GRAPHICAL COMBINATORICS AND A DISTRIBUTIVE LAW FOR MODULAR OPERADS

SOPHIE RAYNOR

Abstract. This work presents a detailed analysis of the combinatorics of modular operads. These are operad-like structures that admit a contraction operation as well as an operadic multiplication. Their combinatorics are governed by graphs that admit cycles, and are known for their complexity. In 2011, Joyal and Kock introduced a powerful graphical formalism for modular operads. In this paper, which extends that work, a monad for modular operads is constructed and a corresponding nerve theorem is proved, using Weber's abstract nerve theory, in the terms originally stated by Joyal and Kock. This is achieved using a distributive law that sheds new light on the combinatorics of modular operads.

Introduction

Modular operads, introduced in [16] to study moduli spaces of Riemann surfaces, are a " 'higher genus' analogue of operads ... in which graphs replace trees in the definition." [16, Abstract].

Roughly speaking, modular operads are $\mathbb{N}$-graded objects $P = \{P(n)\}_{n\in\mathbb{N}}$ that, alongside an operadic multiplication (or composition) $\circ : P(n) \times P(m) \to P(m+n-2)$ for $m, n \geq 1$, admit a contraction operation $\zeta : P(n) \to P(n-2)$, $n \geq 2$. The contraction encodes higher genus structure and trace operations. This work considers a notion of modular operads originally due to Joyal and Kock [23].[1]

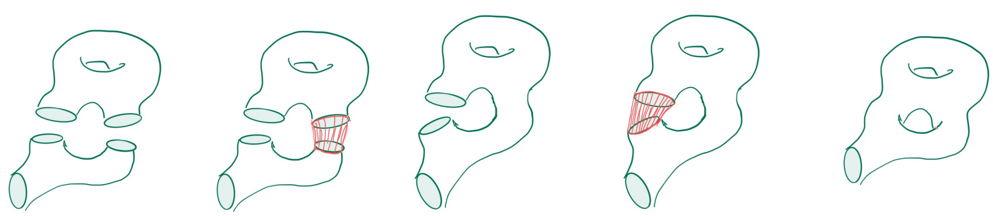

Figure 1. (Self)-gluing of surfaces along boundary components. Moduli of geometric structures – such as Riemann surfaces – are a rich source of examples of modular operads.

Their definition incorporates a broad compass of related structures: including modular operads in the original (undirected) sense [14, 17, 16], but also their directed counterparts. In fact, directed modular operads are equivalent to wheeled properads [19, 40]. More generally, compact closed categories [26] may also be described in terms of modular operads [36, 37]. These are closely related to circuit algebras, used in the study of virtual tangles [1, 2, 12]. As such, modular operads have applications across a range of disciplines.

However, the combinatorics of modular operads are complex. In modular operads equipped with a multiplicative unit, contracting the unit leads to an exceptional 'loop', that can cause an obstruction to

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[1]Joyal and Kock used the term 'compact symmetric multicategories (CSMs)' in [23] to refer to what are here called 'modular operads'. Indeed, I adopted this terminology in a previous version of this paper. In my PhD thesis [35], I used 'CSM' to refer to modular operads with muliplicative units, and reserved 'modular operad' for those without.

proving general results. This paper represents a detailed investigation into the graphical combinatorics of modular operads, and provides a new approach to understanding and working with loops.

In [23], which forms the inspiration for this work, Joyal and Kock construct modular operads as algebras for an endofunctor (Remark 2.5) on a category GS of coloured collections called 'graphical species'. Their machinery is significant in its simplicity. It relies only on minimal data and basic categorical constructions, that give it considerable formal and expressive power.

However, the problem of exceptional loops means that their modular operad endofunctor does not extend to a monad on GS. As a consequence, it does not lead to a precise description of the relationship between modular operads and their graphical combinatorics. (See Section 6 for technical details.)

The following results were first stated in [23], and proved in [35] (where I used similar, though slightly less general methods from those presented here):

Theorem A (Theorem 7.47)**.** *The category* MO *of modular operads is isomorphic to the Eilenberg-Moore category of algebras for a monad* $\mathbb{O}$ *on the category* GS *of graphical species.*

Theorem B (Theorem 8.1)**.** *The category* MO *has a fully faithful subcategory* Ξ *whose objects are graphs. The induced nerve* $N : \mathsf{MO} \to \mathsf{ps}(\Xi)$ *is fully faithful and its essential image is characterised by those presheaves that satisfy a canonical Segal condition.*

An obvious motivation for establishing such a result is provided by the study of weak (up-to-homotopy), or $(\infty, 1)$-modular operads, by weakening the Segal condition of Theorem B (Theorem 8.1), and a number of potential applications of such structures are discussed in the introduction to [21]. To this end, [21, 22] have also recently proved versions of these theorems, by different methods, and used them to obtain a model of $(\infty, 1)$-modular operads that are characterised in terms of a weak Segal condition.

The aim in this work is present a proofs of Theorems A and B, by methods closely based on those of [23], and to use this as a route to fully understanding the underlying combinatorics, and the contraction of multiplicative units in particular.

Neither the construction of the monad $\mathbb{O}$ for modular operads, nor the proof of the nerve theorem B is entirely straightforward. First, the approach of [23], which is closely related to analogous constructions for operads (see Examples 5.1 and, e.g. 6.1 and [19, 27, 31, 34]) does not lead to a well-defined monad. Second, the desired monad, once obtained, does not satisfy the conditions for using the abstract nerve machinery introduced by Weber [39, 6]. To prove the theorems, it is therefore necessary to dissect the problem into its constituent components, thereby rendering the graphical combinatorics of modular operads completely explicit.

Since the obstruction to obtaining a monad in [23] arises from the combination of the modular operadic contraction operation and the multiplicative units (see Section 6), the approach of this work is to first treat these structures seperately – to obtain a monad $\mathbb{T}$ on GS whose algebras are non-unital modular operads, and a monad $\mathbb{D}$ on GS that adjoins distinguished 'unit' elements – and then combine them, using the theory of distributive laws [4].

Theorem A is then a corollary of:

Theorem C (Proposition 7.41 & Theorem 7.47)**.** *There is a distibutive law* λ *of* $\mathbb{T}$ *over* $\mathbb{D}$ *such that the resulting composite monad* $\mathbb{DT}$ *on* GS *is precisely the modular operad monad* $\mathbb{O}$ *of Theorem A.*

The graphical category Ξ, used to define the modular operadic nerve, arises canonically via the unique fully faithful - bijective on objects factorisation of a functor used in the construction of $\mathbb{O}$. Therefore, if the monad $\mathbb{O}$ satisfies certain formal conditions [6], Theorem B follows from the abstract nerve machinery of [39, 6]. (In this case the monad is said to 'have arities'.)

Though these conditions are not satisfied by $\mathbb{O}$ on GS, the distributive law in Theorem C means that we can lift the monad $\mathbb{T}$ to obtain a monad $\mathbb{T}_*$ on the category GS_* of $\mathbb{D}$-algebras. It is then sufficient to show that the Weber machinery can be applied to this new monad $\mathbb{T}_*$.

Lemma C (Lemma 8.13)**.** *The monad $\mathbb{T}_*$ on GS_* satisfies the conditions of Weber's nerve theory.*

I'll conclude this introduction by briefly mentioning three (related) benefits of this abstract approach.

In the first place, the results obtained by this method provide a clear overview of how modular operads fit into the wider framework of operadic structures, and how other general results may be modified to this setting. For example, by Lemma C, $\mathbb{T}_*$ and Ξ satisfy the Assumptions 7.9 of [10] which leads to a suitable notion of weak modular operads via the following corollary:

Corollary C. *There is a model structure on the category of presheaves in simplicial sets (or topological spaces) on Ξ. The fibrant objects are precisely those presheaves that satisfy a weak Segal condition.*

Marcy Robertson and I are comparing this model structure with the model structure for weak modular operads in [21], and exploring whether Ξ, itself, admits a generalised Reedy structure.

Secondly, since this work makes the combinatorics of modular operads – and the tricky bits in particular – completely explicit, it provides a clear road map for working with and extending the theory.

One fruitful direction for extending this work is provided by the use of iterated distributive laws [11] to generalise constructions presented here. In [37], I use an iterated distributive law to construct for a category of modular operads with product – canonically equivalent to the category of small compact categories – as algebras for a composite monad on GS. Once again the distributive laws play an important role in describing the corresponding nerve. Iterated distributive laws may also be used to construct higher modular operads. Using methods similar to [11, Section 3], it is possible to obtain an appropriate notion of n-modular operads for $n \geq 0$. This can be used to give an undirected description of extended cobordism categories.

Finally, in many cases, the complexities of the combinatorics of contractions can provide deep insights into the structures they are intended to model. In current work, also together with ✠......, I'm using these ideas to explore the contraction of spheres to points in the compactification of moduli spaces of algebraic curves.

This work falls into three parts. The opening two sections provide context and background for the rest of the work. An axiomatic definition of modular operads is given in Section 1, and Section 2 is brief review of the basic theory of monads and Weber's abstract nerve theory, that is needed in the rest of the paper. Both these introductory sections include number of examples to motivate the constructions that follow.

The second part, comprising Sections 3 – 5, establishes the graphical and functorial machinery for studying the combinarics of modular operads. In Sections 1 and 2, the (Feynman) graphs of [23] are introduced and discussed in detail in Section 3. Section 4 focuses on local homeomorphisms, or étale morphisms', of graphs. Section 5 is devoted to a description of the monad $\mathbb{T}$ for non-unital CSMs.

The final part of the paper (Sections 6 – 8) deals with the problem of contracting multiplicative units. Though considerable background is needed to understand the details, I note a couple of key points here, as a guide to the reader.

Before jumping into the construction of unital modular operads in Section 7, we pause for a moment in Section 6 to provide some context and motivation by looking in detail at the construction of [23], and the appearance of problematic *loops* in the theory.

- Remarks 6.7 and 6.8 refer to other approaches to the issue of loops. As far as I'm aware, the graphical construction presented in this paper is unique in that it *does not* incorporate some version of the exceptional loop into the graphical calculus, in order to model contractions of units.
- The essense of the problem is revealed in the discussion around Equation (6.6) and after Remark 6.8: the theory appears to incorporate isomorphisms that do not behave well with respect to certain colimits. It seems like there is an arrow in the wrong direction.

The construction of the monad $\mathbb{O}$ for modular operads happens in Section 7. This is the longest and most important section of the work - and where the main contributions can be found. The key points to note are:

- The definition of modular operads Definition 1.22 suggest that an extra morphism should be added to the category of graphs (this is $z : \mathbf{0} \to \S$, see Lemma 7.5). Whereas before, exceptional loops arose from incorporating a formal colimit in the theory, this morphism looks more like a formal limit.
- The monad $\mathbb{D}$ that decribes units (Lemma 7.6), and the distributive law (Proposition 7.41) induce a notion of equivalence, or'similarity', of graphs whereby it is always possible to work with well-behaved representatives, and only quotient by equivalence at the end. This is heavily used in the proof of Theorem 8.1.

Finally, Section 8 contains the proof of the nerve theorem B, Corollary C and a short discussion on weak modular operads.[2]

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1. Definitions and examples

The goal of this subsection is to give an axiomatic definition of modular operads (Definition 1.22), and to provide some motivating examples. As mentioned in the introduction, the term 'modular operad' refers, in this work, to what are called 'compact symmetric multicategories (CSMs)' in [23].

After establishing some basic notional conventions, the section begins with a discussion of Joyal and Kock's category GS of graphical species [23] which generalises various notions of coloured collection used in the study of operads.

1.1. **General notation.** A *presheaf* on a category C is a functor $S : \mathsf{C}^{op} \to \mathsf{Set}$. The corresponding functor category or *presheaf category* is denoted $\mathsf{ps}(\mathsf{C})$. I denote the category of finite sets by $\mathsf{Set}_{\mathsf{f}}$.

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[2] Most of Sections 3 – 5 and Section 8 appeared in my PhD thesis [35]. There are some more substantial changes to Sections 6 and 7, since [35] did not make use of the distributive law.

Let D be a subcategory of a category C, and c and object of C. The *slice category* $\mathsf{D}|c$ *of* D *above* c is the category whose objects are pairs (d, f) with $d \in \mathsf{D}$, and $f \in \mathsf{C}(d, c)$, and whose morphisms $g : (d', f') \to (d, f)$ are morphisms $g \in \mathsf{C}(d', d)$ such that $f' = gf$.

For $n \in \mathbb{N}$, the set $\{1, \ldots, n\}$ is denoted by $\mathbf{n}$, so $\mathbf{0}$ denotes the empty set $\emptyset$.

The groupoid of finite sets and bijections is denoted by $\mathbb{P}$.

Remark 1.1. For each $n \in \mathbb{N}$, let Σ_n denote the permuation group $Aut(\mathbf{n})$. The permutation groupoid Σ with objects $n \in \mathbb{N}$ and morphisms $\Sigma(m, n) = \Sigma_n$ if $m = n$, and $\Sigma(m, n) = \emptyset$ if $m \neq n$, is skeletal in $\mathbb{P}$.

A presheaf $P : \mathbb{P}^{op} \to \mathsf{Set}$ on $\mathbb{P}$ (also called a *species* [24]) determines a presheaf on Σ (or *symmetric sequence*) by restriction.

Conversely, a Σ-presheaf Q may be canonically extended to a $\mathbb{P}$-presheaf $Q_{\mathbb{P}}$, by setting, for all $n \in \mathbb{N}$

$$Q_{\mathbb{P}}(X) \overset{\text{def}}{=} lim_{(n,f) \in \Sigma|X} Q(n).$$

In general, it will be notationally convenient – particularly for describing operations on modular operads – to work with presheaves over (categories containing) $\mathbb{P}$ instead of Σ.

An *involutive set* is a set $\mathfrak{C}$ equipped with an involution $\omega : \mathfrak{C} \to \mathfrak{C}$, $\omega^2 = id$. Given an involutive set $(\mathfrak{C}, \omega)$, the set of ω-orbits in $\mathfrak{C}$ is denoted by $\widetilde{\mathfrak{C}}$, and $\tilde{c} \in \widetilde{\mathfrak{C}}$ is orbit of $c \in \mathfrak{C}$.

1.2. **Graphical species.** Following [23, Section 4], graphical species are presheaves on the category $\mathbb{P}^{\circlearrowleft}$, obtained from $\mathbb{P}$ by adjoining a distinguished involutive object $\S$ according to

$$\begin{array}{rcl} \mathbb{P}^{\circlearrowleft}(\S, \S) & = & \{id, \tau\} \text{ with } \tau^2 = id, \\ \mathbb{P}^{\circlearrowleft}(\S, X) & = & \{ch_x\}_{x \in X} \amalg \{ch_x \circ \tau\}_{x \in X} \\ \mathbb{P}^{\circlearrowleft}(X, Y) & = & \mathbb{P}(X, Y), \text{ where } X \text{ and } Y \text{ are finite sets.} \end{array}$$

Definition 1.2. *A* graphical species *is a presheaf* $S : \mathbb{P}^{\circlearrowleft op} \to \mathsf{Set}$. *The category of graphical species is denoted* $\mathsf{GS} \overset{\text{def}}{=} \mathsf{ps}(\mathbb{P}^{\circlearrowleft})$.

Hence, a graphical species S is described by a species $(S_X)_{X \in \mathsf{Set_f}}$, and an involutive set $(\mathfrak{C}, \omega) = (S_\S, S(\tau))$, together with, for each finite set X, and $x \in X$ a $\mathbb{P}$ -equivariant projection $S(ch_x) : S_X \to \mathfrak{C}$.

In this case S is called a $(\mathfrak{C}, \omega)$*-coloured graphical species* and elements $c \in \mathfrak{C}$ are *colours of* S. The pair $(\mathfrak{C}, \omega)$ is called the *(involutive) palette of* S. As usual, the set of ω-orbits in $\mathfrak{C}$ is denoted by $\tilde{\mathfrak{C}}$.

If $(\mathfrak{C}, \omega) \cong \{*\}$ is trivial then S is a *monochrome graphical species.*

Remark 1.3. The involution τ on $\S$ is responsible for most of the heavy lifting in the constructions that follow. Initially, its role may seem obscure. I mention two key features here. First, the involution enables us to encode local structure such as boundary type, or spin, that may, or may not be directed, thereby bolstering the expressive power of graphical species. (Directed graphical species are discussed in Example 1.11.)

The second is more fundamental. As we'll see in Section 4, The involution enables us to describe formal compositions in graphical species, described in terms of graphs, as categorical limits. This allows us to derive our results by purely abstract methods. For example, it easily leads to a well-defined notion of *graph substitution*, in terms of diagram colimits, without the need to specify extra data (see Sections 5 and 6, and compare with, e.g. [40, 19]).

Example 1.4. The terminal graphical species Z has trivial palette and $Z_X = \{*\}$ for all finite sets X.

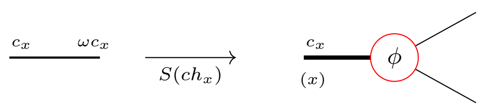
Definition 1.5. *A morphism* $\gamma \in \mathsf{GS}(S, S')$ palette-preserving *if its component at* $\S$ *is the identity on* $S_\S$. *For a fixed palette* $(\mathfrak{C}, \omega)$, $\mathsf{GS}^{(C,\omega)}$ *is the full subcategory of* GS *of* $(\mathfrak{C}, \omega)$*-coloured graphical species and palette-preserving morphisms.*

Definition 1.6. *For each element* $\underline{c} = (c_x)_{x \in X} \in \mathfrak{C}^X$, *the* $\underline{c}$-(coloured) arity $S_{\underline{c}}$ *is the fibre above* $\underline{c} \in \mathfrak{C}^X$ *of the map* $(S(ch_x))_{x \in X} : S_X \to \mathfrak{C}^X$.

If $\gamma \in \mathsf{GS}(S, S')$ is a palette-preserving morphism of $(\mathfrak{C}, \omega)$-coloured graphical species, then $\gamma(S_{\underline{c}}) \subset S'_{\underline{c}}$ for all finite sets X and all $\underline{c} \in \mathfrak{C}^X$.

Remark 1.7. Elements $\phi \in S_{\underline{c}} \subset S_X$, where S is a $(\mathfrak{C}, \omega)$-coloured graphical species and $\underline{c} = (c_x)_{x \in X} \in \mathfrak{C}^X$ for some finite set X, may be visualised as a corolla (or *star graph* or *spider*), with vertex *decorated by* ϕ, and whose legs are bijectively *coloured* by $c_x, x \in X$.

It's also helpful to visualise an element c of $\mathfrak{C} = S_\S$ as a line with one end labelled by c, and the other by ωc (see also the directed edge in Figure 2).



Example 1.8. For any pallette $(\mathfrak{C}, \omega)$, the terminal $(\mathfrak{C}, \omega)$-coloured graphical species $Z^{(C,\omega)}$ in $\mathsf{GS}^{(C,\omega)}$ is described by

$$Z^{(C,\omega)}_X = \mathfrak{C}^X, \text{ with } Z^{(C,\omega)}_{\underline{c}} = \{*\}, \text{ for all finite sets } X \text{ and all } \underline{c} \in \mathfrak{C}^X.$$

Any presheaf $P : \mathsf{C}^{op} \to \mathsf{Set}$ on a category C has an associated *element category* $\int_{\mathsf{C}} P$ given by $\int_{\mathsf{C}} P \overset{\text{def}}{=} (*|P)^{op}$, where $* : \mathsf{C} \to \mathsf{Set}$ is the terminal presheaf on C.

So, objects of $\int_{\mathsf{C}} P$ are pairs (c, α) $c \in \mathsf{C}, \alpha \in P(c)$, and morphisms $(c, \alpha) \to (c', \alpha')$ are morphisms $f \in \mathsf{C}(c, c')$ such that $P(f)(\alpha') = \alpha$. Of course, $\int_{\mathsf{C}} * \cong \mathsf{C}$ canonically.

Definition 1.9. *The category* $\mathsf{el}(S)$ *of* elements of a graphical species S *is defined as*

$$\mathsf{el}(S) \overset{\text{def}}{=} \int_{\mathbb{P}^{\circlearrowleft}} S.$$

Remark 1.10. I only use the term 'category of elements' specifically to refer to categories of the form $\int_{\mathbb{P}^{\circlearrowleft}} S$ where S is a graphical species. In Section 4.4, we'll see that graphical species may be extended to presheaves on a larger category $\mathsf{Gr} \supset \mathbb{P}^{\circlearrowleft}$ of graphs. To distinguish the two viewpoints, objects in $\int_{\mathsf{Gr}} S$ will be called 'S-structured graphs'.

For any graphical species R, there is a canonical isomorphism $\mathsf{GS}|R \cong \mathsf{ps}(\mathsf{el}(R))$ given by:

$$(S, \gamma : S \to R) \longleftrightarrow S^\gamma : \mathsf{el}(R)^{op} \to \mathsf{Set} \text{ defined by } \begin{cases} (\S, c) \mapsto \gamma_\S^{-1}(c), & c \in R_\S \\ (X, \phi) \mapsto \gamma_X^{-1}(\phi), & \phi \in R_X, \ X \in \mathsf{Set_f} \end{cases}$$

where, for each object $\mathcal{C}$ of $\mathbb{P}^{\circlearrowleft}$, $\gamma_{\mathcal{C}}$ denotes the component of γ at $\mathcal{C}$.

Example 1.11. Consider the unique non-identity involution $\sigma_{\mathfrak{Di}}$ on the set $\mathfrak{Di} \overset{\text{def}}{=} \{\text{in}, \text{out}\}$.

The terminal $(\mathfrak{Di}, \sigma_{\mathfrak{Di}})$-coloured graphical species $Di \overset{\text{def}}{=} Z^{(\mathfrak{Di}, \sigma_{\mathfrak{Di}})}$ takes each finite set X to the set $\{\text{in}, \text{out}\}^X$ of ordered partitions $X = X_{\text{in}} \amalg X_{\text{out}}$ of X into 'input' and 'output' sets. We'll think of the objects of $\mathsf{el}(Di)$ as directed corollas $C_{X;Y}$ and the directed exceptional edge $(\downarrow)$. (See Figure 2.)

Note that $(\downarrow)$ has no non-trivial endomorphisms in $\mathsf{el}(Di)$. So, $\mathsf{el}(Di)$ is equivalent to the category $\mathbb{P}^{(\downarrow)}$, obtained from $\mathbb{P} \times \mathbb{P}^{op}$ by adjoining a distinguished element $\downarrow$ and, for all $x \in X$ and $y \in Y$, morphisms $i_x, o_y : (\downarrow) \to (X, Y)$ (induced by $Di(ch_x)$ and $Di(ch_y \circ \tau)$ in $\mathbb{P}^{\circlearrowleft}(\S, X \amalg Y)$)).

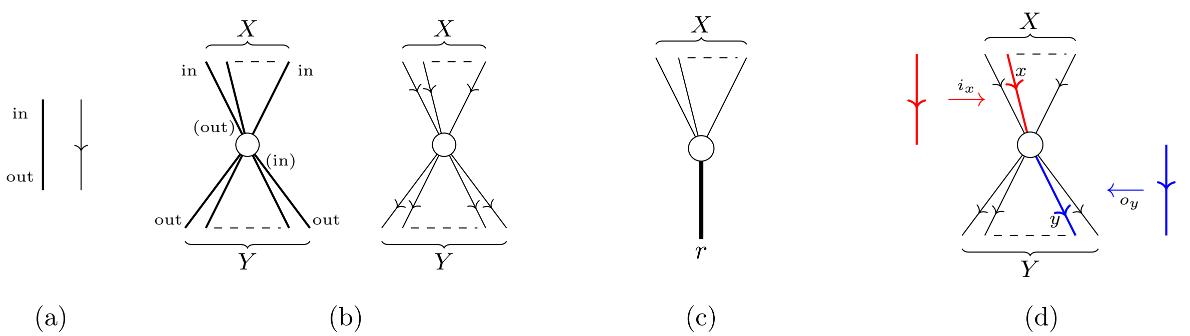


FIGURE 2. (a) the directed exceptional edge (↓), (b) the directed corolla $C_{X;Y}$, (c) the rooted corolla t_X, (d) input and output morphisms in $\mathbb{P}^{\downarrow}$.

The rooted corollas $t_X \stackrel{\text{def}}{=} C_{X;\{r\}}$ will be particularly important in Examples 1.17, 2.13, 5.1.

The category $\overrightarrow{\mathsf{GS}}$ of *directed graphical species* is the slice category $\stackrel{\text{def}}{=} \mathsf{GS}|Di \cong \mathsf{ps}(\mathsf{el}(Di))$.

1.3. **Multiplication and contraction on graphical species.** Intuitively, a multiplication $\diamond$ on a graphical species S is a rule for combining (gluing) distinct elements of S along pairs of legs ('ports') with dual colouring as in Figure 3:

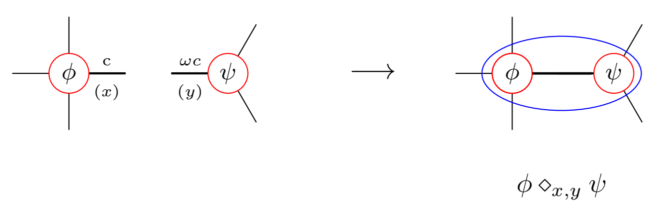


FIGURE 3. Multiplication

Definition 1.12. *Let S be a $(\mathfrak{C},\omega)$- coloured graphical species. A* multiplication $\diamond$ *on S is given by a family of partial maps*

$$- \diamond_{x,y}^{X,Y} - : S_{X\amalg\{x\}} \times S_{Y\amalg\{y\}} \twoheadrightarrow S_{X\amalg Y},$$

defined (for all X, Y), whenever $\phi \in S_{X\amalg\{x\}}, \psi \in S_{Y\amalg\{y\}}$ satisfy $S(ch_x)(\phi) = S(ch_y \circ \tau_{(\tilde{e})})(\psi)$.

Wherever it is defined, the multiplication $\diamond$ satisfies the following conditions:

(m1) (Commutativity axiom.)

$$\psi \diamond_{y,x}^{Y,X} \phi = \phi \diamond_{x,y}^{X,Y} \psi$$

(m2) (Equivariance axiom.)

The multiplication $\diamond$ respects the action of $\mathbb{P}$ on S: For all bijections $\sigma : X \xrightarrow{\cong} W$ and $\rho : Y \xrightarrow{\cong} Z$ that extend to bijections $\hat{\sigma} : X \amalg \{x\} \xrightarrow{\cong} W \amalg \{w\}$ and $\hat{\rho} : Y \amalg \{y\} \xrightarrow{\cong} Z \amalg \{z\}$,

$$S(\sigma \sqcup \rho)(\phi \diamond_{w,z}^{W,Z} \psi) = S(\hat{\sigma})(\phi) \diamond_{x,y}^{X,Y} S(\hat{\rho})(\psi),$$

(where $\sigma \sqcup \rho : X \amalg Y \xrightarrow{\cong} W \amalg Z$ is the block permutation).

A unit for the multiplication $\diamond$ *is a map $\epsilon : \mathfrak{C} \to S_{\mathbf{2}}$, $c \mapsto \epsilon_c$ such that, for all X and all $\phi \in S_{X\amalg\{x\}}$ with $S(ch_x) = c$,*

$$\phi \diamond_{x,2}^{X,\{1\}} \epsilon_c = \epsilon_c \diamond_{2,x}^{\{1\},X} \phi = \phi.$$

A multiplication $\diamond$ is called unital *if it has a unit ϵ. In this case ϵ_c is the corresponding* c-coloured unit for $\diamond$.

If $(\diamond, \epsilon : \mathfrak{C} \to S_{\mathbf{2}})$ is a unital multiplication on a $(\mathfrak{C}, \omega)$-coloured graphical species S, then $\epsilon_c \in S_{(c, \omega c)}$ for all $c \in \mathfrak{C}$. So a unit map $\epsilon : \mathfrak{C} \to S_{\mathbf{2}}$ is injective. It is also unique:

Lemma 1.13. *If S is a $(\mathfrak{C}, \omega)$-coloured graphical species with a unital multiplication $\diamond$, then the unit $\epsilon : \mathfrak{C} \to S_{\mathbf{2}}$ is unique. Moreover, ϵ is compatible with the involutions ω on $\mathfrak{C}$, and the unique non-identity automorphism $\sigma_{\mathbf{2}} \in \mathbb{P}^{\circlearrowleft}(\mathbf{2}, \mathbf{2})$ in that*

$$\epsilon \circ \omega = S(\sigma_{\mathbf{2}}) \circ \epsilon : \mathfrak{C} \to S_{\mathbf{2}}. \tag{1.14}$$

Proof. Let $\lambda : \mathfrak{C} \to S_{\mathbf{2}}$, $c \mapsto \lambda_c$ be another unit for $\diamond$. Then, for all $c \in \mathfrak{C}$,

$$\epsilon_c = \epsilon_c \diamond_{2,2}^{\{1\},\{1\}} \lambda_c = \lambda_c \diamond_{2,2}^{\{1\},\{1\}} \epsilon_c = \lambda_c,$$

whence ϵ is unique.

The second statement follows from the defining properties of multiplication. Namely,

$$S(\sigma_{\mathbf{2}})(\epsilon_c) = S(\sigma_{\mathbf{2}})(\epsilon_c) \diamond_{2,2}^{\{1\},\{1\}} \epsilon_{\omega c} = S(\hat{\sigma}_{\mathbf{2}} \sqcup id_{\{1\}})(\epsilon_c \diamond_{1,2}^{\{2\},\{1\}} \epsilon_{\omega c}) = \epsilon_c \diamond_{1,2}^{\{2\},\{1\}} \epsilon_{\omega c} = \epsilon_{\omega c} \diamond_{1,2}^{\{2\},\{1\}} \epsilon_c = \epsilon_{\omega c}.$$

□

Remark 1.15. (See also Remark 1.19.) A multiplication $\diamond$ on $(\mathfrak{C}, \omega)$-coloured graphical species S corresponds also to a family of maps

$$- \diamond_c^{\underline{c}, \underline{d}} - : S_{(\underline{c}, c)} \times S_{(\underline{d}, \omega c)} \to S_{(\underline{cd})},$$

for $c \in \mathfrak{C}$, $\underline{c} \in \mathfrak{C}^X$, $\underline{d} \in \mathfrak{C}^Y$. This notation is often clearer and more convenient than the notation $\diamond_{x,y}^{X,Y}$ introduced in Definition 1.12 (which is clunky but has the advantage of greater precision, particularly in describing the symmetry actions). Both forms will be used in this paper. Sometimes, if the context is clear, the superscripts will even be dropped altogether.

Before defining the contraction operation for modular operads, we'll discuss some examples of graphical species with multiplication. These may already be quite familiar. As well as aiding intuition, they highlight the elegance and expressive power of Joyal and Kock's involutive palettes.

As one would expect, a multiplication $\diamond$ on a graphical species S is called 'associative' if the result of multiplying multiple elements does not depend on the order of multiplication. This is stated precisely in condition (C1) of the definition of modular operads 1.22. (See also the figure in Definition 1.22 (C1), for a visualisation of the associativity axiom.)

Example 1.16. A graphical species O equipped with a unital, associative multiplication $\diamond$ is a cyclic operad in the sense of [15].

The value of the involutive, graphical species approach to cyclic operads is dicussed in [15] and [22, Introduction].

Example 1.17. The category Op of (coloured symmetric) operads (see e.g.[7]) admits a description in terms of directed graphical species equipped with a unital multiplication.

Let $RC \subset Di$ be the $(\mathfrak{Di}, \sigma_{\mathfrak{Di}})$-coloured (directed) graphical species consisting of the rooted corollas $t_X = C_{X;r}$ with leaves decorated by X.

So $RC_{\mathbf{0}} = \emptyset$ and $RC_Z \cong Z$ is a choice of element $r \in Z$, and the restriction to $Aut(X)$ of the symmetric action on $RC_{X \amalg \{r\}}$ permutes the inputs of t_X.

The category Op of (symmetric) operads is canonically equivalent to the category whose objects are objects of $\mathsf{GS}|RC$ (or *species of rooted corollas*) equipped with an associative unital multiplication, and whose morphisms are morphisms in $\mathsf{GS}|RC$ that preserve the multiplication.

Just as a while loop in a simple piece of code creates the possibility of infinite runtime, the presence of a contractions operation means that modular operads encode very difference algebraic structures – such as trace and duality – from operads.

Intuitively, a contraction ζ on a graphical species S may be thought of as a rule 'self-gluing' single elements of S along pairs of ports with dual colouring (Figure 4).

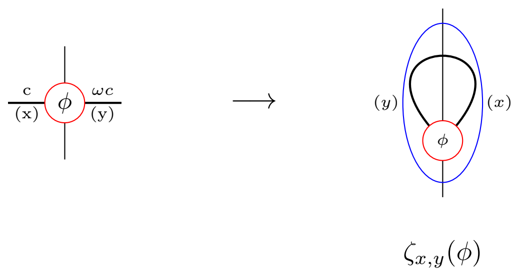


FIGURE 4. Contraction

Definition 1.18. *A* contraction ζ *on* S *is given by a family of partial maps*

$$\zeta^X_{x,y} : S_{X \amalg \{x,y\}} \nrightarrow S_X$$

defined for all finite sets X *and all* $\phi \in S_{X \amalg \{x,y\}}$*, such that* $S(ch_x)(\phi) = \omega S(ch_y \circ \tau_{(\tilde{e})})(\phi)$.

These maps are equivariant with respect to the action of $\mathbb{P}$ *on* S*: If* $\hat{\sigma} : X \amalg \{x,y\} \xrightarrow{\cong} Z \amalg \{w,z\}$ *such that* $\hat{\sigma}(x) = w, \hat{\sigma}(y) = z$*, is extends the bijection* $\sigma : X \xrightarrow{\cong} Z$*, and for* $\phi \in S_{Z \amalg \{w,z\}}$*,*

$$S(\sigma)\left(\zeta^Z_{w,z}(\phi)\right) = \zeta^X_{x,y}\left(S(\hat{\sigma})(\phi)\right).$$

If ζ is a contraction on S, then by, equivariance, $\zeta^X_{x,y}(\phi) = \zeta^X_{y,x}(\phi)$ wherever defined.

Remark 1.19. (See also Remark 1.15.)

A contraction ζ on a $(\mathcal{C}, \omega)$-coloured graphical species S also describes a family of maps

$$\zeta^{\underline{c}}_c : S_{(\underline{c}, c, \omega c)} \to S_{\underline{c}}$$

for $c \in \mathfrak{C}$, and $\underline{c} \in \mathfrak{C}^X$, and this notation (or even ζ_c, where the context is clear) will often be used in place of the more precise, but clunkier $\zeta^X_{x,y}$.

Let S be a $(\mathfrak{C}, \omega)$-coloured graphical species equipped with a unital multiplication $(\diamond, \epsilon)$ and contraction ζ. For all $c \in \mathfrak{C}$, $\epsilon_c \in S_{(c,\omega c)}$, so, by Lemma 1.13, there is a *contracted unit map*

$$o \overset{\text{def}}{=} \zeta\epsilon : \mathfrak{C} \to S_{\mathbf{0}}, \text{ satisfying } \zeta_c(\epsilon_c) = \zeta_{\omega c}(\epsilon_{\omega c}) \text{ for all } c \in \mathfrak{C}. \tag{1.20}$$

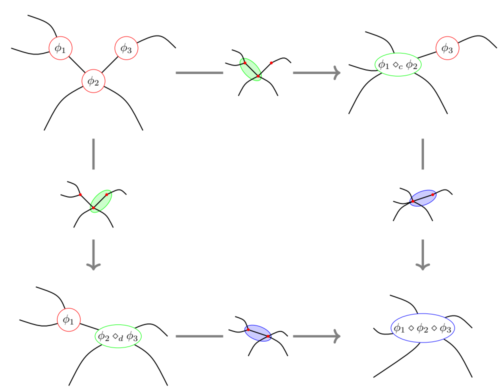
Remark 1.21. We shall see in Section 6 and Section 7 that the contracted units $o : S_{\S} \to S_{\mathbf{0}}$ present the main challenge for describing the combinatorics of modular operads.

1.4. **Modular operads: definition and examples.** Modular operads are graphical species with multiplication and contraction operations that satisfy the nicest possible (mutual) coherence axioms.

Definition 1.22. *A* $(\mathfrak{C},\omega)$-coloured modular operad *is a* $(\mathfrak{C},\omega)$*-coloured graphical species* S *together with a unital multiplication* $\diamond$*,* $\epsilon : \mathfrak{C} \to S_{\mathbf{2}}$*,and a contraction* ζ*, satisfying the following four* coherence axioms *governing their composition:*

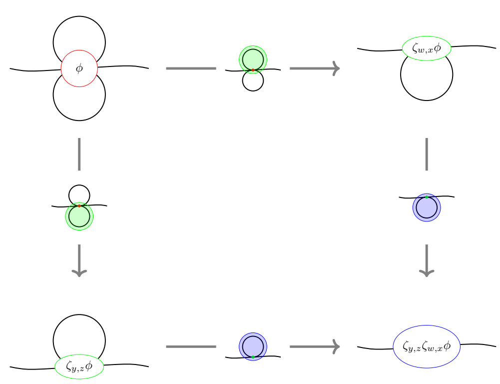
(C1) Multiplication is associative
For all $\underline{b}, \underline{c}, \underline{d}$ *and all* $c, d \in \mathfrak{C}$*:*

$$\begin{tikzcd} S_{(\underline{b},c)} \times S_{(\underline{c},\omega c,d)} \times S_{(\underline{d},\omega d)} \arrow[r, "\diamond_c \times id"] \arrow[d, "id \times \diamond_d"'] & S_{(\underline{bc},d)} \times S_{(\underline{d},\omega d)} \arrow[d, "\diamond_d"] \\ S_{(\underline{b},c)} \times S_{(\underline{c},\omega c,\underline{d})} \arrow[r, "\diamond_c"'] & S_{\underline{bcd}}. \end{tikzcd}$$



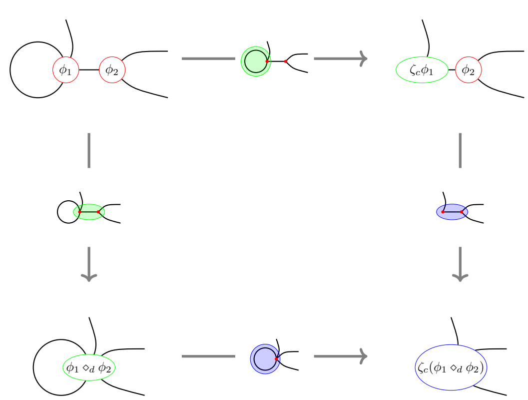
(C2) The order of contraction doesn't matter.
For a finite set X*,* $\underline{c} \in \mathfrak{C}^X$ *and* $c, d \in \mathfrak{C}$*, the following diagram commutes:*

$$\begin{tikzcd} S_{(\underline{c},c,\omega c,d,\omega d)} \arrow[r, "\zeta_c"] \arrow[d, "\zeta_d"'] & S_{(\underline{c},d,\omega d)} \arrow[d, "\zeta_d"] \\ S_{(\underline{c},c,\omega c)} \arrow[r, "\zeta_c"'] & S_{\underline{c}}. \end{tikzcd}$$



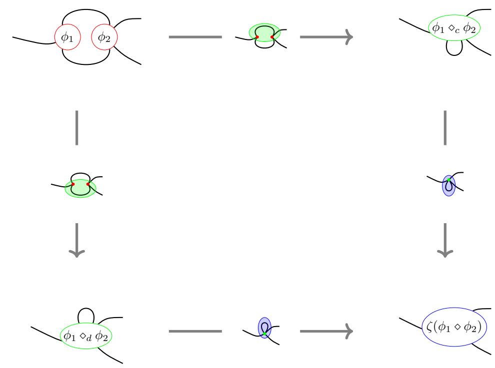
(C3) Multiplication and contraction commute.
For finite sets X_1 *and* X_2*,* $\underline{c} \in \mathfrak{C}^{X_1}$*,* $\underline{d} \in \mathfrak{C}^{X_2}$ *and* $c, d \in \mathfrak{C}$*, the following diagram commutes.*

$$\begin{tikzcd} S_{(\underline{c},c,\omega c,d)} \times S_{(\underline{d},\omega d)} \arrow[r, "\zeta_c \times id"] \arrow[d, "\diamond_d"'] & S_{(\underline{c},d)} \times S_{(\underline{d},\omega d)} \arrow[d, "\diamond_d"] \\ S_{(\underline{c},c,\omega c,\underline{d})} \arrow[r, "\zeta_c"'] & S_{\underline{cd}} \end{tikzcd}$$



(C4) Parallel gluing of distinct elements.
For finite sets X_1, X_2, $\underline{c} \in C^{X_1}$, $\underline{d} \in C^{X_2}$, *and* $c, d \in C$, *the following digram commutes.*

$$\begin{tikzcd}
S_{(\underline{c},c,d)} \times S_{(\underline{d},\omega c,\omega d)} \arrow[r, "\diamond_c"] \arrow[d, "\diamond_d"'] & S_{(\underline{c},d,\underline{d},\omega d)} \arrow[d, "\zeta_d"] \\
S_{(\underline{c},c,\underline{d},\omega c)} \arrow[r, "\zeta_c"'] & S_{\underline{cd}}
\end{tikzcd}$$



Modular operads form a category MO *whose morphisms are morphisms of the underlying graphical species that preserve multiplication, contraction and multiplicative units.*

Informally, the multiplication and contraction operations describe rules for *collapsing* internal edges of graphs that represent formal compositions of contraction and multiplication. Then the coherence axioms (C1) – (C4) say that the order in which this is done doesn't matter.

Remark 1.23. A *non-unital modular operad* $(S, \diamond, \zeta)$ is a graphical species S equipped with a multiplication $\diamond$ and contraction ζ satisfying (C1) – (C4) but without the requirement of a multiplicative unit. Morphisms in the category MO^- of non-unital modular operads are morphisms in GS that preserve the multiplication and contraction operations. Such structures are the subject of Section 5.

To provide context and motivation for the constructions that follow, the remainder of this section is devoted to examples.

Example 1.24. **Getzler-Kapranov modular operads.** We may equip the monochrome graphical species M given by $M_{\mathbf{n}} = \mathbb{N}$ for all $n \in \mathbb{N}$, with a unital multiplication $(+, 0 \in M_{\mathbf{2}})$ induced by addition of natural numbers,

$$+ : M_{\mathbf{m}} \times M_{\mathbf{n}} \to M_{\mathbf{m+n-2}}, \ (g_m, g_n) \mapsto g_m + g_n, \text{ for } m, n \geq 1$$

and a contraction t induced by the successor operation

$$t : M_{\mathbf{n}} \to M_{\mathbf{n-2}}, g_n \mapsto g_n + 1, \text{ for } n \geq 2.$$

Since the topological type of a compact oriented surface with boundary is determined only by its genus and number of boundary components, M models topological gluing of surfaces along boundary components. Monochrome objects (S, γ) of the slice category $\mathsf{MO}|M$ describe a bigraded set $S^{\gamma}(g, n)$ with operations

$$+^S : S(g_1, n_1) \times S(g_2, n_2) \to S(g_1 + g_2, n_1 + n_2 - 2) \text{ for } n_1, n_2 \geq 1,$$

and $t^S : S(g, n) \to S(g + 1, n - 2)$, for $n \geq 2$, and may encode extra geometric structure on surfaces. For example $S(g, n)$ could be the space of conformal structures on a genus g surface Γ with n boundary components.

Getzler and Kapranov [16] originally defined modular operads in terms of the restriction to the *stable part* $M^{st} \subset M$ of the graphical species M. An oriented surface Γ is stable if and only if $2g + n - 2 > 0$. So $M^{st}_{\mathbf{n}} = M_{\mathbf{n}}$ for $n > 2$ but $M^{st}_{\mathbf{0}} = \{2, 3, 4, \dots\}$ and $M^{st}_{\mathbf{1}} = M^{st}_{\mathbf{2}} = \{1, 2, 3, 4, \dots\}$.

In particular, since $0 \notin M^{st}_{\mathbf{2}}$, modular operads in the original sense of [16] are non-unital.

These ideas may be extended to many-coloured cases, for example gluing of surfaces along *open and closed* boundary components describes 2-coloured modular operads (see e.g. [17]). Open-closed modular operads are considered in [14], and a sequel paper [13] gives many of the same constructions, but this time in terms of open-closed 2 cobordism categories (see Example 1.26 below).

Example 1.25. **Wheeled properads are directed modular operads.** Wheeled properads have been studied extensively in [19], [40]. They describe the connected part wheeled props of that have had applications in geometry, deformation theory, and other areas [33, 31]. Since the terminal directed graphical species Di (see Example 1.11), trivially admits the structure of a modular operad, the slice category $\mathsf{MO}|Di$ of *directed modular operads* is well defined. This is canonically equivalent to the category of (Set-valued) wheeled properads. (This correspondence is extended to wheeled props in [36], and explored, in the case of wheeled props and circuit algebras, in [12].)

Example 1.26. **Compact closed categories**, introduced in [25], are symmetric monoidal categories $(\mathsf{C}, \otimes, e)$ for which every object has a symmetric *categorical dual* (see [5, 26]): There is an involutive functor $* : \mathsf{C}^{op} \to \mathsf{C}$, and natural transformations $\cup : \mathsf{C} \otimes \mathsf{C}^* \Rightarrow e$ and $\cap : e \Rightarrow \mathsf{C}^* \otimes \mathsf{C}$ such that, for all $c \in \mathsf{C}$,

$$(\cup_c \otimes id_c) \circ (id_c \otimes \cap_c) = id_c = (\cap_{c^*} \otimes id_c) \circ (id_c \otimes \cup_{c^*}.)$$

Finite dimensional vector spaces, and, more generally, finite dimensional projective modules of a commutative ring R, provide the canonical examples of compact closed categories. Cobordism categories provide more examples.

There is a canonical adjunction, whereby a compact closed category $(\mathsf{C}, \otimes, e, *)$ with object set C_0 is mapped to a $(\mathsf{C}_0, *)$-coloured modular operad S^{C} with

$$\mathsf{C}(c_1 \otimes \cdots \otimes c_m, d_1 \otimes \cdots \otimes d_n) \mapsto S^{\mathsf{C}}_(c_1, \ldots, c_m, d_1^*, \ldots, d_n^*)$$

and the categorical composition, together with $\cup$ and $\cap$ induce the modular operad structure.

This even becomes an adjoint equivalence if we consider a modified category $\mathsf{MO}^{\boxtimes}$ of modular operads equipped with an extra monoidal product (the slice $\mathsf{MO}^{\boxtimes}|Di$ is then equivalent to wheeled props). In particular, there is a precise sense in which modular operads (with product) may be thought of as *undirected* compact closed categories. (This is discussed in [36]. In [37], I prove a Weber style nerve theorem for these structures, via an iterated distributive law.)

This perspective also provides rich possibilities for relaxing the definition of modular operads, for example by replacing the symmetric action with a braid action. ✠Related ideas are being explored by Dansco and Robertson in their work on ????

2. Abstract nerve theorems and the role of graphs

The purpose of this largely formal section is to review some basics on monads, and provide an overview of Weber's abstract nerve theory. The simplicial nerve for categories, and the dendroidal nerve for operads provide motivating examples.

2.1. Monads and their algebras.

We begin by recalling some details on monads.

A monad $(M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ on a category C is an endofunctor $M : \mathsf{C} \to \mathsf{C}$ together with natural transformations $\mu^{\mathbb{M}} : M^2 \Rightarrow M$ (the *monad multiplication*) and $\eta^{\mathbb{M}} : id \Rightarrow M$ (the *monad unit*), such that, for all $c \in ob(\mathsf{C})$, the following diagrams commute in C:

(2.1)
$$\begin{tikzcd} M^3c \arrow[r, "\mu^{\mathbb{M}}Mc"] \arrow[d, "M\mu^{\mathbb{M}}c"'] & M^2c \arrow[d, "\mu^{\mathbb{M}}c"] \\ M^2c \arrow[r, "\mu^{\mathbb{M}}c"'] & Mc, \end{tikzcd}$$

(2.2)
$$\begin{tikzcd} Mc \arrow[r, "M\eta^{\mathbb{M}}c"] \arrow[dr, equal] & M^2c \arrow[d, "\mu^{\mathbb{M}}c"] & Mc \arrow[l, "\eta^{\mathbb{M}}Mc"'] \arrow[dl, equal] \\ & Mc & \end{tikzcd}$$

Any pair $L : \mathsf{D} \leftrightarrows \mathsf{C} : R$ of adjoint functors induces a monad (RL, μ, η) on D where $\eta : 1 \Rightarrow RL$ is the unit for the adjunction and, if $\epsilon : LR \Rightarrow 1$ is the counit, $\mu = R\epsilon L : (RL)^2 \Rightarrow L$.

In fact, every monad, arises from an adjunction as follows:

An algebra for a monad $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ on a category C is a pair (c, θ) with $c \in ob(\mathsf{C})$ and $\theta \in \mathsf{C}(Mc, c)$ so that the following diagrams commute in C.

(2.3)
$$\begin{tikzcd} M^2c \arrow[r, "\mu^{\mathbb{M}}c"] \arrow[d, "M\theta"'] & Mc \arrow[d, "\theta"] \\ Mc \arrow[r, "\theta"'] & c, \end{tikzcd}$$

(2.4)
$$\begin{tikzcd} c \arrow[r, "\eta^{\mathbb{M}}c"] \arrow[dr, equal] & Mc \arrow[d, "\theta"] \\ & c \end{tikzcd}$$

The *Eilenberg-Moore (EM) category* $\mathsf{C}^{\mathbb{M}}$ *of algebras for the monad* $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ has $\mathbb{M}$-algebras as objects and morphisms in $\mathsf{C}^{\mathbb{M}}((c, \theta), (c, \theta))$ are morphisms $f \in \mathsf{C}(c, c)$ such that

$$\begin{tikzcd} Mc \arrow[r, "Mf"] \arrow[d, "\theta"'] & Mc \arrow[d, "\theta"] \\ c \arrow[r, "f"'] & c. \end{tikzcd}$$

In particular, there is a forgetful functor $U : \mathsf{C}^{\mathbb{M}} \to \mathsf{C}$ and this has a left adjoint, the free functor $F : \mathsf{C} \to \mathsf{C}^{\mathbb{M}}$ given by $c \mapsto (Mc, \mu^{\mathbb{M}}c)$. In this case, $\mathbb{M}$ is the monad that arises from the adjunction UF, and the forgetful functor U is said to be *strictly monadic*.

Remark 2.5. More generally, a *pointed endofunctor* on a category C is a pair (E, η^E) of an endofunctor E on C, and natural transformation $\eta^E : 1_{\mathsf{C}} \Rightarrow E$. We may define an *algebra for the pointed endofunctor* (E, η^E) as a pair (c, θ) of an object c of C and a morphism $\theta \in (Ec, e)$ such that $\theta\eta^{\mathsf{C}}_c = id \in \mathsf{C}(c, c)$.

If (E, η^E) is a pointed endofunctor on C, and C^E its category of algebras, then, unlike the monad case, there is, in general, no canonical functor $\mathsf{C} \to \mathsf{C}^E$, $c \mapsto (Ec, m_c)$.

2.2. **Monads with arities and abstract nerve theory.** A subcategory $\iota : \mathsf{D} \hookrightarrow \mathsf{C}$ is *replete* if any object $c \in \mathsf{C}$ such that $c \cong \iota(d)$ for some $d \in \mathsf{D}$, is, itself, an object of D. The *essential image* $im^{es}(F)$ of a functor $F : \mathsf{E} \to \mathsf{C}$ is the smallest replete subcategory $im^{es}(F) \hookrightarrow \mathsf{C}$ containing the image $im(f)$ of f.

An embedding $\iota : \mathsf{D} \hookrightarrow \mathsf{C}$ induces a *nerve functor*

$$N : \mathsf{C} \to \mathsf{ps}(\mathsf{D}), \ c \longmapsto (d \mapsto \mathsf{C}(\iota(d), c)), \text{ for all } c \in \mathsf{C}, \ d \in \mathsf{D}.$$

This is fully faithful if and only if, for all $c \in \mathsf{C}$, the colimit $colim_{\mathsf{D}|c}\ \iota$ exists in C, and is canonically equal to c. In which case, the inclusion $\iota : \mathsf{D} \to \mathsf{C}$ is called *dense* and D is a *dense subcategory* of C.

Every functor $F : \mathsf{E} \to \mathsf{C}$ admits an (up to isomorphism) unique *'bo-ff' factorisation* as a bijective on objects - followed by a fully faithful functor.

Example 2.6. For example, for any monad $\mathbb{M}$ on a category C, the free functor $\mathsf{C} \to \mathsf{C}^{\mathbb{M}}$ has bo-ff factorisation $\mathsf{C} \to \mathsf{C}_{\mathbb{M}} \to \mathsf{C}^{\mathbb{M}}$ where $\mathsf{C}_{\mathbb{M}}$ is the *Kleisli category* of $\mathbb{M}$. (See e.g. [29, Section VI.5].)

Let $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ be a monad on a category C with dense subcategory $\iota : \mathsf{D} \hookrightarrow \mathsf{C}$, and let $\Theta_{\mathbb{M}} \hookrightarrow \mathsf{C}^{\mathbb{M}}$ be the fully faithful subcategory that arises from the canonical bo-ff factorisation of the induced functor $\mathsf{D} \to \mathsf{C}^{\mathbb{M}}$.

(2.7)
$$\begin{tikzcd}
\Theta_{\mathbb{M}} \arrow[r, "\text{f.f.}"] & \mathsf{C}^{\mathbb{M}} \arrow[r, "N_{\mathbb{M}}"] \arrow[d, shift left] & \mathsf{ps}(\Theta_{\mathbb{M}}) \arrow[d, "j^*"] \\
\mathsf{D} \arrow[u, "\text{b.o.}", "j"'] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{C} \arrow[r, "\nu_{\mathsf{D}}", "\text{f.f.}"'] \arrow[u, shift left] & \mathsf{ps}(\mathsf{D})
\end{tikzcd}$$

It is natural to ask under what conditions the fully faithful functor $\Theta_{\mathbb{M}} \to \mathsf{C}^{\mathbb{M}}$ is dense, and hence induces a fully faithful nerve. By [39, Section 4], this is certainly the case if the monad $\mathbb{M}$ *has arities* D:[3] for each $c \in \mathsf{C}$, the functor $\nu_{\mathsf{D}} \circ T : \mathsf{C} \to \mathsf{ps}(\mathsf{D})$ maps the canonical cocone $\mathsf{D}|c$ in C to a colimit cocone in $\mathsf{ps}(\mathsf{D})$. (See [6, Section 2] for details.)

Moreover, if $\mathbb{M}$ has arities D and hence the induced nerve $N_{\mathbb{M}} : \mathsf{C}^{\mathbb{M}} \to \mathsf{ps}(\Theta_{\mathbb{M}})$ is fully faithful, its essential image is the full subcategory of $\mathsf{ps}(\Theta_{\mathbb{M}})$ on presheaves P that satisfy a corresponding *Segal condition*. In case $\mathsf{C} = \mathsf{ps}(\mathcal{E})$ and $\mathcal{E} \hookrightarrow \mathsf{D}$ is a dense subcategory, then $\mathsf{D} \hookrightarrow \mathsf{C}$ is fully faithful and the Segal condition has the form:

$$P(jd) = lim_{(e,f)\in \mathcal{E}|d}\ j^*(P)(e). \tag{2.8}$$

Remark 2.9. The condition that $\mathbb{M}$ has arities $\mathsf{D} \hookrightarrow \mathsf{C}$ is sufficient, but not necessary, for the induced nerve $\mathsf{C}^{\mathbb{M}} \to \mathsf{ps}(\Theta_{\mathbb{M}})$ to be fully faithful.

In fact, the modular operad monad $\mathbb{O}$ on the category of graphical species, together with full dense subcategory $\mathsf{Gr} \hookrightarrow \mathsf{GS}$ of connected graphs and étale morphisms (see Section 4), provides an example of a monad that does not have arities, but for which the nerve theorem holds.

Necessary conditions on the monad $\mathbb{M}$ and $\mathsf{D} \hookrightarrow \mathsf{C}$, to induce a fully faithful nerve are described in [9].

Example 2.10. The classical nerve theorem for categories provides the canonical example of a Weber nerve.

Directed graphs $\mathfrak{s}, \mathfrak{t} : E \rightrightarrows V$ are presheaves over the small diagram category $\mathcal{E} \overset{\text{def}}{=} \bullet \rightrightarrows \bullet$.[4]

Let $\Delta_0 \hookrightarrow \mathsf{ps}(\mathcal{E})$ be the full subcategory of directed linear graphs $[n]$, $n \in \mathbb{N}$

$$[n] \overset{\text{def}}{=} \overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n}{\bullet} \tag{2.11}$$

and successor preserving morphisms. Then $\mathcal{E}$ embeds in Δ_0 as the full subcategory on the objects [0] and [1].

The canonical free-forgetful adjunction $F^{Cat} : \mathsf{ps}(\mathcal{E}) \leftrightarrows \mathsf{Cat} : U^{Cat}$ is monadic, and the simplex category Δ of finite non-zero ordinals and order preserving maps is obtained in the bo-ff factorisation of the induced functor $\Delta_0 \to \mathsf{Cat}$.

(2.12)
$$\begin{tikzcd}
 & \Delta \arrow[r, hook] & \mathsf{Cat} \arrow[r, "N"] \arrow[d, shift left, "U^{Cat}"] & \mathsf{sSet} \arrow[d, "j^*"] \\
\mathcal{E} \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \Delta_0 \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \mathsf{ps}(\mathcal{E}) \arrow[r, hook] \arrow[u, shift left, "F^{Cat}"] & \mathsf{ps}(\Delta_0).
\end{tikzcd}$$

[3]The notion of a monad with arities was introduced in [39, Section 4], on a suggestion of Steve Lack.

[4]This notion of graph is dual to the one introduced in Section 3: here vertices function as 'objects' and connections between them as 'morphisms'. This will be reversed in Example 2.13 below, and the rest of the paper.

(Here $\mathsf{sSet} \overset{\mathrm{def}}{=} \mathsf{ps}(\Delta)$ denotes the category of Δ-presheaves, or *simplicial sets*.)

The monad has arities Δ_0. Hence, the categorical nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful and its essential image consists of precisely those $P \in \mathsf{sSet}$ that satisfy the classical Segal condition (originally formulated in [38]): for $n > 1$, the set P_n of n-simplices is isomorphic to the n-fold fibred product

$$P_n \cong \underbrace{P_1 \times_{P_0} \cdots \times_{P_0} P_1}_{n \text{ times}}.$$

Example 2.13. In Example 1.17, we defined the graphical species RC of rooted corollas and described operads in terms of a multiplicative structure on $\mathsf{ps}(\mathsf{el}(RC)) \cong \mathsf{GS}|RC$.

Let Ω_0 be the category whose objects are rooted symmetric trees (see e.g. [34]), and whose morphisms $\mathbf{S} \to \mathbf{T}$ are (up to symmetric isomorphism) inclusions of rooted trees that preserve vertex valency (see Example 2.13 (a)). Since any tree may be described by grafting rooted corollas along internal edges, the canonical full functor $\mathsf{el}(RC) \to \Omega_0$ is also dense.

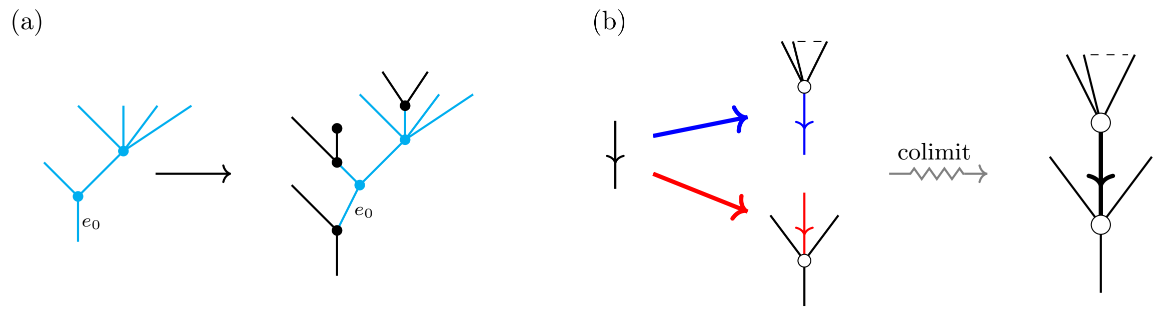


FIGURE 5. (a) Subtree inclusion, (b) grafting of rooted corollas to form a rooted tree

Hence the induced embedding $\Omega_0 \to \mathsf{ps}(\mathsf{el}(RC))$ is fully faithful. It is also dense (it's easy to show that the nerve $\nu : \mathsf{ps}(\mathsf{el}(RC)) \to \mathsf{ps}(\Omega_0)$ is fully faithful as in Section 4.4).

The free operad functor F^{Op} (see also Example 5.1) is left adjoint to the forgetful functor U^{Op} from the category Op of symmetric operads to $\mathsf{ps}(\mathsf{el}(RC))$, and hence the following diagram commutes:

(2.14)
$$\begin{tikzcd}
 & \Omega \arrow[r, hook] & \mathsf{Op} \arrow[r, "N"] \arrow[d, shift left, "U^{Op}"] & \mathsf{ps}(\Omega) \arrow[d, "j^*"] \\
\mathsf{el}(RC) \arrow[r, hook] & \Omega_0 \arrow[r, hook] \arrow[u, "j", "\text{b.o.}"'] & \mathsf{ps}(\mathsf{el}(RC)) \arrow[r, hook] \arrow[u, shift left, "F^{Op}"] & \mathsf{ps}(\Omega_0).
\end{tikzcd}$$

Moreover, the induced operad on $\mathsf{ps}(\mathsf{el}(RC))$ has arities Ω_0, whereby the nerve functor $N : \mathsf{Op} \to \mathsf{ps}(\Omega)$ is fully faithful and its essential image consists of those Ω- presheaves (or *dendroidal sets*) $O : \Omega^{op} \to \mathsf{Set}$ that satisfy the dendroidal Segal condition:

$$O(\mathbf{T}) = lim_{(t,i) \in (\mathsf{el}(RC)|\mathbf{T})} j^*(\mathbf{T}), \quad \text{for all symmetric rooted trees } \mathbf{T}. \tag{2.15}$$

In particular, since Δ_0 is the full subcategory of linear trees in Ω_0, the simplicial nerve theorem for categories is a special case of the dendroidal nerve theorem for operads.

2.3. **Distributive laws.** With these examples in mind, let us return to the case of modular operads. Recall that graphical species are presheaves on the category $\mathbb{P}^{\circlearrowleft}$ and that modular operads are graphical species equipped with certain operations.

Informally we can view monads as encoding, via their algebras, (algebraic) structures on categories. It is the combination of the contraction structure ζ, and the multiplicative unit structure ϵ that provides an obstruction to obtaining a monad from the modular operad endofunctor defined in [23] (see Section 6). So, one approach to constructing the modular operad monad $\mathbb{O}$ on GS could be to find monads for the modular operadic multiplication, contraction, and unital structures separately, and then attempt to combine them.

In general, however, monads do not compose. Given $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}}), \mathbb{M}' = (M', \mu^{\mathbb{M}'}, \eta^{\mathbb{M}'})$ on a category C, there is no canonical choice of natural transformation $\mu : (MM')^2 = MM'MM' \Rightarrow MM'$ defining a monadic multiplication for the endofunctor MM' on C.

However, any natural transformation, $\lambda : M'M \Rightarrow MM'$, induces a map

$$\mu_\lambda : \ (MM')^2 \xRightarrow{M\lambda M'} M^2 M'^2 \xRightarrow{\mu^{\mathbb{M}}\mu^{\mathbb{M}'}} MM'.$$

A *distributive law* λ for $\mathbb{M}, \mathbb{M}'$, is a natural transformation $\lambda : M'M \Rightarrow MM'$ that satisfies four axioms [4, Section 1]. These axioms ensure that the triple $(MM', (\mu^{\mathbb{M}}\mu^{\mathbb{M}'}) \circ (M\lambda M'), \eta^{\mathbb{M}}\eta^{\mathbb{M}'})$ defines a monad $\mathbb{M}\mathbb{M}'$ on C. In this case, λ determines how the $\mathbb{M}$-structures and $\mathbb{M}'$-structures on C interact to form the structure encoded by the composite $\mathbb{M}\mathbb{M}'$.

Example 2.16. The category monad on $\mathsf{ps}(\mathcal{E})$ (Example 2.10) may be obtained as a composite of the *semi-category monad*, which governs associative composition, and the *reflexive graph monad* that adjoins a distinguished loop at each vertex of a graph $G \in \mathsf{ps}(\mathcal{E})$. The corresponding distributive law encodes the property that the adjoined loops provide identities for the semi-categorical composition.

By [4, Section 3], given a distributive law $\lambda : M'M \Rightarrow MM'$ for monads $\mathbb{M}, \mathbb{M}'$ on C, there is a commuting square of strict monadic adjunctions (where, as usual $\mathsf{C}^{\mathbb{M}}$ is the EM category of $\mathbb{M}$-algebras):

(2.17)
$$\begin{tikzcd}
\mathsf{C}^{\mathbb{M}'} \arrow[r, shift right] \arrow[d, shift right] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\vdash"] & \mathsf{C}^{\mathbb{M}\mathbb{M}'} \arrow[l, shift right] \arrow[d, shift right] \arrow[d, phantom, "\vdash"] \\
\mathsf{C} \arrow[u, shift right] \arrow[r, shift right] \arrow[r, phantom, "\top"] & \mathsf{C}^{\mathbb{M}}. \arrow[l, shift right] \arrow[u, shift right]
\end{tikzcd}$$

So, there is a monad $\tilde{\mathbb{M}}$ on $\mathsf{C}^{\mathbb{M}'}$, and a monad $\mathbb{M}'_*$ on $\mathsf{C}^{\mathbb{M}}$ such that the corresponding EM categories of algebras are each canonically isomorphic to the EM category of algebras for $\mathbb{M}\mathbb{M}'$.

Along similar lines to Example 2.16, In Section 7 we'll construct the modular operad monad $\mathbb{O}$ on GS as a composite $\mathbb{D}\mathbb{T}$, where $\mathbb{T}$ on GS is the monad that governs the contraction and (non-unital) multiplication, and $\mathbb{D}$ adjoins distinguished elements to graphical species according to the defining properties of multiplicative units.

Then, by Equation (2.17), there exists a monad $\mathbb{T}_*$ on the EM category GS_* of $\mathbb{D}$-algebras, such that $\mathsf{GS}_*^{\mathbb{T}_*} \cong \mathsf{MO}$.

Now, the category Gr of connected Feynman graphs and étale morphisms defined in [23] (see Section 4) fits into a chain of fully faithful dense embeddings $\mathbb{P}^{\circlearrowleft} \hookrightarrow \mathsf{Gr} \hookrightarrow \mathsf{GS}$. Putting everything together, we get a commuting diagram of functors

(2.18)
$$\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}"] & \mathsf{ps}(\Xi) \arrow[d, "j^*"] \\
\mathbb{P}^{\circlearrowleft}_* \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr}_* \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS}_* \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{ps}(\mathsf{Gr}_*) \arrow[d] \\
\mathbb{P}^{\circlearrowleft} \arrow[u, "\text{b.o.}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr} \arrow[u, "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] & \mathsf{ps}(\mathsf{Gr})
\end{tikzcd}$$

in which the categories $\mathbb{P}^{\circlearrowleft}_*$, Gr_* and Ξ are obtained via bo-ff factorisations.

In Section 8, we'll see that $\mathbb{T}_*$ has arities Gr_*, and hence that $N : \mathsf{MO} \to \mathsf{ps}(\Xi)$ is fully faithful, and then use the fact that $\mathbb{P}^{\circlearrowleft}$ is dense in Gr_* to establish the desired Segal condition for modular operads/

3. Graphs and their morphisms

This section is an introduction to the (Feynman) graphs that will be used to encode the ideas of Section 1.Feynman graphs were originally defined in [23], and most of this section and the next, stays close to the original construction there. Since [23] was just a short note, it contained very few proofs, so relevant results are proved in full here. Where possible, definitions and examples are presented in a way that builds on Section 1 and highlights similarities with familiar concepts in basic topology.

This section deals with basic definitions, examples and morphisms of Feynman graphs. In the following section, we'll look in more detail at the topology by considering étale morphisms of graphs.

3.1. **Graph-like diagrams and Feynman graphs.** A graph consists of a set of vertices V, and a set of connections $\widetilde{E}$. The graphs we are interested in are finite graphs that are allowed to have loops, parallel edges, and loose ends (ports). In order to glue the vertices and connections together, we view $\widetilde{E}$ as the orbit space of an involutive set (E, τ) of edges (or edge ends).

Example 3.1. In [16] and [8] (among others), a graph G is described by sets V and E, an involution $\hat{\tau} : E \to E$, and a map $g : E \to V$. The ports of G are the fixed points of the involution $\hat{\tau}$. A formal exceptional edge graph η – such that maps $\eta \to G$ are choices $\{*\} \to E$ of elements of E – is also allowed.

[3, Section 15] provides a nice overview of graph definitions that appear in the operad literature, and a proof of the equivalence of the objects described.

Feynman graphs are defined similarly to Example 3.1, except that the incidence map $g : E \nrightarrow V$ is allowed to be a partial map, and the involution τ must be fixed-point free. These subtle differences make it possible to encode the whole calculus of Feynman graphs in terms of the formal theory of diagrams in finite sets.

Precisely, the category of *graph-like diagrams* is the category of functors $\mathtt{D}^{op} \to \mathsf{Set}_{\mathsf{f}}$, where $\mathtt{D}$ is the small category $\circlearrowright \bullet \longleftarrow \bullet \longrightarrow \bullet$, and $\mathsf{Set}_{\mathsf{f}}$ is the category of finite sets and all maps.

Since $\mathsf{Set}_{\mathsf{f}}$ admits finite (co)limits, so does $\mathsf{ps}_{\mathsf{f}}(\mathtt{D})$, and these are computed pointwise. The initial object in $\mathsf{ps}_{\mathsf{f}}(\mathtt{D})$ is the empty graph-like diagram

$$\oslash = \quad \circlearrowright \mathbf{0} \longleftarrow \mathbf{0} \longrightarrow \mathbf{0},$$

and the terminal object $\bigstar$ is the trivial diagram on singletons

$$\bigstar = \quad \circlearrowright \mathbf{1} \longleftarrow \mathbf{1} \longrightarrow \mathbf{1}.$$

Feynman graphs, introduced in [23], are graph-like diagrams satisfying extra properties:

Definition 3.2. *A* (Feynman) graph *is a graph-like diagram* $\mathcal{G} \in ob(\mathsf{ps_f}(\mathtt{D}))$

$$\mathcal{G} = \ \tau \circlearrowright E \xleftarrow{\quad s \quad} H \xrightarrow{\quad t \quad} V$$

such that $s : H \to E$ *is injective and* $\tau : E \to E$ *is an involution without fixed points.*

If $\mathcal{G}$ *is a graph, a* strong subgraph $\mathcal{H} \hookrightarrow \mathcal{G}$ *is a subdiagram that inherits a graph structure from* $\mathcal{G}$.

The full subcategory on graphs in $\mathsf{ps_f}(\mathtt{D})$ *is denoted* $\mathsf{Gr}^\sharp$.

As before, V is the vertex set of $\mathcal{G}$ and E is the *edge set* of $\mathcal{G}$. The set H of *half-edges* of $\mathcal{G}$ and the maps (s,t) encode a partial incidence map $E \twoheadrightarrow V$ as above. A half edge $h \in H$ will often be written as an ordered pair $h = (s(h), t(h))$. In the terminology above, the set $\tilde{E}$ of τ-orbits in E, where $\tilde{e} \in \tilde{E}$ is the orbit of $e \in E$, is the set of connections in $\mathcal{G}$.

Remark 3.3. A graph $\mathcal{G}$ may be realised geometrically by a one-dimensional space $|\mathcal{G}|$ where the set $\{*_v\}_{v \in V}$ is the set of 0-cells of $|\mathcal{G}|$ and, for each $e \in E$, we take a copy $[0, \frac{1}{2}]_e$ of the interval $[0, \frac{1}{2}]$ and identify

- $0_{s(h)} \sim *_{t(h)}$ for $h \in H$,
- $(\frac{1}{2})_e \sim (\frac{1}{2})_{\tau e}$ for all $e \in E$. (So each orbit of the involution is represented by a closed interval $[0,1]$.)

Example 3.4. (See also Figure 6(a.).) The *stick graph* $(\shortmid)$ has no vertices and edge set $\mathbf{2} = \{1,2\}$.

$$(\shortmid) \overset{\mathrm{def}}{=} \quad \circlearrowright \mathbf{2} \longleftarrow \mathbf{0} \longrightarrow \mathbf{0}.$$

In general, a stick graph is a graph that is isomorphic to $(\shortmid)$.

For any set X, $X^\dagger \cong X$ denotes its formal involution.

Example 3.5. (See also Figure 6(b.), (c.).) The *(Feynman)* X*-corolla* $\mathcal{C}_X$ associated to a finite set X has the form

$$\mathcal{C}_X : \qquad \dagger \circlearrowright X \amalg X^\dagger \xleftarrow{\text{incl.}} X^\dagger \longrightarrow \{*\}.$$

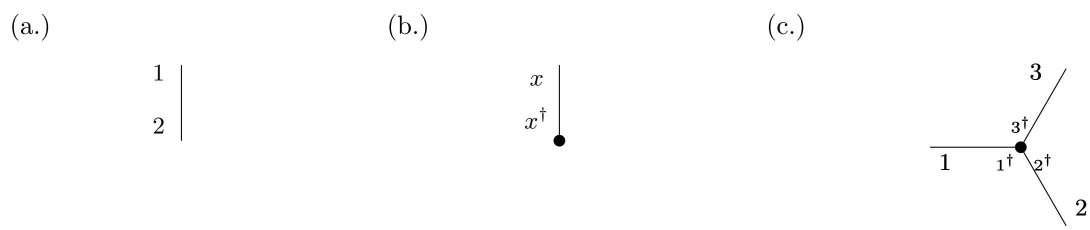


FIGURE 6. (a.) the stick graph $(\shortmid)$, and (b.), (c.) the corollas $\mathcal{C}_{\{x\}}$, $\mathcal{C}_{\mathbf{3}}$.

The set $E_\bullet \subset E$ of *inner edges* of $\mathcal{G}$ is the maximum subset of $im(s) \subset E$ that is closed under τ. So, an *inner edge* of $\mathcal{G}$ is an element $e \in E$ such that $e \in im(s)$ and $\tau e \in im(s)$. Elements of the set $\tilde{E}_\bullet$ of *inner* τ*-orbits* are of the form $\tilde{e}$ for $e \in E_\bullet$. Inner edges are preserved under graph morphisms: $f(E_\bullet(\mathcal{G})) \subset E_\bullet(\mathcal{G}')$ for all graphs $\mathcal{G}, \mathcal{G}'$ and all $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$.

The set $E_0 = E - im(s)$ is the *boundary* of $\mathcal{G}$ and elements $e \in E_0$ are *ports* of $\mathcal{G}$.

Remark 3.6. The fundamental difference between the Feynman graph formalism (Definition 3.2) and the graphs discussed in Example 3.1 is that the exceptional graph η has trivial endmorphism monoid, whereas $\mathsf{Gr}^\sharp(\shortmid, \shortmid) \cong \Sigma_2$. This involutive property underlies all the constructions that follow, as well as the difficulties described in Section 6.

In general, I will refer to Feynman graphs simply as 'graphs', unless I wish to emphasise a point that is specific to the formalism of Feynman graphs.

Example 3.7. Let X, Y be finite sets. We can consider the disjoint union of the corollas $\mathcal{C}_{X \amalg \{x_0\}}$ and $\mathcal{C}_{Y \amalg \{y_0\}}$ in $\mathsf{ps_f}(\mathtt{D})$ and identify the edges $x_0 \sim \tau y_0, y_0 \sim \tau x_0$ to obtain a graph $\mathcal{M}^{X,Y}_{x_0,y_0}$ with two vertices and one inner edge orbit, illustrated in Figure 7.

$$\tau \circlearrowright \begin{array}{c}(X \amalg Y) \amalg (X \amalg Y)^{\dagger} \\ \amalg \{x_0, y_0\}\end{array} \xleftarrow{\quad s \quad} \begin{array}{c}\{(x^{\dagger}, v_X)\}_{x \in X} \amalg \{(y^{\dagger}, v_Y)\}_{y \in Y} \\ \{(x_0, v_Y), (y_0, v_X)\}\end{array} \xrightarrow{\quad t \quad} \{v_X, v_Y\}$$

with the involution τ described by $z \leftrightarrow z^{\dagger}$ for $z \in X \amalg Y$ and $x_0 \leftrightarrow y_0$. The maps s and t are the obvious projections. These graphs are used in the construction of modular operads to encode formal multiplications in graphical species.

Example 3.8. Formal contractions in graphical species are encoded by graphs of the form $\mathcal{N}^{X}_{x_0,y_0}$ for X a finite set (see Figure 7). This is the quotient of the corolla $\mathcal{C}_{X \amalg \{x_0, y_0\}}$ obtained by identifying the edge pairs $x_0 \sim \tau y_0$ and $y_0 \sim \tau x_0$. The graph has boundary $E_0 = X$, one inner τ-orbit $\{x_0, y_0\}$, and one vertex v.

$$\tau \circlearrowright (X \amalg X^{\dagger} \amalg \{x_0, y_0\}) \xleftarrow{s} (\{(x^{\dagger}, v)\}_{x \in X} \amalg \{(x_0, v), (y_0, v)\}) \xrightarrow{t} \{v.\}$$

$|\mathcal{M}^{X,Y}_{x_0,y_0}|$ $|\mathcal{N}^{X}_{x_0,y_0}|$

FIGURE 7. Realisations of $\mathcal{M}^{X,Y}_{x_0,y_0}$ and $\mathcal{N}^{X}_{x_0,y_0}$ for $X \cong \mathbf{2},\ Y \cong \mathbf{3}$.

Since $\mathsf{Gr}^{\sharp}$ is full in $\mathsf{ps_f}(\mathtt{D})$, (co)limits in $\mathsf{Gr}^{\sharp}$, when they exist, correspond to (co)limits in $\mathsf{ps_f}(\mathtt{D})$.

Example 3.9. The empty graph-like diagram $\oslash$ is trivially a graph, and is therefore initial in $\mathsf{Gr}^{\sharp}$. However, there is no non-trivial involution on a singleton set, so the terminal diagram $\bigstar$ in $\mathsf{ps_f}(\mathtt{D})$ is not a graph. Hence, $\mathsf{Gr}^{\sharp}$ is not closed under finite limits. (By Example 3.25, $\mathsf{Gr}^{\sharp}$ is also not closed under finite colimits.)

Given a graph $\mathcal{G}$, it is useful to associate a grading by natural numbers to the sets V, H and E.

For each vertex $v \in V$, define $H\{v\} \overset{\text{def}}{=} t^{-1}(v) \subset H$ to be the fibre of t at v. The set $E\{v\} \subset E$ of edges *incident on* v is then $E\{v\} \overset{\text{def}}{=} s(H\{v\})$.This induces a map $|\cdot| : V \to \mathbb{N}$ by $v \mapsto |v| \overset{\text{def}}{=} |H\{v\}|$, where $|v|$ is called the *valency* of v. The set of n-valent vertices in V is denoted by V_n.

We'll be particularly interested in bivalent vertices v such that $|v| = 2$, and, vertices with valency 0, called *isolated vertices*. A graph $\mathcal{G}$ is *bivalent* if $V = V_2$.

For $n \geq 1$, define $E_n \overset{\text{def}}{=} s(H_n)$ where $H_n \overset{\text{def}}{=} t^{-1}(V_n)$. Then $E = \coprod_{n \in \mathbb{N}} E_n$, since $s(H) = E - E_0 = \coprod_{n \geq 1} E_n$.

Example 3.10. (See Example 3.4.) Since $H(\shortmid)$ is empty, all edges of $(\shortmid)$ are ports: $E(\shortmid) = E_0(\shortmid)$.

The corolla $\mathcal{C}_X$ (see Example 3.5) with vertex $*$ has $X \cong E\{*\} = H\{*\}$. If $|X| = k$ then, $|*| = k$, so $V = V_k$, and $E = E_k \amalg E_0$ with $E_i \cong X$ for $i = 0, k$.

Example 3.11. (See Examples 3.7, 3.8.) For finite sets X and Y, the graph $\mathcal{M}^{X,Y}_{x_0,y_0}$ has $E\{v_X\} = X^\dagger \amalg \{y_0\}$ and $E\{v_Y\} = Y^\dagger \amalg \{x_0\}$. If $X \cong \mathbf{n}$ for some $n \in \mathbb{N}$, then $v_X \in V_{n+1}$, and $X^\dagger \amalg \{y\} \subset E_{n+1}$.

The graph $\mathcal{N}^X_{x_0,y_0}$ has $E\{v\} = X^\dagger \amalg \{x_0, y_0\}$, and $E\{v\} = X^\dagger \amalg \{x_0, y_0\} \cong H$, so $V = V_{n+2}$ when $X \cong \mathbf{n}$.

Since $\mathsf{Gr}^\sharp$ is full in the diagram category $\mathsf{ps_f}(\mathtt{D})$, morphisms $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$ are commuting diagrams in $\mathsf{Set_f}$ of the form

$$
\begin{tikzcd}
\mathcal{G} \arrow[d, "f"] & E \arrow[r, leftrightarrow, "\tau"] \arrow[d, "f_E"'] & E \arrow[d, "f_E"'] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "f_H"'] & V \arrow[d, "f_V"] \\
\mathcal{G}' & E' \arrow[r, leftrightarrow, "\tau'"'] & E' & H' \arrow[l, "s'"] \arrow[r, "t'"'] & V'
\end{tikzcd}
\tag{3.12}
$$

Lemma 3.13. *(See also* [27, Proposition 1.1.11]*.) For any morphism $f = (f_E, f_H, f_V) \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$, the map f_H is completely determined by f_E. Moreover if $\mathcal{G}$ has no isolated vertices, then f_E also determines f_V, and hence f.*

Conversely, if $\mathcal{G}$ has no stick coponents or isolated vertices, then f is completely determined by f_H.

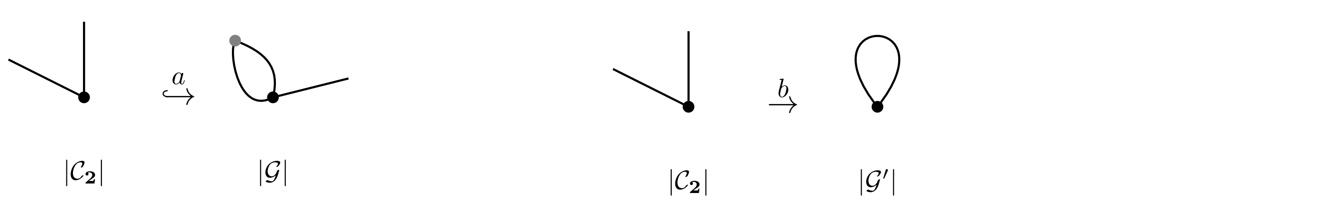
Proof. By injectivity of s, $f_H(h) = s'^{-1} f_E s(h)$ is well-defined for $h \in H$. If $V_0 = \emptyset$, then for each $v \in V$, $H\{v\}$ is non-empty and $f_V(v) = t' s'^{-1} f_E s(h)$ does not depend on the choice of $h \in H\{v\}$.

Since $s : H \to E$ is injective by definition, if $\mathcal{G}$ has no stick components then, for each $e \in E$, there is an $h \in H$ such that $e = s(h)$ or $e = \tau s(h)$, and the last statement of the lemma follows from the first. $\square$

Definition 3.14. *A morphism $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$ is* locally injective *if, for all $v \in V$, the induced map $f_v : E\{v\} \to E\{f(v)\}$ is injective, and* locally surjective *if $f_v : E\{v\} \to E\{f(v)\}$ is surjective for all $v \in V$.*

Locally bijective or 'étale' morphisms are the subject of the next section 4.

Example 3.15. The figure illustrates two examples of morphisms in $\mathsf{Gr}^\sharp$, described below. Both morphisms are locally injective, and (b.) is also surjective and locally surjective.



In each example, the horizontal maps are the obvious projections, and the columns in the edge sets represent the orbits of the involution.

(a.) The map a is determined by $e_i \mapsto ae)i$ for $j = 1, 2$.

$$
\begin{tikzcd}
\mathcal{C}_{\mathbf{2}} \arrow[d, "a"'] & \tau_{\mathbf{2}} \circlearrowright \left\{ \begin{array}{ll} e_1, & e_2, \\ \tau_{\mathbf{2}} e_1, & \tau_{\mathbf{2}} e_2 \end{array} \right\} \arrow[d] & \{(e_1, v), (e_2, v)\} \arrow[l] \arrow[r] \arrow[d] & \{v\} \arrow[d] & v \arrow[d, mapsto] \\
\mathcal{G} & \tau \circlearrowright \left\{ \begin{array}{lll} ae_1, & ae_2, & e_3', \\ \tau ae_1, & \tau ae_2, & \tau e_3' \end{array} \right\} & \left\{ \begin{array}{lll} (ae_1, v_1), & (ae_2, v_1), & (e_3', v_1) \\ (\tau ae_1, v_2), & (\tau ae_2, v_2), & \end{array} \right\} \arrow[l, "s"'] \arrow[r, "t"] & \{v_1, v_2\} & v_1
\end{tikzcd}
$$

(b.) The map b is determined by $e_1, \tau_{\mathbf{2}} e_2 \mapsto be_1$ and $e_2, \tau_{\mathbf{2}} e_1 \mapsto be_2$.

$$
\begin{tikzcd}
\mathcal{C}_{\mathbf{2}} \arrow[d, "b"] & \tau_{\mathbf{2}} \circlearrowright \left\{ \begin{matrix} e_1, & e_2, \\ \tau_{\mathbf{2}} e_1, & \tau_{\mathbf{2}} e_2 \end{matrix} \right\} \arrow[d] & \{(e_1, v), (e_2, v)\} \arrow[l] \arrow[r] \arrow[d] & \{v\} \arrow[d] \\
\mathcal{G}' & \tau' \circlearrowright \{be_1, be_2\} & \{(be_1, v'), (be_2, v')\} \arrow[l, "s'"'] \arrow[r, "t'"] & \{v'\}
\end{tikzcd}
$$

Example 3.16. Recall Examples 3.7, 3.8, above. For finite sets X and Y, there are canonical local injections

$$\mathcal{C}_X \longrightarrow \mathcal{M}^{X,Y}_{x_0,y_0} \longleftarrow \mathcal{C}_Y \qquad \text{and} \qquad \mathcal{C}_X \longrightarrow \mathcal{N}^X_{x_0,y_0}.$$

The canonical maps

$$\mathcal{C}_{X \amalg \{x_0\}} \longrightarrow \mathcal{M}^{X,Y}_{x_0,y_0} \longleftarrow \mathcal{C}_{Y \amalg \{y_0\}}$$

are locally injective and locally surjective but neither is surjective.

However, the canonical map $\mathcal{C}_{X \amalg \{x_0, y_0\}} \to \mathcal{N}^X_{x_0,y_0}$ is locally injective, locally surjective and surjective. When X is empty, this is isomorphic to the morphism (b) in Example 3.15.

Example 3.17. The map $X \mapsto \mathcal{C}_X$ describes a full embedding of $\mathsf{Set_f}$ into $\mathsf{Gr}^\sharp$. Since $\mathsf{Gr}^\sharp(\shortmid, \shortmid)$ and $\mathbb{P}^{\circlearrowleft}(\S, \S)$, are canonically isomorphic, it follows that the assignments $X \mapsto \mathcal{C}_X$ and $\S \mapsto (\shortmid)$ exhibit $\mathbb{P}^{\circlearrowleft}$ as the subcategory of local bijections in $\mathsf{Gr}^\sharp$ between objects of the form $\mathcal{C}_X$, and $(\shortmid)$.

Remark 3.18. Henceforth, $\mathbb{P}^{\circlearrowleft}$ will be viewed both in terms of finite sets and $\S$ and as a subcategory of graphs. The choice of notation for objects – $(\shortmid)$ or $\S$, X or $\mathcal{C}_X$ – will depend on the context. However, the same notation will be used for morphisms in $\mathbb{P}^{\circlearrowleft}$ and their image in $\mathsf{Gr}^\sharp$. So $f \in \mathbb{P}^{\circlearrowleft}(\S, X)$ (or $g \in \mathbb{P}^{\circlearrowleft}(X, Y)$) and $f \in \mathsf{Gr}^\sharp((\shortmid), \mathcal{C}_X)$ (or $g \in \mathsf{Gr}^\sharp(\mathcal{C}_X, \mathcal{C}_Y)$).

3.2. **Connected components of graphs.** The cocartesian monoidal structure on $\mathsf{Set_f}$ is inherited by $\mathsf{ps_f(D)}$ and $\mathsf{Gr}^\sharp$, so that these are strict symmetric monoidal categories under pointwise disjoint union $\amalg$, with monoidal unit given by the empty graph $\oslash$.

Definition 3.19. *A non-empty graphlike diagram $\mathcal{G}$ is* connected *if, for each $\mathsf{f} \in \mathsf{ps_f(D)}(\mathcal{G}, \bigstar \amalg \bigstar)$, the pullback of f along the inclusion $\bigstar \overset{incl_1}{\hookrightarrow} \bigstar \amalg \bigstar$ is either the empty graph-like diagram $\oslash$ or $\mathcal{G}$ itself. A graph $\mathcal{G}$ is connected if it is connected as a graphlike diagram.*

A(connected) component *of a graph $\mathcal{G}$ is a maximal connected strong subdiagram of $\mathcal{G}$.*

The following lemma is immediate from the definition of graphs.

Lemma 3.20. *A connected component of a graph $\mathcal{G}$ inherits a subgraph structure from $\mathcal{G}$.*

If $\mathcal{H} \hookrightarrow \mathcal{G}$ is a strong subgraph of $\mathcal{G}$, then so is its complement $\mathcal{G} - \mathcal{H}$.

Therefore, every graph is the disjoint union of its connected components.

Remark 3.21. A graph $\mathcal{G}$ is connected if and only if its realisation $|\mathcal{G}|$ is a connected space.

Example 3.22. (See Figure 8.) For $k \geq 0$, the *line graph* $\mathcal{L}^k$ is the connected bivalent graph with boundary $E_0 = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\}$, and

- ordered set of k vertices $V(\mathcal{L}^k) = (v_i)_{i=1}^k$, together with
- ordered set of edges $E(\mathcal{L}^k) = (l_j)_{j=0}^{2k+1}$ where $l_0 = 1_{\mathcal{L}^k} \in E_0$ and $l_{2k+1} = 2_{\mathcal{L}^k} \in E_0$, and $E\{v_i\} = \{l_{2i-1}, l_{2_i}\}$ for $1 \leq i \leq k$:

For any graph $\mathcal{G}$ with edge set E, there is a (canonical up to unique isomorphism) bijection $\mathsf{Gr}^\sharp(\shortmid, \mathcal{G}) \cong E$. The morphism $1 \mapsto e \in E$ in $\mathsf{Gr}^\sharp(\shortmid, \mathcal{G})$ that *chooses* $e \in E$ is denoted ch_e, or sometimes $ch_e^{\mathcal{G}}$.

A graph $\mathcal{S} = \mathcal{S}(E)$ with $E(\mathcal{S}) = E$, and $V(\mathcal{S}) = \emptyset$ (and hence, also $H(\mathcal{S}) = \emptyset$) is isomorphic to disjoint union of stick graphs, indexed by the set $\tilde{E}$ of orbits of the involution on E. Following [27], such a graph $\mathcal{S}$ is called a *shrub*.

Example 3.23. (See Figure 8.) For $m \geq 1$, the *wheel graph* $\mathcal{W}^m$ is the connected bivalent graph with empty boundary $E_0 = \emptyset$, and

- m *cyclically* ordererd vertices $V(\mathcal{W}^m) = (v_i)_{i=1}^m)$, together with
- $2k$ cyclically ordered edges $E(\mathcal{W}^m) = (a_j)_{j=0}^{2m-1}$ such that $E\{v_i\} = \{a_{2i-1}, a_{2_i}\}$ for $1 \leq i \leq m$:

$$\mathcal{W}^m \quad = \quad \circlearrowright \mathbf{2m} \longleftarrow \mathbf{2m} \longrightarrow \mathbf{m}\ .$$

When $m = 1$, $\mathcal{W}^m$ will often be denoted simply by $\mathcal{W}$.

$$\mathcal{W} = \quad \circlearrowright \{a, \tau a\} \longleftarrow \{a, \tau a\} \longrightarrow \{*\}\ . \tag{3.24}$$

$\mathcal{L}^4$ $\mathcal{W}^4$ $\mathcal{W} = \mathcal{W}^1$

FIGURE 8. Line and wheel graphs.

In Proposition 4.27, we'll see that a connected bivalent graph is isomorphic to $\mathcal{L}^k$ or $\mathcal{W}^m$ for some $k \geq 0$ or $m \geq 1$.

Example 3.25. For all graphs $\mathcal{G}$, $\mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{W})$ is non-empty. Namely, by Lemma 3.13, a morphism $\mathcal{G} \to \mathcal{W}$ is determined by a projection $\mathcal{S}(E) \twoheadrightarrow (\shortmid) \to \mathcal{W}$. So, $\mathcal{W}$ is weakly terminal in $\mathsf{Gr}^\sharp$ – every diagram in $\mathsf{Gr}^\sharp$ forms a cocone over $\mathcal{W}$ – and there are precisely $2^{|E|} \geq 1$ morphisms $\mathcal{G} \to \mathcal{W}$. In particular, $\mathsf{Gr}^\sharp(\mathcal{W}, \mathcal{W}) = \{id_{\mathcal{W}}, \tau_{(\mathcal{W})}\} \cong \mathsf{Gr}^\sharp(\shortmid, \shortmid)$, where $\tau_{(\mathcal{W})}$ is induced by the involution on $E(\mathcal{W}) = \{a, \tau a\}$.

However, the coequaliser of $id_{\mathcal{W}}, \tau_{(\mathcal{W})} : \mathcal{W} \to \mathcal{W}$ in $\mathsf{ps_f}(\mathsf{D})$ is the terminal diagram $\bigstar$. Therefore, since $\bigstar$ is not a graph, $\mathsf{Gr}^\sharp$ is not closed under finite colimits.

The observation that $\mathcal{W}$ is weakly terminal in $\mathsf{Gr}^\sharp$ leads to other characterisations of connectedness.

Proposition 3.26. *The following are equivalent:*

(1) A graph $\mathcal{G} \in ob(\mathsf{Gr})$ is connected,
(2) $\mathcal{G}$ is non-empty and, for every morphism $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{W} \amalg \mathcal{W})$, the pullback in $\mathsf{ps_f}(\mathsf{D})$ of f along the inclusion $inc_1 : \mathcal{W} \hookrightarrow \mathcal{W} \amalg \mathcal{W}$ is isomorphic to $\mathcal{G}$ or the emty graph $\oslash$,
(3) for every finite sum of graphs $\coprod_{i=1}^n \mathcal{H}_i$,

$$\mathsf{Gr}^\sharp(\mathcal{G}, \sum_{i=1}^n \mathcal{H}_i) \cong \coprod_{i=1}^n \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{H}_i). \tag{3.27}$$

Proof. (1) $\Leftrightarrow$ (2)
Since $\mathcal{W}$ is weakly terminal, any morphism $\mathsf{f} \in \mathsf{ps_f}(\mathsf{D})(\mathcal{G}, \bigstar \amalg \bigstar)$ factors as a morphism $\tilde{f} \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{W} \amalg \mathcal{W})$ followed by the componentwise projection $\mathcal{W} \amalg \mathcal{W} \to \bigstar \amalg \bigstar$ in $\mathsf{ps_f}(\mathsf{D})$.

(1) $\Rightarrow$ (3)
For any $\coprod_{i=1}^k \mathcal{H}_i \in ob(\mathsf{Gr}^\sharp)$, and $1 \leq j \leq k$, let $p_j \in \mathsf{ps_f}(\mathsf{D})(\coprod_{i=1}^k \mathcal{H}_i, \bigstar \amalg \bigstar)$ be the morphism that projects $\mathcal{H}_j$ onto the first summand, and $\coprod_{i \neq j} \mathcal{H}_i$ onto the second summand. Then, for any graph $\mathcal{G}$ and any $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \coprod_{i=1}^k \mathcal{H}_i)$, the diagram

$$\begin{tikzcd} \mathcal{P}_j \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "f"] \\ \mathcal{H}_j \arrow[r, hook, "inc_j"'] \arrow[d] & \coprod_{i=1}^k \mathcal{H}_i \arrow[d, "p_j"] \\ \bigstar \arrow[r, hook, "inc_1"'] & \bigstar \amalg \bigstar \end{tikzcd} \tag{3.28}$$

where the top square is a pullback, commutes in $\mathsf{ps_f}(\mathsf{D})$. Since the lower square is a pullback by construction, so is the outer rectangle.

In particular, if $\mathcal{G}$ is connected, then $\mathcal{P}_j$ is either empty or isomorphic to $\mathcal{G}$ itself. But this implies that there is some unique $1 \leq j \leq k$ such that f factors through the inclusion $inc_j \in \mathsf{Gr}^\sharp(\mathcal{H}_j, \coprod_{i=1}^k \mathcal{H}_k)$. In other words, $\mathsf{Gr}^\sharp(\mathcal{G}, \coprod_{i=1}^k \mathcal{H}_i) = \coprod_{i=1}^k \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{H}_i)$.

(3) $\Rightarrow$ (2)
If $\mathcal{G}$ satisfies condition 3.27, then $\mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{W} \amalg \mathcal{W}) \cong \mathsf{Gr}^\sharp(G, \mathcal{W}) \amalg \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{W})$. So, taking $\coprod_{i=1}^k \mathcal{H}_k = \mathcal{W} \amalg \mathcal{W}$ in Diagram 3.28, we have $\mathcal{P}_j = \emptyset$ or $\mathcal{P}_j \cong \mathcal{G}$ for $j = 1, 2$. $\square$

Example 3.29. Let $\mathcal{G}$ be a graph with edge set E. The shrub $\mathcal{S}(E)$ is a strong subgraph of $\mathcal{G}$ under the identity map id_E.

Each τ-orbit $\tilde{e}$ in $\widetilde{E}$ describes a component of $\mathcal{S}(E) = \coprod_{\tilde{e} \in \tilde{E}} (\shortmid_{\tilde{e}})$

$$(\shortmid_{\tilde{e}}) \stackrel{\text{def}}{=} \quad \circlearrowright \{e, \tau e\} \longleftarrow \emptyset \longrightarrow \emptyset,$$

and the canonical morphism $\iota^{\mathcal{G}}_{\tilde{e}}$ or $\iota_{\tilde{e}} : (\shortmid_{\tilde{e}}) \to \mathcal{G}$, induced by inclusion of edges is called the *essential morphism at* $\tilde{e}$ *(for* $\mathcal{G}$*)*.

If e and τe are both in the boundary E_0, then $(\shortmid_{\tilde{e}}) \hookrightarrow \mathcal{G}$ is a connected component of $\mathcal{G}$.

Example 3.30. Recall that, for each $v \in V$, $E\{v\} \stackrel{\text{def}}{=} s(t^1(v))$ is the set of edges incident on v.

To each vertex v, we associate the corolla $\mathcal{C}_{P\{v\}} \stackrel{\text{def}}{=} \mathcal{C}_{E\{v\}^\dagger}$

$$\mathcal{C}_{P\{v\}} \stackrel{\text{def}}{=} \quad \circlearrowright \left(E\{v\} \amalg E\{v\}^\dagger\right) \xleftarrow{s} H\{v\} \xrightarrow{t} \{v\}.$$

The inclusion $E\{v\} \hookrightarrow E$ induces a morphism $\iota^{\mathcal{G}}_v$ or $\iota_v : \mathcal{C}_{P\{v\}} \to \mathsf{Gr}$ called the *essential morphism at* v *for* $\mathcal{G}$. Observe that, whenever there is an edge e such that $e \in E\{v\}$ and $\tau e \in E\{v\}$, then ι_v is not injective on edges.

If $H\{v\} = E\{v\}$ is empty, so $\mathcal{C}_{P\{v\}}$ is an isolated vertex, then $\mathcal{C}_{P\{v\}} \hookrightarrow \mathcal{G}$ is a connected component of $\mathcal{G}$. In particular, a graph $\mathcal{G}$ such that $H(\mathcal{G}) = \emptyset$ is a disjoint union of stick graphs and isolated vertices.

Example 3.31. Recall, from Examples 3.7, 3.8, and 3.16, the graphs $\mathcal{M}^{X,Y}_{x_0,y_0}$, and $\mathcal{N}^X_{x_0,y_0}$.

For all finite sets X and Y, the canonical morphism $\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}} \to \mathcal{M}^{X,Y}_{x_0,y_0}$ is surjective and locally bijective. Moreover, $\mathcal{M}^{X,Y}_{x_0,y_0}$ is easily seen to be the colimit of the pair of parallel morphisms

$$ch_{x_0}, ch_{y_0} \circ \tau : (\shortmid) \rightrightarrows \mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}}.$$

And, $\mathcal{N}^X_{x_0,y_0}$ is the colimit of a pair of parallel morphisms

$$ch_{x_0},\ ch_{y_0} \circ \tau : (\shortmid) \rightrightarrows \mathcal{C}_{X \amalg \{x_0,y_0\}}.$$

(See also Example 5.15 and Figure 12.)

In Section 4, we'll see that all graphs can be constructed canonically as colimits of diagrams in the image of the embedding $\mathbb{P}^{\circlearrowleft} \hookrightarrow \mathsf{Gr}^\sharp$.

3.3. **Paths and loops.** The section concludes with some more topologically flavoured examples.

Example 3.32. For any graph $\mathcal{G}$, a morphism $p \in \mathsf{Gr}^\sharp(\mathcal{L}^k, \mathcal{G})$ describes a *path of length* k in $\mathcal{G}$. Given distinct elements $x_1, x_2 \in E \amalg V$, a path $p \in \mathsf{Gr}^\sharp(\mathcal{L}^k, \mathcal{G})$ *connects* x_1 *and* x_2 *in* $\mathcal{G}$ if x_1 and x_2 are in the image of p.

Given any path $p \in \mathsf{Gr}^\sharp(\mathcal{L}^k, \mathcal{G})$ connecting x_1 and x_2, we may always (by a Eulerian argument) 'cut out loops and folds', to obtain an injective path $p_i \in \mathsf{Gr}^\sharp(\mathcal{L}^{k_i}, \mathcal{G})$, $k_i \leq k$ connecting x_1, x_2.

Example 3.33. A *cycle* in $\mathcal{G}$ is a morphism $c \in \mathsf{Gr}^\sharp(\mathcal{W}^m, \mathcal{G})$ for some $l \geq 1$. A cycle c is non-trivial if it does not factor as $c : \mathcal{W}^m \to \mathcal{L}^k \xrightarrow{p} \mathcal{G}$. If $m = 2k+1$ for some $k \in \mathbb{N}$, then c is necessarily non-trivial.

A non-trivial cycle $c \in \mathsf{Gr}^\sharp(\mathcal{W}^m, \mathcal{G})$ exists if and only if there is a $1 \leq l \leq m$ and a pointwise injective morphism $f \in \mathsf{Gr}^\sharp(\mathcal{W}^l, \mathcal{G})$. In this case, $\mathcal{G}$ is said to *admit a non-trivial cycle*, and hence have *non-trivial genus*.

A connected graph $\mathcal{G}$ is *simply connected* if it does not admit a non-trivial cycle. This is equivalent to its geometric realisation $|\mathcal{G}|$ being simply connected.

If $m = 2k$ for $k \geq 1$, then $\mathsf{Gr}^\sharp(\mathcal{W}^m, \mathcal{L}^{k+1})$ is non-empty. hence, the existence, for some $k \in \mathbb{N}$, of a morphism $f \in \mathsf{Gr}^\sharp(\mathcal{W}^{2k}, \mathcal{G})$ does not imply that $\mathcal{G}$ has non-trivial genus. (The genus of $\mathcal{G}$ is defined to be the genus of its geometric realisation $|\mathcal{G}|$.) In fact, the set $E_\bullet$ of inner edges of a graph $\mathcal{G}$ is non-empty if and only if $\mathsf{Gr}^\sharp(\mathcal{W}^2, \mathcal{G})$ is non-empty.

A graph $\mathcal{G}$ is *path connected* if, for each pair of distinct elements $x_1, x_2 \in E \amalg V$, there exists some $k \in \mathbb{N}$ and a path $p \in \mathsf{Gr}^\sharp(\mathcal{L}^k, \mathcal{G})$ connecting x_1 and x_2 in $\mathcal{G}$.

Corollary 3.34 (Corollary to Proposition 3.26)**.** *A graph $\mathcal{G}$ is connected if and only if it is path connected.*

Proof. Since $\mathcal{L}^k$ is connected (for all $k \in \mathbb{N}$), path connectedness implies connectedness. By Proposition 3.26, if there exists a morphism $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{W}_1 \amalg \mathcal{W}_2)$ that does not factor through an inclusion $\mathcal{W}_i \hookrightarrow \mathcal{W}_1 \amalg \mathcal{W}_2$, then there are $x_1, x_2 \in V \amalg E$ such that $f(x_i) \in \mathcal{W}_i$ for $i = 1, 2$, and hence, since $\mathcal{L}^k$ is connected, there is no $p \in \mathsf{Gr}^\sharp(\mathcal{L}^k, \mathcal{G})$ connecting x_1, x_2. □

4. The étale site of graphs

By Example 3.17, there is a canonical embedding $\mathbb{P}^{\circlearrowleft} \hookrightarrow \mathsf{Gr}^\sharp$ whose image consists of the exceptional graph $(\shortmid)$, the corollas $\mathcal{C}_X$, and the local bijections between them. The goal of this section is to construct the category $\mathsf{Gr} \hookrightarrow \mathsf{Gr}^\sharp$ of connected graphs and étale morphisms ([23, Section 3]), and establish a chain

$$\mathbb{P}^{\circlearrowleft} \hookrightarrow \mathsf{Gr} \hookrightarrow \mathsf{GS}$$

of dense fully faithful embeddings as discussed in Section 2.

In what follows, we'll work in slightly greater generality and consider the category $\mathsf{Gr}^\sharp_{et} \subset \mathsf{Gr}^\sharp$ of *all* graphs and étale morphisms, of which Gr is the full subcategory on the connected graphs. (From Section 5.3, it will be assumed that all graphs are connected.)

Definition 4.1. *A morphism $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$ is* étale *if the righthand square in the defining diagram 3.12 is a pullback of finite sets.*

The category $\mathsf{Gr}^\sharp_{et} \subset \mathsf{Gr}^\sharp$ is the subcategory of all graphs and étale morphisms.

The full subcategory of connected graphs and étale morphisms is denoted by Gr.

The following proposition, which follows directly from Definition 4.1, tells us that the étale morphisms are precisely the local bijections.

Proposition 4.2. *A morphism $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$ is étale if and only if, for all $v \in V$, the restriction of f to $H\{v\}$ induces an isomorphism $H\{v\} \xrightarrow{\cong} H\{f(v)\}$. In particular, for composable morphisms f, g in $\mathsf{Gr}^\sharp$, if any two of f, g and $g \circ f$ are étale, then so is the third.*

Example 4.3. Any morphism $f : \mathcal{S} \to \mathcal{G}$ from a shrub $\mathcal{S}$ is trivially étale. In particular, morphisms from stick graphs are étale.

For any graph $\mathcal{G}$ and all $v \in V(\mathcal{G})$, the morphism $\iota_v : \mathcal{C}_{P\{v\}} \to \mathcal{G}$ defined in Example 3.30 is étale. In particular, by 4.2, a morphism $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$ is étale if and only if f induces an isomorphism $\mathcal{C}_{P\{v\}} \cong \mathcal{C}_{P\{f(v)\}}$ for all $v \in V$.

Example 4.4. In the Example 3.15, a is not étale since the vertex v of $\mathcal{C}_{\mathbf{2}}$ has valency 2 whereas $a(v) = v_1 \in V_3$. The morphism b is étale.

The canonical maps $\mathcal{C}_X \to \mathcal{M}^{X,Y}_{x_0,y_0} \leftarrow \mathcal{C}_Y$ and $\mathcal{C}_X \to \mathcal{N}^X_{x_0,y_0}$ (Example 3.11) are not étale, whereas the cannoical maps $\mathcal{C}_{X \amalg \{x_0\}} \to \mathcal{M}^{X,Y}_{x_0,y_0} \leftarrow \mathcal{C}_{Y \amalg \{y_0\}}$ and $\mathcal{C}_{X \amalg \{x_0, y_0\}} \to N^X$ are étale.

Example 4.5. Recall the line and wheel graphs $\mathcal{L}^k$ and $\mathcal{W}^m$, introduced in Examples 3.22, and 3.23.

For $k, n \geq 0$, a morphism in $\mathsf{Gr}^\sharp_{et}(\mathcal{L}^k, \mathcal{L}^n)$ may be viewed, informally, as an oriented inclusion of $\mathcal{L}^k$ in $\mathcal{L}^n$. Precisely,

$$\mathsf{Gr}^\sharp_{et}(\mathcal{L}^k, \mathcal{L}^n) \cong \begin{cases} 2(\mathbf{n} - \mathbf{k} + \mathbf{1}) & n \geq k \\ \emptyset, & n < k. \end{cases}$$

A morphism $f \in \mathsf{Gr}^\sharp_{et}(\mathcal{L}^k, \mathcal{W}^m)$, $m \geq 1$, is fixed by $f(1_{\mathcal{L}^k}) \in E(\mathcal{W}^m)$. Hece,

$$\mathsf{Gr}^\sharp_{et}(\mathcal{L}^k, \mathcal{W}^m) \cong E(\mathcal{W}^m) \cong 2(\mathbf{m}).$$

Most interesting are étale morphisms between wheel graphs. Let $l, m \geq 1$ be positive natural numbers. When m divides l, a morphism $f \in \mathsf{Gr}^\sharp_{et}(\mathcal{W}^l, \mathcal{W}^m)$ is fixed by $f(v_1) \in V(\mathcal{W}^m)$, $v_1 \in V(\mathcal{W}^m)$ and a choice of 'winding direction'. So,

$$\mathsf{Gr}^\sharp_{et}(\mathcal{W}^l, \mathcal{W}^m) \cong \begin{cases} 2(\mathbf{m}) & \text{if } \frac{m}{l} = d \in \mathbb{N} \\ \emptyset, & \text{otherwise.} \end{cases}$$

4.1. **Pullbacks and monomorphisms in $\mathsf{Gr}^\sharp$.** By Proposition 4.2, étale maps of graphs are local homeomorphisms. As such, they have similar properties to open maps of toplogical spaces, and it is useful to have this comparison in mind in what follows. However, unlike topological spaces, though epimorphisms of graphs are pointwise surjective, monomorphisms in $\mathsf{Gr}^\sharp$ are not always pointwise (strong) injections.

Let $\mathcal{S}$ be a shrub and $\mathcal{G}$ a graph without stick components.

Definition 4.6. *A morphism $i \in \mathsf{Gr}^\sharp((\mathcal{G} \amalg \mathcal{S}), \mathcal{G}')$ is a* weak injection, *if the following conditions hold:*

(i) the restriction $i : \mathcal{S} \to \mathcal{G}'$ is injective (hence an isomorphism),
(ii) i is injective on $V(\mathcal{G})$ and $H(\mathcal{G})$, but not necessarily on $E(\mathcal{G})$,
(iii) the images $i(\mathcal{G})$ and $i(\mathcal{S})$ are disjoint in $\mathcal{G}'$.

If a weak injection i is defined by inclusions $H \subset H'$, $V \subset V'$, and $s(H) \subset s'(H')$, then i is called a weak inclusion, *and $\mathcal{G} \amalg \mathcal{S}$ is a* weak subgraph *of $\mathcal{G}'$.*

The symbol '$\rightarrowtail$' is used to specify weak injections.

By Proposition 4.13 below, weak injections in $\mathsf{Gr}^\sharp$ are precisely the monomorphisms in $\mathsf{Gr}^\sharp$.

By *Lemma* 3.13 a weak injection $i \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$, is either injective on E or, *glues* pairs of ports of $\mathcal{G}$ in $\mathcal{G}'$. More precisely, if i is not injective on E, then there exist pairs of ports $e_1, e_2 \in E_0(\mathcal{G})$ such that

- $\tau e_1, \tau e_2 \in s(H)$, and hence $e_2 \neq \tau e_1$, and
- $\tau' i(e_2) = i(e_1) \in E'_I$ form a τ'-involutive pair of inner edges of $\mathcal{G}'$.

Example 4.7. Let $\mathcal{W}$ be the wheel graph with single vertex $v \in V(\mathcal{W})$. The étale morphism $\iota_v : \mathcal{C}_{P\{v\}} \to \mathcal{W}$ is both (pointwise) surjective and weakly injective.

Example 4.8. For all finite sets X and Y, the canonical étale morphisms $\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}} \to \mathcal{M}^{X,Y}_{x_0,y_0}$ and $\mathcal{C}_{X \amalg \{x_0, y_0\}} \to \mathcal{N}^X_{x_0,y_0}$ are surjective and not pointwise injective. But they are weak subgraphs.

Remark 4.9. Weak injections of Feynman graphs correspond to pointwise injections in the formalism of Example 3.1.

Lemma 4.10. *The graph categories $\mathsf{Gr}^\sharp$ and $\mathsf{Gr}^\sharp_{et}$ admit pullbacks. Moreover, étale morphisms are preserved under pullbacks in $\mathsf{Gr}^\sharp$.*

Proof. Let $\mathcal{P} = (\underline{E}, \underline{H}, \underline{V}, \underline{s}, \underline{t}, \underline{\tau})$ be the pullback in $\mathsf{ps_f}(\mathsf{D})$ of morphisms $f_1 \in \mathsf{Gr}^\sharp(\mathcal{G}_1, \mathcal{G})$ and $f_2 \in \mathsf{Gr}^\sharp(\mathcal{G}_2, \mathcal{G})$. Since pullbacks in $\mathsf{ps_f}(\mathsf{D})$ are computed pointwise, $\underline{\tau}$ is a fixed-point free involution, and $\underline{s}$ is injective. So, $\mathcal{P}$ is a graph, and $\mathsf{Gr}^\sharp$ admits pullbacks.

Étale morphisms pull back to étale morphisms since limits commute, and therefore, by symmetry, $\mathsf{Gr}^\sharp_{et}$ admits pullbacks. □

Definition 4.11. *For any morphism $f \in \mathsf{Gr}^\sharp(\mathcal{G}, \mathcal{G}')$, not necessarily étale, and any weak subgraph $i : \mathcal{H} \to \mathcal{G}'$, the* preimage $f^{-1}(\mathcal{H}) \subset \mathcal{G}$ of $\mathcal{H}$ under f *is defined as the pullback*

$$\begin{tikzcd} f^{-1}(\mathcal{H}) \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "f"] \\ \mathcal{H} \arrow[r, "i"'] & \mathcal{G}'. \end{tikzcd}$$

In particular, by Lemma 4.10, the preimage of an étale subgraph is an étale subgraph.

Example 4.12. (See also Example 4.5.) Let l, m and d be positive natural numbers such that $l = dm$. By Example 4.5, there exists an étale morphism $f \in \mathsf{Gr}^\sharp_{et}(\mathcal{W}^l, \mathcal{W}^m)$. If $i : \mathcal{L}^k \hookrightarrow \mathcal{W}^m$ is a weak étale inclusion (so $0 \leq k \leq m$), then the preimage $f^{-1}(\mathcal{L}^k) \hookrightarrow \mathcal{W}^m$ is isomorphic to the disjoint union $d\mathcal{L}^k$ of d copies of $\mathcal{L}^k$.

This subsection concludes with a charaterisation of the monomorphisms in $\mathsf{Gr}^\sharp$.

Proposition 4.13. *Let $f \in \mathsf{Gr}^\sharp(\mathcal{H}, \mathcal{G})$ be a morphism. The following are equivalent.*

1. *f is a monomorphism,*
2. *f is a weak injection,*
3. *for every weak subgraph $i : \mathcal{G}' \rightarrowtail \mathcal{G}$ where $\mathcal{G}'$ has no inner edges, the induced morphism $f^{-1}(\mathcal{G}') \to \mathcal{G}'$ is a strong inclusion (injective on edges and vertices).*

Proof. (1) $\Rightarrow$ (2)
If there are distinct vertices v_1, v_2 of $\mathcal{H}$ such that $f(v_1) = f(v_2) = v \in V(\mathcal{G})$, then there are parallel morphisms $i_{v_1}, i_{v_2} : \mathcal{C}_\mathbf{0} \to \mathcal{H}$, with image v_1, resp. v_2 such that $f \circ i_{v_1} = f \circ i_{v_2}$. Similarly, if there are distinct $h_1 = (e_1, v_1), h_2 = (e_2, v_2) \in H(\mathcal{H})$, such that $f(h_1) = f(h_2) = h \in H(\mathcal{G})$ then $ch_{e_1} \neq ch_{e_2} : (\shortmid) \rightrightarrows \mathcal{H}$, but $ch_e = f \circ ch_{e_1} = f \circ ch_{e_2} : (\shortmid) \to \mathcal{G}$. So, if f is not a weak injection, it is not a monomorphism.

(2) $\Rightarrow$ (3)
Let $i : \mathcal{G}' \rightarrowtail \mathcal{G}$ be a weak subgraph such that $\mathcal{G}'$ has no inner edges. Then $f^{-1}(\mathcal{G}') \to \mathcal{G}'$ is a strong injection if $f \in \mathsf{Gr}^\sharp(\mathcal{H}, \mathcal{G})$ is a weak injection.

(3) $\Rightarrow$ (1)
Let f satisfy (iii), and let $g, h : \mathcal{H}' \rightrightarrows \mathcal{H}$ be parallel morphisms such that $f \circ g = f \circ h$. If $i : \mathcal{G}' \rightarrowtail \mathcal{G}$ is as in (iii), then since $f^{-1}(\mathcal{G}') \to \mathcal{G}'$ is a strong injection, the preimages $(f \circ g)^{-1}(\mathcal{G}')$, and $(f \circ h)^{-1}(\mathcal{G}')$ have no inner edges and the induced maps to $f^{-1}(\mathcal{G}')$ agree. By considering the strong subgraphs of the form $\mathcal{C}_0 \xrightarrow{v} \mathcal{H} \xrightarrow{f} \mathcal{G}$ for $v \in V(\mathcal{H})$ and $(\shortmid_{\tilde{e}}) \xrightarrow{ch_e} \mathcal{G}' \xrightarrow{f} \mathcal{G}$, we see immediately that $g = h : \mathcal{H}' \to \mathcal{H}$, and hence f is a monomorphism in $\mathsf{Gr}^\sharp$. $\square$

The following corollary is immediate from the definitions.

Corollary 4.14. *A morphism in* $\mathsf{Gr}^\sharp$ *that is both an epi- and a monomorphism, is étale.*

4.2. **Graph neighbourhoods and the essential category** $\mathsf{es}(\mathcal{G})$**.** A family of morphisms $\mathfrak{U} = \{f_i \in \mathsf{Gr}^\sharp_{et}(\mathcal{G}_i, \mathcal{G})\}_{i \in I}$ is *jointly surjective* on $\mathcal{G}$ if $\mathcal{G} = \bigcup_{i \in I} im(f_i)$. By Lemma 4.10, jointly surjective families of morphims $\{f_i \in \mathsf{Gr}^\sharp_{et}(\mathcal{G}_i, \mathcal{G}')\}_{i \in I}$, define the covers at $\mathcal{G}$ for the canonical *étale topology* J on $\mathsf{Gr}^\sharp_{et}$.

So the sheaves for this topology are those presheaves $P : {\mathsf{Gr}^\sharp_{et}}^{op} \to \mathsf{Set}$ such that, for all graphs $\mathcal{G}$, and all covers $\mathfrak{U} = \{f_i \in \mathsf{Gr}^\sharp_{et}(\mathcal{G}_i, \mathcal{G})\}_{i \in I}$ at $\mathcal{G}$,

$$P(\mathcal{G}) = lim_{f_i \in \mathfrak{U}} P(\mathcal{G}_i).$$

In Proposition 4.34 we'll see that sheaves for the étale site on $\mathsf{Gr}^\sharp_{et}$ are canonically equivalent to graphical species (Definition 1.2). First we establish some more properties of étale morphisms. As usual, it's useful to draw on the analogy between étale morphisms of graphs and open maps of topological spaces for intuition.

Definition 4.15. *A* neighbourhood *of a weak subgraph* $i : \mathcal{G}' \rightarrowtail \mathcal{G}$ *is a weak étale subgraph* $u : \mathcal{U} \rightarrowtail \mathcal{G}$ *such that* $\tilde{i} : \mathcal{G}' \rightarrowtail \mathcal{U}$ *is a weak subgraph and* $i = u \circ \tilde{i} : \mathcal{G}' \to \mathcal{U} \to \mathcal{G}$.

A neighbourhood $(\mathcal{U}, u)$ *of* $i : \mathcal{G}' \rightarrowtail \mathcal{G}$ *is* minimal *if every neighbourhood* $(\mathcal{U}', u')$ *of* $i : \mathcal{G}' \rightarrowtail \mathcal{G}$ *is also a neighbourhood of* $(\mathcal{U}, u)$.

Observe that, since vertices $v \in V$ correspond to subgraphs $i : \mathcal{C}_\mathbf{0} \to \mathcal{G}$ (not generally étale), and edges $e \in E$ to subgraphs $ch_e : (\shortmid) \hookrightarrow \mathcal{G}$, we may also refer to neighbourhoods of vertices and edges. Moreover, since $u : \mathcal{U} \rightarrowtail \mathcal{G}$ is a neighbourhood of $e \in E$ if and only if it is a neighbourhood of $\iota_{\tilde{e}} : (\shortmid_{\tilde{e}}) \to \mathcal{G}$, there is no loss of generality in referring to neighbourhoods of τ-orbits $\tilde{e} \in \tilde{E}$.

For any graph $\mathcal{G}$, let $\mathcal{S}(E_\bullet)$ be shrub on the inner edges of $\mathcal{G}$, with components $(\shortmid_{\tilde{e}})$ for $\tilde{e} \in \tilde{E}_\bullet$. The poset of subsets of $\tilde{E}_\bullet$ is isomorphic to the poset of subgraphs of $\mathcal{S}(E_\bullet)$

$$\tilde{E}_\bullet \supset I \longleftrightarrow \mathcal{I}, \text{ where } \mathcal{I} \overset{\text{def}}{=} \coprod_{\tilde{e} \in I} (\shortmid_{\tilde{e}}) \subset \mathcal{G}.$$

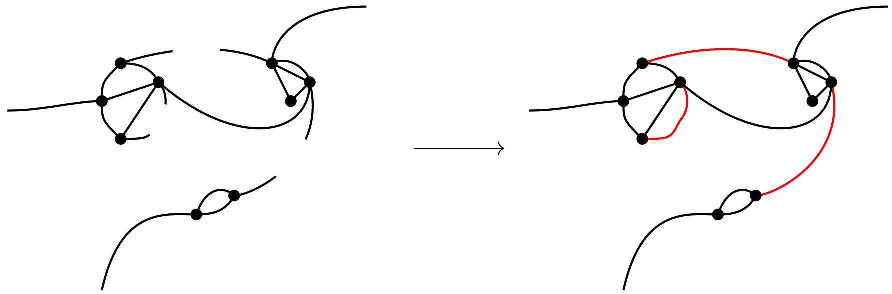


FIGURE 9. Graphs (left) $\mathcal{G}_{/\mathcal{I}}$ and, (right) $\mathcal{G}$, with subgraph $\mathcal{I}$ indicated in red.

For each such subgraph $\mathcal{I} \hookrightarrow \mathcal{S}(E_\bullet)$, we may construct a graph $\mathcal{G}_{\widehat{\mathcal{I}}}$, and a canonical surjective weak inclusion $i_{/\mathcal{I}} : \mathcal{G}_{\widehat{\mathcal{I}}} \rightarrowtail \mathcal{G}$, as in Figure 9, by 'breaking the edges in $\mathcal{I}$':

(4.16)
$$\begin{tikzcd} \mathcal{G}_{\widehat{\mathcal{I}}} \arrow[d, "i_{/\mathcal{I}}"'] & E \amalg (E(\mathcal{I}))^\dagger \arrow[r, leftrightarrow, "\tau_{/\mathcal{I}}"] \arrow[d, two heads] & E \amalg (E(\mathcal{I}))^\dagger \arrow[d, two heads] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, Rightarrow] & V \arrow[d, Rightarrow] \\ \mathcal{G} & E \arrow[r, leftrightarrow, "\tau"'] & E & H \arrow[l, "s"] \arrow[r, "t"'] & V, \end{tikzcd}$$

where

- $(E(\mathcal{I}))^\dagger$ is the formal involution $e \mapsto e^\dagger$ of the set $E(\mathcal{I})$ of edges of $\mathcal{I}$,
- the involution $\tau_{/\mathcal{I}}$ on $E \amalg (E(\mathcal{I}))^\dagger$ is given by

$$e \mapsto \begin{cases} \tau e, & e \in E - E(\mathcal{I}) \\ e^\dagger, & e \in E(\mathcal{I}), \end{cases}$$

So, $\mathcal{G}_{\widehat{\mathcal{I}}}$ has inner edges $E_\bullet(\mathcal{G}_{\widehat{\mathcal{I}}}) = E_\bullet - E(\mathcal{I})$, and boundary $E_0(\mathcal{G}_{\widehat{\mathcal{I}}}) = E_0 \amalg (E(\mathcal{I}))^\dagger$, and the surjection $E \amalg (E(\mathcal{I}))^\dagger \twoheadrightarrow E$ is the identity on E and $e^\dagger \mapsto \tau e$ on $(E(\mathcal{I}))^\dagger$.

Now, for each $\tilde{e} \in I$, the canonical morphism $\iota_{\tilde{e}} : (\shortmid_{\tilde{e}}) \hookrightarrow \mathcal{G}$ factors in two ways through $\mathcal{G}_{\widehat{\mathcal{I}}}$,

$$\begin{tikzcd} (\shortmid_{\tilde{e}}) \arrow[r, shift left, "(e,\tau e)\mapsto(e,e^\dagger)"] \arrow[r, shift right, "(e,\tau e)\mapsto((\tau e)^\dagger,\tau e)"'] & \mathcal{G}_{\widehat{\mathcal{I}}} \arrow[r, tail, "i_{/\mathcal{I}}"] & \mathcal{G}. \end{tikzcd} \tag{4.17}$$

Hence there exist parallel morphisms $\mathcal{I} \rightrightarrows \mathcal{G}_{/\mathcal{I}}$ such that the diagram

$$\begin{tikzcd} \mathcal{I} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{G}_{\widehat{\mathcal{I}}} \arrow[r, "i_{/\mathcal{I}}"] & \mathcal{G} \end{tikzcd} \tag{4.18}$$

commutes, and, moreover, describes a coequaliser diagram in $\mathsf{Gr}^\sharp$. (Of course the choice of pairs of morphisms $\mathcal{I} \rightrightarrows \mathcal{G}_{/\mathcal{I}}$ is not unique (there are $2^{|I|}$ choices), but it is unique up to isomorphism.)

In particular, for each $\mathcal{I}$, the set of components of $\mathcal{I} \amalg \mathcal{G}_{\widehat{\mathcal{I}}}$ defines an étale cover $\mathfrak{U}_{\mathcal{I}} = \{(\mathcal{U}_j, u_j)\}_{j \in J}$ at $\mathcal{G}$, where each u_j is the composition of the inclusion of the component $\mathcal{U}_j$ in $\mathcal{I} \amalg \mathcal{G}_{\widehat{\mathcal{I}}}$ with the canonical morphism to $\mathcal{G}$.

By definition, any surjective weak subgraph $\mathcal{G}' \rightarrowtail \mathcal{G}$ is of the form $(\mathcal{G}_{\widehat{\mathcal{I}}}, i_{/\mathcal{I}})$ for some $I \subset \tilde{E}$. So the collection $\{(\mathcal{G}_{\widehat{\mathcal{I}}}, i_{/\mathcal{I}})\}_{\mathcal{I} \hookrightarrow \mathcal{S}(E_\bullet)} \subset \mathsf{Gr}^\sharp_{et} | \mathcal{G}$ a canonical poset structure with for which the graph $\mathcal{G}_{\widehat{\mathcal{S}(E_\bullet)}}$, obtained by breaking all inner edges of $\mathcal{G}$, is initial. We have proved the following:

Lemma 4.19. *A neighbourhood $(\mathcal{U}, u)$ of a weak subgraph $i \in \mathsf{Gr}^\sharp(\mathcal{G}', \mathcal{G})$ is minimal if and only if each component of $\mathcal{U}$ intersects non-trivially with $\mathcal{G}'$, and $E_\bullet(\mathcal{U}) = E_\bullet(\mathcal{G}')$.*

Hence, for all vertices $v \in V$, and all edges $e \in E$, $\iota_v : \mathcal{C}_{P\{v\}} \rightarrowtail \mathcal{G}$ is the minimal neighbourhood of v and $\iota_{\tilde{e}} : (\shortmid_{\tilde{e}}) \rightarrowtail \mathcal{G}$ is the minimal neighbourhood of e.

In case $\mathcal{G}$ has no stick components, one readily checks that $\mathcal{G}_{\widehat{\mathcal{S}(E_\bullet)}} = \coprod_{v \in V} \mathcal{C}_{P\{v\}}$. In particular, for any graph $\mathcal{G}$, there is a canonical choice of *essential cover* $\mathfrak{Es}_{\mathcal{G}}$ of $\mathcal{G}$ by minimal étale subgraphs $\iota_{\tilde{e}}, \tilde{e} \in \tilde{E}$, $\iota_v, v \in V$ of $\mathcal{G}$. The essential category of $\mathcal{G}$ is the full subcategory of the slice category $\mathsf{Gr}^\sharp_{et}|\mathcal{G}$ on the objects of $\mathfrak{Es}_{\mathcal{G}}$.

Definition 4.20. *Let $\mathcal{G}$ be a graph. The* essential category of $\mathsf{es}(\mathcal{G})$ of $\mathcal{G}$ *is the full subcategory of $\mathsf{Gr}^\sharp_{et}|\mathcal{G}$ on the objects*

$$\iota_{\tilde{e}} : (\shortmid_{\tilde{e}}) \rightarrowtail \mathcal{G},\ \tilde{e} \in \tilde{E},\ \textit{and}\ \iota_v : \mathcal{C}_{P\{v\}} \rightarrowtail \mathcal{G},\ v \in V.$$

There is a canonical bijection between non-identity morphisms in $\mathsf{es}(\mathcal{G})$ and half edges of $\mathcal{G}$, whereby $h = (e, v) \in H$ corresponds to a morphism $\delta_h \in \mathsf{es}(\mathcal{G})(\shortmid_{\tilde{e}}, \mathcal{C}_{P\{v\}})$. Namely, if $h = (e, v) \in H$, then $s(h) = e$ is the unique element in the intersection $E(\shortmid_{\tilde{e}}) \cap E(\mathcal{C}_{P\{v\}}) \cap E$. So, δ_h is the unique morphism in $\mathsf{Gr}^\sharp_{et}(\shortmid_{\tilde{e}}, \mathcal{C}_{P\{v\}})$ above $\mathcal{G}$ that fixes e. And, since the objects of $\mathsf{es}(\mathcal{G})$ are indexed by $\tilde{E} \amalg V$, there are no other non-identity morphisms in $\mathsf{es}(\mathcal{G})$.

Lemma 4.21. *Each graph $\mathcal{G}$ is canonically the colimit of the forgetful functor $(\mathcal{C}, b) \mapsto \mathcal{C}$, $\mathsf{es}(\mathcal{G}) \to \mathsf{Gr}^\sharp_{et}$.*

A presheaf $P \in \mathsf{ps}(\mathsf{Gr}^\sharp_{et})$ is a sheaf for the étale topology on $\mathsf{Gr}^\sharp_{et}$ if and only if for all $\mathcal{G}$,

$$P(\mathcal{G}) = lim_{(\mathcal{C},b) \in \mathsf{es}(\mathcal{G})} P(\mathcal{C}). \tag{4.22}$$

Proof. If $e \in E_0$ is a port of $\mathcal{G}$, then there is, at most, one non-trivial morphism $\delta_h = \delta_{(\tau e, v)}$ in $\mathsf{es}(\mathcal{G})$ with source $\shortmid_{\tilde{e}}$. In this case, δ_h has the form $\delta_{(\tau e, v)}$ for some $v \in V$, and $\mathcal{C}_{P\{v\}}$ is the colimit of the diagram $\shortmid_{\tilde{e}} \xrightarrow{\delta_h} \mathcal{C}_{P\{v\}}$. The result then follows from 4.17 and 4.18.

The second statement is immediate since the essential cover $\mathfrak{Es}_{\mathcal{G}}$ refines every étale cover $\mathfrak{U}$ of $\mathcal{G}$. □

Lemma 4.21 will be used in the proof of Proposition 4.34 to establish that sheaves for the étale topology on $\mathsf{Gr}^\sharp_{et}$ are equivalent to graphical species.

Before doing so, it's worth collecting together a number or results on (étale) graph morphisms that will be useful in the rest of this work.

4.3. **Boundary preserving étale morphisms.** In general, morphisms $f \in \mathsf{Gr}^\sharp_{et}(\mathcal{G}', \mathcal{G})$ do not satisfy $f(E'_0) \subset E_0$. Those that do are componentwise surjective *graphical covering maps* in the sense of Proposition 4.23 below. In particular, monomorphisms $f \in \mathsf{Gr}^\sharp_{et}(\mathcal{G}', \mathcal{G})$ such that $f(E'_0) = E_0$ are componentwise isomorphisms.

Proposition 4.23. *For any étale morphism $f \in \mathsf{Gr}^\sharp_{et}(\mathcal{G}', \mathcal{G})$, $f(E'_0) \subset E_0$ if and only if there exists an étale covering $\mathfrak{U} = \{\mathcal{U}_i, u_i\}_{i \in I}$ of $\mathcal{G}$, such that, for all i, $f^{-1}(\mathcal{U}_i)$ is isomorphic to a disjoint union of $k(\mathcal{U}_i, f) \in \mathbb{N}$ copies of $\mathcal{U}_i$.*

In this case, $k(\mathcal{U}_i, f) = k_f \in \mathbb{N}$ is constant on connected components of $\mathcal{G}$.

Proof. If $u : \mathcal{U} \rightarrowtail \mathcal{G}$ is a weak étale subgraph for which there is a $k \in \mathbb{N}$ such that $f^{-1}(\mathcal{U}) \cong k(\mathcal{U})$, then also $f^{-1}(\mathcal{V}) \cong k\mathcal{V}$ for all weak subgraphs $\mathcal{V} \hookrightarrow \mathcal{U}$. So, we may assume, without loss of generality, that $\mathfrak{U} = \mathfrak{Es}_{\mathcal{G}}$ is the essential covering of $\mathcal{G}$.

Observe first that, if $f(E'_0) \not\subset E_0$, then there exists an $e' \in E'_0$ and $v \in V$ such that $f(e') \in E\{v\}$, and hence $(\shortmid_{e'})$ is a connected component of $f^{-1}(\mathcal{C}_{P\{v\}})$.

For the converse, let $(\mathcal{U}, u)$ be a neighbourhood of $\mathcal{G}$, and consider it's preimage $f^{-1}(\mathcal{U})$ in $\mathsf{Gr}^{\sharp}_{et}$. If $(\mathcal{U}, u) = (\shortmid_{\tilde{e}}, \iota_{\tilde{e}})$, for some $e \in E$, then it is immediate that $f^{-1}(\mathcal{U}) = \coprod_{e' \in E', f(e') = e} (\shortmid_{e'})$, so $k((\shortmid_{\tilde{e}}), f) = |f^{-1}(e)| \in \mathbb{N}$.

So, let $(\mathcal{U}, u) = (\mathcal{C}_{P\{v\}}, \iota_v)$ for some $v \in V$. Since f is étale, by Proposition 4.2, $\mathcal{C}_{P\{v\}} \cong \mathcal{C}_{P\{v'\}}$ for all $v' \in V'$ such that $f(v') = v'$. By the universal property of pullbacks, the weak étale inclusion $\coprod_{v' : f(v') = v} \mathcal{C}_{P\{v\}} \rightarrowtail \mathcal{G}'$ factors through $f^{-1}(\mathcal{C}_{P\{v\}}) \to \mathcal{G}'$. Therefore,

$$f^{-1}(\mathcal{C}_{P\{v\}}) \cong \left(\coprod_{v' : f(v') = v} \mathcal{C}_{P\{v'\}} \right) \amalg \mathcal{S}, \quad \text{where } \mathcal{S} \text{ is a shrub.}$$

We prove that $\mathcal{S}$ is empty. By construction, a component of $\mathcal{S}$ corresesponds to a neighbourhood of $\mathcal{G}'$ of the form $(\shortmid_{e'})$ for some port $e' \in E'_0$ such that $f(e') = e \in E\{v\}$. Otherwise, if $e' \in E'$ is not a port, then $(\shortmid_{\tilde{e}}) \hookrightarrow \mathcal{C}_{P\{t'(e')\}} \in f^{-1}(\mathcal{C}_{P\{v\}})$. But, $f(E'_0) \subset E_0$ by assumption so there is no such e' and $\mathcal{S} = \oslash$ is the empty graph.

Hence, the first statement of the proposition is proved.

By condition (3) of Proposition 3.26, it is sufficient to verify the second part of the proposition componentwise on $\mathcal{G}$. Therefore, we may assume, without loss of generality, that $\mathcal{G}$ is connected.

Let $f \in \mathsf{Gr}^{\sharp}_{et}(\mathcal{G}', \mathcal{G})$ satisfy $f(E'_0) \subset E_0$.

If $\mathcal{G} \cong (\shortmid)$, there is nothing to prove. Otherwise, the following diagram commutes for all $v \in V$:

$$\begin{tikzcd}
\coprod_{e' \in f^{-1}(e)} \shortmid_{\tilde{e}} \arrow[r, "\coprod_{h' \in f^{-1}(h)} \delta_{h'}"] \arrow[d] & \coprod_{v' \in f^{-1}(v)} \mathcal{C}_{P\{v\}} \arrow[r, "\coprod_h \iota_v"] \arrow[d] & \mathcal{G}' \arrow[d, "f"] \\
(\shortmid_{\tilde{e}}) \arrow[r, "\delta_h"'] & \mathcal{C}_{P\{v\}} \arrow[r, "\iota_v"'] & \mathcal{G}.
\end{tikzcd}$$

By the first part of the proof, both squares are pullbacks. In particular, if $f^{-1}(\mathcal{C}_{P\{v\}})$ is isomorphic to $k_v = k(\mathcal{C}_{P\{v\}}, \iota_v)$ copies of $\mathcal{C}_{P\{v\}}$ then $f^{-1}(\shortmid_{\tilde{e}}) \cong k_v(\shortmid_{\tilde{e}})$ for all $e \in E\{v\}$. Hence $\mathcal{C}_{P\{v\}} \mapsto k_v$ extends to a functor K from $\mathsf{es}(\mathcal{G})$ to the discrete category $\mathbb{N}$. Since $\mathcal{G}$ is connected, K must be constant. ☐

Definition 4.24. *A morphism $f \in \mathsf{Gr}^{\sharp}_{et}(\mathcal{G}, \mathcal{G}')$ is* boundary preserving *if it restricts to an isomorphism $f_{E_0} : E_0 \xrightarrow{\cong} E'_0$.*

The following is an immediate corollary of Proposition 4.23.

Corollary 4.25. *If $\mathcal{G}$ is connected, and $\mathcal{G}'$ is non-empty, then any étale morphism $f \in \mathsf{Gr}^{\sharp}_{et}(\mathcal{G}', \mathcal{G})$ such that $f(E'_0) \subset E_0$, is surjective.*

If $\mathcal{G}'$ is also connected and $E'_0 \neq \emptyset$, f is boundary preserving if and only if it is an isomorphism.

Remark 4.26. The condition that $E'_0 \neq \emptyset$ is necessary in the statement of Corollary 4.25. For example, for any $m > 1$, each of the two étale morphisms $\mathcal{W}^m \to \mathcal{W}$ (see Example 4.5) is trivially boundary preserving, but certainly not an isomorphism.

Recall from Example 3.22 that the line graph $\mathcal{L}^k$ has totally ordered edge set $E(\mathcal{L}^k) = (l_j)_{j=0}^{2k+1}$ with ports $l_0 = 1_{\mathcal{L}^k}$ and $l_{2k+1} = 2_{\mathcal{L}^k}$. For each vertex $w_i \in V(\mathcal{L}^k)$, $E\{w_i\} = \{l_{2i-1}, l_{2i}\}$. So, $\tau(2_{\mathcal{L}^k}) = \tau l_{2k+1} = l_{2k}$ (and $\tau 1_{\mathcal{L}^k} = \tau l_0 = l_1$) in $E(\mathcal{L}^k)$.

Proposition 4.27. *Let $\mathcal{G}$ be a connected graph with only bivalent vertices. Then $\mathcal{G} = \mathcal{L}^k$ or $\mathcal{G} = \mathcal{W}^m$ for some $k \geq 0$ or $m \geq 1$.*

Proof. Since $\mathcal{G}$ is bivalent, every weak injection $\mathcal{L}^k \rightarrowtail \mathcal{G}$ is étale, for all $k \in \mathbb{N}$.

If $\mathcal{G} \cong \mathcal{L}^0$ is a stick graph, then the result holds trivially.

Assume therefore that $V = V_2$ is non-empty. Then, for each $v \in V$, a choice of isomorphism $\mathcal{L}^1 \xrightarrow{\cong} \mathcal{C}_{P\{v\}}$ describes a weak étale injection $\mathcal{L}^1 \rightarrowtail \mathcal{G}$. Since $\mathcal{G}$ is finite, there is a maximum $M \geq 1$ for which there exists a weak étale injection $f : \mathcal{L}^M \rightarrowtail \mathcal{G}$.

Let $f \in \mathsf{Gr}(\mathcal{L}^M, \mathcal{G})$ be such a map. By definition of weak morphisms, f is either injective on edges or $f(l_0) = f(l_{2M}) \in E$. We'll show that $f(E_0(\mathcal{L}^M)) \subset E_0$ if f is injective on edges, and that $E_0 = \emptyset$ if not. Let $e_1 = f(1_{\mathcal{L}^M})$, and $e_2 = f(2_{\mathcal{L}^M})$. For $j = 1, 2$, if e_j is not a port of $\mathcal{G}$, then $e_j \in E\{v\}$ for some vertex $v \in V_2$. This vertex v is not in the image of f whenever f is injective on edges. But, this means we can factor f through weak injections $\mathcal{L}^M \rightarrowtail \mathcal{L}^{M+1} \rightarrowtail \mathcal{G}$, thereby contradicting maximality of M. So $e_j \in E_0$ for $j = 1, 2$.

Otherwise, if f is not injective on edges then it must be the case that $f(l_0) = f(l_{2M})$. But then f factors as $\mathcal{L}^M \to \mathcal{W}^M \to \mathcal{G}$. Since f is étale weakly injective, the second map $\mathcal{W}^M \to \mathcal{G}$ is étale, weakly injective and boundary preserving. Hence, by Corollary 4.25, $\mathcal{G} \cong \mathcal{W}^M$. □

Example 4.28. (See also Examples 4.5 and 4.12.) For $m \geq 1$, let $\mathcal{W}^m$ be the wheel graph with $m \geq 1$ vertices. By Proposition 4.27, if $i \in \mathsf{Gr}(\mathcal{G}, \mathcal{W})$ is a weak étale subgraph then $\mathcal{G} \cong \mathcal{W}^m$ or $\mathcal{G} \cong \mathcal{L}^k$ for some $0 \leq k \leq m$.

The extra complexity of the combinatorics of modular operads, compared with cyclic operads is explained by the following corollaries that imply that étale morphisms of simply connected graphs are either subgraph inclusions or isomorphisms. Proposition 4.42 gives analogous results for directed acyclic graphs (Definition 4.41).

Corollary 4.29. *If $\mathcal{G}$ is simply connected, and $\mathsf{Gr}^{\sharp}_{et}(\mathcal{G}', \mathcal{G})$ is non-empty, then $\mathcal{G}'$ is simply connected. In particular, any monomorphism of simply connected graphs is a pointwise injection.*

Proof. Let $\mathcal{G}, \mathcal{G}'$ be graphs and $f : \mathcal{G}' \to \mathcal{G}$ an étale morphism. Assume that $\mathcal{G}'$ admits a pointwise injective cycle $c : \mathcal{W}^m \to \mathcal{G}'$. Then $f \circ c : \mathcal{W}^m \to \mathcal{G}$ is locally injective, and hence $f \circ c$ factors as $\mathcal{W}^m \to \mathcal{W}^l \hookrightarrow \mathcal{G}$ where l divides m and $\mathcal{W}^l \hookrightarrow \mathcal{G}$ is injective. So $\mathcal{G}$ admits a non-trivial cycle. □

Corollary 4.30. *Let $\mathcal{G}$ be simply connected and $\mathcal{G}'$ be connected. An étale morphism $f : \mathcal{G}' \to \mathcal{G}$ satisfies $f(E'_0) \subset E_0$ if and only if f an isomorphism.*

Proof. By Proposition 4.23, it suffices to show that $k(f) = 1$.

The result is clear if $\mathcal{G}'$ (and hence $\mathcal{G}$) is an isolated vertex. So, assume that $E' = E(G')$ is non-empty, $\mathcal{G}'$ is connected, and $f \in \mathsf{Gr}^{\sharp}_{et}(\mathcal{G}', \mathcal{G})$ satisfies $f(E'_0) \subset E_0$. Let e_1, e_2 be distinct edges in E'. Since $\mathcal{G}'$ is path connected, there is a path $p \in \mathsf{Gr}^{\sharp}(\mathcal{L}^k, \mathcal{G}')$, which we may assume to be injective, connecting e_1 and e_2 in $\mathcal{G}'$. If $e_1 \neq e_2$, we may further assume that p is minimal in the sense that $p(1_{\mathcal{L}^k}) \in \{e_1, \tau e_1\}$ and $p(2_{\mathcal{L}^k}) \in \{e_2, \tau e_2\}$.

Since p is injective and f is étale, $f \circ p : \mathcal{L}^k \to \mathcal{G}$ is locally injective. Hence, if $f(e_1) = f(e_2) = e \in E$, then $f \circ p$ factors as $\mathcal{L}^k \to \mathcal{W}^m \hookrightarrow \mathcal{G}$ where m divides k and $\mathcal{W}^m \hookrightarrow \mathcal{G}$ is injective. Hence $\mathcal{G}$ admits a non-trivial cycle.

So, if $\mathcal{G}$ is connected then $k = 1$ and f is an isomorphism by Corollary 4.25. □

4.4. **Étale sheaves on $\mathsf{Gr}^{\sharp}_{et}$.** We'll see that the category GS of graphical species (Definition 1.2) is equivalent to the category of sheaves for the canonical étale topology J on $\mathsf{Gr}^{\sharp}_{et}$ (and its restriction to Gr) induced by jointly surjective collections $(u_i : \mathcal{U}_i \to \mathcal{G})_{i \in I}$ of étale morphisms.

Recall that a graphical species S is a presheaf on the category $\mathbb{P}^{\circlearrowright}$ and that $\Phi : \mathbb{P}^{\circlearrowright} \hookrightarrow \mathsf{Gr}^{\sharp}_{et}$ is a full subcategory under the maps $\S \to (\shortmid)$ and $X \to \mathcal{C}_X$. It's easy to check that a connected graph with two or more vertices has inner edges and therefore:

Lemma 4.31. *The essential image of $\mathbb{P}^{\circlearrowright}$ in $\mathsf{Gr}^{\sharp}_{et}$ is the full subcategory elGr of connected graphs with no inner edges.*

The equivalence $\mathsf{sh}(\mathsf{Gr}^{\sharp}_{et}, J) \simeq \mathsf{GS}$ will follow from:

Lemma 4.32. *The inclusion $\Phi : \mathbb{P}^{\circlearrowright} \hookrightarrow \mathsf{Gr}^{\sharp}_{et}$ is dense. Hence the functor $Y_{et} : \mathsf{Gr}^{\sharp}_{et} \to \mathsf{GS}$, $\mathcal{G} \longmapsto (X \mapsto \mathsf{Gr}^{\sharp}_{et}(\mathcal{C}_X, \mathcal{G}))$ (and its restriction $Y : \mathsf{Gr} \to \mathsf{GS}$) is fully faithful.*

Proof. By Lemma 4.21, it suffices to show that $\mathsf{es}(\mathcal{G})$ is a skeleton for $\mathsf{elGr}|\mathcal{G}$.

The essential category $\mathsf{es}(\mathcal{G})$ is skeletal by definition. Each $(\mathcal{C}, b) \in ob(\mathsf{elGr}|\mathcal{G})$ factors uniquely as an isomorphism followed by some ι_x in $\mathsf{es}(\mathcal{G})$ so the inclusion is essentially surjective on objects. It is full, and therefore also an equivalence, by Definition 4.20 of $\mathsf{es}(\mathcal{G})$. □

The category of elements $\mathsf{el}(S) \overset{\text{def}}{=} \int_{\mathbb{P}^{\circlearrowright}} S$ of a graphical species S was defined in Definition 1.9.

Remark 4.33. By an application of the Yoneda lemma, the category $\mathsf{el}(Y_{et}\mathcal{G})$ of elGr-elements, is canonically isomorphic to the slice category $\mathsf{el}(\mathcal{G}) \overset{\text{def}}{=} \mathbb{P}^{\circlearrowright}|\mathcal{G}$, and I will write $\mathsf{el}(\mathcal{G})$ for both. In general, since $\mathsf{Gr}^{\sharp}_{et}$ is a full subcategory of GS under Y_{et}, I will henceforth write $\mathcal{G} \in ob(\mathsf{GS})$ rather than $Y_{et}\mathcal{G} \in ob(\mathsf{GS})$ where there is no risk of confusion.

Proposition 4.34. *There is a canonical equivalence of categories $\mathsf{sh}(\mathsf{Gr}^{\sharp}_{et}, J) \simeq GS$.*

Proof. This is straightforward from the definitions. More precisely, the fully faithful inclusion $\Phi : \mathbb{P}^{\circlearrowright} \to \mathsf{Gr}^{\sharp}_{et}$ induces an *essential geometric morphism* (e.g. by [30, Theorem 2, page 359]) between the presheaf categories $\mathsf{ps}(\mathbb{P}^{\circlearrowright})$ and $\mathsf{ps}(\mathsf{Gr}^{\sharp}_{et})$. So, the pullback $\Phi^* : \mathsf{ps}(\mathsf{Gr}) \to \mathsf{ps}(\mathbb{P}^{\circlearrowright})$ has fully faithful right and left adjoints $\Phi_* \vdash \Phi^* \vdash \Phi_!$ (by [30, pages 377 and 378]), and

$$\Phi_* : \mathsf{ps}(\mathbb{P}^{\circlearrowright}) \to \mathsf{ps}(\mathsf{Gr}), \quad S \longmapsto (\mathcal{G} \mapsto lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} S(\mathcal{C})). \tag{4.35}$$

Hence, a presheaf P on $\mathsf{Gr}^{\sharp}_{et}$ satisfies $\Phi_*\Phi^*(P) \cong P$ if and only if P is a sheaf, in which case, for all graphs $\mathcal{G}$ (with Yoneda embedding $y\mathcal{G} \in \mathsf{ps}(\mathsf{Gr}^{\sharp}_{et})$)

$$P(G) \cong \mathsf{ps}(\mathsf{Gr}^{\sharp}_{et})(y\mathcal{G}, P) = \mathsf{ps}(\mathsf{Gr}^{\sharp}_{et})(y\mathcal{G}, \Phi_*\Phi^*P) = \mathsf{GS}(Y_{et}\mathcal{G}, P).$$

□

As an immediate consequence:

Corollary 4.36. *The restriction of the étale topology J to Gr induces an equivalence $\mathsf{sh}(\mathsf{Gr}, J) \simeq \mathsf{GS}$.*

(Refer to Remark 1.10 for a discussion of element categories.)

Definition 4.37. *Let S be a graphical species. An S-*structured graph *is an object of the element category $\mathsf{Gr}^{\sharp}_{et}(S) \overset{\text{def}}{=} \int_{\mathsf{Gr}^{\sharp}_{et}} S$. Given a graph $\mathcal{G}$, $S(\mathcal{G})$ is called* the set of S-structures on $\mathcal{G}$. *The category $\mathsf{Gr}(S) \overset{\text{def}}{=} \int_{\mathsf{Gr}} S$ of* connected S-structured graphs *is the full subcategory of $\mathsf{Gr}^{\sharp}_{et}(S)$ on the objects $(\mathcal{G}, \alpha)$ with $\mathcal{G} \in \mathsf{Gr}$ connected.*

Remark 4.38. By the Yoneda lemma, $S(\mathcal{G}) \cong \mathsf{GS}(Y\mathcal{G}, S)$ canonically, and the same notation α (or $(\mathcal{G}, \alpha)$) will be used for both an S-structure $\alpha \in S(\mathcal{G})$, and the corresponding morphism $\alpha \in \mathsf{GS}(\mathcal{G}, S)$.

4.5. **Directed graphs.** By way of example, and to provide extra context, this section ends with a discussion of directed graphs. In particular, we see that, if Di is the terminal directed graphical species defined in Example 1.11, then $\mathsf{Gr}^\sharp_{et}(Di)$ is precisely the category of directed graphs and étale morphisms used in [27], to prove a Weber style nerve theorem for properads.

For any graph $\mathcal{G}$, a Di-structure $\xi \in Di(\mathcal{G})$ is a partition $E = E_{\text{in}} \amalg E_{\text{out}}$, where $E_{\text{in}} \ni e \Leftrightarrow \tau e \in E_{\text{out}}$. Hence τ induces bijjections $E_{\text{in}} \cong \widetilde{E} \cong E_{\text{out}}$.

So, an object $(\mathcal{G}, \xi) \in ob(\mathsf{Gr}(Di))$, called an *orientation on* $\mathcal{G}$, is given by a diagram of finite sets

$$\widetilde{E} \xleftarrow{\widetilde{s_{\text{in}}}} H_{\text{in}} \xrightarrow{t_{\text{in}}} V \xleftarrow{t_{\text{out}}} H_{\text{out}} \xrightarrow{\widetilde{s_{\text{out}}}} \widetilde{E}\ , \tag{4.39}$$

where the maps $\widetilde{s_{\text{in}}}, \widetilde{s_{\text{out}}}$, respectively $t_{\text{in}}, t_{\text{out}}$ denote the appropriate (quotients of) restrictions of $s : H \to E$, respectively $t : H \to V$. Morphisms in $\mathsf{Gr}^\sharp_{et}(Di)$ correspond to quadruples of finite set maps making the obvious diagrams commute, and such that the outer left and right squares are pullbacks. This is the definition of the category of directed graphs and étale morphisms in [27, Section 1.5].

Example 4.40. The line graphs $\mathcal{L}^k$ with $E(\mathcal{L}^k) = \{l_i\}_{i=0}^{2k+1}$ and wheel graphs $\mathcal{W}^m$ with $E(\mathcal{W}^m) = \{a_j\}_{j=0}^{2m-1}$ admit a canonical distinguished choice of orientation $\theta_{\mathcal{L}^k} \in Di(\mathcal{L}^k)$, $\theta_{\mathcal{W}^m} \in Di(\mathcal{W}^m)$, given by

$$\theta_{\mathcal{L}^k} : E(\mathcal{L}^k) \to \{\text{in}, \text{out}\},\ l_{2i+1} \mapsto \text{in},\ \text{and}\ l_{2i} \mapsto \text{out},\ \text{for}\ 0 \leq i \leq k,$$

$$\theta_{\mathcal{L}^k} : E(\mathcal{L}^k) \to \{\text{in}, \text{out}\},\ a_{2j-1} \mapsto \text{in},\ \text{and}\ e_{2j} \mapsto \text{out},\ \text{for}\ 1 \leq j \leq m,$$

A morphism $\gamma : (\mathcal{L}^k, \theta_{\mathcal{L}^k}) \to (\mathcal{G}, \xi)$ in $\mathsf{Gr}^\sharp_{et}(Di)$ is a *directed path of length k in* $(\mathcal{G}, \xi)$, and a *directed cycle of length m in* $(\mathcal{G}, \xi)$ is a morphism $\rho : (\mathcal{W}^m, \theta_{\mathcal{W}^m}) \to (\mathcal{G}, \xi)$ in $\mathsf{Gr}^\sharp_{et}(Di)$.

Directed paths and cycles are induced by locally injective morphisms of undirected graphs. So, if a directed graph $(\mathcal{G}, \xi)$ admits a directed cycle, then the underlying graph $\mathcal{G}$ admits a non-trivial cycle (Example 3.33).

Definition 4.41. *A* directed acyclic graph (DAG) *is a directed graph* $(\mathcal{G}, \xi)$ *without directed cycles. In other words,* $\mathsf{Gr}^\sharp_{et}(Di)\,((\mathcal{W}^m, \theta_{\mathcal{W}^m}), (\mathcal{G}, \xi)) = \emptyset$ *for all* $m \geq 1$.

There are directed versions of Corollaries 4.29 and 4.30:

Proposition 4.42. *For all morphisms* $f \in \mathsf{Gr}^\sharp_{et}(Di)\,((\mathcal{G}', \xi'), (\mathcal{G}, \xi))$ *between connected DAGs, the corresponding morphism* $f \in \mathsf{Gr}^\sharp_{et}(\mathcal{G}, \mathcal{G})$ *is a monomorphism.*

Moreover, if $(\mathcal{G}, \xi)$ *is a DAG and* $\mathsf{Gr}^\sharp_{et}(Di)\,((\mathcal{G}', \xi'), (\mathcal{G}, \xi))$ *is non-empty, then* $(\mathcal{G}', \xi')$ *is a DAG, so morphisms to DAGs in* $\mathsf{Gr}^\sharp_{et}(Di)$ *are monomorphisms.*

The proof follows from the undirected versions, and Equation (4.39). Since it is not necessary for what follows, I leave it as an exercise to the interested reader.)

5. Non-unital modular operads

The goal of the current section is to construct a monad $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ on GS whose EM category of algebras $\mathsf{GS}^{\mathbb{T}}$ is equivalent to the category MO^- of non-unital modular operads (Remark 1.23).

The endofunctor $T : \mathsf{GS} \to \mathsf{GS}$ for this monad is closely related to Joyal and Kock's modular operad endofunctor [23, Section 5] (see also Section 6). Having defined T and its unit $\eta^{\mathbb{T}}$, the remainder of the section is devoted to establishing some technical results needed to obtain the multiplication $\mu^{\mathbb{T}}$, and the proof of the equivalence $\mathsf{GS}^{\mathbb{T}} \simeq \mathsf{MO}^-$.

Example 5.1. Recall, from Examples 1.17,2.13, the category of rooted corollas $\mathsf{el}(RC)$ whose objects are of the form $(\downarrow)$ or t_X where X is a finite set, and the dendroidal category Ω that governs the combinatorics of operads. [34]

The operad endofunctor M^{Op} on $\mathsf{ps}(\mathsf{el}(RC))$ takes $P : \mathsf{el}(RC)^{op} \to \mathsf{Set}$ to the presehaves of rooted corollas with the same palette, and with $M^{Op}P(t_X)$ defined to be the set of formal compositions – with X inputs –of elements of P. These are rooted tress $\mathbf{T} \in \Omega$, whose leaves are bijectively labelled by X and whose vertices v are decorated by elements of $P(t_{\partial v})$ (where ∂v here denotes the rooted boundary of the minimal neighbourhood of v in $\mathbf{T}$). (See e.g. [34].)

The monadic unit $\eta^{\mathbb{M}^{Op}}$ is induced by the inclusion of rooted corollas in Ω, $\eta^{\mathbb{M}^{Op}}(\phi) = (t_X, \phi)$ for all $\phi \in P(t_X)$. Applying the monad twice amounts to 'tree nesting': replacing the neighbourhoods of vertices v with trees $\mathbf{T}_v$ according to $\partial v \cong \partial(\mathbf{T}_v)$, as in Figure 10. The multiplication $\mu^{\mathbb{M}^{Op}}$ for M^{Op}, is described by erasing the inner nesting (coloured blue in Figure 10 below), and it's clear that this always results in a well defined tree.

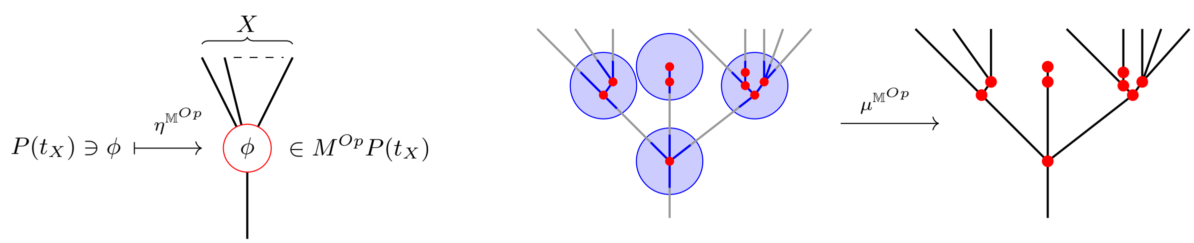


FIGURE 10. Visualising the unit and mulitplication for the operad monad on rooted corollas.

If (P, h) is an algebra for $\mathbb{M}^{Op}$ then h describes a rule for collapsing the inner edges of each P-decorated tree according to the axioms of operadic composition.

Likewise, the non-unital modular operad monad $\mathbb{T}$ on GS will take a graphical species S to a species whose elements are represented by S-structured graphs that encode formal multiplications and contractions in S.

5.1. **X-graphs and an endofunctor for non-unital modular operads.** We first establish conventions for bijectively labelling graph boundaries by finite sets.

Definition 5.2. *Let X be a finite set. An X-*graph *is a pair $\mathcal{X} = (\mathcal{G}, \rho)$, where $\mathcal{G} \in ob(\mathsf{Gr})$ is a connected graph such that $V \neq \emptyset$ and $\rho : E_0 \xrightarrow{\cong} X$ is a bijection of finite sets called an X-*labelling *for $\mathcal{G}$.*

*Given X-graphs $\mathcal{X} = (\mathcal{G}, \rho), \mathcal{X}' = (\mathcal{G}', \rho')$, an X-*isomorphism* $\mathcal{X} \to \mathcal{X}'$ is an isomorphism $g \in \mathsf{Gr}(\mathcal{G}, \mathcal{G}')$ that preserves the X-labelling. That is, $\rho' \circ g_{E_0} = \rho : E_0 \to X$.*

The groupoid $X\mathsf{Gr}_{iso}$ is the groupoid of X-graphs and X-isomorphisms.

Remark 5.3. It is often convenient to use the same notation for labelled and unlabelled graphs. In particular, an X-graph $\mathcal{X} = (\mathcal{G}, \rho)$ is denoted simply by $\mathcal{G}$ when the labelling ρ is trivial or completely canonical. For example, for all finite sets X, the corolla $\mathcal{C}_X$ canonically defines an X-graph $\mathcal{C}_X = (\mathcal{C}_X, id)$.

Example 5.4. The line graph $\mathcal{L}^k$, with $E_0(\mathcal{L}^k) = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\}$, $k \geq 0$ is canonically labelled by $1_{\mathcal{L}^k} \mapsto 1 \in \mathbf{2}$ and therefore has the structure of a $\mathbf{2}$-graph when $k \geq 1$.

Note, however, that $\mathcal{L}^0 = (\shortmid)$ is not a $\mathbf{2}$-graph since its vertex set is empty. (This will be discussed in detail in Sections 6 and 7.)

Definition 5.5. *Let S be a graphical species and X a finite set. Objects of the groupoid $X\mathsf{Gr}_{iso}(S)$ of* S-structured X-graphs *are elements $\alpha \in S(\mathcal{X})$ for $\mathcal{X} \in ob(X\mathsf{Gr}_{iso})$ and morphisms in $X\mathsf{Gr}_{iso}(S)(\alpha, \alpha')$ are isomorphisms $g \in X\mathsf{Gr}_{iso}(\mathcal{X}, \mathcal{X}')$ such that $S(g)(\alpha') = \alpha \in S(\mathcal{G})$.*

Let $Aut_X(\mathcal{X}) \overset{\text{def}}{=} X\mathsf{Gr}_{iso}(\mathcal{X}, \mathcal{X})$ be the automorphism group of $\mathcal{X} \in ob(X\mathsf{Gr}_{iso})$. Given parallel X-isomorphisms $g, g' \in X\mathsf{Gr}_{iso}(\mathcal{X}, \mathcal{X}')$, there is some $\sigma \in Aut_X(\mathcal{X})$ and $\sigma' \in Aut_X(\mathcal{X}')$ such that $g' = \sigma' g \sigma$.

It follows that there is a completely canonical (independent of $g \in X\mathsf{Gr}_{iso}(\mathcal{X}, \mathcal{X}')$) choice of natural (in $\mathcal{X}$) isomorphism

$$\frac{S(\mathcal{X})}{Aut_X(\mathcal{X})} \cong \frac{S(\mathcal{X}')}{Aut_X(\mathcal{X}')}, \quad [\alpha] \mapsto [g(\alpha)], \text{ for } \alpha \in S(\mathcal{X}). \tag{5.6}$$

We can now define the non-unital modular operad endofunctor T on GS, that takes a graphical species S to equivalence classes of graphs decorated by S.

Definition 5.7. *The* non-unital CSM endofunctor $T : \mathsf{GS} \to \mathsf{GS}$ *is defined for all $S \in \mathsf{GS}$, by*

$$\begin{array}{rcll} TS_{\S} & = & S_{\S}, & \\ TS_X & = & colim_{\mathcal{X} \in X\mathsf{Gr}_{iso}} S(\mathcal{X}) & \textit{for all finite sets } X. \end{array} \tag{5.8}$$

We check that the assignment $S \mapsto TS$ defines an endofunctor on GS: Using Equation (5.6), we obtain

$$\begin{array}{rcl} TS_X & = & \coprod_{[\mathcal{X}] \in \pi_0(X\mathsf{Gr}_{iso})} \frac{S(\mathcal{X})}{Aut_X(\mathcal{X})} \\ & & \\ & = & \pi_0(X\mathsf{Gr}_{iso}(S)) \end{array} \tag{5.9}$$

where, for $\mathcal{X} = (\mathcal{G}, \rho)$, $[\mathcal{X}] \in \pi_0(X\mathsf{Gr}_{iso})$ is the connected component of $\mathcal{X}$ in $X\mathsf{Gr}_{iso}$.

So, elements of TS_X may be viewed as isomorphism classes of S-structured X-graphs and two X-labelled S-structured graphs $(\mathcal{X}, \alpha)$ and $(\mathcal{X}', \alpha')$ represent the same class $[\mathcal{X}, \alpha] \in TS_X$ precisely when there is an isomorphism $g \in X\mathsf{Gr}_{iso}(\mathcal{X}, \mathcal{X}')$ such that $S(g)(\alpha') = \alpha$.

Isomorphisms of corollas induce relabelings of graph ports in TS.

To describe the projections $TS(ch_x) : TS_X \to S_{\S}$, let $\mathcal{X}$ be an X-graph and

$$ch_x^{\mathcal{X}} \overset{\text{def}}{=} ch_{\rho^{-1}(x)} \in \mathsf{Gr}(\shortmid, \mathcal{G}), \quad 1 \mapsto \rho^{-1}(x) \in E_0(\mathcal{G}).$$

Then $TS(ch_x)$ is induced by

$$[\mathcal{X}, \alpha] \mapsto S(ch_x^{\mathcal{X}})(\alpha).$$

If $(\mathcal{X}, \alpha)$ and $(\mathcal{X}', \alpha')$ represent the same element of TS_X, then there is a $g \in X\mathsf{Gr}_{iso}(\mathcal{X}, \mathcal{X}')$ such that $S(g)(\alpha') = \alpha \in S_X$. Therefore $ch_x^{\mathcal{X}'} = g \circ ch_x^{\mathcal{X}} \in \mathsf{Gr}(\shortmid, \mathcal{G}')$, and

$$S(ch_x^{\mathcal{X}'})(\alpha') = S(ch_x^{\mathcal{X}}) \circ S(g)(\alpha') = S(ch_x^{\mathcal{X}})(\alpha) \ \in S(\shortmid).$$

So T is a well-defined functor on GS.

The natural transformation $\eta^{\mathbb{T}} : id_{\mathsf{GS}} \Rightarrow T$ induced by $S_X \xrightarrow{\cong} S(\mathcal{C}_X)$ (and $S_{\S} \xrightarrow{\cong} S(\shortmid)$), provides a unit for T.

5.2. **Gluing constructions.** The monadic multiplication for T will be defined in terms of colimits of a certain class of diagrams in Gr. However, since Gr does not admit general colimits (see Example 3.25), we'll need to do a little extra work.

Example 5.10. Another example, of particular significance to this work, of a diagram in Gr that does not admit a coequaliser in Gr is given by the pair of parallel morphisms $id, \tau : (\shortmid) \rightrightarrows (\shortmid)$. In $\mathsf{GrShape}$, the coequaliser of these morphisms is the *exceptional loop* $\bigcirc$:

$$\bigcirc \stackrel{\text{def}}{=} \begin{tikzcd} \mathbf{1} \arrow[loop left] & \mathbf{0} \arrow[l] \arrow[r] & \mathbf{0} \end{tikzcd}.$$

Clearly $\bigcirc \notin ob(\mathsf{Gr}^\sharp)$ since a singleton set does not admit a non-trivial involution. (See Section 6 for a detailed discussion of this diagram.)

For any graphical species S' and finite set X, elements of TS'_X are represented by S' decorated X-graphs $(\mathcal{X}, \alpha)$, $\alpha \in S'(\mathcal{X})$. So, if $S' = TS$ where S is a $(\mathfrak{C}, \omega)$-coloured graphical species, elements of $T^2 S_X$ have representatives of the form $\beta \in TS(\mathcal{X})$. In other words, there are represented by X-graphs $\mathcal{X}$ whose vertices are decorated by elements of $\mathsf{Gr}(S)$ and whose edges are coloured by $\mathfrak{C}$.

More precisely, $\beta \in TS(\mathcal{X})$, that can be thought of as a functor

$$el(\beta) : \mathsf{el}(\mathcal{X}) \to \mathsf{el}(TS), \quad (\mathcal{C}_{X_b}, b) \mapsto (\mathcal{C}_{X_b}, S(b)(\beta)), \quad \text{where } S(b)(\beta) \in TS_{X_b}$$

that, in an appropriate sense, respects the shape of $\mathcal{X}$.

Then, as in Figures 10 and 11, we'd like to 'forget' the vertices of the original graph $\mathcal{X}$ to obtain an element of TS_X.

Definition 5.11. *Let $\mathcal{G}$ be a Feynman graph. A* $\mathcal{G}$-shaped graph of graphs *is a functor $\mathbf{\Gamma} : \mathsf{el}(\mathcal{G}) \to \mathsf{Gr}^\sharp_{et}$ (or $\mathbf{\Gamma}^{\mathcal{G}}$) such that*

$$\begin{array}{ll} \mathbf{\Gamma}(a) = (\shortmid) & \textit{for all } (\shortmid, a) \in \mathsf{el}(\mathcal{G}), \\ E_0(\mathbf{\Gamma}(b)) = X & \textit{for all } (\mathcal{C}_X, b) \in \mathsf{el}(\mathcal{G}), \end{array}$$

and, for all $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$ and all $x \in X_b$,

$$\mathbf{\Gamma}(ch_x) = ch_x^{\mathbf{\Gamma}(b)} \in \mathsf{Gr}^\sharp_{et}(\shortmid, \mathbf{\Gamma}(b)).$$

A $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma} : \mathsf{el}(\mathcal{G}) \to \mathsf{Gr}^\sharp_{et}$ is non-degenerate *if, for all $v \in V$, $\mathbf{\Gamma}(\iota_v)$ has no stick components. Otherwise, $\mathbf{\Gamma}$ is called* degenerate.

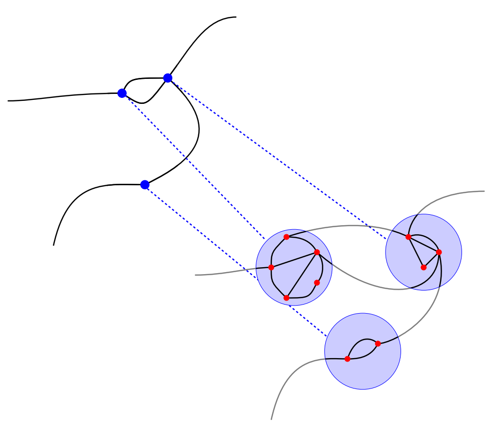


FIGURE 11. A $\mathcal{G}$ shaped graph of graphs $\mathbf{\Gamma}$ describes *graph substitution* in which each vertex v of $\mathcal{G}$ is replaced by a graph $\mathcal{G}_v$ according to a bijection $\delta(\mathcal{C}_{P\{v\}}) \to \delta(\mathcal{C}_v)$. When $\mathbf{\Gamma}$ is non-degenerate, taking its colimit corresponds to erasing the inner (blue) nesting.

Informally, a non-degenerate $\mathcal{G}$-shaped graph of graphs is a rule for substituting graphs into vertices of $\mathcal{G}$ as in Figure 11. However, this intuitive description of a graph of graphs in terms of graph insertion does not always apply to degenerate graphs of graphs (see also Section 6, and Section 7).

By Lemma 4.21, every graph $\mathcal{G}$ is the colimit of a non-degenerate graph of graphs, namely the *identity $\mathcal{G}$-shaped graph of graphs* given by the forgetful functor $(\mathcal{C}, b) \to \mathcal{C}$, $\mathsf{el}(\mathcal{G}) \to \mathsf{Gr}^{\sharp}_{et}$. And, by the discussion around 4.18, if $\mathcal{G}$ has no stick components, this is equivalent to the statement that $\mathcal{G}$ is the coequaliser of the canonical diagram

$$\mathcal{S}(E_\bullet) \rightrightarrows \coprod_{v \in V} \mathcal{C}_{P\{v\}} \xrightarrow{\coprod(\iota_v)} \mathcal{G}. \tag{5.12}$$

To prove that non-degenerate graphs of graphs admit a colimit in $\mathsf{Gr}^{\sharp}_{et}$, we'll generalise this observation using a modification of the *gluing data of directed graphs* defined in [27, Section 1.5.1].

Definition 5.13. *Let $\mathcal{S} = \coprod_{i \in I} (\shortmid_i)$ be a shrub, and let $\mathcal{G}$ be a graph without stick components. A pair of parallel morphisms $\delta_1, \delta_2 : \mathcal{S} \rightrightarrows \mathcal{G}$ is a* gluing datum *if*

- *δ_1, δ_2 are injective and have disjoint images in $\mathcal{G}$,*
- *for all $i \in I$, $\delta_1(1_l) = e^1_l \in E_0(\mathcal{G})$ and $\delta_2(2_l) = e^2_l \in E_0(\mathcal{G})$ are ports.*

Lemma 5.14. *Gluing data admit coequalisers in $\mathsf{Gr}^{\sharp}_{et}$.*

Proof. Let $\mathcal{G}$ be a graph without stick components and $\delta_1, \delta_2 : \mathcal{S} = \coprod_{i \in I} (\shortmid_i) \rightrightarrows \mathcal{G}$ be a gluing datum with coequaliser $\overline{p} : \mathcal{G} \to \overline{\mathcal{G}} = (\overline{E}, \overline{H}, \overline{V}, \overline{s}, \overline{t}, \overline{\tau})$ in $\mathsf{ps_f}(\mathsf{D})$. So, for $X = E, H, V$, elements of $\overline{X}$ are equivalence classes of elements of X.

We show that $\overline{P}$ is a graph. Then the diagram has a coequaliser in $\mathsf{Gr}^{\sharp}$, and this is a coequaliser in $\mathsf{Gr}^{\sharp}_{et}$ by Proposition 4.2.

Observe first that $\overline{H} = H$, since $h, h' \in H(\mathcal{G})$ are identified in $\overline{H}$ if and only if there is an edge $l \in E(\mathcal{S})$ such that $\delta_1(l) = s(h)$ and $\delta_2(l) = s(h')$ which contradicts the conditions of Definition 5.13. Moreover $e, e' \in E(\mathcal{G})$ are identified in $\overline{E}$ if and only if there is an $l \in E(\mathcal{S})$ such that $\delta_1(l) = e$ and $\delta_2(l) = e'$ (or vice versa). So, since $\mathcal{G}$ has no stick components, and δ_1, δ_2 have disjoint images, we may assume that $e \in E_0$ is a port and $e' \in s(H)$. In particular, $\overline{s} =: H \to \overline{E}$ is injective.

Since δ_1, δ_2 have disjoint images, $\overline{\tau} : \overline{E} \to \overline{E}$ is a fixed-point free involution, and $\overline{P}$ is a graph. $\square$

Example 5.15. In Example 3.31, the graphs $\mathcal{M}^{X,Y}_{x,y}$ and $\mathcal{N}^{X}_{x,y}$ (Examples 3.7, 3.8) were constructed as coequalisers of gluing data:

$$\big(ch_x,\ ch_y \circ \tau : (\shortmid)\ \rightrightarrows\ (\mathcal{C}_{X \amalg \{x\}} \amalg \mathcal{C}_{Y \amalg \{y\}})\big) \longrightarrow \mathcal{M}^{X,Y}_{x,y}, \quad \big(ch_x,\ ch_y \circ \tau : (\shortmid)\ \rightrightarrows \mathcal{C}_{X \amalg \{x,y\}}\big) \longrightarrow \mathcal{N}^{X}_{x,y}.$$

This is visualised in Figure 12.

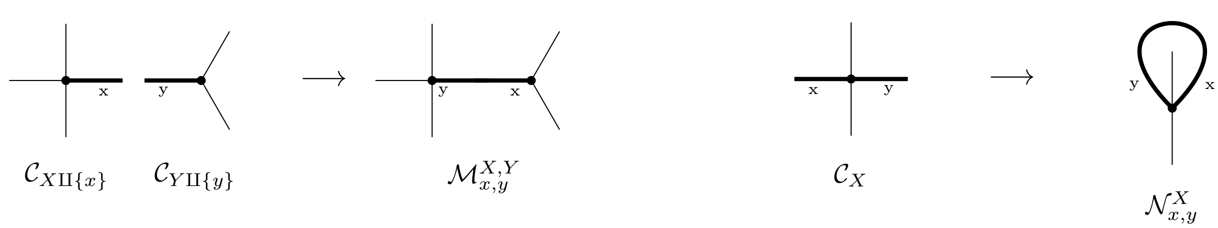


FIGURE 12. Construction of $\mathcal{M}^{X,Y}_{x,y}$ and $\mathcal{N}^{X}_{x,y}$ as coequalisers of gluing data.

Proposition 5.16. *A non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma} : \mathsf{el}(\mathcal{G}) \to \mathsf{Gr}^{\sharp}_{et}$ admits a colimit $\mathbf{\Gamma}(\mathcal{G})$ in $\mathsf{Gr}^{\sharp}_{et}$.*

Proof of Proposition 5.16. For all graphs $\mathcal{G}$, $\mathsf{el}(\mathcal{G})$ is a connected category if and only if $\mathcal{G}$ is a connected graph, so, the colimit $\mathbf{\Gamma}(\mathcal{G})$ of a $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma} : \mathsf{el}(\mathcal{G}) \to \mathsf{Gr}^\sharp_{et}$, if it exists, may be constructed component-wise.

A non-degenerate $(\shortmid)$-shaped graph of graphs is just an isomorphism $(\shortmid) \xrightarrow{\cong} (\shortmid)$.

Assume therefore, that $\mathcal{G}$ is a graph without stick components and let $\mathcal{S}(E_\bullet) \overset{\text{def}}{=} \coprod_{\tilde{e} \in \tilde{E_\bullet}} (\shortmid_{\tilde{e}})$ be the shrub on the inner edges of $\mathcal{G}$.

Since $\mathbf{\Gamma}$ preserves objectwise boundaries and incidence data, we may apply $\mathbf{\Gamma}$ to all components of Equation (5.12) to obtain a diagram

$$\coprod_{\tilde{e} \in \tilde{E_\bullet}} (\shortmid_{\tilde{e}}) \rightrightarrows \coprod_{v \in V} \mathbf{\Gamma}(\mathcal{C}_{P\{v\}}, \iota_v), \tag{5.17}$$

in $\mathsf{Gr}^\sharp_{et}$. As $\mathbf{\Gamma}$ is non-degenerate, 5.17 satisfies that conditions of Lemma 5.14 and hence has a colimit $\overline{\mathcal{G}}$ in $\mathsf{Gr}^\sharp_{et}$.

To show that the colimit of $\mathbf{\Gamma}$ exists and is equal to $\overline{\mathcal{G}}$, observe that the inclusions

$$\begin{array}{ll} \mathbf{\Gamma}(\iota_{v'}) \longrightarrow \coprod_{v \in V} \mathbf{\Gamma}(\mathcal{C}_{P\{v\}}, \iota_v) & \text{for all } v' \in V, \\ \mathbf{\Gamma}(\delta_{\tau e, w}) : \mathbf{\Gamma}(\iota_{\tilde{e}}) \longrightarrow \mathcal{C}_{P\{w\}} \coprod_{v \in V} \mathbf{\Gamma}(\mathcal{C}_{P\{v\}}, \iota_v) & \text{for } e \in E_0, \text{ such that } \tau e \in E\{w\}, \\ \mathbf{\Gamma}(\iota_{\tilde{e}}) \longrightarrow \coprod_{\tilde{e} \in \tilde{E_\bullet}} (\shortmid_{\tilde{e}}), & \text{for all } e \in E_\bullet \end{array}$$

induce a functor $\mathsf{es}(\mathcal{G})$ to 5.17, and hence a cocone of $\mathbf{\Gamma}$ above $\overline{\mathcal{G}}$ in $\mathsf{Gr}^\sharp_{et}$.

Conversely, Equation (5.17) forms a cocone above the colimit in $\mathsf{ps}_\mathsf{f}(\mathsf{D})$ of $\mathbf{\Gamma}$. Hence, by Lemma 5.14, and the universal properties of the colimit, $\mathbf{\Gamma}(\mathcal{G}) = colim_{\mathsf{el}(\mathcal{G})} \mathbf{\Gamma}$ exists in $\mathsf{Gr}^\sharp_{et}$ and is equal to $\overline{\mathcal{G}}$. □

Remark 5.18. In fact, as will follow from Proposition 7.16, all graphs of graphs admit a colimit in Gr. However, the non-degeneracy condition simplifies the proof of the proposition, and is all that is needed for now.

There is a canonical inclusion of the edge set E of a graph $\mathcal{G}$ into the set of edges of the colimit $\mathbf{\Gamma}(\mathcal{G})$ of a non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$. The construction of the multiplication $\mu^{\mathbb{T}}$ for T relies on the fact that this inclusion restricts to an identity between their boundaries.

Corollary 5.19. *If $\mathcal{G}$ is a graph, and $\mathbf{\Gamma}$ is a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\mathbf{\Gamma}(\mathcal{G})$, then $\mathbf{\Gamma}$ induces an identity $E_0(\mathcal{G}) \xrightarrow{=} E_0(\mathbf{\Gamma}(\mathcal{G}))$, and, for each $(\mathcal{C}, b) \in ob(\mathsf{el}(\mathcal{G}))$, the universal map $\mathbf{\Gamma}(b) \to \mathbf{\Gamma}(\mathcal{G})$ is a weak inclusion.*

In particular,

$$E(\mathbf{\Gamma}(\mathcal{G})) \cong E(\mathcal{G}) \amalg \coprod_{v \in V} E_\bullet(\mathbf{\Gamma}(\iota_v)).$$

Proof. The final statement follows directly from the first two.

By the proof of Proposition 5.16, only the inner edges of $\mathcal{G}$, and, the edges $(E - E_\bullet)(\mathbf{\Gamma}(b)) \cong E(\mathcal{C})$, $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, are involved in forming the colimit $\mathbf{\Gamma}(\mathcal{G})$ of $\mathbf{\Gamma}$. Hence the induced map

$$\coprod_{\tilde{e} \in \widetilde{E}} (\shortmid_{\tilde{e}}) \xrightarrow{\cong} \coprod_{\tilde{e} \in \widetilde{E}} \mathbf{\Gamma}(\iota_{\tilde{e}}) \hookrightarrow \mathbf{\Gamma}(\mathcal{G})$$

is a strict inclusion and restricts to an identity $E_0(\mathcal{G}) = E_0(\mathbf{\Gamma}(\mathcal{G}))$ of ports. The second statement is immediate. □

Corollary 5.20. *(Corresponds to* [27, Lemma 1.5.12]*.) If $\mathbf{\Gamma}$ is non-degenerate and $\mathbf{\Gamma}(\mathcal{C}, b) \in ob(\mathsf{Gr})$ is connected for all $(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})$, then $\mathbf{\Gamma}(\mathcal{G}) = colim(\mathbf{\Gamma}) \in ob(\mathsf{Gr}^\sharp_{et})$ is connected if and only if $\mathcal{G}$ is connected.*

Proof. A $(\shortmid)$-shaped graph of graphs is, up to isomorphism, the identity functor $(\shortmid) \mapsto (\shortmid)$ with colimit $(\shortmid)$. So, we may assume that $\mathcal{G}$ has no stick components.

Let $\mathbf{\Gamma} : \mathsf{el}(\mathcal{G}) \to \mathsf{Gr}$ be a non-degenerate $\mathcal{G}$-shaped graph of graphs with colimit $\mathbf{\Gamma}(\mathcal{G})$.

A morphism $\gamma \in \mathsf{GrShape}(\mathbf{\Gamma}(\mathcal{G}), \bigstar \amalg \bigstar)$ is equivalently a cocone in $\mathsf{GrShape}$ under a gluing datum

$$\mathcal{S}(E_\bullet) \rightrightarrows \coprod_{v\in V(\mathcal{G})} \mathbf{\Gamma}(\iota_v) \longrightarrow \bigstar \amalg \bigstar. \tag{5.21}$$

If $\mathbf{\Gamma}(\iota_v)$ is connected for each $v \in V$, then all maps $\mathbf{\Gamma}(\iota_v) \to \bigstar \amalg \bigstar$ are constant and hence morphisms $\coprod_{v\in V} \mathbf{\Gamma}(\iota_v) \to \bigstar \amalg \bigstar$ are in one-to-one correspondence with morphisms $\sum_{v\in V} \mathcal{C}_{P\{v\}} \to \bigstar \amalg \bigstar$.

So (by the proof of Proposition 5.16), there is a bijection between morphisms $\mathbf{\Gamma}(\mathcal{G}) \to \bigstar \amalg \bigstar$ in $\mathsf{GrShape}$ and morphisms $\mathcal{G} \to \bigstar \amalg \bigstar$ in $\mathsf{GrShape}$. It then follows from Proposition 3.26 that $\mathbf{\Gamma}(\mathcal{G})$ is connected if and only if $\mathcal{G}$ is connected. □

As usual, let Z be the terminal graphical species.

Definition 5.22. *For a graph $\mathcal{G}$, and graphical species S, a* (non-degenerate) $\mathcal{G}$-shaped graph of S-structured graphs *is a functor $\mathbf{\Gamma}_S : \mathsf{el}(\mathcal{G}) \to \mathsf{Gr}(S)$ such that the canonical (forgetful) functor*

$$\mathbf{\Gamma} :\ \mathsf{elGr}\mathcal{G} \xrightarrow{\ \mathbf{\Gamma}_S\ } \mathsf{Gr}^\sharp_{et}(S) \twoheadrightarrow \mathsf{Gr}^\sharp_{et}(Z) = \mathsf{Gr}^\sharp_{et}\ ,$$

is a (non-degenerate) graph of graphs

Definition 5.23. *The category of* non-degenerate $\mathcal{G}$-shaped graphs of S-structured graphs $\mathsf{Gr}^{(\mathcal{G})}(S)$ *is the subcategory of the functor category $\mathsf{Fun}(\mathsf{el}(\mathcal{G}), \mathsf{Gr}(S))$ whose objects are non-degenerate $\mathcal{G}$-shaped graphs of S-structured graphs $\mathbf{\Gamma}_S : \mathsf{el}(\mathcal{G}) \to \mathsf{Gr}(S)$ and whose morphisms are natural transformations.*

Lemma 5.24. *For $\mathcal{G} \ncong \mathcal{C}_{\mathbf{0}}$, two $\mathcal{G}$-shaped graphs of S-structured graphs $\mathbf{\Gamma}^1_S, \mathbf{\Gamma}^2_S \in ob(\mathsf{Gr}^{(\mathcal{G})})$ are in the same connected component if and only if $\mathbf{\Gamma}^1_S(\mathcal{C}, b) \cong \mathbf{\Gamma}^2_S(\mathcal{C}, b)$ for all $(\mathcal{C}, b) \in ob(\mathsf{el}(\mathcal{G}))$.*

In particular, if $\mathbf{\Gamma}^1_S$ and $\mathbf{\Gamma}^2_S$ are in the same connected component of $\mathsf{Gr}^{(\mathcal{G})}(S)$, then $\mathbf{\Gamma}^1_S$ and $\mathbf{\Gamma}^2_S$ have isomorphic colimits in $\mathsf{Gr}(S)$.

Proof. Let $\phi : \mathbf{\Gamma}^1_S \Rightarrow \mathbf{\Gamma}^2_S$ be a morphism in $\mathsf{Gr}^{(\mathcal{G})}(S)$. For each $(\mathcal{C}_X, b) \in ob(\mathsf{el}(\mathcal{G}))$, the component $\phi_{(\mathcal{C}_X, b)}$ of ϕ at b, is, by definition a boundary preserving morphism in $\mathsf{Gr}(S)$. Since $\mathcal{G} \ncong \mathcal{C}_{\mathbf{0}}$, the boundary of $\mathbf{\Gamma}^j_S(b)$ is non-empty for $j = 1, 2$. So, by Corollary 4.25, $\phi_{(\mathcal{C}, b)}$ is an isomorphism of S-structured graphs.

The converse is immediate, as is the final statement. □

5.3. **Multiplication for the monad $\mathbb{T}$.** We're now in a position to define a natural transformation $\mu^{\mathbb{T}} : T^2 \Rightarrow T$ induced by taking colimits of graphs of graphs.

From now on, all graphs will be connected, unless explicitly stated otherwise.

Example 5.25. Let X be a finite set and $\mathcal{X} = (\mathcal{G}, \rho)$ an X-graph. If $\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathcal{X}} : \mathsf{el}(\mathcal{X}) \to \mathsf{Gr}$ is a non-degenerate $\mathcal{X}$-shaped graph of graphs, then the colimit $\mathbf{\Gamma}(\mathcal{X}) = colim_{\mathsf{el}(\mathcal{X})}\mathbf{\Gamma}$ exists by Proposition 5.16 and, by Corollary 5.19, it inherits the X-labelling ρ of $\mathcal{X}$.

Given a graphical species S and finite set X, elements of T^2S_X are represented by $\mathcal{X}$-shaped graphs of S-structured graphs for X-graphs $\mathcal{X}$.

If $(\mathcal{X}^j, \mathbf{\Gamma}^j_S) : \mathsf{el}(\mathcal{X}^j) \to \mathsf{Gr}(S)$ both represent $[\mathcal{X}, \beta] \in T^2S_X$. Then, $\mathcal{X}^1 \cong \mathcal{X}^2$ in $X\mathsf{Gr}_{iso}$ and,

$$colim_{\mathsf{el}(\mathcal{X}^1)}\mathbf{\Gamma}^1_S \cong colim_{\mathsf{el}(\mathcal{X}^2)}\mathbf{\Gamma}^2_S \in X\mathsf{Gr}_{iso}(S) \tag{5.26}$$

by Lemma 5.24. Denote this colimit by $\mathbf{\Gamma}_S(\mathcal{X}) = (\mathbf{\Gamma}(\mathcal{X}), \alpha^{\mathbf{\Gamma}_S})$. Then the assignment

$$\mu^{\mathbb{T}}S : T^2S \to TS, \quad [\mathcal{X}, \beta] \mapsto [\mathbf{\Gamma}(\mathcal{X}), \alpha^{\mathbf{\Gamma}_S}]$$

is well-defined.

To see that $\mu^{\mathbb{T}}S$ defines a morphism $T^2S \to TS$ in GS, let $[\mathcal{X}, \beta] \in T^2S_X$ be represented by an $\mathcal{X}$ indexed graph of S-structured graphs $\mathbf{\Gamma}_{\mathcal{X}} : \mathsf{el}(\mathcal{X}) \to \mathsf{Gr}(S)$ with colimit $\mathbf{\Gamma}_S(\mathcal{X}) = (\mathbf{\Gamma}(\mathcal{X}), \alpha) \in X\mathsf{Gr}_{iso}(S)$.

By Corollary 5.19, there is a canonical inclusion $E(\mathcal{X}) \hookrightarrow E(\mathbf{\Gamma}(\mathcal{X}))$ and, for each $e \in E(\mathcal{X})$,

$$S(ch_e^{\mathbf{\Gamma}(\mathcal{X})})(\alpha) = S(ch_e^{\mathcal{X}})(\beta) \in S(\shortmid).$$

Hence, for all $x \in X$, there is a commutative diagram of sets

$$\begin{tikzcd} T^2S_X \arrow[rr, "\mu^{\mathbb{T}}S_X"] \arrow[dr, "T^2S(ch_x)"'] & & TS_X \arrow[dl, "TS(ch_x)"] \\ & T^2S(\shortmid) = TS(\shortmid). & \end{tikzcd}$$

Whereby $\mu^{\mathbb{T}}S$ defines a morphism in $\mathsf{GS}(T^2S, TS)$.

Naturality of $\mu^{\mathbb{T}}S$ in S is immediate from the definition and, [27, page 172] admits a straightforward modification to show that $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ satisfies the axioms Equation (2.1) and Equation (2.2) for a monad.

Remark 5.27. For all graphical species S, $\mu^{\mathbb{T}}S$ and $\eta^{\mathbb{T}}S$ are palette-preserving morphisms in GS. So $\mathbb{T}$ restricts to a monad $\mathbb{T}^{(\mathfrak{C},\omega)}$ on $\mathsf{GS}^{(C,\omega)}$, for all $(\mathfrak{C}, \omega)$. If $A \in ob(\mathsf{GS}^{(C,\omega)})$ and $h \in \mathsf{GS}(TA, A)$, then (A, h) is a $\mathbb{T}$-algebra if and only if it is a $\mathbb{T}^{(\mathfrak{C},\omega)}$-algebra.

Example 5.28. Since Z is the terminal graphical species, $\mathsf{Gr}(Z) \cong \mathsf{Gr}$ and hence elements of TZ are boundary preserving isomorphism classes of graphs in Gr. The unique morphism $r \in \mathsf{GS}(TZ, Z)$ makes Z into an algebra for $\mathbb{T}$ by assigning to $\mathcal{G} \in ob(X\mathsf{Gr}_{iso})$ the corolla $\mathcal{C}_X$.

Likewise, for any palette $(\mathfrak{C}, \omega)$, the unique palette-preserving morphism $r^{(\mathfrak{C},\omega)} \in \mathsf{GS}^{(C,\omega)}(TZ^{(C,\omega)}, Z^{(C,\omega)})$ makes the terminal $(\mathfrak{C}, \omega)$-coloured graphical species $Z^{(C,\omega)}$ into an algebra for $\mathbb{T}$.

5.4. **$\mathbb{T}$-algebras are non-unital modular operads.** Finally in this section, we prove that $\mathbb{T}$-algebras (see Section 2) are non-unital modular operads.

Lemma 5.29. *A $\mathbb{T}$-algebra (A, h) admits a multiplication ${}_h\diamond$ and contraction ${}_h\zeta$. The induced operations are preserved by morphisms in $\mathsf{GS}^{\mathbb{T}}$.*

Proof. Let X and Y be finite sets and let $\mathcal{M}_{x,y}^{X,Y}$ be the $X \amalg Y$-graph described in Examples 3.7 and 5.15 obtained by gluing the corollas $\mathcal{C}_{X \amalg \{x\}}$ and $\mathcal{C}_{Y \amalg \{y\}}$ along ports x and y.

Given a $(\mathfrak{C}, \omega)$-coloured graphical species S, and $\underline{c} \in \mathfrak{C}^X$, $\underline{d} \in \mathfrak{C}^Y$ and $c \in \mathfrak{C}$, ordered pairs $(\phi, \psi) \in S_{(\underline{c},c)} \times S_{(\underline{d},\omega c)}$ determine elements $\mathcal{M}_c(\phi, \psi)$ of $S(\mathcal{M}_{x,y}^{X,Y})$.

The canonical map $S(\mathcal{M}_{x,y}^{X,Y}) \twoheadrightarrow TS_{X \amalg Y}$ is injective unless $X = Y = \emptyset$, in which case $[\mathcal{M}_c(\phi_1, \psi_1)] = [\mathcal{M}_c(\phi_2, \psi_2)]$ in $TS_{\mathbf{0}}$ if and only if $(\phi_2, \psi_2) = (\psi_1, \phi_1)$.

If (A, h) is a $(\mathfrak{C}, \omega)$-coloured $\mathbb{T}$-algebra, the family of maps defined by the composition

$${}_h\diamond : \ S_{(\underline{c},c)} \times S_{(\underline{d},\omega c)} \xrightarrow{[\mathcal{M}(\cdot,\cdot)]} TS_{\underline{cd}} \xrightarrow{\quad h \quad} S_{\underline{cd}},$$

defines a multiplication on A (see Figure 13).

Similarly, for a finite set X, let $\mathcal{N}_{x,y}^X$ be the X-graph described in Examples 3.8, 4.8, 5.15, obtained by gluing the ports of $\mathcal{C}_{X \amalg \{x,y\}}$ labelled by x and y.

Each $\phi \in S_{(\underline{c},c,\omega c)} \subset S_{X \amalg \{x,y\}}$, with $\underline{c} \in \mathfrak{C}^X$ and $c \in \mathfrak{C}$ determines an element $\mathcal{N}_c^S(\phi) \in S(\mathcal{N}_{x,y}^X)$. The only non-trivial boundary preserving automorphism of $\mathcal{N}_{x,y}^X$ is the permutation $\sigma_{x,y} \in Aut(X \amalg \{x,y\})$

that swaps x and y and leaves the other elements unchanged, so $[\mathcal{N}_c^S(\phi)] = [\mathcal{N}_c^S(\psi)]$ in TS_X if and only if $S(\sigma_{x,y})(\phi) = \psi$.

If (A, h) is a $(\mathfrak{C}, \omega)$-coloured algebra for T, the family of maps defined by the composition

$$_h\zeta : \ A_{(\underline{c}, c, \omega c)} \xrightarrow{[\mathcal{N}^A(\cdot)]} TA_{\underline{c}} \xrightarrow{\quad h \quad} A_{\underline{c}}$$

defines an equivariant contraction on A (see Figure 13). □

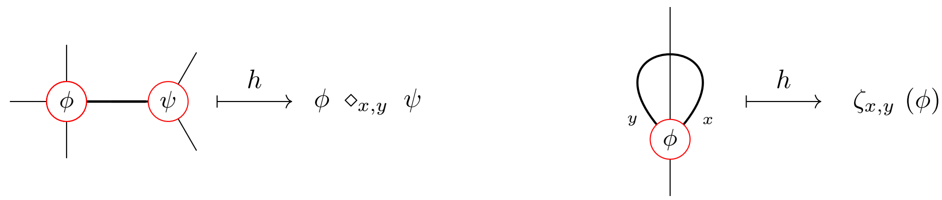


FIGURE 13. If (A, h) is a $\mathbb{T}$-algebra, h induces a multiplication and contraction on A.

We're now able to prove that algebras for the monad $\mathbb{T}$ on GS are precisely non-unital modular operads.

Proposition 5.30. *There is a canonical isomorphism of categories* $\mathsf{GS}^{\mathbb{T}} \cong \mathsf{MO}^-$.

Proof. A $\mathbb{T}$-algebra structure $h : TA \to A$ on a graphical species A equips A with a multiplication $\diamond =_h \diamond$, and contraction $\zeta =_h \zeta$ as in Lemma 5.29. We must show that $(A, \diamond, \zeta)$ satisfies conditions (C1)-(C4) of Definition 1.22.

The key observation for this result is that, up to choice of boundary E_0, any connected graph with two inner edge orbits has one of the forms illustrated in Figures 14 - 17.

To show condition (C1), let $\phi_1 \in A_{(\underline{b}, c)}, \phi_2 \in A_{(\underline{c}, \omega c, d)}, \phi_3 \in A_{(\underline{d}, \omega d)}$. By Lemma 5.29, and the monad algebra axioms 2.3, 2.4,

$$\begin{array}{rcl} (\phi_1 \diamond_c \phi_2) \diamond_d \phi_3 & = & h\left[\mathcal{M}_d^A\left((\phi_1 \diamond_c \phi_2), \phi_3\right)\right] \\ & = & h\left[\mathcal{M}_d^A\left(h[\mathcal{M}_c(\phi_1, \phi_2)], h\eta^{\mathbb{T}}\phi_3\right)\right] \\ & = & h\mu^{\mathbb{T}}\left[\mathcal{M}_d^{TA}\left([\mathcal{M}_c(\phi_1, \phi_2)], \eta^{\mathbb{T}}\phi_3\right)\right], \end{array}$$

and, likewise

$$\phi_1 \diamond_c (\phi_2 \diamond_d \phi_3) = h\mu^{\mathbb{T}}\left[\mathcal{M}_c^{TA}\left(\eta^{\mathbb{T}}A\phi_1, [\mathcal{M}_d^A(\phi_2, \phi_3)]\right)\right].$$

So, it suffices to show that, for all ϕ_1, ϕ_2, ϕ_3 as above,

$$\mu^{\mathbb{T}}\left[\mathcal{M}_d^{TA}\left([\mathcal{M}_c(\phi_1, \phi_2)], \eta^{\mathbb{T}}\phi_3\right)\right] = \mu^{\mathbb{T}}\left[\mathcal{M}_c^{TA}\left(\eta^{\mathbb{T}}A\phi_1, [\mathcal{M}_d^A(\phi_2, \phi_3)]\right)\right],$$

and this follows immediately from the observation that by Example 5.15, since colimits commute,

$$coeq_{\mathsf{Gr}}\left(ch_y, ch_z \circ \tau : (\shortmid) \rightrightarrows \mathcal{M}_{w,x}^{X_1,(X_2 \amalg \{y\})} \amalg \mathcal{C}_{X_3 \amalg \{z\}}\right) = coeq_{\mathsf{Gr}}\left(ch_w, ch_x \circ \tau : (\shortmid) \rightrightarrows \mathcal{C}_{X_1 \amalg \{w\}} \amalg \mathcal{M}_{y,z}^{(X_2 \amalg \{x\}), X_3}\right).$$

This is illustrated in Figure 14. The coherence conditions (C2)-(C4) all follow in the same way from the defining axioms 2.3 and 2.4 of monad algebras. Figures 15 - 17 illustrate each condition.

By the method of Lemma 5.29, the assignment $(A, h) \mapsto (A,_h \diamond,_h \zeta)$ clearly extends to a functor $\mathsf{GS}^{\mathbb{T}} \to \mathsf{MO}^-$.

The proof of the converse closely resembles [16, Theorem 3.7].

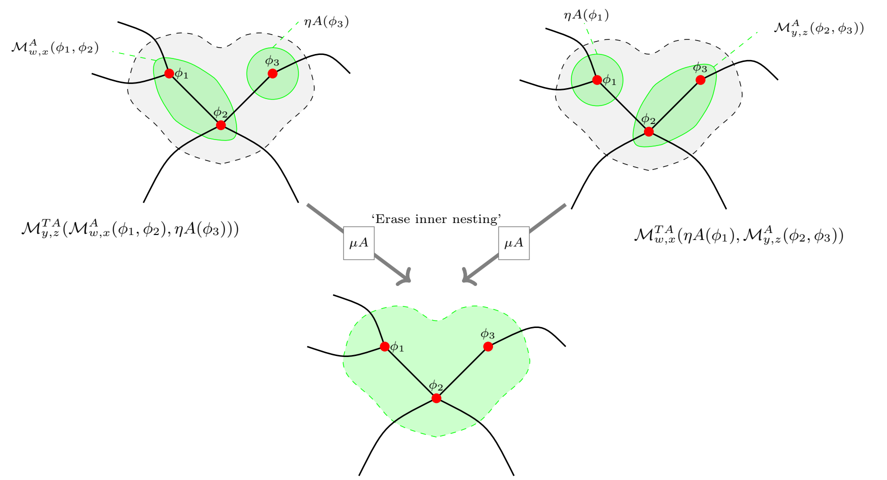


FIGURE 14. Coherence condition (C1) Applying $\mu^{\mathbb{T}}A : T^2A \to A$ amounts to *erasing inner nesting.*

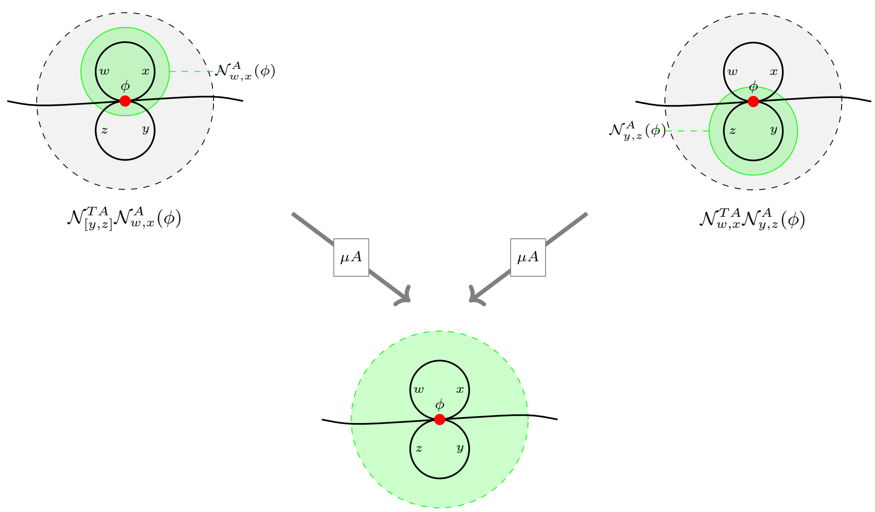


FIGURE 15. Coherence condition (C2)

Namely, let $(S, \diamond, \zeta)$ satisfy the conditions in the statement of the proposition. We construct a structure morphism $h \in \mathsf{GS}(TS, S)$ by successively using $\diamond$ and ζ to remove (or *collapse* inner edge orbits of S-structured graphs $(\mathcal{G}, \alpha)$ to obtain a finite sequence of S-structured graphs terminating in an S-structured corolla.

As usual, let X be a finite set and $(\mathcal{X}, \alpha)$ a representative of $[\mathcal{X}, \alpha] \in TS_X$.

If $\mathcal{G}$ has no inner edges, then $\mathcal{X} = \mathcal{C}_X$, and so $[\mathcal{X}, \alpha] = \eta^{\mathbb{T}}S(\phi)$ for some $\phi \in S_X$. In this case, define

$$h[\mathcal{X}, \alpha] \overset{\text{def}}{=} \phi \in S_X. \tag{5.31}$$

Otherwise, let $\tilde{e} \in \tilde{E}_\bullet$ be the orbit of $e, \tau e \in E_\bullet$. $\mathcal{G}_{/\tilde{e}}$ be the graph obtained from $\mathcal{G}$ by removing the τ-orbit $\{e, \tau e\}$ and identifying the endpoints $t(e) \sim t(\tau e)$:

CASE 1: $t(e) \neq t(\tau e)$
Let $t(e) = v_1$ and $t(\tau e) = v_2$ be disjoint vertices in V. Then

$$\mathcal{G}_{/\tilde{e}} \overset{\text{def}}{=} \quad \tau \circlearrowright (E - \{e, \tau e\}) \xleftarrow{\ s\ } (H - s^{-1}\{e, \tau e\}) \xrightarrow{\ \bar{t}\ } V/(v_1 \sim v_2)\ ,$$

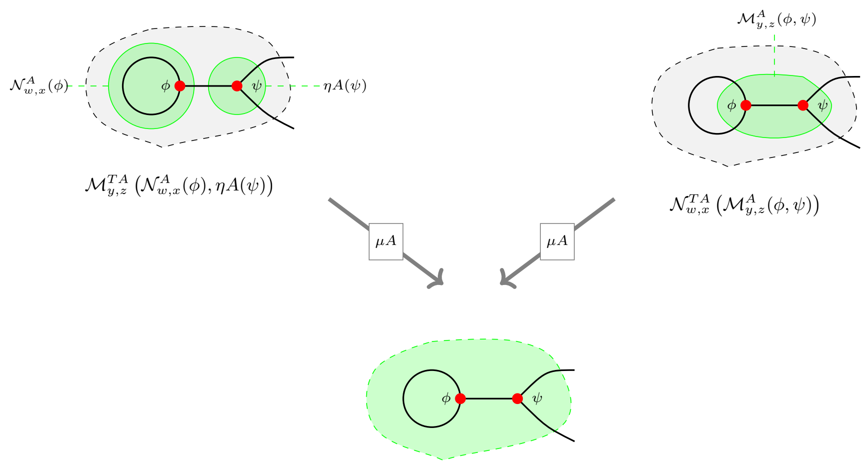


FIGURE 16. Coherence condition (C3).

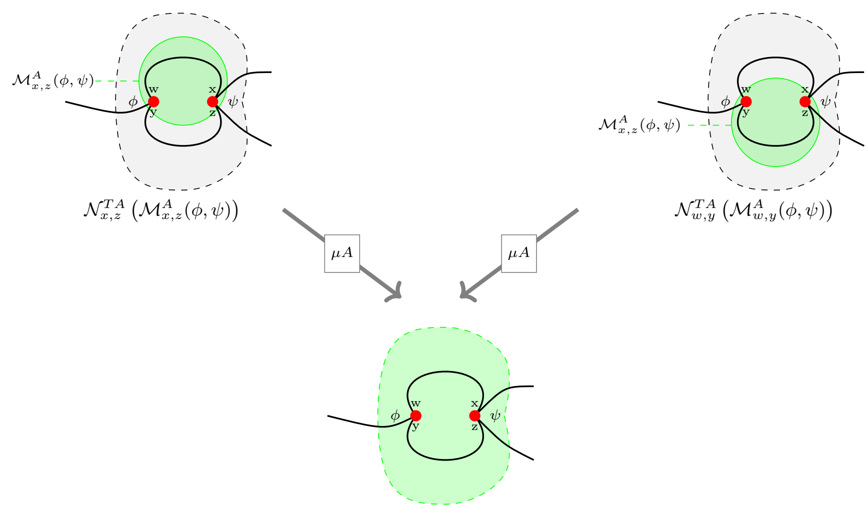


FIGURE 17. Coherence condition (C4).

where $\bar{t}$ is the composition of $t = t : H \to V$ with the quotient $V \twoheadrightarrow V/(v_1 \sim v_2)$.

The class of v_1, v_2 in $V/(v_1 \sim v_2)$ is given by $\overline{v}$. If $(\mathcal{C}_{X_1 \amalg \{x_1\}}, b_1) \in \mathsf{el}(\mathcal{G})$ is a neighbourhood of v_1 such that $b_1(x_1) = \tau e$ and $(\mathcal{C}_{X_2 \amalg \{x_2\}}, b_2) \in \mathsf{el}(\mathcal{G})$ is a neighbourhood of v_2 such that $b_2(x_2) = e$, then the minimal neighbourhood of v_1, v_2 and $\tilde{e}$ is of the form $b_{x_1,x_2} : \mathcal{M}^{X_1,X_2}_{x_1,x_2} \to \mathcal{G}$,.

Observe also that collapsing $\tilde{e}$ induces a minimal neighbourhood

$$\bar{b} : \left(\mathcal{M}^{X_1,X_2}_{x_1,x_2}\right)_{/\tilde{e}} \cong \mathcal{C}_{X_1 \amalg X_2} \to \mathcal{G}_{/\tilde{e}}$$

is a of $\overline{v}$ in $\mathcal{G}_{/\tilde{e}}$.

Let $\phi_i \overset{\text{def}}{=} S(b_1)(\alpha) \in S_{X_i \amalg \{x_i\}}$ for $i = 1, 2$, and define the induced S-structure $\alpha_{\tilde{e}}$ on $\mathcal{G}_{/\tilde{e}}$ by

$$S(b)\alpha_{\tilde{e}} = \begin{cases} S(b)(\alpha) & \text{whenever } (\mathcal{C}_Y, b) \text{ is a neighbourhood of } w \in V - \{v_1, v_2\}, \\ \phi_1 \diamond_{x_1,x_2} \phi_2 & \text{if } b = \bar{b}. \end{cases}$$

Case 2: $t(e) = t(\tau e)$
Otherwise, if $t(e) = t(\tau e) = v$, the graph $\mathcal{G}_{/\tilde{e}}$ obtained from $\mathcal{G}$ by collapsing $\{e, \tau e\}$ has the form

$$\mathcal{G}_{/\tilde{e}} \stackrel{\text{def}}{=} \quad \tau \circlearrowright (E - \{e, \tau e\}) \xleftarrow{\quad s \quad} (H - s^{-1}\{e, \tau e\}) \xrightarrow{\qquad \bar{t} \qquad} V\ .$$

In particular $V(\mathcal{G}_{/\tilde{e}}) = V(\mathcal{G})$.

If $(\mathcal{C}_{X \amalg \{x,y\}}, b) \in \mathsf{el}(\mathcal{G})$ is a neighbourhood of v such that $b(x) = \tau e$ and $b(y) = e$, then the minimal neighbourhood of v and $\tilde{e}$ in $\mathcal{G}$ is of the form $b_{x,y} : \mathcal{N}^X_{x,y} \to \mathcal{G}$.

Collapsing $\tilde{e}$ induces a minimal neighbourhood

$$\bar{b} : \left(\mathcal{N}^X_{x,y}\right)_{/\tilde{e}} \cong \mathcal{C}_X \to \mathcal{G}_{/\tilde{e}}$$

is a neighbourhood of v in $\mathcal{G}_{/\tilde{e}}$.

Let $\phi \stackrel{\text{def}}{=} S(b)(\alpha) \in S_{X \amalg \{x,y\}}$, and define the induced S-structure $\alpha_{\tilde{e}}$ on $\mathcal{G}_{/\tilde{e}}$ by

$$S(b)\alpha_{\tilde{e}} = \begin{cases} S(b)(\alpha) & \text{whenever } (\mathcal{C}_Y, b) \text{ is a neighbourhood of } w \in V - \{v\}, \\ \zeta_{x,y}(\phi) & \text{if } b = \bar{b}. \end{cases}$$

Hence, an ordering $(\tilde{e}_1, \dots, \tilde{e}_N)$ of the set $\widetilde{E_\bullet}$ of inner τ orbits of $\mathcal{G}$, defines a terminating sequence of S-structured X-graphs

$$(\mathcal{X}, \alpha) \mapsto (\mathcal{X}_{\tilde{e}_1}, \alpha_{\tilde{e}_1}) \mapsto ((\mathcal{X}_{\tilde{e}_1})_{\tilde{e}_2}, (\alpha_{\tilde{e}_1})_{\tilde{e}_2}) \mapsto \dots \mapsto (((\mathcal{X}_{\tilde{e}_1}) \dots)_{\tilde{e}_N}, ((\alpha)_{\tilde{e}_1} \dots)_{\tilde{e}_N}).$$

Since $((\mathcal{X}_{\tilde{e}_1}) \dots)_{\tilde{e}_N}$ has no inner edges, and is therefore an X-corolla, there exists an element $\phi_{(\mathcal{X},\alpha)} \in S_X$ such that

$$((\alpha)_{\tilde{e}_1} \dots)_{\tilde{e}_N} = \eta^{\mathbb{T}} S(\phi_{(\mathcal{X},\alpha)}) \in TS_X.$$

The coherence conditions (C1)-(C4) are equivalent to the statement that $\phi_{(\mathcal{X},\alpha)} \in S_X$ so obtained is unchanged if the order of collapse of consecutive pairs $\tilde{e}_j, \tilde{e}_{j+1} \in \widetilde{E_\bullet}$ of inner τ-orbits is switched. In other words, $\phi_{(\mathcal{X},\alpha)}$ is independent of the choice of ordering of $E_\bullet$.

It is independent of the choice of representative of $(\mathcal{X}, \alpha)$ by definition of TS.

To complete the proof of the proposition, it remains to establish that h satisfies the monad algebra axioms 2.3, 2.4 for $\mathbb{T}$. Compatibility of h with $\eta^{\mathbb{T}}$ (Equation (2.4)) is immediate from Equation (5.31). Compatibility of h with $\mu^{\mathbb{T}}$ (Equation (2.3)), follows since the coherence conditions (C1) – (C4) ensure that $h[\mathcal{X}, \alpha]$ is independent of the order of collapse of the inner edges of $\mathcal{X}$.

So $(S, \diamond, \zeta)$ defines a $\mathbb{T}$-algebra (S, h), and this assignment extends in the obvious way to a functor $\mathsf{MO}^- \to \mathsf{GS}^{\mathbb{T}}$ that, by construction, is inverse to the functor $\mathsf{GS}^{\mathbb{T}} \to \mathsf{MO}^-$ defined above. $\square$

6. The problem of loops

Before constructing the monad $\mathbb{D}$ that encodes the combinatorics of units, it is worthwhile to pause, and provide some details on the obstruction to obtaining a monad in the construction outlined in [23].

Example 6.1. In Example 5.1, I sketched the idea of the operad monad $\mathbb{M}^{Op}$, whose underlying endofunctor takes a species P of rooted corollas to the species of formal compositions in P, encoded as P-decorated rooted trees. However, I didn't say anything there about the units for the operadic composition.

These are provided by allowing *degenerate substitution* of the exceptional directed tree $(\downarrow)$ with no vertices, into the vertex of the corolla $t_{\mathbf{1}}$ with one leaf, as in Figure 18, in the definition of $\mathbb{M}^{Op}$.

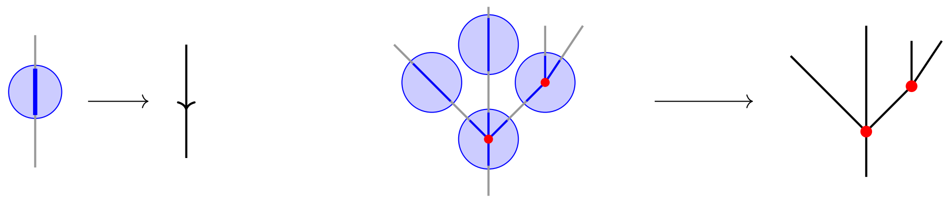


FIGURE 18. Degenerate substitution of the exceptional tree.

The modular operad endofunctor in [23] is similarly obtained by slightly modifying the non-unital modular operad endofunctor $T : \mathsf{GS} \to \mathsf{GS}$, to allow degenerate substitions.

relies on a small adjustment to the Definition 5.2 of X-graphs to allow degenerate substitutions.

Precisely, if S is a graphical species and X a finite set, then the groupoid $X\mathsf{Gr}^{(\shortmid)}_{iso}(S)$ is obtained from $X\mathsf{Gr}_{iso}$ by dropping the condition that X-graphs must have non-empty vertex set. So,

$$X\mathsf{Gr}^{(\shortmid)}_{iso} = X\mathsf{Gr}_{iso} \text{ for } X \ncong \mathbf{2}, \quad \text{and} \quad \mathbf{2}\mathsf{Gr}^{(\shortmid)}_{iso} \cong \mathbf{2}\mathsf{Gr}_{iso} \amalg \{(\shortmid, c)\}_{c\in\mathfrak{C}},$$

(where the boundary of $(\shortmid)$ has the identity $\mathbf{2}$-labelling).

Let $T^{(\shortmid)} : \mathsf{GS} \to \mathsf{GS}$ be the endofunctor (defined in [23, Section 5]) defined on objects by

$$\begin{aligned} T^{(\shortmid)}S_{\S} &\overset{\text{def}}{=} S_{\S}, \\ T^{(\shortmid)}S_X &\overset{\text{def}}{=} colim_{\mathcal{X}\in X\mathsf{Gr}^{(\shortmid)}_{iso}} S(\mathcal{X}), \qquad \text{for all finite sets } X. \end{aligned}$$

Then $T \subset T^{(\shortmid)}$ and the inclusion makes $\eta^{\mathbb{T}}$ into a unit $\eta^{(\shortmid)}$ for $T^{(\shortmid)}$.

Proposition 6.2 ([23])**.** *Algebras for the pointed endofunctor $(T^{(\shortmid)}, \eta^{(\shortmid)})$ on GS are modular operads.*

Proof. Since $T \subset T^{(\shortmid)}$, algbras for $T^{(\shortmid)}$ have the structure of unpointed modular operads by Proposition 5.30. If (A, h) is an algebra for $T^{(\shortmid)}$, then, for each $c \in A_{\S}$, $h(\shortmid, c) \in A_{\mathbf{2}}$ provides a c-coloured unit for the induced multiplication. □

For all graphical species S, an element of ${T^{(\shortmid)}}^2 S_X$ is represented by an X-graph X and a (possibly degenerate) $\mathcal{X}$-shaped graph of S-structured connected graphs $\mathbf{\Gamma}_S : \mathsf{el}(\mathcal{X}) \to \mathsf{Gr}(S)$, and, a multiplication $\mu^{(\shortmid)} : {T^{(\shortmid)}}^2 \Rightarrow T^{(\shortmid)}$ for $T^{(\shortmid)}$ should extend the $\mathbb{T}$-multiplication $\mu^{\mathbb{T}}$.

Example 6.3. As usual, let $\mathcal{W}$ be the wheel graph with one vertex v and edges $\{a, \tau a\}$. Its category of elements $\mathsf{el}(\mathcal{W})$ has skeletal subcategory

$$\begin{tikzcd} (\shortmid) \arrow[rr, shift left, "ch_1^{\mathcal{C}_\mathbf{2}}"] \arrow[rr, shift right, "ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau"'] \arrow[dr, "ch_a"'] & & \mathcal{C}_\mathbf{2} \arrow[dl, "1_{\mathcal{C}_\mathbf{2}} \mapsto a"] \\ & \mathcal{W} & \end{tikzcd} \tag{6.4}$$

If S is a $(\mathfrak{C}, \omega)$-coloured graphical species and $c \in \mathfrak{C}$, then there is a $\mathcal{W}$-shaped graph of S-structured graphs $\Lambda_{S,c}$ given by

$$\begin{tikzcd} \mathsf{el}(\mathcal{W}) \arrow[d, "\Lambda_{S,c}"'] & (\shortmid) \arrow[rr, shift left, "ch_1^{\mathcal{C}_\mathbf{2}}"] \arrow[rr, shift right, "ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau"'] \arrow[d, mapsto, "\Lambda_{S,c}(ch_a)"'] & & \mathcal{C}_\mathbf{2} \arrow[d, mapsto, "\Lambda_{S,c}(1_{\mathcal{C}_\mathbf{2}} \mapsto a)"] \\ \mathsf{Gr}(S) & (\shortmid, c) \arrow[rr, shift left, "\Lambda_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}})"] \arrow[rr, shift right, "\Lambda_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau)"'] & & (\shortmid, c). \end{tikzcd} \tag{6.5}$$

So,

$$\Lambda_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}})\,(\Lambda_{S,c}(1_{\mathcal{C}_\mathbf{2}} \mapsto a)) = \Lambda_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}})(\shortmid, c) = c,$$
$$\Lambda_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau)\,(\Lambda_{S,c}(1_{\mathcal{C}_\mathbf{2}} \mapsto a)) = \Lambda_{S,c}(ch_1^{\mathcal{C}_\mathbf{2}} \circ \tau)(\shortmid, c) = (\shortmid, c).$$

Hence the coequalier $\Lambda_{S,c}(\mathcal{W})$ exists in $\mathsf{Gr}(S)$ and is given by $id_c : (\shortmid, c) \to (\shortmid, c)$.

In the first place, this a little surprising since $E_0(\shortmid) \neq E_0(\mathcal{W})$ so Corollary 5.19 does not hold for $\Lambda_{S,c}$.

Observe, moreover, that $\mathcal{W}$-shaped graph of S-structured graphs $\Lambda_{S,\omega c}$

$$ch_a \mapsto (\shortmid, \omega c) \text{ and } (1_{\mathcal{C}_\mathbf{2}} \mapsto a) \longmapsto (\shortmid, \omega c),$$

has colimit $(\shortmid, \omega c)$. But $\mathcal{W}$ admits a unique non-trivial – but trivially boundary fixing – automorphism $\tau_{(\mathcal{W})} : \mathcal{W} \to \mathcal{W}$, such that $\Lambda_{S,c}$ and $\Lambda_{S,\omega c}$ are related by

$$\Lambda_{S,\omega c}(\mathcal{C}, b) = \Lambda_{S,c}(\mathcal{C}, \tau_{(\mathcal{W})} \circ b) \text{ for all } (\mathcal{C}, b) \in \mathsf{el}(W). \tag{6.6}$$

In particular, $\Lambda_{S,c}$ and $\Lambda_{S,\omega c}$ represent the same element of ${T^{(\shortmid)}}^2 S_\mathbf{0}$. So isomorphisms in $X\mathsf{Gr}_{iso}$ don't respect colimits of graphs of graphs!

Hence, if we wish to extend $\mu^{\mathbb{T}} S$ to a well-defined natural transformation $\mu^{(\shortmid)} S : {T^{(\shortmid)}}^2 S \Rightarrow T^{(\shortmid)} S$, we must include an extra element of $T^{(\shortmid)} S_\mathbf{0}$ that identifies $(\shortmid, c)$ and $(\shortmid, \omega c)$.

Remark 6.7. The description of Feynman graphs $\mathcal{G}$ as formal colimits of their element categories $\mathsf{el}(\mathcal{G})$, and therefore also the definition of graph substitution in terms of colimits of graphs of graphs, relies heavily on the involution τ on $(\shortmid)$.

To my knowledge, the construction that I present in the next section is unique among graphical descriptions of unital modular operads (or wheeled prop(erad)s), in that all others include some version of the exceptional loop as a graph. (See e.g. [31, 33, 19, 40, 3].)

In the formalism of [16, 8] described in Example 3.1 (as well as in, for example [19, 31, 40]), where graph ports are defined to be the fixed points of edge involution, the graph substitution is not defined in terms of a functorial construction, but simply by 'removing neighbourhoods of vertices and gluing in graphs'. Therefore, the exceptional loop arises more intuitively from substitution as in Figure 19.

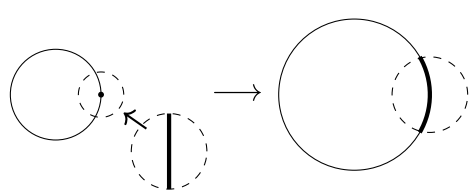


FIGURE 19. Constructing an exceptional loop by removing a vertex and substituting the stick graph.

Remark 6.8. Hackney, Robertson and Yau [22, Definition 1.1] solve the issue, within the framework of Feynman graphs, by including extra *boundary* data in their definition of graphs. For them, a graph is a pair $(\mathcal{G}, \eth(\mathcal{G}))$ of a Feynman graph $\mathcal{G}$, and subset $\eth(\mathcal{G}) \subset E_0$ of ports, that satisfies certain conditions. So, in their formalism, $(\shortmid, \mathbf{2})$ and $(\shortmid, \mathbf{0})$ are different graphs.

As the goal of this work is, *precisely*, to unpick and understand this issue, let's dig a little deeper.

In Example 6.3, we saw that taking colimits in $X\mathsf{Gr}_{iso}^{(\shortmid)}$ of degenerate graphs of S-structured graphs doesn't always lead to a well defined class of S-structured graphs, let alone one in the correct arity.

The issue is that the parallel morphisms $id_{(\shortmid)}, \tau : (\shortmid) \rightrightarrows (\shortmid)$ don't have a coequaliser in $\mathsf{Gr}^{\sharp}_{et}$. In $\mathsf{ps_f(D)}$, their coequaliser is the exceptional loop $\bigcirc \notin ob(\mathsf{Gr})$ described in Example 5.10. So, an obvious first attempt at a resolution of this problem would be to enlarge Gr to include the exceptional loop $\bigcirc$.

To this end, let $\mathsf{Gr}^{\bigcirc}$ be the category of *fully generalised Feynman graphs* and étale morphisms, obtained from Gr by adding the object $\bigcirc$ and a unique morphism $(\shortmid) \to \bigcirc$. In other words, objects of $\mathsf{Gr}^{\bigcirc}$ are $\mathsf{Set_f}$-diagrams $\tau \circlearrowright E \xleftarrow{s} H \xrightarrow{t} V$, such that $s : H \to E$ is injective, and the involution $\tau : E \to E$ is allowed to have fixed points in $E_0 = E - im(s)$ but not in $im(s)$.

By definition, $\bigcirc$ is the coequaliser of $id, \tau : (\shortmid) \rightrightarrows (\shortmid)$, so we can define the category of elements of $\bigcirc$ in the obvious way $\mathsf{el}(\bigcirc) \overset{\text{def}}{=} \mathbb{P}^{\circlearrowleft} | \bigcirc$, and thereby extend any graphical species $S : \mathbb{P}^{\circlearrowleft op} \to P$ to a presheaf on $\mathsf{Gr}^{\bigcirc}$ according to $\mathcal{G} \longmapsto lim_{\mathsf{el}(\mathcal{G})} S$.

Observe, however, that $\mathsf{el}(\bigcirc) \cong \mathsf{el}(\mathcal{G})(\shortmid)$, and hence $S(\bigcirc) \cong S(\shortmid)$ canonically for all graphical species S. So the the inclusion $\mathbb{P}^{\circlearrowleft} \hookrightarrow \mathsf{Gr}^{\bigcirc}$ is not dense and $\mathsf{Gr}^{\bigcirc}$ does not embed in GS. (See also [19, 20] for a discussion of some of these issues.)

If we try to construct a multiplication for the obvious endofunctor $T^{\bigcirc} : \mathsf{GS} \to \mathsf{GS}$

$$\begin{aligned} T^{\bigcirc} S_{\S} &\overset{\text{def}}{=} S_{\S}, \\ T^{\bigcirc} S_X &\overset{\text{def}}{=} colim_{(\mathcal{G},\rho) \in X\mathsf{Gr}^{\bigcirc}_{iso}} S(\mathcal{G}), \end{aligned}$$

then the graph of graph functors $\Lambda_{S,c}$ and $\Lambda_{S,\omega c}$ described in Example 6.3 represent the same element $[\mathcal{W}, \beta] \in T^{\bigcirc^2} S_{\mathbf{0}}$. But, since $S(\bigcirc) \cong S(\shortmid) = C$, $[\bigcirc, c], [\bigcirc, \omega c] \in T^{\bigcirc} S_{\mathbf{0}}$ are distinct whenever $c \neq \omega c$. It follows that $\mu^{\mathbb{T}}$ does not extend to a multiplication $\mu^{\bigcirc} : T^{\bigcirc^2} \Rightarrow T^{\bigcirc}$.

Indeed, this is not surprising, since contraction of an element $\phi \in S_{\mathbf{2}}$ quotients out the Σ_2 action on $S_{\mathbf{2}}$ and so results in a loss of information. The map $(\shortmid) \to \bigcirc$ in $\mathsf{Gr}^{\bigcirc}$ would seem to be *in the wrong direction*!

In the next section, we'll examine the combinatorics of contracted units more closely. The problem of loops discussed in this section, will be resolved by adjoining a map that acts as a formal *equaliser*, rather than a coequaliser, of $id, \tau : (\shortmid) \rightrightarrows (\shortmid)$.

7. Modular operads with unit

Proposition 5.30 identifies the category of non-unital CSMs with the EM category of algebras for the monad $\mathbb{T}$ on GS. The goal of this section is to modify this in order to obtain (unital) CSMs. Some potential obstacles have been discussed in Section 6, where it was also explained why the 'obvious' modification of the operad or properad monad (see Example 5.1) doesn't work for unital modular operads.

In this section, we return to the definition 1.22 of modular operads and take a more detailed look at the combinatorics of (contracted) units. We'll then build the monad $\mathbb{D}$ on GS that encodes this information. From here it's a small step to defining the distributive law that provides the modular operad monad.

A central feature of the construction of the modular operad monad in Section 7.5 is that it allows us to perform constructions on nice representatives, and then take equivalence classes, thereby avoiding the issues of Section 6.

7.1. **Pointed graphical species.** By definition, if $(S, \diamond, \zeta, \epsilon)$ is a modular operad, then the unit $\epsilon : S_{\S} \to S_{\mathbf{2}}$ is an injective map such that

$$\epsilon\omega = S(\sigma_{\mathbf{2}})\epsilon : S_{\S} \to S_{\mathbf{2}}. \tag{7.1}$$

But, the contraction ζ means that there is also a map

$$o = \zeta\epsilon : S_{\S} \to S_{\mathbf{0}} \text{that is well defined and fixed by } \S_{\tau} : S_{\S} \to S_{\S}. \tag{7.2}$$

So, we can shift our focus to graphical species that are equipped with this extra structure. Throughout the key point to bear in mind is that the combination of a unit and a contraction implies structure in arity $\mathbf{0}$ as well as in arity $\mathbf{2}$.

Definition 7.3. *Objects of the category* GS_* *of* pointed graphical species *are triples* $S_* = (S, \epsilon, o)$ *(or* (S, ϵ^S, o^S)*) where* S *is a graphical species and the unit map* $\epsilon : S_\S \to S_\mathbf{2}$*, and contracted unit map* $o : S_\S \to S_\mathbf{0}$ *satisfy conditions 7.1 and 7.2 above. Morphisms in* GS_* *are morphisms in* GS *that preserve the additional structure.*

Example 7.4. Given a palette $(\mathfrak{C}, \omega)$, $\mathsf{GS}_*^{(C,\omega)} \subset \mathsf{GS}_*$ is the category of $(\mathfrak{C}, \omega)$-*coloured pointed graphical species* and palette-preserving morphisms. The terminal $(\mathfrak{C}, \omega)$-coloured graphical species $Z^{(C,\omega)}$ has a pointed structure and is terminal in $\mathsf{GS}_*^{(C,\omega)}$.

In fact, GS_* is a presheaf category.

Namely, we can obtain the category $\mathbb{P}^{\circlearrowleft}_*$ from $\mathbb{P}^{\circlearrowleft}$ by formally adjoining morphisms $u : \mathbf{2} \to \S$ and $z : \mathbf{0} \to \S$, subject to the relations

- $u \circ ch_1 = id \in \mathbb{P}^{\circlearrowleft}(\S, \S)$ and $u \circ ch_2 = \tau \in \mathbb{P}^{\circlearrowleft}(\S, \S)$,
- $\tau \circ u = u \circ \sigma_\mathbf{2} \in \mathbb{P}^{\circlearrowleft}(\mathbf{2}, \S)$, and
- $z = \tau \circ z \in \mathbb{P}^{\circlearrowleft}(\mathbf{0}, \S)$.

It is easy to check directly that $\mathbb{P}^{\circlearrowleft}_*$ is completely described by

- $\mathbb{P}^{\circlearrowleft}_*(\S, \S) = \mathbb{P}^{\circlearrowleft}(\S, \S)$ and $\mathbb{P}^{\circlearrowleft}_*(Y, X) = \mathbb{P}^{\circlearrowleft}(Y, X)$ whenever $Y \ncong \mathbf{0}, Y \ncong \mathbf{2}$,
- $\mathbb{P}^{\circlearrowleft}_*(\mathbf{0}, \S) = \{z\}$, and $\mathbb{P}^{\circlearrowleft}_*(\mathbf{0}, X) = \mathbb{P}^{\circlearrowleft}(\mathbf{0}, X) \amalg \{ch_x \circ z\}_{x \in X}$,
- $\mathbb{P}^{\circlearrowleft}_*(\mathbf{2}, \S) = \{u, \tau \circ u\}$, and $\mathbb{P}^{\circlearrowleft}_*(\mathbf{2}, X) = \mathbb{P}^{\circlearrowleft}(\mathbf{0}, X) \amalg \{ch_x \circ u, ch_x \circ \tau \circ u\}_{x \in X}$,

and so:

Lemma 7.5. *The following are equivalent:*

(1) S_* *is a presheaf on* $\mathbb{P}^{\circlearrowleft}_*$ *that restricts to a graphical species* S *on* $\mathbb{P}^{\circlearrowleft}$*,*
(2) (S, ϵ, o)*, with* $\epsilon = S_*(u)$ *and* $o = S_*(z)$ *is a pointed graphical species.*

The category GS_* *of* pointed graphical species *is the category* $\mathsf{ps}(\mathsf{elGr}_*)$ *of presheaves on* elGr_**.*

Henceforth, by Lemma 7.5, the notation S_* and (S, ϵ, o) will be used interchangeably to denote the same pointed graphical species.

The forgetful functor $\mathsf{GS}_* \to \mathsf{GS}$ has a left adjoint $(\cdot)^+$ that takes a graphical species S to its left Kan extension $S^+ = (DS, \epsilon^+, o^+)$ along the inclusion $\mathsf{elGr}^{op} \hookrightarrow \mathsf{elGr}_*^{op}$. The monad induced by the adjunction does nothing more than formally adjoin elements $\{\epsilon^+_c\}_{c \in S_\S}$ to $S_\mathbf{2}$ and $\{o_{\tilde{c}}\}_{\tilde{c} \in \tilde{S}_\S}$ to $S_\mathbf{0}$ according to the combinatorics of contracted units: $\epsilon^+ \stackrel{\text{def}}{=} S^+(u) : S_\S \to S_\mathbf{2}$, $o^+ = S^+(z)$. More precisely:

Lemma 7.6. *The forgetful functor* $\mathsf{GS}_* \to \mathsf{GS}$ *is strictly monadic. In other words,* GS_* *is the EM category of algebras for the monad* $\mathbb{D} = (D, \mu^\mathbb{D}, \eta^\mathbb{D})$ *on* GS *induced by the adjunction.*

The monadic unit $\eta^\mathbb{D}$ is provided by the inclusion $S \hookrightarrow DS$, and the multiplication $\mu^\mathbb{D}$ is induced by the canonical projections $D^2 S_\mathbf{2} \to DS_\mathbf{2}$.

Remark 7.7. Some care must be taken with the notation ϵ^+, o^+. In case it's necessary to specify the precise pointed graphical species in which the (contracted) units live, I'll always do so, by using ϵ^{DS}, o^{DS} instead.

For example, for each graphical species S and each $c \in S_\S$, $\epsilon^{DS}_c \in DS_{\mathbf{2}}$ is an element of $D^2 S_{\mathbf{2}}$ (since $DS_{\mathbf{2}} \subset D^2 S_{\mathbf{2}}$), but $\epsilon^{D^2 S} = (DS)^+(u)$, and *not* ϵ^{DS} provides units for the free pointed graphical species $(DS)^+$.

Definition 7.8. *A* pointed element of a pointed graphical species S_* *is an object* $(\mathcal{C}, \phi)$, $\mathcal{C} \in \mathbb{P}^{\circlearrowleft}_*$, $\phi \in S(\mathcal{C})$ *of the element category* $\mathsf{el}_*(S) \overset{\text{def}}{=} \int_{\mathbb{P}^{\circlearrowleft}_*} S_*$.

So, the forgetful functor $\mathsf{GS}_* \to \mathsf{GS}$ induces identity on objects inclusions $\mathsf{el}(S) \to \mathsf{el}_*(S_*)$ of categories for all pointed graphical species $S_* = (S, \epsilon, o)$.

7.2. **Pointed graphs.** Let Gr_* be the category obtained in the bo-ff factorisation of $(Y-)^+ : \mathsf{Gr} \hookrightarrow \mathsf{GS} \to \mathsf{GS}_*$.

$$\begin{tikzcd}
\mathbb{P}^{\circlearrowleft}_* \arrow[r, "\text{dense}"] \arrow[rr, bend left, "\text{f.f.}"] & \mathsf{Gr}_* \arrow[r, "Y_*", "\text{f.f.}"'] & \mathsf{GS}_* \arrow[d, shift left, "\text{forget}"] \\
\mathbb{P}^{\circlearrowleft} \arrow[u, "\text{b.o.}"] \arrow[r, hook] & \mathsf{Gr} \arrow[u, "\text{b.o.}"] \arrow[r, hook, "Y"'] & \mathsf{GS} \arrow[u, shift left, "(\cdot)_*"]
\end{tikzcd} \tag{7.9}$$

The inclusion $\mathbb{P}^{\circlearrowleft}_* \to \mathsf{Gr}_*$ is fully faithful – by uniqueness of bo-ff factorisation – and also dense, since the induced nerve $Y_* : \mathsf{Gr}_* \to \mathsf{GS}_*$ is fully faithful by construction.

Let $\mathcal{G} \in \mathsf{Gr}$ be a graph. For each edge $e \in E$, the ch_e-coloured unit for $Y_*\mathcal{G}$ is defined, by Lemma 7.5, as $\epsilon^{\mathcal{G}}_e \overset{\text{def}}{=} ch_e \circ u \in \mathsf{Gr}_*(\mathcal{C}_{\mathbf{2}}, \mathcal{G})$, and $o^{\mathcal{G}}_e = ch_e \circ z \in \mathsf{Gr}_*(\mathcal{C}_{\mathbf{0}}, \mathcal{G})$ is the corresponding contracted unit.

For all graphs $\mathcal{G}$, $\mathsf{el}_*(Y_*\mathcal{G})$ is canonically isomorphic to the slice category $\mathsf{elGr}_*|\mathcal{G}$ in Gr_*, by an application of the Yoneda lemma. Therefore, we'll usually identify these and write $\mathsf{el}_*(\mathcal{G}) \cong \mathsf{el}_*(Y_*\mathcal{G})$ for the *category of pointed elements of a graph* $\mathcal{G}$.

Lemma 7.10. *For all graphs $\mathcal{G}$, the inclusion $\mathsf{el}(\mathcal{G}) \hookrightarrow \mathsf{el}_*(\mathcal{G})$ is final. In other words, a functor $\Phi : \mathsf{el}_*(\mathcal{G}) \to \mathsf{C}$ has a colimit if and only if its restriction to $\mathsf{el}(\mathcal{G})$ has a colimit, in which case these colimits agree. In particular, every graph $\mathcal{G}$ is the colimit of the forgetful functor $\mathsf{el}_*(\mathcal{G}) \to \mathsf{Gr}_*$, $(\mathcal{C}, b) \to \mathcal{C}$.*

Proof. By definition, $\mathsf{el}_*(\mathcal{G})$ is obtained from $\mathsf{el}(\mathcal{G})$ by adjoining, for each $e \in E$, the objects $(\mathbf{2}, ch_e \circ u)$ and $(\mathbf{0}, ch_e \circ u) = (\mathbf{0}, ch_{\tau e} \circ u)$. Hence, for all $(\mathcal{C}, b) \in \mathsf{el}_*(\mathcal{G})$, the slice category $b|\mathsf{el}(\mathcal{G})$ is connected and non-empty, and the lemma follows by e.g.[29, Section 3, Chapter IX]. □

Example 7.11. A surprising consequence of the definitions is the morphism set $\mathsf{Gr}_*(\mathcal{W}, (\mid))$ is non-empty. Recall the degenerate $\mathcal{W}$-indexed graph of graphs $\Lambda : \mathsf{el}(\mathcal{W}) \to \mathsf{Gr}$, of Example 6.3. On components, this describes precisely the constant $\mathbb{P}^{\circlearrowleft}_*$-cocone of $\mathsf{el}(\mathcal{W})$ above $(\mid)$ illustrated in Diagram 7.12.

Hence, there are two morphisms $\kappa, \tau \circ \kappa \in \mathsf{Gr}_*(\mathcal{W}, (\mid))$ corresponding to the diagrams

$$\begin{tikzcd}
 & \mathcal{W} & \\
(\mid) \arrow[ur, "ch_a"] \arrow[rr, shift left, "ch_1"] \arrow[rr, shift right, "ch_2 \circ \tau"'] \arrow[dr, equal] & & \mathcal{C}_{\mathbf{2}} \arrow[ul, "1_{\mathcal{C}_{\mathbf{2}}} \mapsto a"'] \arrow[dl, "u"] \\
 & (\mid) &
\end{tikzcd} \tag{7.12}$$

$$\begin{tikzcd}
 & \mathcal{W} & \\
(\mid) \arrow[ur, "ch_a"] \arrow[rr, shift left, "ch_1"] \arrow[rr, shift right, "ch_2 \circ \tau"'] \arrow[d, equal] & & \mathcal{C}_{\mathbf{2}} \arrow[ul, "1_{\mathcal{C}_{\mathbf{2}}} \mapsto a"'] \arrow[d, "\sigma_{\mathbf{2}}"] \\
(\mid) \arrow[rr, shift left, "ch_2"] \arrow[rr, shift right, "ch_1 \circ \tau"'] \arrow[dr, equal] & & \mathcal{C}_{\mathbf{2}} \arrow[dl, "u"] \\
 & (\mid). &
\end{tikzcd} \tag{7.13}$$

It follows that, for all graphs $\mathcal{G} \not\cong \mathcal{W}$,

$$\mathsf{Gr}_*(\mathcal{W}, \mathcal{G}) \cong E(\mathcal{G}) \text{ by } ch_e \circ \kappa \mapsto e. \tag{7.14}$$

These morphisms play a crucial role in the proof of the nerve theorem, Theorem 8.1.

Let $\mathcal{G} \in ob(\mathsf{Gr}_*)$ be any connected graph and $W \subset V_2$ a subset of bivalent vertices.

Definition 7.15. *A* vertex deletion functor (for W) *is a $\mathcal{G}$-shaped (degenerate) graph of graphs $\mathbf{\Lambda}^{\mathcal{G}}_{/W}$: $\mathsf{el}(\mathcal{G}) \to \mathsf{Gr}_*$ such that for $(\mathcal{C}_X, b) \in \mathsf{el}(\mathcal{G})$,*

$$\mathbf{\Lambda}^{\mathcal{G}}_{/W}(b) = \begin{cases} (\mid) & \textit{if } (\mathcal{C}_X, b) \textit{ is a neighbourhood of } v \in W, \\ \mathcal{C}_X & \textit{otherwise.} \end{cases}$$

If $\mathbf{\Lambda}^{\mathcal{G}}_{/W}$, admits a colimit $\mathcal{G}_{/W}$ in Gr_*, then the induced morphism $\mathsf{del}_{/W} \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}_{/W})$ is called the vertex deletion morphism corresponding to W.

It follows immediately from the definition that if $\mathsf{del}_{/W} : \mathcal{G} \to \mathcal{G}_{/W}$ exists in Gr_*, and $W = W_1 \amalg W_2$, then $\mathsf{del}_{/W_1} : \mathcal{G} \to \mathcal{G}_{/W_1}$ and

$$\mathsf{del}^{\mathcal{G}_{/W_1}}_{/W_2} : \mathcal{G}_{/W_1} \to (\mathcal{G}_{/W_1})_{/W_2} = \mathcal{G}_{/W}$$

also exist in Gr_* and $\mathsf{del}_{/W} = \mathsf{del}^{\mathcal{G}_{/W_1}}_{/W_2} \circ \mathsf{del}_{/W_1}$.

Proposition 7.16. *For all $\mathcal{G} \in ob(\mathsf{Gr}_*)$ and all $W \subset V_2$, the colimit $\mathcal{G}_{/W}$ of $\mathbf{\Lambda}^{\mathcal{G}}_{/W}$ exists in Gr_*.*

Moreover, $E_0(\mathcal{G}) = E_0(\mathcal{G}_{/W})$ unless $\mathcal{G} = \mathcal{W}^m$ and $W = V$ for some $l \geq 1$.

Example 7.17. For $\mathcal{G} = \mathcal{C}_{\mathbf{2}}$ and $W = V = \{*\}$, $\mathbf{\Lambda}^{\mathcal{G}}_{/W}$ is the constant functor induced by the cocone of $\mathsf{el}(\mathcal{C}_{\mathbf{2}})$ above $(\mid)$ in Gr_*:

$$\begin{tikzcd} (\mid) \arrow[r, "ch_1"] \arrow[dr, "id_{(\mid)}"'] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & (\mid) \arrow[l, "ch_2\circ\tau"'] \arrow[dl, "id_{(\mid)}"] \\ & (\mid) & \end{tikzcd} . \tag{7.18}$$

So, trivially, $\mathbf{\Lambda}^{\mathcal{G}}_{/W}$ has colimit $(\mid)$ in Gr_* and $\mathsf{del}_{/W} = u \in \mathsf{Gr}_*(\mathcal{C}_{\mathbf{2}}, \mid)$.

It follows that, for $\mathcal{G} = \mathcal{L}^k$, and $W = V$, then $\mathbf{\Lambda}^{\mathcal{G}}_{/W}$ is also the constant functor to $\mathcal{G}_{/W} = (\mid)$, and $u^k \overset{\text{def}}{=} \mathsf{del}_{/W} : \mathcal{L}^k \to \mathsf{Gr}_*$ is induced by the Gr_*-cocone under $\mathsf{el}_*(\mathcal{G})$:

$$\begin{tikzcd} (\mid) \arrow[r, "ch_1"] \arrow[dr, "id_{(\mid)}"'] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & \arrow[l, "ch_2\circ\tau"'] \ldots \arrow[r, "ch_1"] & \mathcal{C}_{\mathbf{2}} \arrow[d, "u"] & (\mid) \arrow[l, "ch_2\circ\tau"'] \arrow[dl, "id_{(\mid)}"] \\ & (\mid) \arrow[r, equal] & \ldots \arrow[r, equal] & (\mid). & \end{tikzcd}$$

Observe in particular, that $u^1 = u : \mathcal{C}_{\mathbf{2}} \to (\mid)$ and u^0 is just the identity on $(\mid)$.

For any graph $\mathcal{G}$, a pointwise étale injection $\iota \in \mathsf{Gr}(\mathcal{L}^k, \mathcal{G})$ includes $W = V(\mathcal{L}^k)$ as a subset of bivalent vertices in $V = V(\mathcal{G})$. Hence, $\mathsf{del}_{/W} \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}_{/W})$ exists in Gr_* and there is a commuting diagram

$$\begin{tikzcd} \mathcal{L}^k \arrow[r, "\iota"] \arrow[d, "u^k"'] & \mathcal{G} \arrow[d, "\mathsf{del}_{/W}"] \\ (\mid) \arrow[r, "ch_e"'] & \mathcal{G}_{/W}, \end{tikzcd} \tag{7.19}$$

where $e = \iota(1_{\mathcal{L}^k}) \in E$ and the horizontal morphisms are étale morphism in Gr.

Example 7.20. By Example 7.11, $\mathcal{W}_{/\{v\}} = colim_{\mathsf{el}(\mathcal{W})}\mathbf{\Lambda}^{\mathcal{W}}_{/\{v\}}$ exists and is isomorphic to $(\shortmid)$ in Gr. (See also Section 6.) The induced morphism $\mathsf{del}_{/\{v\}}$ is precisely $\kappa : \mathcal{W} \to (\shortmid)$.

More generally, let $\mathcal{W}^m$ be the wheel graph with m cyclically ordered vertices $(v_i)_{i=1}^m$, and let $\iota \in \mathsf{Gr}(\mathcal{L}^{m-1}, \mathcal{W}^m)$ be a pointwise étale inclusion that respects the (cyclic) ordering of vertices. So, if W is the image of $V(\mathcal{L}^{m-1})$ in $V(\mathcal{W}^m)$, then $V(\mathcal{W}^m) \cong W \amalg \{*\}$. By 7.19, ι defines a vertex deletion morphism $\mathsf{del}_{/W} \in \mathsf{Gr}_*(\mathcal{W}^m, \mathcal{W})$. Therefore $\kappa^m \overset{\text{def}}{=} \mathsf{del}_{/V(\mathcal{W}^m)}$ is given by the composite $\kappa \circ \mathsf{del}_{/W} : \mathcal{W}^m \to (\shortmid)$.

In particular, for all $m \geq 1$, there are two distinct pointed morphisms κ^m, and $\tau \circ \kappa^m$ in $\mathsf{Gr}_*(\mathcal{W}^m, \shortmid)$. Hence, for all $\mathcal{G}$,

$$\mathsf{Gr}_*(\mathcal{W}^m, \mathcal{G}) = \mathsf{Gr}(\mathcal{W}^m, \mathcal{G}) \amalg \{ch_e \circ \kappa^m\}_{e \in E(\mathcal{G})} \cong \mathsf{Gr}_*(\mathcal{W}, \mathcal{G}).$$

Proof of Proposition 7.16. If W is empty, then $\mathcal{G}_{/W} = \mathcal{G}$ and $\mathsf{del}_{/W}$ is the identity on $\mathcal{G}$. On the other hand, If $W = V$ then, by Proposition 4.27, $\mathcal{G} = \mathcal{L}^k$ or $\mathcal{G} = \mathcal{W}^m$ for some $k \geq 0$, $l \geq 1$. In these cases, it follows from the Examples 7.17 and 7.20 that $\mathcal{G}_{/W} = (\shortmid)$ in Gr_*. For $\mathcal{G} = \mathcal{L}^k$, the vertex deletion morphism $u^k : \mathcal{L}^k \to (\shortmid)$ induces a bijection on boundaries.

So, assume that $W \neq V$ and $W \neq \emptyset$.

Let $\mathcal{G}^W \subset \mathcal{G}$ be the subgraph with $V(\mathcal{G}^W) = W$, $H(\mathcal{G}^W) = \coprod_{w \in W} H\{w\}$. The edge set $E(\mathcal{G}^W)$ is the τ-closure of $\coprod_{w \in W} E\{w\}$.(See Figure 20.)

By connectedness of $\mathcal{G}$, and Proposition 4.27, $\mathcal{G}^W = \coprod_{i=1}^m \mathcal{L}_i^{k_i}$ is a disjoint union of line graphs, where $k_i \geq 1$ for all i. In particular the boundary $E_0(\mathcal{G}^W) = \coprod_{i=1}^m \{1_{\mathcal{L}^{k_i}}, 2_{\mathcal{L}^{k_i}}\} \cong m(\mathbf{2})$ and the sets $\coprod_{i=1}^m \{1_{\mathcal{L}^{k_i}}\}$ and $\coprod_{i=1}^m \{2_{\mathcal{L}^{k_i}}\}$ are disjoint in E. For $1 \leq i \leq m$, let $e_i = 1_{\mathcal{L}^{k_i}} \in E$.

The graph $\mathcal{G}_{/W}$ is then obtained by applying $u^{k_i} : \mathcal{L}^{k_i} \to (\shortmid)$ on each component $\mathcal{L}^{k_i}$ of $\mathcal{G}^W \subset \mathcal{G}$. Equation (7.19) gives a commuting diagram

$$\begin{tikzcd}
\mathcal{G}^W \arrow[r, "\iota"] \arrow[d, "\coprod_{i=1}^m u^{k_i}"'] & \mathcal{G} \arrow[d, "\mathsf{del}_{/W}"] \\
\coprod_{i=1}^m (\shortmid) \arrow[r, "\coprod_i ch_{e_i}"'] & \mathcal{G}_{/W},
\end{tikzcd}$$

and hence the coproduct $\mathsf{del}_{/W} : \mathcal{G} \to \mathcal{G}_{/W}$ exists in Gr_*.

Let

$$\begin{array}{rcl cl}
V_{/W} & \overset{\text{def}}{=} & V - W, & & \\
H_{/W} & \overset{\text{def}}{=} & H - H(\mathcal{G}^W) & = & H - \coprod_{v \in W} H\{v\}, \\
E_{/W} & \overset{\text{def}}{=} & E - \big(E(\mathcal{G}^W) - E_0(\mathcal{G}^W)\big) & = & E - \coprod_{v \in W} E\{v\},
\end{array}$$

As $V \neq W$, the involution $\tau_{/W} : E_{/W} \to E_{/W}$ given by

$$\begin{array}{rcll}
\tau_{/W}(e) & = & \tau e & \text{for } e \in E - E(\mathcal{G}^W), \\
\tau_{/W}(1_{\mathcal{L}^{k_i}}) & = & 2_{\mathcal{L}^{k_i}} & \text{for } 1 \leq i \leq m
\end{array}$$

is fixed point free. Then $\mathcal{G}_{/W}$ has the explicit description

$$\mathcal{G}_{/W} = \quad \tau_{/W} \circlearrowright E_{/W} \xleftarrow{s_{/W}} H_{/W} \xrightarrow{t_{/W}} V_{/W},$$

where $s_{/W}, t_{/W}$ are just the restrictions of s and t. (Figure 20.)

Since $\coprod_{i=1}^m \{1_{\mathcal{L}^{k_i}}\}$ and $\coprod_{i=1}^m \{2_{\mathcal{L}^{k_i}}\}$ are disjoint in E when $W \neq V$, $\mathsf{del}_{/W} \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}_{/W})$ restricts to an identity on boundaries. Hence the final statement, that $E_0(\mathcal{G}) = E_0(\mathcal{G}_{/W})$ except when $\mathcal{G} = \mathcal{W}^m$ and $W = V$ for some $l \geq 1$ follows immediately. $\square$

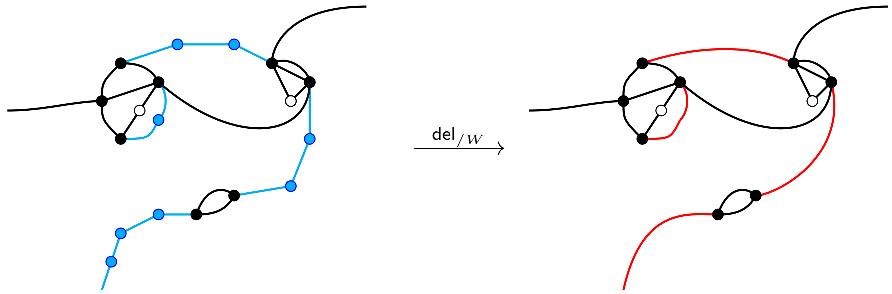


FIGURE 20. Vertex deletion $\mathsf{del}_{/W} : \mathcal{G} \to \mathcal{G}_{/W}$, with $\mathcal{G}^W \subset \mathcal{G}$ and $W \subset V_2$, and $\coprod_{i=1}^{3} u^{k_i}(\mathcal{G}^W) \subset \mathcal{G}_{/W}$.

Now, let $\mathcal{G} \not\cong \mathcal{C}_\mathbf{0}$ and $\mathcal{G}'$ be connected graphs and let $f \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}')$ be any morphism. If $W_f \subset V_2$ is the set of bivalent vertices such that $f(v) = \tilde{e}' \in \widetilde{E'}$, then $v \in W_f$ if and only if each minimal neighbourhood $(\mathcal{C}_\mathbf{2}, b)$ of v in $\mathcal{G}$ is described by

$$f \circ b = ch_{e'} \circ u, \text{ or } f \circ b = ch_{\tau' e'} \circ u.$$

The following corollary follows immediately from Proposition 7.16 and Corollary 4.25.

Corollary 7.21. *Let $\mathcal{G} \in ob(\mathsf{Gr}_*)$ with $\mathcal{G} \not\cong \mathcal{C}_\mathbf{0}$, and let $f \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}')$. Then f factors uniquely as a vertex deletion morphism $\mathsf{del}_{/W_f} \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}_{/W_f})$ followed by a morphism $f_{/W_f} \in \mathsf{Gr}(\mathcal{G}_{/W_f}, \mathcal{G}')$.*

In particular, if $E_0(\mathcal{G}) \neq \emptyset$ and $f \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}')$ induces an isomorphism $E_0 \xrightarrow{\cong} E_0'$ (i.e. f is boundary preserving), then $f_{/W_f} \in \mathsf{Gr}(\mathcal{G}_{/W_f}, \mathcal{G}')$ is an isomorphism.

In other words, there is an orthogonal ('generic-free' [39]) factorisation on Gr_* such that the left class of morphisms consists of $z : \mathcal{C}_\mathbf{0} \to (\shortmid)$, the vertex deletion morphisms, and the isomorphisms, and the right class consists of morphisms in Gr.

Definition 7.22. *Morphisms in the left class of the factorisation on Gr_* are called* similarity morphisms.

Example 7.23. By Corollary 7.21, any morphism in Gr_* with target $(\shortmid)$ is a similarity morphism and (up to isomorphism) has one of the forms $u^k \in \mathsf{Gr}_*(\mathcal{L}^k, \shortmid)$, $k \geq 0$, or $z \in \mathsf{Gr}_*(\mathcal{C}_\mathbf{0}, \shortmid)$, or $\kappa^m \in \mathsf{Gr}_*(\mathcal{W}^m, \shortmid)$ for $m \geq 1$.

Example 7.24. For all graphs $\mathcal{G}$ and all $k \in \mathbb{N}$,

$$\mathsf{Gr}_*(\mathcal{L}^k, \mathcal{G}) \cong \coprod_{j=0}^{k} \binom{k}{j} \mathsf{Gr}(\mathcal{L}^j, \mathcal{G}).$$

The proof of Proposition 7.16 implies that we may characterise morphisms in Gr_* in terms of commuting diagrams in $\mathsf{Set_f}$:

Corollary 7.25. *Morphisms $f \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}')$ are characterised by commuting diagrams of the form 7.26, such that $\mathfrak{f}_V^{-1}(\widetilde{E'}) \subset V_0 \amalg V_2$ is either a single isolated vertex or a (possibly empty) subset of bivalent vertics, and the square 7.27 is a pullback.*

$$\begin{tikzcd} E \arrow[r, leftrightarrow, "\tau"] \arrow[d, "\mathfrak{f}_E"'] & E \arrow[d, "\mathfrak{f}_E"'] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\mathfrak{f}_H"'] & V \arrow[d, "\mathfrak{f}_V"] \\ E' \arrow[r, leftrightarrow, "\tau'"'] & E' & H' \amalg E' \arrow[l, "s' \amalg id'"] \arrow[r, "t' \amalg q'"'] & V' \amalg \widetilde{E'} \end{tikzcd} \tag{7.26}$$

$$\begin{tikzcd} H \arrow[r, "t"] \arrow[d, "\mathfrak{f}_H"'] & (V - V_0) \arrow[d, "\mathfrak{f}_V"] \\ H' \amalg E' \arrow[r, "t' \amalg q'"'] & (V - V_0) \amalg \widetilde{E'} \end{tikzcd} \tag{7.27}$$

If $\mathcal{G} \not\cong \mathcal{C}_\mathbf{0}$, and $f = f_{/W_f} \circ \mathsf{del}_{/W_f}$ is the generic-free factorisation of f, then $W_f = \mathfrak{f}_V^{-1}(\widetilde{E'}) \subset V_2$.

Example 7.28. The diagrams 7.29 satisfy the axioms for $z : \mathbf{0} \to \S$ and $u : \mathbf{2} \to \S$ in $\mathbb{P}_*^{\circlearrowleft} \subset \mathsf{Gr}_*$:

(7.29)

$$\begin{tikzcd}
\emptyset \arrow[d] & \emptyset \arrow[l] \arrow[r] \arrow[d] & \{v\} \arrow[d] \\
\{1,2\} \arrow[loop left, "\tau"] & \{1,2\} \arrow[l, "id"'] \arrow[r, "q_{\shortmid}"'] & \{\tilde{1}\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\left\{\begin{matrix} e_1, & e_2, \\ \tau_{\mathbf{2}} e_1, & \tau_{\mathbf{2}} e_2 \end{matrix}\right\} \arrow[loop left, "\tau_2"] \arrow[d] & \{(e_1, v),\ (e_2, v)\} \arrow[l] \arrow[r] \arrow[d] & \{v\} \arrow[d] \\
\{1,2\} \arrow[loop left, "\tau"] & \{1,2\} \arrow[l, "id"] \arrow[r, "q_{\shortmid}"] & \{\tilde{1}\}
\end{tikzcd}$$

The follwing, slightly weakened version of Lemma 3.13 implies that most morphisms in Gr_* are completely determined by their action on edges.

Lemma 7.30. *If $\mathcal{G} \ncong \mathcal{C}_\mathbf{0}$ and $\mathcal{G}' \ncong \mathcal{W}$, then $\mathfrak{f}_E$ is sufficient to define $f \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}')$.*

Proof. Assume that $\mathcal{G} \ncong \mathcal{C}_\mathbf{0}$ (hence $E \neq \emptyset$) and $\mathcal{G}' \ncong \mathcal{W}$.

Since $\mathcal{G}' \neq \mathcal{W}$, the edges $e'_1, e'_2 \in E\{v'\}$ incident on a bivalent vertex $v' \in V'_2$ are in distinct τ'-orbits.

So, if $v \in V_2$ with $E\{v\} = \{e_1, e_2\} \subset E_2$ and $\mathfrak{f}_E(e_1) = \mathfrak{f}_E(\tau e_2) \in E'$, then it must be the case that

$$\mathfrak{f}_V(v) = q'(\mathfrak{f}_E(e_1)) = q'(\mathfrak{f}_E(\tau e_2)) \in \widetilde{E'}.$$

Otherwise, if $\mathfrak{f}_E(e_1) \neq \mathfrak{f}_E(\tau e_2)$, then $\mathfrak{f}_V(v) = t' {s'}^{-1}(\mathfrak{f}_E(e_1)) \in V'$. $\square$

Example 7.31. Lemma 7.30 does not hold if $\mathcal{G}' = \mathcal{W}$. For example, there are only two maps $E(\mathcal{W}^2) \to E(\mathcal{W})$ that are compatible with the involution, and these correspond to the two maps in $\mathsf{Gr}(\mathcal{W}^2, \mathcal{W})$. However, $\mathsf{Gr}_*(\mathcal{W}^2, \mathcal{W})$ has 6 distinct elements.

7.3. **S_*-structured graphs.** The étale topology on Gr extends to a topology on Gr_* whose covers at $\mathcal{G}$ are jointly surjective collections $\mathfrak{U} \subset \mathsf{Gr}_* | \mathcal{G}$. It follows that $P_* \in \mathsf{ps}(\mathsf{Gr}_*)$ is a sheaf for this topology if and only if, for all $\mathcal{G}$

$$P_*(\mathcal{G}) \cong lim_{(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})} P(\mathcal{C}), \text{ where } P : \mathsf{Gr}^{op} \to \mathcal{G} \text{ is the restriction to } \mathsf{Gr}.$$

Hence, there is a canonical isomorphism $\mathsf{sh}(\mathsf{Gr}_*, J_*) \cong \mathsf{GS}_*$, and for all pointed species $S_* = (S, \epsilon, o)$, the corresponding J_* sheaf on Gr_* is described by

$$S_*(\mathcal{G}) \overset{\text{def}}{=} lim_{(\mathcal{C}, b) \in \mathsf{el}_*(\mathcal{G})} S_*(\mathcal{C}) = lim_{(\mathcal{C}, b) \in \mathsf{el}(\mathcal{G})} S(\mathcal{C}) \cong S(\mathcal{G}) \text{ for all graphs } \mathcal{G}. \tag{7.32}$$

Let S_* be a pointed graphical species.

Definition 7.33. *(Compare Definition 4.37.) An S_*-*structured (pointed) graph *is an object $(\mathcal{G}, \alpha)$, $\mathcal{G} \in \mathsf{Gr}_*$, $\alpha \in S(\mathcal{G})$ of the element category $\mathsf{Gr}_*(S_*) \overset{\text{def}}{=} \int_{\mathsf{Gr}_*} S_*$.*

An S_*-structured graph $(\mathcal{G}, \alpha)$ is admissible *if $V(\mathcal{G})$ is non-empty.*

Example 7.34. For $k \geq 0$ and $m \geq 1$, the vertex deletion morphisms $u^k : \mathcal{L}^k \to (\shortmid)$ and $\kappa^m : \mathcal{W}^m \to (\shortmid)$ in Gr_* induce injective maps in $\mathsf{Gr}_*(S_*)$:

$$\epsilon^k \overset{\text{def}}{=} S_*(u^k) : \mathfrak{C} \to S(\mathcal{L}^k), \quad \text{and} \quad o^m \overset{\text{def}}{=} S_*(\kappa^k) : \mathfrak{C} \to S(\mathcal{W}^m). \tag{7.35}$$

Observe, in particular, that $\epsilon^1 = \epsilon$, and ϵ^k, o^m factor through $\epsilon : \mathfrak{C} \to S_\mathbf{2}$ for all $k \geq 0, m \geq 1$.

For each $c \in \mathfrak{C}$, we write

$$\mathcal{L}^k(\epsilon_c) \overset{\text{def}}{=} \epsilon^k(c) \in S_*(\mathcal{L}^k) \;\; \text{and} \;\; \mathcal{W}^m(\epsilon_c) \overset{\text{def}}{=} o^m(c) \in S_*(\mathcal{W}^m)$$

and call these the c-coloured unit structures on $\mathcal{L}^k$ and $\mathcal{W}^m$.

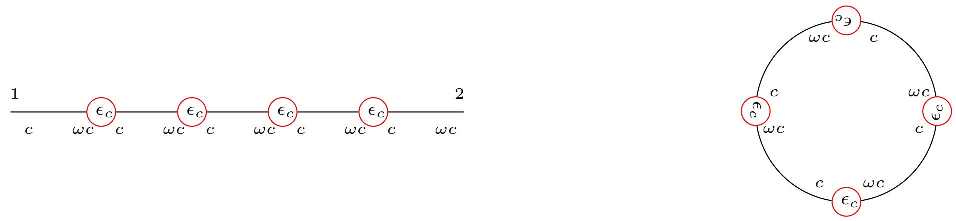


FIGURE 21. The c-coloured unit structures $\mathcal{L}^4(\epsilon_c)$ and $\mathcal{W}^4(\epsilon_c)$.

If $W \subset V_2$ is a subset of bivalent vertices of a graph $\mathcal{G}$, so that $\mathsf{del}_{/W} : \mathcal{G} \to \mathcal{G}_{/W}$ exists in Gr_*, then, for any S_*-structure $\alpha_{/W} \in S(\mathcal{G}_{/W})$, there is a unique S_*-structure $\alpha \in S_*(\mathcal{G})$ such that $\mathsf{del}_{/W} \in \mathsf{Gr}_*(S_*)(\alpha, \alpha_{/W})$. Then, for any minimal neighbourhood $(\mathcal{C}_\mathbf{2}, b) \in \mathsf{el}(\mathcal{G})$ of $v \in W$, there is some $e \in E(\mathcal{G}_{/W})$, such that $\mathsf{del}_{/W} \circ b = ch_e \circ u$, and hence

$$S_*(b)(\alpha) = S_*(\mathsf{del}_{/W} \circ b)(\alpha_{/W}) = S_*(u)S(ch_e)(\alpha_{/W})$$

is in the image of ϵ.

Definition 7.36. *For $(\mathcal{G}, \alpha) \in \mathsf{Gr}_*(S_*)$ with $\mathcal{G} \ncong \mathcal{C}_\mathbf{0}$,*

$$W_\alpha = \{v \mid S(b)(\alpha) \in im(\epsilon) \text{ for each neighbourhood } (\mathcal{C}_\mathbf{2}, b) \text{ of } v\}$$

is the subset of bivalent vertices decorated by units *(for α).*

If $\mathcal{G} = \mathcal{C}_\mathbf{0}$, then the set W_α, of vertices α-decorated by contracted units, *is non-empty (and hence $W_\alpha = V(\mathcal{G})$) if and only if there is a $c \in \mathfrak{C}$ such that $S_*(z)(c) = \alpha$.*

If $W_\alpha = \emptyset$, then $(\mathcal{G}, \alpha)$ is called reduced *in $\mathsf{Gr}_*(S_*)$.*

So, $\mathsf{del}_{/W} \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{G}_{/W})$ describes a morphism from $(\mathcal{G}, \alpha)$ in $\mathsf{Gr}_*(S_*)$ if and only if $W \subset W_\alpha$. Informally, we describe $\mathsf{del}_{/W} \in \mathsf{Gr}_*(S_*)((\mathcal{G}, \alpha), (\mathcal{G}_{/W}, \alpha_{/W}))$ (and $z \in \mathsf{Gr}_*((\mathcal{C}_\mathbf{0}, \alpha), (\shortmid, c)))$ as 'deleting' a subset W of vertices α-decorated by (contracted) units.

If $(\mathcal{G}, \alpha)$ is not of the form $(\mathcal{C}_\mathbf{0}, o_{\tilde{c}})$, then there is a unique reduced S_*-structured graph $(\mathcal{G}^\perp_\alpha, \alpha^\perp) \stackrel{\text{def}}{=} (\mathcal{G}_{/W_\alpha}, \alpha_{/W_\alpha})$ obtained by deleting *all* vertices of $\mathcal{G}$ that are α-decorated by units. For any structure $(\mathcal{G}, \alpha)$, the reduced graph $(\mathcal{G}^\perp_\alpha, \alpha^\perp)$ is admissible if and only if $W_\alpha \neq V$.

7.4. **Similar structures.** The issues that arise from trying to incorporate degenerate substitution by the stick graph into the definition of the modular operad monad have been outlined in Section 6.

Degenerate substitutions – and exceptional loops – can be avoided if there is a suitable notion of equivalence of S_*-structured graphs, for which all constructions can be obtained in terms of admissible representatives.

This principle will inform the construction of the distributive law $\lambda : TD \Rightarrow DT$.

We can use the (contracted) unit of a pointed graphical species S_* to enlarge the categories $X\mathsf{Gr}_{iso}(S)$ (Definition 5.2).

Definition 7.37. *Let S_* pointed graphical species. The category $X\mathsf{Gr}_{sim_*}(S_*)$, of* similar S_*-structured X-graphs *is the category obtained from $X\mathsf{Gr}_{iso}(S)$ (Definition 5.2) by adjoining the similarity morphisms (Definition 7.22), and, in arities **2** and **0**, the objects $(\shortmid, c)$, for all $c \in S_\S$.*

If $S_* = Z$ is the terminal graphical species then $X\mathsf{Gr}_{sim_*} \stackrel{\text{def}}{=} X\mathsf{Gr}_{sim_*}(Z)$ is the category of similar X-graphs.

Structured X-graphs $(\mathcal{X}^1, \alpha^1), (\mathcal{X}^2, \alpha^2) \in ob(X\mathsf{Gr}_{sim_*}(S_*))$ are called similar, *written $(\mathcal{X}^1, \alpha^1) \sim (\mathcal{X}^2, \alpha^2)$ (or just $\alpha^1 \sim \alpha^2$), if they are in the same connected component of $X\mathsf{Gr}_{sim_*}(S_*)$.*

In particular, when $X \cong \mathbf{0}$ or $X \cong \mathbf{2}$, there are distinguished elements $(\shortmid, c) \in X\mathsf{Gr}_{sim_*}(S_*)$ and we assume that these are equipped with the canonical labelling by $\mathbf{2} = E_0(\shortmid)$.

Example 7.38. (Compare Section 6.) If $\mathcal{X} \ncong \mathcal{C}_\mathbf{0}$ and $W_\alpha = V(\mathcal{G})$, then $(\mathcal{X}, \alpha) \in X\mathsf{Gr}_{sim_*}(S_*)$ has the form $\mathcal{L}^k(\epsilon_c)$, or $\mathcal{W}^m(\epsilon_c)$ for some $c \in \mathfrak{C}$. So, the reduced similar structure $(\mathcal{G}^\perp_\alpha, \alpha^\perp) = (\shortmid, c)$ and is not admissible.

Observe that $z \in \mathsf{Gr}_*(S_*)((\mathcal{C}_\mathbf{0}, o_{\tilde{c}}), (\shortmid, c))$ *and* $z \in \mathsf{Gr}_*(S_*)((\mathcal{C}_\mathbf{0}, o_{\tilde{c}}), (\shortmid, \omega c))$ so there is no unique choice of reduced structure similar to $(\mathcal{C}_\mathbf{0}, o_{\tilde{c}})$.

For all $c \in C$ and all $k, n, l, m \geq 1$,

$$\mathcal{L}^k(\epsilon_c) \sim \mathcal{L}^n(\epsilon_c) \in \mathbf{2}\mathsf{Gr}_{sim_*}(S) \text{ and } \mathcal{W}^l(\epsilon_c) \sim \mathcal{W}^m(\epsilon_c) \sim \mathcal{W}^m(\epsilon_{\omega c}) \in \mathbf{0}\mathsf{Gr}_{sim_*}(S).$$

But also, $\mathcal{W}(\epsilon_c) \sim (\mathcal{C}_\mathbf{0}, o_{\tilde{c}})$ in $\mathbf{0}\mathsf{Gr}_{sim_*}(S_*)$, because the similarity maps $\kappa : \mathcal{W} \to (\shortmid) \leftarrow \mathcal{C}_\mathbf{0} : z$ in Gr_*, induce morphisms

$$\mathcal{W}(\epsilon_c) \longrightarrow (\shortmid, c) \longleftarrow (\mathcal{C}_\mathbf{0}, o_{\tilde{c}}) \longrightarrow (\shortmid, \omega c) \longleftarrow \mathcal{W}(\epsilon_{\omega c}) .$$

For all X, S_*, since $X\mathsf{Gr}_{iso}(S) \subset X\mathsf{Gr}_{sim_*}(S_*)$, there is an induced surjection $\pi_0(X\mathsf{Gr}_{iso}(S)) \to \pi_0(X\mathsf{Gr}_{sim_*}(S_*))$ on connected components.

Lemma 7.39. *For all pointed graphical species S_* and all finite sets X, there is a canonical bijection*

$$colim_{(\mathcal{G},\rho) \in X\mathsf{Gr}_{sim_*}} S_*(\mathcal{G}) \cong \pi_0(X\mathsf{Gr}_{sim_*}(S_*)), \tag{7.40}$$

and every component of $X\mathsf{Gr}_{sim_*}(S_*)$ has a representative in $X\mathsf{Gr}_{iso}(S)$.

Proof. The second part is immediate. Therefore $colim_{(\mathcal{G},\rho) \in X\mathsf{Gr}_{sim_*}} S_*(\mathcal{G})$ is a quotient of TS_X and the first part follows from Equation (5.9). $\square$

7.5. **A distributive law for modular operads.** An element of TDS_X is represented by an X-graph $\mathcal{X}$, with a decoration $\alpha \in DS(\mathcal{X}) = S^+(\mathcal{X})$, (where, as before S^+ is the free pointed graphical species on S). The idea is to construct $\lambda : TD \to DT$ that ignores vertices decorated by units. This will be the case if λ is invariant under similarity morphisms $X\mathsf{Gr}_{sim_*}(S_*)$.

Proposition 7.41. *There is a distributive law $\lambda : TD \Rightarrow DT$ such that for all graphical species S and finite sets X, and all $[\mathcal{X}, \alpha]$ and $[\mathcal{X}', \alpha']$ in TDS_X,*

$$\lambda[\mathcal{X}, \alpha] = \lambda[\mathcal{X}', \alpha'] \textit{ in } DTS_X \textit{ if and only if } [\mathcal{X}, \alpha] \sim [\mathcal{X}', \alpha'] \in X\mathsf{Gr}_{sim_*}(S^+).$$

Proof. Since the endofunctor D just adjoins elements, there are canonical inclusions $TS \hookrightarrow DTS$ and $TS \hookrightarrow TDS$. The natural transformation $\lambda : TD \Rightarrow DT$ will restrict to the identity on TS.

For a finite set X, elements of TDS_X are represented by elements $(\mathcal{X}, \alpha) \in X\mathsf{Gr}_{iso}(DS)$, whereas elements of DTS_X are either of the form $\epsilon^{DTS}_c, o^{DTS}_{\tilde{c}}$ for $c \in S_\S$, or are represented by elements $(\mathcal{X}', \alpha') \in X\mathsf{Gr}_{iso}(S)$. Observe also that an object $(\mathcal{X}, \alpha) \in X\mathsf{Gr}_{sim_*}(S^+)$ is reduced and admissible (Definition 7.36) if and only if $(\mathcal{X}, \alpha) \in X\mathsf{Gr}_{iso}(S)$ and hence represents an element of TS_X.

If $X = \mathbf{2}$, and $(\mathcal{X}, \alpha)$ has the form $\mathcal{L}^k(\epsilon_c)$, set

$$\lambda[\mathcal{X}, \alpha] = \epsilon^{DS}_c \in DTS_\mathbf{2}.$$

And, if $X = \mathbf{0}$, and $(\mathcal{X}, \alpha) = \mathcal{W}^m(\epsilon_c)$ or $(\mathcal{X}, \alpha) = (\mathcal{C}_\mathbf{0}, o_{\tilde{c}})$, set

$$\lambda[\mathcal{X}, \alpha] = o^{DTS}_{\tilde{c}} \in DTS_\mathbf{0}.$$

Otherwise, the component of $(\mathcal{X}, \alpha)$ in $X\mathsf{Gr}_{sim_*}(S^+)$ has an admissible terminal object $(\mathcal{X}^{\perp}_{\alpha}, \alpha^{\perp})$, so we can set

$$\lambda[\mathcal{X}, \alpha] = [\mathcal{X}^{\perp}_{\alpha}, \alpha^{\perp}] \in TS_X \subset DTS_X.$$

As morphims of pointed graphical species preserve (contracted) units, λ is clearly natural in S.

The verification that λ satisfies the four axioms [4, Section 1] for a distributive law is straightforward but tedious, so I prove just one here: that Diagram 7.42 of natural transformations commutes.

$$\begin{tikzcd} TD^2 \arrow[r, Rightarrow, "\lambda D"] \arrow[d, Rightarrow, "T(\mu^{\mathbb{D}})"'] & DTD \arrow[r, Rightarrow, "D\lambda"] & D^2T \arrow[d, Rightarrow, "(\mu^{\mathbb{D}})T"] \\ TD \arrow[rr, Rightarrow, "\lambda"'] & & DT \end{tikzcd} \tag{7.42}$$

So, let $[\mathcal{X}, \alpha] \in TD^2S_X$. The result is immediate when $\mathcal{X} = \mathcal{C}_{\mathbf{0}}$. Moreover, all the maps in 7.42 restrict to the identity on $TS \subset TD^2S$.

So we may assume that $[\mathcal{X}, \alpha] \notin TS_X$ and that $\mathcal{X} \ncong \mathcal{C}_{\mathbf{0}}$. For $j = 1, 2$, define the sets W^j, of vertices decorated by distinguished elements adjoined in the j^{th} application of D.

$$W^1 \overset{\text{def}}{=} \{v | v \text{ has a minimal neighbourhood } (\mathcal{C}_X, b) \text{ with } \ D^2S(b)(\alpha) \in im(\epsilon^{DS})\},$$

and

$$W^2 \overset{\text{def}}{=} \{v | v \text{ has a minimal neighbourhood } (\mathcal{C}_X, b) \text{ with } \ D^2S(b)(\alpha) \in im(\epsilon^{D^2S})\}.$$

Then $[\mathcal{X}, \mu^{\mathbb{D}}\alpha] = T\mu^{\mathbb{D}}S[\mathcal{X}, \alpha] \in TDS_X$ is described by

$$\begin{aligned} DS(b)(\mu^{\mathbb{D}}\alpha) &= \epsilon^{DS}_c \text{ if } (\mathcal{C}, b) \text{ is a minimal neighbourhood of } v \in W^1 \amalg W^2, \\ &= D^2S(b)(\alpha) \in S(\mathcal{C}) \text{ otherwise.} \end{aligned}$$

If $W^1 \cup W^2 \neq V$, the diagram gives

$$\begin{tikzcd} {[\mathcal{X}, \alpha]} \arrow[r, mapsto, "\lambda D"] & {[\mathcal{X}_{/W^2}, \alpha_{/W^2}]} \arrow[r, mapsto, "D\lambda"] & {[(\mathcal{X}_{/W^2})_{/W^1}, (\alpha_{/W^2})_{/W^1}]} \arrow[d, equal] \\ {[\mathcal{X}, \alpha]} \arrow[r, mapsto, "T(\mu^{\mathbb{D}})"'] & {[\mathcal{X}, \mu^{\mathbb{D}}\alpha]} \arrow[r, mapsto, "\lambda"'] & {[\mathcal{X}_{/(W^1 \amalg W^2)}, \alpha_{/(W^1 \amalg W^2)}] \in TS_X} \end{tikzcd}.$$

If $W^1 \amalg W^2 = V$, then $T(\mu^{\mathbb{D}})[\mathcal{X}, \alpha]$ has the form $\mathcal{L}^k(\epsilon^{DS}_c)$ or $\mathcal{W}^m(\epsilon^{DS}_c)$ and both paths map to the corresponding (contracted) unit in DTS.

□

Hence, λ induces a composite monad $\mathbb{DT}$ on GS. Therefore, as discussed in Section 2, $\lambda : TD \to DT$ induces a lift $\mathbb{T}_*$ of $\mathbb{DT}$ to GS_*, such that the EM categories $\mathsf{GS}^{\mathbb{DT}}$ and $\mathsf{GS}_*^{\mathbb{T}_*}$ are canonically isomorphic.

The following is immediate from Proposition 7.41 and Lemma 7.39:

Corollary 7.43. *For all graphical species S, and all finite sets X,*

$$\lambda(TDS_X) = colim_{(\mathcal{G}, \rho) \in X\mathsf{Gr}_{sim_*}} S^+(\mathcal{G}).$$

So, unsurprisingly

Corollary 7.44. *The monad $\mathbb{T}_* = (T_*, \mu_*, \eta_*)$ on GS_* is given by*

$$T_*S_{\S} = S_{\S}, \text{ and } T_*S_X = colim_{(\mathcal{G}, \rho) \in X\mathsf{Gr}_{sim_*}} S_*(\mathcal{G}).$$

The unit $\eta_* : 1_{\mathsf{GS}_*} \Rightarrow T_*$ and multiplication $\mu_* : T_*^2 \Rightarrow T_*$ are induced by the unit $\eta^{\mathbb{T}}$ and multiplication $\mu^{\mathbb{T}}$ for $\mathbb{T}$. In other words, if $\mathcal{X}$ is an X-graph and $\alpha \in S(\mathcal{X})$ and $[\mathcal{X}, \alpha]_*$ denotes the class of $[\mathcal{X}, \alpha] \in TS_X$ in T_*S_X, then

$$\eta_*(\phi) = [\eta^{\mathbb{T}}\phi]_*, \text{ and } \mu_*[\mathcal{X}, \beta] = [\mu^{\mathbb{T}}[\mathcal{X}, \beta]]_*.$$

Proof. By [4, Section 3], and the fact that $(\mu^{\mathbb{D}}T) \circ (D\lambda) = (\mu^{\mathbb{D}}\mu^{\mathbb{T}}) \circ (D\lambda T) \circ (DTD\eta^{\mathbb{T}}) : DTD \Rightarrow DT$, for $\mathbb{T}$ and $\mathbb{D}$, the endofunctor T_* is described, for all $S_* = (S, \epsilon, o)$, by the coequaliser

$$\begin{tikzcd} DTDS \arrow[rr, "DTh_{\mathbb{D}}"] \arrow[dr, "D\lambda S"'] & & DTS \arrow[r, "\pi"] & T_*S_*. \\ & D^2TS. \arrow[ur, "\mu^{\mathbb{D}}TS"'] & & \end{tikzcd} \tag{7.45}$$

Observe first that both maps $DTDS \to DTS$ in Diagram 7.45 restrict to the inclusion $TS \hookrightarrow DTS$ on $TS \subset DTDS$.

By the application of $h_{\mathbb{D}}$, the upper path, $\pi : DTS \to T_*S_*$ identifies all occurances of ϵ and ϵ^{DS} in elements of $DTDS$ and, by the application of $\eta^{\mathbb{D}}$ in the lower path, it identifies all occurances of ϵ^{DS} and ϵ^{DTDS}.

All other identifications come from the occurance of λ in the lower path. In particular, by the above, similar elements of $TS \subset DTS$ are identified by the quotient π. Hence $T_*S_X = colim_{(\mathcal{G},\rho) \in X\mathsf{Gr}_{sim_*}} S_*(\mathcal{G})$ as required.

The unit for T_*S_* is provided by the map $[\mathcal{L}^k(\epsilon)]_* : \mathfrak{C} \to T_*S_{\mathbf{2}}$ that takes c to the class of all ϵ_c decorated line graphs $\mathcal{L}^k$, and the contracted unit by $[\mathcal{W}^m(\epsilon)]_* = [\mathcal{C}_{\mathbf{0}}, o]_* : \mathfrak{C} \to T_*S_{\mathbf{0}}$ that takes c to the class of all ϵ_c decorated wheel graphs, and the isolated vertex decorated by ζc.

Let $[\mathcal{X}, \beta]$ represent an element of T^2S_X. And let $\mathsf{del}_{/W} \in X\mathsf{Gr}_{sim_*}((\mathcal{X}, \beta), (\mathcal{X}_{/W}, \beta_{/W}))$. Then, for all neighbourhoods $(\mathcal{C}_{\mathbf{2}}, b) \in \mathsf{el}(\mathcal{X})$ of $v \in W$, let $S(b)(\beta) = [\mathcal{L}^k(\epsilon_c)] \in TS_{\mathbf{2}}$. It is then immediate that $\mu_* = [\mu^{\mathbb{T}}(-)]_*$ is well-defined and provides the multiplication for $\mathbb{T}_*$.

Clearly $\eta_* = [\eta^{\mathbb{T}}(-)]_*$, whereby the result follows immediately. □

It follows, in particular, that to compute $\mu_*(S_*) : T_*^2(S_*)_X \to T_*(S_*)_X$, we only have to consider non-degenerate graphs of graphs. In other words, we can proceed exactly as in the non-unital case (Section 5) and only quotient by similarity at the end.

At last we're ready to prove the first main theorem - that modular operads are $\mathbb{DT}$-algebras in GS or, equivalently $\mathbb{T}_*$-algebras in GS_*.

Observe first that, if $(S, \diamond, \zeta, \epsilon)$ is a modular operad, then $(S, \epsilon, \zeta\epsilon)$ is a pointed graphical species, and $(S, \diamond, \zeta)$ is an unpointed modular operad. Therefore, by Proposition 5.30, S is equipped with a $\mathbb{T}$-algebra structure $p_{\mathbb{T}} = p_{\mathbb{T}}^{\diamond,\epsilon} : TS \to S$.

Lemma 7.46. *Let $(S, \diamond, \zeta, \epsilon)$ be a modular operad, and let $p_{\mathbb{T}} : TS \to S$ denote the corresponding $\mathbb{T}$-algebra structure on S. Objects in the same connected component of $X\mathsf{Gr}_{sim_*}(S, \epsilon, \zeta\epsilon)$ have the same image under the map*

$$X\mathsf{Gr}_{iso}(S) \twoheadrightarrow TS_X \xrightarrow{p_{\mathbb{T}}} S_X.$$

Proof. By definition, $p_{\mathbb{T}}$ satisfies

$$p_{\mathbb{T}}[\mathcal{M}_c(\phi, \epsilon_c)] = \phi \diamond_c \epsilon_c = \phi = p_{\mathbb{T}}(\eta^{\mathbb{T}}\phi) \text{ for all } \phi \in S_{(\underline{c}, c)}.$$

And, $p_{\mathbb{T}}[\mathcal{N}_c(\psi)] = \zeta(\psi)$ for all $\psi \in S_{(\underline{c}, c, \omega c)}$. So, in particular

$$p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}(\epsilon_c)] = \zeta(\epsilon_c).$$

Hence, by the monad algebra axioms 2.3,2.4,

$$p_{\mathbb{T}}[\mathcal{W}^m(\epsilon)] = p_{\mathbb{T}}[\mathcal{C}_{\mathbf{0}}, \zeta\epsilon] : \mathfrak{C} \to S_{\mathbf{0}},$$

and, otherwise, $p_{\mathbb{T}}[\mathcal{X}, \alpha] = p_{\mathbb{T}}[\mathcal{X}_0, \alpha_0]$ where, either $(\mathcal{X}_0, \alpha_0) = (\mathcal{X}^{\perp}{}_{\alpha}, \alpha^{\perp})$ if the latter is admissible, or $(\mathcal{X}_0, \alpha_0) = (\mathcal{C}_{\mathbf{2}}, \epsilon_c)$ if $(\mathcal{X}, \alpha) \sim \mathcal{L}^k(\epsilon)$. □

Theorem 7.47. *The EM category* $\mathsf{GS}^{\mathbb{DT}}$ *of algebras for the composite* $\mathbb{DT}$ *is canonically isomorphic to* MO.

Proof. A $\mathbb{DT}$-algebra (A, h) induces canonical $\mathbb{D}$- and $\mathbb{T}$-structure morphisms

$$h_{\mathbb{D}} \overset{\text{def}}{=} h \circ D\eta^{\mathbb{T}}A : DA \to A, \qquad \text{and} \qquad h_{\mathbb{T}} \overset{\text{def}}{=} h \circ D\eta^{\mathbb{D}}A : TA \to A$$

([4, Proposition, Section 2]). In particular, since $\eta^{\mathbb{D}}$ is just an inclusion, $h_{\mathbb{T}} = h|_{TA} : TA \to A$ is the restriction to $TA \subset DTA$, and by Proposition 5.30, A is equipped with a multiplication $\diamond = h \circ [\mathcal{M}(-,-)]$ and contraction $\zeta = h \circ [\mathcal{N}(-)]$, so that $(A, \diamond, \zeta)$ is a non-unital modular operad.

It remains to show that ϵ provides a unit for the multiplication $\diamond$:

By the monad algebra axioms 2.3, 2.4, there are commuting diagrams

(7.48)
$$\begin{tikzcd} A \arrow[r, "\eta^{\mathbb{D}}\eta^{\mathbb{T}}A"] \arrow[dr, equal] & DTA \arrow[d, "h"] \\ & A, \end{tikzcd}$$

(7.49)
$$\begin{tikzcd} (DT)^2A \arrow[r, "D\lambda TA"] \arrow[d, "DTh"'] & D^2T^2A \arrow[r, "\mu^{\mathbb{D}}\mu^{\mathbb{T}}A"] & DTA \arrow[d, "h"] \\ DTA \arrow[rr, "h"'] & & A. \end{tikzcd}$$

For all $\underline{c} \in \mathfrak{C}^X$, and all $\phi \in A_{\underline{c},c}$, the image of the element $[\mathcal{M}_c(\eta^{\mathbb{T}}(\phi), \epsilon_c^{DTA})] \in (TD)^2A_X$ under the top-right path in Equation (7.49) is just $\phi \in A_X$, since the application of $D\lambda TA$ deletes the unit ϵ_c^{DTA}.

Now $DTh[\mathcal{M}_c(\eta^{\mathbb{T}}(\phi), \epsilon_c^{DTA})] = [\mathcal{M}_c(\phi, \epsilon_c^{DTA})] \in DTA_X$ and diagram 7.49 commutes so $\phi \diamond_c \epsilon_c = h[\mathcal{M}_c(\phi, \epsilon_c)] = \phi$ and ϵ is a unit for $\diamond$ as required.

Conversely, a modular operad $(S, \diamond, \zeta, \epsilon)$ has underlying pointed graphical species $(S, \epsilon, \zeta\epsilon)$, and $(S, \diamond, \zeta)$ is an unpointed modular operad. So, let $\mathbb{D}$- and $\mathbb{T}$-algebra structures $p_{\mathbb{D}} : DS \to S$ and $p_{\mathbb{T}} : TS \to S$ be the corresponding structure maps. In particular, $p_{\mathbb{T}}$ satisfies

$$\diamond = p_{\mathbb{T}} \ \circ [\mathcal{M}(\cdot,\cdot)], \quad \text{and} \quad \zeta = p_{\mathbb{T}} \ \circ [\mathcal{N}(\cdot)].$$

Since ϵ is a unit for $\diamond$, for all $c \in S_{\S}$,

$$p_{\mathbb{T}}[\mathcal{N}(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)]$$

by Lemma 7.46, so $o_{\tilde{c}} \overset{\text{def}}{=} \zeta\epsilon_c = p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)]$ for all $c \in \mathfrak{C}$, $m \geq 1$.

Then $p : DTS \to S$ may be defined by the restriction $p|_{TS_X} = p_{\mathbb{T}} : TS \to S$, together with

$$p(\epsilon^{DTS}) = \epsilon : \mathfrak{C} \to S_{\mathbf{2}}, \text{ and } p(o^{DTS}) = \zeta\epsilon : \mathfrak{C} \to S_{\mathbf{0}},$$

making Diagram 7.48 commute.

It remains to check that 7.49 commutes for (S, p). This is clear for the adjoined (contracted) units $\epsilon^{(TDTS)^+}$, and $o^{(TDTS)^+}$ for $(DT)^2S$. So we must check that the restriction

(7.50)
$$\begin{tikzcd} TDTS \arrow[r, "D\lambda TS"] \arrow[d, "DTp"'] & D^2T^2S \arrow[r, "\mu^{\mathbb{D}}\mu^{\mathbb{T}}S"] & DTS \arrow[d, "p"] \\ DTS \arrow[rr, "p"'] & & S \end{tikzcd}$$

commutes.

To this end, let $[\mathcal{X}, \beta] \in TDTS_X$. Exactly one of the following four conditions holds:

(i) $X = \mathbf{0}$ and $[\mathcal{X}, \beta] = [\mathcal{C}_\mathbf{0}, o^{DTS}_{\tilde{c}}]$, in which case its image under both paths in Diagram 7.50 is $o_{\tilde{c}}$,

(ii) $X = \mathbf{0}$ and $[\mathcal{X}, \beta] = [\mathcal{W}^m(\epsilon^{DTS}_c)]$ for some $m \geq 1$, and $c \in \mathfrak{C}$.

Then the application of λTS in the top-right path means that this path takes $[\mathcal{W}^m(\epsilon^{DTS}_c)]$ to $o_{\tilde{c}} \in S_\mathbf{0}$, whereas the bottom left path takes $[\mathcal{W}^m(\epsilon^{DTS}_c)]$ first to $[\mathcal{W}^m(\epsilon_c)] \in TS_\mathbf{0}$ by applying p inside, and then to

$$p[\mathcal{W}^m(\epsilon_c)] = p[\mathcal{W}(\epsilon_c)] = p_\mathbb{T}[\mathcal{N}(\epsilon_c)] = \zeta\epsilon_c = o_{\tilde{c}}$$

by an application of Lemma 7.46 and the definition of the contraction o

(iii) $X = \mathbf{2}$ and $[\mathcal{X}, \beta] = [\mathcal{L}^k(\epsilon^{DTS}_c)]$ for some $k \geq 1$, and $c \in \mathfrak{C}$.

Then, once again, the application of λTS in the top-right path means that this path takes $[\mathcal{L}^k(\epsilon^{DTS}_c)]$ to $\epsilon_c \in S_\mathbf{2}$, whereas the bottom left path takes $[\mathcal{L}^k(\epsilon^{DTS}_c)]$ first to $[\mathcal{L}^k(\epsilon_c)] \in TS_\mathbf{2}$ by applying p inside, and then to

$$p[\mathcal{L}^k(\epsilon_c)] = p_\mathbb{T}[\mathcal{L}^k(\epsilon_c)] = \epsilon_c$$

by an application of Lemma 7.46.

(iv) Otherwise, a representative of the reduced similar strucutre $(\mathcal{X}^\perp_\beta, \beta^\perp)$ is admissible in $DTS(\mathcal{X})$. This means precisely that $[\mathcal{X}^\perp_\beta, \beta^\perp] \in T^2S_X$.

Then $\lambda T[\mathcal{X}, \beta] = \lambda T[\mathcal{X}^\perp_\beta, \beta^\perp] = [\mathcal{X}^\perp_\beta, \beta^\perp] \in TS_X$. Hence $Tp[\mathcal{X}, \beta] = Tp_\mathbb{T}[\mathcal{X}^\perp_\beta, \beta^\perp]$, and so Diagram 7.49 commutes.

Therefore, (S, p) naturally admits the structure of a $\mathbb{DT}$-algebra. It is straightforward to verify that the functors $\mathsf{MO} \leftrightarrows \mathsf{GS}^{\mathbb{DT}}$ so defined are each others' inverses. □

Remark 7.51. There is also a distributive law in the other direction $DT \Rightarrow TD$. Algebras for the composite monad $\mathbb{TD}$ are just the cofibred coproducts of algebras for $\mathbb{D}$ and $\mathbb{T}$. There is no further relationship between the two structures.

8. A nerve theorem

The goal of this section is to prove the following nerve theorem for modular operads using the abstract machinery described in Section 2.

Theorem 8.1. *The functor $N : \mathsf{MO} \to \mathsf{ps}(\Xi)$ is full and faithful. Its essential image consists of precisely those presheaves P on Ξ whose restriction to $\mathsf{ps}(\Xi)$ is a graphical species. In other words,*

(8.2) $$P(\mathcal{G}) = lim_{(\mathcal{C},b) \in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \text{ for all graphs } \mathcal{G}.$$

Remark 8.3. A version of this theorem was stated in [23], and another version was proved, by different methods in [21, Theorem 3.8]. In [35], I proved this theorem by essentially the same methods, but without the use of the distributive law. In all these versions, the statement of the Segal condition is the same.

By Section 7, there is a commuting diagram of functors

(8.4)
$$\begin{tikzcd}
& \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}^{\mathbb{T}_*}"] & \mathsf{ps}(\Xi) \arrow[d, "j^*"] \\
\mathbb{P}^{\circlearrowleft}_* \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr}_* \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS}_* \arrow[u, shift left, "\text{free}^{\mathbb{T}_*}"] \arrow[r, hook, "\text{f.f.}"'] \arrow[d, shift left, "\text{forget}^{\mathbb{D}}"] & \mathsf{ps}(\mathsf{Gr}_*) \arrow[d] \\
\mathbb{P}^{\circlearrowleft} \arrow[u, "\text{b.o.}"] \arrow[ur, "\text{dense}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr} \arrow[u, "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}^{\mathbb{D}}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{ps}(\mathsf{Gr}).
\end{tikzcd}$$

where Ξ is the category obtained the bo-ff factorisation $\mathsf{Gr} \to \mathsf{GS} \to \mathsf{MO}$, *and* also in the bo-ff factorisation of $\mathsf{Gr}_* \to \mathsf{GS}_* \to \mathsf{MO}$.

As discussed in Section 2, the Theorem 8.1 follows immediately if the monad $\mathbb{DT}$ on GS has arities Gr. Unfortunately, this is not the case. The obstruction, unsurprisingly, relates to the contracted units. (See Remark 8.14.)

However, by Propositions 7.41 and 7.44, the nerve $N : \mathsf{MO} \to \mathsf{ps}(\Xi)$ is fully faithful if the monad $\mathbb{T}_*$ has arities Gr_*. And, because $\mathbb{P}^{\circlearrowleft}$ is dense in Gr_*, the essential image of N is charactersised by Ξ-presheaves P that satisfy the Segal condition 8.2.

The remainder of this work is devoted to showing that $\mathbb{T}_*$ has arities Gr_* (see [6, Definition 1.8]).

The first step is to study the graphical category Ξ in more detail.

8.1. **The category Ξ.** By construction, $\Xi \subset \mathsf{MO}$ is the full subcategory of the free modular operads $\Xi(\mathcal{G})$ on connected graphs $\mathcal{G} \in \mathsf{Gr}$.

Example 8.5 (The free modular operad on a graph). Let $\mathcal{H} = (E, H, V, s, t, \tau)$ be a graph. To streamline the notation, we'll write $T\mathcal{H} \overset{\text{def}}{=} TY\mathcal{H}$ for the free non-unital modular operad on $\mathcal{H}$, and $T_*\mathcal{H} \overset{\text{def}}{=} T_*Y_*\mathcal{H}$ for the corresponding free unital modular operad on $\mathcal{H}$.

Of course, $T_*\mathcal{H}(\shortmid) = \{ch_e\}_{e\in E} = \mathsf{Gr}_*(\shortmid, \mathcal{H})$ with involution $\tau : ch_e \mapsto ch_{\tau e}$.

Recall that the unit for $Y_*\mathcal{H}$ is given by $ch_e \mapsto \epsilon^{\mathcal{H}}_e \overset{\text{def}}{=} ch_e \circ u \in \mathsf{Gr}_*(\mathcal{C}_{\mathbf{2}}, \mathcal{H})$, and the contracted unit for $Y_*\mathcal{H}$ is given by $ch_e \mapsto o^{\mathcal{H}}_{\tilde{e}} \overset{\text{def}}{=} ch_e \circ z \in \mathsf{Gr}_*(\mathcal{C}_{\mathbf{0}}, \mathcal{H})$.

So, $T_*\mathcal{H}$ has (contracted) units

$$ch_e \mapsto \epsilon^{T_*\mathcal{H}}_e \overset{\text{def}}{=} [\eta^{\mathbb{T}}\epsilon^{\mathcal{H}}_e]_* = [ch_e \circ u^k]$$

and

$$ch_e \mapsto o^{T_*\mathcal{H}}_{\tilde{e}} \overset{\text{def}}{=} [\eta^{\mathbb{T}} o^{\mathcal{H}}_e]_* = [ch_e \circ z] = [ch_e \circ \kappa^m].$$

In other words, $\epsilon^{T_*\mathcal{H}}_e$ is represented by morphisms of the form $ch_e \circ u^k \in \mathsf{Gr}_*(\mathcal{L}^k, \mathcal{H})$ for $k \geq 1$, and $o^{T_*\mathcal{H}}_{\tilde{e}}$ is represented by $ch_e \circ z \in \mathsf{Gr}_*(\mathcal{C}_{\mathbf{0}}, \mathcal{H})$, and $ch_e \circ \kappa^m \in \mathsf{Gr}_*(\mathcal{W}^m, \mathcal{H})$.

For X a finite set, a elements of $T_*\mathcal{H}_X$ are represented by pairs $(\mathcal{X}, f)$ where $\mathcal{X} = (\mathcal{G}, \rho)$ is an X-graph and $f \in \mathsf{Gr}_*(\mathcal{G}, \mathcal{H})$. By Corollary 7.44, pairs $(\mathcal{X}^1, f^1)$ and $(\mathcal{X}^1, f^2)$ represent the same element $[\mathcal{X}, f]_* \in T_*\mathcal{H}_X$ if and only if there is a commuting diagram

$$\begin{tikzcd}
\mathcal{X}^1 \arrow[r, "g^1"] \arrow[dr, "f^1"'] & \tilde{\mathcal{X}} \arrow[d, "f"] & \mathcal{X}^2 \arrow[l, "g^2"'] \arrow[dl, "f^2"] \\
& \mathcal{H} &
\end{tikzcd} \tag{8.6}$$

in Gr_* such that g^j are morphisms in $X\mathsf{Gr}_{sim_*}$ for $j = 1, 2$, and $f : \tilde{\mathcal{X}} \to \mathcal{H}$ is an (unpointed) étale morphism in Gr.

In particular, every object in $T_*\mathcal{H}_X$ has a representative of the form $(\mathcal{X}, f')$ where $\mathcal{X}$ is an X-graph (so has non-empty vertex set), and $f \in \mathsf{Gr}_*(\mathcal{X}, \mathcal{H})$. Moreover, outside the (contracted) units, we can always choose an an admissible reduced representative $(\mathcal{X}, f)$ with $f \in \mathsf{Gr}(\mathcal{X}, H)$. The (reduced) morphism $f : \tilde{\mathcal{X}} \to \mathcal{H}$ is of the form ch_e, and therefore not admissible, if and only if $[\mathcal{X}, f]_*$ is a (contracted) unit in T_*S_X.

Observe, in particular, that the following special case of 8.6 commutes in Gr_* for all $e \in E$:

(8.7)
$$\begin{tikzcd} \mathcal{C}_{\mathbf{0}} \arrow[r, "z"] \arrow[dr, "ch_e \circ z"'] & (\shortmid) \arrow[d, "ch_e"] & \mathcal{W}^m \arrow[l, "\kappa"'] \arrow[dl, "ch_e \circ \kappa"] \\ & \mathcal{H}. & \end{tikzcd}$$

This observation is essential in the proof of Theorem 8.1.

By definition, Ξ is the restriction to Gr_* of the Kleisli category of $\mathbb{T}_*$ so

$$\Xi(\mathcal{G}, \mathcal{H}) = \mathsf{GS}_*(\mathcal{G}, T_*\mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G})$$

for all pairs $(\mathcal{G}, \mathcal{H})$ of graphs. Therefore, $\Xi(\mathcal{G}, \mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G})$ has been described in Example 8.5 in case $\mathcal{G}$ is elementary.

For the general case, it follows from Example 8.5 that, since $\Xi(\mathcal{G}, \mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G}) = lim_{(\mathcal{C},b)\in \mathsf{el}_*(\mathcal{G})} T_*(\mathcal{H})(\mathcal{C})$ and $T_*\mathcal{H}$ is a quotient of $T\mathcal{H}$, $\Xi(\mathcal{G}, \mathcal{H})$ is obtained as a quotient of $T\mathcal{H}(\mathcal{G})$. Therefore, a morphism $\gamma \in \Xi(\mathcal{G}, \mathcal{H})$ is represented by a non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$, together with a morphism f in Gr_* from the colimit $\mathbf{\Gamma}(\mathcal{G}) = colim_{\mathsf{el}_*(\mathcal{G})}\mathbf{\Gamma}$, to $\mathcal{H}$.

Recall that every graph $\mathcal{G}$ is trivially the colimit of the identity $\mathcal{G}$-shaped graph of graphs $(\mathcal{C}, b) \mapsto \mathcal{C}$. So the assignment $f \mapsto (id, f) \in \Xi(\mathcal{G}, \mathcal{H})$ induces an inclusion $\mathsf{Gr}_* \hookrightarrow \Xi$.

The following terminology is from [27].

Definition 8.8. *A* (pointed) free map in Ξ *is a morphism in the image of the inclusion* $\mathsf{Gr}_* \hookrightarrow \Xi$. *An* unpointed free map *in* Ξ *is a morphism in the image of* $\mathsf{Gr} \hookrightarrow \mathsf{Gr}_* \hookrightarrow \Xi$. *A* refinement in Ξ *is a morphism in* $\Xi(\mathcal{G}, \mathcal{H})$ *with a representative of the form* $(\mathbf{\Gamma}, id_{\mathcal{H}})$.

So, in particular, a refinement in $\Xi(\mathcal{G}, \mathcal{H})$ is described by a non-degenerate $\mathcal{G}$-shaped graph of graphs $\mathbf{\Gamma}$ with colimit $\mathcal{H}$, and hence $\mathbf{\Gamma}$ induces an identity on boundaries.

Let $\mathcal{G} \ncong \mathcal{C}_{\mathbf{0}}$ and $\mathcal{H}$ be graphs and, for $i = 1, 2$, let $(\mathbf{\Gamma}^i, f^i)$ be a pair with $\mathbf{\Gamma}^i$ a non-degenerate $\mathcal{G}$-shaped graph of graphs, and $f \in \mathsf{Gr}_*(\mathbf{\Gamma}(\mathcal{G}), \mathcal{H})$ where $\mathbf{\Gamma}^i(\mathcal{G})$ is the colimit of $\mathbf{\Gamma}^i$ in Gr.

Lemma 8.9. *Two such pairs* $(\mathbf{\Gamma}^1, f^1), (\mathbf{\Gamma}^2, f^2)$ *represent the same element* α *of* $\Xi(\mathcal{G}, \mathcal{H})$ *if and only if there is representative* $(\mathbf{\Gamma}, f)$ *of* α*, and a commuting diagram in* Gr_**, where morphisms in the top row are vertex deletion morphisms:*

(8.10)
$$\begin{tikzcd} \mathbf{\Gamma}^1(\mathcal{G}) \arrow[r] \arrow[dr, "f^1"'] & \mathbf{\Gamma}(\mathcal{G}) \arrow[d, "f"] & \mathbf{\Gamma}^2(\mathcal{G}) \arrow[l] \arrow[dl, "f^2"] \\ & \mathcal{H}. & \end{tikzcd}$$

Proof. By definition, if two such pairs $(\mathbf{\Gamma}^1, f^1), (\mathbf{\Gamma}^2, f^2)$ represent the same element, then, for all $(\mathcal{C}_{X_b}, b) \in \mathsf{el}(\mathcal{G})$, $(\mathbf{\Gamma}^1(b), f^1 \circ b)$ and $(\mathbf{\Gamma}^2(b), f^2 \circ b)$ are similar in $X_b\mathsf{Gr}_{sim_*}(Y_*\mathcal{H}) \subset \mathsf{Gr}_*|\mathcal{H}$.

Since $\mathbf{\Gamma}^1(b)$ and $\mathbf{\Gamma}^2(b)$ are similar in Gr_* and $\mathcal{G} \ncong \mathcal{C}_{\mathbf{0}}$, there is, for all $(\mathcal{C}_{X_b}, b)$ in $\mathsf{el}(\mathcal{G})$, a pair $(\mathbf{\Gamma}(b), f_b) \in \mathsf{Gr}_*|\mathcal{H}$ *where* $\mathbf{\Gamma}(b)$ *is admissible* and a cospan

$$\begin{tikzcd} \mathbf{\Gamma}^1(b) \arrow[r] \arrow[dr, "f^1 \circ b"'] & \mathbf{\Gamma}(b) \arrow[d, "f_b"] & \mathbf{\Gamma}^2(b \arrow[l] \arrow[dl, "f^2 \circ b"] \\ & \mathcal{H}. & \end{tikzcd}$$

of vertex deletion morphisms above $\mathcal{H}$. Namely, we can choose $\mathbf{\Gamma}(b) = (\mathbf{\Gamma}^i(b))^{\perp}$ if this is admissible. Otherwise, if $(\mathbf{\Gamma}^i(b))^{\perp}$ is not admissable, then $\mathbf{\Gamma}^i \cong \mathcal{L}^{k_i}$, so we can take $\mathbf{\Gamma}_b = \mathcal{C}_{\mathbf{2}}$.

So, the $\mathcal{G}$-shaped graph of $T_*\mathcal{H}$-sructured graphs $\mathbf{\Gamma}_{T_*\mathcal{H}} : \mathsf{el}(\mathcal{G}) \to \mathsf{Gr}(T_*\mathcal{H}), b \mapsto (\mathbf{\Gamma}(b), f_b)$ so obtained is non-degenerate and hence admits a colimit $(\mathbf{\Gamma}(\mathcal{G}), f)$ in $\mathsf{Gr}(T_*\mathcal{H})$such that 8.10 commutes.

The converse follows immediately from the definitions. □

From the discussion above it follows that, for all $\mathcal{G}, \mathcal{H} \in ob(\mathsf{Gr}_*)$ and for all morphisms $\alpha \in \Xi(\mathcal{G}, \mathcal{H})$, α factors as a refinement followed by a free map. Since morphisms in Gr_* also factor as vertex deletion morphisms (or $z : \mathcal{C}_{\mathbf{0}} \to (\shortmid)$) followed by a morphism in Gr, the composite structure of the monad $\mathbb{DT}$ induces a ternary factorisation system on Ξ.

8.2. **Factorisation categories.** More generally, let $\mathsf{GS}_{*\mathbb{T}_*}$ be the Kleisli category of T_*. So, $\mathsf{GS}_{*\mathbb{T}_*}(S_*, S'_*) = \mathsf{GS}_*(S_*, T_*S'_*)$ for all $S_*, S'_* \in \mathsf{GS}_*$.

Elements of T_*S_X correspond to similarity classes of S_*-structured X-graphs $(\mathcal{X}, \alpha)$. So an element $\beta \in \mathsf{GS}_*(Y_*\mathcal{G}, T_*S_*) \cong T_*S_*(\mathcal{G})$ is represented by a non-degenerate $\mathcal{X}$-shaped graph of S-structured graphs $\mathbf{\Gamma}_S$. The colimit of $\mathbf{\Gamma}_S$ describes an S_* structured X-graph $(\mathbf{\Gamma}(\mathcal{X}), \alpha^{\mathbf{\Gamma}})$, where $\mathbf{\Gamma}(\mathcal{X})$ is the colimit of the underlying $\mathcal{X}$-shaped graph of graphs $\mathbf{\Gamma} : \mathsf{el}(\mathcal{X}) \to \mathsf{Gr}(S_*) \to \mathsf{Gr}$.

By [6, Proposition 2.5], the monad $\mathbb{T}_*$ has arities Gr_* if certain categories associated to factorisations of morphisms to free $\mathbb{T}_*$ algebras are connected.

Let $\mathcal{G}$ be a graph, S_* be a pointed graphical species, and let $\beta \in \mathsf{GS}_*(Y_*\mathcal{G}, T_*S)$. Following [6, Section 2.4],

Definition 8.11. *The* factorisation category $\mathsf{fact}(\beta)$ of β *is the category whose objects are pairs* $(\mathbf{\Gamma}, \alpha)$ *where* $\mathbf{\Gamma}$ *is a non-degenerate* $\mathcal{G}$*-shaped graph of graphs with colimit* $\mathbf{\Gamma}(\mathcal{G})$ *and* $\alpha \in S(\mathbf{\Gamma}(\mathcal{G}))$ *is such that* β *is equal to the composition in* GS_**:*

$$Y_*\mathcal{G} \xrightarrow{\quad \mathbf{\Gamma} \quad} Y_*\mathbf{\Gamma}(\mathcal{G}) \xrightarrow{\quad \alpha \quad} S_*.$$

Morphisms in $\mathsf{fact}(\beta)((\mathbf{\Gamma}^1, \alpha^1), (\mathbf{\Gamma}^2, \alpha^2))$ *are commuting diagrams*

(8.12)
$$\begin{tikzcd}
 & \mathbf{\Gamma}^1(\mathcal{G}) \arrow[dd, "g"] \arrow[dr, "\alpha^1"] & \\
\mathcal{G} \arrow[ur] \arrow[dr] & & S_* \\
 & \mathbf{\Gamma}^2(\mathcal{G}) \arrow[ur, "\alpha^2"'] &
\end{tikzcd}$$

in $\mathsf{GS}_{*\mathbb{T}_*}$*, where* g *is a morphism in* $\mathsf{Gr}_*(\mathbf{\Gamma}^1(\mathcal{G}), \mathbf{\Gamma}^2(\mathcal{G}))$*.*

Lemma 8.13. *For all* $\mathcal{G} \in ob(\mathsf{Gr}_*)$ *and all* $\beta \in \mathsf{GS}_*(\mathcal{G}, T_*S)$*, the category* $\mathsf{fact}(\beta)$ *is connected.*

Proof. This follows directly from the discussion above, and in particular Example 8.5.

Let S_* be pointed graphical species. For X is a finite set, S_*-structured X-graphs $(\mathcal{X}^1, \alpha^1), (\mathcal{X}^2, \alpha^2)$ represent the same element of T_*S_X if and only if they are similar in $X\mathsf{Gr}_{sim_*}(S_*) \cong \mathsf{GS}_*(\mathcal{C}_X, T_*S_*)$. To the lemma holds for elementary graphs.

For general $\mathcal{G} \in ob(\mathsf{Gr}_*)$, we may assume that $\mathcal{G} \not\cong \mathcal{C}_{\mathbf{0}}$. Then elements of $\mathsf{GS}_*(Y_*\mathcal{G}, T_*S) \cong T_*S(\mathcal{G})$ are represented by non-degenerate $\mathcal{G}$-shaped graphs of S_*-structured graphs. Since there is no object of the form $(\mathcal{C}_{\mathbf{0}}, b)$ in $\mathsf{el}(\mathcal{G})$, two such S-structured graphs of graphs, $\mathbf{\Gamma}^1_{S_*}, \mathbf{\Gamma}^2_{S_*}$ represent the same element of $T_*S(\mathcal{G})$ if and only if

$$\mathbf{\Gamma}^1_{S_*}(\mathcal{C}_X, b) \sim \mathbf{\Gamma}^2_{S_*}(\mathcal{C}_X, b) \in X\mathsf{Gr}_{sim_*}(S)$$

for all $(\mathcal{C}_X, b) \in \mathsf{el}(\mathcal{G})$ and hence their colimits are also similar. In other words, $\mathsf{fact}(\beta)$ is connected. □

Theorem 8.1 now follows from [6, Sections 1,2].

Proof of Theorem 8.1. The category Gr_* is dense in GS_*. By [6, Proposition 2.5], the statement of Lemma 8.13 is equivalent to the statement that the monad $\mathbb{T}_*$ has arities Gr_*.Therefore, the monad $\mathbb{T}_*$ on GS_* has arities Gr_*.

Hence the induced nerve functor $N : \mathsf{MO} \to \mathsf{ps}(\Xi)$ is fully faithful by [6, Propositions 1.5 and 1.9]. Moreover, by [6, Theorem 1.10] its essential image is the subcategory of those presheaves on Ξ whose restriction to Gr_* are in the image of the fully faithful embedding $\mathsf{GS}_* \hookrightarrow \mathsf{ps}(\mathsf{Gr}_*)$. In other words, the essential image of the inclusion $\mathsf{MO} \hookrightarrow \mathsf{ps}(\Xi)$ consists of precisely those presheaves whose restriction to Gr_* is a sheaf on (Gr_*, J_*).

So $S \in ob(\mathsf{ps}(\Xi))$ is in the image of N if and only if

$$S(\mathcal{G}) = lim_{(\mathcal{C},b)\in \mathsf{el}_*(\mathcal{G})} S(\mathcal{C}),$$

and, by finality of $\mathsf{el}(\mathcal{G}) \subset \mathsf{el}_*(\mathcal{G})$, this is the case precisely if the Segal condition 8.2 is satisfied. $\square$

Remark 8.14. Using the method of [6, Section 2], we can construct the corresponding *unpointed* factorisation categories for the monad $\mathbb{DT}$ on GS.

Let S be a graphical species and $\mathcal{G}$ a graph. We consider morphisms $Y\mathcal{G} \to S$ in the Kleisli category $\mathsf{GS}_{\mathbb{DT}} \xrightarrow{\text{f.f.}} \mathsf{MO}$. Since

$$\mathsf{GS}_{\mathbb{DT}}(\mathcal{G}, S) \cong \mathsf{MO}(DTY\mathcal{G}) \cong \mathsf{GS}_{*\mathbb{T}_*}(Y_*\mathcal{G}, T_*S^+) \cong S^+(\mathcal{G}),$$

a morphism $\beta \in \mathsf{GS}_{\mathbb{DT}}(\mathcal{G}, S)$ is represented by a $\mathcal{G}$-indexed graph of graphs $\mathbf{\Gamma}$ with colimit $\mathbf{\Gamma}(\mathcal{G})$ and a DS-structure $\alpha \in DS(\mathbf{\Gamma}(\mathcal{G}))$.

Such pairs $(\mathbf{\Gamma}, \alpha)$ are the objects of the unpointed factorisation category $\mathsf{fact}(\beta)^-$. Morphisms in $f \in \mathsf{fact}(\beta)^-((\mathbf{\Gamma}, \alpha), (\mathbf{\Gamma}', \alpha'))$ are morphisms in $\mathsf{Gr}(\mathbf{\Gamma}(\mathcal{G}), \mathbf{\Gamma}(\mathcal{G}'))$ making the diagram Equation (8.12) commute.

But let $S = Y(\shortmid)$, so $TS \cong S$ and take $\beta = z = o^{(Y(\shortmid))^+} : \mathcal{C}_\mathbf{0} \to (\shortmid)$. Then $\mathsf{fact}(\beta)^-$ has objects given by $\mathcal{C}_\mathbf{0} \to \mathcal{C}_\mathbf{0} \xrightarrow{z} (\shortmid)$ and $\mathcal{C}_\mathbf{0} \to \mathcal{W} \xrightarrow{\kappa} (\shortmid)$, and since $\mathcal{C}_\mathbf{0}$ is disjoint in Gr, these are disjoint in $\mathsf{fact}(\beta)^-$.

Therefore, by [6, Proposition 2.5], $\mathbb{DT}$ does not have arities Gr.

8.3. **Weak modular operads and related results.** In [21, 22] Hackney, Robertson and Yau have recently proved a version of Theorem 8.1 in terms of a bijective on objects subcategory $U \hookrightarrow \Xi$, constructed precisely so as to have a generalised Reedy structure ([21, Theorem 3.8]). They then localise the Reedy model structure on $\mathsf{Set}^{\Delta^{op}}$-valued presheaves on U at the *Segal maps* $lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \hookrightarrow P(\mathcal{G})$ in order to obtain a model structure in which the fibrant objects are those simplicial presheaves on U satisfying the *weak Segal condition*

$$P(\mathcal{G}) \simeq lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \text{ for all graphs } \mathcal{G}. \tag{8.15}$$

In order to obtain the Reedy structure, the surjections in $\mathsf{Gr} \hookrightarrow \Xi$ that are not isomorphisms (see Section 4.3) are excluded from U as well as all morphisms to $(\shortmid)$ that do not preserve ports (so are of the form $z : \mathcal{C}_\mathbf{0} \to (\shortmid)$ or $\kappa^m : \mathcal{W}^m \to (\shortmid)$).Hence, though $U \subset \mathsf{MO}$ is dense, and hence induces a fully faithful nerve, it is not itself fully faithful in MO.

In [10], Caviglia and Horel describe a general class of rigidification results whereby, given a dense inclusion $\mathsf{D} \hookrightarrow \mathsf{C}$ of categories satisfying certain conditions, an equivalence is established between sSet-valued presheaves on D that satisfy a weak Segal condition, and C objects internal to sSet that are Segal on the nose. As an application, they apply their result to a certain class of monads with arities. Their result can be applied directly to obtain the following corollary to Theorem 8.1.

Corollary 8.16. *There is a model category structure on the category* $\mathsf{Fun}(\Xi^{op}, \mathsf{sSet})$ *of simplicial presheaves on* Ξ *whose fibrant objects are presheaves* P *satisfying the week Segal condition*

$$P(\mathcal{G}) \simeq lim_{(\mathcal{C},b)\in \mathsf{el}(\mathcal{G})} P(\mathcal{C}) \quad \textit{for all } \mathcal{G} \in \mathsf{Gr}. \tag{8.17}$$

Proof. The monad $\mathbb{T}_*$ has arities Gr_* and $\mathbb{P}^{\circlearrowleft}|\mathcal{G}$ is small and connected for all connected graphs $\mathcal{G}$. Therefore assumptions 7.9 [10] are satisfied. By [10, Section 7.5], MO is equivalent to the category of models in Set for the limit sketch $L = (\mathsf{Gr}_*, \{(\mathcal{G}|\mathbb{P}^{\circlearrowleft op})_{\mathcal{G}\in\mathsf{Gr}}\})$ and there is a Segal model structure on the category of sSet valued models for L

Finally, by [10, Proposition 7.1], this can be transferred along a Quillen equivalence to a model structure on $\mathsf{Fun}(\Xi^{op}, \mathsf{sSet})$, whose fibrant objects are those presheaves that satisfy the weak Segal condition. □

In current work with Marcy Robertson, we are comparing the existing models for weak modular operads. It is expected that there is a direct Quillen equivalence between the model structure on $\mathsf{Fun}(\Xi^{op}, \mathsf{sSet})$ of Corollary 8.16 and the model structure on $\mathsf{Fun}(U^{op}, \mathsf{sSet})$ of [21]. I'm also investigating whether there is a Reedy structure on Ξ.

Remark 8.18. The graphical category $\overline{Gr}$ whose morphisms are described in [23, Section 6] (and discussed in Remark Remark 8.18, and the footnote on page 64) is the wide subcategory of Ξ that does not contain the morphisms $z : \mathcal{C}_{\mathbf{0}} \to (\shortmid)$ or any morphisms in $\mathsf{Gr}_*(\mathcal{W}^m, \mathcal{G})$ that factor through some $ch_e \in \mathsf{Gr}_*(\shortmid, \mathcal{G})$. Therefore, $\overline{Gr}$ does not embed fully faithfully in MO.

However, $U \subset \overline{Gr} \subset \Xi$ so $\overline{Gr}$ is also dense in MO, and yields a fully faithful nerve functor whose essential image satisfies the same Segal condition 8.2, whereby the main statement of [23] is established. See also [22, Theorem 3.6 & Section 4] for details.

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