat}$, it follows that there is a diagram of functors

$$\begin{tikzcd}
 & \Delta \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Cat} \arrow[r, "N_\Delta"] \arrow[d, shift left, "\text{forget}"] & \mathsf{sSet} \arrow[d, "j^*"] \\
\mathcal{E} \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \Delta_0 \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{psh}(\mathcal{E}) \arrow[u, shift left, "\text{free}"] \arrow[r, hook, "N_{\Delta_0}", "\text{f.f.}"'] & \mathsf{psh}(\Delta_0).
\end{tikzcd} \tag{2.7}$$

where the left square commutes, and the right square commutes up to natural isomorphism.

It is straightforward to prove – using for example [5, Sections 1 & 2] – that the category monad on $\mathsf{psh}(\mathcal{E})$ has arities Δ_0. Hence $N_\Delta \colon \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful, and a simplicial set P is in its essential image if and only if j^*P is in the essential image of N_{Δ_0}. Segal's condition (2.6) follows from the fact that $\mathcal{E}$ is dense in Δ_0.

The classical Segal condition (2.6) may be generalised as follows:

As before, let $\mathsf{D} \subset \mathsf{C}$ be a dense subcategory, and, as in Example 2.4, let $\mathsf{C} = \mathsf{psh}(\mathcal{E})$ be the category of presheaves on a dense subcategory $\mathcal{E}$ of D. So, the dense inclusion $\mathsf{D} \hookrightarrow \mathsf{C}$ is also full. If D provides arities for a monad $\mathbb{M}$ on C, then by [5, Lemma 3.6], a presheaf $P \colon \Theta^{\mathrm{op}}_{\mathbb{M},\mathsf{D}} \to \mathsf{Set}$ is in the essential image of $N_{\mathbb{M},\mathsf{D}}$ if and only if

$$P(jd) = \lim_{(e,f)\in \mathcal{E}/d} j^*(P)(e) \quad \text{for all } d \in \mathsf{D}. \tag{2.8}$$

Equation (2.8) is called the *Segal condition* for the nerve functor $N_{\mathbb{M},\mathsf{D}}$.

Remark 2.9. The notion of graph in Example 2.4 is, in a suitable sense, dual to the one used in Example 2.10, and in the rest of this paper (see Section 3), where edges function as 'objects' and connections between them as 'morphisms'. This is also the case in Example 2.10.

Example 2.10. Recall from Example 1.18 that an operad is a presheaf O on $\mathbf{B}^{\downarrow}$, equipped with a unital composition operation satisfying certain axioms. The forgetful functor $\mathsf{Op} \to \mathsf{psh}(\mathbf{B}^{\downarrow})$ is monadic, so every presheaf O on $\mathbf{B}^{\downarrow}$ freely generates an operad. Let $\mathbb{M}_{\mathsf{Op}}$ be the induced monad.

Objects of $\mathbf{B}^{\downarrow}$ are the rooted corollas t_X for finite X, and the directed exceptional edge $(\downarrow)$. *Rooted trees* $\mathfrak{T}$ are obtained as formal colimits of finite diagrams in $\mathbf{B}^{\downarrow}$, that describe *grafting* of objects of $\mathbf{B}^{\downarrow}$ root-to-leaf as in Figure 7(b).

Let Ω_0 be the category whose objects are such rooted trees $\mathfrak{T}$ and whose morphisms $\mathfrak{S} \to \mathfrak{T}$ are (up to isomorphism) inclusions of rooted trees that preserve vertex valency (as in Figure 7(a)). Then $\mathbf{B}^{\downarrow} \subset \Omega_0$ is the full, and dense, subcategory of rooted trees with zero or one vertex.

(a) (b)

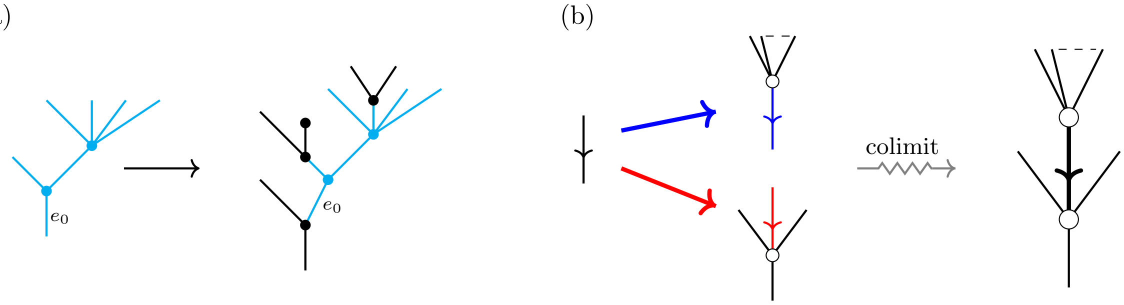


FIGURE 7. (a) Subtree inclusion, (b) grafting of rooted corollas to form a rooted tree.

Hence, the induced nerve $\Omega_0 \to \mathsf{psh}(\mathbf{B}^{\downarrow})$ is full and faithful, and $\mathbf{B}^{\downarrow}$ canonically induces a topology on Ω_0 whose sheaves are precisely $\mathbf{B}^{\downarrow}$-presheaves. In particular, $\Omega_0 \to \mathsf{psh}(\mathbf{B}^{\downarrow})$ is also dense (see e.g. Section 4.4 for comparison), and there is a diagram of functors

$$\begin{tikzcd} & \Omega \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Op} \arrow[r, "N_\Omega"] \arrow[d, shift right, "\text{free}"', leftarrow] \arrow[d, shift left, "\text{forget}"] & \mathsf{psh}(\Omega) \arrow[d, "j^*"] \\ \mathbf{B}^{\downarrow} \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \Omega_0 \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{psh}(\mathbf{B}^{\downarrow}) \arrow[r, hook, "N_{\Omega_0}", "\text{f.f.}"'] & \mathsf{psh}(\Omega_0) \end{tikzcd} \tag{2.11}$$

in which the left square commutes and the right square commutes up to natural isomorphism. The full subcategory of Op induced by the bo-ff factorisation of the functor $\Omega_0 \to \mathsf{psh}(\mathbf{B}^{\downarrow}) \to \mathsf{Op}$ is the *dendroidal category* Ω of free operads on rooted trees, described in [34]. There it was established (by different methods) that the full inclusion $\Omega \hookrightarrow \mathsf{Op}$ is dense, and hence the *dendroidal nerve* N_Ω is fully faithful.

It is easy to show, e.g. using methods similar to those described in Section 8, that the monad $\mathbb{M}_{\mathsf{Op}}$ on $\mathsf{psh}(\mathbf{B}^{\downarrow})$ has arities Ω_0. Hence, the abstract nerve theory of [5] may also be used to show that the nerve functor $N_\Omega \colon \mathsf{Op} \to \mathsf{psh}(\Omega)$ is fully faithful and its essential image consists of those Ω-presheaves (or *dendroidal sets*) $O \colon \Omega^{\mathrm{op}} \to \mathsf{Set}$ that satisfy the *dendroidal Segal condition* first proved in [11, Corollary 2.6]:

$$O(\mathfrak{T}) = \lim_{(t,i)\in(\mathbf{B}^{\downarrow}/\mathfrak{T})} j^*O(t) \quad \text{for all symmetric rooted trees } \mathfrak{T}. \tag{2.12}$$

In particular, since Δ_0 is the full subcategory of linear trees in Ω_0, the simplicial nerve theorem for categories is a special case of the dendroidal nerve theorem for operads.

Definition 2.13. *A* pointed endofunctor *on a category* C *is an endofunctor* E *on* C *together with a natural transformation* $\eta^E \colon 1_{\mathsf{C}} \Rightarrow E$*. An* algebra for a pointed endofunctor (E, η^E) *on* C *is a pair* (c, θ) *of an object* c *of* C *and a morphism* $\theta \in \mathsf{C}(Ec, c)$ *such that* $\theta \circ \eta^{\mathsf{C}}_c = id_c \in \mathsf{C}(c, c)$*.*

For example, modular operads are algebras for the pointed endofunctor on GS described in [22]. However, as discussed in Section 6, the abstract nerve machinery of [5] cannot be modified for algebras of (pointed) endofunctors:

Namely, for any monad $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ on a category C, the EM category $\mathsf{C}^{\mathbb{M}}$ of $\mathbb{M}$-algebras embeds canonically in the category C^M of algebras for the pointed endofunctor $(M, \eta^{\mathbb{M}})$. However, the free functor $C \to \mathsf{C}^{\mathbb{M}} \to \mathsf{C}^M$, $c \mapsto (Mc, \mu^{\mathbb{M}}c)$ depends crucially on the monadic multiplication $\mu^{\mathbb{M}} \colon M^2 \Rightarrow M$ of $\mathbb{M}$.

By contrast, for an arbitrary pointed endofunctor (E, η^E) is on C, there is, in general, no canonical choice of functor $\mathsf{C} \to \mathsf{C}^E$.

2.2. Distributive laws.

With Examples 2.4 and 2.10 in mind, let us return to the case of modular operads. Recall that graphical species are presheaves on the category $\mathbf{B}^{\S}$ and that modular operads are graphical species equipped with certain operations.

Informally, monads are gadgets that encode, via their algebras, (algebraic) structure on objects of categories. In [22], it is the combination of the contraction structure ζ, and the multiplicative unit structure ϵ that provides an obstruction to extending the modular operad endofunctor on GS to a monad (see Section 6). So, one approach to constructing the modular operad monad $\mathbb{O}$ on GS could be to find monads for the modular operadic multiplication, contraction, and unital structures separately, and then attempt to combine them.

In general, monads do not compose. Given monads $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ and $\mathbb{M}' = (M', \mu^{\mathbb{M}'}, \eta^{\mathbb{M}'})$ on a category C, there is no obvious choice of natural transformation $\mu \colon (MM')^2 \Rightarrow MM'$ defining a monadic multiplication for the endofunctor MM' on C.

Observe, however, that any natural transformation $\lambda\colon M'M \Rightarrow MM'$ induces a natural transformation

$$\mu_\lambda\colon\ (MM')^2 \xRightarrow{M\lambda M'} M^2M'^2 \xRightarrow{\mu^{\mathbb{M}}\mu^{\mathbb{M}'}} MM'.$$

Definition 2.14. *A* distributive law [3] *for* $\mathbb{M}$ *and* $\mathbb{M}'$ *is a natural transformation* $\lambda\colon M'M \Rightarrow MM'$ *such that the triple* $(MM', \mu_\lambda, \eta^{\mathbb{M}}\eta^{\mathbb{M}'})$ *defines a monad* $\mathbb{M}\mathbb{M}'$ *on* C.

A distributive law $\lambda\colon M'M \Rightarrow M'M$ determines how the $\mathbb{M}$-structures and $\mathbb{M}'$-structures on C interact to form the structure encoded by the composite monad $\mathbb{M}\mathbb{M}'$.

Example 2.15. The category monad on $\mathsf{psh}(\mathcal{E})$ (Example 2.4) may be obtained as a composite of the *semi-category monad*, which governs associative composition, and the *reflexive graph monad* that adjoins a distinguished loop at each vertex of a graph $\mathbf{G} \in \mathsf{psh}(\mathcal{E})$. The corresponding distributive law encodes the property that the adjoined loops provide identities for the semi-categorical composition.

(There is also a distributive law in the other direction, but the two structures do not interact in the composite. See also Remark 7.44.)

As usual, let $\mathsf{C}^{\mathbb{M}}$ denote the EM category of algebras for a monad $\mathbb{M}$ on C.

By [3, Section 3], given monads $\mathbb{M}, \mathbb{M}'$ on C, and a distributive law $\lambda\colon M'M \Rightarrow MM'$, there is a commuting square of strict monadic adjunctions:

(2.16)
$$\begin{tikzcd}
\mathsf{C}^{\mathbb{M}'} \arrow[d, shift right] \arrow[d, phantom, "\vdash"] \arrow[r, shift right] \arrow[r, phantom, "\top"] & \mathsf{C}^{\mathbb{M}\mathbb{M}'} \arrow[l, shift right] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] \\
\mathsf{C} \arrow[u, shift right] \arrow[r, shift right] \arrow[r, phantom, "\top"] & \mathsf{C}^{\mathbb{M}}. \arrow[l, shift right] \arrow[u, shift right]
\end{tikzcd}$$

In Section 4, it is shown that the category CetGr of connected Feynman graphs and étale morphisms (first defined in [22]) fits into a chain $\mathbf{B}^{\S} \hookrightarrow \mathsf{CetGr} \hookrightarrow \mathsf{GS}$ of fully faithful dense embeddings. And, in Section 7, the modular operad monad $\mathbb{O}$ on GS is constructed as a composite $\mathbb{D}\mathbb{T}$ of monads $\mathbb{T}$ (that governs contraction and non-unital multiplication) and $\mathbb{D}$ (that governs multiplicative units) on GS.

Hence, by (2.16), there is a monad $\mathbb{T}_*$ on the EM category GS_* of $\mathbb{D}$-algebras, such that $\mathsf{GS}_*^{\mathbb{T}_*} \cong \mathsf{MO}$ and a diagram of functors

(2.17)
$$\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}"] & \mathsf{psh}(\Xi) \arrow[d, "j^*"] \\
\mathbf{B}^{\S}_* \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{CetGr}_* \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{GS}_* \arrow[u, shift left, "\text{free}"] \arrow[r, hook, "\text{f.f.}"'] \arrow[d, shift left, "\text{forget}"] & \mathsf{psh}(\mathsf{CetGr}_*) \arrow[d] \\
\mathbf{B}^{\S} \arrow[u, "\text{b.o.}"] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{CetGr} \arrow[u, "\text{b.o.}"'] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{psh}(\mathsf{CetGr})
\end{tikzcd}$$

in which the categories $\mathbf{B}^{\S}_*$, CetGr_* and Ξ are obtained via bo-ff factorisations.

In Section 8, it is shown that $\mathbb{T}_*$ has arities CetGr_* (see Section 2). Whence it follows that the induced nerve $N\colon \mathsf{MO} \to \mathsf{psh}(\Xi)$ is fully faithful and its essential image is characterised in terms of $\mathbf{B}^{\S}_*$.

3. Graphs and their morphisms

This section is an introduction to Feynman graphs as defined in [22]. Most of this section and the next stay close to the original constructions there. Since [22] was just a short note, it contained very few proofs, and so relevant results are proved in full here. Extensive examples are also given. Where possible, definitions and examples are presented in a way that builds on Section 1 and highlights similarities with familiar concepts in basic topology.

This section deals with basic definitions and examples. The following section is devoted to a more detailed study of the topology of Feyman graphs, in terms of their étale morphisms.

3.1. **Graph-like diagrams and Feynman graphs.** In this paper, all graphs are finite, and may have loops, parallel edges, and loose ends (ports). Roughly speaking, a graph consists of a finite set of vertices V, and a finite set of connections $\widetilde{E}$, together with an incidence relation: if $\widetilde{E}$ is the orbit space of a set E under an involution τ, then the incidence is a partial function $E \rightharpoonup V$ that attaches connections to vertices.

Example 3.1. Section 15 of [2] provides a nice overview of various graph definitions that appear in the operad literature. The definition that is perhaps most familiar is that found in, for example, [16] and [7]. There, a graph G is described by sets V of vertices and E of edges, an involution $\hat{\tau}\colon E \to E$, and an incidence function $\hat{t}\colon E \to V$. The ports of G are the fixed points of the involution $\hat{\tau}$. A formal exceptional edge graph η is also allowed. Morphisms $\eta \to G$ are choices $\{*\} \to E$ of elements of E.

Feynman graphs are defined similarly to the graphs described in Example 3.1, except the involution on E must be fixed-point free, while the incidence is allowed to be a partial map $E \rightharpoonup V$. These subtle differences make it possible to encode the whole calculus of Feynman graphs in terms of the formal theory of diagrams in finite sets.

The category of *graph-like diagrams* is the category $\mathsf{psh_f}(\mathtt{D})$ of functors $\mathtt{D}^{\mathrm{op}} \to \mathsf{Set_f}$, where $\mathtt{D}$ is the small category $\circlearrowright \bullet \longleftarrow \bullet \longrightarrow \bullet$, and $\mathsf{Set_f}$ is the category of finite sets and all maps.

The initial object in $\mathsf{psh_f}(\mathtt{D})$ is the empty graph-like diagram:

$$\oslash = \quad \circlearrowright \mathbf{0} \longleftarrow \mathbf{0} \longrightarrow \mathbf{0},$$

and the terminal object $\bigstar$ is the trivial diagram of singletons:

$$\bigstar = \quad \circlearrowright \mathbf{1} \longleftarrow \mathbf{1} \longrightarrow \mathbf{1}.$$

Feynman graphs, introduced in [22], are graph-like diagrams satisfying extra properties:

Definition 3.2. *A* Feynman graph *is a graph-like diagram*

$$\mathcal{G} = \ \tau \circlearrowright E \xleftarrow{\ s\ } H \xrightarrow{\ t\ } V$$

such that $s\colon H \to E$ is injective and $\tau\colon E \to E$ is an involution without fixed points.

A (strong) subgraph $\mathcal{H} \hookrightarrow \mathcal{G}$ *of a Feynman graph $\mathcal{G}$ is a subdiagram that inherits a Feynman graph structure from $\mathcal{G}$.*

The full subcategory on graphs in $\mathsf{psh_f}(\mathtt{D})$ *is denoted by* $\mathsf{Grpsh_f}(\mathtt{D})$.

Elements of V are *vertices* of $\mathcal{G}$ and elements of E are called *edges* of $\mathcal{G}$. For each edge e, $\tilde{e}$ is the τ-orbit of e, and $\widetilde{E}$ is the set of τ-orbits in E. Elements of H are *half-edges* of $\mathcal{G}$, and, together with the maps s and t, H encodes a partial map $E \rightharpoonup V$ describing the incidence for the graph. A half-edge $h \in H$ may also be written as the ordered pair $h = (s(h), t(h))$.

In general, unless I wish to emphasise a point that is specific to the formalism of Feynman graphs, I will refer to Feynman graphs simply as 'graphs'.

Remark 3.3. A graph $\mathcal{G}$ may be realised geometrically by a one-dimensional space $|\mathcal{G}|$ obtained from the discrete space $\{*_v\}_{v\in V}$, and, for each $e \in E$, a copy $[0, \frac{1}{2}]_e$ of the interval $[0, \frac{1}{2}]$ subject to the identifications $0_{s(h)} \sim *_{t(h)}$ for $h \in H$, and $(\frac{1}{2})_e \sim (\frac{1}{2})_{\tau e}$ for all $e \in E$.

Example 3.4. (See also Figure 8(a).) The graph $(\shortmid)$ has no vertices and edge set $\mathbf{2} = \{1, 2\}$.

$$(\shortmid) \overset{\text{def}}{=} \quad \circlearrowright \mathbf{2} \longleftarrow \mathbf{0} \longrightarrow \mathbf{0}.$$

A *stick graph* is a graph that is isomorphic to $(\shortmid)$.

For any set X, $X^\dagger \cong X$ denotes its formal involution.

Example 3.5. (See also Figure 8(b), (c).) The X-*corolla* $\mathcal{C}_X$ associated to a finite set X has the form

$$\mathcal{C}_X : \qquad {}^{\dagger}\circlearrowright X \amalg X^\dagger \xhookleftarrow{\text{inc}} X^\dagger \longrightarrow \{*\}.$$

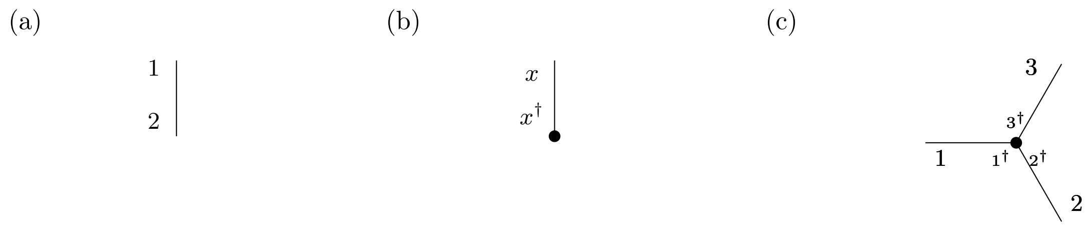


FIGURE 8. Realisations of (a) the stick graph $(\shortmid)$, and the corollas (b) $\mathcal{C}_{\{x\}}$ and (c) $\mathcal{C}_{\mathbf{3}}$.

Definition 3.6. *An* inner edge *of $\mathcal{G}$ is an element $e \in E$ such that $e \in im(s)$ and $\tau e \in im(s)$. The set $E_\bullet \subset E$ of inner edges of $\mathcal{G}$ is the maximal subset of $im(s) \subset E$ that is closed under τ. The set of* inner τ-orbits *$\tilde{e} \in \widetilde{E}$ with $e \in E_\bullet$ is denoted by $\widetilde{E_\bullet}$.*

The set $E_0 = E \setminus im(s)$ is the boundary *of $\mathcal{G}$ and elements $e \in E_0$ are* ports *of $\mathcal{G}$.*

A stick component *of a graph $\mathcal{G}$ is a pair $\{e, \tau e\}$ of edges of $\mathcal{G}$ such that e and τe are both ports.*

Graph morphisms preserve inner edges by definition. The stick graph $(\shortmid)$ has $E_0(\shortmid) = E(\shortmid) = \mathbf{2}$, and, for all finite sets X, the X-corolla $\mathcal{C}_X$ has boundary $E_0(\mathcal{C}_X) = X$.

Since $\mathsf{Set_f}$ admits finite (co)limits, so does $\mathsf{psh_f}(\mathsf{D})$, and these are computed pointwise. And, since $\mathsf{Grpsh_f}(\mathsf{D})$ is full in $\mathsf{psh_f}(\mathsf{D})$, (co)limits in $\mathsf{Grpsh_f}(\mathsf{D})$, when they exist, correspond to (co)limits in $\mathsf{psh_f}(\mathsf{D})$.

Example 3.7. The empty graph-like diagram $\oslash$ is trivially a graph, and is therefore initial in $\mathsf{Grpsh_f}(\mathsf{D})$. However, there is no non-trivial involution on a singleton set, so the terminal diagram $\bigstar$ in $\mathsf{psh_f}(\mathsf{D})$ is not a graph. Hence, $\mathsf{Grpsh_f}(\mathsf{D})$ is not closed in $\mathsf{psh_f}(\mathsf{D})$ under finite limits. (By Examples 3.16 and 3.33, $\mathsf{Grpsh_f}(\mathsf{D})$ is also not closed under finite colimits in $\mathsf{psh_f}(\mathsf{D})$.)

The cocartesian monoidal structure on $\mathsf{Set_f}$ is inherited by $\mathsf{psh_f}(\mathsf{D})$ and $\mathsf{Grpsh_f}(\mathsf{D})$, making these into strict symmetric monoidal categories under pointwise disjoint union $\amalg$, and with monoidal unit given by the empty graph $\oslash$.

Example 3.8. Let X and Y be finite sets. The graph $\mathcal{M}^{X,Y}_{x_0,y_0}$, illustrated in Figure 9, has two vertices and one inner edge orbit (highlighed in bold-face in Figure 9). It is obtained from the disjoint union $\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}}$ by identifying the τ-orbits of the ports x_0 and y_0 according to $x_0 \sim \tau y_0, y_0 \sim \tau x_0$. So,

$$\mathcal{M}^{X,Y}_{x_0,y_0} = \quad {}^{\tau}\circlearrowright \big((X \amalg Y) \amalg (X \amalg Y)^\dagger \amalg \{x_0, y_0\}\big) \xhookleftarrow{\quad s \quad} \big((X \amalg Y)^\dagger \amalg \{x_0, y_0\}\big) \xrightarrow{\ t\ } \{v_X, v_Y\},$$

where s is the obvious inclusion, and the involution τ is described by $z \leftrightarrow z^\dagger$ for $z \in X \amalg Y$ and $x_0 \leftrightarrow y_0$. The map t is described by $t^{-1}(v_X) = X^\dagger \amalg \{y_0\}$ and $t^{-1}(v_Y) = Y^\dagger \amalg \{x_0\}$.

In the construction of modular operads, graphs of the form $\mathcal{M}^{X,Y}_{x_0,y_0}$ are used to encode formal multiplications in graphical species.

Example 3.9. Formal contractions in graphical species are encoded by graphs of the form $\mathcal{N}^X_{x_0,y_0}$ (see Figure 9). For X a finite set, the graph $\mathcal{N}^X_{x_0,y_0}$ is the quotient of the corolla $\mathcal{C}_{X\amalg\{x_0,y_0\}}$ obtained by identifying the τ-orbits of the ports x_0 and y_0 according to $x_0 \sim \tau y_0$ and $y_0 \sim \tau x_0$. It has boundary $E_0 = X$, one inner τ-orbit $\{x_0, y_0\}$ (bold-face in Figure 9), and one vertex v. So,

$$\mathcal{N}^X_{x_0,y_0} = \quad {}^{\tau}\circlearrowright (X \amalg X^\dagger \amalg \{x_0, y_0\}) \xleftarrow{\ s\ } (X^\dagger \amalg \{x_0, y_0\}) \xrightarrow{\ t\ } \{v\}.$$

$|\mathcal{M}^{X,Y}_{x_0,y_0}|$ $\qquad$ $|\mathcal{N}^X_{x_0,y_0}|$

FIGURE 9. Realisations of $\mathcal{M}^{X,Y}_{x_0,y_0}$ and $\mathcal{N}^X_{x_0,y_0}$ for $X \cong \mathbf{2},\ Y \cong \mathbf{3}$.

Let $\mathcal{G}$ be a graph with vertex and edge sets V and E respectively. For each vertex v, define ${}^{H}\!/\!_{v} \overset{\text{def}}{=} t^{-1}(v) \subset H$ to be the fibre of t at v.

Definition 3.10. *Edges in the set* ${}^{E}\!/\!_{v} \overset{\text{def}}{=} s({}^{H}\!/\!_{v}) \subset E$ *are said to be* incident *on* v.

The map $|\cdot|\colon V \to \mathbb{N},\ v \mapsto |v| \overset{\text{def}}{=} |{}^{H}\!/\!_{v}|$, *defines the* valency *of* v *and* $V_n \subset V$ *is the set of* n-valent *vertices of* $\mathcal{G}$. *A* bivalent graph *is a graph* $\mathcal{G}$ *with* $V = V_2$.

A vertex v *is* bivalent *if* $|v| = 2$. *An* isolated vertex *of* $\mathcal{G}$ *is a vertex* $v \in V(\mathcal{G})$ *such that* $|v| = 0$.

Bivalent and isolated vertices are particularly important in Section 7.

Vertex valency also induces an $\mathbb{N}$-grading on the edge set E (and half-edge set H) of $\mathcal{G}$: For $n \geq 1$, define $H_n \overset{\text{def}}{=} t^{-1}(V_n)$ and $E_n \overset{\text{def}}{=} s(H_n)$. Since $s(H) = E \setminus E_0 = \coprod_{n\geq 1} E_n$,

$$E = \coprod_{n\in\mathbb{N}} E_n.$$

Example 3.11. (See Example 3.4.) Since $H(\shortmid)$ is empty, both edges of $(\shortmid)$ are ports: $E(\shortmid) = E_0(\shortmid)$.

The corolla $\mathcal{C}_X$ (see Example 3.5) with vertex $*$ has $X \cong {}^{E}\!/\!_{*} = {}^{H}\!/\!_{*}$. If $|X| = k$, then $|*| = k$, so $V = V_k$, and $E = E_k \amalg E_0$ with $E_i \cong X$ for $i = 0, k$.

Example 3.12. (See Examples 3.8, 3.9.) For finite sets X and Y, the graph $\mathcal{M}^{X,Y}_{x_0,y_0}$ has ${}^{E}\!/\!_{v_X} = X^\dagger \amalg \{y_0\}$ and ${}^{E}\!/\!_{v_Y} = Y^\dagger \amalg \{x_0\}$. If $X \cong \mathbf{n}$ for some $n \in \mathbb{N}$, then $v_X \in V_{n+1}$, and ${}^{E}\!/\!_{v_X} \subset E_{n+1}$.

The graph $\mathcal{N}^X_{x_0,y_0}$ has ${}^{E}\!/\!_{v} = X^\dagger \amalg \{x_0, y_0\} \cong H$, so $V = V_{n+2}$ when $X \cong \mathbf{n}$.

Since $\mathsf{Grpsh_f}(\mathsf{D})$ is full in the diagram category $\mathsf{psh_f}(\mathsf{D})$, morphisms $f \in \mathsf{Grpsh_f}(\mathsf{D})(\mathcal{G}, \mathcal{G}')$ are commuting diagrams in $\mathsf{Set_f}$ of the form

$$(3.13)\qquad \begin{tikzcd} \mathcal{G} \arrow[d, "f"'] \\ \mathcal{G}' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, leftrightarrow, "\tau"] \arrow[d, "f_E"'] & E \arrow[d, "f_E"'] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "f_H"'] & V \arrow[d, "f_V"] \\ E' \arrow[r, leftrightarrow, "\tau'"'] & E' & H' \arrow[l, "s'"] \arrow[r, "t'"'] & V'. \end{tikzcd}$$

Lemma 3.14. *For any morphism $f = (f_E, f_H, f_V) \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{G}')$, the map f_H is completely determined by f_E. Moreover if $\mathcal{G}$ has no isolated vertices, then f_E also determines f_V, and hence f.*

If $\mathcal{G}$ has no stick components or isolated vertices, then f is completely determined by f_H.

See also [27, Proposition 1.1.11].

Proof. By injectivity of s, The map $f_H \colon H \to H'$ given by $h \mapsto (s')^{-1} f_E s(h)$ is well defined since s is injective. If $\mathcal{G}$ has no isolated vertices, then, for each $v \in V$, ${}^{H}\!/\!{}_{v}$ is non-empty and the map $f_V \colon V \to V'$ given by $v \mapsto t'(s')^{-1} f_E s(h)$ does not depend on the choice of $h \in {}^{H}\!/\!{}_{v}$.

If $\mathcal{G}$ has no stick components then, for each $e \in E$, there is an $h \in H$ such that $e = s(h)$ or $e = \tau s(h)$, and the last statement of the lemma follows from the first. $\square$

Example 3.15. For any graph $\mathcal{G}$ with edge set E, there is a canonical (up to unique isomorphism) bijection $\mathsf{Grpsh_f}(\mathtt{D})(\shortmid, \mathcal{G}) \cong E$. The morphism $1 \mapsto e \in E$ in $\mathsf{Grpsh_f}(\mathtt{D})(\shortmid, \mathcal{G})$ that *chooses* $e \in E$ is denoted ch_e, or $ch_e^{\mathcal{G}}$.

Example 3.16. By Example 3.15, the stick graph $(\shortmid)$ has endomorphisms $ch_1 = id$ and $ch_2 = \tau$ in $\mathsf{Grpsh_f}(\mathtt{D})$. The coequaliser of $id, \tau \colon (\shortmid) \rightrightarrows (\shortmid)$ in the category $\mathsf{psh_f}(\mathtt{D})$ of graph-like diagrams is the *exceptional loop* $\bigcirc$:

$$\bigcirc \overset{\text{def}}{=} \circlearrowleft \mathbf{1} \longleftarrow \mathbf{0} \longrightarrow \mathbf{0} \,.$$

Clearly $\bigcirc$ is not a graph since a singleton set does not admit a non-trivial involution. Hence $\mathsf{Grpsh_f}(\mathtt{D})$ does not admit all finite colimits. This example is the subject of Section 6.

Definition 3.17. *A morphism $f \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{G}')$ is* locally injective *if, for all $v \in V$, the induced map $f_v \colon {}^{E}\!/\!{}_{v} \to {}^{E'}\!/\!{}_{f(v)}$ is injective, and* locally surjective *if $f_v \colon {}^{E}\!/\!{}_{v} \to {}^{E'}\!/\!{}_{f(v)}$ is surjective for all $v \in V$.*

Locally bijective morphisms are called étale.

Étale morphisms of graphs are studied in detail in Section 4.

Example 3.18. The following display illustrates two examples of morphisms in $\mathsf{Grpsh_f}(\mathtt{D})$, described below. Both morphisms are locally injective, and (b) is also surjective and locally surjective, hence étale.

In each example, the horizontal maps are the obvious projections, and the columns in the edge sets represent the orbits of the involution.

(a) The map f_a is determined by the image of the edges e_1 and e_2.

$$\begin{array}{ccccc}
\mathcal{C}_\mathbf{2} & \tau_\mathbf{2} \circlearrowright \left\{ \begin{array}{cc} e_1, & e_2, \\ \tau_\mathbf{2} e_1, & \tau_\mathbf{2} e_2 \end{array} \right\} & \hookleftarrow & \{(e_1, v), (e_2, v)\} & \longrightarrow \{v\} \\
\big\downarrow f_a & \big\downarrow & & \big\downarrow & \big\downarrow {\scriptstyle v \mapsto v_1} \\
\mathcal{G} & \tau \circlearrowright \left\{ \begin{array}{ccc} f_a(e_1), & f_a(e_2), & e_3', \\ \tau f_a(e_1), & \tau f_a(e_2), & \tau e_3' \end{array} \right\} & \hookleftarrow & \left\{ \begin{array}{ccc} (f_a(e_1), v_1), & (f_a(e_2), v_1), & (e_3', v_1) \\ (\tau f_a(e_1), v_2), & (\tau f_a(e_2), v_2), & \end{array} \right\} & \longrightarrow \{v_1, v_2\}.
\end{array}$$

(b) The map f_b is determined by $e_1, \tau_{\mathbf{2}} e_2 \mapsto f_b(e_1)$ and $e_2, \tau_{\mathbf{2}} e_1 \mapsto f_b(e_2)$.

$$\begin{tikzcd}
\mathcal{C}_{\mathbf{2}} \arrow[d, "f_b"'] & \left\{ \begin{array}{ll} e_1, & e_2, \\ \tau_{\mathbf{2}} e_1, & \tau_{\mathbf{2}} e_2 \end{array} \right\} \arrow[loop left, "\tau_{\mathbf{2}}"] \arrow[d] & \{(e_1, v), (e_2, v)\} \arrow[l] \arrow[r] \arrow[d] & \{v\} \arrow[d] \\
\mathcal{W} & \{f_b(e_1), f_b(e_2)\} \arrow[loop left, "\tau'"] & \{(f_b(e_1), v'), (f_b(e_2), v')\} \arrow[l] \arrow[r] & \{v'\}
\end{tikzcd}$$

Example 3.19. Recall Examples 3.8 and 3.9, above. For finite sets X and Y, the canonical morphisms $\mathcal{C}_X \longrightarrow \mathcal{M}^{X,Y}_{x_0,y_0} \longleftarrow \mathcal{C}_Y$ and $\mathcal{C}_X \longrightarrow \mathcal{N}^{X}_{x_0,y_0}$ are locally injective, but not locally surjective.

The canonical morphisms $\mathcal{C}_{X \amalg \{x_0\}} \longrightarrow \mathcal{M}^{X,Y}_{x_0,y_0} \longleftarrow \mathcal{C}_{Y \amalg \{y_0\}}$ are locally injective and locally surjective (hence étale), but neither is surjective. However, the canonical morphism $\mathcal{C}_{X \amalg \{x_0, y_0\}} \to \mathcal{N}^X_{x_0,y_0}$ is étale and surjective. (See Example 3.18(b) for the case $X = \emptyset$.)

Example 3.20. The assignment $X \mapsto \mathcal{C}_X$ describes a full embedding of $\mathsf{Set}_{\mathsf{f}}$ into $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$. Since $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})(\shortmid, \shortmid) \cong \mathbf{B}^{\S}(\S, \S)$ canonically, and any morphism in $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$ with domain $(\shortmid)$ is étale, it follows that $\mathbf{B}^{\S}$ embeds in $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$ as the subcategory of étale morphisms between the corollas $\mathcal{C}_X$ ($X \in \mathsf{Set}_{\mathsf{f}}$), and $(\shortmid)$.

Remark 3.21. By Example 3.20, $\mathbf{B}^{\S}$ will henceforth also be viewed as a subcategory of $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$. The choice of notation for objects – $(\shortmid)$ or $\S$, X or $\mathcal{C}_X$ – will depend on the context. The same notation will be used for morphisms in $\mathbf{B}^{\S}$ and their image in $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$. So $ch_x \in \mathbf{B}^{\S}(\S, X)$ (or $f \in \mathbf{B}^{\S}(X, Y)$) may also be written as $ch_x \in \mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})((\shortmid), \mathcal{C}_X)$ (or $f \in \mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})(\mathcal{C}_X, \mathcal{C}_Y)$).

Example 3.22. For all finite sets X and Y, the diagram of parallel morphisms

$$ch_{x_0}, ch_{y_0} \circ \tau \colon (\shortmid) \rightrightarrows \left(\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}}\right). \tag{3.23}$$

is in the image of the inclusion $\mathbf{B}^{\S} \hookrightarrow \mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$. It has colimit $\mathcal{M}^{X,Y}_{x_0,y_0}$ in $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$.

The graph $\mathcal{N}^X_{x_0,y_0}$ is the colimit in $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$ of the diagram

$$ch_{x_0},\ ch_{y_0} \circ \tau \colon (\shortmid) \rightrightarrows \mathcal{C}_{X \amalg \{x_0, y_0\}} \tag{3.24}$$

of parallel morphisms in the image of $\mathbf{B}^{\S}$ in $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$. (See also Example 5.12 and Figure 15.)

As will be shown in Section 4, all graphs may be constructed canonically as colimits of diagrams in the image of $\mathbf{B}^{\S}$ in $\mathsf{Grpsh}_{\mathsf{f}}(\mathtt{D})$.

3.2. **Connected components of graphs.** A graph is connected if it cannot be written as a disjoint union of non-empty connected graphs. Precisely:

Definition 3.25. *A non-empty graph-like diagram $\mathcal{G}$ is* connected *if, for each $\mathsf{f} \in \mathsf{psh}_{\mathsf{f}}(\mathtt{D})(\mathcal{G}, \bigstar \amalg \bigstar)$, the pullback of f along the inclusion $\bigstar \overset{incl_1}{\hookrightarrow} \bigstar \amalg \bigstar$ is either the empty graph-like diagram $\oslash$ or $\mathcal{G}$ itself. A graph $\mathcal{G}$ is connected if it is connected as a graph-like diagram.*

A (connected) component *of a graph $\mathcal{G}$ is a maximal connected subdiagram of $\mathcal{G}$.*

By Definition 3.2, a subdiagram $\mathcal{H} \hookrightarrow \mathcal{G}$ is a subgraph precisely when $E(\mathcal{H}) \subset E$ is closed under $\tau \colon E \to E$. Hence:

Lemma 3.26. *A connected component of a graph $\mathcal{G}$ inherits a subgraph structure from $\mathcal{G}$. If $\mathcal{H} \hookrightarrow \mathcal{G}$ is a subgraph of $\mathcal{G}$, then so is its complement $\mathcal{G} \setminus \mathcal{H}$.*

Therefore, every graph is the disjoint union of its connected components.

Remark 3.27. A graph $\mathcal{G}$ is connected if and only if its realisation $|\mathcal{G}|$ is a connected space.

Example 3.28. Following [27], a *shrub* $\mathcal{S}$ is a graph that is isomorphic to a disjoint union of stick graphs. Hence a shrub $\mathcal{S} = \mathcal{S}(J)$ is determined by a set J of edges, together with a fixed-point free involution $\tau_J \colon J \xrightarrow{\cong} J$. If f is a morphism in $\mathsf{Grpsh_f}(\mathtt{D})$ whose domain is a shrub, then f is trivially étale.

Given any graph $\mathcal{G}$, the shrub $\mathcal{S}(E)$ determined by (E, τ) is canonically a subgraph of $\mathcal{G}$. Components of $\mathcal{S}(E)$ are of the form

$$(\shortmid_{\tilde{e}}) \overset{\text{def}}{=} \quad \circlearrowright \{e, \tau e\} \longleftarrow \emptyset \longrightarrow \emptyset,$$

for each $\tilde{e} \in \widetilde{E}$. This is a subgraph of $\mathcal{G}$ under the *essential morphism* $\iota_{\tilde{e}} \colon (\shortmid_{\tilde{e}}) \hookrightarrow \mathcal{G}$ *at* $\tilde{e}$ *(for* $\mathcal{G}$*)* induced by the inclusion $\{e, \tau e\} \hookrightarrow E$.

Recall from Definition 3.6 that a stick component of $\mathcal{G}$ is a τ-orbit $\{e, \tau e\}$ in the boundary E_0 of $\mathcal{G}$. In particular, $\{e, \tau e\}$ is a stick component of $\mathcal{G}$ if and only if $(\shortmid_{\tilde{e}})$ is a connected component of $\mathcal{G}$.

Example 3.29. Recall that, for each $v \in V$, ${}^{E}\!/\!_{v} \overset{\text{def}}{=} s(t^{-1}(v))$ is the set of edges incident on v. Let $\mathbf{v} = ({}^{E}\!/\!_{v})^{\dagger}$ denote its formal involution. Then the corolla $\mathcal{C}_{\mathbf{v}}$ is given by

$$\mathcal{C}_{\mathbf{v}} = \quad \circlearrowright \left({}^{E}\!/\!_{v} \amalg ({}^{E}\!/\!_{v})^{\dagger}\right) \xleftarrow{\ s\ } {}^{H}\!/\!_{v} \xrightarrow{\ t\ } \{v\}.$$

The inclusion ${}^{E}\!/\!_{v} \hookrightarrow E$ induces a morphism $\iota_v^{\mathcal{G}}$ or $\iota_v \colon \mathcal{C}_{\mathbf{v}} \to \mathcal{G}$ called the *essential morphism at* v *for* $\mathcal{G}$. Whenever there is an edge e such that both e and τe are incident on v, then ι_v is not injective on edges.

If ${}^{E}\!/\!_{v}$ is empty – so $\mathcal{C}_{\mathbf{v}}$ is an isolated vertex – then $\mathcal{C}_{\mathbf{v}} \hookrightarrow \mathcal{G}$ is a connected component of $\mathcal{G}$.

Example 3.30. For $k \geq 0$, the *line graph* $\mathcal{L}^k$ is the connected bivalent graph (illustrated in Figure 10) with boundary $E_0 = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\}$, and

- ordered set of edges $E(\mathcal{L}^k) = (l_j)_{j=0}^{2k+1}$ where $l_0 = 1_{\mathcal{L}^k} \in E_0$ and $l_{2k+1} = 2_{\mathcal{L}^k} \in E_0$, and the involution is given by $\tau(l_{2i}) = l_{2i+1}$, for $0 \leq i \leq k$,
- ordered set of k vertices $V(\mathcal{L}^k) = (v_i)_{i=1}^k$, such that ${}^{E}\!/\!_{v_i} = \{l_{2i-1}, l_{2_i}\}$ for $1 \leq i \leq k$.

So, $\mathcal{L}^k$ is described by a diagram of the form $\circlearrowright \mathbf{2} \amalg 2(\mathbf{k}) \longleftarrow 2(\mathbf{k}) \longrightarrow \mathbf{k}$.

Example 3.31. The *wheel graph* $\mathcal{W} = \mathcal{W}^1$ with one vertex is the graph

$$\mathcal{W} \overset{\text{def}}{=} \quad \circlearrowright \{a, \tau a\} \longleftarrow \{a, \tau a\} \longrightarrow \{*\} \tag{3.32}$$

obtained as the coequaliser in $\mathsf{Grpsh_f}(\mathtt{D})$ of the morphisms $ch_1, ch_2 \circ \tau \colon (\shortmid) \rightrightarrows \mathcal{C}_{\mathbf{2}}$ (see Example 3.18(b)).

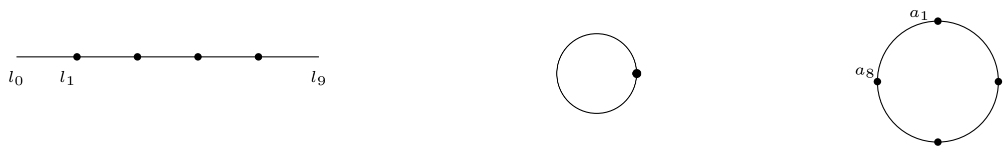
More generally, for $m \geq 1$, the wheel graph $\mathcal{W}^m$ (illustrated in Figure 10) is the connected bivalent graph obtained as the coequaliser in $\mathsf{Grpsh_f}(\mathtt{D})$ of the morphisms $ch_{1_{\mathcal{L}^m}}, ch_{2_{\mathcal{L}^m}} \circ \tau \colon (\shortmid) \rightrightarrows \mathcal{L}^m$. So $\mathcal{W}^m$ has empty boundary and

- $2m$ cyclically ordered edges $E(\mathcal{W}^m) = (a_j)_{j=1}^{2m}$, such that the involution satisfies $\tau(a_{2i}) = a_{2i+1}$ for $0 \leq i \leq m$ (where $a_0 = a_{2m}$),
- m cyclically ordered vertices $V(\mathcal{W}^m) = (v_i)_{i=1}^m$, that ${}^{E}\!/\!_{v_i} = \{a_{2i-1}, a_{2i}\}$ for $1 \leq i \leq m$.

So $\mathcal{W}^m$ is described by a diagram of the form $\circlearrowright 2(\mathbf{m}) \longleftarrow 2(\mathbf{m}) \longrightarrow \mathbf{m}$.

In Proposition 4.23, it will be shown that a connected bivalent graph is isomorphic to $\mathcal{L}^k$ or $\mathcal{W}^m$ for some $k \geq 0$ or $m \geq 1$.

Example 3.33. The wheel graph $\mathcal{W}$ with one vertex is weakly terminal in $\mathsf{Grpsh_f}(\mathtt{D})$: By Lemma 3.14, since $\widetilde{E}(\mathcal{W}) \cong V(\mathcal{W}) \cong \{*\}$, morphisms in $\mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{W})$ are in canonical bijection with projections $\mathsf{Grpsh_f}(\mathtt{D})(\mathcal{S}(E), \shortmid)$. Hence, for all graphs $\mathcal{G}$, there are precisely $2^{|\widetilde{E}|} \geq 1$ morphisms $\mathcal{G} \to \mathcal{W}$ in $\mathsf{Grpsh_f}(\mathtt{D})$ and every diagram in $\mathsf{Grpsh_f}(\mathtt{D})$ forms a cocone over $\mathcal{W}$.



$\mathcal{L}^4$ $\quad$ $\mathcal{W} = \mathcal{W}^1$ $\quad$ $\mathcal{W}^4$

FIGURE 10. Line and wheel graphs.

In particular,

$$\mathsf{Grpsh_f}(\mathtt{D})(\mathcal{W},\mathcal{W}) = \{id_{\mathcal{W}}, \tau_{\mathcal{W}}\} \cong \mathsf{Grpsh_f}(\mathtt{D})(\shortmid, \mathcal{W}) \cong \mathsf{Grpsh_f}(\mathtt{D})(\shortmid, \shortmid).$$

The morphisms $id_{\mathcal{W}}, \tau_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{W}$ do not admit a coequaliser in $\mathsf{Grpsh_f}(\mathtt{D})$ since their coequaliser in $\mathsf{psh_f}(\mathtt{D})$ is the terminal diagram $\bigstar$, which is not a graph.

Example 3.33 leads to another characterisation of connectedness:

Proposition 3.34. *The following are equivalent:*

(1) A graph $\mathcal{G}$ is connected;
(2) $\mathcal{G}$ is non-empty and, for every morphism $f \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{W} \amalg \mathcal{W})$, the pullback in $\mathsf{psh_f}(\mathtt{D})$ of f along the inclusion $inc_1 \colon \mathcal{W} \hookrightarrow \mathcal{W} \amalg \mathcal{W}$ is either the empty graph $\oslash$ or isomorphic to $\mathcal{G}$ itself;
(3) for every finite disjoint union of graphs $\coprod_{i=1}^k \mathcal{H}_i$,

$$\mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \coprod_{i=1}^k \mathcal{H}_i) \cong \coprod_{i=1}^k \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{H}_i). \tag{3.35}$$

Proof. (1) $\Leftrightarrow$ (2): Since $\mathcal{W}$ is weakly terminal, any morphism $f \in \mathsf{psh_f}(\mathtt{D})(\mathcal{G}, \bigstar \amalg \bigstar)$ factors as a morphism $\tilde{f} \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{W} \amalg \mathcal{W})$ followed by the componentwise projection $\mathcal{W} \amalg \mathcal{W} \to \bigstar \amalg \bigstar$ in $\mathsf{psh_f}(\mathtt{D})$.

(1) $\Rightarrow$ (3): For any finite disjoint union of graphs $\coprod_{i=1}^k \mathcal{H}_i$, and each $1 \leq j \leq k$, let $p_j \in \mathsf{psh_f}(\mathtt{D})(\coprod_{i=1}^k \mathcal{H}_i, \bigstar \amalg \bigstar)$ be the morphism that projects $\mathcal{H}_j$ onto the first summand, and $\coprod_{i \neq j} \mathcal{H}_i$ onto the second summand. Then, for any graph $\mathcal{G}$ and any $f \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \coprod_{i=1}^k \mathcal{H}_i)$, the diagram

$$\begin{tikzcd}
\mathcal{P}_j \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "f"] \\
\mathcal{H}_j \arrow[r, hook, "inc_j"'] \arrow[d] & \coprod_{i=1}^k \mathcal{H}_i \arrow[d, "p_j"] \\
\bigstar \arrow[r, hook, "inc_1"'] & \bigstar \amalg \bigstar
\end{tikzcd} \tag{3.36}$$

where the top square is a pullback, commutes in $\mathsf{psh_f}(\mathtt{D})$. Since the lower square is a pullback by construction, so is the outer rectangle.

In particular, if $\mathcal{G}$ is connected, then $\mathcal{P}_j$ is either empty or isomorphic to $\mathcal{G}$ itself. But this implies that there is some unique $1 \leq j \leq k$ such that f factors through the inclusion $inc_j \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{H}_j, \coprod_{i=1}^k \mathcal{H}_k)$. In other words, $\mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \coprod_{i=1}^k \mathcal{H}_i) \cong \coprod_{i=1}^k \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{H}_i)$.

(3) $\Rightarrow$ (2): If $\mathcal{G}$ satisfies condition (3.35), then $\mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{W} \amalg \mathcal{W}) \cong \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{W}) \amalg \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}, \mathcal{W})$. So, taking $\coprod_{i=1}^k \mathcal{H}_k = \mathcal{W} \amalg \mathcal{W}$ in (3.36), we have $\mathcal{P}_j = \oslash$ or $\mathcal{P}_j \cong \mathcal{G}$ for $j = 1, 2$. $\square$

3.3. **Paths and cycles.** Paths and cycles in a graph $\mathcal{G}$ may be defined using line and wheel graphs (Examples 3.30 and 3.31).

Definition 3.37. *For any graph $\mathcal{G}$, a morphism $p \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{L}^k, \mathcal{G})$ is called a* path of length k *in $\mathcal{G}$. Given any pair $x_1, x_2 \in E \amalg V$, x_1 and x_2 are* connected by a path $p \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{L}^k, \mathcal{G})$ *if $\{x_1, x_2\} \subset im(p)$.*

A non-empty graph $\mathcal{G}$ is path connected *if each pair of distinct elements $x_1, x_2 \in E \amalg V$ is connected by a path in $\mathcal{G}$.*

Example 3.38. The isolated vertex $\mathcal{C}_\mathbf{0}$ is trivially path connected.

Since $\mathsf{Grpsh_f}(\mathtt{D})(\mathcal{L}^k, \mathcal{C}_\mathbf{1})$ is non-empty only when $k = 0$ or $k = 1$, only the unique path $\mathcal{L}^1 = \mathcal{C}_\mathbf{2} \to \mathcal{C}_\mathbf{1}$ connects the unique vertex v of $\mathcal{C}_\mathbf{1}$ with either edge $e \in E(\mathcal{C}_\mathbf{1})$.

Corollary 3.39 (Corollary to Proposition 3.34)**.** *A graph $\mathcal{G}$ is connected if and only if it is path connected.*

Proof. A morphism $f \colon \mathcal{G} \to \mathcal{W}_1 \amalg \mathcal{W}_2$ that does not factor through an inclusion $\mathcal{W} \hookrightarrow \mathcal{W}_1 \amalg \mathcal{W}_2$ exists if and only if there are distinct $x_1, x_2 \in E \amalg V$ such that $f(x_1) \in \mathcal{W}_1$ and $f(x_2) \in \mathcal{W}_2$. By Proposition 3.34, since $\mathcal{L}^k$ is connected for all k, this is the case if and only if there is no $p \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{L}^k, \mathcal{G})$ connecting x_1 and x_2. □

Lemma 3.40. *If e_1 and e_2 are distinct edges of a connected graph $\mathcal{G}$, then there is a locally injective path connecting e_1 and e_2 in $\mathcal{G}$.*

Proof. For all edges e of $\mathcal{G}$, $ch_e \colon (\shortmid_{\tilde{e}}) = \mathcal{L}^0 \to \mathcal{G}$ describes an injective path connecting e and τe.

So, let e_1 and $e_2 \neq \tau e_1$ be distinct edges of a connected graph $\mathcal{G}$. By Corollary 3.39, there is a path $p \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{L}^k, \mathcal{G})$ connecting e_1 and e_2 in $\mathcal{G}$. Moreover, we may assume, without loss of generality, that, for $i = 1, 2$, $p(i_{\mathcal{L}^k}) \in \{e_i, \tau e_i\}$: if not, we may replace p with a path $p \circ \iota$ – where $\iota \colon \mathcal{L}^{k'} \to \mathcal{L}^k$ $(1 \leq k' < k)$ is injective – for which this holds.

If p is not locally injective then there is some $1 \leq j \leq k$, such that $p(l_{2j-1}) = p(l_{2j}) \in E(\mathcal{G})$.

In this case, if $j = 1$, then p may be replaced by a path $p_{\hat{1}} \colon \mathcal{L}^{k-1} \to \mathcal{G}$ obtained by precomposing p with the étale inclusion $\mathcal{L}^{k-1} \hookrightarrow \mathcal{L}^k$, $l'_i \mapsto l_{i+2}$, $0 \leq i \leq 2k-1$.

For $1 < j < k$ (so $k > 2$), $p(l_{2j-1}) = p(l_{2j})$ implies that $p(v_{j-1}) = p(v_{j+1})$. Therefore, p may be replaced with a path $p_{\hat{j}} \colon \mathcal{L}^{k-2}$ of length $k-2$ given by

$$p_{\hat{j}}(l'_i) = \begin{cases} p(l_i) & \text{for } 0 \leq i \leq 2j-3, \\ p(l_{i+4}) & \text{for } 2j-2 \leq i \leq 2k-3. \end{cases}$$

Finally, if $j = k$, then replace p with the path $p_{\hat{k}} \colon \mathcal{L}^{k-1} \to \mathcal{G}$ obtained by precomposing p with the inclusion $\mathcal{L}^{k-1} \hookrightarrow \mathcal{L}^k$, $l'_i \mapsto l_i$, $1 \leq i \leq k-1$.

By iterating this process (always starting with the lowest value of j for which the path p is not injective at v_j), we obtain a unique, locally injective path p_I connecting e_1 and e_2. □

Morphisms from wheel graphs $\mathcal{W}^m$ describe the higher genus structure of graphs (see Remark 3.43).

Definition 3.41. *A* cycle *in $\mathcal{G}$ is a morphism $c \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{W}^m, \mathcal{G})$ for some $m \geq 1$.*

A connected graph $\mathcal{G}$ is simply connected *if it has no locally injective cycles.*

A cycle $c \colon \mathcal{W}^m \to \mathcal{G}$ is trivial *if there is a simply connected graph $\mathcal{H}$ such that c factors through $\mathcal{H}$.*

Using the cyclic ordering on the edges of each $\mathcal{W}^m$, it is straightforward to verify that a graph $\mathcal{G}$ is simply connected if and only if its geometric realisation $|\mathcal{G}|$ is.

Example 3.42. For all finite sets X, the corolla $\mathcal{C}_X$ is trivially simply connected since it has no inner edges and therefore does not admit any cycles.

Since the edge sets of the line graphs $\mathcal{L}^k$ are totally ordered for all k, there can be no locally injective morphism $\mathcal{W}^m \to \mathcal{L}^k$. Hence $\mathcal{L}^k$ is simply connected. However, for all $k \geq 1$, there are morphisms $\mathsf{Grpsh_f(D)}(\mathcal{W}^{2k}, \mathcal{L}^{k+1})$ that are surjective on vertices and inner edges. Namely, let $V(\mathcal{W}^{2k}) = (w_i)_{i=1}^{2k}$ and $V(\mathcal{L}^{k+1}) = (v_j)_{j=1}^{k+1}$ be canonically ordered as in Examples 3.30, and 3.31. Then the assignment $w_i \mapsto v_i$ for $1 \leq i \leq 2k+1$, and $w_{k+1+j} \mapsto v_{k+1-j}$ for $1 \leq j < k$ induces a morphism $q\colon \mathcal{W}^{2k} \to \mathcal{L}^{2k+1}$ that *flattens* $\mathcal{W}^{2k}$ (Figure 11(a)). This fails to be a local injection at w_1 and w_k.

More generally, by flattening $\mathcal{W}^2$ as above, we see that, for any graph $\mathcal{G}$, the set $E_\bullet$ of inner edges of $\mathcal{G}$ is non-empty if and only if $\mathsf{Grpsh_f(D)}(\mathcal{W}^2, \mathcal{G})$ is (see Figure 11(b)).

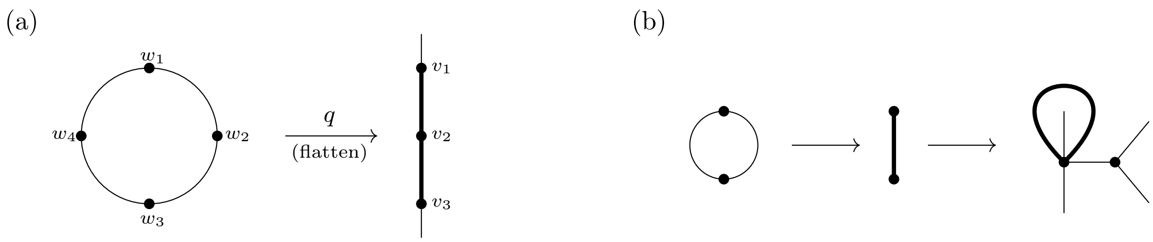


FIGURE 11. (a) A morphism $q\colon \mathcal{W}^4 \to \mathcal{L}^3$ in $\mathsf{Grpsh_f(D)}$ that flattens $\mathcal{W}^4$. (b) If a graph $\mathcal{G}$ has an inner edge, then $\mathsf{Grpsh_f(D)}(\mathcal{W}^2, \mathcal{G})$ is non-empty.

Remark 3.43. For any graph $\mathcal{G}$, we may define an equivalence relation of *path homotopy* on paths in $\mathcal{G}$. Two paths in $\mathcal{G}$ are *homotopic* if applying the proof of Lemma 3.40 to each, leads to the same locally injective path p_I in $\mathcal{G}$. When $E_\bullet \neq \emptyset$, this relation extends to an equivalence relation on cycles in $\mathcal{G}$. If $\mathcal{G}$ is also connected, the set of equivalence classes of cycles has a canonical group structure that is isomorphic to the fundamental group $\pi_i(|\mathcal{G}|)$ of the geometric realisation of $\mathcal{G}$.

The fundamental group construction can be extended, using Proposition 7.17, to all graphs $\mathcal{G}$ without isolated vertices. These ideas are not developed in the current work.

4. The étale site of graphs

By Example 3.20, there is a canonical embedding $\mathbf{B}^\S \hookrightarrow \mathsf{Grpsh_f(D)}$ whose image consists of the exceptional graph $(\shortmid)$, the corollas $\mathcal{C}_X$, and the étale (locally bijective) morphisms between them.

Local bijections are preserved under composition so étale morphisms form a subcategory etGr of $\mathsf{Grpsh_f(D)}$. The full subcategory of etGr on the connected graphs is denoted by CetGr.

The goal of this section is to describe the categories CetGr and etGr in detail, and establish the chain

$$\mathbf{B}^\S \hookrightarrow \mathsf{CetGr} \hookrightarrow \mathsf{GS}$$

of dense fully faithful embeddings discussed in Section 2.

The following is immediate from Definition 3.17 and the universal property of pullbacks of sets:

Proposition 4.1. *A morphism $f \in \mathsf{Grpsh_f(D)}(\mathcal{G}, \mathcal{G}')$ is* étale *if and only if the right square in the defining diagram (3.13) is a pullback of finite sets.*

Example 4.2. For any graph $\mathcal{G}$ and all $v \in V(\mathcal{G})$, the essential morphism $\iota_v\colon \mathcal{C}_{\mathrm{v}} \to \mathcal{G}$ defined in Example 3.29 is étale. Indeed, a morphism $f \in \mathsf{Grpsh_f(D)}(\mathcal{G}, \mathcal{G}')$ is étale if and only if f induces an isomorphism $\mathcal{C}_{\mathrm{v}} \xrightarrow{\cong} \mathcal{C}_{\mathbf{f(v)}}$ for all $v \in V$.

Example 4.3. As discussed in Example 3.42, there are no étale morphism $\mathcal{W}^m \to \mathcal{L}^k$, for any $k \geq 0, m \geq 1$.

All étale morphisms between line graphs are pointwise injective: for $k, n \in \mathbb{N}$,

$$\mathsf{etGr}(\mathcal{L}^k, \mathcal{L}^n) \cong \begin{cases} 2(\mathbf{n-k+1}) & n \geq k \\ \emptyset, & n < k. \end{cases}$$

For $m \geq 1$, a morphism $f \in \mathsf{etGr}(\mathcal{L}^k, \mathcal{W}^m)$ is pointwise injective precisely when $k < m$. For all $k \geq 0$, f is fixed by $f(1_{\mathcal{L}^k}) \in E(\mathcal{W}^m)$. Hence, $\mathsf{etGr}(\mathcal{L}^k, \mathcal{W}^m) \cong E(\mathcal{W}^m) \cong 2(\mathbf{m})$.

Étale morphisms between wheel graphs are surjective: For $l, m \geq 1$

$$\mathsf{etGr}(\mathcal{W}^l, \mathcal{W}^m) \cong \begin{cases} 2(\mathbf{m}) & \text{if } \frac{l}{m} \in \mathbb{N} \\ \emptyset, & \text{otherwise.} \end{cases}$$

If m divides l, a morphism in $\mathsf{etGr}(\mathcal{W}^l, \mathcal{W}^m)$ is fixed by $w_1 = f(v_1) \in V(\mathcal{W}^m)$, and a choice $f(a_1) \in {}^{E}/_{w_1}$ of 'winding direction'.

4.1. Pullbacks and monomorphisms in $\mathsf{Grpsh_f}(\mathtt{D})$.

By Proposition 4.1, étale maps of graphs are local homeomorphisms and, as such, have similar properties to open maps of topological spaces.

Lemma 4.4. *The graph categories* $\mathsf{Grpsh_f}(\mathtt{D})$ *and* etGr *admit pullbacks. Moreover, étale morphisms are preserved under pullbacks in* $\mathsf{Grpsh_f}(\mathtt{D})$.

Proof. Recall that graphs are objects of the presheaf category $\mathsf{psh_f}(\mathtt{D})$ of graph-like diagrams in $\mathsf{Set_f}$. Let $\mathcal{P} = (\underline{E}, \underline{H}, \underline{V}, \underline{s}, \underline{t}, \underline{\tau})$ be the pullback in $\mathsf{psh_f}(\mathtt{D})$ of morphisms $f_1 \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}_1, \mathcal{G})$ and $f_2 \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{G}_2, \mathcal{G})$. Since pullbacks in $\mathsf{psh_f}(\mathtt{D})$ are computed pointwise, $\underline{\tau}$ is a fixed-point free involution, and $\underline{s}$ is injective. So, $\mathcal{P}$ is a graph, and $\mathsf{Grpsh_f}(\mathtt{D})$ admits pullbacks.

Étale morphisms pull back to étale morphisms since limits commute with limits, and therefore, by symmetry, etGr admits pullbacks. □

Definition 4.5. *For any morphism* $f \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{H}, \mathcal{G})$, *not necessarily étale, and any morphism* $w \colon \mathcal{G}' \to \mathcal{G}$, *the* preimage $f^{-1}(\mathcal{G}') \to \mathcal{H}$ of $\mathcal{G}'$ under f *is defined by the pullback*

$$\begin{tikzcd} f^{-1}(\mathcal{G}') \arrow[r] \arrow[d] & \mathcal{H} \arrow[d, "f"] \\ \mathcal{G}' \arrow[r, "w"'] & \mathcal{G}. \end{tikzcd}$$

In particular, by Lemma 4.4, if $w \colon \mathcal{G}' \to \mathcal{G}$ is étale, then so is the preimage $f^{-1}(\mathcal{G}') \to \mathcal{H}$.

Epimorphisms in $\mathsf{Grpsh_f}(\mathtt{D})$ are pointwise surjections. However, the following example shows that monomorphisms in $\mathsf{Grpsh_f}(\mathtt{D})$, and in etGr, are not necessarily pointwise injective. In particular, étale morphisms do not admit a unique epi- mono factorisation.

Example 4.6. Let $\mathcal{W}$ be the wheel graph with vertex $v \in V(\mathcal{W})$. The essential morphism $\iota_v \colon \mathcal{C}_{\mathsf{v}} \to \mathcal{W}$ is pointwise surjective, and easily shown to be a monomorphism. In fact, for all $k \geq 1$, the canonical morphism $\mathcal{L}^k \to \mathcal{W}^k$ (Example 3.31) is both an epimorphism and a monomorphism.

To characterise monomorphisms in $\mathsf{Grpsh_f}(\mathtt{D})$, observe that any (possibly empty) graph $\mathcal{H}$ has the form $\mathcal{H}' \amalg \mathcal{S}$, where $\mathcal{H}'$ is a graph without stick components and $\mathcal{S}$ is a shrub (Example 3.28).

Lemma 4.7. *If* $f \in \mathsf{Grpsh_f}(\mathtt{D})(\mathcal{H}, \mathcal{G})$ *is a monomorphism, then*

(i) the images $f(\mathcal{H}')$ *and* $f(\mathcal{S})$ *are disjoint in* $\mathcal{G}$*;*
(ii) the restriction of f *to* $\mathcal{S}$ *is injective;*

(iii) f is injective on $V(\mathcal{H})$ and $H(\mathcal{H})$ (but not necessarily on $E(\mathcal{H})$).

Hence, any monomorphism $f\colon \mathcal{H}\to\mathcal{G}$ is either pointwise injective or, there exists a pair of ports $e_1, e_2 \in E_0(\mathcal{H})$ such that

- *$\tau_{\mathcal{H}} e_1, \tau_{\mathcal{H}} e_2 \in s(H)$, and hence $e_2 \neq \tau_{\mathcal{H}} e_1$ (where $\tau_{\mathcal{H}}$ is the involution on $E(\mathcal{H})$),*
- *$\tau_{\mathcal{G}} f(e_2) = f(e_1) \in E_\bullet(\mathcal{G})$ so $\{f(e_1), f(e_2)\}$ forms a $\tau_{\mathcal{G}}$-orbit of inner edges of $\mathcal{G}$.*

If $e_1, e_2 \in E_0(\mathcal{H})$ and $\tau_{\mathcal{G}} f(e_2) = f(e_1) \in E_\bullet(\mathcal{G})$, then we say that f *glues e_1 and e_2 in $\mathcal{G}$.*

Proof. Pairs of edges $e' \in E(\mathcal{H}')$ and $l \in E(\mathcal{S})$ of $\mathcal{H}$ such that $f(e') = f(l) \in E(\mathcal{G})$ correspond to pairs of parallel morphisms $ch_{e'}, ch_l \colon (\shortmid) \rightrightarrows \mathcal{H}$ such that $f \circ ch_l = f \circ ch_{e'}$. So monomorphisms in $\mathsf{Grpsh_f(D)}$ satisfy (i). Similar arguments show that monomorphisms also satisfy (ii) and (iii). The last statement follows immediately from conditions (i)-(iii). $\square$

The converse to Lemma 4.7 also holds:

Proposition 4.8. *Let $f \in \mathsf{Grpsh_f(D)}(\mathcal{H}, \mathcal{G})$ be a morphism. The following are equivalent:*

(1) f is a monomorphism in $\mathsf{Grpsh_f(D)}$;
(2) f satisfies the conditions of Lemma 4.7;
(3) if $\mathcal{G}'$ has no inner edges, and $w\colon \mathcal{G}'\to\mathcal{G}$ satisfies the conditions of Lemma 4.7, then the induced morphism $f^{-1}(\mathcal{G}') \to \mathcal{G}'$ is pointwise injective.

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 4.7.

For (2) $\Rightarrow$ (3), let $w\colon \mathcal{G}'\to\mathcal{G}$ and $f\colon \mathcal{H} \to \mathcal{G}$ satisfy the conditions of Lemma 4.7. Then $f^{-1}(\mathcal{G}')$ has no inner edges and the defining morphism $f^{-1}(\mathcal{G}') \to \mathcal{G}'$ is pointwise injective by the last statement of Lemma 4.7.

For (3) $\Rightarrow$ (1), assume that (3) holds and let $f \in \mathsf{Grpsh_f(D)}(\mathcal{H}, \mathcal{G})$, and $w\colon \mathcal{G}'\to\mathcal{G}$ satisfy (3). So $f^{-1}(\mathcal{G}') \to \mathcal{G}'$ is a pointwise injection. If $g, h\colon \mathcal{H}' \rightrightarrows \mathcal{H}$ are parallel morphisms such that $f \circ g = f \circ h$, then, since $f^{-1}(\mathcal{G}')$ has no inner edges, neither does $(f \circ g)^{-1}(\mathcal{G}') = (f \circ h)^{-1}(\mathcal{G}')$ and the induced maps to $f^{-1}(\mathcal{G}')$ agree. By taking $w\colon \mathcal{G}'\to\mathcal{G}$ of the form $\mathcal{C}_{\mathbf{0}} \xrightarrow{v} \mathcal{G}$ for all $v \in V$, or $\iota_{\tilde{e}}\colon (\shortmid_{\tilde{e}})\to\mathcal{G}$ for $\tilde{e} \in \widetilde{E}$, we see immediately that $g = h\colon \mathcal{H}' \to \mathcal{H}$. Hence f is a monomorphism in $\mathsf{Grpsh_f(D)}$. $\square$

Example 4.9. For all finite sets X and Y, the canonical étale morphisms $\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}} \to \mathcal{M}^{X,Y}_{x_0,y_0}$ and $\mathcal{C}_{X \amalg \{x_0, y_0\}} \to \mathcal{N}^{X}_{x_0,y_0}$ are surjective and not pointwise injective. But they are monomorphisms.

4.2. **Graph neighbourhoods and the essential category $\mathsf{es}(\mathcal{G})$.** A family of morphisms $\mathfrak{U} = \{f_i \in \mathsf{etGr}(\mathcal{G}_i, \mathcal{G})\}_{i \in I}$ is *jointly surjective* on $\mathcal{G}$ if $\mathcal{G} = \bigcup_{i \in I} im(f_i)$. By Lemma 4.4, etGr admits pullbacks, and jointly surjective families of étale morphisms $\{f_i \in \mathsf{etGr}(\mathcal{G}_i, \mathcal{G})\}_{i \in I}$ define the covers at $\mathcal{G}$ for a canonical *étale topology* J on etGr. Sheaves for this topology are those presheaves $P\colon \mathsf{etGr}^{\mathrm{op}} \to \mathsf{Set}$ such that, for all graphs $\mathcal{G}$, and all covers $\mathfrak{U} = \{f_i \in \mathsf{etGr}(\mathcal{G}_i, \mathcal{G})\}_{i \in I}$ at $\mathcal{G}$,

$$P(\mathcal{G}) \cong \lim_{f_i \in \mathfrak{U}} P(\mathcal{G}_i). \tag{4.10}$$

As will be shown in Proposition 4.26, the category $\mathsf{sh}(\mathsf{etGr}, J)$ of sheaves for the étale site (etGr, J) is canonically equivalent to the category GS of graphical species (Definition 1.3).

As motivation for this result, we will first establish more properties of étale morphisms. As usual, the analogy between étale morphisms of graphs and open maps of topological spaces provides some intuition.

Definition 4.11. *A* neighbourhood *of a monomorphism $w\colon \mathcal{G}'\to\mathcal{G}$ is an étale monomorphism $u\colon \mathcal{U}\to\mathcal{G}$ such that $w = u \circ \tilde{w}\colon \mathcal{G}'\to\mathcal{U}\to\mathcal{G}$, for some monomorphism $\tilde{w}\colon \mathcal{G}'\to\mathcal{U}$.*

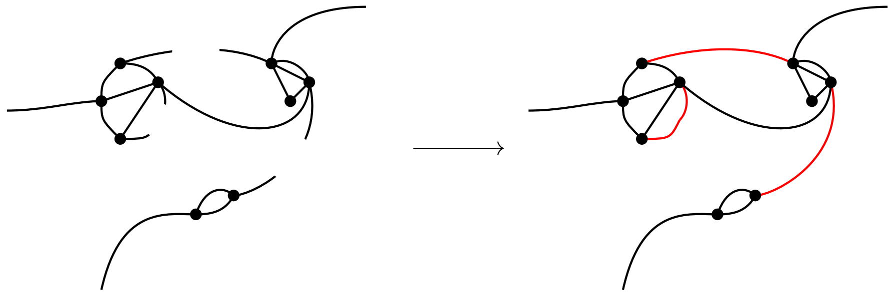


FIGURE 12. Graphs (left) $\mathcal{G}_{\widehat{\mathcal{I}}}$ and, (right) $\mathcal{G}$, with subgraph $\mathcal{I}$ indicated in red.

A neighbourhood $(\mathcal{U}, u)$ of $w\colon \mathcal{G}' \to \mathcal{G}$ is minimal *if every other neighbourhood $(\mathcal{U}', u')$ of $w\colon \mathcal{G}' \to \mathcal{G}$ is also a neighbourhood of $(\mathcal{U}, u)$.*

Since vertices v of $\mathcal{G}$ correspond to subgraphs $v\colon \mathcal{C}_{\mathbf{0}} \to \mathcal{G}$, and edges e of $\mathcal{G}$ are in bijection with subgraphs $ch_e\colon (\shortmid) \hookrightarrow \mathcal{G}$, we may also refer to neighbourhoods of vertices and edges. Moreover, since $u\colon \mathcal{U} \to \mathcal{G}$ is a neighbourhood of $e \in E$ if and only if it is a neighbourhood of $\iota_{\tilde{e}}\colon (\shortmid_{\tilde{e}}) \to \mathcal{G}$, there is no loss of generality in referring to neighbourhoods of τ-orbits $\tilde{e} \in \widetilde{E}$.

Let $\mathcal{S}(E_\bullet) = \coprod_{\tilde{e} \in \widetilde{E_\bullet}} (\shortmid_{\tilde{e}})$ be the shrub on the inner edges of a graph $\mathcal{G}$. Given any subgraph $\mathcal{I} \hookrightarrow \mathcal{S}(E_\bullet)$, there is a graph $\mathcal{G}_{\widehat{\mathcal{I}}}$ and a canonical surjective monomorphism $i_{\widehat{\mathcal{I}}}\colon \mathcal{G}_{\widehat{\mathcal{I}}} \to \mathcal{G}$ (see Figure 12):
(4.12)

$$\begin{tikzcd}
\mathcal{G}_{\widehat{\mathcal{I}}} \arrow[d, "i_{\widehat{\mathcal{I}}}"'] & E \amalg (E(\mathcal{I}))^\dagger \arrow[r, leftrightarrow, "\tau_{\widehat{\mathcal{I}}}"] \arrow[d, two heads] & E \amalg (E(\mathcal{I}))^\dagger \arrow[d, two heads] & E \arrow[l, hook'] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, equal] & V \arrow[d, equal] \\
\mathcal{G} & E \arrow[r, leftrightarrow, "\tau"'] & E & & H \arrow[ll, "s"] \arrow[r, "t"'] & V,
\end{tikzcd}$$

where

- $(E(\mathcal{I}))^\dagger$ is the formal involution $e \mapsto e^\dagger$ of the set $E(\mathcal{I})$ of edges of $\mathcal{I}$,
- the involution $\tau_{\widehat{\mathcal{I}}}$ on $E \amalg (E(\mathcal{I}))^\dagger$ is defined by

$$e \mapsto \begin{cases} \tau e, & e \in E \setminus E(\mathcal{I}) \\ e^\dagger, & e \in E(\mathcal{I}), \end{cases}$$

- the surjection $E \amalg (E(\mathcal{I}))^\dagger \twoheadrightarrow E$ is the identity on E and $e^\dagger \mapsto \tau e$, $e \in E(\mathcal{I})$.

So, $\mathcal{G}_{\widehat{\mathcal{I}}}$ has inner edges $E_\bullet(\mathcal{G}_{\widehat{\mathcal{I}}}) = E_\bullet \setminus E(\mathcal{I})$, and boundary $E_0(\mathcal{G}_{\widehat{\mathcal{I}}}) = E_0 \amalg (E(\mathcal{I}))^\dagger$.

Informally, $\mathcal{G}_{\widehat{\mathcal{I}}}$ is the graph obtained from $\mathcal{G}$ by 'breaking the edges' of $\mathcal{I}$, as in Figure 12.

For $\tilde{e} \in \widetilde{E}(\mathcal{I})$, the essential morphism $\iota_{\tilde{e}}\colon (\shortmid_{\tilde{e}}) \hookrightarrow \mathcal{G}$ (Example 3.28) factors in two ways through $\mathcal{G}_{\widehat{\mathcal{I}}}$:

$$(\shortmid_{\tilde{e}}) \underset{(e,\tau e)\mapsto((\tau e)^\dagger, \tau e)}{\overset{(e,\tau e)\mapsto(e,e^\dagger)}{\rightrightarrows}} \mathcal{G}_{\widehat{\mathcal{I}}} \xrightarrow{i_{\widehat{\mathcal{I}}}} \mathcal{G}. \tag{4.13}$$

Hence there exist parallel morphisms $\mathcal{I} \rightrightarrows \mathcal{G}_{\widehat{\mathcal{I}}}$, and a coequaliser diagram in $\mathsf{Grpsh_f(D)}$:

$$\mathcal{I} \rightrightarrows \mathcal{G}_{\widehat{\mathcal{I}}} \xrightarrow{i_{\widehat{\mathcal{I}}}} \mathcal{G}. \tag{4.14}$$

(The choice of morphisms $\mathcal{I} \rightrightarrows \mathcal{G}_{\widehat{\mathcal{I}}}$ in (4.14) is not unique – there are $2^{|\widetilde{E}(\mathcal{I})|}$ pairs – but it is unique up to isomorphism.)

For each $\mathcal{I} \subset \mathcal{S}(E_\bullet)$, there is étale cover at $\mathcal{G}$ defined by the set of components of $\mathcal{I} \amalg \mathcal{G}_{\widehat{\mathcal{I}}}$, together with the canonical monomorphisms to $\mathcal{G}$.

The collection $\{(\mathcal{G}_{\widehat{\mathcal{I}}}, i_{\widehat{\mathcal{I}}})\}_{\mathcal{I} \subset \mathcal{S}(E_\bullet)} \subset \mathsf{etGr}/\mathcal{G}$ inherits a poset structure from the poset of subgraphs of $\mathcal{S}(E_\bullet)$, and the graph $\mathcal{G}_{\widehat{\mathcal{S}(E_\bullet)}}$ with no inner edges is initial in this poset. Moreover, any surjective monomorphism $\mathcal{G}' \to \mathcal{G}$ factors as $\mathcal{G}' \xrightarrow{\cong} \mathcal{G}_{\widehat{\mathcal{I}}} \xrightarrow{i_{\widehat{\mathcal{I}}}} \mathcal{G}$ for some unique $\mathcal{I} \subset \mathcal{S}(E_\bullet)$. Hence, we have proved the following:

Lemma 4.15. *A neighbourhood $(\mathcal{U}, u)$ of a monomorphism $w \in \mathsf{Grpsh_f(D)}(\mathcal{G}', \mathcal{G})$ is minimal if and only if $E_\bullet(\mathcal{U}) = E_\bullet(\mathcal{G}')$ and the monomorphism $\mathcal{G}' \to \mathcal{U}$ induces a surjection on connected components.*

For all edges $e \in E$, and all vertices $v \in V$, the essential morphisms $\iota_{\tilde{e}} \colon (\shortmid_{\tilde{e}}) \to \mathcal{G}$, $\tilde{e} \in \widetilde{E}$, and $\iota_v \colon \mathcal{C}_{\mathbf{v}} \to \mathcal{G}$, $v \in V$ describe minimal neighbourhoods of e and v respectively.

When $\mathcal{G}$ has no stick components, one readily checks that $\mathcal{G}_{\widehat{\mathcal{S}(E_\bullet)}} = \coprod_{v \in V} \mathcal{C}_{\mathbf{v}}$. In particular, for any graph $\mathcal{G}$, there is a canonical choice of *essential cover* $\mathfrak{Es}_\mathcal{G}$ at $\mathcal{G}$ by the essential morphisms $\iota_{\tilde{e}}$ ($\tilde{e} \in \widetilde{E}$) and ι_v ($v \in V$).

Definition 4.16. *Let $\mathcal{G}$ be a graph. The* essential category $\mathsf{es}(\mathcal{G})$ of $\mathcal{G}$ *is the full subcategory of* $\mathsf{etGr}/\mathcal{G}$ *on the essential monomorphisms $\iota_{\tilde{e}} \colon (\shortmid_{\tilde{e}}) \to \mathcal{G}$, $\tilde{e} \in \widetilde{E}$, and $\iota_v \colon \mathcal{C}_{\mathbf{v}} \to \mathcal{G}$, $v \in V$.*

Since $\mathsf{es}(\mathcal{G})$ has no non-trivial isomorphisms by definition, there is a canonical bijection $h = (e, v) \leftrightarrow (\delta_h \colon \iota_{\tilde{e}} \to \iota_v)$ between half-edges h of $\mathcal{G}$, and non-identity morphisms δ in $\mathsf{es}(\mathcal{G})$.

Lemma 4.17. *Each graph $\mathcal{G}$ is canonically the colimit of the forgetful functor* $\mathsf{es}(\mathcal{G}) \to \mathsf{etGr}$.

A presheaf $P \in \mathsf{psh}(\mathsf{etGr})$ is a sheaf for the étale site (etGr, J) if and only if for all $\mathcal{G}$,

$$P(\mathcal{G}) \cong \lim_{(\mathcal{C}, b) \in \mathsf{es}(\mathcal{G})} P(\mathcal{C}). \tag{4.18}$$

Proof. If $e \in E_0$ is a port of $\mathcal{G}$, then there is, at most, one non-trivial morphism $\delta_h = \delta_{(\tau e, v)}$ with domain $\shortmid_{\tilde{e}}$ in $\mathsf{es}(\mathcal{G})$. In this case, $\mathcal{C}_{\mathbf{v}}$ is the colimit of the diagram $\shortmid_{\tilde{e}} \xrightarrow{\delta_h} \mathcal{C}_{\mathbf{v}}$. The first statement then follows from (4.13) and (4.14). The second statement is immediate since, by Lemma 4.15, the essential cover $\mathfrak{Es}_\mathcal{G}$ refines every étale cover $\mathfrak{U}$ of $\mathcal{G}$. $\square$

4.3. **Boundary-preserving étale morphisms.** In general, morphisms $f \in \mathsf{etGr}(\mathcal{G}', \mathcal{G})$ do not satisfy $f(E_0') \subset E_0$. Those that do are componentwise surjective *graphical covering morphisms* in the sense of Proposition 4.19 below. In particular, monomorphisms $f \in \mathsf{etGr}(\mathcal{G}', \mathcal{G})$ such that $f(E_0') = E_0$ are componentwise isomorphisms.

Proposition 4.19. *For any étale morphism $f \in \mathsf{etGr}(\mathcal{G}', \mathcal{G})$, $f(E_0') \subset E_0$ if and only if there exists an étale cover $\mathfrak{U} = \{\mathcal{U}_i, u_i\}_{i \in I}$ of $\mathcal{G}$, such that, for all i, $f^{-1}(\mathcal{U}_i)$ is isomorphic to a disjoint union of $k(\mathcal{U}_i, f) \in \mathbb{N}$ copies of $\mathcal{U}_i$.*

In this case, $k(\mathcal{U}_i, f) = k_f \in \mathbb{N}$ is constant on connected components of $\mathcal{G}$.

Proof. If $u \colon \mathcal{U} \to \mathcal{G}$ is an étale monomorphism for which there is a $k \in \mathbb{N}$ such that $f^{-1}(\mathcal{U}) \cong k(\mathcal{U})$, then also $f^{-1}(\mathcal{V}) \cong k(\mathcal{V})$ for all monomorphisms $\mathcal{V} \to \mathcal{U}$. So, we may assume, without loss of generality, that $\mathfrak{U} = \mathfrak{Es}_\mathcal{G}$ is the essential cover of $\mathcal{G}$.

Observe first that, if $f(E_0') \not\subset E_0$, there exists a vertex v of $\mathcal{G}$ and a port e' of $\mathcal{G}'$ such that $f(e') \in {}^{E}\!/v$. Hence $(\shortmid_{\tilde{e}'}) \ncong \mathcal{C}_{\mathbf{v}}$ is a connected component of $f^{-1}(\mathcal{C}_{\mathbf{v}})$.

For the converse, let $v \in V$ be a vertex of $\mathcal{G}$. Since f is étale, $\mathcal{C}_{\mathbf{v}} \cong \mathcal{C}_{\mathbf{v}'}$ for all $v' \in V'$ such that $f(v') = v$. By the universal property of pullbacks, the canonical monomorphism $\coprod_{v' \colon f(v') = v} \mathcal{C}_{\mathbf{v}} \to \mathcal{G}'$

factors through $f^{-1}(\mathcal{C}_{\mathbf{v}}) \to \mathcal{G}'$, and therefore

$$f^{-1}(\mathcal{C}_{\mathbf{v}}) \cong \left(\coprod_{v' \colon f(v') = v} \mathcal{C}_{\mathbf{v}'} \right) \amalg \mathcal{S}, \quad \text{for some shrub } \mathcal{S}.$$

Then, by construction, a connected component of $\mathcal{S}$ must be of the form $\iota_{\tilde{e}'} \colon (\shortmid_{\tilde{e}'}) \to \mathcal{G}'$ for some port e' of $\mathcal{G}'$ satisfying $f(e') = e \in {}^{E}\!/v$. But $f(E_0') \subset E_0$ by assumption, so there is no such port. Hence $\mathcal{S} = \oslash$ is the empty graph, and $f^{-1}(\mathcal{C}_{\mathbf{v}}) \cong \coprod_{v' \colon f(v') = v} \mathcal{C}_{\mathbf{v}}$ so $k(\mathcal{C}_{\mathbf{v}}, f) = |f^{-1}(v)| \in \mathbb{N}$.

It is immediate that $f^{-1}(\shortmid_{\tilde{e}}) \cong \coprod_{e' \in E', f(e') = e} (\shortmid_{\tilde{e}'})$ for all $e \in E$, so $k(\shortmid_{\tilde{e}}, f) = |f^{-1}(e)| \in \mathbb{N}$, and the first statement of the proposition is proved.

By condition (3) of Proposition 3.34, it is sufficient to verify the second part of the proposition componentwise on $\mathcal{G}$. Therefore we may assume, without loss of generality, that $\mathcal{G}$ is connected.

Let $f \in \mathsf{etGr}(\mathcal{G}', \mathcal{G})$ satisfy $f(E_0') \subset E_0$.

If $\mathcal{G} \cong (\shortmid)$ is a stick, there is nothing to prove. Otherwise, the following diagram commutes for all vertices v, and all half edges $h = (e, v)$ of $\mathcal{G}$:

$$\begin{tikzcd}
\coprod_{e' \in f^{-1}(e)} (\shortmid_{\tilde{e}}) \arrow[r, "\amalg_{h' \in f^{-1}(h)} \delta_{h'}"] \arrow[d] & \coprod_{v' \in f^{-1}(v)} \mathcal{C}_{\mathbf{v}'} \arrow[r, "\amalg_{v'} \iota_{v'}"] \arrow[d] & \mathcal{G}' \arrow[d, "f"] \\
(\shortmid_{\tilde{e}}) \arrow[r, "\delta_h"'] & \mathcal{C}_{\mathbf{v}} \arrow[r, "\iota_v"'] & \mathcal{G}.
\end{tikzcd}$$

The first part of the proof implies that both squares are pullbacks. So, if $f^{-1}(\mathcal{C}_{\mathbf{v}})$ is isomorphic to $k_v = k(\mathcal{C}_{\mathbf{v}}, \iota_v)$ copies of $\mathcal{C}_{\mathbf{v}}$ then $f^{-1}(\shortmid_{\tilde{e}}) \cong k_v (\shortmid_{\tilde{e}})$ for all $e \in {}^{E}\!/v$. Hence $\mathcal{C}_{\mathbf{v}} \mapsto k_v$ extends to a functor $k_{\mathcal{G}}$ from $\mathsf{es}(\mathcal{G})$ to the discrete category $\mathbb{N}$. Since $\mathcal{G}$ is connected, so is $\mathsf{es}(\mathcal{G})$ and therefore $k_{\mathcal{G}}$ is constant. □

Definition 4.20. *A morphism $f \in \mathsf{etGr}(\mathcal{G}', \mathcal{G})$ is called* boundary-preserving *if it restricts to an isomorphism $f_{E_0} \colon E_0 \xrightarrow{\cong} E_0'$.*

The following is an immediate corollary of Proposition 4.19.

Corollary 4.21. *If $\mathcal{G}$ is connected, and $\mathcal{G}'$ is non-empty, then an étale morphism $f \in \mathsf{etGr}(\mathcal{G}', \mathcal{G})$ such that $f(E_0') \subset E_0$ is surjective. If $\mathcal{G}'$ is also connected and its boundary E_0' is non-empty, then f is boundary-preserving if and only if it is an isomorphism.*

Remark 4.22. The condition that E_0' is non-empty is necessary in the statement of Corollary 4.21. For example, for any $m > 1$, each of the two étale morphisms $\mathcal{W}^m \to \mathcal{W}$ (Example 4.3) is trivially boundary-preserving, but certainly not an isomorphism.

Recall from Example 3.30 that the line graph $\mathcal{L}^k$ has totally ordered edge set $E(\mathcal{L}^k) = (l_j)_{j=0}^{2k+1}$ with ports $l_0 = 1_{\mathcal{L}^k}$ and $l_{2k+1} = 2_{\mathcal{L}^k}$. For each vertex $w_i \in V(\mathcal{L}^k)$, ${}^{E}\!/w_i = \{l_{2i-1}, l_{2i}\}$.

Proposition 4.23. *Let $\mathcal{G}$ be a connected graph with only bivalent vertices. Then $\mathcal{G} = \mathcal{L}^k$ or $\mathcal{G} = \mathcal{W}^m$ for some $k \geq 0$ or $m \geq 1$.*

Proof. Since $\mathcal{G}$ is bivalent, every monomorphism $\mathcal{L}^k \rightarrowtail \mathcal{G}$ from a line graph is étale, for all $k \in \mathbb{N}$.

The result holds trivially if $\mathcal{G} \cong \mathcal{L}^0$ is a stick graph.

Otherwise, if $V = V_2$ is non-empty, then, for each $v \in V$, a choice of isomorphism $\mathcal{L}^1 \xrightarrow{\cong} \mathcal{C}_{\mathbf{v}}$ describes a monomorphism $\mathcal{L}^1 \rightarrowtail \mathcal{G}$. Since $\mathcal{G}$ is finite, there is a maximum $M \geq 1$ such that there exists a monomorphism $f \colon \mathcal{L}^M \rightarrowtail \mathcal{G}$.

Let $f \in \mathsf{CetGr}(\mathcal{L}^M, \mathcal{G})$ be such a map. By Proposition 4.8, f is either injective on edges, or $f(2_{\mathcal{L}^M}) = \tau f(1_{\mathcal{L}^M}) \subset E_\bullet$. Let $e_1 = f(1_{\mathcal{L}^M})$, and $e_2 = f(2_{\mathcal{L}^M})$. Assume that e_j is not a port of $\mathcal{G}$ for some $j = 1, 2$. Then $e_j \in {}^{E}/_{v}$ for some vertex $v \in V_2$.

If f is injective on edges then v is not in the image of f. But then, since v is bivalent, this means that f factors through a monomorphism $\mathcal{L}^M \rightarrow \mathcal{L}^{M+1}$, contradicting maximality of M. So $e_j \in E_0$ for $j = 1, 2$. Therefore, $\{e_1, e_2\} \subset E_0$ so f is surjective and boundary-preserving, hence an isomorphism $\mathcal{L}^M \xrightarrow{\cong} \mathcal{G}$ by Corollary 4.21.

Otherwise, if f is not injective on edges, then, by Proposition 4.8, it must be the case that $f(l_0) = f(l_{2M})$. Therefore, f factors through a cycle $f\colon \mathcal{L}^M \to \mathcal{W}^M \to \mathcal{G}$. Since f is an étale monomorphism, the second map $\mathcal{W}^M \to \mathcal{G}$ is étale and monomorphic. Hence, f is boundary-preserving and $\mathcal{G} \cong \mathcal{W}^M$ by Corollary 4.21. ☐

Étale morphisms of simply connected graphs are either subgraph inclusions or isomorphisms. (This is why the combinatorics of cyclic operads are much simpler than those of modular operads.)

Corollary 4.24. *Let $\mathcal{G}$ be simply connected. If $f \in \mathsf{Grpsh}_{\mathsf{f}}(\mathsf{D})(\mathcal{G}', \mathcal{G})$ is locally injective, then f is pointwise injective on connected components of $\mathcal{G}'$. Hence, if $\mathcal{G}'$ is connected, it is simply connected.*

It follows that any étale morphism of simply connected graphs is a pointwise injection,

Proof. We may assume, without loss of generality, that $\mathcal{G}'$ and $\mathcal{G}$ are connected and – since the result holds trivially when either $\mathcal{G}'$, or $\mathcal{G}$ is an isolated vertex – that both graphs have non-trivial edge sets.

Let $f\colon \mathcal{G}' \to \mathcal{G}$ be a local injection. For any locally injective path $p\colon \mathcal{L}^k \to \mathcal{G}'$ in $\mathcal{G}'$, $f \circ p\colon \mathcal{L}^k \to \mathcal{G}$, is a locally injective path in $\mathcal{G}$. If $f \circ p$ is not pointwise injective, then either $f \circ p$ factors through a locally injective cycle in $\mathcal{G}$ – and hence $\mathcal{G}$ is not simply connected – or there are $1 \leq i < j \leq k$ such that $f \circ p(v_i) = f \circ p(v_j) \in V(\mathcal{G})$.

So, let $1 \leq i < j \leq k$ be such that $f \circ p(v_i) = f \circ p(v_j) \in V(\mathcal{G})$. We may assume that either $j = i + 1$ or $f \circ p(v_{i+1}) \neq f \circ p(v_{j-1})$: If not, keep replacing i and j with $i + 1$ and $j - 1$ until this holds. This is possible since otherwise there exists $i \leq m \leq j$ such that $f \circ p(v_{m-1}) = f \circ p(v_{m+1})$ in which case $f \circ p$ is not injective at v_m.

Let $L = j - i$. Then there is a cycle $c\colon \mathcal{W}^L \to \mathcal{G}$ described, on edges of $\mathcal{W}^L$, by $c(a_{2L-1}) = f \circ p(l_{2j-1})$, $c(a_{2L}) = f \circ p(l_{2i})$, and $c(a_{2s}) = f \circ p(l_{2(s+i)})$ and hence $c(a_{2s-1}) = f \circ p(l_{2(s+i)-1})$ for $1 \leq s < L$. In particular, for $1 \leq s < L$, c is injective at the vertex w_s since $f \circ p$ is locally injective. And c is injective at w_L since, by assumption $c(a_{2L}) = f \circ p(l_{2i}) \neq f \circ p(l_{2j-1}) = c(a_{2L-1})$. Therefore c is locally injective.

It follows that any local injection $f\colon \mathcal{G}' \to \mathcal{G}$ from a connected graph $\mathcal{G}'$ to a simply connected graph $\mathcal{G}$ is pointwise injective, and, if such a map exists, then $\mathcal{G}'$ is also simply connected.

The final statement is immediate since étale morphisms are locally injective by definition. ☐

Proposition 4.34 gives analogous results for directed acyclic graphs (Definition 4.33).

4.4. **Étale sheaves on** etGr. Recall that graphical species are presheaves on the category $\mathbf{B}^{\S}$, and that there is a full inclusion $\Phi\colon \mathbf{B}^{\S} \hookrightarrow \mathsf{etGr}$. We prove that Φ induces an equivalence $\mathsf{GS} \simeq \mathsf{sh}(\mathsf{etGr}, J)$ between graphical species and sheaves for the étale topology on CetGr.

Lemma 4.25. *The inclusion $\Phi\colon \mathbf{B}^{\S} \hookrightarrow \mathsf{etGr}$ is dense.*

Proof. It is easy to check that any connected graph without inner edges is isomorphic to $\S$ or $\mathcal{C}_X$ for some finite set X, and therefore, the essential image $im^{es}(\Phi)$ of $\mathbf{B}^{\S}$ in etGr is the full subcategory of connected graphs with no inner edges. Moreover, it follows immediately from the definition of $\mathsf{es}(\mathcal{G})$ (Definition 4.16) that the canonical inclusion $\mathsf{es}(\mathcal{G}) \hookrightarrow im^{es}(\Phi)/\mathcal{G}$ is full and essentially surjective on objects, and hence an equivalence of categories.

Therefore $\mathsf{es}(\mathcal{G}) \simeq \mathbf{B}^{\S}/\mathcal{G}$, and the lemma follows from Lemma 4.17. $\square$

In particular, etGr is a full subcategory of GS under the induced nerve functor $\Upsilon \overset{\text{def}}{=} N_{\mathsf{etGr}} \colon \mathsf{etGr} \to \mathsf{GS}$, and I will write $\mathcal{G}$ rather than $\Upsilon\mathcal{G}$, where there is no risk of confusion. The category $\mathsf{el}(\Upsilon\mathcal{G}) = \mathbf{B}^{\S}/\mathcal{G}$, whose objects are *elements of* $\mathcal{G}$, will be denoted by $\mathsf{el}(\mathcal{G})$.

Proposition 4.26. *There is a canonical equivalence of categories* $\mathsf{sh}(\mathsf{etGr}, J) \simeq \mathsf{GS}$.

Proof. This is straightforward from the definitions and Lemma 4.25. Namely, the inclusion $\Phi \colon \mathbf{B}^{\S} \to \mathsf{etGr}$ induces an *essential geometric morphism* between the presheaf categories $\mathsf{psh}(\mathbf{B}^{\S}) = \mathsf{GS}$ and $\mathsf{psh}(\mathsf{etGr})$. The right adjoint to the pullback $\Phi^* \colon \mathsf{psh}(\mathsf{etGr}) \to \mathsf{GS}$ is given by

$$\Phi_* \colon \mathsf{GS} = \mathsf{psh}(\mathbf{B}^{\S}) \to \mathsf{psh}(\mathsf{CetGr}), \quad S \longmapsto (\mathcal{G} \mapsto \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} S(\mathcal{C})). \tag{4.27}$$

By Lemmas 4.17 and 4.25, a presheaf P on etGr is a sheaf for the canonical étale topology J on etGr if and only if, for all graphs $\mathcal{G}$,

$$P(\mathcal{G}) \cong \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}), \tag{4.28}$$

and therefore Φ_* factors through the inclusion $\mathsf{sh}(\mathsf{etGr}, J) \to \mathsf{psh}(\mathsf{etGr})$.

Moreover, since Φ is fully faithful, so is Φ_* (e.g. by [31, Section VII.2]), whereby Φ_* induces the equivalence $\mathsf{GS} \simeq \mathsf{sh}(\mathsf{etGr}, J)$. $\square$

I will use the same notation to denote a graphical species S and the corresponding sheaf on (etGr, J). So, for any graph $\mathcal{G}$, $S(\mathcal{G}) \overset{\text{def}}{=} \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} S(\mathcal{C})$.

Definition 4.29. *An S-*structured graph $(\mathcal{G}, \alpha)$ *(or simply α) is a graph $\mathcal{G}$ together with an element $\alpha \in S(\mathcal{G})$ (or $\alpha \in \mathsf{GS}(\mathcal{G}, S)$). The category of S-structured graphs will be denoted by* etGr/S.

The following is an immediate consequence of the Proposition 4.26:

Corollary 4.30. *The restriction of the étale topology J to the category* CetGr *of connected graphs and étale morphisms induces an equivalence* $\mathsf{sh}(\mathsf{CetGr}, J) \simeq \mathsf{GS}$.

4.5. **Directed graphs.** By way of example, and to provide extra context, this section ends with a discussion of directed graphs. In particular, we see that, if Di is the terminal directed graphical species defined in Example 1.8 (see also Example 1.10), then etGr/Di defines a category of directed Feynman graphs and étale morphisms.

For any graph $\mathcal{G}$, a Di-structure $\xi \in Di(\mathcal{G})$ is precisely a partition $E = E_{\text{in}} \amalg E_{\text{out}}$, where $e \in E_{\text{in}}$ if and only if $\tau e \in E_{\text{out}}$. So τ induces bijections $E_{\text{in}} \cong \widetilde{E} \cong E_{\text{out}}$, and an object $(\mathcal{G}, \xi)$ of etGr/Di – called an *orientation on* $\mathcal{G}$ – is given by a diagram of finite sets

$$\widetilde{E} \xleftarrow{\widetilde{s_{\text{in}}}} H_{\text{in}} \xrightarrow{t_{\text{in}}} V \xleftarrow{t_{\text{out}}} H_{\text{out}} \xrightarrow{\widetilde{s_{\text{out}}}} \widetilde{E}\ , \tag{4.31}$$

where the maps $\widetilde{s_{\text{in}}}, \widetilde{s_{\text{out}}}$, and $t_{\text{in}}, t_{\text{out}}$ denote the appropriate (quotients of) restrictions of $s \colon H \to E$, respectively $t \colon H \to V$. Then morphisms in etGr/Di are quadruples of finite set maps making the obvious diagrams commute, and such that the outer left and right squares are pullbacks. This is the definition of the category of directed graphs and étale morphisms used in [27, Section 1.5] to prove a nerve theorem for properads in the style of [5].

Example 4.32. The line graphs $\mathcal{L}^k$ with $E(\mathcal{L}^k) = \{l_i\}_{i=0}^{2k+1}$ admit a distinguished choice of orientation $\theta_{\mathcal{L}^k} \in Di(\mathcal{L}^k)$ given by

$$\theta_{\mathcal{L}^k} \colon E(\mathcal{L}^k) \to \{\text{in}, \text{out}\}, \ l_{2i} \mapsto (\text{in}) \text{ and } l_{2i+1} \mapsto (\text{out}) \text{ for } 0 \leq i \leq k.$$

For $m \geq 1$, the canonical morphism $\mathcal{L}^m \to \mathcal{W}^m$ induces an orientation $\theta_{\mathcal{W}^m}$ (with $a_{2j} \mapsto (\text{in})$) on the wheel graph $\mathcal{W}^m$.

Definition 4.33. *A* directed path of length k in $(\mathcal{G}, \xi)$ *is a path* $p \colon \mathcal{L}^k \to \mathcal{G}$ *in* $\mathcal{G}$ *such that, for all* $l \in E(\mathcal{L}^k)$,

$$Di(ch_l)(\theta_{\mathcal{L}^k}) = Di(ch_{p(l)})(\xi) \in \{\text{in}, \text{out}\}.$$

A directed cycle of length m in $(\mathcal{G}, \xi)$ *is a cycle* $c \colon \mathcal{W}^m \to \mathcal{G}$ *in* $\mathcal{G}$ *such that the induced morphism* $\mathcal{L}^m \to \mathcal{W}^m \to \mathcal{G}$ *is a directed path.*

A directed acyclic graph (DAG) *is a directed graph* $(\mathcal{G}, \xi)$ *without directed cycles.*

It is straightforward to verify that any directed path or cycle in a directed graph $(\mathcal{G}, \xi)$ is locally injective. Hence, if $(\mathcal{G}, \xi)$ admits a directed cycle, $\mathcal{G}$ is not simply connected. The converse is not true.

The following directed version of Corollary 4.24 is not necessary for the constructions of this paper, so I leave its proof as an exercise for the interested reader:

Proposition 4.34. *For all étale morphisms $f \colon (\mathcal{G}', \xi') \to (\mathcal{G}, \xi)$ between connected DAGs, the underlying morphism $f \colon \mathcal{G}' \to \mathcal{G}$ is an étale monomorphism.*

Moreover, if $(\mathcal{G}, \xi)$ is a DAG and the set of morphisms $(\mathcal{G}', \xi') \to (\mathcal{G}, \xi)$ in etGr/Di is non-empty, then $(\mathcal{G}', \xi')$ is a DAG. Hence, any morphism to a DAG in etGr/Di is a pointwise injection (on connected components).

A consequence of Proposition 4.34 is that the combinatorics of properads, which are governed by DAGs, are much simpler than those of wheeled properads or modular operads.

5. Non-unital modular operads

The goal of the current section is to construct a monad $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ on GS whose EM category of algebras $\mathsf{GS}^{\mathbb{T}}$ is isomorphic to the category MO^{-} of non-unital modular operads (Remark 1.25).

To provide context for this section, consider the following example:

Example 5.1. Recall, from Example 1.18, the category $\mathbf{B}^{\downarrow}$, whose objects are either the directed exceptional edge $(\downarrow)$, or rooted corollas t_X, with X a finite set.

The operad endofunctor M_{Op} on $\mathsf{psh}(\mathbf{B}^{\downarrow})$ from Example 2.10 is described in detail in [6, Section 3]. It takes a presheaf $P \colon \mathbf{B}^{\downarrow\mathrm{op}} \to \mathsf{Set}$ to the presheaf $M_{\mathsf{Op}}P$ on $\mathbf{B}^{\downarrow}$ with $M_{\mathsf{Op}}P(\downarrow) = P(\downarrow)$, and where $M_{\mathsf{Op}}P(t_X)$ is a set of formal operadic compositions (i.e. root-to-leaf graftings of decorated corollas as in Figure 7(b)) of elements of P: Elements of $M_{\mathsf{Op}}P(t_X)$ are represented by rooted trees $\mathfrak{T} \in \Omega$, whose leaves are bijectively labelled by X, together with a decoration of vertices v by elements of $P(t_{\partial v})$ – where $t_{\partial v}$ denotes the minimal neighbourhood of v in $\mathfrak{T}$ – that determines a colouring of edges of $\mathfrak{T}$ by $P(\downarrow)$.

The monadic unit $\eta^{\mathbb{M}_{\mathsf{Op}}}$ is induced by the inclusion of rooted corollas as trees with one vertex in Ω. So, $\eta^{\mathbb{M}_{\mathsf{Op}}}(\phi) = (t_X, \phi)$ for all $\phi \in P(t_X)$ (Figure 13, left side). Applying the monad twice describes a nesting of trees, and the multiplication $\mu^{\mathbb{M}_{\mathsf{Op}}}$ for M_{Op} is induced by erasing the inner nesting (the blue circles in the right hand side of Figure 13).

If (P, h) is an algebra for $\mathbb{M}_{\mathsf{Op}}$, then h describes a rule – that satisfies the axioms of operadic composition – for collapsing the inner edges of each P-decorated tree.

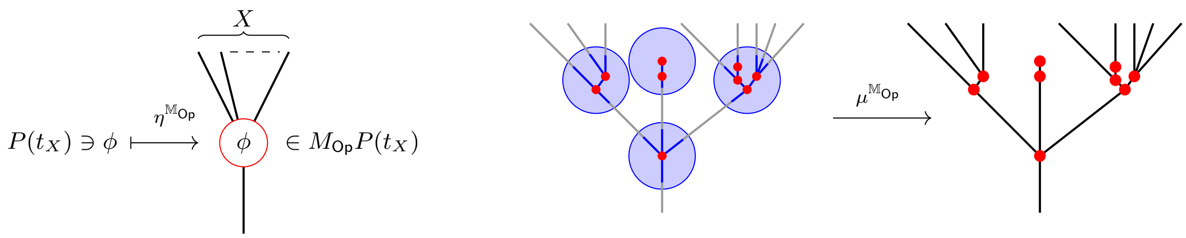


FIGURE 13. Visualising the unit and multiplication for the operad monad on rooted corollas.

Just as the operad endofunctor takes a $\mathbf{B}^{\downarrow}$-presheaf O to trees decorated by O, the non-unital modular operad endofunctor T on GS takes a graphical species S to the graphical species TS whose elements are formal multiplications and contractions in S, represented by S-structured connected graphs.

5.1. **X-graphs and an endofunctor for non-unital modular operads.** The first step in defining the endofunctor $T \colon \mathsf{GS} \to \mathsf{GS}$ is to bijectively label graph boundaries by finite sets.

Definition 5.2. *Let X be a finite set. An* (admissible) X-graph *is a pair $\mathcal{X} = (\mathcal{G}, \rho)$, where $\mathcal{G}$ is a connected graph such that $V \neq \emptyset$ and $\rho \colon E_0 \xrightarrow{\cong} X$ is a bijection, or* X-labelling *for $\mathcal{G}$.*

An X-isomorphism *$\mathcal{X} \to \mathcal{X}'$ of X-graphs $\mathcal{X} = (\mathcal{G}, \rho)$ and $\mathcal{X}' = (\mathcal{G}', \rho')$, is an isomorphism $g \in \mathsf{CetGr}(\mathcal{G}, \mathcal{G}')$ that preserves the X-labelling: $\rho' \circ g_{E_0} = \rho \colon E_0 \to X$.*

The groupoid of X-graphs and X-isomorphisms is denoted by $X\text{-}\mathsf{CGr}_{\mathsf{iso}}$.

Remark 5.3. It is convenient to use the same notation for labelled and unlabelled graphs. In particular, an X-graph $\mathcal{X} = (\mathcal{G}, \rho)$ is denoted simply by $\mathcal{G}$ when the labelling ρ is trivial or canonical. For example, for all finite sets X, the corolla $\mathcal{C}_X$ canonically defines an X-graph $\mathcal{C}_X = (\mathcal{C}_X, id)$.

Example 5.4. The line graph $\mathcal{L}^k$, with $E_0(\mathcal{L}^k) = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\}$, $k \geq 0$ is labelled by $1_{\mathcal{L}^k} \mapsto 1 \in \mathbf{2}$ and therefore has the structure of a $\mathbf{2}$-graph when $k \geq 1$. However, $\mathcal{L}^0 = (\shortmid)$ has empty vertex set and is therefore not a $\mathbf{2}$-graph.

For all finite sets X, there is a canonical functor $X\text{-}\mathsf{CGr}_{\mathsf{iso}} \to \mathsf{CetGr} \hookrightarrow \mathsf{GS}$. A graphical species S defines a presheaf on $X\text{-}\mathsf{CGr}_{\mathsf{iso}}$ with $S(\mathcal{X}) = S(\mathcal{G})$ for $\mathcal{X} = (\mathcal{G}, \rho)$. Objects of the element category $X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S$ are called *S-structured X-graphs.*

We can now define the non-unital modular operad endofunctor T on GS, that takes a graphical species S to equivalence classes of S-structured graphs.

For all graphical species S, let TS be defined on objects by

$$
\begin{array}{rcll}
TS_{\S} & = & S_{\S}, & \\
TS_X & = & \operatorname{colim}_{\mathcal{X} \in X\text{-}\mathsf{CGr}_{\mathsf{iso}}} S(\mathcal{X}) & \text{for all finite sets } X.
\end{array}
\tag{5.5}
$$

Let $Aut_X(\mathcal{X}) \overset{\text{def}}{=} X\text{-}\mathsf{CGr}_{\mathsf{iso}}(\mathcal{X}, \mathcal{X})$ be the automorphism group of an X-graph $\mathcal{X}$. If $g, g' \in X\text{-}\mathsf{CGr}_{\mathsf{iso}}(\mathcal{X}, \mathcal{X}')$ are parallel X-isomorphisms, then there are $\sigma \in Aut_X(\mathcal{X})$ and $\sigma' \in Aut_X(\mathcal{X}')$ such that $g' = \sigma' g \sigma$. Therefore, there is a completely canonical (independent of $g \in X\text{-}\mathsf{CGr}_{\mathsf{iso}}(\mathcal{X}, \mathcal{X}')$) choice of natural (in $\mathcal{X}$) isomorphism

$$
\frac{S(\mathcal{X})}{Aut_X(\mathcal{X})} \xrightarrow{\cong} \frac{S(\mathcal{X}')}{Aut_X(\mathcal{X}')}, \quad [\alpha] \mapsto [g(\alpha)], \text{ for } \alpha \in S(\mathcal{X}).
\tag{5.6}
$$

It follows from (5.6), that

$$
\begin{aligned}
TS_X &= \coprod_{[\mathcal{X}]\in\pi_0(X\text{-}\mathsf{CGr}_{\mathsf{iso}})} \tfrac{S(\mathcal{X})}{Aut_X(\mathcal{X})}\\
&= \pi_0(X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S)
\end{aligned} \tag{5.7}
$$

where, for $\mathcal{X}=(\mathcal{G},\rho)$, $[\mathcal{X}]\in\pi_0(X\text{-}\mathsf{CGr}_{\mathsf{iso}})$ is the connected component of $\mathcal{X}$ in $X\text{-}\mathsf{CGr}_{\mathsf{iso}}$.

Hence, elements of TS_X may be viewed as isomorphism classes of S-structured X-graphs, and two X-labelled S-structured graphs $(\mathcal{X},\alpha)$ and $(\mathcal{X}',\alpha')$ represent the same class $[\mathcal{X},\alpha]\in TS_X$ precisely when there is an isomorphism $g\in X\text{-}\mathsf{CGr}_{\mathsf{iso}}(\mathcal{X},\mathcal{X}')$ such that $S(g)(\alpha')=\alpha$.

Since bijections $f\colon X\xrightarrow{\cong}Y$ of finite sets induce isomorphisms $Y\text{-}\mathsf{CGr}_{\mathsf{iso}}/S\xrightarrow{\cong}X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S$, the action of TS on isomorphisms in $\mathbf{B}^{\S}$ is the obvious one.

The projections $TS(ch_x)\colon TS_X\to TS_{\S}=S_{\S}$ are induced by the projections

$$X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S\to S_{\S}\text{ given by }\ (\mathcal{X},\alpha)\mapsto S(ch_x^{\mathcal{X}})(\alpha),$$

where $ch_x^{\mathcal{X}}\in\mathsf{CetGr}(\shortmid,\mathcal{G})$ is the map $ch_{\rho^{-1}(x)}$ defined by $1\mapsto\rho^{-1}(x)\in E_0(\mathcal{G})$.

This is well-defined since, if $(\mathcal{X},\alpha)$ and $(\mathcal{X}',\alpha')$ represent the same element of TS_X, then there is an X-isomorphism $g\colon\mathcal{X}\to\mathcal{X}'$ such that $S(g)(\alpha')=\alpha\in S_X$ and hence

$$S(ch_x^{\mathcal{X}'})(\alpha')=S(ch_x^{\mathcal{X}})\circ S(g)(\alpha')=S(ch_x^{\mathcal{X}})(\alpha).$$

So TS describes a graphical species. Moreover, it is clear from the definition that the assignment $S\mapsto TS$ extends to an endofunctor T on GS, with unit $\eta^{\mathbb{T}}\colon id_{\mathsf{GS}}\Rightarrow T$ given by the canonical maps $S_X\xrightarrow{\cong}S(\mathcal{C}_X)\to TS_X$ for all X.

5.2. **Gluing constructions.** A monadic multiplication $\mu^{\mathbb{T}}$ for the pointed endofunctor $(T,\eta^{\mathbb{T}})$ will be defined in terms of colimits of a certain class of diagrams in CetGr. However, since CetGr does not admit general colimits (see Examples 3.16 and 3.33), a small amount of preparation is necessary.

Let S be a graphical species and Y a finite set. Since, elements of TS_Y are represented by S-structured Y-graphs, it follows that, for all finite sets X, elements of T^2S_X are represented by X-graphs $\mathcal{X}$ that are decorated by S-structured graphs. In other words, each $[\mathcal{X},\beta]\in T^2S_X$ is represented by a functor

$$el(\beta)\colon\mathsf{el}(\mathcal{X})\to\mathsf{el}(TS),\ \begin{cases}(\mathcal{C}_{X_b},b) & \mapsto(\mathcal{C}_{X_b},S(b)(\beta)),\quad\text{where }S(b)(\beta)\in TS_{X_b}\\ (\shortmid,ch_e) & \mapsto(\shortmid,c)\qquad\qquad\qquad\quad c\in S_{\S}\end{cases}$$

such that

$$el(\beta)(ch_{x_b})(S(b)(\beta))=el(\beta)(ch_e)\in S(\shortmid)$$

for all morphisms in $\mathsf{el}(\mathcal{X})$ of the form

$$\begin{tikzcd}(\shortmid)\arrow[rr,"ch_{x_b}"]\arrow[dr,"ch_e"'] & & \mathcal{C}_{X_b}\arrow[dl,"b"]\\ & \mathcal{X}. & \end{tikzcd}$$

Then, as in the operad case (Example 5.1, Figure 13), we would like to think of the monad multiplication as forgetting the vertices of the original graph $\mathcal{X}$ to obtain an element of TS_X (Figure 14).

Graphs of graphs are functors that encode this idea:

Definition 5.8. *Let $\mathcal{G}$ be a graph. A $\mathcal{G}$-shaped graph of graphs is a functor $\mathbf{\Gamma}\colon\mathsf{el}(\mathcal{G})\to\mathsf{etGr}$ such that*

$$
\begin{aligned}
&\mathbf{\Gamma}(ch_e)=(\shortmid) && \textit{for all } (\shortmid,ch_e)\in\mathsf{el}(\mathcal{G}),\\
&E_0(\mathbf{\Gamma}(b))=X_b && \textit{for all } (\mathcal{C}_{X_b},b)\in\mathsf{el}(\mathcal{G}),
\end{aligned}
$$

and, for all $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$ *and all* $x_b \in X_b$,

$$\mathbf{\Gamma}(ch_{x_b}) = ch_{x_b}^{\mathbf{\Gamma}(b)} \in \mathsf{etGr}(\shortmid, \mathbf{\Gamma}(b)).$$

A $\mathcal{G}$*-shaped graph of graphs* $\mathbf{\Gamma}\colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ *is* non-degenerate *if, for all* $v \in V$, $\mathbf{\Gamma}(\iota_v)$ *has no stick components. Otherwise,* $\mathbf{\Gamma}$ *is* degenerate.

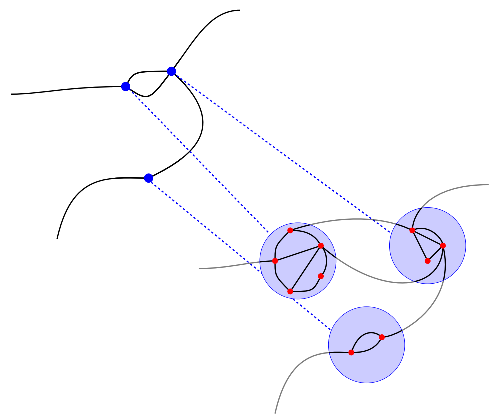


FIGURE 14. A $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$ describes *graph substitution*: each vertex v of $\mathcal{G}$ is replaced by a graph $\mathcal{G}_v$ according to a bijection $E_0(\mathcal{G}_v) \xrightarrow{\cong} ({}^{E}/{}_{v})^{\dagger}$. When $\mathbf{\Gamma}$ is non-degenerate, taking its colimit in etGr corresponds to erasing the inner (blue) nesting.

Informally, a non-degenerate $\mathcal{G}$-shaped graph of graphs is a rule for substituting graphs into vertices of $\mathcal{G}$ as in Figure 14. However, this intuitive description of a graph of graphs in terms of graph insertion does not always apply in the degenerate case (see Sections 6 and 7).

By Lemma 4.17, every graph $\mathcal{G}$ is the colimit of the (non-degenerate) *identity* $\mathcal{G}$*-shaped graph of graphs* $\mathbf{I}^{\mathcal{G}}$ given by the forgetful functor $\mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$, $(\mathcal{C}, b) \mapsto \mathcal{C}$. It follows from Section 4.2 that, if $\mathcal{G}$ has no stick components, this is equivalent to the statement that $\mathcal{G}$ is the coequaliser of the canonical diagram

$$\mathcal{S}(E_\bullet) \rightrightarrows \coprod_{v \in V} \mathcal{C}_{\mathbf{v}} \xrightarrow{\coprod(\iota_v)} \mathcal{G}. \tag{5.9}$$

To prove that all non-degenerate graphs of graphs admit a colimit in etGr, we generalise this observation using a modification of the *gluing data* (for directed graphs) from [27, Section 1.5.1].

Definition 5.10. *Let* $\mathcal{S} = \coprod_{i \in I}(\shortmid_i)$ *be a shrub, and let* $\mathcal{G}$ *be a (not necessarily connected) graph without stick components. A pair of parallel morphisms* $\delta_1, \delta_2 \colon \mathcal{S} \rightrightarrows \mathcal{G}$ *such that*

- δ_1, δ_2 *are injective and have disjoint images in* $\mathcal{G}$*; and*
- *for all* $i \in I$, $\delta_1(1_i)$ *and* $\delta_2(2_i)$ *are ports of* $\mathcal{G}$,

is called a gluing datum *in* etGr.

Lemma 5.11. *Gluing data admit coequalisers in* etGr.

Proof. Let $\mathcal{G}$ be a graph without stick components and $\delta_1, \delta_2 \colon \mathcal{S} = \coprod_{i \in I}(\shortmid_i) \rightrightarrows \mathcal{G}$ a gluing datum with coequaliser $\overline{p}\colon \mathcal{G} \to \overline{\mathcal{G}} = (\overline{E}, \overline{H}, \overline{V}, \overline{s}, \overline{t}, \overline{\tau})$ in the category $\mathsf{psh_f}(\mathtt{D})$ of graph-like diagrams. Hence, if $\overline{\mathcal{G}}$ is a graph, then it is the coequaliser of δ_1, δ_2 in $\mathsf{Grpsh_f}(\mathtt{D})$.

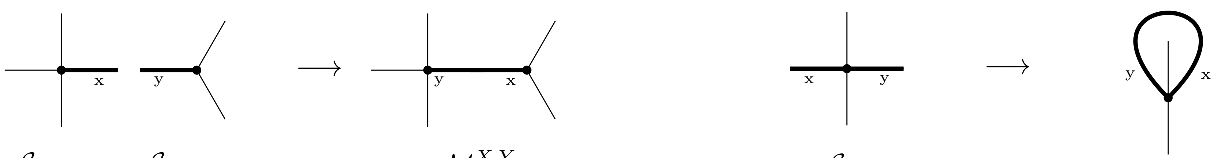
Half-edges h and h' of $\mathcal{G}$ are identified in $\overline{H}$ if and only if there is an edge $l \in E(\mathcal{S})$ such that $\delta_1(l) = s(h)$ and $\delta_2(l) = s(h')$. This contradicts the conditions of Definition 5.10, whereby $\overline{H} = H$. Moreover $e, e' \in E(\mathcal{G})$ are identified in $\overline{E}$ if and only if there is an $l \in E(\mathcal{S})$ such that $\delta_1(l) = e$ and $\delta_2(l) = e'$ (or vice versa). So, since $\mathcal{G}$ has no stick components, and δ_1, δ_2 have disjoint images, we may assume that e is a port and $e' \in s(H)$. In particular, $\overline{s}\colon \overline{H} \to \overline{E}$ is injective, and since δ_1 and δ_2 have disjoint images, $\overline{\tau}\colon \overline{E} \to \overline{E}$ is a fixed-point free involution. Hence $\overline{\mathcal{G}}$ is a graph.

In particular, $\overline{V} = V$ since $\mathcal{S}$ is a shrub, and we have shown that $\overline{H} = H$. It follows that $\overline{p}\colon \mathcal{G} \to \overline{\mathcal{G}}$ is an étale monomorphism, and the lemma is proved. □

Example 5.12. In Example 3.22, the graphs $\mathcal{M}^{X,Y}_{x,y}$ and $\mathcal{N}^{X}_{x,y}$ (Examples 3.8, 3.9) were constructed as coequalisers of gluing data (3.23), (3.24):

$$\big(ch_x,\ ch_y \circ \tau\colon (\shortmid) \ \rightrightarrows\ (\mathcal{C}_{X \amalg \{x\}} \amalg \mathcal{C}_{Y \amalg \{y\}})\big) \longrightarrow \mathcal{M}^{X,Y}_{x,y}, \qquad \big(ch_x,\ ch_y \circ \tau\colon (\shortmid) \ \rightrightarrows \mathcal{C}_{X \amalg \{x,y\}}\big) \longrightarrow \mathcal{N}^{X}_{x,y}.$$

This is visualised in Figure 15.



$\mathcal{C}_{X \amalg \{x\}}$ $\mathcal{C}_{Y \amalg \{y\}}$ $\mathcal{M}^{X,Y}_{x,y}$ $\mathcal{C}_X$ $\mathcal{N}^{X}_{x,y}$

FIGURE 15. Construction of $\mathcal{M}^{X,Y}_{x,y}$ and $\mathcal{N}^{X}_{x,y}$ as coequalisers of gluing data.

Proposition 5.13. *A non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}\colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$ admits a colimit $\mathbf{\Gamma}(\mathcal{G})$ in* etGr.

Proof. For all graphs $\mathcal{G}$, $\mathsf{el}(\mathcal{G})$ is a connected category if and only if $\mathcal{G}$ is a connected graph. So, the colimit $\mathbf{\Gamma}(\mathcal{G})$ of a $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}\colon \mathsf{el}(\mathcal{G}) \to \mathsf{etGr}$, if it exists, may be constructed componentwise. In particular, we may assume that $\mathcal{G}$ is connected.

A non-degenerate $(\shortmid)$-shaped graph of graphs is just an isomorphism $(\shortmid) \xrightarrow{\cong} (\shortmid)$.

Assume therefore, that $\mathcal{G} \ncong (\shortmid)$ is a connected graph. Since $\mathbf{\Gamma}$ preserves incidence data of $\mathcal{G}$, and preserves boundaries objectwise on $\mathsf{el}(\mathcal{G})$, we may apply $\mathbf{\Gamma}$ to all components of (5.9) to obtain a diagram in etGr:

$$\coprod_{\tilde{e} \in \widetilde{E_\bullet}} (\shortmid_{\tilde{e}}) \rightrightarrows \coprod_{v \in V} \mathbf{\Gamma}(\mathcal{C}_{\mathbf{v}}, \iota_v). \tag{5.14}$$

This is a gluing datum since $\mathbf{\Gamma}$ is non-degenerate. Therefore (5.14) has a colimit $\overline{\mathcal{G}}$ in etGr by Lemma 5.11.

For vertices $v' \in V$, and inner edges $e' \in E_\bullet$, there are canonical inclusions

$$\mathbf{\Gamma}(\iota_{v'}) \hookrightarrow \coprod_{v \in V} \mathbf{\Gamma}(\iota_v), \quad \text{and} \quad \mathbf{\Gamma}(\iota_{\tilde{e}'}) \hookrightarrow \coprod_{\tilde{e} \in \widetilde{E_\bullet}} \mathbf{\Gamma}(\iota_{\tilde{e}}) \xrightarrow{\cong} \coprod_{\tilde{e} \in \widetilde{E_\bullet}} (\shortmid_{\tilde{e}}). \tag{5.15}$$

Since $\mathcal{G}$ is connected and $\mathcal{G} \ncong (\shortmid)$, for any port $e \in E_0(\mathcal{G})$, there is a unique half-edge $(\tau e, w) \in H(\mathcal{G})$ and the morphism $\mathbf{\Gamma}(\delta_{(\tau e, w)})\colon \mathbf{\Gamma}(\iota_{\tilde{e}}) \hookrightarrow \mathbf{\Gamma}(\iota_w)$ induces an inclusion

$$\mathbf{\Gamma}(\iota_{\tilde{e}}) \hookrightarrow \mathbf{\Gamma}(\iota_w) \hookrightarrow \coprod_{v \in V} \mathbf{\Gamma}(\iota_v). \tag{5.16}$$

In particular, the inclusions (5.15) and (5.16) describe a functor from $\mathsf{es}(\mathcal{G})$ to (5.14), and hence a cocone of $\mathbf{\Gamma}$ above $\overline{\mathcal{G}}$.

Conversely, $\mathbf{\Gamma}$ has a colimit $\mathbf{\Gamma}(\mathcal{G})$ in the category $\mathsf{psh}_{\mathsf{f}}(\mathsf{D})$ of graph-shaped diagrams, and the colimit cocone of $\mathbf{\Gamma}$ above $\mathbf{\Gamma}(\mathcal{G})$ factors through (5.14). Therefore, $\mathbf{\Gamma}(\mathcal{G}) = \overline{\mathcal{G}}$ by the universal properties of colimits, and $\mathbf{\Gamma}(\mathcal{G}) = \overline{\mathcal{G}}$ is a graph. Moreover, since $\overline{\mathcal{G}}$ defines a coequaliser in etGr of (5.14), it follows immediately that $\mathbf{\Gamma}$ has a colimit in etGr. □

Remark 5.17. In fact, as will follow from Proposition 7.17, all graphs of graphs admit a colimit in CetGr. However, the non-degeneracy condition simplifies the proof of Proposition 5.13, and is all that is needed for now.

Corollary 5.18. *If $\mathcal{G}$ is a graph, and $\mathbf{\Gamma}$ is a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\mathbf{\Gamma}(\mathcal{G})$, then the induced map $E(\mathcal{G}) \to E(\mathbf{\Gamma})$ is injective and restricts to the identity $E_0(\mathcal{G}) \xrightarrow{=} E_0(\mathbf{\Gamma}(\mathcal{G}))$ on ports. For each $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, the universal map $\mathbf{\Gamma}(b) \to \mathbf{\Gamma}(\mathcal{G})$ is an étale monomorphism. In particular,*

$$E(\mathbf{\Gamma}(\mathcal{G})) \cong E(\mathcal{G}) \amalg \coprod_{v \in V} E_\bullet(\mathbf{\Gamma}(\iota_v)).$$

Proof. The final statement follows directly from the first two.

By the proof of Proposition 5.13, only the inner edges of $\mathcal{G}$, and, for all $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, the τ-orbits of ports of $\mathbf{\Gamma}(b)$ are involved in forming the colimit $\mathbf{\Gamma}(\mathcal{G})$ of $\mathbf{\Gamma}$. Hence $\mathbf{\Gamma}$ induces a strict inclusion

$$\textstyle\coprod_{\tilde{e} \in \widetilde{E}} (\shortmid_{\tilde{e}}) \xrightarrow{\cong} \coprod_{\tilde{e} \in \widetilde{E}} \mathbf{\Gamma}(\iota_{\tilde{e}}) \hookrightarrow \mathbf{\Gamma}(\mathcal{G})$$

that restricts to an identity $E_0(\mathcal{G}) = E_0(\mathbf{\Gamma}(\mathcal{G}))$ of ports. The second part is immediate. □

Corollary 5.19. *(Corresponds to* [27, Lemma 1.5.12]*.) If $\mathbf{\Gamma}$ is non-degenerate and, for each $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, the graph $\mathbf{\Gamma}(\mathcal{C}, b)$ is connected, then $\mathbf{\Gamma}(\mathcal{G}) = colim(\mathbf{\Gamma})$ is a connected graph if and only if $\mathcal{G}$ is.*

Proof. Since a $(\shortmid)$-shaped graph of graphs is isomorphic to the identity functor $(\shortmid) \mapsto (\shortmid)$ and hence has colimit $(\shortmid)$, we assume that $\mathcal{G}$ has no stick components.

Let $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{G}) \to \mathsf{CetGr}$ be a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\mathbf{\Gamma}(\mathcal{G})$.

A morphism $\gamma \in \mathsf{psh_f}(\mathsf{D})(\mathbf{\Gamma}(\mathcal{G}), \bigstar \amalg \bigstar)$ is equivalently described by a diagram in $\mathsf{psh_f}(\mathsf{D})$:

$$\mathcal{S}(E_\bullet) \rightrightarrows \textstyle\coprod_{v \in V(\mathcal{G})} \mathbf{\Gamma}(\iota_v) \longrightarrow \bigstar \amalg \bigstar. \tag{5.20}$$

If $\mathbf{\Gamma}(\iota_v)$ is connected for each $v \in V$, then all maps $\mathbf{\Gamma}(\iota_v) \to \bigstar \amalg \bigstar$ are constant. Hence morphisms $\coprod_{v \in V} \mathbf{\Gamma}(\iota_v) \to \bigstar \amalg \bigstar$ are in bijection with morphisms $\coprod_{v \in V} \mathcal{C}_{\mathbf{v}} \to \bigstar \amalg \bigstar$.

So, $\mathsf{psh_f}(\mathsf{D})(\mathcal{G}, \bigstar \amalg \bigstar) \cong \mathsf{psh_f}(\mathsf{D})(\mathbf{\Gamma}(\mathcal{G}), \bigstar \amalg \bigstar)$, and it follows from Proposition 3.34 that $\mathbf{\Gamma}(\mathcal{G})$ is connected if and only if $\mathcal{G}$ is connected. □

Let $\mathcal{G}$ be a graph and S any graphical species. As usual, let K be the terminal graphical species.

Definition 5.21. *A* (non-degenerate) $\mathcal{G}$-shaped graph of S-structured graphs *is a functor $\mathbf{\Gamma}_S \colon \mathsf{el}(\mathcal{G}) \to \mathsf{CetGr}/S$ such that the forgetful functor induced by the unique morphism $S \to K$:*

$$\mathbf{\Gamma}\colon\ \mathsf{el}(\mathcal{G}) \xrightarrow{\mathbf{\Gamma}_S} \mathsf{etGr}/S \longrightarrow \mathsf{etGr}/K = \mathsf{etGr} \tag{5.22}$$

is a (non-degenerate) $\mathcal{G}$-shaped graph of graphs.

For a connected graph $\mathcal{G}$, the category $(\mathsf{CetGr}/S)^{(\mathcal{G})}$ of non-degenerate $\mathcal{G}$-shaped graphs of S-structured graphs is the category whose objects are non-degenerate $\mathcal{G}$-shaped graphs of S-structured connected graphs $\mathbf{\Gamma}_S \colon \mathsf{el}(\mathcal{G}) \to \mathsf{CetGr}/S$ and whose morphisms are natural transformations.

Lemma 5.23. *For $\mathcal{G}$ connected, $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$, two $\mathcal{G}$-shaped graphs of S-structured graphs $\mathbf{\Gamma}^1_S, \mathbf{\Gamma}^2_S$ are in the same connected component of $(\mathsf{CetGr}/S)^{(\mathcal{G})}$ if and only if, for all $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$, $\mathbf{\Gamma}^1_S(b)$ and $\mathbf{\Gamma}^2_S(b)$ are in the same connected component of X_b-$\mathsf{CGr_{iso}}/S$.*

In particular, if $\mathbf{\Gamma}^1_S$ and $\mathbf{\Gamma}^2_S$ are in the same connected component of $(\mathsf{CetGr}/S)^{(\mathcal{G})}$, then $\mathbf{\Gamma}^1_S$ and $\mathbf{\Gamma}^2_S$ have isomorphic colimits in CetGr/S.

Proof. Let $\phi\colon \mathbf{\Gamma}^1_S \Rightarrow \mathbf{\Gamma}^2_S$ be a morphism in $(\mathsf{CetGr}/S)^{(\mathcal{G})}$. For each element $(\mathcal{C}_{X_b}, b)$ of $\mathcal{G}$, the component $\phi_{(b)}$ of ϕ at b, is, by definition a boundary-preserving morphism in CetGr/S between X_b-graphs with non-empty vertex sets. Since $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$, if $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$, then X_b is non-empty. So, by Corollary 4.21, ϕ_b is an isomorphism in CetGr/S of S-structured X_b-graphs.

The converse is immediate, as is the final statement. $\square$

5.3. **Multiplication for the monad $\mathbb{T}$.** The aim of this section is to describe the multiplication $\mu^{\mathbb{T}}\colon T^2 \Rightarrow T$ in terms of colimits of graphs of graphs.

From now on, all graphs will be connected, unless explicitly stated otherwise.

Let X be a finite set and $\mathcal{X} = (\mathcal{G}, \rho)$ an X-graph. If $\mathbf{\Gamma}\colon \mathsf{el}(\mathcal{X}) \to \mathsf{CetGr}$ is a non-degenerate $\mathcal{X}$-shaped graph of graphs, then the colimit $\mathbf{\Gamma}(\mathcal{X}) = \mathrm{colim}_{\mathsf{el}(\mathcal{X})}\mathbf{\Gamma}$ exists by Proposition 5.13 and it inherits the X-labelling ρ of $\mathcal{X}$, by Corollary 5.18.

Given a graphical species S and finite set X, elements $[\mathcal{X}, \beta]$ of $T^2 S_X$ are represented by pairs $(\mathcal{X}, \mathbf{\Gamma}_S)$ where $\mathcal{X}$ is an X-graph, and $\mathbf{\Gamma}_S$ is an $\mathcal{X}$-shaped graphs of S-structured graphs. Let $(\mathbf{\Gamma}(\mathcal{X}), \alpha)$ be the colimit of $\mathbf{\Gamma}_S$ in CetGr/S, where $\alpha \in S(\mathbf{\Gamma}(\mathcal{X}))$ and $\mathbf{\Gamma}(\mathcal{X})$ is the colimit of the underlying $\mathcal{X}$-shaped graph of graphs $\mathbf{\Gamma}$ defined as in (5.22).

If, for $j = 1, 2$, $(\mathcal{X}^j, \mathbf{\Gamma}^j_S)\colon \mathsf{el}(\mathcal{X}^j) \to \mathsf{CetGr}/S$ both represent the same element $[\mathcal{X}, \beta]$ of $T^2 S_X$, then $\mathcal{X}^1 \cong \mathcal{X}^2$ in $X\text{-}\mathsf{CGr}_{\mathsf{iso}}$ by definition of T, and by Lemma 5.23,

$$\mathrm{colim}_{\mathsf{el}(\mathcal{X}^1)}\mathbf{\Gamma}^1_S \cong \mathrm{colim}_{\mathsf{el}(\mathcal{X}^2)}\mathbf{\Gamma}^2_S \in X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S. \tag{5.24}$$

The monad multiplication $\mu^{\mathbb{T}}\colon T^2 \Rightarrow T$ for $(T, \eta^{\mathbb{T}})$ is induced by the assignments

$$[\mathcal{X}, \beta] \mapsto [\mathbf{\Gamma}(\mathcal{X}), \alpha] \tag{5.25}$$

This is well-defined by (5.24).

To see that (5.25) extends to a morphism $\mu^{\mathbb{T}}S\colon T^2 S \to TS$ of graphical species, let $[\mathcal{X}, \beta] \in T^2 S_X$ be represented by an $\mathcal{X}$-shaped graph of S-structured graphs $\mathbf{\Gamma}_S\colon \mathsf{el}(\mathcal{X}) \to \mathsf{CetGr}/S$ with colimit $\mathbf{\Gamma}_S(\mathcal{X}) = (\mathbf{\Gamma}(\mathcal{X}), \alpha) \in X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S$.

By Corollary 5.18, there is a canonical inclusion $E(\mathcal{X}) \hookrightarrow E(\mathbf{\Gamma}(\mathcal{X}))$ of edge sets, and for each $e \in E(\mathcal{X})$,

$$S(ch_e^{\mathbf{\Gamma}(\mathcal{X})})(\alpha) = S(ch_e^{\mathcal{X}})(\beta) \in S(\shortmid).$$

Hence, for all $x \in X$, there is a commuting diagram of sets

$$\begin{tikzcd} T^2 S_X \arrow[rr, "\mu^{\mathbb{T}} S_X"] \arrow[dr, "T^2 S(ch_x)"'] & & TS_X \arrow[dl, "TS(ch_x)"] \\ & T^2 S(\shortmid) = TS(\shortmid). & \end{tikzcd}$$

Naturality of $\mu^{\mathbb{T}}S$ in S is immediate from the definition and, by a straightforward modification of [27, Section 2.2], it may be shown that $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ satisfies the two axioms for a monad.

Remark 5.26. For all graphical species S, $\mu^{\mathbb{T}}S$ and $\eta^{\mathbb{T}}S$ are palette-preserving morphisms in GS. So $\mathbb{T}$ restricts to a monad $\mathbb{T}^{(\mathfrak{C},\omega)}$ on $\mathsf{GS}^{(\mathfrak{C},\omega)}$, for all $(\mathfrak{C}, \omega)$. If A is a $(\mathfrak{C}, \omega)$-coloured graphical species and $h \in \mathsf{GS}(TA, A)$, then (A, h) is a $\mathbb{T}$-algebra if and only if it is a $\mathbb{T}^{(\mathfrak{C},\omega)}$-algebra.

Example 5.27. Since K is the terminal graphical species, $\mathsf{CetGr}/K \cong \mathsf{CetGr}$ and hence elements of TK are boundary-preserving isomorphism classes of graphs in CetGr. The unique morphism $! \in \mathsf{GS}(TK, K)$ makes K into an algebra for $\mathbb{T}$. Likewise, for any palette $(\mathfrak{C}, \omega)$, the terminal $(\mathfrak{C}, \omega)$-coloured graphical species $K^{(\mathfrak{C},\omega)}$ is a $\mathbb{T}$-algebra together with the unique palette-preserving morphism $!^{(\mathfrak{C},\omega)}\colon TK^{(\mathfrak{C},\omega)} \to K^{(\mathfrak{C},\omega)}$.

5.4. **$\mathbb{T}$-algebras are non-unital modular operads.** Having constructed the monad $\mathbb{T}$, it just remains to prove that $\mathbb{T}$-algebras are non-unital modular operads.

Lemma 5.28. *A $\mathbb{T}$-algebra (A, h) admits a multiplication ${}_h\diamond$ and contraction ${}_h\zeta$, that are natural with respect to morphisms in $\mathsf{GS}^{\mathbb{T}}$.*

Proof. Let X and Y be finite sets and let $\mathcal{M}^{X,Y}_{x,y}$ be the $X \amalg Y$-graph (described in Examples 3.8 and 5.12) obtained by gluing the corollas $\mathcal{C}_{X\amalg\{x\}}$ and $\mathcal{C}_{Y\amalg\{y\}}$ along ports x and y.

Given a $(\mathfrak{C}, \omega)$-coloured graphical species S, a choice of $\underline{c} \in \mathfrak{C}^X$, $\underline{d} \in \mathfrak{C}^Y$ and $c \in \mathfrak{C}$, together with an ordered pair $(\phi, \psi) \in S_{(\underline{c},c)} \times S_{(\underline{d},\omega c)}$, determines an element $\mathcal{M}_c(\phi, \psi)$ of $S(\mathcal{M}^{X,Y}_{x,y})$.

The canonical map $S(\mathcal{M}^{X,Y}_{x,y}) \twoheadrightarrow TS_{X\amalg Y}$ is injective unless $X = Y = \emptyset$, in which case $[\mathcal{M}_c(\phi_1, \psi_1)] = [\mathcal{M}_c(\phi_2, \psi_2)]$ if and only if $(\phi_2, \psi_2) = (\psi_1, \phi_1)$.

If (A, h) is a $(\mathfrak{C}, \omega)$-coloured $\mathbb{T}$-algebra, then the family of maps given by the compositions

$$ {}_h\diamond\colon\ S_{(\underline{c},c)} \times S_{(\underline{d},\omega c)} \xrightarrow{[\mathcal{M}(\cdot,\cdot)]} TS_{\underline{cd}} \xrightarrow{\quad h \quad} S_{\underline{cd}} $$

defines a multiplication on A (see Figure 16).

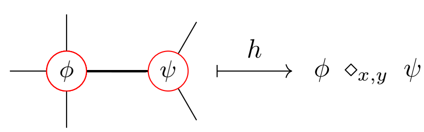
Similarly, for a finite set X, let $\mathcal{N}^X_{x,y}$ be the X-graph (described in Examples 3.9 and 5.12) obtained by gluing the ports of $\mathcal{C}_{X\amalg\{x,y\}}$ labelled by x and y.

For $\underline{c} \in \mathfrak{C}^X$ and $c \in \mathfrak{C}$, each $\phi \in S_{(\underline{c},c,\omega c)} \subset S_{X\amalg\{x,y\}}$ determines an element $\mathcal{N}^S_c(\phi) \in S(\mathcal{N}^X_{x,y})$. The only non-trivial boundary-preserving automorphism of $\mathcal{N}^X_{x,y}$ is the permutation $\sigma_{x,y} \in Aut(X \amalg \{x, y\})$ that switches x and y and leaves the other elements unchanged, so $[\mathcal{N}^S_c(\phi)] = [\mathcal{N}^S_c(\psi)]$ in TS_X if and only if $S(\sigma_{x,y})(\phi) = \psi$.

If (A, h) is a $(\mathfrak{C}, \omega)$-coloured algebra for T, the family of maps given by the compositions

$$ {}_h\zeta\colon\ A_{(\underline{c},c,\omega c)} \xrightarrow{[\mathcal{N}^A(\cdot)]} TA_{\underline{c}} \xrightarrow{\quad h \quad} A_{\underline{c}} $$

defines an equivariant contraction on A (see Figure 16). Naturality of ${}_h\diamond$ and ${}_h\zeta$ is immediate from the construction. □



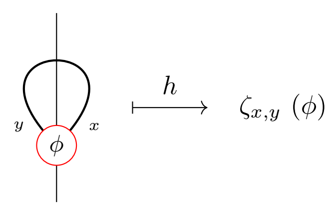


FIGURE 16. If (A, h) is a $\mathbb{T}$-algebra, h induces a multiplication and contraction on A.

We are now able to show that algebras for the monad $\mathbb{T}$ on GS are precisely non-unital modular operads.

Proposition 5.29. *There is a canonical isomorphism of categories $\mathsf{GS}^{\mathbb{T}} \cong \mathsf{MO}^-$.*

Proof. A $\mathbb{T}$-algebra structure $h\colon TA \to A$ equips a graphical species A with a multiplication $\diamond = {}_h\diamond$, and contraction $\zeta = {}_h\zeta$ as in Lemma 5.28. We must show that $(A, \diamond, \zeta)$ satisfies conditions (M1)-(M4) of Definition 1.24.

The key observation for this result is that, up to choice of boundary E_0, any connected graph with two inner edge orbits has one of the forms illustrated in Figures 17-20.

Condition (M1) is illustrated in Figure 17. Precisely, let $\phi_1 \in A_{(\underline{b},c)}, \phi_2 \in A_{(\underline{c},\omega c,d)}$ and $\phi_3 \in A_{(\underline{d},\omega d)}$. By Lemma 5.28, and the monad algebra axioms,

$$\begin{array}{rcl}(\phi_1 \diamond_c \phi_2) \diamond_d \phi_3 & = & h\left[\mathcal{M}_d^A\left((\phi_1 \diamond_c \phi_2), \phi_3\right)\right] \\ & = & h\left[\mathcal{M}_d^A\left(h[\mathcal{M}_c(\phi_1, \phi_2)], h\eta^{\mathbb{T}} A(\phi_3)\right)\right] \\ & = & h\mu^{\mathbb{T}}\left[\mathcal{M}_d^{TA}\left([\mathcal{M}_c(\phi_1, \phi_2)], \eta^{\mathbb{T}} A(\phi_3)\right)\right],\end{array}$$

and, likewise

$$\phi_1 \diamond_c (\phi_2 \diamond_d \phi_3) = h\mu^{\mathbb{T}}\left[\mathcal{M}_c^{TA}\left(\eta^{\mathbb{T}} A(\phi_1), [\mathcal{M}_d^A(\phi_2, \phi_3)]\right)\right].$$

So, it suffices to show that, for all ϕ_1, ϕ_2, ϕ_3 as above,

$$\mu^{\mathbb{T}}\left[\mathcal{M}_d^{TA}\left([\mathcal{M}_c(\phi_1, \phi_2)], \eta^{\mathbb{T}} A(\phi_3)\right)\right] = \mu^{\mathbb{T}}\left[\mathcal{M}_c^{TA}\left(\eta^{\mathbb{T}} A(\phi_1), [\mathcal{M}_d^A(\phi_2, \phi_3)]\right)\right].$$

By Example 5.12, since colimits commute,

$$\mathrm{coeq}_{\mathsf{CetGr}}\left(ch_y, ch_z \circ \tau \colon (\shortmid) \rightrightarrows \mathcal{M}_{w,x}^{X_1,(X_2 \amalg \{y\})} \amalg \mathcal{C}_{X_3 \amalg \{z\}}\right) = \mathrm{coeq}_{\mathsf{CetGr}}\left(ch_w, ch_x \circ \tau \colon (\shortmid) \rightrightarrows \mathcal{C}_{X_1 \amalg \{w\}} \amalg \mathcal{M}_{y,z}^{(X_2 \amalg \{x\}), X_3}\right).$$

The coherence conditions (M2)-(M4) all follow in the same way from the defining axioms of monad algebras. Figures 18-20 illustrate each condition.

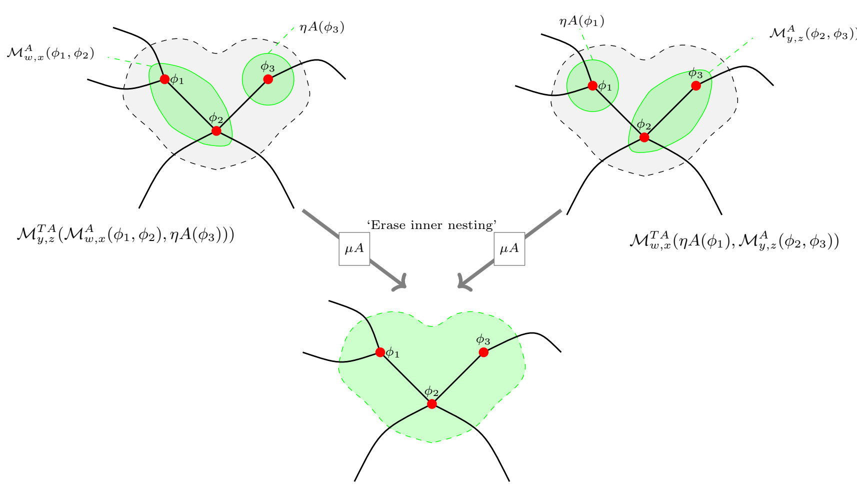


FIGURE 17. Coherence condition (M1) Applying $\mu^{\mathbb{T}} A \colon T^2 A \to A$ amounts to *erasing inner nesting.*

The assignment $(A, h) \mapsto (A, \diamond, \zeta)$ clearly extends to a functor $\mathsf{GS}^{\mathbb{T}} \to \mathsf{MO}^-$.

The proof of the converse closely resembles that of [16, Theorem 3.7]. Namely, let $(S, \diamond, \zeta)$ satisfy the conditions in the statement of the proposition. We construct a morphism $h \in \mathsf{GS}(TS, S)$ by successively using $\diamond$ and ζ to *collapse* inner edge orbits of S-structured X-graphs $(\mathcal{X}, \alpha)$. This results in a finite sequence of S-structured X-graphs that terminates in an S-structured corolla $(\mathcal{C}_X, \phi)$.

As usual, let X be a finite set and let $(\mathcal{X}, \alpha)$ be a representative of $[\mathcal{X}, \alpha] \in TS_X$.

If $\mathcal{X}$ has no inner edges, then $\mathcal{X} = \mathcal{C}_X$, and so $[\mathcal{X}, \alpha] = \eta^{\mathbb{T}} S(\phi)$ for some $\phi \in S_X$. In this case, define

$$h[\mathcal{X}, \alpha] \stackrel{\text{def}}{=} \phi \in S_X. \tag{5.30}$$

Otherwise, let $\mathcal{X}$ have vertex set V, edge set E, and let $\tilde{e} \in \widetilde{E_\bullet}$ be the orbit of a pair $e, \tau e$ of inner edges of $\mathcal{X}$. Write $t(e) \stackrel{\text{def}}{=} ts^{-1}(e)$ for the vertex v with $e \in {}^E/_v$.

There are two possibilities: either $t(e) = t(\tau e)$ or $t(e) \neq t(\tau e)$.

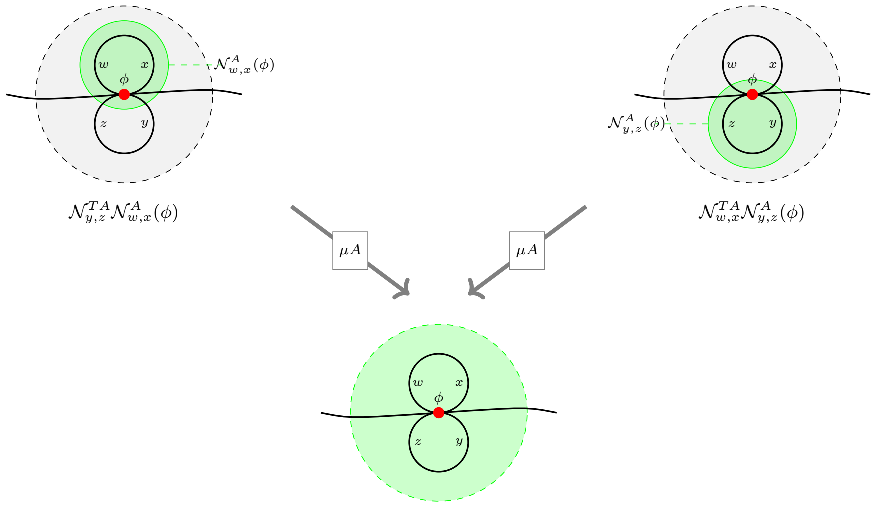


FIGURE 18. Coherence condition (M2)

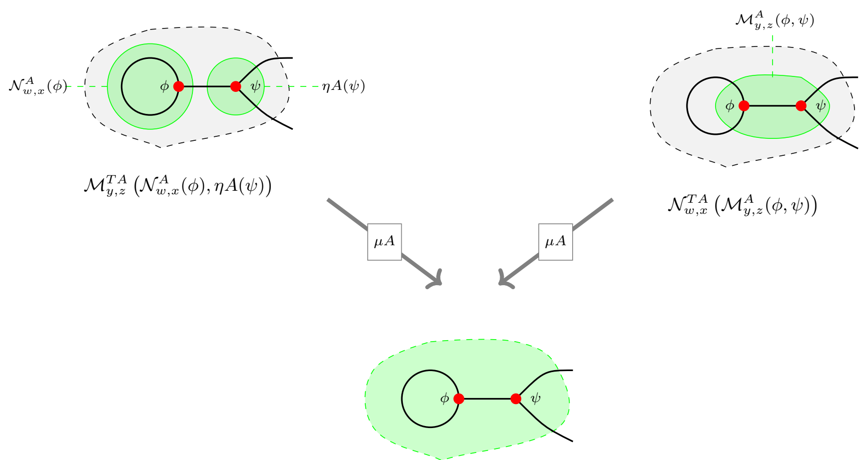


FIGURE 19. Coherence condition (M3).

CASE 1: $t(e) = v_1$ and $t(\tau e) = v_2$ are distinct vertices of $\mathcal{X}$.
Let $\mathcal{X}_{/\tilde{e}}$ be the graph obtained from $\mathcal{X}$ by removing the τ-orbit $\{e, \tau e\}$ and identifying v_1 and v_2 to a vertex $\overline{v} \in {}^{V}\!/_{(v_1 \sim v_2)}$:

$$\mathcal{X}_{/\tilde{e}} \overset{\text{def}}{=} \quad \tau \circlearrowright (E \setminus \{e, \tau e\}) \xleftarrow{s} (H \setminus s^{-1}\{e, \tau e\}) \xrightarrow{\bar{t}} {}^{V}\!/_{(v_1 \sim v_2)},$$

where $\bar{t}$ is the composition of $t\colon H \to V$ with the quotient $V \twoheadrightarrow {}^{V}\!/_{(v_1 \sim v_2)}$. So, for each vertex $v \in V \setminus \{v_1, v_2\}$ and each minimal neighbourhood $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{X})$ of a vertex v in $\mathcal{X}$, $(\mathcal{C}_{X_b}, b)$ also describes a minimal neighbourhood of v in $\mathcal{X}_{/\tilde{e}}$.

Let $(\mathcal{C}_{X_1 \amalg \{x_1\}}, b_1) \in \mathsf{el}(\mathcal{X})$ be a neighbourhood of v_1 in $\mathcal{X}$ such that $b_1(x_1) = \tau e$, and let $(\mathcal{C}_{X_2 \amalg \{x_2\}}, b_2) \in \mathsf{el}(\mathcal{X})$ be a neighbourhood of v_2 such that $b_2(x_2) = e$. Then $b_{x_1,x_2}\colon \mathcal{M}^{X_1,X_2}_{x_1,x_2} \to \mathcal{X}$ is a minimal neighbourhood containing v_1, v_2, and $\tilde{e}$ in $\mathcal{X}$.

Hence, the monomorphism $\overline{b}\colon \left(\mathcal{M}^{X_1,X_2}_{x_1,x_2}\right)_{/\tilde{e}} \to \mathcal{X}_{/\tilde{e}}$, obtained by collapsing $\tilde{e}$ in $\mathcal{M}^{X_1,X_2}_{x_1,x_2}$ describes a minimal neighbourhood of $\overline{v}$, and therefore $\left(\mathcal{M}^{X_1,X_2}_{x_1,x_2}\right)_{/\tilde{e}} = \mathcal{C}_{X_1 \amalg X_2} \cong \mathcal{C}_{\overline{\mathbf{v}}}$ by Lemma 4.15.

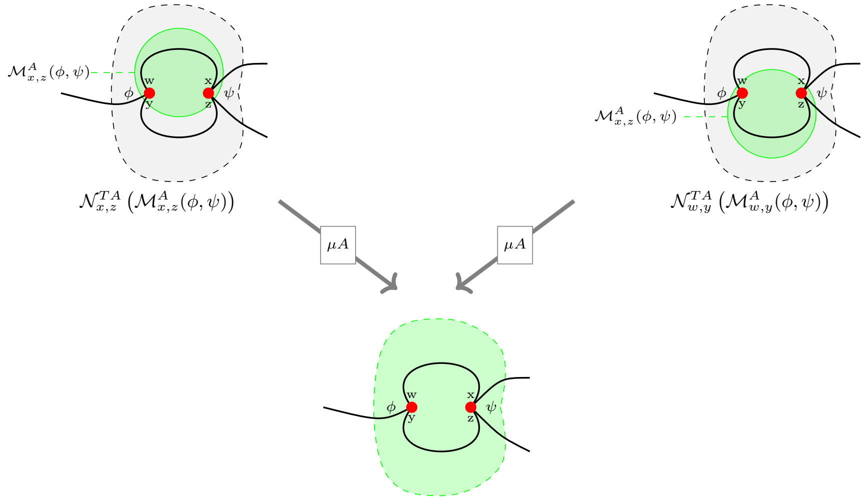


FIGURE 20. Coherence condition (M4).

For $i = 1, 2$, let $\phi_i \overset{\text{def}}{=} S(b_1)(\alpha) \in S(\mathcal{C}_{X_i \amalg \{x_i\}})$. There is an S-structure $\alpha_{/\tilde{e}}$ on $\mathcal{X}_{/\tilde{e}}$ such that, for all $(\mathcal{C}_Z, b') \in \mathsf{el}(\mathcal{X}_{/\tilde{e}})$,

$$S(b')(\alpha_{/\tilde{e}}) = \begin{cases} \phi_1 \diamond_{x_1,x_2} \phi_2 & \text{if } b' = \overline{b} \colon \mathcal{C}_{X_1 \amalg X_2} \to \mathcal{X}_{/\tilde{e}}, \\ S(b)(\alpha) & \text{if } (\mathcal{C}_Z, b') \text{ is given by a neighbourhood } (\mathcal{C}_{X_b}, b) \text{ in } \mathcal{X} \text{ of } v \notin \{v_1, v_2\}. \end{cases}$$

CASE 2: $t(e) = t(\tau e) = v \in V$.
In this case, the graph $\mathcal{X}_{/\tilde{e}}$ obtained from $\mathcal{X}$ by collapsing $\{e, \tau e\}$ has the form

$$\mathcal{X}_{/\tilde{e}} \overset{\text{def}}{=} \quad \tau \circlearrowright (E \setminus \{e, \tau e\}) \xleftarrow{\quad s \quad} (H \setminus s^{-1}\{e, \tau e\}) \xrightarrow{\quad \bar{t} \quad} V \ .$$

If $(\mathcal{C}_{X_v \amalg \{x,y\}}, b) \in \mathsf{el}(\mathcal{X})$ is a neighbourhood of v such that $b(x) = \tau e$ and $b(y) = e$, then the minimal neighbourhood containing v and $\tilde{e}$ in $\mathcal{X}$ is of the form $b_{x,y} \colon \mathcal{N}^{X_v}_{x,y} \to \mathcal{X}$.

The canonical monomorphism $\overline{b} \colon \left(\mathcal{N}^{X_v}_{x,y}\right)_{/\tilde{e}} \to \mathcal{X}_{/\tilde{e}}$ obtained by collapsing $\tilde{e}$ is a minimal neighbourhood of v in $\mathcal{X}_{/\tilde{e}}$. Hence, $\left(\mathcal{N}^{X_v}_{x,y}\right)_{/\tilde{e}} \cong \mathcal{C}_{X_v}$ by Lemma 4.15.

Let $\phi \overset{\text{def}}{=} S(b)(\alpha) \in S_{X_v \amalg \{x,y\}}$. So, there is an S-structure $\alpha_{/\tilde{e}}$ on $\mathcal{X}_{/\tilde{e}}$ defined by

$$S(b')(\alpha_{/\tilde{e}}) = \begin{cases} \zeta_{x,y}(\phi) & \text{if } b' = \overline{b} \colon \mathcal{C}_{X_v} \to \mathcal{X}_{/\tilde{e}}, \\ S(b)(\alpha) & \text{if } (\mathcal{C}_Z, b') \text{ is given by a neighbourhood } (\mathcal{C}_{X_b}, b) \text{ in } \mathcal{X} \text{ of } v' \neq v. \end{cases}$$

It follows that an ordering $(\tilde{e}_1, \dots, \tilde{e}_N)$ of the set $\widetilde{E_\bullet}$ of inner τ -orbits of $\mathcal{X}$ defines a terminating sequence of S-structured X-graphs

$$(\mathcal{X}, \alpha) \mapsto (\mathcal{X}_{/\tilde{e}_1}, \alpha_{/\tilde{e}_1}) \mapsto ((\mathcal{X}_{/\tilde{e}_1})_{/\tilde{e}_2}, (\alpha_{/\tilde{e}_1})_{/\tilde{e}_2}) \mapsto \dots \mapsto (((\mathcal{X}_{/\tilde{e}_1}) \dots)_{/\tilde{e}_N}, (\alpha_{/\tilde{e}_1}) \dots)_{/\tilde{e}_N}).$$

Since $((\mathcal{X}_{/\tilde{e}_1}) \dots)_{/\tilde{e}_N} = \mathcal{C}_X$ has no inner edges, there is a $\phi_{(\mathcal{X},\alpha)} \in S_X$ such that

$$(\alpha_{/\tilde{e}_1} \dots)_{/\tilde{e}_N} = \eta^{\mathbb{T}} S(\phi_{(\mathcal{X},\alpha)}) \in TS_X.$$

The coherence conditions (M1)-(M4) are equivalent to the statement that $\phi_{(\mathcal{X},\alpha)} \in S_X$ so obtained is unchanged if the order of collapse of consecutive pairs $\tilde{e}_j, \tilde{e}_{j+1} \in \widetilde{E_\bullet}$ of inner τ-orbits is switched. In other words, $\phi_{(\mathcal{X},\alpha)}$ is independent of the choice of ordering of $\widetilde{E_\bullet}$. Moreover, it is independent of the choice of representative of $(\mathcal{X}, \alpha)$ by the definition of TS.

To complete the proof of the proposition, it remains to establish that h satisfies the monad algebra axioms for $\mathbb{T}$. Compatibility of h with $\eta^{\mathbb{T}}$ is immediate from equation (5.30). Compatibility of h with $\mu^{\mathbb{T}}$ follows since the coherence conditions (M1)-(M4) ensure that $h[\mathcal{X}, \alpha]$ is independent of the order of collapse of the inner edges of $\mathcal{X}$.

So $(S, \diamond, \zeta)$ defines a $\mathbb{T}$-algebra (S, h), and this assignment extends in the obvious way to a functor $\mathsf{MO}^- \to \mathsf{GS}^{\mathbb{T}}$ that, by construction, is inverse to the functor $\mathsf{GS}^{\mathbb{T}} \to \mathsf{MO}^-$ defined above. □

6. The problem of loops

Before constructing the monad $\mathbb{D}$ that encodes the combinatorics of units in Section 7.1, let us first pause to discuss the obstruction to obtaining a monadic multiplication for the modular operad endofunctor in the construction outlined in [22].

Example 6.1. In Example 5.1, I sketched the idea behind the construction of the operad monad $\mathbb{M}_{\mathsf{Op}}$ on $\mathsf{psh}(\mathbf{B}^{\downarrow})$, whose underlying endofunctor takes a $\mathfrak{D}$-coloured presheaf P to the $\mathfrak{D}$-coloured presheaf of formal operadic compositions in P, encoded as P-decorated rooted trees. However, I did not describe how the units for the operadic composition are obtained.

Unlike T on GS, the definition of M_{Op} on $\mathsf{psh}(\mathbf{B}^{\downarrow})$ allows *degenerate substitution* of the exceptional directed tree $(\downarrow)$ into the vertex of the corolla $t_{\mathbf{1}}$ with one leaf (Figure 21). Since grafting an exceptional directed edge $(\downarrow)$ onto the leaf or root of any tree $\mathfrak{T}$ leaves $\mathfrak{T}$ unchanged (see Figure 7(b)), if (P, θ) is a $\mathfrak{D}$-coloured algebra for $\mathbb{M}_{\mathsf{Op}}$, and hence a $\mathfrak{D}$ coloured operad, then the elements $\theta(\downarrow, d) \in P(t_{\mathbf{1}})$ (with $d \in \mathfrak{D}$) provide units for the operadic composition.

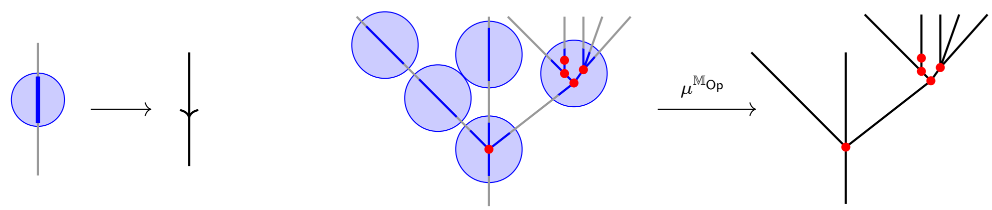


FIGURE 21. The combinatorics of the operadic unit are represented graphically by the degenerate substitution of the exceptional tree into $t_{\mathbf{1}}$. Applying the monad multiplication $\mu^{\mathbb{M}_{\mathsf{Op}}}P$ to nested trees in ${M_{\mathsf{Op}}}^2 P$ *deletes* vertices decorated by elements of $P(\downarrow)$. (See also Figure 13.)

The modular operad endofunctor $T^{\mathrm{ds}} \colon \mathsf{GS} \to \mathsf{GS}$ defined in [22, Section 5], whose algebras are modular operads, is obtained by a slight modification of the non-unital modular operad endofunctor T, to allow degenerate substitutions as in Example 6.1.

If S is a graphical species and X a finite set, then let $X\text{-}\mathsf{Gr}^{\mathrm{ds}}_{\mathsf{iso}}/S$ be the groupoid obtained from $X\text{-}\mathsf{CGr}_{\mathsf{iso}}$ by dropping the condition that X-graphs must have non-empty vertex set. So,

$$X\text{-}\mathsf{Gr}^{\mathrm{ds}}_{\mathsf{iso}} = X\text{-}\mathsf{CGr}_{\mathsf{iso}} \text{ for } X \ncong \mathbf{2} \quad \text{and} \quad \mathbf{2}\text{-}\mathsf{Gr}^{\mathrm{ds}}_{\mathsf{iso}} \cong \mathbf{2}\text{-}\mathsf{CGr}_{\mathsf{iso}} \amalg \{(\shortmid, c)\}_{c \in S_{\S}},$$

(where the boundary of $(\shortmid)$ has the identity $\mathbf{2}$-labelling).

The endofunctor $T^{\mathrm{ds}} \colon \mathsf{GS} \to \mathsf{GS}$ is defined pointwise by

$$\begin{aligned} T^{\mathrm{ds}} S_{\S} &\overset{\mathrm{def}}{=} S_{\S}, \\ T^{\mathrm{ds}} S_X &\overset{\mathrm{def}}{=} \operatorname{colim}_{\mathcal{X} \in X\text{-}\mathsf{Gr}^{\mathrm{ds}}_{\mathsf{iso}}} S(\mathcal{X}) \quad \text{ for all finite sets } X, \end{aligned}$$

together with the obvious extension of T on morphisms in $\mathbf{B}^{\S}$.

Clearly $\eta^{\mathbb{T}}$, together with the inclusion $TS \subset T^{\mathrm{ds}} S$ (for all S), induces a unit η^{ds} for T^{ds}.

Proposition 6.2. *Algebras for the pointed endofunctor $(T^{\mathrm{ds}}, \eta^{\mathrm{ds}})$ on GS are modular operads.*

Proof. Since $T \subset T^{\mathrm{ds}}$, algebras for T^{ds} have the structure of unpointed modular operads by Proposition 5.29. If (A, h) is an algebra for $(T^{\mathrm{ds}}, \eta^{\mathrm{ds}})$ (see Definition 2.13), then for each $c \in A_{\S}$, $h(\shortmid, c) \in A_{\mathbf{2}}$ provides a c-coloured unit for the induced multiplication. $\square$

For all graphical species S, an element of ${T^{\mathrm{ds}}}^2 S_X$ is represented by an X-graph $\mathcal{X}$ and a (possibly degenerate) $\mathcal{X}$-shaped graph of S-structured connected graphs $\mathbf{\Gamma}_S \colon \mathsf{el}(\mathcal{X}) \to \mathsf{CetGr}/S$.

In particular, if T^{ds} admits a monad multiplication $\mu^{\mathrm{ds}} \colon {T^{\mathrm{ds}}}^2 \Rightarrow T^{\mathrm{ds}}$, then, for all graphical species S, the restriction of $\mu^{\mathrm{ds}} S$ to $T^2 S$ is given by $\mu^{\mathbb{T}} S$. But this creates problems, as the following example shows:

Example 6.3. As usual, let $\mathcal{W}$ be the wheel graph with one vertex v and edges $\{a, \tau a\}$. Its category of elements $\mathsf{el}(\mathcal{W})$ has skeletal subcategory

$$\begin{tikzcd} (\shortmid) \arrow[rr, shift left, "ch_1^{\mathcal{C}_\mathbf{2}}"] \arrow[rr, shift right, "ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau"'] \arrow[dr, "ch_a"'] & & \mathcal{C}_\mathbf{2} \arrow[dl, "(1_{\mathcal{C}_\mathbf{2}} \mapsto a)"] \\ & \mathcal{W} & \end{tikzcd} \tag{6.4}$$

So, if S is a $(\mathfrak{C}, \omega)$-coloured graphical species and $c \in \mathfrak{C}$, then there is a (degenerate) $\mathcal{W}$-shaped graph of S-structured graphs $\mathbf{\Lambda}_{S,c}$ given by

$$\begin{tikzcd} \mathsf{el}(\mathcal{W}) \arrow[d, "\mathbf{\Lambda}_{S,c}"'] & (\shortmid) \arrow[r, shift left, "ch_1^{\mathcal{C}_\mathbf{2}}"] \arrow[r, shift right, "ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau"'] \arrow[d, mapsto, "\mathbf{\Lambda}_{S,c}(ch_a)"'] & \mathcal{C}_\mathbf{2} \arrow[d, mapsto, "\mathbf{\Lambda}_{S,c}(1_{\mathcal{C}_\mathbf{2}} \mapsto a)"] \\ \mathsf{CetGr}/S & (\shortmid, c) \arrow[r, shift left, "\mathbf{\Lambda}_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}})"] \arrow[r, shift right, "\mathbf{\Lambda}_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau)"'] & (\shortmid, c). \end{tikzcd} \tag{6.5}$$

In particular, $\mathbf{\Lambda}_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}}) = id_{(\shortmid, c)} = \mathbf{\Lambda}_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau)$.

Hence $\mathbf{\Lambda}_{S,c}$ has colimit $id_c \colon (\shortmid, c) \to (\shortmid, c)$ in CetGr/S.

In the first place, this a little surprising since $E_0(\shortmid) \neq E_0(\mathcal{W})$ so Corollary 5.18 does not hold for $\mathbf{\Lambda}_{S,c}$.

Moreover, $\mathcal{W}$ admits a unique non-trivial – but trivially boundary fixing – automorphism $\tau_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{W}$. So, if $\mathbf{\Lambda}_{S,\omega c}$ is the $\mathcal{W}$-shaped graph of S-structured graphs

$$(\shortmid, ch_a \mapsto (\shortmid, \omega c)) \text{ and } (\mathcal{C}_\mathbf{2}, (1_{\mathcal{C}_\mathbf{2}} \mapsto a)) \longmapsto (\shortmid, \omega c),$$

then $\mathbf{\Lambda}_{S,c}$ and $\mathbf{\Lambda}_{S,\omega c}$ represent the same element of ${T^{\mathrm{ds}}}^2 S_{\mathbf{0}}$ since

$$\mathbf{\Lambda}_{S,\omega c}(\mathcal{C}, b) = \mathbf{\Lambda}_{S,c}(\mathcal{C}, \tau_{\mathcal{W}} \circ b) \text{ for all } (\mathcal{C}, b) \in \mathsf{el}(\mathcal{W}), \tag{6.6}$$

and therefore $\tau_{\mathcal{W}} \in \mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{iso}}(\mathcal{W}, \mathcal{W})$.

But $\mathbf{\Lambda}_{S,c}$ has colimit $(\shortmid, c)$ in CetGr/S while $\mathbf{\Lambda}_{S,\omega c}$ has colimit $(\shortmid, \omega c) \in \mathsf{CetGr}/S$, and these are distinct if $c \neq \omega c$.

So X-isomorphisms do not respect colimits of graphs of graphs on the nose.

As Example 6.3 shows, taking colimits in $X\text{-}\mathsf{Gr}^{\mathrm{ds}}_{\mathsf{iso}}$ of degenerate graphs of S-structured graphs does not always lead to a well-defined class of S-structured graphs, let alone one in the correct arity.

The issue is that the coequaliser in $\mathsf{psh_f}(\mathsf{D})$ of the parallel morphisms $id_{(\shortmid)}, \tau\colon (\shortmid) \rightrightarrows (\shortmid)$ is the exceptional loop $\bigcirc$, which is not a graph (Example 3.16). Therefore, an obvious first attempt at resolving this problem in order to extend $\mu^{\mathbb{T}} S$ to a well-defined morphism $\mu^{\mathrm{ds}} S\colon T^{\mathrm{ds}^2} S \to T^{\mathrm{ds}} S$, is to enlarge CetGr to include $\bigcirc$, and a morphism $(\shortmid) \to \bigcirc$. (See Figure 2.)

Remark 6.7. In the formalism of [16, 7] described in Example 3.1 (as well as in, for example [18, 32, 42]), where graph ports are defined to be the fixed points of edge involution, the graph substitution is not defined in terms of a functorial construction, but by 'removing neighbourhoods of vertices and gluing in graphs'. Therefore, the exceptional loop arises from substitution as in Figure 22.

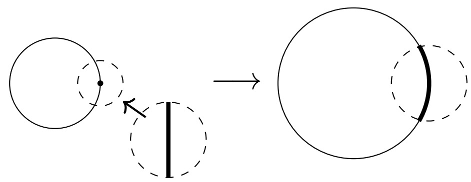


FIGURE 22. Constructing an exceptional loop by removing a vertex and substituting the stick graph.

Since the goal of this work is to unpick and understand this issue of loops, let us dig a little deeper.

Let $\mathsf{CetGr}^{\bigcirc}$ be the category of *fully generalised Feynman graphs* and étale morphisms, obtained from CetGr by adding the object $\bigcirc$ and a unique morphism $(\shortmid) \to \bigcirc$. In other words, objects of $\mathsf{CetGr}^{\bigcirc}$ are $\mathsf{Set_f}$-diagrams $\tau \circlearrowright E \xleftarrow{s} H \xrightarrow{t} V$, such that $s\colon H \to E$ is injective, and the involution τ on E is allowed to have fixed points in $E_0 = E \setminus im(s)$ but not in $im(s)$.

By definition, $\bigcirc$ is the coequaliser of $id, \tau\colon (\shortmid) \rightrightarrows (\shortmid)$, so we can define the category of elements $\mathsf{el}(\bigcirc) \stackrel{\mathrm{def}}{=} \mathbf{B}^{\S}/\bigcirc$, and thereby extend any graphical species $S\colon \mathbf{B}^{\S^{\mathrm{op}}} \to \mathsf{Set}$ to a presheaf on $\mathsf{CetGr}^{\bigcirc}$ according to $\mathcal{G} \longmapsto \lim_{\mathsf{el}(\mathcal{G})} S$. But then $\mathsf{el}(\bigcirc) \cong \mathsf{el}(\shortmid)$, and hence $S(\bigcirc) \cong S(\shortmid)$ for all graphical species S.

It follows that $\mathsf{CetGr}^{\bigcirc}$ does not embed densely or fully in GS. (See also [18, 19] for a discussion of some of these issues.) In particular, there is no monad $\mathbb{M}$ on GS with arities $\mathsf{CetGr}^{\bigcirc}$ (see Section 2).

Let $T^{\bigcirc}\colon \mathsf{GS} \to \mathsf{GS}$ be the endofunctor

$$
\begin{aligned}
T^{\bigcirc} S_{\S} &\stackrel{\mathrm{def}}{=} S_{\S},\\
T^{\bigcirc} S_{X} &\stackrel{\mathrm{def}}{=} \mathrm{colim}_{(\mathcal{G},\rho) \in X\text{-}\mathsf{Gr}^{\bigcirc}_{\mathsf{iso}}} S(\mathcal{G}).
\end{aligned}
$$

So, the graph of graph functors $\mathbf{\Lambda}_{S,c}$ and $\mathbf{\Lambda}_{S,\omega c}$ described in Example 6.3 represent the same element $[\mathcal{W}, \beta] \in T^{\bigcirc^2} S_{\mathbf{0}}$. But, since $S(\bigcirc) \cong S(\shortmid) = S_{\S}$,

$$[\bigcirc, c] \neq [\bigcirc, \omega c] \in T^{\bigcirc} S_{\mathbf{0}} \text{ whenever } c \neq \omega c \in S_{\S}$$

It follows that $\mu^{\mathbb{T}}$ cannot be extended to a multiplication $\mu^{\bigcirc}\colon T^{\bigcirc^2} \Rightarrow T^{\bigcirc}$.

Indeed, this is not surprising: for each $c \in S_{\S}$, the contraction of $\phi \in S_{c,\omega c} \subset S_{\mathbf{2}}$ factors through the quotient $\widetilde{S_{\mathbf{2}}}$ of $S_{\mathbf{2}}$ under the action of $Aut(\mathbf{2})$. Hence, $\zeta(\phi)$ loses data relative to ϕ. The map $(\shortmid) \to \bigcirc$ in $\mathsf{CetGr}^{\bigcirc}$ would seem to be *in the wrong direction*!

The combinatorics of contracted units are examined more closely in the next section. The *problem of loops* discussed in this section will be resolved by adjoining a map that acts as a formal *equaliser*, rather than a coequaliser, of $id, \tau\colon (\shortmid) \rightrightarrows (\shortmid)$ (see Figures 2 and 25).

Remark 6.8. Hackney, Robertson and Yau [21, Definition 1.1] are able to construct the modular operad monad on GS, within the framework of Feynman graphs, by including extra *boundary* data in their definition of graphs. For them, a graph is a pair $(\mathcal{G}, \eth(\mathcal{G}))$ of a Feynman graph $\mathcal{G}$, and subset $\eth(\mathcal{G}) \subset E_0$ of ports, that satisfies certain conditions. So, in their formalism, $(\shortmid, \mathbf{2})$ and $(\shortmid, \mathbf{0})$ are different graphs, and $(\shortmid, \mathbf{0})$ plays the the role of the exceptional loop $\bigcirc$.

However, this approach does not result in a suitable dense functor from the graph category to GS. Therefore, though they construct a fully faithful nerve for modular operads in terms of a dense subcategory $U \hookrightarrow \mathsf{MO}$ of graphs, the inclusion is not fully faithful, and so U does not fully describe the graphical combinatorics of modular operads.

Remark 6.9. To my knowledge, the construction that I present in Section 7 is unique among graphical descriptions of unital modular operads (or wheeled prop(erad)s), in that all others include some version of the exceptional loop as a graph. (See e.g. [32, 33, 18, 42, 2].)

7. Modular operads with unit

Proposition 5.29 identifies the category of non-unital modular operads with the EM category of algebras for the monad $\mathbb{T}$ on GS. The goal of this section is to modify this in order to obtain (unital) modular operads. Some potential obstacles have been discussed in Section 6, where it was also explained why the 'obvious' modification of the operad monad (Examples 6.1 and 5.1) does not work for unital modular operads.

This section begins by returning to the definition of modular operads in Definition 1.24 and looking in more detail at the combinatorics of (contracted) units. This combinatorial information can be encoded in a monad $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ on GS.

Once $\mathbb{D}$ is defined, it is a small step to obtaining the distributive law $\lambda\colon TD \Rightarrow DT$, whose construction provides us with an explicit description of the modular operad monad $\mathbb{DT}$ in terms of equivalence classes of graphs structured by graphical species. Moreover, as discussed in Section 7.5, the construction of $\mathbb{DT}$ is such that it is always possible to work with nice (non-degenerate and with well-behaved colimits) representatives of these classes, thereby avoiding the problem of loops (Section 6).

7.1. **Pointed graphical species.** By definition, if $(S, \diamond, \zeta, \epsilon)$ is a modular operad, then the unit $\epsilon\colon S_\S \to S_\mathbf{2}$ is an injective map such that

$$\epsilon \circ S_\tau = S(\sigma_\mathbf{2}) \circ \epsilon. \tag{7.1}$$

The key point is that the combination of a unit and a contraction implies that – as well as the structure in arity $\mathbf{2}$ provided by $\epsilon\colon S_\S \to S_\mathbf{2}$ – modular operads are equipped with structure in arity $\mathbf{0}$. Namely, as in (1.23), there is a well-defined contracted unit map

$$o = \zeta \circ \epsilon\colon S_\S \to S_\mathbf{0} \tag{7.2}$$

is well-defined and invariant under $S_\tau\colon S_\S \to S_\S$.

Definition 7.3. *Objects of the category* GS_* *of* pointed graphical species *are triples* $S_* = (S, \epsilon, o)$ *(or* (S, ϵ^S, o^S)*) where* S *is a graphical species and* $\epsilon\colon S_\S \to S_\mathbf{2}$*, and* $o\colon S_\S \to S_\mathbf{0}$ *are maps satisfying conditions (7.1) and (7.2) above. Morphisms in* GS_* *are morphisms in* GS *that preserve the additional structure.*

Example 7.4. For any palette $(\mathfrak{C}, \omega)$, the terminal $(\mathfrak{C}, \omega)$-coloured graphical species $K^{(\mathfrak{C}, \omega)}$ is trivially pointed and hence terminal in the category of $(\mathfrak{C}, \omega)$-*coloured pointed graphical species* and palette-preserving morphisms.

The category GS_* is also a presheaf category: Namely, we can obtain the category $\mathbf{B}^\S_*$ from $\mathbf{B}^\S$ by formally adjoining morphisms $u\colon \mathbf{2} \to \S$ and $z\colon \mathbf{0} \to \S$, subject to the relations

- $u \circ ch_1 = id \in \mathbf{B}^{\S}(\S, \S)$ and $u \circ ch_2 = \tau \in \mathbf{B}^{\S}(\S, \S)$,
- $\tau \circ u = u \circ \sigma_{\mathbf{2}} \in \mathbf{B}^{\S}(\mathbf{2}, \S)$, and
- $z = \tau \circ z \in \mathbf{B}^{\S}(\mathbf{0}, \S)$.

Lemma 7.5. *The following are equivalent:*

(1) S_* is a presheaf on $\mathbf{B}^{\S}_*$ that restricts to a graphical species S on $\mathbf{B}^{\S}$;
(2) (S, ϵ, o), with $\epsilon = S_*(u)$ and $o = S_*(z)$ is a pointed graphical species.

Proof. It is easy to check directly that $\mathbf{B}^{\S}_*$ is completely described by

- $\mathbf{B}^{\S}_*(\S, \S) = \mathbf{B}^{\S}(\S, \S)$ and $\mathbf{B}^{\S}_*(Y, X) = \mathbf{B}^{\S}(Y, X)$ whenever $Y \not\cong \mathbf{0}$ and $Y \not\cong \mathbf{2}$,
- $\mathbf{B}^{\S}_*(\mathbf{0}, \S) = \{z\}$, and $\mathbf{B}^{\S}_*(\mathbf{0}, X) = \mathbf{B}^{\S}(\mathbf{0}, X) \amalg \{ch_x \circ z\}_{x \in X}$,
- $\mathbf{B}^{\S}_*(\mathbf{2}, \S) = \{u, \tau \circ u\}$, and $\mathbf{B}^{\S}_*(\mathbf{2}, X) = \mathbf{B}^{\S}(\mathbf{0}, X) \amalg \{ch_x \circ u, ch_x \circ \tau \circ u\}_{x \in X}$,

and the lemma follows immediately. □

As a consequence of Lemma 7.5, the notation S_* and (S, ϵ, o) will be used interchangeably to denote the same pointed graphical species. To reduce clutter, the category of elements of a pointed graphical species S_* will be denoted by $\mathsf{el}_*(S_*) \overset{\text{def}}{=} \mathsf{el}_{\mathbf{B}^{\S}_*}(S_*)$.

Lemma 7.6. *The forgetful functor $\mathsf{GS}_* \to \mathsf{GS}$ is strictly monadic: it has a left adjoint $\mathsf{GS} \to \mathsf{GS}_*$, and GS_* is the EM category of algebras for the monad $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ on GS induced by the adjunction.*

Proof. The left adjoint $(\cdot)^+$ to the forgetful functor $\mathsf{GS}_* \to \mathsf{GS}$ takes a graphical species S to its left Kan extension S^+ along the inclusion $(\mathbf{B}^{\S})^{\mathrm{op}} \hookrightarrow (\mathbf{B}^{\S}_*)^{\mathrm{op}}$. This does nothing more than formally adjoin elements $\{\epsilon^+_c\}_{c \in S_\S}$ to $S_{\mathbf{2}}$ and $\{o^+_{\tilde{c}}\}_{\tilde{c} \in \widetilde{S_\S}}$ to $S_{\mathbf{0}}$ according to the combinatorics of contracted units. So S^+ is described by (DS, ϵ^+, o^+) where $DS_{\mathbf{2}} = S_{\mathbf{2}} \amalg \{\epsilon^+_c\}_{c \in S_\S}$, $DS_{\mathbf{0}} = S_{\mathbf{0}} \amalg \{o^+_{\tilde{c}}\}_{\tilde{c} \in \widetilde{S_\S}}$, and $DS_X = S_X$ for $X \not\cong \mathbf{2}, X \not\cong \mathbf{0}$. By definition, the canonical pointed structure on DS is given by $(DS, \epsilon^{DS}, o^{DS}) = (DS, \epsilon^+, o^+)$.

The monadic unit $\eta^{\mathbb{D}}$ is provided by the inclusion $S \hookrightarrow DS$, and the multiplication $\mu^{\mathbb{D}}$ is induced by the canonical projections $D^2 S_{\mathbf{2}} \to DS_{\mathbf{2}}$. □

7.2. **Pointed graphs.** Let CetGr_* be the category obtained in the bo-ff factorisation of $(\Upsilon -)^+ \colon \mathsf{CetGr} \hookrightarrow \mathsf{GS} \to \mathsf{GS}_*$, making the following diagram commute:

(7.7)
$$\begin{tikzcd}
\mathbf{B}^{\S}_* \arrow[r, "\text{dense}", "\text{f.f.}"'] \arrow[rr, bend left, "\text{f.f.}"] & \mathsf{CetGr}_* \arrow[r, "\Upsilon_*", "\text{f.f.}"'] & \mathsf{GS}_* \arrow[d, shift left, "\text{forget}"] \\
\mathbf{B}^{\S} \arrow[u, "\text{b.o.}"] \arrow[r, hook] & \mathsf{CetGr} \arrow[u, "\text{b.o.}"] \arrow[r, hook, "\Upsilon"'] & \mathsf{GS}. \arrow[u, shift left, "(\cdot)^+"]
\end{tikzcd}$$

The inclusion $\mathbf{B}^{\S}_* \to \mathsf{CetGr}_*$ is fully faithful (by uniqueness of bo-ff factorisation), and also dense, since the induced nerve $\Upsilon_* \colon \mathsf{CetGr}_* \to \mathsf{GS}_*$ is fully faithful by construction.

Let $\mathcal{G} \in \mathsf{CetGr}$ be a graph. By Lemma 7.5, for each edge $e \in E$, the ch_e-coloured unit for $\Upsilon_* \mathcal{G}$ is defined, as $\epsilon^{\mathcal{G}}_e = ch_e \circ u \in \mathsf{CetGr}_*(\mathcal{C}_{\mathbf{2}}, \mathcal{G})$, and the corresponding contracted unit is given by $o^{\mathcal{G}}_{\tilde{e}} = ch_e \circ z \in \mathsf{CetGr}_*(\mathcal{C}_{\mathbf{0}}, \mathcal{G})$.

Since the functor Υ_* embeds CetGr_* as a full subcategory of GS_*, I will write $\mathcal{G}$, rather than $\Upsilon_* \mathcal{G}$ where there is no risk of confusion. In particular, the element category $\mathsf{el}_*(\Upsilon_* \mathcal{G})$ is denoted simply by $\mathsf{el}_*(\mathcal{G})$ and called the *category of pointed elements of a graph $\mathcal{G}$*.

The forgetful functor $\mathsf{GS}_* \to \mathsf{GS}$ induces injective-on-objects inclusions $\mathsf{el}(S) \to \mathsf{el}_*(S_*)$ for all pointed graphical species S_*. In particular $\mathsf{el}(\mathcal{G}) \hookrightarrow \mathsf{el}_*(\mathcal{G})$.

Recall [30, Section IX.3] that a functor $\Psi\colon \mathsf{C} \to \mathsf{D}$ is called *final* if the slice category $d/\Psi \overset{\text{def}}{=} \Psi^{\text{op}}/d$ is non-empty and connected for all $d \in \mathsf{D}$, and that $\Psi\colon \mathsf{C} \to \mathsf{D}$ is final if and only if for any functor $\Phi\colon \mathsf{D} \to \mathsf{E}$ such that $\mathrm{colim}_{\mathsf{C}}(\Phi \circ \Psi)$ exists in E, $\mathrm{colim}_{\mathsf{D}}\Phi$ also exists in E and the two colimits agree.

Lemma 7.8. *For all graphs $\mathcal{G}$, the inclusion $\mathsf{el}(\mathcal{G}) \hookrightarrow \mathsf{el}_*(\mathcal{G})$ is final. Therefore, $\mathcal{G}$ is the colimit of the forgetful functor $\mathsf{el}_*(\mathcal{G}) \to \mathsf{CetGr}_*$, $(\mathcal{C}, b) \to \mathcal{C}$. And, for all pointed graphical species $S_* = (S, \epsilon, o)$,*

$$\lim_{(\mathcal{C}',b')\in \mathsf{el}_*(\mathcal{G})} S_*(\mathcal{C}') = \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} S(\mathcal{C}) = S(\mathcal{G}).$$

Proof. By definition, $\mathsf{el}_*(\mathcal{G})$ is obtained from $\mathsf{el}(\mathcal{G})$ by adjoining, for each $e \in E$, the objects $(\mathbf{2}, ch_e \circ u)$ and $(\mathbf{0}, ch_e \circ u) = (\mathbf{0}, ch_{\tau e} \circ u)$. Hence, for all $(\mathcal{C}, b) \in \mathsf{el}_*(\mathcal{G})$, the slice category $b/\mathsf{el}(\mathcal{G})$ is connected and non-empty. $\square$

By Lemma 7.8, a morphism $f \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}')$ is described by a functor $\mathsf{el}(\mathcal{G}) \to \mathsf{el}_*(\mathcal{G}')$ such that for each $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, $(\mathcal{C}, b) \mapsto (\mathcal{C}, f \circ b)$, and there is a commuting diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "g"] \arrow[dr, "f\circ b"'] & & \mathcal{C}' \arrow[dl, "b'"] \\
& \mathcal{G}' &
\end{tikzcd}$$

such that $g \in \mathbf{B}_*^{\S}(\mathcal{C}, \mathcal{C}')$ and $b' \in \mathsf{CetGr}(\mathcal{C}', \mathcal{G}')$, so $(\mathcal{C}', b') \in \mathsf{el}(\mathcal{G}')$ is an (unpointed) element of $\mathcal{G}'$.

Example 7.9. (Compare Example 6.3.) A surprising consequence of the definitions is that the morphism set $\mathsf{CetGr}_*(\mathcal{W}, \shortmid)$ is non-empty. There are two morphisms $\kappa, \tau \circ \kappa \in \mathsf{CetGr}_*(\mathcal{W}, \shortmid)$:

(7.10)
$$\begin{tikzcd}
& \mathcal{W} & \\
(\shortmid) \arrow[ur, "ch_a"] \arrow[rr, shift left, "ch_1"] \arrow[rr, shift right, "ch_2\circ\tau"'] \arrow[dr, equal] & & \mathcal{C}_\mathbf{2} \arrow[ul, "1_{\mathcal{C}_\mathbf{2}}\mapsto a"'] \arrow[dl, "u"] \\
& (\shortmid) &
\end{tikzcd}$$

(7.11)
$$\begin{tikzcd}
& \mathcal{W} & \\
(\shortmid) \arrow[ur, "ch_a"] \arrow[rr, shift left, "ch_1"] \arrow[rr, shift right, "ch_2\circ\tau"'] \arrow[d, equal] & & \mathcal{C}_\mathbf{2} \arrow[ul, "1_{\mathcal{C}_\mathbf{2}}\mapsto a"'] \arrow[d, "\sigma_\mathbf{2}"] \\
(\shortmid) \arrow[rr, shift left, "ch_2"] \arrow[rr, shift right, "ch_1\circ\tau"'] \arrow[dr, equal] & & \mathcal{C}_\mathbf{2} \arrow[dl, "u"] \\
& (\shortmid). &
\end{tikzcd}$$

Hence, $\mathsf{CetGr}_*(\mathcal{W}, \shortmid) \cong \mathsf{CetGr}_*(\shortmid, \shortmid) \cong \mathsf{CetGr}_*(\mathcal{W}, \mathcal{W})$. In particular, for all graphs $\mathcal{G} \ncong \mathcal{W}$,

$$\mathsf{CetGr}_*(\mathcal{W}, \mathcal{G}) \cong E(\mathcal{G}) \ \text{ by } \ ch_e \circ \kappa \mapsto e.$$

These morphisms play a crucial role in the proof of the nerve theorem, Theorem 8.2.

Now, let $W \subset V_2$ be a subset of bivalent vertices of a connected graph $\mathcal{G}$.

Definition 7.12. *A* vertex deletion functor (for W) *is a $\mathcal{G}$-shaped graph of graphs $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}\colon \mathsf{el}(\mathcal{G}) \to \mathsf{CetGr}_*$ such that for $(\mathcal{C}_X, b) \in \mathsf{el}(\mathcal{G})$,*

$$\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}(b) = \begin{cases} (\shortmid) & \text{if } (\mathcal{C}_X, b) \text{ is a neighbourhood of } v \in W, \\ \mathcal{C}_X & \text{otherwise.} \end{cases}$$

If $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ admits a colimit $\mathcal{G}_{\setminus W}$ in CetGr_*, then the induced morphism $\mathsf{del}_{\setminus W} \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ is called the vertex deletion morphism corresponding to W.

Note, in particular, that a vertex deletion functor $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ is non-degenerate if and only if $W = \emptyset$ in which case $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ is the identity graph of graphs $\mathbf{I}^{\mathcal{G}} \colon (\mathcal{C}, b) \mapsto \mathcal{C}$ (Section 5.2).

It follows immediately that, if $W = W_1 \amalg W_2$ and $\mathsf{del}_{\setminus W} = \mathsf{del}^{\mathcal{G}}_{\setminus W} \colon \mathcal{G} \to \mathcal{G}_{\setminus W}$ exists in CetGr_*, then

$$\mathsf{del}^{\mathcal{G}}_{\setminus W_1} \colon \mathcal{G} \to \mathcal{G}_{\setminus W_1} \quad \text{and} \quad \mathsf{del}^{\mathcal{G}_{\setminus W_1}}_{\setminus W_2} \colon \mathcal{G}^{\mathcal{G}}_{\setminus W_1} \to (\mathcal{G}_{\setminus W_1})_{\setminus W_2} = \mathcal{G}^{\mathcal{G}}_{\setminus W}$$

also exist in CetGr_* and $\mathsf{del}_{\setminus W} = \mathsf{del}^{\mathcal{G}_{\setminus W_1}}_{\setminus W_2} \circ \mathsf{del}^{\mathcal{G}}_{\setminus W_1}$.

Example 7.13. For $\mathcal{G} = \mathcal{C}_{\mathbf{2}}$ and $W = V = \{*\}$, $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ is the constant functor induced by the cocone of $\mathsf{el}(\mathcal{C}_{\mathbf{2}})$ over $(\shortmid)$ in CetGr_*:

(7.14)
$$\begin{tikzcd} \mathsf{el}_*(\mathcal{C}_{\mathbf{2}}) \arrow[d] & \simeq & (\shortmid) \arrow[r, "ch_1"] \arrow[dr, "id_{(\shortmid)}"'] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & (\shortmid) \arrow[l, "ch_2 \circ \tau"'] \arrow[dl, "id_{(\shortmid)}"] \\ \mathsf{el}(\shortmid) & & & (\shortmid) & \end{tikzcd}.$$

So, trivially, $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ has colimit $(\shortmid)$ in CetGr_* and $\mathsf{del}_{\setminus W} = u \in \mathsf{CetGr}_*(\mathcal{C}_{\mathbf{2}}, \shortmid)$.

More generally, if $\mathcal{G} = \mathcal{L}^k$, and $W = V$, then $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ is also the constant functor to $\mathcal{G}_{\setminus W} = (\shortmid)$, and $u^k \stackrel{\text{def}}{=} \mathsf{del}_{\setminus W} \colon \mathcal{L}^k \to \mathsf{CetGr}_*$ is induced by the CetGr_*-cocone under $\mathsf{el}_*(\mathcal{L}^k)$:

$$\begin{tikzcd} \mathsf{el}_*(\mathcal{L}^k) \arrow[d] & \simeq & (\shortmid) \arrow[r, "ch_1"] \arrow[dr, "id_{(\shortmid)}"'] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & \arrow[l, "ch_2 \circ \tau"'] & \dots \arrow[r, "ch_1"] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & (\shortmid) \arrow[l, "ch_2 \circ \tau"'] \arrow[dl, "id_{(\shortmid)}"] \\ \mathsf{el}(\shortmid) & & & (\shortmid) \arrow[r, equal] & & \dots \arrow[r, equal] & (\shortmid). & \end{tikzcd}$$

(So $u^1 = u \colon \mathcal{C}_{\mathbf{2}} \to (\shortmid)$ and u^0 is just the identity on $(\shortmid)$.)

For any graph $\mathcal{G}$, a pointwise étale injection $\iota \in \mathsf{CetGr}(\mathcal{L}^k, \mathcal{G})$ includes $W = V(\mathcal{L}^k)$ as a subset $W \subset V_2(\mathcal{G})$ of bivalent vertices of $\mathcal{G}$. Hence, $\mathsf{del}_{\setminus W} \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ exists in CetGr_* and there is a commuting diagram

(7.15)
$$\begin{tikzcd} \mathcal{L}^k \arrow[r, "\iota"] \arrow[d, "u^k"'] & \mathcal{G} \arrow[d, "\mathsf{del}_{\setminus W}"] \\ (\shortmid) \arrow[r, "ch_{e_{\setminus W}}"'] & \mathcal{G}_{\setminus W}, \end{tikzcd}$$

where $e_{\setminus W} = \mathsf{del}_{\setminus W}(\iota(1_{\mathcal{L}^k}))$.

Example 7.16. By Example 7.9, $\mathcal{W}_{\setminus \{v\}} = \operatorname{colim}_{\mathsf{el}(\mathcal{W})} \mathbf{\Lambda}^{\mathcal{W}}_{\setminus \{v\}}$ exists and is isomorphic to $(\shortmid)$ in CetGr. (See also Section 6.) The induced morphism $\mathsf{del}_{\setminus \{v\}}$ is precisely $\kappa \colon \mathcal{W} \to (\shortmid)$.

Let $(v_i)_{i=1}^m$ be the (cyclically) ordered vertices of the wheel graph $\mathcal{W}^m$, and let $\iota \in \mathsf{CetGr}(\mathcal{L}^{m-1}, \mathcal{W}^m)$ be a pointwise étale inclusion that respects the (cyclic) ordering of vertices. If W is the image of $V(\mathcal{L}^{m-1})$ in $V(\mathcal{W}^m)$, then $V(\mathcal{W}^m) \cong W \amalg \{*\}$, and by (7.15), ι induces a vertex deletion morphism $\mathsf{del}_{\setminus W} \in \mathsf{CetGr}_*(\mathcal{W}^m, \mathcal{W})$. Therefore $\kappa^m \stackrel{\text{def}}{=} \mathsf{del}_{\setminus V(\mathcal{W}^m)}$ is given by the composite $\kappa^m = \kappa \circ \mathsf{del}_{\setminus W} \colon \mathcal{W}^m \to \mathcal{W} \to (\shortmid)$.

In particular, for all $m \geq 1$, there are precisely two distinct morphisms $\mathcal{W}^m \to (\shortmid)$ in CetGr_*: κ^m and $\tau \circ \kappa^m$. Hence, for all $\mathcal{G}$,

$$\mathsf{CetGr}_*(\mathcal{W}^m, \mathcal{G}) = \mathsf{CetGr}(\mathcal{W}^m, \mathcal{G}) \amalg \{ch_e \circ \kappa^m\}_{e \in E(\mathcal{G})}.$$

Proposition 7.17. *For all graphs $\mathcal{G}$ and all $W \subset V_2$, the colimit $\mathcal{G}_{\setminus W}$ of $\mathbf{\Lambda}^{\mathcal{G}}_{\setminus W}$ exists in CetGr_*.*

Moreover, $E_0(\mathcal{G}) = E_0(\mathcal{G}_{\setminus W})$ unless $\mathcal{G} = \mathcal{W}^m$ $(m \geq 1)$ and $W = V$.

Proof. If W is empty, then $\mathcal{G}_{\setminus W} = \mathcal{G}$ and $\mathsf{del}_{\setminus W}$ is the identity on $\mathcal{G}$. On the other hand, if $W = V$ then, by Proposition 4.23, $\mathcal{G} = \mathcal{L}^k$ or $\mathcal{G} = \mathcal{W}^m$ for some $k \geq 0$, $m \geq 1$, and hence it follows from Examples 7.13 and 7.16 that $\mathcal{G}_{\setminus W} = (\shortmid)$ in CetGr_*. For $\mathcal{G} = \mathcal{L}^k$, the vertex deletion morphism $u^k \colon \mathcal{L}^k \to (\shortmid)$ induces a bijection on boundaries. So the proposition is proved when $W = V$ or $W = \emptyset$.

Assume therefore, that $\emptyset \neq W \subsetneq V$ is a proper, non-empty subset of (bivalent) vertices of $\mathcal{G}$.

Let $\mathcal{G}^W \subset \mathcal{G}$ be the subgraph with vertices $V(\mathcal{G}^W) = W$, half-edges $H(\mathcal{G}^W) = \coprod_{v \in W} {}^{H}\!/v$ and whose edge set $E(\mathcal{G}^W)$ is the τ-closure of $\coprod_{v \in W} {}^{E}\!/v$. (See Figure 23.)

By Proposition 4.23, $\mathcal{G}^W = \coprod_{i=1}^m \mathcal{L}^{k_i}$ is a disjoint union of line graphs, where $k_i \geq 1$ for all i. So,

$$E_0(\mathcal{G}^W) = \coprod_{i=1}^{m} \{1_{\mathcal{L}^{k_i}}, 2_{\mathcal{L}^{k_i}}\}, \quad \text{and} \quad \left(\coprod_{i=1}^{m} \{1_{\mathcal{L}^{k_i}}\}\right) \cap \left(\coprod_{i=1}^{m} \{2_{\mathcal{L}^{k_i}}\}\right) = \emptyset \ \text{ in } E(\mathcal{G}). \tag{7.18}$$

For $1 \leq i \leq m$, let $e_i \in E(\mathcal{G})$ be the edge corresponding to $1_{\mathcal{L}^{k_i}}$.

The graph $\mathcal{G}_{\setminus W}$ is obtained by applying $u^{k_i} \colon \mathcal{L}^{k_i} \to (\shortmid)$ on each component $\mathcal{L}^{k_i}$ of $\mathcal{G}^W \subset \mathcal{G}$. Since $W \neq V$ and components of $\mathcal{G}^W$ are disjoint in $\mathcal{G}$, there is a commuting diagram

$$\begin{tikzcd}
\mathcal{G}^W \arrow[r, equal] & \coprod_{i=1}^m \mathcal{L}^{k_i} \arrow[r, "\iota"] \arrow[d, "\coprod_{i=1}^m u^{k_i}"'] & \mathcal{G} \arrow[d, "\mathsf{del}_{\setminus W}"] \\
& \coprod_{i=1}^m (\shortmid) \arrow[r, "\coprod_i ch_{e_{\setminus W_i}}"'] & \mathcal{G}_{\setminus W},
\end{tikzcd}$$

and therefore $\mathsf{del}_{\setminus W} \colon \mathcal{G} \to \mathcal{G}_{\setminus W}$ exists in CetGr_*.

The graph $\mathcal{G}_{\setminus W}$ (see Figure 23) is described explicitly (when $W \neq V$) by:

$$\mathcal{G}_{\setminus W} = \quad \tau_{\setminus W} \circlearrowright E_{\setminus W} \xleftarrow{s_{\setminus W}} H_{\setminus W} \xrightarrow{t_{\setminus W}} V_{\setminus W},$$

where

$$\begin{array}{rcl}
V_{\setminus W} & = & V \setminus W, \\
H_{\setminus W} & = & H \setminus H(\mathcal{G}^W) = H \setminus \left(\coprod_{v \in W} {}^{H}\!/v\right), \\
E_{\setminus W} & = & E \setminus \left(\coprod_{v \in W} {}^{E}\!/v\right),
\end{array}$$

$s_{\setminus W}, t_{\setminus W}$ are just the restrictions of s and t, and the involution $\tau_{\setminus W} \colon E_{\setminus W} \to E_{\setminus W}$ is given by

$$\begin{array}{rcll}
\tau_{\setminus W}(e) & = & \tau e & \text{for } e \in E \setminus E(\mathcal{G}^W), \\
\tau_{\setminus W}(1_{\mathcal{L}^{k_i}}) & = & 2_{\mathcal{L}^{k_i}} & \text{for } 1 \leq i \leq m.
\end{array}$$

This is fixed point free by (7.18).

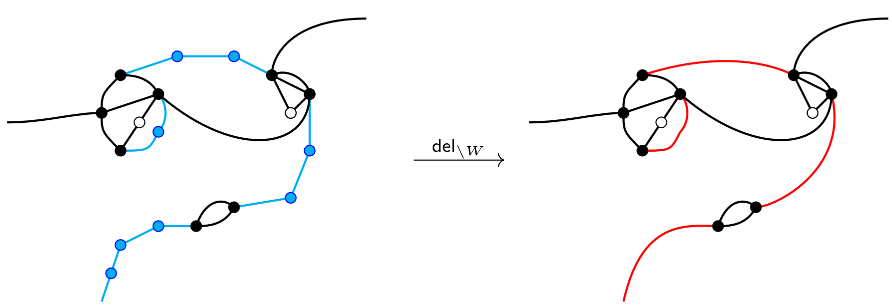


FIGURE 23. Vertex deletion $\mathsf{del}_{\setminus W} \colon \mathcal{G} \to \mathcal{G}_{\setminus W}$, with $\mathcal{G}^W \subset \mathcal{G}$ and $W \subset V_2$, and $\coprod_{i=1}^{3} u^{k_i}(\mathcal{G}^W) \subset \mathcal{G}_{\setminus W}$.

The final statement, that $E_0(\mathcal{G}) = E_0(\mathcal{G}_{\setminus W})$ except when $\mathcal{G} = \mathcal{W}^m$ and $W = V$ for some $m \geq 1$, follows since $\mathsf{del}_{\setminus W} \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ restricts to an identity on boundaries when $W \neq V$ by (7.18). □

Definition 7.19. *The* similarity category $\mathsf{CGr_{sim}} \hookrightarrow \mathsf{CetGr}_*$ *is the identity-on-objects subcategory of* CetGr_* *whose morphisms are generated under composition by* $z \colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$*, the vertex deletion morphisms, and graph isomorphisms. Morphisms in* $\mathsf{CGr_{sim}}$ *are called* similarity morphisms*, and connected components of* $\mathsf{CGr_{sim}}$ *are* similarity classes*. Graphs in the same connected component of* $\mathsf{CGr_{sim}}$ *are* similar*.*

Example 7.20. Up to isomorphism, the only morphisms in CetGr_* with codomain $(\shortmid)$ are similarity morphisms of the form $z \colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$, $\kappa^m \colon \mathcal{W}^m \to (\shortmid)$ $(m \geq 1)$, or $u^k \colon \mathcal{L}^k \to (\shortmid)$ $(k \geq 0)$.

Corollary 7.21 (Corollary to Proposition 7.17)**.** *The pair* $(\mathsf{CGr_{sim}}, \mathsf{CetGr})$ *of subcategories of* CetGr_* *defines an orthogonal factorisation system on* CetGr_**.*

In particular, assume that $E_0(\mathcal{G}) \neq \emptyset$ *(hence* $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$*) and let* $f \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}')$ *factor as* $f_{\setminus W_f} \circ \mathsf{del}_{\setminus W_f}$ *with* $f_{\setminus W_f} \in \mathsf{CetGr}(\mathcal{G}_{\setminus W_f}, \mathcal{G}')$*. If* f *is boundary preserving, then* $f_{\setminus W_f}$ *is an isomorphism.*

Proof. The only morphisms in CetGr_* with domain $\mathcal{C}_{\mathbf{0}}$ are of the form $id_{\mathcal{C}_{\mathbf{0}}}$ or $ch_e \circ z \colon \mathcal{C}_{\mathbf{0}} \to \mathcal{G}$ for some graph $\mathcal{G}$ and edge e of $\mathcal{G}$. So, let $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$ be a connected graph, and let $f \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}')$. The first statement follows by taking W_f to be the set of bivalent vertices w of $\mathcal{G}$ such that, if $(\mathcal{C}_{\mathbf{2}}, b)$ is a minimal neighbourhood of w, then there is an edge e' of $\mathcal{G}'$ such that $f \circ b = ch_{e'} \circ u \colon \mathcal{C}_{\mathbf{2}} \to \mathcal{G}'$.

The second statement follows immediately from Corollary 4.21. □

Example 7.22. For all graphs $\mathcal{G}$ and all $k \in \mathbb{N}$, Corollary 7.21 gives

$$\mathsf{CetGr}_*(\mathcal{L}^k, \mathcal{G}) \cong \coprod_{j=0}^{k} \binom{k}{j} \mathsf{CetGr}(\mathcal{L}^j, \mathcal{G}).$$

Example 7.23. By Corollary 7.21, for all graphical species S and all graphs $\mathcal{G}$ with no isolated vertices,

$$DS(\mathcal{G}) = \coprod_{W \subset V_2} S(\mathcal{G}_{\setminus W}). \tag{7.24}$$

By Corollary 7.21, a morphism $f \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}')$ is uniquely characterised by a commuting diagram of the form (7.25), and such that $\mathfrak{f}_V^{-1}(\widetilde{E'}) \subset V_0 \amalg V_2$ is either a single isolated vertex or a (possibly empty) subset of bivalent vertices, and the induced square 7.26 is a pullback.

(7.25)
$$\begin{tikzcd}
E \arrow[r, leftrightarrow, "\tau"] \arrow[d, "\mathfrak{f}_E"'] & E \arrow[d, "\mathfrak{f}_E"'] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\mathfrak{f}_H"'] & V \arrow[d, "\mathfrak{f}_V"] \\
E' \arrow[r, leftrightarrow, "\tau'"'] & E' & H' \amalg E' \arrow[l, "s' \amalg id"] \arrow[r, "t' \amalg q'"'] & V' \amalg \widetilde{E'}
\end{tikzcd}$$

(7.26)
$$\begin{tikzcd}
H \arrow[r, "t"] \arrow[d, "\mathfrak{f}_H"'] & (V \setminus V_0) \arrow[d, "\mathfrak{f}_V"] \\
H' \amalg E' \arrow[r, "t' \amalg q'"'] & (V \setminus V_0) \amalg \widetilde{E'}
\end{tikzcd}$$

If $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$, and $f = f_{\setminus W_f} \circ \mathsf{del}_{\setminus W_f}$ with $f_{\setminus W_f}$ in CetGr, then $W_f = \mathfrak{f}_V^{-1}(\widetilde{E'}) \subset V_2$.

Example 7.27. Diagram (7.28) describes $z \colon \mathbf{0} \to \S$, and (7.29) describes $u \colon \mathbf{2} \to \S$ in $\mathbf{B}_*^{\S} \hookrightarrow \mathsf{CetGr}_*$:

(7.28)
$$\begin{tikzcd}
\emptyset \arrow[d] & \emptyset \arrow[l] \arrow[r] \arrow[d] & \{v\} \arrow[d] \\
\{1,2\} \arrow[loop left, "\tau"] & \{1,2\} \arrow[l, "id"] \arrow[r, "q_{\shortmid}"'] & \{\tilde{1}\}
\end{tikzcd}$$

(7.29)
$$\begin{tikzcd}
\left\{ \begin{matrix} e_1, & e_2, \\ \tau_{\mathbf{2}} e_1, & \tau_{\mathbf{2}} e_2 \end{matrix} \right\} \arrow[loop left, "\tau_2"] \arrow[d] & \{(e_1, v),\ (e_2, v)\} \arrow[l] \arrow[r] \arrow[d] & \{v\} \arrow[d] \\
\{1,2\} \arrow[loop left, "\tau"] & \{1,2\} \arrow[l, "id"'] \arrow[r, "q_{\shortmid}"'] & \{\tilde{1}\}.
\end{tikzcd}$$

Lemma 3.14 admits the following extension to CetGr_*, that says that most morphisms in CetGr_* are completely determined by their action on edges:

Lemma 7.30. *If $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$ and $\mathcal{G}' \not\cong \mathcal{W}$, then $\mathfrak{f}_E$ is sufficient to define $f \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}')$.*

Proof. Assume that $\mathcal{G}$ is not an isolated vertex, hence $E \neq \emptyset$. If $\mathcal{G}' \not\cong \mathcal{W}$, then

$$\tau' e'_1 \neq e'_2 \text{ for all } e'_1, e'_2 \in E' \text{ such that } \{e'_1, e'_2\} = {}^{E'}\!/_{v'}, v' \in V'_2.$$

So, $v \in V_2$ is a bivalent vertex of $\mathcal{G}$ with ${}^{E}\!/_{v} = \{e_1, e_2\} \subset E_2$, and

$$\mathfrak{f}_E(e_1) = \mathfrak{f}_E(\tau e_2) \text{ if and only if } \mathfrak{f}_V(v) = q'(\mathfrak{f}_E(e_1)) = q'(\mathfrak{f}_E(\tau e_2)) \in \widetilde{E'}.$$

Otherwise, if $\mathfrak{f}_E(e_1) \neq \mathfrak{f}_E(\tau e_2)$, then $\mathfrak{f}_V(v) = t's'^{-1}(\mathfrak{f}_E(e_1)) \in V'$. □

Example 7.31. Lemma 7.30 does not hold if $\mathcal{G}' = \mathcal{W}$. For example, there are only two maps of edges $E(\mathcal{W}^2) \to E(\mathcal{W})$ that are compatible with the involution, and these correspond to the two maps in $\mathsf{CetGr}(\mathcal{W}^2, \mathcal{W})$. However, there are six distinct morphisms $\mathcal{W}^2 \to \mathcal{W}$ in CetGr_*.

7.3. S_*-structured graphs.

The étale topology on CetGr extends to a topology on CetGr_* whose covers at $\mathcal{G}$ are jointly surjective collections $\mathfrak{U} \subset \mathsf{CetGr}_*/\mathcal{G}$.

So, $P_* \in \mathsf{psh}(\mathsf{CetGr}_*)$ is a sheaf for this topology if and only if, for all graphs $\mathcal{G}$,

$$P_*(\mathcal{G}) \cong \lim_{(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})} P(\mathcal{C}), \text{ where } P \colon \mathsf{CetGr}^{\mathrm{op}} \to \mathcal{G} \text{ is the restriction to } \mathsf{CetGr}.$$

In particular, there is a canonical equivalence $\mathsf{sh}(\mathsf{CetGr}_*, J_*) \simeq \mathsf{GS}_*$.

Definition 7.32. *(Compare Definition 4.29.) For all pointed graphical species S_* and all connected graphs $\mathcal{G}$, an S_** structure α (or $(\mathcal{G}, \alpha)$) *on $\mathcal{G}$ is an element of $S_*(\mathcal{G}) \cong \mathsf{GS}_*(\mathcal{G}, S_*)$. The category CetGr_*/S_* is called the category of S_*-structured graphs.*

An S_*-structured graph $(\mathcal{G}, \alpha)$ is called admissible *if $\mathcal{G} \not\cong (\shortmid)$ is not a stick graph.*

Example 7.33. For $k \geq 0$ and $m \geq 1$, the vertex deletion morphisms $u^k \colon \mathcal{L}^k \to (\shortmid)$ and $\kappa^m \colon \mathcal{W}^m \to (\shortmid)$ in CetGr_* induce injective maps in CetGr_*/S_*:

$$\epsilon^k \overset{\text{def}}{=} S_*(u^k) \colon S_\S \to S(\mathcal{L}^k), \quad \text{and} \quad o^m \overset{\text{def}}{=} S_*(\kappa^m) \colon S_\S \to S(\mathcal{W}^m), \tag{7.34}$$

where $\epsilon^1 = \epsilon$, and ϵ^k, o^m factor through $\epsilon \colon S_\S \to S_{\mathbf{2}}$ for all $k \geq 0, m \geq 1$.

For each $c \in S_\S$, let

$$\mathcal{L}^k(\epsilon_c) \overset{\text{def}}{=} \epsilon^k(c) \in S_*(\mathcal{L}^k) \text{ and } \mathcal{W}^m(\epsilon_c) \overset{\text{def}}{=} o^m(c) \in S_*(\mathcal{W}^m)$$

be the *c-coloured unit structures* on $\mathcal{L}^k$ and $\mathcal{W}^m$, as pictured in Figure 24.

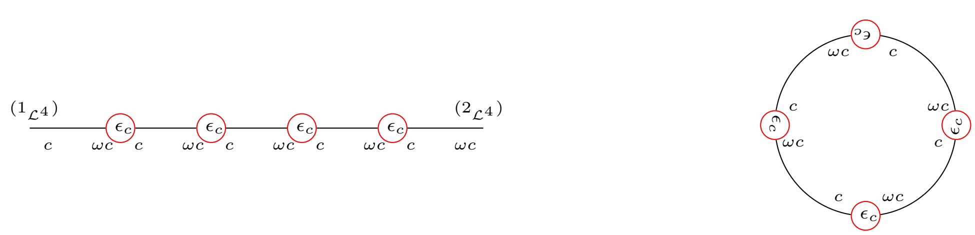


FIGURE 24. The c-coloured unit structures $\mathcal{L}^4(\epsilon_c)$ and $\mathcal{W}^4(\epsilon_c)$.

For any subset $W \subset V_2$ of bivalent vertices of a graph $\mathcal{G}$, and any S_*-structure $\alpha_{\setminus W} \in S(\mathcal{G}_{\setminus W})$, there is a unique S_*-structure $\alpha \in S(\mathcal{G})$ such that $\mathsf{del}_{\setminus W} \in \mathsf{CetGr}_*/S_*(\alpha, \alpha_{\setminus W})$. For, let $(\mathcal{C}_{\mathbf{2}}, b) \in \mathsf{el}(\mathcal{G})$ be a minimal neighbourhood of $v \in W$. Then there is some $e \in E(\mathcal{G}_{\setminus W})$, such that $\mathsf{del}_{\setminus W} \circ b = ch_e \circ u$, and hence

$$S_*(b)(\alpha) = S_*(\mathsf{del}_{\setminus W} \circ b)(\alpha_{\setminus W}) = S_*(u) S(ch_e)(\alpha_{\setminus W}) = \epsilon\left(S(ch_e)(\alpha_{\setminus W})\right).$$

Definition 7.35. *Let $(\mathcal{G}, \alpha)$ be an S_*-structured graph. The set*

$$W_\alpha = \{v \mid \text{there is a neighbourhood } (\mathcal{C}, b) \text{ of } v \text{ such that } S(b)(\alpha) \in im(\epsilon) \cup im(o)\} \subset V_0 \amalg V_2,$$

is the subset of vertices α-decorated by (contracted) units*.*

An S_*-structure $(\mathcal{G}, \alpha)$ is called reduced *if $W_\alpha = \emptyset$.*

For all $\mathcal{G} \ncong \mathcal{C}_\mathbf{0}$, and all subsets $W \subset V_2$ of bivalent vertices of $\mathcal{G}$, there is an S_*-structure α_W such that $\mathsf{del}_{\setminus W} \in \mathsf{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ describes a morphism in $\mathsf{CetGr}_*/S_*((\mathcal{G}, \alpha), (\mathcal{G}_{\setminus W}, \alpha_{\setminus W}))$ if and only if $W \subset W_\alpha$. And $(\mathcal{G}_{\setminus W}, \alpha_{\setminus W})$ is reduced if and only if $W = W_\alpha$.

7.4. **Similar structures.** The issues that can arise from trying to incorporate degenerate substitution by the stick graph into the definition of the modular operad monad in order to obtain multiplicative units have been outlined in Section 6.

Degenerate substitutions – and therefore exceptional loops – can be avoided if there is a suitable notion of equivalence of S_*-structured graphs, for which all constructions can be obtained in terms of admissible representatives.

This principle will inform the construction of the distributive law $\lambda \colon TD \Rightarrow DT$.

By Proposition 7.17, any similarity morphism that is not of the form $z \colon \mathcal{C}_\mathbf{0} \to (\shortmid)$ or $\kappa^m \colon \mathcal{W}^m \to (\shortmid)$ is boundary preserving. So, let $\mathcal{G}$ be a connected graph with non-empty vertex set and non-empty boundary $E_0 \cong X$, and let $f \in \mathsf{CGr}_{\mathsf{sim}}(\mathcal{G}, \mathcal{G}')$ be a similarity morphism. Then f, together with an X-labelling $\rho \colon E_0 \to X$ of $\mathcal{G}$, induces an X-labelling on $\mathcal{G}'$. The category $X\text{-}\mathsf{CGr}_{\mathsf{sim}}$ is obtained by adjoining to the category $X\text{-}\mathsf{CGr}_{\mathsf{iso}}$ of (admissible) X-graphs and port-preserving isomorphisms, all similarity morphisms from admissible X-graphs, and their codomains, equipped with the induced labelling.

- If $X \ncong \mathbf{0}$, $X \ncong \mathbf{2}$, then $X\text{-}\mathsf{CGr}_{\mathsf{sim}}$ is the category whose objects are connected X-graphs and whose morphisms are similarity morphisms that preserve the labelling of the ports.
- For $X = \mathbf{2}$, $\mathbf{2}\text{-}\mathsf{CGr}_{\mathsf{sim}}$ contains the morphisms $\mathsf{del}_{\setminus V} \colon \mathcal{L}^k \to (\shortmid)$, and hence the labelled stick graphs $(\shortmid, id)$ and $(\shortmid, \tau)$. There are no non-trivial morphisms out of these special graphs, and $\mathcal{X}$ is in the same connected component as $(\shortmid, id)$ if and only if $\mathcal{X} = \mathcal{L}^k$ (with the identity labelling) for some $k \in \mathbb{N}$. In particular, $\tau \colon (\shortmid) \to (\shortmid)$ does not induce a morphism in $\mathbf{2}\text{-}\mathsf{CGr}_{\mathsf{sim}}$.
- Finally, when $X = \mathbf{0}$, the morphisms $\mathsf{del}_{\setminus V} \colon \mathcal{W}^m \to (\shortmid)$, and $z \colon \mathcal{C}_\mathbf{0} \to (\shortmid)$ are not boundary-preserving, and, in particular, do not equip $(\shortmid)$ with any labelling of its ports. So, the objects of $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}$ are the admissible $\mathbf{0}$-graphs, and $(\shortmid)$.

  In particular, $\mathcal{W}^m, \mathcal{C}_\mathbf{0}$ and $(\shortmid)$ are in the same connected component of $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}$. Since $(\shortmid)$ is not admissible, there are no non-trivial morphisms in $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}$ with $(\shortmid)$ as domain.

  In what follows, to simplify notation, we'll write $\mathcal{C}_{\mathbf{0}\setminus V} \overset{\text{def}}{=} (\shortmid)$ and $\mathsf{del}_{\setminus V} = z \colon \mathcal{C}_\mathbf{0} \to (\shortmid)$.

Definition 7.36. *The slice category $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$ induced by the canonical functor $X\text{-}\mathsf{CGr}_{\mathsf{sim}} \to \mathsf{CetGr}_* \hookrightarrow \mathsf{GS}_*$ is the category of* similar S_*-structured X-graphs*.*

Admissible S_*-structured X-graphs $(\mathcal{X}^1, \alpha^1), (\mathcal{X}^2, \alpha^2)$ are called similar*, written $(\mathcal{X}^1, \alpha^1) \sim (\mathcal{X}^2, \alpha^2)$ (or just $\alpha^1 \sim \alpha^2$), if they are in the same connected component of $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$.*

For all $(\mathcal{X}, \alpha) \in \mathsf{CetGr}_*/S_*$, if $W_\alpha \neq V(\mathcal{X})$, then the unique reduced structure $(\mathcal{X}^\perp_\alpha, \alpha^\perp)$ similar to $(\mathcal{X}, \alpha)$ is admissible. If $\mathcal{X} \ncong \mathcal{C}_\mathbf{0}$ and $W_\alpha = V(\mathcal{X})$, then $(\mathcal{X}, \alpha) \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$ has the form $\mathcal{L}^k(\epsilon_c)$, or $\mathcal{W}^m(\epsilon_c)$ for some $c \in S_\S$. So, the reduced similar structure $(\mathcal{X}^\perp_\alpha, \alpha^\perp)$ has the form $(\shortmid, c)$ and is not admissible.

Whenever $c \neq \omega c$, $(\shortmid, c) \not\sim (\shortmid, \omega c)$ in $\mathbf{2}\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$. Hence $(\shortmid, c)$ is the unique reduced structure similar to $\mathcal{L}^k(\epsilon_c)$ for all $c \in S_\S$ and all $k \in \mathbb{N}$. By contrast, since $z\colon \mathcal{C}_\mathbf{0} \to (\shortmid)$ induces morphisms $(\mathcal{C}_\mathbf{0}, o_{\tilde{c}}) \to (\shortmid, c)$ and $(\mathcal{C}_\mathbf{0}, o_{\tilde{c}}) \to (\shortmid, \omega c)$ in CetGr_*/S_*. Hence, for all $c \in S_\S$, $(\shortmid, c) \sim (\shortmid, \omega c)$ in $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$.

Moreover, the similarity maps $\kappa\colon \mathcal{W} \to (\shortmid) \leftarrow \mathcal{C}_\mathbf{0} \colon z$ in CetGr_* induce morphisms

$$\mathcal{W}(\epsilon_c) \longrightarrow (\shortmid, c) \longleftarrow (\mathcal{C}_\mathbf{0}, o_{\tilde{c}}) \longrightarrow (\shortmid, \omega c) \longleftarrow \mathcal{W}(\epsilon_{\omega c}) \; . \tag{7.37}$$

So, $\mathcal{W}(\epsilon_c) \sim (\mathcal{C}_\mathbf{0}, o_{\tilde{c}})$ and there is a double-cone shaped diagram in $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$ (Figure 25).

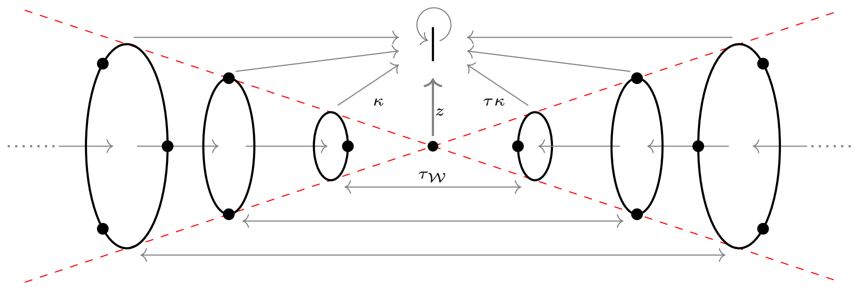


FIGURE 25. The contraction of units is described in terms of a commuting diagram in $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}$ that is strongly suggestive of a conical singularity.

In particular, for all $c \in S_\S$, $(\shortmid, c)$ and $(\shortmid, \omega c)$ are in the same connected component of $\mathbf{0}\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$ but are in disjoint components of $\mathbf{2}\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$ whenever $c \neq \omega c$.

Lemma 7.38. *For all pointed graphical species S_* and all finite sets X, there is a canonical bijection*

$$\operatorname{colim}_{(\mathcal{G},\rho) \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}} S_*(\mathcal{G}) \cong \pi_0(X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*). \tag{7.39}$$

Proof. Since $X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S \subset X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$, there is a surjection of connected components $\pi_0(X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S) \to \pi_0(X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*)$. So every component of $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$ has a representative in $X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S$, and the result follows from the definition and equation (5.7). □

7.5. **A distributive law for modular operads.** Let S be a graphical species and X a finite set. An element of TDS_X is represented by an X-graph $\mathcal{X}$, with a decoration $\alpha \in DS(\mathcal{X}) = S^+(\mathcal{X})$. The idea is to construct a distributive law $\lambda\colon TD \Rightarrow DT$ so that λTDS is invariant under morphisms in $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S^+$.

Proposition 7.40. *There is a distributive law $\lambda\colon TD \Rightarrow DT$ such that for all graphical species S and finite sets X, and all $[\mathcal{X}, \alpha]$ and $[\mathcal{X}', \alpha']$ in TDS_X,*

$$\lambda[\mathcal{X}, \alpha] = \lambda[\mathcal{X}', \alpha'] \text{ in } DTS_X \text{ if and only if } [\mathcal{X}, \alpha] \sim [\mathcal{X}', \alpha'] \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S^+.$$

Proof. Since the endofunctor D just adjoins elements, there are canonical inclusions $TS \hookrightarrow DTS$ and $TS \hookrightarrow TDS$. The natural transformation $\lambda\colon TD \Rightarrow DT$ will restrict to the identity on TS.

For a finite set X, elements of TDS_X are represented by DS-structured X-graphs $(\mathcal{X}, \alpha)$, whereas elements of DTS_X are either of the form $\epsilon_c^{DTS}, o_{\tilde{c}}^{DTS}$ for $c \in S_\S$, or are represented by S-structured X-graphs $(\mathcal{X}', \alpha')$. Observe also that an object $(\mathcal{X}, \alpha) \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S^+$ is reduced and admissible if and only if $(\mathcal{X}, \alpha) \in X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S$, and hence $[\mathcal{X}, \alpha] \in TS_X$.

Let $(\mathcal{X}, \alpha) \in X\text{-}\mathsf{CGr}_{\mathsf{iso}}/DS$. If $X = \mathbf{2}$, and $(\mathcal{X}, \alpha)$ has the form $\mathcal{L}^k(\epsilon_c)$, set

$$\lambda[\mathcal{X}, \alpha] = \epsilon_c^{DS} \in DTS_\mathbf{2}.$$

And, if $X = \mathbf{0}$, and $(\mathcal{X}, \alpha) = \mathcal{W}^m(\epsilon_c)$ or $(\mathcal{X}, \alpha) = (\mathcal{C}_\mathbf{0}, o_{\tilde{c}})$, set

$$\lambda[\mathcal{X}, \alpha] = o_{\tilde{c}}^{DTS} \in DTS_\mathbf{0}.$$

Otherwise, the component of $(\mathcal{X}, \alpha)$ in $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S^+$ has an admissible and reduced (hence terminal) object $(\mathcal{X}^\perp_\alpha, \alpha^\perp)$, so we can set

$$\lambda[\mathcal{X}, \alpha] = [\mathcal{X}^\perp_\alpha, \alpha^\perp] \in TS_X \subset DTS_X.$$

The assignment so defined clearly extends to a natural transformation $\lambda\colon TD \Rightarrow DT$.

The verification that λ satisfies the four axioms [3, Section 1] for a distributive law is straightforward but tedious, so I prove just one here, that the following diagram of natural transformations commutes:

(7.41)
$$\begin{tikzcd} TD^2 \arrow[r, Rightarrow, "\lambda D"] \arrow[d, Rightarrow, "T(\mu^{\mathbb{D}})"'] & DTD \arrow[r, Rightarrow, "D\lambda"] & D^2T \arrow[d, Rightarrow, "(\mu^{\mathbb{D}})T"] \\ TD \arrow[rr, Rightarrow, "\lambda"'] & & DT \end{tikzcd}$$

So, let $[\mathcal{X}, \alpha] \in TD^2S_X$. The result is immediate when $\mathcal{X} = \mathcal{C}_\mathbf{0}$. Moreover, all the maps in (7.41) restrict to the identity on $TS \subset TD^2S$.

Therefore, we may assume that $[\mathcal{X}, \alpha] \notin TS_X$ and that $\mathcal{X} \ncong \mathcal{C}_\mathbf{0}$. For $j = 1, 2$, define the sets W^j, of vertices decorated by distinguished elements adjoined in the j^{th} application of D.

$$W^j \overset{\text{def}}{=} \{v | v \text{ has a minimal neighbourhood } (\mathcal{C}_X, b) \text{ with } \; D^2S(b)(\alpha) \in im(\epsilon^{D^jS})\} \subset V_2.$$

Then $[\mathcal{X}, \mu^{\mathbb{D}}\alpha] = T\mu^{\mathbb{D}}S[\mathcal{X}, \alpha] \in TDS_X$ is described by

$$DS(b)(\mu^{\mathbb{D}}\alpha) = \begin{cases} \epsilon_c^{DS} & \text{if } (\mathcal{C}, b) \text{ is a minimal neighbourhood of } v \in W^1 \amalg W^2, \\ D^2S(b)(\alpha) \in S(\mathcal{C}) & \text{otherwise.} \end{cases}$$

If $W^1 \cup W^2 \neq V$, the diagram gives

$$\begin{tikzcd} {[\mathcal{X}, \alpha]} \arrow[r, mapsto, "\lambda D"] & {[\mathcal{X}_{\setminus W^2}, \alpha_{\setminus W^2}]} \arrow[r, mapsto, "D\lambda"] & {[(\mathcal{X}_{\setminus W^2})_{\setminus W^1}, (\alpha_{\setminus W^2})_{\setminus W^1}]} \arrow[d, equal] \\ {[\mathcal{X}, \alpha]} \arrow[r, mapsto, "T(\mu^{\mathbb{D}})"'] & {[\mathcal{X}, \mu^{\mathbb{D}}\alpha]} \arrow[r, mapsto, "\lambda"'] & {[\mathcal{X}_{\setminus (W^1 \amalg W^2)}, \alpha_{\setminus (W^1 \amalg W^2)}] \in TS_X} \end{tikzcd}.$$

If $W^1 \amalg W^2 = V$, then $T(\mu^{\mathbb{D}})[\mathcal{X}, \alpha]$ has the form $\mathcal{L}^k(\epsilon_c^{DS})$ or $\mathcal{W}^m(\epsilon_c^{DS})$ and both paths map to the corresponding (contracted) unit in DTS. □

It follows that there is a composite monad $\mathbb{DT}$ on GS, induced by λ.

By [3, Section 3], $\lambda\colon TD \Rightarrow DT$ induces a lift $\mathbb{T}_*$ of $\mathbb{T}$ to GS_*, such that the EM categories $\mathsf{GS}^{\mathbb{DT}}$ and $\mathsf{GS}_*^{\mathbb{T}_*}$ are canonically isomorphic. (See also Section 2.2.)

Corollary 7.42. *The monad $\mathbb{T}_* = (T_*, \mu^{\mathbb{T}_*}, \eta^{\mathbb{T}_*})$ on GS_* is given by*

$$T_*S_\S = S_\S, \text{ and } T_*S_X = \mathrm{colim}_{(\mathcal{G}, \rho) \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}} S(\mathcal{G}).$$

The unit $\eta^{\mathbb{T}_*}\colon 1_{\mathsf{GS}_*} \Rightarrow T_*$ and multiplication $\mu^{\mathbb{T}_*}\colon T_*^2 \Rightarrow T_*$ are induced by the unit $\eta^{\mathbb{T}}$ and multiplication $\mu^{\mathbb{T}}$ for $\mathbb{T}$. In other words, if $\mathcal{X}$ is an X-graph and $\alpha \in S(\mathcal{X})$ and $[\mathcal{X}, \alpha]_*$ denotes the class of $[\mathcal{X}, \alpha] \in TS_X$ in T_*S_X, then

$$\eta^{\mathbb{T}_*}(\phi) = [\eta^{\mathbb{T}}\phi]_* \text{ and } \mu^{\mathbb{T}_*}[\mathcal{X}, \beta] = [\mu^{\mathbb{T}}[\mathcal{X}, \beta]]_*.$$

Proof. Let S be a graphical species with $\mathbb{D}$-algebra structure $h_{\mathbb{D}}\colon DS \to S$ so that $S_* = (S, \epsilon, o)$, with $\epsilon = h_{\mathbb{D}}(\epsilon^+)$ and $o = h_{\mathbb{D}}(o^+)$, is a pointed graphical species.

Observe first that

$$(\mu^{\mathbb{D}}T) \circ (D\lambda) = (\mu^{\mathbb{D}}\mu^{\mathbb{T}}) \circ (D\lambda T) \circ (DTD\eta^{\mathbb{T}}) \colon DTD \Rightarrow DT,$$

so, by [3, Section 3], $T_*(S_*)$ is described by the coequaliser

$$\begin{tikzcd} DTDS \arrow[rr, "DTh_{\mathbb{D}}"] \arrow[dr, "D\lambda S"'] & & DTS \arrow[r, "\pi"] & T_*S_*. \\ & D^2TS. \arrow[ur, "\mu^{\mathbb{D}}TS"'] & & \end{tikzcd} \tag{7.43}$$

On $TS \subset DTDS$, both maps in (7.43) restrict to the inclusion $TS \hookrightarrow DTS$. By the application of $h_{\mathbb{D}}$, the upper path, $\pi \colon DTS \to T_*S_*$ identifies all occurrences of ϵ and ϵ^{DS} in elements of $DTDS$ and, by the application of $\mu^{\mathbb{D}}$ in the lower path, it identifies all occurrences of ϵ^{DS} and ϵ^{DTDS}. The occurrence of λ means that similar elements of $TDS \hookrightarrow DTDS$ are identified by the lower path. Hence, similar elements of $TS \subset DTS$ are identified by the quotient π and $T_*S_X = \mathrm{colim}_{(\mathcal{G},\rho)\in X\text{-}\mathsf{CGr}_{\mathsf{sim}}} S_*(\mathcal{G})$ as required.

The unit for T_*S_* is provided by the map $[\mathcal{L}^k(\epsilon)]_* \colon S_\S \to T_*S_{\mathbf{2}}$ that takes c to the class of all ϵ_c decorated line graphs $\mathcal{L}^k$, and the contracted unit by $[\mathcal{W}^m(\epsilon)]_* = [\mathcal{C}_{\mathbf{0}}, o]_* \colon S_\S \to T_*S_{\mathbf{0}}$ that takes c to the class of all ϵ_c decorated wheel graphs, and the isolated vertex decorated by $\zeta(\epsilon_c) = o_{\tilde{c}}$.

Let $(\mathcal{X}, \beta)$ represent an element of T^2S_X, and let $\mathsf{del}_{\backslash W} \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}((\mathcal{X}, \beta), (\mathcal{X}_{\backslash W}, \beta_{\backslash W}))$. then $(\mathcal{C}_{\mathbf{2}}, b) \in \mathsf{el}(\mathcal{X})$ is a neighbourhood of $v \in W$ if and only if there is a $k \geq 1$, and a $c \in S_\S$, such that $S(b)(\beta) = [\mathcal{L}^k(\epsilon_c)]$ in $TS_{\mathbf{2}}$. It follows that $\mu^{\mathbb{T}_*} = [\mu^{\mathbb{T}}(-)]_*$ is well-defined and provides the multiplication for $\mathbb{T}_*$.

Clearly $\eta^{\mathbb{T}_*} = [\eta^{\mathbb{T}}(-)]_*$, whereby the result follows immediately. $\square$

So, elements of T_*S_X may be viewed as similarity classes of S_*-structured X-graphs. Two X-labelled S_*-structured graphs $(\mathcal{X}, \alpha)$ and $(\mathcal{X}', \alpha')$ represent the same class $[\mathcal{X}, \alpha] \in TS_X$ precisely when there is a similarity morphism $g \in X\text{-}\mathsf{CGr}_{\mathsf{sim}}(\mathcal{X}, \mathcal{X}')$ such that $S(g)(\alpha') = \alpha$.

In particular, to compute the image of $[\mathcal{X}, \beta] \in T_*^2S_*$ under the multiplication $\mu^{\mathbb{T}_*}(S_*) \colon T_*^2S_* \to T_*S_*$, it is sufficient to chose a non-degenerate graph of graphs and quotient by similarity at the end.

Remark 7.44. There is also a distributive law for composing $\mathbb{D}$ and $\mathbb{T}$ in the other direction $DT \Rightarrow TD$. Algebras for the composite monad $\mathbb{TD}$ are just the cofibred coproducts of algebras for $\mathbb{D}$ and $\mathbb{T}$. There is no further relationship between the two structures. (See also Example 2.15.)

7.6. **$\mathbb{DT}$-algebras are modular operads.** At last we are ready to prove the first main theorem – that modular operads are $\mathbb{DT}$-algebras in GS or, equivalently, $\mathbb{T}_*$-algebras in GS_*.

Observe first that, if $(S, \diamond, \zeta, \epsilon)$ is a modular operad, then $S_* = (S, \epsilon, \zeta\epsilon)$ is a pointed graphical species, and $(S, \diamond, \zeta)$ is an unpointed modular operad. Therefore, by Proposition 5.29, S is equipped with a $\mathbb{T}$-algebra structure $p_{\mathbb{T}} = p_{\mathbb{T}}^{\diamond,\epsilon} \colon TS \to S$.

Lemma 7.45. *Admissible objects in the same connected component of $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$ have the same image under the defining functor $X\text{-}\mathsf{CGr}_{\mathsf{iso}}/S \twoheadrightarrow TS_X \xrightarrow{p_{\mathbb{T}}} S_X$.*

Proof. Since $(S, \diamond, \zeta, \epsilon)$ is a modular operad, $p_{\mathbb{T}}$ satisfies

$$p_{\mathbb{T}}[\mathcal{M}_c(\phi, \epsilon_c)] = \phi \diamond_c \epsilon_c = \phi = p_{\mathbb{T}}(\eta^{\mathbb{T}}\phi) \text{ wherever defined.} \tag{7.46}$$

So, let $(\mathcal{X}, \alpha)$ be an admissible object in $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$. If $(\mathcal{X}, \alpha) \sim (\mathcal{C}_{\mathbf{2}}, \epsilon_c)$ in $\mathbf{2}\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$, then $p_{\mathbb{T}}[\mathcal{X}, \alpha] = p_{\mathbb{T}}[\mathcal{C}_{\mathbf{2}}, \epsilon_c] = \epsilon_c$ by (7.46). And, since $p_{\mathbb{T}}[\mathcal{N}_c(\psi)] = \zeta(\psi)$ for all $\psi \in S_{(\underline{c}, c, \omega c)}$,

$$p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}(\epsilon_c)] = \zeta(\epsilon_c) = p_{\mathbb{T}}[\mathcal{C}_{\mathbf{0}}, o_{\tilde{c}}].$$

Otherwise, the connected component of $(\mathcal{X}, \alpha)$ in $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$ contains a unique admissible reduced – and hence terminal – element $(\mathcal{X}^{\perp}_{\alpha}, \alpha^{\perp})$, and $p_{\mathbb{T}}[\mathcal{X}, \alpha] = p_{\mathbb{T}}[\mathcal{X}^{\perp}_{\alpha}, \alpha^{\perp}]$ by (7.46). □

It is now straightforward to prove that $\mathbb{DT}$ is the desired modular operad monad on GS.

Theorem 7.47. *The EM category* $\mathsf{GS}^{\mathbb{DT}}$ *of algebras for* $\mathbb{DT}$ *is canonically isomorphic to* MO.

Proof. A $\mathbb{DT}$-algebra (A, h) induces $\mathbb{D}$- and $\mathbb{T}$-structure morphisms

$$h_{\mathbb{D}} \overset{\text{def}}{=} h \circ (D\eta^{\mathbb{T}} A) \colon DA \to A, \qquad \text{and} \qquad h_{\mathbb{T}} \overset{\text{def}}{=} h \circ (D\eta^{\mathbb{D}} A) \colon TA \to A$$

by [3, Section 2], where, since $\eta^{\mathbb{D}}$ is just an inclusion, $h_{\mathbb{T}} = h|_{TA} \colon TA \to A$ is the restriction of h to $TA \subset DTA$. By Proposition 5.29, A is equipped with a multiplication $\diamond = h \circ [\mathcal{M}(-,-)]$ and contraction $\zeta = h \circ [\mathcal{N}(-)]$, so that $(A, \diamond, \zeta)$ is a non-unital modular operad.

It remains to show that ϵ provides a unit for the multiplication $\diamond$. By the monad algebra axioms, there are commuting diagrams

(7.48)
$$\begin{tikzcd} A \arrow[r, "\eta^{\mathbb{D}}\eta^{\mathbb{T}}A"] \arrow[dr, equal] & DTA \arrow[d, "h"] \\ & A, \end{tikzcd}$$

(7.49)
$$\begin{tikzcd} (DT)^2A \arrow[r, "D\lambda TA"] \arrow[d, "DTh"'] & D^2T^2A \arrow[r, "\mu^{\mathbb{D}}\mu^{\mathbb{T}}A"] & DTA \arrow[d, "h"] \\ DTA \arrow[rr, "h"'] & & A. \end{tikzcd}$$

For all finite sets X, all $\underline{c} \in (A_{\S})^X$, and all $\phi \in A_{\underline{c},c}$, the image of the element $[\mathcal{M}_c(\eta^{\mathbb{T}}(\phi), \epsilon_c^{DTA})] \in (TD)^2 A_X$ under the top-right path in (7.49) is just $\phi \in A_X$, since the application of $D\lambda TA$ deletes the unit ϵ_c^{DTA}.

Now $DTh[\mathcal{M}_c(\eta^{\mathbb{T}}(\phi), \epsilon_c^{DTA})] = [\mathcal{M}_c(\phi, \epsilon_c^{DTA})] \in DTA_X$ by construction, and, since (7.49) commutes,

$$\phi \ \diamond_c \ \epsilon_c = h[\mathcal{M}_c(\phi, \epsilon_c)] = \phi.$$

Hence, ϵ is a unit for $\diamond$.

Conversely, a modular operad $(S, \diamond, \zeta, \epsilon)$ induces a pointed graphical species $S_* = (S, \epsilon, \zeta\epsilon)$. And, by Proposition 5.29, since $(S, \diamond, \zeta)$ is an unpointed modular operad, it is a $\mathbb{T}$-algebra with structure map $p_{\mathbb{T}} \colon TS \to S$ satisfying

$$\diamond = p_{\mathbb{T}} \ \circ [\mathcal{M}(\cdot,\cdot)] \ \text{ and } \ \zeta = p_{\mathbb{T}} \ \circ [\mathcal{N}(\cdot)].$$

By Lemma 7.45, since ϵ is a unit for $\diamond$, for all $c \in S_{\S}$ and all $m \geq 1$,

$$p_{\mathbb{T}}[\mathcal{N}(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)].$$

So, $o_{\tilde{c}} \overset{\text{def}}{=} \zeta\epsilon_c = p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)]$.

Then (7.48) commutes when $h = p \colon DTS \to S$ is defined by $p_{\mathbb{T}} \colon TS \to S$ on TS, and

$$p(\epsilon^{DTS}) = \epsilon \colon S_{\S} \to S_{\mathbf{2}}, \text{ and } p(o^{DTS}) = \zeta\epsilon \colon S_{\S} \to S_{\mathbf{0}}.$$

It remains to check that (7.49) commutes for S and $h = p$. This is clear for the adjoined (contracted) units $\epsilon^{(DT)^2S}$, and $o^{(DT)^2S}$ for $(DT)^2S$. So we must check that the restriction

(7.50)
$$\begin{tikzcd} TDTS \arrow[r, "\lambda TS"] \arrow[d, "Tp"'] & DT^2S \arrow[r, "D\mu^{\mathbb{T}}S"] & DTS \arrow[d, "p"] \\ TS \arrow[rr, "p"'] & & S \end{tikzcd}$$

of (7.49) commutes.

To this end, let $[\mathcal{X}, \beta] \in TDTS_X$. Exactly one of the following four conditions holds:

(i) If $X = \mathbf{0}$ and $[\mathcal{X}, \beta] = [\mathcal{C}_\mathbf{0}, o_{\tilde{c}}^{DTS}]$, then it is clear that its image under both paths in (7.50) is $o_{\tilde{c}}$;

(ii) If $X = \mathbf{0}$ and $[\mathcal{X}, \beta] = [\mathcal{W}^m(\epsilon_c^{DTS})]$ for some $m \geq 1$, and $c \in S_\S$, then the application of λTS in the top-right path means that this path takes $[\mathcal{W}^m(\epsilon_c^{DTS})]$ to $o_{\tilde{c}} \in S_\mathbf{0}$.

The bottom left path takes $[\mathcal{W}^m(\epsilon_c^{DTS})]$ first to $[\mathcal{W}^m(\epsilon_c)] \in TS_\mathbf{0}$ by applying p to ϵ^{DTS}, and then (by an application of Lemma 7.45) to

$$p[\mathcal{W}^m(\epsilon_c)] = p[\mathcal{W}(\epsilon_c)] = p_\mathbb{T}[\mathcal{N}(\epsilon_c)] = \zeta\epsilon_c = o_{\tilde{c}};$$

(iii) If $X = \mathbf{2}$ and $[\mathcal{X}, \beta] = [\mathcal{L}^k(\epsilon_c^{DTS})]$ for some $k \geq 1$, and $c \in S_\S$, then, once again, the application of λTS in the top-right path means that this path takes $[\mathcal{L}^k(\epsilon_c^{DTS})]$ to $\epsilon_c \in S_\mathbf{2}$.

The bottom left path takes $[\mathcal{L}^k(\epsilon_c^{DTS})]$ first to $[\mathcal{L}^k(\epsilon_c)] \in TS_\mathbf{2}$ by applying p inside, and then, by an application of Lemma 7.45,

$$p[\mathcal{L}^k(\epsilon_c)] = p_\mathbb{T}[\mathcal{L}^k(\epsilon_c)] = \epsilon_c;$$

(iv) Otherwise, the reduced structure $(\mathcal{X}_\beta^\perp, \beta^\perp)$ similar to $(\mathcal{X}, \beta)$ is admissible in $DTS(\mathcal{X})$. Hence $[\mathcal{X}_\beta^\perp, \beta^\perp] \in T^2 S_X$, and therefore $Tp[\mathcal{X}, \beta] = Tp_\mathbb{T}[\mathcal{X}^\perp, \beta^\perp]$ by Lemma 7.45. Then (7.50) commutes, since $(S, p_\mathbb{T})$ is an algebra for $\mathbb{T}$.

Therefore, (S, p) naturally admits the structure of a $\mathbb{DT}$-algebra. It is straightforward to verify that the functors $\mathsf{MO} \leftrightarrows \mathsf{GS}^{\mathbb{DT}}$ so defined are each others' inverses. □

Remark 7.51. In particular, $\mathbb{DT}$ is the algebraically free monad [25] on the endofunctor T^{ds} in Section 6.

8. A nerve theorem for modular operads

By Theorem 7.47 and [3], there is a diagram of functors

(8.1)
$$\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\mathrm{forget}^{\mathbb{T}_*}"] & \mathsf{psh}(\Xi) \arrow[d, "j^*"] \\
\mathbf{B}^\S_* \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{CetGr}_* \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{GS}_* \arrow[u, shift left, "\mathrm{free}^{\mathbb{T}_*}"] \arrow[r, hook, "\text{f.f.}"'] \arrow[d, shift left, "\mathrm{forget}^{\mathbb{D}}"] & \mathsf{psh}(\mathsf{CetGr}_*) \arrow[d] \\
\mathbf{B}^\S \arrow[u, "\text{b.o.}"] \arrow[ur, "\text{dense}"] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{CetGr} \arrow[u, "\text{b.o.}"'] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\mathrm{free}^{\mathbb{D}}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{psh}(\mathsf{CetGr}).
\end{tikzcd}$$

where Ξ is the category obtained in the bo-ff factorisation of $\mathsf{CetGr} \to \mathsf{GS} \to \mathsf{MO}$, *and* also in the bo-ff factorisation of $\mathsf{CetGr}_* \to \mathsf{GS}_* \to \mathsf{MO}$.

The goal of this section is to prove the following nerve theorem for modular operads using the abstract machinery described in Section 2.

Theorem 8.2. *The functor $N\colon \mathsf{MO} \to \mathsf{psh}(\Xi)$ is full and faithful. Its essential image consists of precisely those presheaves P on Ξ whose restriction to $\mathsf{psh}(\Xi)$ are graphical species. In other words,*

(8.3)
$$P(\mathcal{G}) = \lim_{(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \text{ for all graphs } \mathcal{G}.$$

Remark 8.4. A version of this theorem was stated in [22], and another version was proved, by different methods, in [20, Theorem 3.8]. In [35], I proved this theorem by essentially the same methods, but without the use of the distributive law. In all these versions, the statement of the Segal condition (8.3) is the same.

As discussed in Section 2, Theorem 8.2 follows immediately if the monad $\mathbb{DT}$ on GS has arities CetGr (see [5, Definition 1.8]). Unfortunately, this is not the case. The obstruction, unsurprisingly, relates to the contracted units. (See Remark 8.12.)

The remainder of this work is devoted to showing that $\mathbb{T}_*$ has arities CetGr_*. In this case, the nerve $N\colon \mathsf{MO} \to \mathsf{psh}(\Xi)$ is fully faithful. Moreover, because $\mathbf{B}^{\S}$ is dense in CetGr_*, the essential image of N is characterised by the Ξ-presheaves P that satisfy the Segal condition (8.3).

The first step is to study the graphical category Ξ in more detail. By construction, $\Xi \subset \mathsf{MO}$ is the full subcategory on the modular operads $\Xi(\mathcal{G})$ free on connected graphs $\mathcal{G} \in \mathsf{CetGr}$.

8.1. **The free modular operad on a graph.** Let $\mathcal{H} = (E, H, V, s, t, \tau)$ be a graph. To streamline the notation, let $T\mathcal{H} \overset{\text{def}}{=} T\Upsilon\mathcal{H}$ denote the free non-unital modular operad on $\mathcal{H}$, and $T_*\mathcal{H} \overset{\text{def}}{=} T_*\Upsilon_*\mathcal{H}$ the corresponding free unital modular operad on $\mathcal{H}$.

Of course, $T_*\mathcal{H}(\shortmid) = \{ch_e\}_{e\in E} = \mathsf{CetGr}_*(\shortmid, \mathcal{H})$. Recall that the unit for $\Upsilon_*\mathcal{H}$ is given by $ch_e \mapsto \epsilon^{\mathcal{H}}_e \overset{\text{def}}{=} ch_e \circ u \in \mathsf{CetGr}_*(\mathcal{C}_{\mathbf{2}}, \mathcal{H})$, and the contracted unit for $\Upsilon_*\mathcal{H}$ is given by $ch_e \mapsto o^{\mathcal{H}}_{\tilde{e}} \overset{\text{def}}{=} ch_e \circ z \in \mathsf{CetGr}_*(\mathcal{C}_{\mathbf{0}}, \mathcal{H})$.

So, by Proposition 7.40, $T_*\mathcal{H}$ has units

$$ch_e \mapsto \epsilon^{T_*\mathcal{H}}_e \overset{\text{def}}{=} [\eta^{\mathbb{T}}\epsilon^{\mathcal{H}}_e]_* = [ch_e \circ u^k],\ k \geq 1.$$

and contracted units

$$ch_e \mapsto o^{T_*\mathcal{H}}_{\tilde{e}} \overset{\text{def}}{=} [\eta^{\mathbb{T}} o^{\mathcal{H}}_e]_* = [ch_e \circ z] = [ch_e \circ \kappa^m],\ m \geq 1.$$

Let X be a finite set. By Corollary 7.42, elements of $T_*\mathcal{H}_X$ are represented by pairs $(\mathcal{X}, f)$ where $\mathcal{X}$ is an admissible X-graph and $f \in \mathsf{CetGr}_*(\mathcal{X}, \mathcal{H})$, and pairs $(\mathcal{X}^1, f^1)$ and $(\mathcal{X}^1, f^2)$ represent the same element $[\mathcal{X}, f]_* \in T_*\mathcal{H}_X$ if and only if there is a commuting diagram

$$\begin{tikzcd} \mathcal{X}^1 \arrow[r, "g^1"] \arrow[dr, "f^1"'] & \mathcal{X}^\perp \arrow[d, "f^\perp"] & \mathcal{X}^2 \arrow[l, "g^2"'] \arrow[dl, "f^2"] \\ & \mathcal{H} & \end{tikzcd} \tag{8.5}$$

in CetGr_* such that, for $j = 1, 2$, g^j is a morphism in X-$\mathsf{CGr}_{\mathsf{sim}}$, and $\mathcal{X}^\perp$ is reduced, so $f^\perp\colon \mathcal{X}^\perp \to \mathcal{H}$ is an (unpointed) étale morphism in CetGr.

Outside the (contracted) units, $\mathcal{X}^\perp$ is admissible. Otherwise $f^\perp = ch_e \in \mathsf{CetGr}(\shortmid, \mathcal{H})$ for some $e \in E$. Observe, in particular, that, for all $e \in E$, and all $m \geq 1$, the following special case of (8.5) commutes in CetGr_*:

$$\begin{tikzcd} \mathcal{C}_{\mathbf{0}} \arrow[r, "z"] \arrow[dr, "ch_e \circ z"'] & (\shortmid) \arrow[d, "ch_e"] & \mathcal{W}^m \arrow[l, "\kappa^m"'] \arrow[dl, "ch_e \circ \kappa^m"] \\ & \mathcal{H}. & \end{tikzcd} \tag{8.6}$$

This will be essential in the proof of Theorem 8.2.

8.2. **The category Ξ.** By Equation (8.1), Ξ is the restriction to CetGr_* of the Kleisli category of $\mathbb{T}_*$. So, for all pairs $(\mathcal{G}, \mathcal{H})$ of graphs

$$\Xi(\mathcal{G}, \mathcal{H}) = \mathsf{GS}_*(\mathcal{G}, T_*\mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G}).$$

In particular, for $\mathcal{G} \cong \mathcal{C}_X$ or $\mathcal{G} \cong (\shortmid)$, $\Xi(\mathcal{G}, \mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G})$ has been described in Section 8.1.

For the general case, it follows from Section 8.1 that, $\Xi(\mathcal{G},\mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G})$ is obtained as a quotient of $T\mathcal{H}(\mathcal{G})$. Hence, a morphism $\gamma \in \Xi(\mathcal{G},\mathcal{H})$ is represented by a non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$ with colimit $\mathbf{\Gamma}(\mathcal{G})$, together with a morphism $f \in \mathsf{CetGr}_*(\mathbf{\Gamma}(\mathcal{G}),\mathcal{H})$.

Moreover, since every graph $\mathcal{G}$ is trivially the colimit of the identity $\mathcal{G}$-shaped graph of graphs $\mathbf{I}^{\mathcal{G}} \colon (\mathcal{C}, b) \mapsto \mathcal{C}$ (Section 5.2), the assignment $f \mapsto [\mathbf{I}^{\mathcal{G}}, f] \in \Xi(\mathcal{G},\mathcal{H})$ induces an inclusion $\mathsf{CetGr}_* \hookrightarrow \Xi$ of categories.

So, there is weak factorisation system on Ξ where the left class consists of (boundary preserving) morphisms $[\mathbf{\Gamma}] \colon \mathcal{G} \to \mathbf{\Gamma}(\mathcal{G})$ represented by non-degenerate graphs of graphs $\mathbf{\Gamma}$, and the right class is induced by morphisms in CetGr_*. In particular, since the pair $(\mathsf{CGr}_{\mathsf{sim}}, \mathsf{CetGr})$ defines a factorisation system on CetGr_* (Corollary 7.21), the composite structure of the monad $\mathbb{DT}$ induces a ternary factorisation system on Ξ.

Let $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$ and $\mathcal{H}$ be graphs and, for $i = 1, 2$, let $\mathbf{\Gamma}^i$ be a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\mathbf{\Gamma}^i(\mathcal{G})$, and $f^i \in \mathsf{CetGr}_*(\mathbf{\Gamma}^i(\mathcal{G}),\mathcal{H})$. For each $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, let $\iota^i_b \colon \mathbf{\Gamma}^i(b) \to \mathbf{\Gamma}^i(\mathcal{G})$ denote the canonical monomorphism.

Lemma 8.7. *The pairs $(\mathbf{\Gamma}^1, f^1), (\mathbf{\Gamma}^2, f^2)$ represent the same element $\alpha \in \Xi(\mathcal{G},\mathcal{H})$ if and only if there is a non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$ with colimit $\mathbf{\Gamma}(\mathcal{G})$, a morphism $f \in \mathsf{CetGr}_*(\mathbf{\Gamma}(\mathcal{G}),\mathcal{H})$ and a commuting diagram in CetGr_* where the morphisms in the top row are vertex deletion morphisms:*

$$\begin{tikzcd} \mathbf{\Gamma}^1(\mathcal{G}) \arrow[r] \arrow[dr, "f^1"'] & \mathbf{\Gamma}(\mathcal{G}) \arrow[d, "f"] & \mathbf{\Gamma}^2(\mathcal{G}) \arrow[l] \arrow[dl, "f^2"] \\ & \mathcal{H}. & \end{tikzcd} \tag{8.8}$$

Proof. By definition, if $(\mathbf{\Gamma}^1, f^1)$ and $(\mathbf{\Gamma}^2, f^2)$ represent the same element, then, for all $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$, $(\mathbf{\Gamma}^1(b), f^1 \circ \iota^1_b)$ and $(\mathbf{\Gamma}^2(b), f^2 \circ \iota^2_b)$ are similar in X_b-$\mathsf{CGr}_{\mathsf{sim}}/\Upsilon_*\mathcal{H}$. Therefore, by Section 8.1, since $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$, there is an admissible graph $\mathbf{\Gamma}(b)$ and a morphism $f_b \in \mathsf{CetGr}_*(\mathbf{\Gamma}(b),\mathcal{H})$, such that the following diagram – in which the horizontal morphisms are vertex deletion morphisms between graphs with non-empty boundaries – commutes in CetGr_*:

$$\begin{tikzcd} \mathbf{\Gamma}^1(b) \arrow[r] \arrow[dr, "f^1 \circ \iota^1_b"'] & \mathbf{\Gamma}(b) \arrow[d, "f_b"] & \mathbf{\Gamma}^2(b \arrow[l] \arrow[dl, "f^2 \circ \iota^2_b"] \\ & \mathcal{H}. & \end{tikzcd}$$

If $(\mathbf{\Gamma}(\mathcal{G}), f)$ is the colimit of the non-degenerate $\mathcal{G}$-shaped graph of $\Upsilon_*\mathcal{H}$-structured graphs defined by $(\mathcal{C}_{X_b}, b) \mapsto (\mathbf{\Gamma}(b), f_b)$ for all $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$, then (8.8) commutes by construction. The converse follows immediately from the definitions. □

8.3. **Factorisation categories.** The graphical category Ξ is the full subcategory on graphs of the Kleisli category $\mathsf{GS}_{*\mathbb{T}_*}$ of T_* with $\mathsf{GS}_{*\mathbb{T}_*}(S_*, S'_*) = \mathsf{GS}_*(S_*, T_*S'_*)$ for all $S_*, S'_* \in \mathsf{GS}_*$.

Since elements of T_*S_X correspond to similarity classes of S_*-structured X-graphs $(\mathcal{X}, \alpha)$, each element $\beta \in \mathsf{GS}_*(\mathcal{G}, T_*S_*)$ is represented by a non-degenerate $\mathcal{X}$-shaped graph of S-structured graphs $\mathbf{\Gamma}_S$. The colimit of $\mathbf{\Gamma}_S$ describes an S_* structured X-graph $(\mathbf{\Gamma}(\mathcal{X}), \alpha)$, where $\mathbf{\Gamma}(\mathcal{X})$ is the colimit of the underlying $\mathcal{X}$-shaped graph of graphs $\mathbf{\Gamma} \colon \mathsf{el}(\mathcal{X}) \to \mathsf{CetGr}/S \to \mathsf{CetGr}$.

By [5, Proposition 2.5], the monad $\mathbb{T}_*$ has arities CetGr_* if certain categories associated to factorisations of morphisms in the Kleisli category $\mathsf{GS}_{*\mathbb{T}_*}$ ([5, Section 2.4]) are connected.

Let S_* be a pointed graphical species, $\mathcal{G}$ a graph, and let $\beta \in \mathsf{GS}_*(\mathcal{G}, T_*S)$.

Definition 8.9. *The* factorisation category $\mathsf{fact}_*(\beta)$ of β *is the category whose objects are pairs* $(\mathbf{\Gamma}, \alpha)$, *where* $\mathbf{\Gamma}$ *is a non-degenerate* $\mathcal{G}$*-shaped graph of graphs with colimit* $\mathbf{\Gamma}(\mathcal{G})$ *and* $\alpha \in \mathsf{GS}_*(\mathbf{\Gamma}(\mathcal{G}), S) \cong S(\mathbf{\Gamma}(\mathcal{G}))$ *is such that* β *is given by the composition of morphisms in* $\mathsf{GS}_{*\mathbb{T}_*}$*:*

$$\mathcal{G} \xrightarrow{\quad [\mathbf{\Gamma}] \quad} \mathbf{\Gamma}(\mathcal{G}) \xrightarrow{\quad \alpha \quad} S_*.$$

Morphisms in $\mathsf{fact}_*(\beta)((\mathbf{\Gamma}^1, \alpha^1), (\mathbf{\Gamma}^2, \alpha^2))$ *are commuting diagrams in* $\mathsf{GS}_{*\mathbb{T}_*}$

(8.10)
$$\begin{tikzcd} & \mathbf{\Gamma}^1(\mathcal{G}) \arrow[dd, "g"] \arrow[dr, "\alpha^1"] & \\ \mathcal{G} \arrow[ur, "{[\mathbf{\Gamma}^1]}"] \arrow[dr, "{[\mathbf{\Gamma}^2]}"'] & & S_* \\ & \mathbf{\Gamma}^2(\mathcal{G}) \arrow[ur, "\alpha^2"'] & \end{tikzcd}$$

such that g *is a morphism in* $\mathsf{CetGr}_* \hookrightarrow \mathsf{GS}_{*\mathbb{T}_*}$.

Lemma 8.11. *For all graphs* $\mathcal{G}$ *and all* $\beta \in \mathsf{GS}_*(\mathcal{G}, T_*S)$*, the category* $\mathsf{fact}_*(\beta)$ *is connected.*

Proof. This follows directly from the discussion above, and in particular Section 8.1.

Let S_* be a pointed graphical species. For X a finite set, S_*-structured X-graphs $(\mathcal{X}^1, \alpha^1), (\mathcal{X}^2, \alpha^2)$ represent the same element of T_*S_X if and only if they are similar in $X\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_* \cong \mathsf{GS}_*(\mathcal{C}_X, T_*S_*)$. So the lemma holds for the stick graph $(\shortmid)$ and all corollas $\mathcal{C}_X$ (including $\mathcal{C}_{\mathbf{0}}$).

Now, let $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$ be any connected graph. Elements of $\mathsf{GS}_*(\mathcal{G}, T_*S) \cong T_*S(\mathcal{G})$ are represented by non-degenerate $\mathcal{G}$-shaped graphs of S_*-structured graphs. Since there is no object of the form $(\mathcal{C}_{\mathbf{0}}, b)$ in $\mathsf{el}(\mathcal{G})$, two such non-degenerate S_*-structured graphs of graphs, $\mathbf{\Gamma}^1_{S_*}, \mathbf{\Gamma}^2_{S_*}$ represent the same element of $T_*S(\mathcal{G})$ if and only if for all $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$, $\mathbf{\Gamma}^1_{S_*}(\mathcal{C}_{X_b}, b) \sim \mathbf{\Gamma}^2_{S_*}(\mathcal{C}_{X_b}, b)$ in $X_b\text{-}\mathsf{CGr}_{\mathsf{sim}}/S_*$, whereby the colimits $\mathbf{\Gamma}^1_{S_*}(\mathcal{G})$ and $\mathbf{\Gamma}^2_{S_*}(\mathcal{G})$ are also similar in CetGr_*/S_*. Hence, $\mathsf{fact}_*(\beta)$ is connected by Corollary 7.42. □

Theorem 8.2 now follows from [5, Sections 1 & 2].

Proof of Theorem 8.2. The category CetGr_* is dense in GS_*. By [5, Proposition 2.5], the statement of Lemma 8.11 is equivalent to the statement that the monad $\mathbb{T}_*$ has arities CetGr_*.

Therefore, the monad $\mathbb{T}_*$ on GS_* has arities $\mathsf{CetGr}_*$and the induced nerve functor $N \colon \mathsf{MO} \to \mathsf{psh}(\Xi)$ is fully faithful by [5, Propositions 1.5 & 1.9].

Moreover, by [5, Theorem 1.10] its essential image is the subcategory of those presheaves on Ξ whose restriction to CetGr_* are in the image of the fully faithful embedding $\mathsf{GS}_* \hookrightarrow \mathsf{psh}(\mathsf{CetGr}_*)$.

So a presheaf P on Ξ is in the essential image of N if and only if

$$P(\mathcal{G}) = \lim_{(\mathcal{C},b) \in \mathsf{el}_*(\mathcal{G})} P(\mathcal{C}),$$

and, by finality of $\mathsf{el}(\mathcal{G}) \subset \mathsf{el}_*(\mathcal{G})$, this is the case precisely if, for all graphs $\mathcal{G}$,

$$P(\mathcal{G}) = \lim_{(\mathcal{C},b) \in \mathsf{el}(\mathcal{G})} P(\mathcal{C}).$$

□

Remark 8.12. Using the method of [5, Section 2], we can construct the corresponding *unpointed* factorisation categories for the monad $\mathbb{DT}$ on GS.

For any graphical species S and graph $\mathcal{G}$, $\mathsf{GS}_{\mathbb{DT}}(\Upsilon\mathcal{G}, S) \cong \mathsf{GS}_*(\Upsilon_*\mathcal{G}, T_*S^+) \cong S^+(\mathcal{G})$ canonically.

So, a morphism $\beta\colon \Upsilon\mathcal{G} \to S$ in the Kleisli category $\mathsf{GS}_{\mathbb{DT}}$ is represented by a $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$ with colimit $\mathbf{\Gamma}(\mathcal{G})$, and a DS-structure $\alpha \in \mathsf{GS}(\mathbf{\Gamma}(\mathcal{G}), DS) \cong DS(\mathbf{\Gamma}(\mathcal{G}))$.

Such pairs $(\mathbf{\Gamma}, \alpha)$ are the objects of the unpointed factorisation category $\mathsf{fact}(\beta)$. Morphisms in $\mathsf{fact}(\beta)((\mathbf{\Gamma}, \alpha), (\mathbf{\Gamma}', \alpha'))$ are morphisms in $\mathsf{CetGr}(\mathbf{\Gamma}(\mathcal{G}), \mathbf{\Gamma}(\mathcal{G}'))$ making the diagram (8.10) commute.

By [5, Proposition 2.5], $\mathbb{DT}$ has arities CetGr if and only if $\mathsf{fact}(\beta)$ is connected for all S, $\mathcal{G}$, and β.

To see that this is not the case, let $S = \Upsilon(\shortmid)$, and so $TS \cong S$. Let $\mathcal{G} = (\shortmid)$ and let $\beta = o = z\colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$. Then the diagrams $\mathcal{C}_{\mathbf{0}} \to \mathcal{C}_{\mathbf{0}} \xrightarrow{z} (\shortmid)$, and $\mathcal{C}_{\mathbf{0}} \to \mathcal{W} \xrightarrow{\kappa} (\shortmid)$ describe objects in $\mathsf{fact}(\beta)$. Since there are no non-trivial morphisms in CetGr with domain or codomain $\mathcal{C}_{\mathbf{0}}$, these objects are in disjoint components of $\mathsf{fact}(\beta)$. Therefore, $\mathbb{DT}$ does not have arities CetGr.

8.4. **Weak modular operads.** In [20, 21], Hackney, Robertson and Yau have proved a version of Theorem 8.2 in terms of a bijective-on-objects subcategory U of Ξ that was constructed precisely so as to have a generalised Reedy structure. The category U does not contain those morphisms in $\mathsf{CetGr}_* \hookrightarrow \Xi$ that factor through $z\colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$ or $\kappa^m\colon \ \to (\shortmid)$, $m \geq 1$, nor does it contain any morphisms of CetGr that are not monomorphic. Therefore, the inclusion $U \hookrightarrow \mathsf{MO}$ is not fully faithful. However, by [21, Theorem 3.6], it is dense and hence induces a fully faithful nerve.

In [20, Theorem 3.8], they have shown that the simplicial presheaf category $\mathsf{psh}_{\mathsf{sSet}}(U)$ admits a cofibrantly generated model structure, obtained by localising the Reedy model structure at the *Segal maps*

$$\lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \longrightarrow P(\mathcal{G}),$$

and that the fibrant objects for this model structure are those simplicial presheaves on U that satisfy the *weak Segal condition*

$$P(\mathcal{G}) \simeq \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}), \quad \text{for all graphs } \mathcal{G} \in U. \tag{8.13}$$

In [9], Caviglia and Horel describe a general class of rigidification results whereby, given a dense inclusion $\mathsf{D} \hookrightarrow \mathsf{C}$ of categories satisfying certain conditions, an equivalence is established between sSet-valued presheaves on D that satisfy a weak Segal condition, and C objects internal to sSet that satisfy the Segal condition on the nose. Furthermore, they apply their result to a certain class of monads with arities. This leads directly to the following corollary of Theorem 8.2:

Corollary 8.14. *There is a model category structure on the category* $\mathsf{psh}_{\mathsf{sSet}}(\Xi)$ *of functors* $P\colon \Xi^{\mathrm{op}} \to \mathsf{sSet}$ *whose fibrant objects are those* P *that satisfy the weak Segal condition:*

$$P(\mathcal{G}) \simeq \lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \quad \textit{for all graphs } \mathcal{G} \in \mathsf{CetGr}. \tag{8.15}$$

Proof. The monad $\mathbb{T}_*$ has arities CetGr_* and $\mathsf{el}(\mathcal{G})$ is connected and essentially small for all connected graphs $\mathcal{G}$. Therefore the assumptions of [9, Assumptions 7.9] are satisfied. By [9, Section 7.5], MO is equivalent to the category of models in Set of the limit sketch $L = (\mathsf{CetGr}_*, \{(\mathcal{G}/\mathbf{B}^{\S^{\mathrm{op}}})_{\mathcal{G}\in\mathsf{CetGr}}\})$. There is a Segal model structure on the category of sSet valued models for L.

By [9, Proposition 7.1], this can be transferred along a Quillen equivalence to a model structure on $\mathsf{psh}_{\mathsf{sSet}}(\Xi)$, whose fibrant objects are those presheaves that satisfy the weak Segal condition. □

In current work with M. Robertson, we are comparing the existing models for weak modular operads. We expect that there is a direct Quillen equivalence between the model structure on $\mathsf{psh}_{\mathsf{sSet}}(\Xi)$ of Corollary 8.14 and the model structure on $\mathsf{psh}_{\mathsf{sSet}}(U)$ of [20].

Remark 8.16. The main statements of [22] were formulated in terms of the graphical category $\overline{Gr}$, whose morphisms are described in [22, Section 6]. This is the bijective-on-objects subcategory of Ξ that does not contain any morphisms in CetGr_* that factor through $z\colon \mathcal{C}_{\mathbf{0}} \to (\shortmid)$ or $\kappa\colon \mathcal{W} \to (\shortmid)$. Therefore, $\overline{Gr}$ does not embed fully faithfully in MO.

There are inclusions $U \subset \overline{Gr} \subset \Xi$. So, since Ξ and U are both dense in MO, $\overline{Gr}$ is also dense in MO, and yields a fully faithful nerve functor whose essential image satisfies the same Segal condition (8.3); whence we may deduce the main statement of [22]. (See also [21, Theorem 3.6 & Section 4] for details.)

References

[1] Dror Bar-Natan and Zsuzsanna Dancso. Finite type invariants of w-knotted objects II: tangles, foams and the Kashiwara-Vergne problem. *Math. Ann.*, 367(3-4):1517–1586, 2017.

[2] M. A. Batanin and C. Berger. Homotopy theory for algebras over polynomial monads. *Theory Appl. Categ.*, 32:Paper No. 6, 148–253, 2017.

[3] Jon Beck. Distributive laws. In *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 119–140. Springer, Berlin, 1969.

[4] James C. Becker and Daniel Henry Gottlieb. A history of duality in algebraic topology. In *History of topology*, pages 725–745. North-Holland, Amsterdam, 1999.

[5] Clemens Berger, Paul-André Melliès, and Mark Weber. Monads with arities and their associated theories. *J. Pure Appl. Algebra*, 216(8-9):2029–2048, 2012.

[6] Clemens Berger and Ieke Moerdijk. Resolution of coloured operads and rectification of homotopy algebras. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 31–58. Amer. Math. Soc., Providence, RI, 2007.

[7] Dennis V. Borisov and Yuri I. Manin. Generalized operads and their inner cohomomorphisms. In *Geometry and dynamics of groups and spaces*, volume 265 of *Progr. Math.*, pages 247–308. Birkhäuser, Basel, 2008.

[8] John Bourke and Richard Garner. Monads and theories. *Adv. Math.*, 351:1024–1071, 2019.

[9] Giovanni Caviglia and Geoffroy Horel. Rigidification of higher categorical structures. *Algebr. Geom. Topol.*, 16(6):3533–3562, 2016.

[10] Eugenia Cheng. Iterated distributive laws. *Math. Proc. Cambridge Philos. Soc.*, 150(3):459–487, 2011.

[11] Denis-Charles Cisinski and Ieke Moerdijk. Dendroidal Segal spaces and ∞-operads. *J. Topol.*, 6(3):675–704, 2013.

[12] Pierre-Louis Curien and Jovana Obradović. Categorified cyclic operads. *Appl. Categ. Structures*, 28(1):59–112, 2020.

[13] Zsuzsanna Dancso, Iva Halacheva, and Marcy Robertson. Circuit algebras are wheeled props, 2020.

[14] Gabriel C. Drummond-Cole and Philip Hackney. Dwyer–Kan homotopy theory for cyclic operads. 2018.

[15] E. Getzler and M. M. Kapranov. Cyclic operads and cyclic homology. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 167–201. Int. Press, Cambridge, MA, 1995.

[16] E. Getzler and M. M. Kapranov. Modular operads. *Compositio Math.*, 110(1):65–126, 1998.

[17] Jeffrey Giansiracusa. Moduli spaces and modular operads. *Morfismos*, 17(2):101–125, 2013.

[18] Philip Hackney, Marcy Robertson, and Donald Yau. *Infinity properads and infinity wheeled properads*, volume 2147 of *Lecture Notes in Mathematics*. Springer, Cham, 2015.

[19] Philip Hackney, Marcy Robertson, and Donald Yau. On factorizations of graphical maps. *Homology Homotopy Appl.*, 20(2):217–238, 2018.

[20] Philip Hackney, Marcy Robertson, and Donald Yau. A graphical category for higher modular operads. *Adv. Math.*, 365:107044, 2020.

[21] Philip Hackney, Marcy Robertson, and Donald Yau. Modular operads and the nerve theorem. *Adv. Math.*, 370:107206, 39, 2020.

[22] A. Joyal and J. Kock. Feynman graphs, and nerve theorem for compact symmetric multicategories (extended abstract). *Electronic Note in Theoretical Computer Science*, 270(2):105 – 113, 2011. Proceedings of the 6th International Workshop on Quantum Physics and Logic (QPL 2009).

[23] André Joyal. Une théorie combinatoire des séries formelles. *Adv. in Math.*, 42(1):1–82, 1981.

[24] G. M. Kelly. Many-variable functorial calculus. I. In *Coherence in categories*, pages 66–105. Lecture Notes in Math., Vol. 281. 1972.

[25] G. M. Kelly. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bull. Austral. Math. Soc.*, 22(1):1–83, 1980.

[26] G. M. Kelly and M. L. Laplaza. Coherence for compact closed categories. *J. Pure Appl. Algebra*, 19:193–213, 1980.

[27] Joachim Kock. Graphs, hypergraphs, and properads. *Collect. Math.*, 67(2):155–190, 2016.

[28] Joachim Lambek. *Completions of categories*. Seminar lectures given in 1966 in Zürich. Lecture Notes in Mathematics, No. 24. Springer-Verlag, Berlin-New York, 1966.

[29] Saunders Mac Lane. Categorical algebra. *Bull. Amer. Math. Soc.*, 71:40–106, 1965.

[30] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[31] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[32] M. Markl, S. Merkulov, and S. Shadrin. Wheeled PROPs, graph complexes and the master equation. *J. Pure Appl. Algebra*, 213(4):496–535, 2009.

[33] Sergei A. Merkulov. Wheeled props in algebra, geometry and quantization. In *European Congress of Mathematics*, pages 83–114. Eur. Math. Soc., Zürich, 2010.

[34] Ieke Moerdijk and Ittay Weiss. Dendroidal sets. *Algebr. Geom. Topol.*, 7:1441–1470, 2007.

[35] S. Raynor. *Compact Symmetric Multicategories and the problem of loops*. PhD thesis, University of Aberdeen, 2018.

[36] S. Raynor. Modular operads, and a graphical calculus and nerve for circuit algebras. *In preparation*, 2021.

[37] S. Raynor. Modular operads, and a graphical calculus and nerve for compact closed categories. *In preparation.*, 2021.

[38] S. Raynor. An operadic model structure for compact closed categories. *In preparation.*, 2021.

[39] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.

[40] Bruno Vallette. A Koszul duality for PROPs. *Trans. Amer. Math. Soc.*, 359(10):4865–4943, 2007.

[41] Mark Weber. Familial 2-functors and parametric right adjoints. *Theory Appl. Categ.*, 18:No. 22, 665–732, 2007.

[42] Donald Yau and Mark W. Johnson. *A foundation for PROPs, algebras, and modules*, volume 203 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.