

GRAPHICAL COMBINATORICS AND A DISTRIBUTIVE LAW FOR MODULAR OPERADS

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ABSTRACT. This work presents a detailed analysis of the combinatorics of modular operads. These are operad-like structures that admit a contraction operation as well as an operadic multiplication. Their combinatorics are governed by graphs that admit cycles, and are known for their complexity. In 2011, Joyal and Kock introduced a powerful graphical formalism for modular operads. This paper extends that work. A monad for modular operads is constructed and a corresponding nerve theorem is proved, using Weber’s abstract nerve theory, in the terms originally stated by Joyal and Kock. This is achieved using a distributive law that sheds new light on the combinatorics of modular operads.

INTRODUCTION

Modular operads, introduced in [16] to study moduli spaces of Riemann surfaces, are a “‘higher genus’ analogue of operads . . . in which graphs replace trees in the definition.” [16, Abstract].

Roughly speaking, modular operads are \mathbb{N} -graded objects $P = \{P(n)\}_{n \in \mathbb{N}}$ that, alongside an operadic multiplication (or composition) $\circ: P(n) \times P(m) \rightarrow P(m + n - 2)$ for $m, n \geq 1$, admit a contraction operation $\zeta: P(n) \rightarrow P(n - 2)$, $n \geq 2$. For example, as in Figure 1, we may multiply two oriented surfaces by gluing them along chosen boundary components, or contract a single surface by gluing together two distinct boundary components.



FIGURE 1. Gluing (multiplication) and self-gluing (contraction) of surfaces along boundary components. Moduli of geometric structures – such as Riemann surfaces – provide many examples of modular operads.

This work considers a notion of modular operads originally due to Joyal and Kock [23],¹ that incorporates a broad compass of related structures, including modular operads in the original sense of [16] and their coloured counterparts [17], but also wheeled properads [18, 43] (see Example 1.30). More generally, compact closed categories [27] also provide examples of modular operads [37, 38] (see Example 1.28). These are closely related to circuit algebras that are used in the study of finite-type knot invariants [1, 13] (see Example 1.29). As such, modular operads have applications across a range of disciplines.

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¹Joyal and Kock used the term ‘compact symmetric multicategories (CSMs)’ in [23] to refer to what are here called ‘modular operads’. Indeed, I adopted this terminology in a previous version of this paper and in [35].

However, the combinatorics of modular operads are complex. In modular operads equipped with a multiplicative unit, contracting this unit leads to an exceptional ‘loop’, that can obstruct the proof of general results. This paper undertakes a detailed investigation into the graphical combinatorics of modular operads, and provides a new approach to understanding these loops.

In [23], which forms the inspiration for this work, Joyal and Kock construct modular operads as algebras for an endofunctor (Section 6) on a category \mathbf{GS} of coloured collections called ‘graphical species’. Their machinery is significant in its simplicity. It relies only on minimal data and basic categorical constructions, that lend it considerable formal and expressive power.

However, the presence of exceptional loops means that their modular operad endofunctor does not extend to a monad on \mathbf{GS} . As a consequence, it does not lead to a precise description of the relationship between modular operads and their graphical combinatorics. (See Section 6 for technical details.)

This paper contains proofs of the following statements that first appeared in [23] (and were proved – by similar, though slightly less general methods than those presented here – in my PhD thesis [35]):

Theorem 0.1 (Monad existence Theorem 7.47). *The category \mathbf{MO} of modular operads is isomorphic to the Eilenberg-Moore category of algebras for a monad \mathbb{O} on the category \mathbf{GS} of graphical species.*

In particular, \mathbb{O} is the *algebraically free monad* [26] on the endofunctor of [23]. Theorem 0.2 – the ‘nerve theorem’ – characterises modular operads in terms of presheaves on a category Ξ of graphs.

Theorem 0.2 (Nerve Theorem 8.1). *The category \mathbf{MO} has a full subcategory Ξ whose objects are graphs. The induced (nerve) functor N from \mathbf{MO} to the category $\mathbf{ps}(\Xi)$ of presheaves on Ξ is fully faithful.*

There is a canonical (restriction) functor $F^: \mathbf{ps}(\Xi) \rightarrow \mathbf{GS}$, and the essential image of N consists of precisely those presheaves $P: \Xi^{\text{op}} \rightarrow \mathbf{Set}$ that satisfy the so-called ‘Segal condition’:*

*P is in the essential image of N if and only if it is completely determined by the graphical species F^*P .*

An obvious motivation for establishing such results is provided by the study of weak (up-to-homotopy), or $(\infty, 1)$ -modular operads, by weakening the Segal condition of Theorem 0.2. A number of potential applications of such structures are discussed in the introduction to [20]. To this end, Hackney, Robertson and Yau have also recently proved versions of Theorems 0.1 and 0.2, by different methods, and used them to obtain a model of $(\infty, 1)$ -modular operads that are characterised in terms of a weak Segal condition [20, 21].

The aim of this work is to prove Theorems 0.1 and 0.2 in the manner originally proposed by [23] – using the abstract nerve machinery introduced by Weber [42, 5] (see Section 2) – and to use these proofs as a route to a full understanding of the underlying combinatorics, and the contraction of multiplicative units in particular. This places strict requirements on the relationship between the modular operad monad \mathbb{O} and the graphical category Ξ . In fact, to apply the results of [5], the category Ξ must – in a sense that will be made precise in Section 2 – arise naturally from the definition of \mathbb{O} .

Neither the construction of the monad \mathbb{O} for modular operads, nor the proof of Theorem 0.2 is entirely straightforward. First, the method of [23], which is closely related to analogous constructions for operads (Examples 5.1, 6.1, c.f. [18, 28, 32, 34]) does not lead to a well-defined monad. Second, as a consequence of the contracted units, the desired monad, once obtained, does not satisfy the conditions for applying the machinery of [5]. To prove Theorems 0.1 and 0.2, it is therefore necessary to break the problem into smaller pieces, thereby rendering the graphical combinatorics of modular operads completely explicit.

Since the obstruction to obtaining a monad in [23] arises from the combination of the modular operadic contraction operation and the multiplicative units (see Section 6), the approach of this work is to first treat these structures separately – via a monad \mathbb{T} on \mathbf{GS} whose algebras are non-unital modular operads,

and a monad \mathbb{D} on \mathbf{GS} that adjoins distinguished ‘unit’ elements – and then combine them, using the theory of distributive laws [3].

Theorem 0.1 is then a corollary of:

Theorem 0.3 (Proposition 7.41 & Theorem 7.47). *There is a distributive law λ for \mathbb{T} over \mathbb{D} such that the resulting composite monad \mathbb{DT} on \mathbf{GS} is precisely the modular operad monad \mathbb{O} of Theorem 0.1.*

The graphical category Ξ , used to define the modular operadic nerve, arises canonically via the unique fully faithful–bijective on objects factorisation of a functor used in the construction of \mathbb{O} . Therefore, if the monad \mathbb{O} satisfies certain formal conditions – if it ‘has arities’ (see [5]) – then Theorem 0.2 follows from [5, Section 1].

Though the monad \mathbb{O} on \mathbf{GS} does not have arities, the distributive law in Theorem 0.3 implies that there is a monad \mathbb{T}_* , on the category \mathbf{GS}_* of \mathbb{D} -algebras, whose algebras are modular operads. Moreover, Theorem 0.2 follows from:

Lemma 0.4 (Lemma 8.12). *The monad \mathbb{T}_* on \mathbf{GS}_* has arities, and hence satisfies the conditions of [5, Theorem 1.10].*

I conclude this introduction by briefly mentioning three (related) benefits of this abstract approach.

In the first place, the results obtained by this method provide a clear overview of how modular operads fit into the wider framework of operadic structures, and how other general results may be modified to this setting. For example, by Lemma 0.4, \mathbb{T}_* and Ξ satisfy the Assumptions 7.9 of [9], which leads to a suitable notion of weak modular operad via the following corollary:

Corollary 0.5 (Corollary 8.15). *There is a model structure on the category of presheaves in simplicial sets on Ξ . The fibrant objects are precisely those presheaves that satisfy a weak Segal condition.*

Second, since this work makes the combinatorics of modular operads – including the tricky bits – completely explicit, it provides a clear road map for working with and extending the theory.

One fruitful direction for extending this work is to use iterated distributive laws [10] to generalise constructions presented here. In [36], an iterated distributive law is used to construct circuit operads – modular operads with an extra product operation, closely related to small compact closed categories – as algebras for a composite monad on \mathbf{GS} (Example 1.29). Once again, the distributive laws play an important role in describing the corresponding nerve. The approach of [10, Section 3] may also be used to construct higher (or (n, k) -) modular operads. This can be used to give a modular operadic description of extended cobordism categories.

Finally, the complexities of the combinatorics of contractions can provide new insights into the structures they are intended to model. In current work, also together with L. Bonatto, S. Chettih, A. Linton, M. Robertson, N. Wahl, I am using these ideas to explore singular curves in the compactification of moduli spaces of algebraic curves. (See also Example 1.27, and consider Figure 2.)

This work owes its existence to the ideas of A. Joyal and J. Kock and I thank Joachim for taking time to speak with me about it. P. Hackney, M. Robertson and D. Yau’s work has been an invaluable resource. Conversations with Marcy have been particularly helpful. I gratefully acknowledge the anonymous reviewer whose insights have not only improved the paper, but also increased my appreciation of the mathematics.

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Overview of the paper. The opening two sections provide context and background for the rest of the work. An axiomatic definition of modular operads is given in Section 1. Section 2 provides a brief review of Weber’s abstract nerve theory, that provides a framework for the later sections. Both these introductory sections include a number of examples to motivate the constructions that follow.

Section 3 provides a detailed introduction to the (Feynman) graphs of [23], and Section 4 focuses on their étale morphisms. The monad \mathbb{T} for non-unital modular operads is constructed in Section 5.

Section 6 acts as a short intermezzo in which the appearance of exceptional loops in the theory, and why they are problematic in the construction of [23], is explained.

The construction of the monad \mathbb{O} for modular operads happens in Section 7. This is the longest and most important section of the work, and contains most of the new contributions. Finally, Section 8 contains the proof of the Nerve Theorem 0.2, as well as a short discussion on weak modular operads.²

Though considerable background is needed to understand the details of Sections 6-8, I note a couple of key points here, as a guide to the reader:

- Remarks 6.7 and 6.8 refer to other approaches in the literature to the issue of loops. As far as I am aware, the graphical construction presented in this paper (Section 7) is unique in that it *does not* incorporate some version of the exceptional loop into the graphical calculus, in order to model contractions of units. (Remark 6.9.)
- The approaches referenced in Remarks 6.7 and 6.8 describe contraction of units by adjoining a formal colimit of a diagram $id, \tau: (i) \rightrightarrows (i)$ of graphs, resulting in the exceptional loop (Example 3.16). By contrast, in Section 7, we see that, the definition of modular operads (Definition 1.25) implies that the contracted units are described by adjoining a morphism (specifically, $z: \mathcal{C}_0 \rightarrow (i)$, Section 7.1) that acts as a formal limit of the very same diagram $id, \tau: (i) \rightrightarrows (i)$.

Consequently, the unit contraction is described graphically in terms of a ‘double cone’ of graphs as illustrated in Figure 2 (see also Section 7.4, (7.38)), and not by an exceptional loop.

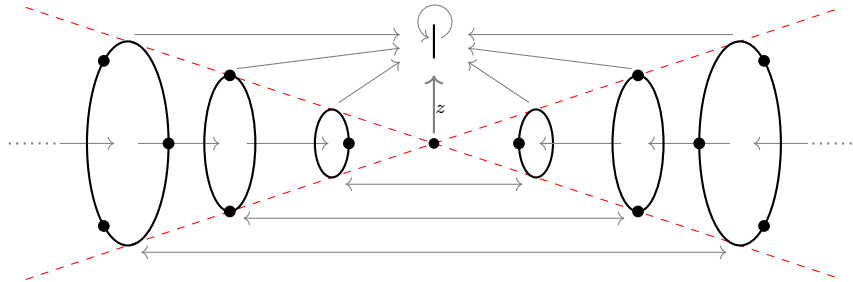


FIGURE 2. In Section 7, the contraction of units is described in terms of a commuting diagram of ‘shrinking wheels’ in the category \mathbf{OGr}_{sim_*} (defined in Section 7.4) that is strongly suggestive of a conical singularity.

- The graph morphisms illustrated in Figure 2 are encoded by the monad \mathbb{D} on \mathbf{GS} (Lemma 7.6) which, together with the distributive law, induces a subtle notion of equivalence, or ‘similarity’, of graphs (Proposition 7.41). In particular, all constructions may be performed using well-behaved representatives of similarity classes of graphs. This is important in the proof of Theorem 8.1.

²Most of Sections 3-5 and Section 8 appeared in my PhD thesis [35]. There are some more substantial differences between [35] and Sections 6 and 7, since [35] did not make use of the distributive law.

1. DEFINITIONS AND EXAMPLES

The goal of this section is to give an axiomatic definition of modular operads (Definition 1.25), and to provide some motivating examples. As mentioned in the introduction, the term ‘modular operad’ refers here to what are called ‘compact symmetric multicategories (CSMs)’ in [23].

After establishing some basic notional conventions, the section begins with a discussion of Joyal and Kock’s category \mathbf{GS} of graphical species [23] which generalises various notions of coloured collection used in the study of operads.

1.1. Graphical species. A *presheaf* on a category \mathbf{C} is a functor $P: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$. The corresponding functor category is denoted $\mathbf{ps}(\mathbf{C})$.

The notion of elements of a presheaf will be used extensively in what follows:

Definition 1.1. An element of a presheaf $P: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ is a pair (c, x) where c is an object of \mathbf{C} and $x \in P(c)$. The elements of P are the objects of a category $\mathbf{El}_P(\mathbf{C})$ whose morphisms $(c, x) \rightarrow (d, y)$ are given by morphisms $f \in \mathbf{C}(c, d)$ such that $P(f)(y) = x$.

If a presheaf P on \mathbf{C} is of the form $\mathbf{C}(-, c)$, then $\mathbf{El}_P(\mathbf{C})$ is the *slice category* \mathbf{C}/c whose objects are pairs (d, f) where $f \in \mathbf{C}(d, c)$, and morphisms $(d, f) \rightarrow (d', f')$ are given by commuting triangles in \mathbf{C} :

$$\begin{array}{ccc} d & \xrightarrow{g} & d' \\ & \searrow f \quad \swarrow f' & \\ & c. & \end{array}$$

Given a functor $\iota: \mathbf{D} \rightarrow \mathbf{C}$, and an object c of \mathbf{C} , the pullback $\iota^*\mathbf{C}(-, c)$, $d \mapsto \mathbf{C}(\iota(d), c)$ defines a presheaf on \mathbf{D} , and $\iota/c \stackrel{\text{def}}{=} \mathbf{El}_{\iota^*\mathbf{C}(-, c)}(\mathbf{D})$ is called the *slice category of ι over c* .

The groupoid of finite sets and bijections is denoted by \mathbf{B} . For $n \in \mathbb{N}$, the set $\{1, \dots, n\}$ is denoted by \mathbf{n} . So $\mathbf{0} = \emptyset$ is the empty set.

Remark 1.2. Let $\Sigma \subset \mathbf{B}$ denote the skeletal subgroupoid on the objects \mathbf{n} , for $n \in \mathbb{N}$. A presheaf $P: \mathbf{B}^{\text{op}} \rightarrow \mathbf{Set}$ on \mathbf{B} , also called a (*monochrome* or *single-sorted*) *species* [24], determines a presheaf on Σ by restriction. Conversely, a Σ -presheaf Q may always be extended to a \mathbf{B} -presheaf $Q_{\mathbf{B}}$, by setting

$$Q_{\mathbf{B}}(X) \stackrel{\text{def}}{=} \lim_{(\mathbf{n}, f) \in \Sigma/X} Q(\mathbf{n}) \quad \text{for all } n \in \mathbb{N}.$$

Graphical species, defined in [23, Section 4], are a *coloured* or *multi-sorted* version of species.

Let the category \mathbf{B}° be obtained from \mathbf{B} by adjoining a distinguished object \S that satisfies

- $\mathbf{B}^{\circ}(\S, \S) = \{id, \tau\}$ with $\tau^2 = id$,
- for each finite set X and each element $x \in X$, there is a morphism $ch_x \in \mathbf{B}^{\circ}(\S, X)$ that ‘chooses’ x , and $\mathbf{B}^{\circ}(\S, X) = \{ch_x, ch_x \circ \tau\}_{x \in X}$,
- for all finite sets X and Y , $\mathbf{B}^{\circ}(X, Y) = \mathbf{B}(X, Y)$, and morphisms are equivariant with respect to the action of \mathbf{B} : $ch_{f(x)} = f \circ ch_x \in \mathbf{B}^{\circ}(\S, Y)$ for all $x \in X$ and all bijections $f: X \xrightarrow{\cong} Y$.

Definition 1.3. A graphical species is a presheaf $S: \mathbf{B}^{\circ \text{op}} \rightarrow \mathbf{Set}$. The category of graphical species is denoted $\mathbf{GS} \stackrel{\text{def}}{=} \mathbf{ps}(\mathbf{B}^{\circ})$.

Hence, a graphical species S is described by a species $(S_X)_{X \in \mathbf{B}}$, and a set S_{\S} with involution $S_{\tau}: S_{\S} \rightarrow S_{\S}$, together with, for each finite set X , and $x \in X$ a \mathbf{B} -equivariant projection $S(ch_x): S_X \rightarrow S_{\S}$.

Definition 1.4. Given a graphical species S , the pair (S_{\S}, S_{τ}) is called the (involutive) palette of S and elements $c \in S_{\S}$ are colours of S . If S_{\S} is trivial then S is a monochrome graphical species.

For each element $\underline{c} = (c_x)_{x \in X} \in S_{\S}^X$, the \underline{c} -(coloured) arity $S_{\underline{c}}$ is the fibre above $\underline{c} \in S_{\S}^X$ of the map $(S(ch_x))_{x \in X}: S_X \rightarrow S_{\S}^X$.

Remark 1.5. The involution τ on \S is responsible for much of the heavy lifting in the constructions that follow. Initially however, its role may seem obscure. I mention two key features here. First, the involution provides the expressive power necessary to describe composition rules involving colours, such as particle spin, that may have an orientation, or direction. (Directed graphical species are discussed in Example 1.11.)

The second is more fundamental. The involution enables us to encode formal compositions in graphical species – described in terms of graphs – as categorical limits, and thereby derive the results of this paper by purely abstract methods. For example, the involution underlies a well-defined notion of graph nesting, or substitution, in terms of diagram colimits, without the need to specify extra data (see Sections 5 and 6, and compare with, e.g. [43, 18]).

Example 1.6. The terminal graphical species K has trivial palette and $K_X = \{*\}$ for all finite sets X .

Definition 1.7. A morphism $\gamma \in \mathbf{GS}(S, S')$ is palette-preserving if its component γ_{\S} at \S is the identity on S_{\S} . For a fixed palette (\mathfrak{C}, ω) , $\mathbf{GS}^{(\mathfrak{C}, \omega)}$ is the subcategory of \mathbf{GS} on the (\mathfrak{C}, ω) -coloured graphical species and palette-preserving morphisms.

Example 1.8. For any palette (\mathfrak{C}, ω) , the terminal (\mathfrak{C}, ω) -coloured graphical species $K^{(\mathfrak{C}, \omega)}$ in $\mathbf{GS}^{(\mathfrak{C}, \omega)}$ is described by

$$K_X^{(\mathfrak{C}, \omega)} = \mathfrak{C}^X, \text{ with } K_{\underline{c}}^{(\mathfrak{C}, \omega)} = \{*\}, \text{ for all finite sets } X \text{ and all } \underline{c} \in \mathfrak{C}^X.$$

In particular, let $\sigma_{\mathfrak{D}\mathfrak{i}}$ be the unique non-identity involution on the set $\mathfrak{D}\mathfrak{i} \stackrel{\text{def}}{=} \{\text{in}, \text{out}\}$. A *monochrome directed graphical species* is a graphical species with palette $(\mathfrak{D}\mathfrak{i}, \sigma_{\mathfrak{D}\mathfrak{i}})$. The terminal monochrome directed graphical species is denoted by $Di \stackrel{\text{def}}{=} K^{(\mathfrak{D}\mathfrak{i}, \sigma_{\mathfrak{D}\mathfrak{i}})}$. See also Example 1.11.

Remark 1.9. It is helpful represent a finite set X as a *corolla* or *star graph* with legs in bijection with X (Figure 3 left side), and the distinguished object \S as a loose edge with its endpoints decorated by \blacklozenge and \blacklozenge . For each $x \in X$, the map $ch_x: \S \rightarrow X$ corresponds to the inclusion $\blacklozenge \mapsto x$.

Given a graphical species S and $\underline{c} = (c_x)_{x \in X}$, $c_x \in S_{\S}$, an element $\phi \in S_{\underline{c}}$ may be visualised, as in the right hand side of Figure 3, as a corolla with vertex decorated by ϕ and legs bijectively coloured by $(c_x)_{x \in X}$. (See also Figure 4.)

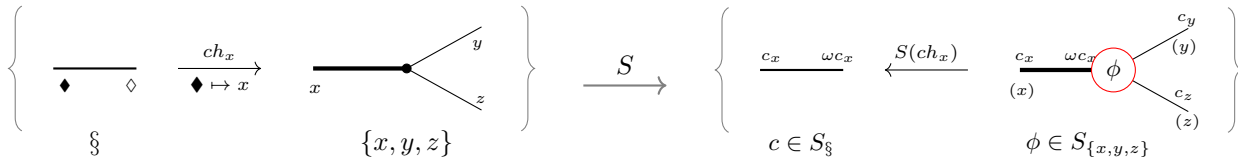


FIGURE 3. Graphical species may be represented graphically: $\phi \in S_{\{x, y, z\}}$ is visualised as a $\{x, y, z\}$ -corolla with vertex decorated by ϕ and x -leg coloured by $S(ch_x)$.

Recall Definition 1.1 of elements of a presheaf.

Definition 1.10. The category $\mathbf{el}(S)$ of elements of a graphical species S is defined as $\mathbf{el}(S) \stackrel{\text{def}}{=} \mathbf{El}_S(\mathbf{B}^{\odot})$.

For any graphical species R , there is a canonical isomorphism $\mathbf{GS}/R \cong \mathbf{ps}(\mathbf{el}(R))$ given by:

$$(S, \gamma: S \rightarrow R) \longleftrightarrow S^\gamma: \mathbf{el}(R)^{\text{op}} \rightarrow \mathbf{Set}, \quad \begin{cases} (\S, c) \mapsto \gamma_\S^{-1}(c), & c \in R_\S \\ (X, \phi) \mapsto \gamma_X^{-1}(\phi), & \phi \in R_X, X \in \mathbf{B}. \end{cases}$$

Example 1.11. A *PROP* [29] is a strict symmetric monoidal category $(\mathbf{E}, +, 0)$ whose objects are natural numbers and whose monoidal product $+$ is addition on objects. Equivalently, \mathbf{E} is described by a presheaf $P = P_\mathbf{E}: (\mathbf{B} \times \mathbf{B}^{\text{op}})^{\text{op}} \rightarrow \mathbf{Set}$ with $P(X; Y) = \mathbf{E}(|X|, |Y|)$, together with the distinguished element $id_1 \in P(\mathbf{1}; \mathbf{1}) = \mathbf{E}(1, 1)$, and maps defining the composition and monoidal product.

More generally, for any set \mathfrak{D} , a \mathfrak{D} -coloured *PROP* is a strict symmetric monoidal category whose monoid of objects is freely generated by \mathfrak{D} .

Namely, let $(\mathbf{B} \times \mathbf{B}^{\text{op}})^\downarrow$ be obtained from $\mathbf{B} \times \mathbf{B}^{\text{op}}$ by adjoining a distinguished element (\downarrow) with trivial endomorphism group, and, for all pairs (X, Y) of finite sets, and all elements $x \in X$ and $y \in Y$, *input* morphisms $i_x: (\downarrow) \rightarrow (X, Y)$, and *output* morphisms $o_y: (\downarrow) \rightarrow (X, Y)$ that are compatible with the action of $\mathbf{B} \times \mathbf{B}^{\text{op}}$ (see Figure 4(d)).

So, a \mathfrak{D} -coloured *PROP* $(\mathbf{E}^\mathfrak{D}, \oplus, \emptyset)$ is equivalently a presheaf $P^\mathfrak{D}$ on $(\mathbf{B} \times \mathbf{B}^{\text{op}})^\downarrow$ with $P^\mathfrak{D}(\downarrow) = \mathfrak{D}$ and, for all pairs (X, Y) of finite sets,

$$P^\mathfrak{D}(X; Y) = \coprod_{(\underline{c}, \underline{d}) \in \mathfrak{D}^{|X|} \times \mathfrak{D}^{|Y|}} \mathbf{E}^\mathfrak{D}(\underline{c}, \underline{d}),$$

together with composition and monoidal product maps, and an injection $P^\mathfrak{D}(\downarrow) \hookrightarrow P^\mathfrak{D}(\mathbf{1}; \mathbf{1})$ that induces the identities for composition.

Following Remark 1.9, the objects (X, Y) of $(\mathbf{B} \times \mathbf{B}^{\text{op}})^\downarrow$ may be visualised, as in Figure 4(b), as *directed corollas* and the distinguished element (\downarrow) as a *directed exceptional edge* (Figure 4(a)).

If $Y = \{r\}$ is a singleton, then the object $(X, \{*\})$ describes a *rooted corolla* as in Figure 4(c). See also Examples 1.19, 2.10, 5.1 on operads.

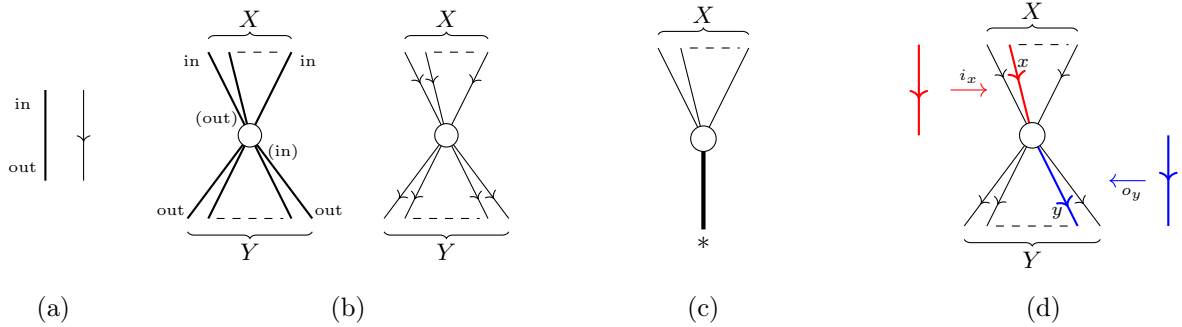


FIGURE 4. (a) the directed exceptional edge (\downarrow) , (b) the pair $(X, Y) \in (\mathbf{B} \times \mathbf{B}^{\text{op}})^\downarrow$ describes a directed corolla, (c) $(X, \{*\})$ describes a rooted corolla, (d) input and output morphisms in $(\mathbf{B} \times \mathbf{B}^{\text{op}})^\downarrow$.

Let Di be the terminal $(\mathfrak{D}i, \sigma_{\mathfrak{D}i})$ -coloured graphical species defined in Example 1.8. Then, for each finite set X , $Di_X = \{\text{in}, \text{out}\}^X$ is the set of partitions $X = X_{\text{in}} \amalg X_{\text{out}}$ of X into *input* and *output* sets, with blockwise action of the partition-preserving isomorphisms in $\mathbf{el}(Di)$. Hence, $\mathbf{el}(Di) \simeq (\mathbf{B} \times \mathbf{B}^{\text{op}})^\downarrow$ canonically and so $\mathbf{GS}/Di \cong \mathbf{ps}(\mathbf{el}(Di))$ is equivalent to the category $\mathbf{ps}((\mathbf{B} \times \mathbf{B}^{\text{op}})^\downarrow)$ of *directed graphical species*. The subcategory $\mathbf{GS}^{(\mathfrak{D}i, \sigma_{\mathfrak{D}i})}/Di$ of *monochrome directed graphical species* is equivalent to $\mathbf{ps}(\mathbf{B} \times \mathbf{B}^{\text{op}})$. In particular, *PROPs* may be described in terms of graphical species.

1.2. Multiplication and contraction on graphical species. Intuitively, a multiplication \diamond on a graphical species S is a rule for combining (gluing) distinct elements of S along pairs of legs (called ‘ports’) with dual colouring as in Figure 5:

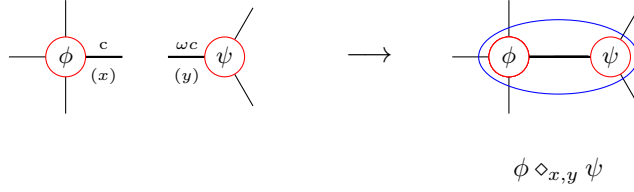


FIGURE 5. Multiplication

The notation ‘ \dashv ’ denotes a partial map of sets: $f: A \dashv B$ is given by a function $A' \rightarrow B$ with $A' \subset A$.

Definition 1.12. A multiplication \diamond on a graphical species S is given by a family of partial maps

$$(1.13) \quad - \diamond_{x,y}^{X,Y} -: S_{X \amalg \{x\}} \times S_{Y \amalg \{y\}} \dashv S_{X \amalg Y},$$

defined (for all X, Y) when $\phi \in S_{X \amalg \{x\}}, \psi \in S_{Y \amalg \{y\}}$ satisfy $S(ch_x)(\phi) = S(ch_y \circ \tau)(\psi)$.

The multiplication \diamond satisfies the following conditions:

(m1) (Commutativity axiom.)

Wherever $\diamond_{x,y}^{X,Y}$ is defined,

$$\psi \diamond_{y,x}^{Y,X} \phi = \phi \diamond_{x,y}^{X,Y} \psi$$

(m2) (Equivariance axiom.)

For all bijections $\sigma: X \xrightarrow{\cong} W$ and $\rho: Y \xrightarrow{\cong} Z$ that extend to bijections $\hat{\sigma}: X \amalg \{x\} \xrightarrow{\cong} W \amalg \{w\}$ and $\hat{\rho}: Y \amalg \{y\} \xrightarrow{\cong} Z \amalg \{z\}$,

$$S(\sigma \sqcup \rho)(\phi \diamond_{w,z}^{W,Z} \psi) = S(\hat{\sigma})(\phi) \diamond_{x,y}^{X,Y} S(\hat{\rho})(\psi),$$

(where $\sigma \sqcup \rho: X \amalg Y \xrightarrow{\cong} W \amalg Z$ is the block permutation).

A unit for the multiplication \diamond is a map $\epsilon: S_{\S} \rightarrow S_2$, $c \mapsto \epsilon_c$ such that, for all X and all $\phi \in S_{X \amalg \{x\}}$ with $S(ch_x) = c \in S_{\S}$,

$$\phi \diamond_{x,2}^{X,\{1\}} \epsilon_c = \epsilon_c \diamond_{2,x}^{\{1\},X} \phi = \phi.$$

A multiplication \diamond is called unital if it has a unit ϵ . In this case ϵ_c is a c -coloured unit for \diamond .

If $(\diamond, \epsilon: \mathfrak{C} \rightarrow S_2)$ is a unital multiplication on a (\mathfrak{C}, ω) -coloured graphical species S , then $\epsilon_c \in S_{(c, \omega c)}$ for all $c \in \mathfrak{C}$. So a unit map $\epsilon: \mathfrak{C} \rightarrow S_2$ is injective. Let $\sigma_2 \in \mathbf{B}^\cup(2, 2)$ be the unique non-identity endomorphism.

Lemma 1.14. If \diamond admits a unit $\epsilon: \mathfrak{C} \rightarrow S_2$, it is unique. Moreover, ϵ is compatible with the involutions $\omega = S_\tau$ on \mathfrak{C} and $S(\sigma_2)$ in that

$$(1.15) \quad \epsilon \circ \omega = \epsilon \circ S_\tau = S(\sigma_2) \circ \epsilon: \mathfrak{C} \rightarrow S_2.$$

Proof. Let $\lambda: \mathfrak{C} \rightarrow S_2$, $c \mapsto \lambda_c$ be another unit for \diamond . Then, for all $c \in \mathfrak{C}$,

$$\epsilon_c = \epsilon_c \diamond_{2,2}^{\{1\},\{1\}} \lambda_c = \lambda_c \diamond_{2,2}^{\{1\},\{1\}} \epsilon_c = \lambda_c,$$

whence ϵ is unique. The second statement follows from the defining properties of multiplication. Namely,

$$S(\sigma_2)(\epsilon_c) = S(\sigma_2)(\epsilon_c) \diamond_{2,2}^{\{1\},\{1\}} \epsilon_{\omega c} = S(\hat{\sigma}_2 \sqcup id_{\{1\}})(\epsilon_c \diamond_{1,2}^{\{2\},\{1\}} \epsilon_{\omega c}) = \epsilon_c \diamond_{1,2}^{\{2\},\{1\}} \epsilon_{\omega c} = \epsilon_{\omega c} \diamond_{1,2}^{\{2\},\{1\}} \epsilon_c = \epsilon_{\omega c}.$$

□

Remark 1.16. Equivalently, a multiplication \diamond on (\mathfrak{C}, ω) -coloured graphical species S is a family of maps

$$(1.17) \quad - \diamond_{\underline{c}, \underline{d}}^{\underline{c}, \underline{d}} -: S_{(\underline{c}, c)} \times S_{(\underline{d}, \omega c)} \rightarrow S_{(\underline{cd})}, \quad \text{for } c \in \mathfrak{C}, \underline{c} \in \mathfrak{C}^X, \underline{d} \in \mathfrak{C}^Y,$$

Both (1.13) and (1.17) are used in what follows. Where the context is clear, the superscripts may be dropped altogether.

As one would expect, a multiplication \diamond on a graphical species S is called ‘associative’ if the result of several consecutive multiplications does not depend on their order. This is stated precisely in condition (C1) of Definition 1.25, and visualised in the figure therein.

Example 1.18. A graphical species O equipped with a unital, associative multiplication (\diamond, ϵ) is a cyclic operad in the sense of [14]. When the involution is trivial, these are the *entries-only* cyclic operads of [12] (see there for a comparison with cyclic operads as introduced in [15]).

Some advantages of the involutive, graphical species approach to cyclic operads are discussed in [14] and [21, Introduction].

Example 1.19. To describe operads in terms of graphical species, recall the graphical species Di introduced in Example 1.8, and that $\text{el}(Di) \simeq (\mathbf{B} \times \mathbf{B}^{\text{op}})^{\downarrow}$ by Example 1.11, where objects of $(\mathbf{B} \times \mathbf{B}^{\text{op}})^{\downarrow}$ are either the exceptional directed edge (\downarrow) , or pairs (X, Y) of finite sets.

If $Y \cong \{*\}$ is a singleton, then $(X, \{*\})$ is called a rooted corolla, and denoted by t_X . Let $\mathbf{B}^{\downarrow} \subset (\mathbf{B} \times \mathbf{B}^{\text{op}})^{\downarrow}$ be the full subcategory on (\downarrow) and the rooted corollas t_X (for all finite sets X).

Presheaves $O: \mathbf{B}^{\downarrow \text{op}}$ are described by a set $\mathfrak{D} = O(\downarrow)$ and sets $O(\underline{c}; d)$, defined for all $d \in \mathfrak{D}$ and $\underline{c} \in \mathfrak{D}^X$ (for all finite sets X). The action of \mathbf{B} on O induces isomorphisms $O((c_x)_x; d) \cong O((c_{f(x)})_x; d)$ for all $f: X \xrightarrow{\cong} Y$. So, a \mathfrak{D} -coloured operad is a \mathbf{B}^{\downarrow} presheaf O , together with an operadic composition, and unit elements $1_d \in O(d; d)$ for all $d \in \mathfrak{D} = O(\downarrow)$.

The graphical species $RC \subset Di$ is given by essential image of $\mathbf{B}^{\downarrow} \subset (\mathbf{B} \times \mathbf{B}^{\text{op}})^{\downarrow}$ under the equivalence $(\mathbf{B} \times \mathbf{B}^{\text{op}})^{\downarrow} \simeq \text{el}(Di)$. So, $RC_{\S} = Di_{\S} = \{\text{in}, \text{out}\}$, $RC_{\mathbf{0}} = \emptyset$ and, for all finite sets X , $RC_{X \amalg \{*\}}$ consists of those $\phi \in Di_{X \amalg \{*\}}$ such that

$$Di(ch_x)(\phi) = (\text{in}) \text{ for all } x \in X, \text{ and } Di(ch_*)(\phi) = (\text{out}).$$

Clearly, RC inherits the trivial unital multiplication from Di . Moreover, a presheaf $O: \mathbf{B}^{\downarrow \text{op}} \rightarrow \mathbf{Set}$ has the structure of an operad precisely when the corresponding graphical species $O^{\text{GS}} \in \mathbf{GS}/RC$ is equipped with an associative unital multiplication. Hence, the category \mathbf{Op} of (symmetric) operads is equivalent to the category whose objects are objects of \mathbf{GS}/RC equipped with an associative unital multiplication, and whose morphisms are morphisms in \mathbf{GS}/RC that preserve the multiplication and units.

Remark 1.20. Examples 1.18 and 1.19 highlight the expressive power of graphical species. The involution τ on \S means that (undirected) cyclic operads and (directed) operads may be expressed in terms of presheaves on the same underlying category. (See also Examples 1.11 and 1.30.)

Intuitively, a contraction ζ on a graphical species S may be thought of as a rule for ‘self-gluing’ single elements of S along pairs of ports with dual colouring (Figure 6). The presence of a contraction operation enables modular operads to encode algebraic structures – such as those involving trace – that ordinary operads cannot [32, 33].

Definition 1.21. A contraction ζ on S is given by a family of partial maps

$$(1.22) \quad \zeta_{x,y}^X: S_{X \amalg \{x,y\}} \rightharpoonup S_X$$

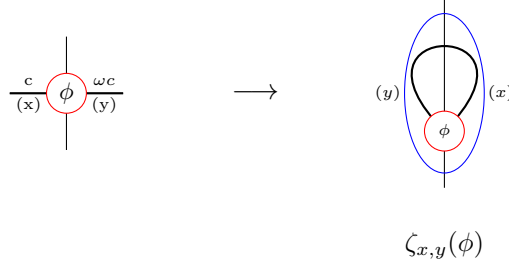


FIGURE 6. Contraction

defined for all finite sets X and all $\phi \in S_{X \amalg \{x,y\}}$, such that $S(ch_x)(\phi) = S(ch_y \circ \tau)(\phi)$, and equivariant with respect to the action of \mathbf{B} on S : If $\hat{\sigma}: X \amalg \{x,y\} \xrightarrow{\cong} Z \amalg \{w,z\}$ extends the bijection $\sigma: X \xrightarrow{\cong} Z$ by $\hat{\sigma}(x) = w, \hat{\sigma}(y) = z$, then for any $\phi \in S_{Z \amalg \{w,z\}}$, we have

$$S(\sigma)(\zeta_{w,z}^Z(\phi)) = \zeta_{x,y}^X(S(\hat{\sigma})(\phi)).$$

If ζ is a contraction on S , then by, equivariance, $\zeta_{x,y}^X(\phi) = \zeta_{y,x}^X(\phi)$ wherever defined.

Remark 1.23. A contraction ζ on a (\mathfrak{C}, ω) -coloured graphical species S is equivalently a family of maps

$$\zeta_c^{\underline{c}}: S_{(\underline{c}, c, \omega c)} \rightarrow S_{\underline{c}}$$

for $c \in \mathfrak{C}$, and $\underline{c} \in \mathfrak{C}^X$. Depending on context, both $\zeta_c^{\underline{c}}$ (and even ζ_c) and $\zeta_{x,y}^X$ will be used.

Let S be a (\mathfrak{C}, ω) -coloured graphical species equipped with a unital multiplication (\diamond, ϵ) and contraction ζ . By Lemma 1.14, there is a *contracted unit* map

$$(1.24) \quad o \stackrel{\text{def}}{=} \zeta \epsilon: \mathfrak{C} \rightarrow S_0, \text{ satisfying } \zeta_c(\epsilon_c) = \zeta_{\omega c}(\epsilon_{\omega c}) \text{ for all } c \in \mathfrak{C}.$$

As will be explained in Sections 6 and 7, the contracted units $o: S_{\mathfrak{s}} \rightarrow S_0$ present the main challenge for describing the combinatorics of modular operads.

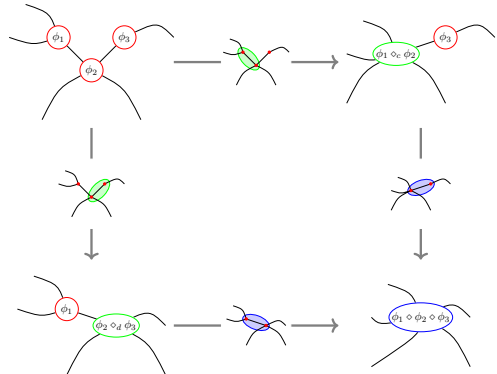
1.3. Modular operads: definition and examples. Modular operads are graphical species with multiplication and contraction operations that satisfy the nicest possible (mutual) coherence axioms.

Definition 1.25. A modular operad is a graphical species S , with palette (\mathfrak{C}, ω) , say, together with a unital multiplication (\diamond, ϵ) , and a contraction ζ , satisfying the following four coherence axioms governing their composition:

(C1) Multiplication is associative.

For all $\underline{b} \in \mathfrak{C}^{X_1}, \underline{c} \in \mathfrak{C}^{X_2}, \underline{d} \in \mathfrak{C}^{X_3}$ and all $c, d \in \mathfrak{C}$, the following square commutes:

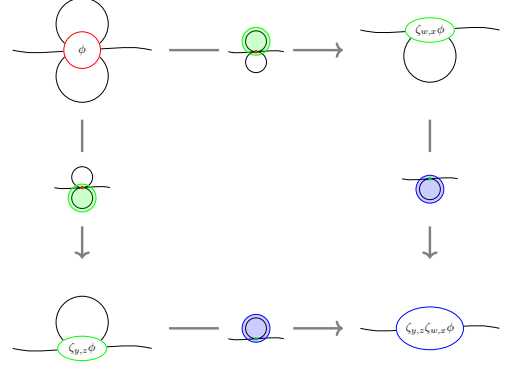
$$\begin{array}{ccc} S_{(\underline{b}, c)} \times S_{(\underline{c}, \omega c, d)} \times S_{(\underline{d}, \omega d)} & \xrightarrow{\diamond_c \times id} & S_{(\underline{bc}, d)} \times S_{(\underline{d}, \omega d)} \\ \downarrow id \times \diamond_d & & \downarrow \diamond_d \\ S_{(\underline{b}, c)} \times S_{(\underline{c}, \omega c, \underline{d})} & \xrightarrow{\diamond_c} & S_{\underline{bcd}}. \end{array}$$



(C2) Order of contraction does not matter.

For all $\underline{c} \in \mathfrak{C}^X$ and $c, d \in \mathfrak{C}$, the following square commutes:

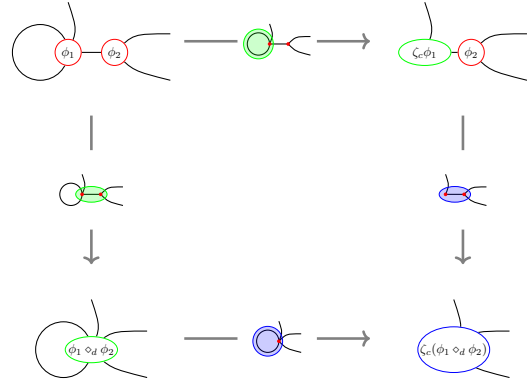
$$\begin{array}{ccc} S_{(\underline{c}, c, \omega c, d, \omega d)} & \xrightarrow{\zeta_c} & S_{(\underline{c}, d, \omega d)} \\ \downarrow \zeta_d & & \downarrow \zeta_d \\ S_{(\underline{c}, c, \omega c)} & \xrightarrow{\zeta_c} & S_{\underline{c}} \end{array}$$



(C3) Multiplication and contraction commute.

For all $\underline{c} \in \mathfrak{C}^{X_1}$, $\underline{d} \in \mathfrak{C}^{X_2}$ and $c, d \in \mathfrak{C}$, the following square commutes.

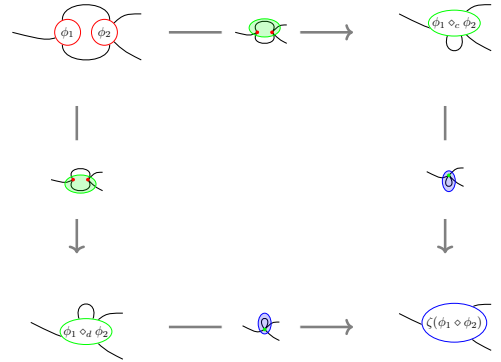
$$\begin{array}{ccc} S_{(\underline{c}, c, \omega c, d)} \times S_{(\underline{d}, \omega d)} & \xrightarrow{\zeta_c \times id} & S_{(\underline{c}, d)} \times S_{(\underline{d}, \omega d)} \\ \downarrow \diamond_d & & \downarrow \diamond_d \\ S_{(\underline{c}, c, \omega c, \underline{d})} & \xrightarrow{\zeta_c} & S_{\underline{cd}} \end{array}$$



(C4) ‘Parallel multiplication’ of pairs.

For all $\underline{c} \in \mathfrak{C}^{X_1}$, $\underline{d} \in \mathfrak{C}^{X_2}$, and $c, d \in \mathfrak{C}$, the following square commutes:

$$\begin{array}{ccc} S_{(\underline{c}, c, d)} \times S_{(\underline{d}, \omega c, \omega d)} & \xrightarrow{\diamond_c} & S_{(\underline{c}, d, d, \omega d)} \\ \downarrow \diamond_d & & \downarrow \zeta_d \\ S_{(\underline{c}, c, \underline{d}, \omega c)} & \xrightarrow{\zeta_c} & S_{\underline{cd}} \end{array}$$



Modular operads form a category \mathbf{MO} whose morphisms are morphisms of the underlying graphical species that preserve multiplication, contraction and multiplicative units.

Informally, the multiplication and contraction operations describe rules for *collapsing* inner edges of graphs that represent formal compositions of elements. The coherence axioms (C1)-(C4) say that this is independent of the order in which the edges are collapsed.

Remark 1.26. A *non-unital modular operad* (S, \diamond, ζ) is a graphical species S equipped with a multiplication \diamond and contraction ζ satisfying (C1)-(C4), but without the requirement of a multiplicative unit. These form a category \mathbf{MO}^- whose morphisms are morphisms in \mathbf{GS} that preserve the multiplication and contraction operations. Non-unital modular operads are the subject of Section 5.

To provide context and motivation for the constructions that follow, the remainder of this section is devoted to examples.

Example 1.27. Getzler-Kapranov modular operads. We may equip the monochrome graphical species $M : M_X = \mathbb{N}$ (for all X), with a unital multiplication $(+, 0 \in M_2)$ induced by addition in \mathbb{N} :

$$+ : M_{\mathbf{m}} \times M_{\mathbf{n}} \rightarrow M_{\mathbf{m}+\mathbf{n}-2}, (g_m, g_n) \mapsto g_m + g_n \text{ for } m, n \geq 1,$$

and a contraction t induced by the successor operation:

$$t : M_{\mathbf{n}} \rightarrow M_{\mathbf{n}-2}, g_n \mapsto g_n + 1 \text{ for } n \geq 2.$$

Since a compact oriented surface with boundary is determined, up to homeomorphism, by its genus and number of boundary components, $(M, +, s)$ models gluing of topological surfaces along boundary components. Monochrome objects (S, γ) of the slice category \mathbf{MO}/M describe a bigraded set $S^\gamma(g, n)$ with operations

$$+^S : S^\gamma(g_1, n_1) \times S^\gamma(g_2, n_2) \rightarrow S^\gamma(g_1 + g_2, n_1 + n_2 - 2) \text{ for } n_1, n_2 \geq 1,$$

$$t^S : S^\gamma(g, n) \rightarrow S^\gamma(g + 1, n - 2) \text{ for } n \geq 2,$$

and may encode (moduli spaces) of geometric structures on surfaces. For example, the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of genus g smooth algebraic curves with n marked points, may be described, via Belyi's Theorem, in terms of the space of genus g Riemann surfaces with n nodes, and the spaces $\overline{\mathcal{M}}_n \stackrel{\text{def}}{=} \coprod_{g \in \mathbb{N}} \overline{\mathcal{M}}_{g,n}$ form a monochrome modular operad. (See also [16, Example 6.2].)

Getzler and Kapranov [16] originally defined modular operads in terms of the restriction to the *stable part* $M^{st} \subset M$ of the graphical species M . An oriented surface Γ is stable if and only if $2g + n - 2 > 0$. So $M_n^{st} = M_n$ for $n > 2$ but $M_0^{st} = \{2, 3, 4, \dots\}$ and $M_1^{st} = M_2^{st} = \{1, 2, 3, 4, \dots\}$.

In particular, since $0 \notin M_2^{st}$, modular operads in the original sense of [16] are non-unital.

These ideas may be extended to many-coloured cases: for example, one can describe a 2-coloured modular operad for gluing surfaces along open and closed subsets of their boundaries. (See, e.g. [17].)

Example 1.28. Compact closed categories, introduced in [25], are symmetric monoidal categories (\mathbf{C}, \otimes, e) for which every object $c \in \mathbf{C}$ has a symmetric *categorical dual* (see [4, 27]): there is an object $c^* \in \mathbf{C}$, and morphisms $\cap_c : e \rightarrow c^* \otimes c$ and $\cup_c : c \otimes c^* \rightarrow e$ such that

$$(\cup_c \otimes id_c) \circ (id_c \otimes \cap_c) = id_c = (\cap_{c^*} \otimes id_c) \circ (id_c \otimes \cup_{c^*}).$$

Examples of compact closed categories include categories of finite dimensional vector spaces over a given field, or, more generally, finite dimensional projective modules over a commutative ring. Cobordism categories provide other important examples.

The category of small compact closed categories contains an equivalent full subcategory \mathbf{Comp} of those categories \mathbf{C} such that $c = c^{**}$ for all objects, and there is a canonical monadic adjunction $\mathbf{MO} \rightleftarrows \mathbf{Comp}$. The right adjoint takes a small compact closed category $(\mathbf{C}, \otimes, e, *)$ with object set \mathbf{C}_0 to a $(\mathbf{C}_0, *)$ -coloured modular operad $S^{\mathbf{C}}$ with coloured arities

$$S_{(c_1, \dots, c_m, d_1^*, \dots, d_n^*)}^{\mathbf{C}} = \mathbf{C}(c_1 \otimes \dots \otimes c_m, d_1 \otimes \dots \otimes d_n)$$

and the modular operad structure on $S^{\mathbf{C}}$ is induced by composition in \mathbf{C} together with \cup and \cap . The left adjoint $\mathbf{MO} \rightarrow \mathbf{Comp}$ is induced by the free monoid functor on palettes and arities.

These observations underly the proof, in [37], of an ‘operadic’ nerve theorem for compact closed categories in the style of Section 8.

Example 1.29. Circuit algebras – so named because of their resemblance to electronic circuits – are a symmetric version of Jones’s planar algebras, introduced to study finite-type invariants in low-dimensional topology [1, 13].

Objects of the category \mathbf{CO} of *circuit operads* [36] are modular operads equipped, via a monadic adjunction $\mathbf{MO} \rightleftharpoons \mathbf{CO}$, with an extra ‘external product’ operation. Circuit operads in a linear category \mathcal{E} are precisely the circuit algebras in \mathcal{E} .

The adjunction between modular operads and compact closed categories in Example 1.28 factors through the adjunction $\mathbf{MO} \rightleftharpoons \mathbf{CO}$. Moreover, every compact closed category in \mathbf{Comp} is equivalent to the free compact closed category on a circuit operad.

This formal perspective on modular operads, circuit algebras, and compact closed categories leads to interesting questions in a number of directions. For example, we can study the analogous relationships if the definition of modular operads is relaxed by replacing the symmetric action with a braiding, or by considering higher dimensional versions. Related ideas are being explored by Dansco and Robertson in their work on algebraic and categorical structures in low-dimensional topology.

Example 1.30. Wheeled properads. Wheeled properads have been studied extensively in [18] and [43]. They describe the *connected part* (c.f. [41, Introduction]) of wheeled PROPs – coloured PROPs with a contraction – that have applications in geometry, deformation theory, and other areas [33, 32].

The category \mathbf{WP} of (Set-valued) wheeled properads is canonically equivalent to the slice category \mathbf{MO}/\mathbf{Di} of *directed modular operads*. This is well-defined since the terminal directed graphical species \mathbf{Di} trivially admits the structure of a modular operad (see Example 1.11).

An equivalence between wheeled PROPs in linear categories and directed circuit algebras is established in [13]. The equivalence between \mathbf{MO}/\mathbf{Di} and \mathbf{WP} is extended to an equivalence between wheeled PROPs and circuit operads in [36].

2. ABSTRACT NERVE THEOREMS AND THE ROLE OF GRAPHS

The purpose of this largely formal section is to review some basic theory of distributive laws, and provide an overview of Weber’s abstract nerve theory. The simplicial nerve for categories, and the dendroidal nerve for operads provide motivating examples for the latter.

For an overview of monads and their Eilenberg–Moore (EM) categories of algebras, see for example [30, Chapter VI].

2.1. Monads with arities and abstract nerve theory. For any locally small category \mathbf{C} , a functor $F: \mathbf{D} \rightarrow \mathbf{C}$ induces a *nerve functor* $N_{\mathbf{D}}: \mathbf{C} \rightarrow \mathbf{ps}(\mathbf{D})$ by $N_{\mathbf{D}}(c)(d) = \mathbf{C}(Fd, c)$ for all $c \in \mathbf{C}$ and $d \in \mathbf{D}$. If the nerve $N_{\mathbf{D}}$ is fully faithful, and F and \mathbf{D} are suitably nice, then $N_{\mathbf{D}}$ provides a useful tool for studying \mathbf{C} .

In the crudest sense, monads with arities are monads whose EM category of algebras may be characterised in terms of a purely abstract fully faithful nerve. The aim of this section is to explain, without proofs, the key points of this abstract nerve. This theory motivates the framework of this paper, and underlies the proof of the nerve theorem for modular operads, Theorem 8.1, in Section 8. Details may be found in [5, Sections 1-3].

Recall that every functor admits an (up to isomorphism) unique *bo-ff factorisation* as a bijective on objects functor followed by a fully faithful functor. For example, if \mathbb{M} is a monad on a category \mathbf{C} , and $\mathbf{C}^{\mathbb{M}}$ is the EM category of algebras for \mathbb{M} by $\mathbf{C}^{\mathbb{M}}$, then the free functor $\mathbf{C} \rightarrow \mathbf{C}^{\mathbb{M}}$ has bo-ff factorisation $\mathbf{C} \rightarrow \mathbf{C}_{\mathbb{M}} \rightarrow \mathbf{C}^{\mathbb{M}}$, where $\mathbf{C}_{\mathbb{M}}$ is the Kleisli category of free \mathbb{M} -algebras (see e.g. [30, Section VI.5]).

Hence, for any subcategory D of C , the bo-ff factorisation of the canonical functor $D \hookrightarrow C \xrightarrow{\text{free}} C^{\mathbb{M}}$ factors through the full subcategory $\Theta_{\mathbb{M},D}$ of $C^{\mathbb{M}}$ with objects from D .

By construction, the defining functor $\Theta_{\mathbb{M},D} \rightarrow C^{\mathbb{M}}$ is fully faithful. So, it is natural to ask if there are conditions on D and \mathbb{M} that ensure that the induced nerve $N_{\mathbb{M},D}: C^{\mathbb{M}} \rightarrow \mathbf{ps}(\Theta_{\mathbb{M},D})$ is fully faithful. This is the motivation for describing monads with arities.

Definition 2.1. *As usual, a subcategory $D \hookrightarrow C$ is called replete if any object $c \in C$ such that $c \cong d$ for some $d \in D$, is, itself, an object of D . The essential image $\text{im}^{es}(F)$ of a functor $F: E \rightarrow C$ is the smallest replete subcategory $\text{im}^{es}(F) \hookrightarrow C$ containing the image $\text{im}(F)$ of F in C .*

A subcategory $\iota: D \hookrightarrow C$ is a dense subcategory (and ι is a dense functor) if the induced nerve $N_D: C \rightarrow \mathbf{ps}(D)$ is full and faithful.

Once again, let $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ be a monad on C . Let $\iota: D \hookrightarrow C$ be a dense subcategory, and let $\Theta_{\mathbb{M},D}$ be obtained in the bo-ff factorisation of $D \rightarrow C^{\mathbb{M}}$. There is an induced diagram of functors

$$(2.2) \quad \begin{array}{ccccc} \Theta_{\mathbb{M},D} & \xrightarrow{\text{f.f.}} & C^{\mathbb{M}} & \xrightarrow{N_{\mathbb{M},D}} & \mathbf{ps}(\Theta_{\mathbb{M},D}) \\ \text{b.o.} \uparrow j & & \updownarrow & & \downarrow j^* \\ D & \xrightarrow{\text{dense}} & C & \xrightarrow[N_{\mathbb{M},D}]{\text{f.f.}} & \mathbf{ps}(D). \end{array}$$

where j^* is the pullback of the bijective on objects functor $j: D \rightarrow \Theta_{\mathbb{M},D}$. The left square of (2.2) commutes by definition, and the right square commutes up to natural isomorphism.

Now, by [22, 40], the inclusion $\iota: D \hookrightarrow C$ is dense if and only if every object c of C is given canonically by the colimit of the functor $D/c \rightarrow C$, $(d, f) \mapsto \iota(d)$.

The monad \mathbb{M} has arities D if $N_D \circ M$ takes the canonical colimit cocones D/c in C to colimit cocones in $\mathbf{ps}(D)$. In this case, by [42, Section 4], the full inclusion $\Theta_{\mathbb{M},D} \rightarrow C^{\mathbb{M}}$ is dense, and the essential image of the induced fully faithful nerve $N_{\mathbb{M},D}: C^{\mathbb{M}} \rightarrow \mathbf{ps}(\Theta_{\mathbb{M},D})$ is the full subcategory of $\mathbf{ps}(\Theta_{\mathbb{M},D})$ on those presheaves P whose restriction j^*P to D is in the essential image of $N_D: C \rightarrow \mathbf{ps}(D)$.

Remark 2.3. The condition that \mathbb{M} has arities $D \hookrightarrow C$ is sufficient, but not necessary, for the induced nerve $C^{\mathbb{M}} \rightarrow \mathbf{ps}(\Theta_{\mathbb{M},D})$ to be fully faithful.

In fact, by Theorem 8.1 and Remark 8.13, the modular operad monad \mathbb{O} on the category of graphical species, together with the full dense subcategory $\mathbf{CetGr} \hookrightarrow \mathbf{GS}$ of connected graphs and étale morphisms (see Section 4), provides an example of a monad that does not have arities, but for which the nerve theorem holds.

Necessary conditions on \mathbb{M} and $D \hookrightarrow C$, for the induced nerve to be fully faithful are described in [8].

Example 2.4. Recall that directed graphs $\mathbf{G} = (\mathfrak{s}, \mathfrak{t}: E \rightrightarrows V)$ are presheaves over the small diagram category $\mathcal{E} \stackrel{\text{def}}{=} \bullet \rightrightarrows \bullet$, and that the canonical forgetful functor from \mathbf{Cat} to $\mathbf{ps}(\mathcal{E})$ – that takes a small category C to the directed graph \mathbf{G}_C , with vertex set V_C indexed by objects of C , and edge set E_C indexed by morphisms of C – is monadic. So every directed graph freely generates a small category.

Let $\mathbb{M}_{\mathbf{Cat}}$ be the corresponding monad on $\mathbf{ps}(\mathcal{E})$.

For all $n \in \mathbb{N}$, the n -simplex $\Delta(n)$ is the free category on the finite ordinal $[n]$, viewed as a directed linear graph

$$(2.5) \quad [n] = \overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n}{\bullet}.$$

Let Δ be the *simplex category* of *simplices* $\Delta(n)$, $n \in \mathbb{N}$, and functors between them, and let $\mathbf{sSet} \stackrel{\text{def}}{=} \mathbf{ps}(\Delta)$ denote the category of Δ -presheaves, or *simplicial sets*.

The *classical nerve theorem* states that the induced nerve functor $N_\Delta: \mathbf{Cat} \rightarrow \mathbf{sSet}$ is fully faithful. Moreover, its essential image consists of precisely those $P \in \mathbf{sSet}$ that satisfy the classical Segal condition (originally formulated in [39]): a simplicial set P is the nerve of a small category if and only if, for $n > 1$, the set P_n of n -simplices is isomorphic to the n -fold fibred product

$$(2.6) \quad P_n \cong \underbrace{P_1 \times_{P_0} \cdots \times_{P_0} P_1}_{n \text{ times}}.$$

The nerve theorem and Segal condition (2.6) may be derived using the abstract nerve theory of [5]:

Let $\Delta_0 \subset \mathbf{ps}(\mathcal{E})$ be the full subcategory on the directed linear graphs $[n]$. Morphisms of Δ_0 are *successor-preserving*: $f \in \Delta_0([m], [n])$ satisfies $f(i+1) = f(i) + 1$ for all $0 \leq i < m$. Then \mathcal{E} embeds in Δ_0 as the full subcategory on the objects $[0]$ and $[1]$. Moreover the full inclusion $\Delta_0 \hookrightarrow \mathbf{ps}(\mathcal{E})$ is the nerve induced by the inclusion $\mathcal{E} \hookrightarrow \Delta_0$. Hence \mathcal{E} is dense in Δ_0 . Furthermore, N_{Δ_0} is fully faithful (e.g. by [31, Section VII.2]), so Δ_0 is also dense in $\mathbf{ps}(\mathcal{E})$.

Since Δ is the category obtained in the bo-ff factorisation of $\Delta_0 \rightarrow \mathbf{ps}(\mathcal{E}) \rightarrow \mathbf{Cat}$, it follows that there is a diagram of functors

$$(2.7) \quad \begin{array}{ccccc} \Delta & \xrightarrow{\text{f.f.}} & \mathbf{Cat} & \xrightarrow{N_\Delta} & \mathbf{sSet} \\ \uparrow j \text{ b.o.} & & \uparrow F_{\mathbf{Cat}} \downarrow U_{\mathbf{Cat}} & & \downarrow j^* \\ \mathcal{E} & \xrightarrow[\text{f.f.}]{\text{dense}} & \Delta_0 & \xrightarrow[\text{f.f.}]{\text{dense}} & \mathbf{ps}(\mathcal{E}) & \xrightarrow[\text{f.f.}]{N_{\Delta_0}} & \mathbf{ps}(\Delta_0) \end{array}$$

where the left square commutes, and the right square commutes up to natural isomorphism.

It is straightforward to prove – using for example [5, Sections 1 & 2] – that the category monad on $\mathbf{ps}(\mathcal{E})$ has arities Δ_0 . Hence $N_\Delta: \mathbf{Cat} \rightarrow \mathbf{sSet}$ is fully faithful, and a simplicial set P is in its essential image if and only if j^*P is in the essential image of N_{Δ_0} . Segal’s condition (2.6) follows from the fact that \mathcal{E} is dense in Δ_0 .

Remark 2.8. The notion of graph in Example 2.4 is, in a suitable sense, dual to the one used in Example 2.10, and in the rest of this paper (see Section 3), where edges function as ‘objects’ and connections between them as ‘morphisms’. This is also the case in Example 2.10.

The classical Segal condition (2.6) may be generalised as follows:

As before, let $D \subset C$ be a dense subcategory, and, as in Example 2.4, let $C = \mathbf{ps}(\mathcal{E})$ be the category of presheaves on a dense subcategory \mathcal{E} of D . So, the dense inclusion $D \hookrightarrow C$ is also full. If D provides arities for a monad \mathbb{M} on C , then by [5, Lemma 3.6], $P: \Theta_{\mathbb{M}, D}^{\text{op}} \rightarrow \mathbf{Set}$ is in the essential image of $N_{\mathbb{M}, D}$ if and only if

$$(2.9) \quad P(jd) = \lim_{(e, f) \in \mathcal{E}/d} j^*(P)(e) \quad \text{for all } d \in D.$$

Equation (2.9) is called the *Segal condition* for the nerve functor $N_{\mathbb{M}, D}$.

Example 2.10. A *dendroidal nerve theorem* for operads was originally proved in [34, Section 4], and a corresponding *dendroidal Segal condition* was first given in [11, Corollary 2.6]. These may also be derived using abstract nerve machinery:

Recall from Example 1.19 that an operad is a presheaf $O: \mathbf{B}^{\downarrow \text{op}} \rightarrow \mathbf{Set}$, equipped with a unital composition operation satisfying certain axioms. The forgetful functor $\mathbf{Op} \rightarrow \mathbf{ps}(\mathbf{B}^{\downarrow})$ is monadic, so every presheaf O on \mathbf{B}^{\downarrow} freely generates an operad. Let $\mathbb{M}_{\mathbf{Op}}$ be the induced monad.

Objects of \mathbf{B}^{\downarrow} are the rooted corollas t_X for finite X , and the directed exceptional edge (\downarrow) . *Rooted trees* \mathfrak{T} are obtained as formal colimits of finite diagrams in \mathbf{B}^{\downarrow} , describing *grafting* of objects of \mathbf{B}^{\downarrow} root-to-leaf as in Figure 7(b).

Let Ω_0 be the category whose objects are such rooted trees \mathfrak{T} and whose morphisms $\mathfrak{S} \rightarrow \mathfrak{T}$ are (up to isomorphism) inclusions of rooted trees that preserve vertex valency (Figure 7(a)). Then $\mathbf{B}^\downarrow \subset \Omega_0$ is the full, and dense, subcategory of rooted trees with zero or one vertex.

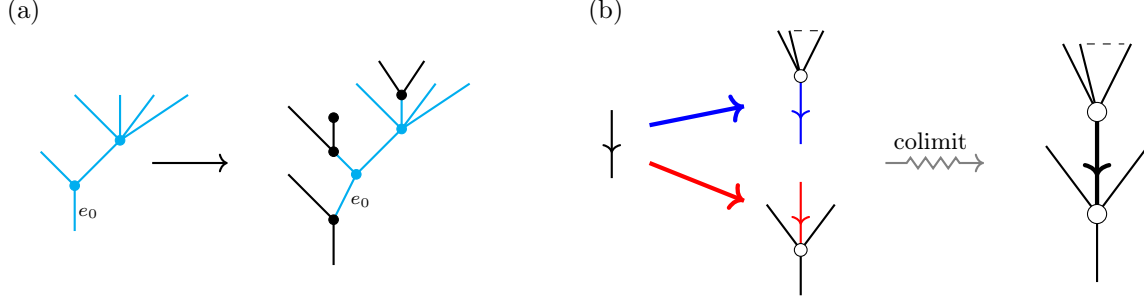


FIGURE 7. (a) Subtree inclusion, (b) grafting of rooted corollas to form a rooted tree.

Hence, the induced nerve $\Omega_0 \rightarrow \mathbf{ps}(\mathbf{B}^\downarrow)$ is full and faithful, and \mathbf{B}^\downarrow canonically induces a topology on Ω_0 whose sheaves are precisely \mathbf{B}^\downarrow -presheaves. In particular, $\Omega_0 \rightarrow \mathbf{ps}(\mathbf{B}^\downarrow)$ is also dense. (See Section 4.4 for comparison).

There is a diagram of functors

$$(2.11) \quad \begin{array}{ccccc} \Omega^\downarrow & \xrightarrow{\text{f.f.}} & \mathbf{Op} & \xrightarrow{N_\Omega} & \mathbf{ps}(\Omega) \\ \uparrow j \text{ b.o.} & & \uparrow F_{\mathbf{Op}} \downarrow U_{\mathbf{Op}} & & \downarrow j^* \\ \mathbf{B}^\downarrow & \xrightarrow[\text{f.f.}]{\text{dense}} & \Omega_0 & \xrightarrow[\text{f.f.}]{\text{dense}} & \mathbf{ps}(\mathbf{B}^\downarrow) & \xrightarrow[\text{f.f.}]{N_{\Omega_0}} & \mathbf{ps}(\Omega_0) \end{array}$$

in which the left square commutes and the right square commutes up to natural isomorphism. The full subcategory of free operads on rooted trees \mathfrak{T} , induced by the bo-ff factorisation of the functor $\Omega_0 \rightarrow \mathbf{ps}(\mathbf{B}^\downarrow) \rightarrow \mathbf{Op}$ is precisely the *dendroidal category* Ω of [34]. There it was established, by different methods, that the full inclusion $\Omega \hookrightarrow \mathbf{Op}$ is dense, and hence the nerve N_Ω is fully faithful.

It is easy to show, e.g. using methods similar to those described in Section 8, that the monad $\mathbb{M}_{\mathbf{Op}}$ on $\mathbf{ps}(\mathbf{B}^\downarrow)$ has arities Ω_0 . Hence, by the abstract nerve theory of [5], the nerve functor $N_\Omega: \mathbf{Op} \rightarrow \mathbf{ps}(\Omega)$ is fully faithful and its essential image consists of those Ω -presheaves (or *dendroidal sets*) $O: \Omega^{\text{op}} \rightarrow \mathbf{Set}$ that satisfy the dendroidal Segal condition [11, Corollary 2.6]:

$$(2.12) \quad O(\mathfrak{T}) = \lim_{(t,i) \in (\mathbf{B}^\downarrow/\mathfrak{T})} j^*(\mathfrak{T}) \text{ for all symmetric rooted trees } \mathfrak{T}.$$

In particular, since Δ_0 is the full subcategory of linear trees in Ω_0 , the simplicial nerve theorem for categories is a special case of the dendroidal nerve theorem for operads.

Definition 2.13. A pointed endofunctor on a category \mathbf{C} is an endofunctor E on \mathbf{C} together with a natural transformation $\eta^E: 1_{\mathbf{C}} \Rightarrow E$. An algebra for a pointed endofunctor (E, η^E) on \mathbf{C} is a pair (c, θ) of an object c of \mathbf{C} and a morphism $\theta \in \mathbf{C}(Ec, c)$ such that $\theta\eta_c^E = \text{id}_c \in \mathbf{C}(c, c)$.

For example, modular operads are algebras for the pointed endofunctor on \mathbf{GS} described in [23]. However, as discussed in Section 6, the abstract nerve machinery of [5] cannot be modified for algebras of (pointed) endofunctors:

Namely, given a monad $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ on a category \mathbf{C} , then the EM category $\mathbf{C}^{\mathbb{M}}$ of algebras for \mathbb{M} embeds canonically in the category of algebras for the pointed endofunctor $(M, \eta^{\mathbb{M}})$ on \mathbf{C} . However,

the free functor $C \rightarrow C^{\mathbb{M}} \rightarrow C^M$, $c \mapsto (Mc, \mu^{\mathbb{M}}c)$ depends crucially on the monadic multiplication $\mu^{\mathbb{M}}: M^2 \Rightarrow M$ of \mathbb{M} .

By contrast, if (E, η^E) is any pointed endofunctor on C , though it is possible for C and C^E to admit dense subcategories on the same set of objects (as is the case in [23]), there is, in general, no canonical choice of functor $C \rightarrow C^E$.

2.2. Distributive laws. With Examples 2.4 and 2.10 in mind, let us return to the case of modular operads. Recall that graphical species are presheaves on the category \mathbf{B}^\odot and that modular operads are graphical species equipped with certain operations.

Informally, monads are gadgets that encode, via their algebras, (algebraic) structure on objects of categories. In [23], it is the combination of the contraction structure ζ , and the multiplicative unit structure ϵ that provides an obstruction to extending the modular operad endofunctor on \mathbf{GS} to a monad (see Section 6). So, one approach to constructing the modular operad monad \mathbb{O} on \mathbf{GS} could be to find monads for the modular operadic multiplication, contraction, and unital structures separately, and then attempt to combine them.

In general, monads do not compose. Given monads $\mathbb{M} = (M, \mu^{\mathbb{M}}, \eta^{\mathbb{M}})$ and $\mathbb{M}' = (M', \mu^{\mathbb{M}'}, \eta^{\mathbb{M}'})$ on a category C , there is no obvious choice of natural transformation $\mu: (MM')^2 \Rightarrow MM'$ defining a monadic multiplication for the endofunctor MM' on C .

Observe, however, that any natural transformation $\lambda: M'M \Rightarrow MM'$ induces a natural transformation

$$\mu_\lambda: (MM')^2 \xrightarrow{M\lambda M'} M^2 M'^2 \xrightarrow{\mu^{\mathbb{M}} \mu^{\mathbb{M}'}} MM'.$$

Definition 2.14. A distributive law [3] for \mathbb{M} and \mathbb{M}' is a natural transformation $\lambda: M'M \Rightarrow MM'$ such that the triple $(MM', \mu_\lambda, \eta^{\mathbb{M}} \eta^{\mathbb{M}'})$ defines a monad $\mathbb{M}\mathbb{M}'$ on C .

A distributive law $\lambda: M'M \Rightarrow MM'$ determines how the \mathbb{M} -structures and \mathbb{M}' -structures on C interact to form the structure encoded by the composite monad $\mathbb{M}\mathbb{M}'$.

Example 2.15. The category monad on $\mathbf{ps}(\mathcal{E})$ (Example 2.4) may be obtained as a composite of the *semi-category monad*, which governs associative composition, and the *reflexive graph monad* that adjoins a distinguished loop at each vertex of a graph $\mathbf{G} \in \mathbf{ps}(\mathcal{E})$. The corresponding distributive law encodes the property that the adjoined loops provide identities for the semi-categorical composition.

(There is also a distributive law in the other direction, but the two structures do not interact in the composite. See also Remark 7.43.)

As usual, let $C^{\mathbb{M}}$ denote the EM category of algebras for a monad \mathbb{M} on C .

By [3, Section 3], given a distributive law $\lambda: M'M \Rightarrow MM'$ for monads \mathbb{M}, \mathbb{M}' on C , there is a commuting square of strict monadic adjunctions:

$$(2.16) \quad \begin{array}{ccc} C^{\mathbb{M}'} & \xleftarrow{\top} & C^{\mathbb{M}\mathbb{M}'} \\ \downarrow \vdash & & \downarrow \vdash \\ C & \xleftarrow{\top} & C^{\mathbb{M}} \end{array}$$

In Section 4, it is shown that the category \mathbf{CetGr} of connected Feynman graphs and étale morphisms (first defined in [23]) fits into a chain $\mathbf{B}^\odot \hookrightarrow \mathbf{CetGr} \hookrightarrow \mathbf{GS}$ of fully faithful dense embeddings. And, in Section 7, the modular operad monad \mathbb{O} on \mathbf{GS} is constructed as a composite \mathbb{DT} of monads \mathbb{T} (that governs contraction and non-unital multiplication) and \mathbb{D} (that governs multiplicative units) on \mathbf{GS} .

Hence, by (2.16), there is a monad \mathbb{T}_* on the EM category \mathbf{GS}_* of \mathbb{D} -algebras, such that $\mathbf{GS}_*^{\mathbb{T}_*} \cong \mathbf{MO}$ and a diagram of functors

$$(2.17) \quad \begin{array}{ccccccc} & & \Xi & \xrightarrow{\text{f.f.}} & \mathbf{MO} & \xrightarrow{N} & \mathbf{ps}(\Xi) \\ & & \uparrow j & \text{b.o.} & \uparrow \text{free} & \downarrow \text{forget} & \downarrow j^* \\ \mathbf{B}_*^\odot & \xrightarrow[\text{f.f.}]{\text{dense}} & \mathbf{CetGr}_* & \xrightarrow[\text{f.f.}]{\text{dense}} & \mathbf{GS}_* & \xrightarrow[\text{f.f.}]{} & \mathbf{ps}(\mathbf{CetGr}_*) \\ \uparrow \text{b.o.} & & \uparrow \text{b.o.} & & \uparrow \text{free} & \downarrow \text{forget} & \downarrow \\ \mathbf{B}^\odot & \xrightarrow[\text{f.f.}]{\text{dense}} & \mathbf{CetGr} & \xrightarrow[\text{f.f.}]{\text{dense}} & \mathbf{GS} & \xrightarrow[\text{f.f.}]{} & \mathbf{ps}(\mathbf{CetGr}) \end{array}$$

in which the categories \mathbf{B}_*^\odot , \mathbf{CetGr}_* and Ξ are obtained via bo-ff factorisations.

In Section 8, it is shown that \mathbb{T}_* has arities \mathbf{CetGr}_* (see Section 2), from whence it follows that the induced nerve $N: \mathbf{MO} \rightarrow \mathbf{ps}(\Xi)$ is fully faithful and its essential image is characterised in terms of \mathbf{B}_*^\odot .

3. GRAPHS AND THEIR MORPHISMS

This section is an introduction to Feynman graphs as defined in [23]. Most of this section and the next stay close to the original constructions there. Since [23] was just a short note, it contained very few proofs, and so relevant results are proved in full here. Extensive examples are also given. Where possible, definitions and examples are presented in a way that builds on Section 1 and highlights similarities with familiar concepts in basic topology.

This section deals with basic definitions and examples. The following section is devoted to a more detailed study of the topology of Feynman graphs, in terms of their étale morphisms.

3.1. Graph-like diagrams and Feynman graphs. In this paper, all graphs are finite, and may have loops, parallel edges, and loose ends (ports). Roughly speaking, a graph consists of a finite set of vertices V , and a finite set of connections \tilde{E} , together with an incidence relation: if \tilde{E} is the orbit space of a set E under an involution τ , then the incidence is a partial function $E \rightarrow V$ that attaches connections to vertices.

Example 3.1. Section 15 of [2] provides a nice overview of various graph definitions that appear in the operad literature. The definition that is perhaps most familiar is that found in, for example [16] and [7]. There, a graph G is described by sets V of vertices and E of edges, an involution $\hat{\tau}: E \rightarrow E$, and an incidence function $\hat{t}: E \rightarrow V$. The ports of G are the fixed points of the involution $\hat{\tau}$. A formal exceptional edge graph η is also allowed. Morphisms $\eta \rightarrow G$ are choices $\{*\} \rightarrow E$ of elements of E .

Feynman graphs are defined similarly to the graphs described in Example 3.1, except the involution on E must be fixed-point free, while the incidence is allowed to be a partial map $E \rightarrow V$. These subtle differences make it possible to encode the whole calculus of Feynman graphs in terms of the formal theory of diagrams in finite sets.

The category of *graph-like diagrams* is the category $\mathbf{ps}_f(\mathbf{D})$ of functors $\mathbf{D}^{\text{op}} \rightarrow \mathbf{Set}_f$, where \mathbf{D} is the small category $\bigcirc \bullet \longleftarrow \bullet \longrightarrow \bullet$, and \mathbf{Set}_f is the category of finite sets and all maps.

The initial object in $\mathbf{ps}_f(\mathbf{D})$ is the empty graph-like diagram:

$$\emptyset = \bigcirc \mathbf{0} \longleftarrow \mathbf{0} \longrightarrow \mathbf{0},$$

and the terminal object \star is the trivial diagram of singletons:

$$\star = \bigcirc \mathbf{1} \longleftarrow \mathbf{1} \longrightarrow \mathbf{1}.$$

Feynman graphs, introduced in [23], are graph-like diagrams satisfying extra properties:

Definition 3.2. A Feynman graph is a graph-like diagram

$$\mathcal{G} = \tau \bigcirc E \xleftarrow{s} H \xrightarrow{t} V$$

such that $s: H \rightarrow E$ is injective and $\tau: E \rightarrow E$ is an involution without fixed points.

A (strong) subgraph $\mathcal{H} \hookrightarrow \mathcal{G}$ of a Feynman graph \mathcal{G} is a subdiagram that inherits a Feynman graph structure from \mathcal{G} .

The full subcategory on graphs in $\mathbf{psf}(\mathbf{D})$ is denoted by $\mathbf{Grpsf}(\mathbf{D})$.

Elements of V are *vertices* of \mathcal{G} and elements of E are called *edges* of \mathcal{G} . For each edge e , \tilde{e} is the τ -orbit of e , and \tilde{E} is the set of τ -orbits in E . The set H of *half-edges* of \mathcal{G} , and the maps s and t encode a partial map $E \rightarrow V$ describing the incidence for the graph. A half-edge $h \in H$ may also be written as the ordered pair $h = (s(h), t(h))$.

In general, unless I wish to emphasise a point that is specific to the formalism of Feynman graphs, I will refer to Feynman graphs simply as ‘graphs’.

Remark 3.3. A graph \mathcal{G} may be realised geometrically by a one-dimensional space $|\mathcal{G}|$ obtained from the discrete space $\{*_v\}_{v \in V}$, and, for each $e \in E$, a copy $[0, \frac{1}{2}]_e$ of the interval $[0, \frac{1}{2}]$ subject to the identifications

- $0_{s(h)} \sim *_v$ for $h \in H$,
- $(\frac{1}{2})_e \sim (\frac{1}{2})_{\tau e}$ for all $e \in E$.

So each orbit of the involution is represented by a closed interval $[0, 1]$.

Example 3.4. (See also Figure 8(a).) The *stick graph* (i) has no vertices and edge set $\mathbf{2} = \{1, 2\}$.

$$(i) \stackrel{\text{def}}{=} \bigcirc \mathbf{2} \xleftarrow{\quad} \mathbf{0} \xrightarrow{\quad} \mathbf{0}.$$

In general, a stick graph is a graph that is isomorphic to (i).

For any set X , $X^\dagger \cong X$ denotes its formal involution.

Example 3.5. (See also Figure 8(b), (c).) The X -corolla \mathcal{C}_X associated to a finite set X has the form

$$\mathcal{C}_X : \quad \dagger \bigcirc X \amalg X^\dagger \xleftarrow{\text{inc}} X^\dagger \longrightarrow \{*\}.$$

(a)



(b)



(c)

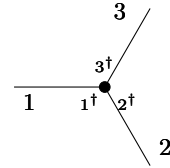


FIGURE 8. Realisations of (a) the stick graph (i), and (b), (c) the corollas $\mathcal{C}_{\{x\}}$ and \mathcal{C}_3 .

Definition 3.6. An inner edge of \mathcal{G} is an element $e \in E$ such that $e \in \text{im}(s)$ and $\tau e \in \text{im}(s)$. The set $E_\bullet \subset E$ of inner edges of \mathcal{G} is the maximum subset of $\text{im}(s) \subset E$ that is closed under τ . The set of inner τ -orbits $\tilde{e} \in \tilde{E}$ with $e \in E_\bullet$ is denoted by \tilde{E}_\bullet .

The set $E_0 = E \setminus \text{im}(s)$ is the boundary of \mathcal{G} and elements $e \in E_0$ are ports of \mathcal{G} . So, the stick graph (i) has $E_0(i) = E(i) = \mathbf{2}$, and, for all finite sets X , the X -corolla \mathcal{C}_X has boundary $E_0(\mathcal{C}_X) = X$.

A stick component of a graph \mathcal{G} is a pair $\{e, \tau e\}$ of edges such that both e , and τe are ports.

Graph morphisms preserve inner edges by definition.

Since \mathbf{Set}_f admits finite (co)limits, so does $\mathbf{ps}_f(\mathbf{D})$, and these are computed pointwise. And, since $\mathbf{Grps}_f(\mathbf{D})$ is full in $\mathbf{ps}_f(\mathbf{D})$, (co)limits in $\mathbf{Grps}_f(\mathbf{D})$, when they exist, correspond to (co)limits in $\mathbf{ps}_f(\mathbf{D})$.

Example 3.7. The empty graph-like diagram \emptyset is trivially a graph, and is therefore initial in $\mathbf{Grps}_f(\mathbf{D})$. However, there is no non-trivial involution on a singleton set, so the terminal diagram \star in $\mathbf{ps}_f(\mathbf{D})$ is not a graph. Hence, $\mathbf{Grps}_f(\mathbf{D})$ is not closed in $\mathbf{ps}_f(\mathbf{D})$ under finite limits. (By Examples 3.16 and 3.33, $\mathbf{Grps}_f(\mathbf{D})$ is also not closed under finite colimits in $\mathbf{ps}_f(\mathbf{D})$.)

The cocartesian monoidal structure on \mathbf{Set}_f is inherited by $\mathbf{ps}_f(\mathbf{D})$ and $\mathbf{Grps}_f(\mathbf{D})$, making these into strict symmetric monoidal categories under pointwise disjoint union \amalg , and with monoidal unit given by the empty graph \emptyset .

Example 3.8. Let X and Y be finite sets. The graph $\mathcal{M}_{x_0, y_0}^{X, Y}$, illustrated in Figure 9, has two vertices and one inner edge orbit (highlighted in bold-face in Figure 9). It is obtained from the disjoint union $\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}}$ by identifying the τ -orbits of the ports x_0 and y_0 according to $x_0 \sim \tau y_0, y_0 \sim \tau x_0$:

$$\tau \bigcirc ((X \amalg Y) \amalg (X \amalg Y)^\dagger \amalg \{x_0, y_0\}) \xleftarrow{s} ((X \amalg Y)^\dagger \amalg \{x_0, y_0\}) \xrightarrow{t} \{v_X, v_Y\},$$

where s is the obvious inclusion, and the involution τ is described by $z \leftrightarrow z^\dagger$ for $z \in X \amalg Y$ and $x_0 \leftrightarrow y_0$. The map t is described by $t^{-1}(v_X) = X^\dagger \amalg \{y_0\}$ and $t^{-1}(v_Y) = Y^\dagger \amalg \{x_0\}$.

In the construction of modular operads, graphs of the form $\mathcal{M}_{x_0, y_0}^{X, Y}$ are used to encode formal multiplications in graphical species.

Example 3.9. Formal contractions in graphical species are encoded by graphs of the form \mathcal{N}_{x_0, y_0}^X (see Figure 9). For X a finite set, the graph \mathcal{N}_{x_0, y_0}^X is the quotient of the corolla $\mathcal{C}_{X \amalg \{x_0, y_0\}}$ obtained by identifying the τ -orbits of the ports x_0 and y_0 according to $x_0 \sim \tau y_0$ and $y_0 \sim \tau x_0$. It has boundary $E_0 = X$, one inner τ -orbit $\{x_0, y_0\}$ (bold-face in Figure 9), and one vertex v .

$$\tau \bigcirc (X \amalg X^\dagger \amalg \{x_0, y_0\}) \xleftarrow{s} (X^\dagger \amalg \{x_0, y_0\}) \xrightarrow{t} \{v\}.$$

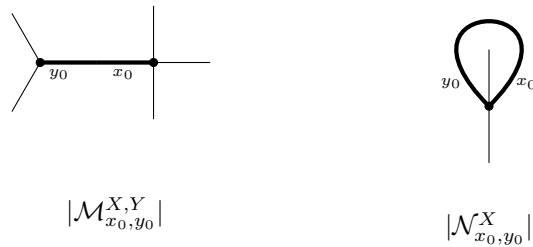


FIGURE 9. Realisations of $\mathcal{M}_{x_0, y_0}^{X, Y}$ and \mathcal{N}_{x_0, y_0}^X for $X \cong \mathbf{2}$, $Y \cong \mathbf{3}$.

Let \mathcal{G} be a graph with vertex and edge sets V and E respectively. For each vertex v , define $H/v \stackrel{\text{def}}{=} t^{-1}(v) \subset H$ to be the fibre of t at v .

Definition 3.10. Edges in the set $E/v \stackrel{\text{def}}{=} s(H/v) \subset E$ are said to be incident on v .

The map $|\cdot|: V \rightarrow \mathbb{N}$, $v \mapsto |v| \stackrel{\text{def}}{=} |H/v|$, defines the valency of v and $V_n \subset V$ is the set of n -valent vertices of \mathcal{G} . A bivalent graph is a graph \mathcal{G} with $V = V_2$.

A vertex v is bivalent if $|v| = 2$. An isolated vertex of \mathcal{G} is a vertex $v \in V(\mathcal{G})$ such that $|v| = 0$.

Bivalent, and isolated vertices are particularly important in what follows.

Vertex valency also induces an \mathbb{N} -grading on the edge set E (and half-edge set H) of \mathcal{G} : For $n \geq 1$, define $H_n \stackrel{\text{def}}{=} t^{-1}(V_n)$ and $E_n \stackrel{\text{def}}{=} s(H_n)$. Since $s(H) = E \setminus E_0 = \coprod_{n \geq 1} E_n$,

$$E = \coprod_{n \in \mathbb{N}} E_n.$$

Example 3.11. (See Example 3.4.) Since $H(\mathfrak{i})$ is empty, all edges of (\mathfrak{i}) are ports: $E(\mathfrak{i}) = E_0(\mathfrak{i})$.

The corolla \mathcal{C}_X (see Example 3.5) with vertex $*$ has $X \cong E/* = H/*$. If $|X| = k$, then $|*| = k$, so $V = V_k$, and $E = E_k \amalg E_0$ with $E_i \cong X$ for $i = 0, k$.

Example 3.12. (See Examples 3.8, 3.9.) For finite sets X and Y , the graph $\mathcal{M}_{x_0, y_0}^{X, Y}$ has $E/v_X = X^\dagger \amalg \{y_0\}$ and $E/v_Y = Y^\dagger \amalg \{x_0\}$. If $X \cong \mathbf{n}$ for some $n \in \mathbb{N}$, then $v_X \in V_{n+1}$, and $E/v_X \subset E_{n+1}$.

The graph \mathcal{N}_{x_0, y_0}^X has $E/v = X^\dagger \amalg \{x_0, y_0\} \cong H$, so $V = V_{n+2}$ when $X \cong \mathbf{n}$.

Since $\mathbf{Grps}_f(\mathbf{D})$ is full in the diagram category $\mathbf{ps}_f(\mathbf{D})$, morphisms $f \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}, \mathcal{G}')$ are commuting diagrams in \mathbf{Set}_f of the form

$$(3.13) \quad \begin{array}{ccccccc} \mathcal{G} & & E & \xleftarrow{\tau} & E & \xleftarrow{s} & H & \xrightarrow{t} & V \\ f \downarrow & & f_E \downarrow & & f_E \downarrow & & f_H \downarrow & & f_V \downarrow \\ \mathcal{G}' & & E' & \xleftarrow{\tau'} & E' & \xleftarrow{s'} & H' & \xrightarrow{t'} & V' \end{array}$$

Lemma 3.14. For any morphism $f = (f_E, f_H, f_V) \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}, \mathcal{G}')$, the map f_H is completely determined by f_E . Moreover if \mathcal{G} has no isolated vertices, then f_E also determines f_V , and hence f .

Conversely, if \mathcal{G} has no stick components or isolated vertices, then f is completely determined by f_H .

See also [28, Proposition 1.1.11].

Proof. By injectivity of s , The map $f_H: H \rightarrow H'$ given by $h \mapsto (s')^{-1} f_E s(h)$ is well defined since s is injective. If \mathcal{G} has no isolated vertices, then, for each $v \in V$, H/v is non-empty and the map $f_V: V \rightarrow V'$ given by $v \mapsto t'(s')^{-1} f_E s(h)$ does not depend on the choice of $h \in H/v$.

If \mathcal{G} has no stick components then, for each $e \in E$, there is an $h \in H$ such that $e = s(h)$ or $e = \tau s(h)$, and the last statement of the lemma follows from the first. \square

Example 3.15. For any graph \mathcal{G} with edge set E , there is a canonical (up to unique isomorphism) bijection $\mathbf{Grps}_f(\mathbf{D})(\mathfrak{i}, \mathcal{G}) \cong E$. The morphism $1 \mapsto e \in E$ in $\mathbf{Grps}_f(\mathbf{D})(\mathfrak{i}, \mathcal{G})$ that chooses $e \in E$ is denoted ch_e , or $ch_e^{\mathcal{G}}$.

Example 3.16. By Example 3.15, the stick graph (\mathfrak{i}) has endomorphisms $ch_1 = id_x$ and $ch_2 = \tau$ in $\mathbf{Grps}_f(\mathbf{D})$. The coequaliser of $id, \tau: (\mathfrak{i}) \rightrightarrows (\mathfrak{i})$ in the category $\mathbf{ps}_f(\mathbf{D})$ of graph-like diagrams is the *exceptional loop* \bigcirc :

$$\bigcirc \stackrel{\text{def}}{=} \begin{array}{c} \curvearrowright \\ \mathbf{1} \end{array} \longleftarrow \mathbf{0} \longrightarrow \mathbf{0}.$$

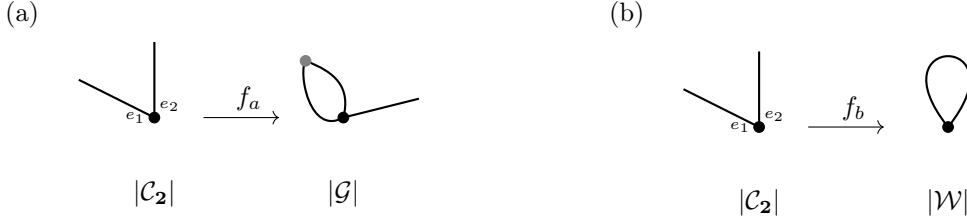
Clearly \bigcirc is not a graph since a singleton set does not admit a non-trivial involution. Hence $\mathbf{Grps}_f(\mathbf{D})$ does not admit all finite colimits. This example is the subject of Section 6.

Definition 3.17. A morphism $f \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}, \mathcal{G}')$ is locally injective if, for all $v \in V$, the induced map $f_v: E/v \rightarrow E'/f(v)$ is injective, and locally surjective if $f_v: E/v \rightarrow E'/f(v)$ is surjective for all $v \in V$.

Locally bijective morphisms are called *étale*.

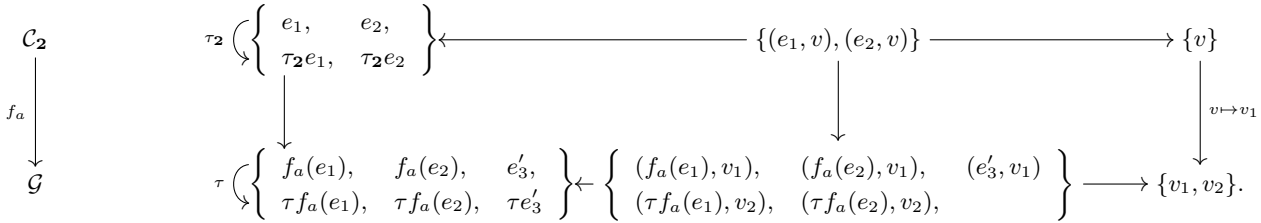
Étale morphisms of graphs are studied in Section 4.

Example 3.18. The following display illustrates two examples of morphisms in $\mathbf{Grps}_f(\mathbf{D})$, described below. Both morphisms are locally injective, and (b) is also surjective and locally surjective, hence étale.

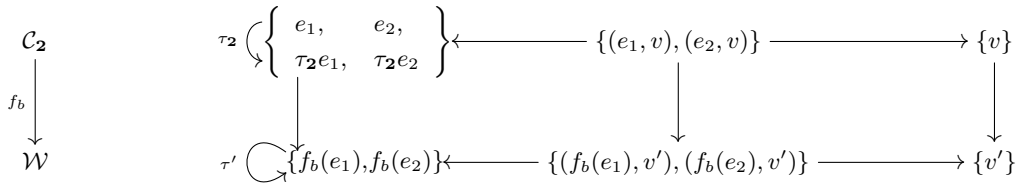


In each example, the horizontal maps are the obvious projections, and the columns in the edge sets represent the orbits of the involution.

(a) The map f_a is determined by the image of the edges e_1 and e_2 .



(b) The map f_b is determined by $e_1, \tau_2 e_2 \mapsto f_b(e_1)$ and $e_2, \tau_2 e_1 \mapsto f_b(e_2)$.



Example 3.19. Recall Examples 3.8 and 3.9, above. For finite sets X and Y , the canonical morphisms $\mathcal{C}_X \longrightarrow \mathcal{M}_{x_0, y_0}^{X, Y} \longleftarrow \mathcal{C}_Y$ and $\mathcal{C}_X \longrightarrow \mathcal{N}_{x_0, y_0}^X$ are locally injective, but not locally surjective.

The canonical morphisms $\mathcal{C}_{X \amalg \{x_0\}} \longrightarrow \mathcal{M}_{x_0, y_0}^{X, Y} \longleftarrow \mathcal{C}_{Y \amalg \{y_0\}}$ are locally injective and locally surjective, hence étale, but neither is surjective. However, the canonical morphism $\mathcal{C}_{X \amalg \{x_0, y_0\}} \rightarrow \mathcal{N}_{x_0, y_0}^X$ is étale and surjective.

Example 3.20. The map $X \mapsto \mathcal{C}_X$ describes a full embedding of \mathbf{Set}_f into $\mathbf{Grps}_f(\mathbf{D})$. Since $\mathbf{Grps}_f(\mathbf{D})(\mathfrak{l}, \mathfrak{l}) \cong \mathbf{B}^\odot(\S, \S)$ canonically, and any morphism in $\mathbf{Grps}_f(\mathbf{D})$ with domain (\mathfrak{l}) is étale, it follows that \mathbf{B}^\odot embeds in $\mathbf{Grps}_f(\mathbf{D})$ as the subcategory of étale morphisms between the corollas \mathcal{C}_X ($X \in \mathbf{Set}_f$), and (\mathfrak{l}) .

Remark 3.21. Henceforth, \mathbf{B}^\odot will also be viewed as a subcategory of $\mathbf{Grps}_f(\mathbf{D})$. The choice of notation for objects – (\mathfrak{l}) or \S , X or \mathcal{C}_X – will depend on the context. However, the same notation will be used for morphisms in \mathbf{B}^\odot and their image in $\mathbf{Grps}_f(\mathbf{D})$. So $ch_x \in \mathbf{B}^\odot(\S, X)$ (or $f \in \mathbf{B}^\odot(X, Y)$) may also be written as $ch_x \in \mathbf{Grps}_f(\mathbf{D})(\mathfrak{l}, \mathcal{C}_X)$ (or $f \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{C}_X, \mathcal{C}_Y)$).

Example 3.22. For all finite sets X and Y , the pair of parallel morphisms

$$(3.23) \quad ch_{x_0}, ch_{y_0} \circ \tau: (1) \rightrightarrows (\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}}).$$

is in the image of the inclusion $\mathbf{B}^\odot \hookrightarrow \mathbf{Grps}_f(\mathbf{D})$, and has colimit $\mathcal{M}_{x_0, y_0}^{X, Y}$ in $\mathbf{Grps}_f(\mathbf{D})$.

The graph \mathcal{N}_{x_0, y_0}^X is the colimit in $\mathbf{Grps}_f(\mathbf{D})$ of the pair

$$(3.24) \quad ch_{x_0}, ch_{y_0} \circ \tau: (1) \rightrightarrows \mathcal{C}_{X \amalg \{x_0, y_0\}}$$

of parallel morphisms in the image of \mathbf{B}^\odot in $\mathbf{Grps}_f(\mathbf{D})$. (See also Example 5.13 and Figure 15.)

As will be shown in Section 4, all graphs may be constructed canonically as colimits of diagrams in the image of the embedding $\mathbf{B}^\odot \hookrightarrow \mathbf{Grps}_f(\mathbf{D})$ in Example 3.20.

3.2. Connected components of graphs. A graph is connected if it cannot be written as a disjoint union of non-empty connected graphs. Precisely:

Definition 3.25. A non-empty graph-like diagram \mathcal{G} is connected if, for each $f \in \mathbf{ps}_f(\mathbf{D})(\mathcal{G}, \star \amalg \star)$, the pullback of f along the inclusion $\star \xrightarrow{\text{incl}_1} \star \amalg \star$ is either the empty graph-like diagram \emptyset or \mathcal{G} itself. A graph \mathcal{G} is connected if it is connected as a graph-like diagram.

A (connected) component of a graph \mathcal{G} is a maximal connected subdiagram of \mathcal{G} .

By Definition 3.2, a subdiagram $\mathcal{H} \hookrightarrow \mathcal{G}$ is a subgraph precisely when $E(\mathcal{H}) \subset E$ is closed under $\tau: E \rightarrow E$. Hence:

Lemma 3.26. A connected component of a graph \mathcal{G} inherits a subgraph structure from \mathcal{G} . If $\mathcal{H} \hookrightarrow \mathcal{G}$ is a subgraph of \mathcal{G} , then so is its complement $\mathcal{G} \setminus \mathcal{H}$.

Therefore, every graph is the disjoint union of its connected components.

Remark 3.27. A graph \mathcal{G} is connected if and only if its realisation $|\mathcal{G}|$ is a connected space.

Example 3.28. Following [28], a *shrub* \mathcal{S} is a graph that is isomorphic to a disjoint union of stick graphs. Hence a shrub $\mathcal{S} = \mathcal{S}(J)$ is determined by a set J of edges, equipped with a fixed-point free involution τ_J . If f is a morphism in $\mathbf{Grps}_f(\mathbf{D})$ whose domain is a shrub, then f is trivially étale.

Given any graph \mathcal{G} , the shrub $\mathcal{S}(E)$ determined by (E, τ) is canonically a subgraph of \mathcal{G} . Components of $\mathcal{S}(E)$ are of the form

$$(\downarrow_{\tilde{e}}) \stackrel{\text{def}}{=} \downarrow \{e, \tau e\} \longleftarrow \emptyset \longrightarrow \emptyset,$$

for each $\tilde{e} \in \tilde{E}$. This is a subgraph of \mathcal{G} under the *essential morphism* $\iota_{\tilde{e}}: (\downarrow_{\tilde{e}}) \hookrightarrow \mathcal{G}$ at \tilde{e} (for \mathcal{G}) induced by the inclusion $\{e, \tau e\} \hookrightarrow E$.

Recall from Definition 3.6 that a stick component of \mathcal{G} is a τ -orbit $\{e, \tau e\}$ in the boundary E_0 of \mathcal{G} . In particular, $\{e, \tau e\}$ is a stick component of \mathcal{G} if and only if $(\downarrow_{\tilde{e}})$ is a connected component of \mathcal{G} .

Example 3.29. Recall that, for each $v \in V$, $E/v \stackrel{\text{def}}{=} s(t^{-1}(v))$ is the set of edges incident on v . Let $\mathbf{v} = (E/v)^\dagger$ denote its formal involution. Then the corolla $\mathcal{C}_{\mathbf{v}}$ is given by

$$\mathcal{C}_{\mathbf{v}} = \bigcirc_{\leftarrow} (E/v \amalg (E/v)^\dagger) \xleftarrow{s} H/v \xrightarrow{t} \{v\}.$$

The inclusion $E/v \hookrightarrow E$ induces a morphism $\iota_v^{\mathcal{G}}$ or $\iota_v: \mathcal{C}_{\mathbf{v}} \rightarrow \mathcal{G}$ called the *essential morphism at v* for \mathcal{G} . Observe that, whenever there is an edge e such that both e and τe are incident on v , then ι_v is not injective on edges.

If E/v is empty – so $\mathcal{C}_{\mathbf{v}}$ is an isolated vertex – then $\mathcal{C}_{\mathbf{v}} \hookrightarrow \mathcal{G}$ is a connected component of \mathcal{G} .

Example 3.30. For $k \geq 0$, the *line graph* \mathcal{L}^k is the connected bivalent graph (illustrated in Figure 10) with boundary $E_0 = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\}$, and

- ordered set of edges $E(\mathcal{L}^k) = (l_j)_{j=0}^{2k+1}$ where $l_0 = 1_{\mathcal{L}^k} \in E_0$ and $l_{2k+1} = 2_{\mathcal{L}^k} \in E_0$, and the involution is given by $\tau(l_{2i}) = l_{2i+1}$, for $0 \leq i \leq k$,
- ordered set of k vertices $V(\mathcal{L}^k) = (v_i)_{i=1}^k$, such that $E/v_i = \{l_{2i-1}, l_{2i}\}$ for $1 \leq i \leq k$.

So, \mathcal{L}^k is described by a diagram of the form $\bigcirc \mathbf{2} \amalg 2(\mathbf{k}) \longleftarrow 2(\mathbf{k}) \longrightarrow \mathbf{k}$.

Example 3.31. The *wheel graph* $\mathcal{W} = \mathcal{W}^1$ with one vertex is the graph

$$(3.32) \quad \mathcal{W} \stackrel{\text{def}}{=} \bigcirc \{a, \tau a\} \longleftarrow \{a, \tau a\} \longrightarrow \{*\}$$

obtained as the coequaliser in $\mathbf{Grps}_f(\mathbf{D})$ of the morphisms $ch_1, ch_2 \circ \tau: (1) \rightrightarrows \mathcal{C}_2$ (see Example 3.22).

More generally, for $m \geq 1$, the wheel graph \mathcal{W}^m (illustrated in Figure 10) is the connected bivalent graph obtained as the coequaliser in $\mathbf{Grps}_f(\mathbf{D})$ of the morphisms $ch_{1_{\mathcal{L}^m}}, ch_{2_{\mathcal{L}^m}} \circ \tau: (1) \rightrightarrows \mathcal{L}^m$. So \mathcal{W}^m has empty boundary and

- $2m$ cyclically ordered edges $E(\mathcal{W}^m) = (a_j)_{j=1}^{2m}$, such that the involution satisfies $\tau(a_{2i}) = a_{2i+1}$ for $0 \leq i \leq m$ (where $a_0 = a_{2m}$),
- m cyclically ordered vertices $V(\mathcal{W}^m) = (v_i)_{i=1}^m$, that $E/v_i = \{a_{2i-1}, a_{2i}\}$ for $1 \leq i \leq m$.

So \mathcal{W}^m is described by a diagram of the form $\bigcirc 2(\mathbf{m}) \longleftarrow 2(\mathbf{m}) \longrightarrow \mathbf{m}$.

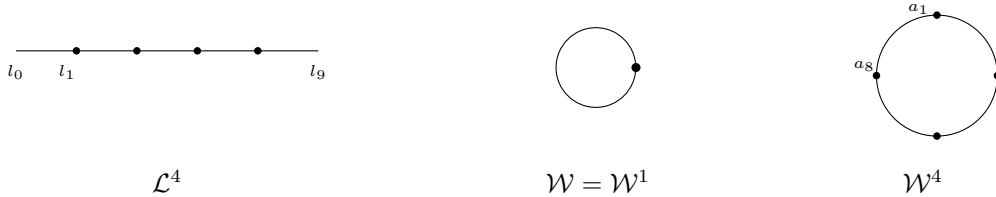


FIGURE 10. Line and wheel graphs.

In Proposition 4.23, it will be shown that a connected bivalent graph is isomorphic to \mathcal{L}^k or \mathcal{W}^m for some $k \geq 0$ or $m \geq 1$.

Example 3.33. The wheel graph \mathcal{W} with one vertex is weakly terminal in $\mathbf{Grps}_f(\mathbf{D})$: By Lemma 3.14, since $\tilde{E}(\mathcal{W}) \cong V(\mathcal{W}) \cong \{*\}$, morphisms in $\mathbf{Grps}_f(\mathbf{D})(\mathcal{G}, \mathcal{W})$ are in canonical bijection with projections $\mathbf{Grps}_f(\mathbf{D})(\mathcal{S}(E), \iota_a)$. Hence, for all graphs \mathcal{G} , there are precisely $2^{|E|} \geq 1$ morphisms $\mathcal{G} \rightarrow \mathcal{W}$ in $\mathbf{Grps}_f(\mathbf{D})$ and every diagram in $\mathbf{Grps}_f(\mathbf{D})$ forms a cocone over \mathcal{W} .

In particular,

$$\mathbf{Grps}_f(\mathbf{D})(\mathcal{W}, \mathcal{W}) = \{id_{\mathcal{W}}, \tau_{\mathcal{W}}\} \cong \mathbf{Grps}_f(\mathbf{D})(1, \mathcal{W}) \cong \mathbf{Grps}_f(\mathbf{D})(1, 1).$$

The morphisms $id_{\mathcal{W}}, \tau_{\mathcal{W}}: \mathcal{W} \rightarrow \mathcal{W}$ do not admit a coequaliser in $\mathbf{Grps}_f(\mathbf{D})$ since their coequaliser in $\mathbf{ps}_f(\mathbf{D})$ is the terminal diagram \star , which is not a graph.

Example 3.33 leads to another characterisation of connectedness:

Proposition 3.34. *The following are equivalent:*

- (1) A graph \mathcal{G} is connected;
- (2) \mathcal{G} is non-empty and, for every morphism $f \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}, \mathcal{W} \amalg \mathcal{W})$, the pullback in $\mathbf{ps}_f(\mathbf{D})$ of f along the inclusion $inc_1: \mathcal{W} \hookrightarrow \mathcal{W} \amalg \mathcal{W}$ is either the empty graph \emptyset or isomorphic to \mathcal{G} itself;

(3) for every finite disjoint union of graphs $\coprod_{i=1}^k \mathcal{H}_i$,

$$(3.35) \quad \text{Grps}_f(\mathcal{D})(\mathcal{G}, \coprod_{i=1}^k \mathcal{H}_i) \cong \prod_{i=1}^k \text{Grps}_f(\mathcal{D})(\mathcal{G}, \mathcal{H}_i).$$

Proof. (1) \Leftrightarrow (2): Since \mathcal{W} is weakly terminal, any morphism $f \in \text{ps}_f(\mathcal{D})(\mathcal{G}, \star \amalg \star)$ factors as a morphism $\tilde{f} \in \text{Grps}_f(\mathcal{D})(\mathcal{G}, \mathcal{W} \amalg \mathcal{W})$ followed by the componentwise projection $\mathcal{W} \amalg \mathcal{W} \rightarrow \star \amalg \star$ in $\text{ps}_f(\mathcal{D})$.

(1) \Rightarrow (3): For any finite disjoint union of graphs $\coprod_{i=1}^k \mathcal{H}_i$, and each $1 \leq j \leq k$, let $p_j \in \text{ps}_f(\mathcal{D})(\coprod_{i=1}^k \mathcal{H}_i, \star \amalg \star)$ be the morphism that projects \mathcal{H}_j onto the first summand, and $\coprod_{i \neq j} \mathcal{H}_i$ onto the second summand. Then, for any graph \mathcal{G} and any $f \in \text{Grps}_f(\mathcal{D})(\mathcal{G}, \coprod_{i=1}^k \mathcal{H}_i)$, the diagram

$$(3.36) \quad \begin{array}{ccc} \mathcal{P}_j & \xrightarrow{\quad} & \mathcal{G} \\ \downarrow & & \downarrow f \\ \mathcal{H}_j & \xrightarrow{\quad inc_j \quad} & \coprod_{i=1}^k \mathcal{H}_i \\ \downarrow & & \downarrow p_j \\ \star & \xrightarrow{\quad inc_1 \quad} & \star \amalg \star \end{array}$$

where the top square is a pullback, commutes in $\text{ps}_f(\mathcal{D})$. Since the lower square is a pullback by construction, so is the outer rectangle.

In particular, if \mathcal{G} is connected, then \mathcal{P}_j is either empty or isomorphic to \mathcal{G} itself. But this implies that there is some unique $1 \leq j \leq k$ such that f factors through the inclusion $inc_j \in \text{Grps}_f(\mathcal{D})(\mathcal{H}_j, \coprod_{i=1}^k \mathcal{H}_i)$. In other words, $\text{Grps}_f(\mathcal{D})(\mathcal{G}, \coprod_{i=1}^k \mathcal{H}_i) \cong \prod_{i=1}^k \text{Grps}_f(\mathcal{D})(\mathcal{G}, \mathcal{H}_i)$.

(3) \Rightarrow (2): If \mathcal{G} satisfies condition (3.35), then $\text{Grps}_f(\mathcal{D})(\mathcal{G}, \mathcal{W} \amalg \mathcal{W}) \cong \text{Grps}_f(\mathcal{D})(\mathcal{G}, \mathcal{W}) \amalg \text{Grps}_f(\mathcal{D})(\mathcal{G}, \mathcal{W})$. So, taking $\coprod_{i=1}^k \mathcal{H}_i = \mathcal{W} \amalg \mathcal{W}$ in (3.36), we have $\mathcal{P}_j = \emptyset$ or $\mathcal{P}_j \cong \mathcal{G}$ for $j = 1, 2$. \square

3.3. Paths and cycles. Using line and wheel graphs (Examples 3.30 and 3.31), it is possible to define paths and cycles in a graph \mathcal{G} .

Definition 3.37. For any graph \mathcal{G} , a morphism $p \in \text{Grps}_f(\mathcal{D})(\mathcal{L}^k, \mathcal{G})$ describes a path of length k in \mathcal{G} . Given any pair $x_1, x_2 \in E \amalg V$, x_1 and x_2 are connected by a path $p \in \text{Grps}_f(\mathcal{D})(\mathcal{L}^k, \mathcal{G})$ if $\{x_1, x_2\} \subset \text{im}(p)$.

A non-empty graph \mathcal{G} is path connected if each pair of distinct elements $x_1, x_2 \in E \amalg V$ is connected by a path in \mathcal{G} .

Example 3.38. The isolated vertex \mathcal{C}_0 is trivially path connected.

Since $\text{Grps}_f(\mathcal{D})(\mathcal{L}^k, \mathcal{C}_1)$ is non-empty only when $k = 0$ or $k = 1$, only the unique path $\mathcal{L}^1 = \mathcal{C}_2 \rightarrow \mathcal{C}_1$ connects the unique vertex v of \mathcal{C}_1 with an edge $e \in E(\mathcal{C}_1)$.

Corollary 3.39 (Corollary to Proposition 3.34). A graph \mathcal{G} is connected if and only if it is path connected.

Proof. A morphism $f \in \text{Grps}_f(\mathcal{D})(\mathcal{G}, \mathcal{W}_1 \amalg \mathcal{W}_2)$ that does not factor through an inclusion $\mathcal{W} \hookrightarrow \mathcal{W}_1 \amalg \mathcal{W}_2$ exists if and only if there are distinct $x_1, x_2 \in E \amalg V$ such that $f(x_1) \in \mathcal{W}_1$ and $f(x_2) \in \mathcal{W}_2$. By Proposition 3.34, since \mathcal{L}^k is connected for all k , this is the case if and only if there is no $p \in \text{Grps}_f(\mathcal{D})(\mathcal{L}^k, \mathcal{G})$ connecting x_1, x_2 . \square

Lemma 3.40. If e_1 and e_2 are edges of a connected graph \mathcal{G} , then there is an injective path connecting e_1 and e_2 in \mathcal{G} .

Proof. For all edges e of \mathcal{G} , $ch_e : (1_e) = \mathcal{L}^0 \rightarrow \mathcal{G}$ describes an injective path connecting e and τe .

So, let e_1 and $e_2 \neq \tau e_1$ be distinct edges of a connected graph \mathcal{G} . By Corollary 3.39, there is a path $p \in \text{Grps}_f(\mathcal{D})(\mathcal{L}^k, \mathcal{G})$ connecting e_1 and e_2 in \mathcal{G} . Moreover, we may assume, without loss of generality, that, for $i = 1, 2$, $p(i_{\mathcal{L}^k}) \in \{e_i, \tau e_i\}$: if not, we may replace p with a path $p \circ \iota$, where $\iota: \mathcal{L}^{k'} \rightarrow \mathcal{L}^k$ ($1 \leq k' < k$) is injective, for which this holds.

Now, let $w \in V(\mathcal{G})$ and $1 \leq i_w < \dots < j_w \leq k$ be such that $p(v_{i_w}) = p(v_{j_w}) = w$. Let $L_w \stackrel{\text{def}}{=} i_w - j_w$.

Then, since $p(l_{2i_w-1}), p(l_{2j_w}) \in E/w$, there is a unique path $p_w: \mathcal{L}^{k-L_w} \rightarrow \mathcal{G}$ that agrees with p on ports (and therefore connects e_1 and e_2) and such that, for all $v'_l \in V(\mathcal{L}^{k-L_w})$, $1 \leq l \leq k - L_w$

$$p_w(v'_l) = \begin{cases} p(v_i) & \text{for } i \leq i_w \\ p(v_{i+L_w}) & \text{for } i > j_w. \end{cases}$$

This process may be iterated to obtain a path $p_I: \mathcal{L}^{k_I} \rightarrow \mathcal{G}$ that connects e_1 and e_2 , and is injective on vertices, and hence injective on inner edges. By assumption $e_2 \neq \tau e_1$ and $p_I(i_{\mathcal{L}^{k_I}}) \in \{e_i, \tau e_i\}$ for $i = 1, 2$. So, p_I is injective. \square

Morphisms from wheel graphs \mathcal{W}^m describe the higher genus structure of graphs (see Remark 3.43).

Definition 3.41. A cycle in \mathcal{G} is a morphism $c \in \text{Grps}_f(\mathcal{D})(\mathcal{W}^m, \mathcal{G})$ for some $m \geq 1$.

A connected graph \mathcal{G} is simply connected if it has no locally injective cycles.

A cycle $c: \mathcal{W}^m \rightarrow \mathcal{G}$ is trivial if there is a simply connected graph \mathcal{H} such that c factors through \mathcal{H} .

It follows from the proof of Lemma 3.40 that \mathcal{G} admits a locally injective cycle if and only if it admits a pointwise injective cycle. Using the cyclic ordering on the edges of each \mathcal{W}^m , it is straightforward to verify that a graph \mathcal{G} is simply connected if and only if its geometric realisation $|\mathcal{G}|$ is.

Example 3.42. For all finite sets X , the corolla \mathcal{C}_X is trivially simply connected since it does not admit any cycles.

Since the edge sets of the line graphs \mathcal{L}^k are totally ordered for all k , there can be no locally injective morphism $\mathcal{W}^m \rightarrow \mathcal{L}^k$, and hence \mathcal{L}^k is simply connected. However, for all $k \geq 1$, there are morphisms $\text{Grps}_f(\mathcal{D})(\mathcal{W}^{2k}, \mathcal{L}^{k+1})$ that are surjective on vertices and inner edges. Namely, let $V(\mathcal{W}^{2k}) = (w_i)_{i=1}^{2k}$ and $V(\mathcal{L}^{k+1}) = (v_j)_{j=1}^{k+1}$ be canonically ordered as in Examples 3.30, and 3.31. Then the assignment $w_i \mapsto v_i$ for $1 \leq i \leq 2k + 1$, and $w_{k+1+j} \mapsto v_{k+1-j}$ for $1 \leq j < k$ induces a morphism $q: \mathcal{W}^{2k} \rightarrow \mathcal{L}^{2k+1}$ that flattens \mathcal{W}^{2k} (Figure 11(a)). This fails to be a local injection at w_1 and w_k .

More generally, by flattening \mathcal{W}^2 as above, we see that, for any graph \mathcal{G} , the set E_\bullet of inner edges of \mathcal{G} is non-empty if and only if $\text{Grps}_f(\mathcal{D})(\mathcal{W}^2, \mathcal{G})$ is (see Figure 11(b)).

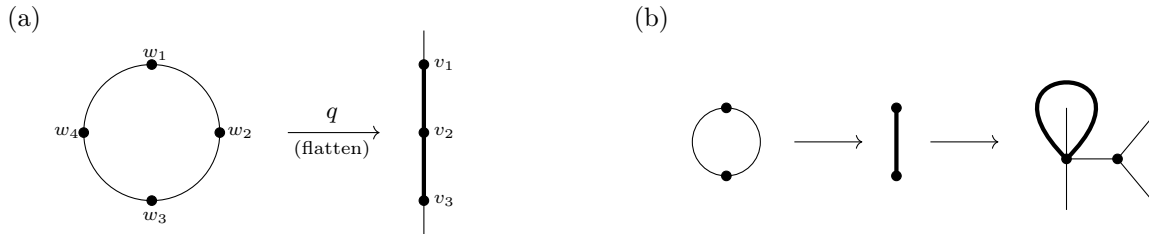


FIGURE 11. (a) A morphism $q: \mathcal{W}^4 \rightarrow \mathcal{L}^3$ in $\text{Grps}_f(\mathcal{D})$ that flattens \mathcal{W}^4 . (b) If a graph \mathcal{G} has an inner edge, then $\text{Grps}_f(\mathcal{D})(\mathcal{W}^2, \mathcal{G})$ is non-empty.

Remark 3.43. Using a modification of the proof of Lemma 3.40, it is quite straightforward to construct an equivalence relation of *homotopy* on paths in a graph \mathcal{G} , such that each path is homotopic to a unique

locally injective path. This extends to an equivalence of cycles in \mathcal{G} if $E_\bullet \neq \emptyset$. If \mathcal{G} is also connected, the set of equivalence classes of cycles has a canonical group structure that is isomorphic to the fundamental group $\pi_1(|\mathcal{G}|)$ of the geometric realisation of \mathcal{G} .

The fundamental group construction can be extended, using the methods introduced in Section 7 (and specifically Proposition 7.14), to all graphs \mathcal{G} without isolated vertices. These ideas are not developed in the current work.

4. THE ÉTALE SITE OF GRAPHS

By Example 3.20, there is a canonical embedding $\mathbf{B}^\circ \hookrightarrow \mathbf{Grps}_f(\mathbf{D})$ whose image consists of the exceptional graph (i), the corollas \mathcal{C}_X , and the étale (locally bijective) morphisms between them.

Local bijections are preserved under composition so étale morphisms form a subcategory \mathbf{etGr} of $\mathbf{Grps}_f(\mathbf{D})$. The full subcategory of \mathbf{etGr} on the connected graphs is denoted by \mathbf{CetGr} .

The goal of this section is to describe the categories \mathbf{CetGr} and \mathbf{etGr} in detail, and establish the chain

$$\mathbf{B}^\circ \hookrightarrow \mathbf{CetGr} \hookrightarrow \mathbf{GS}$$

of dense fully faithful embeddings discussed in Section 2.

The following is immediate from Definition 3.17 and the universal property of pullbacks of sets:

Proposition 4.1. *A morphism $f \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}, \mathcal{G}')$ is étale if and only if the right square in the defining diagram (3.13) is a pullback of finite sets.*

Example 4.2. For any graph \mathcal{G} and all $v \in V(\mathcal{G})$, the essential morphism $\iota_v: \mathcal{C}_v \rightarrow \mathcal{G}$ defined in Example 3.29 is étale. Indeed, a morphism $f \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}, \mathcal{G}')$ is étale if and only if f induces an isomorphism $\mathcal{C}_v \cong \mathcal{C}_{f(v)}$ for all $v \in V$.

Example 4.3. There are no étale morphism $\mathcal{W}^m \rightarrow \mathcal{L}^k$ for all $k \geq 0, m \geq 1$. For example, the flattening morphism $q: \mathcal{W}^{2k} \rightarrow \mathcal{L}^{k+1}$ in Example 3.42 fails to be locally injective (or surjective) at w_1 and w_k .

All étale morphisms between line graphs are pointwise injective: For $k, n \in \mathbb{N}$,

$$\mathbf{etGr}(\mathcal{L}^k, \mathcal{L}^n) \cong \begin{cases} 2(\mathbf{n} - \mathbf{k} + \mathbf{1}) & n \geq k \\ \emptyset, & n < k. \end{cases}$$

For $m \geq 1$, a morphism $f \in \mathbf{etGr}(\mathcal{L}^k, \mathcal{W}^m)$ is pointwise injective precisely when $k \leq m$. For all $k \geq 0$, f is fixed by $f(1_{\mathcal{L}^k}) \in E(\mathcal{W}^m)$. Hence, $\mathbf{etGr}(\mathcal{L}^k, \mathcal{W}^m) \cong E(\mathcal{W}^m) \cong 2(\mathbf{m})$.

Étale morphisms between wheel graphs are surjective: For $l, m \geq 1$

$$\mathbf{etGr}(\mathcal{W}^l, \mathcal{W}^m) \cong \begin{cases} 2(\mathbf{m}) & \text{if } \frac{l}{m} = d \in \mathbb{N} \\ \emptyset, & \text{otherwise.} \end{cases}$$

If m divides l , a morphism in $\mathbf{etGr}(\mathcal{W}^l, \mathcal{W}^m)$ is fixed by $w_1 = f(v_1) \in V(\mathcal{W}^m)$, and a choice $f(a_1) \in E/w_1$ of ‘winding direction’.

4.1. Pullbacks and monomorphisms in $\mathbf{Grps}_f(\mathbf{D})$. By Proposition 4.1, étale maps of graphs are local homeomorphisms and, as such, have similar properties to open maps of topological spaces.

Lemma 4.4. *The graph categories $\mathbf{Grps}_f(\mathbf{D})$ and \mathbf{etGr} admit pullbacks. Moreover, étale morphisms are preserved under pullbacks in $\mathbf{Grps}_f(\mathbf{D})$.*

Proof. Recall that graphs are objects of the presheaf category $\mathbf{ps}_f(\mathbf{D})$ of graph-like diagrams in \mathbf{Set}_f . Let $\mathcal{P} = (\underline{E}, \underline{H}, \underline{V}, \underline{s}, \underline{t}, \underline{\tau})$ be the pullback in $\mathbf{ps}_f(\mathbf{D})$ of morphisms $f_1 \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}_1, \mathcal{G})$ and $f_2 \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}_2, \mathcal{G})$.

Since pullbacks in $\mathbf{psf}(\mathcal{D})$ are computed pointwise, τ is a fixed-point free involution, and \underline{s} is injective. So, \mathcal{P} is a graph, and $\mathbf{Grpsf}(\mathcal{D})$ admits pullbacks.

Étale morphisms pull back to étale morphisms since limits commute with limits. Hence, by symmetry, \mathbf{etGr} admits pullbacks. \square

Definition 4.5. For any morphism $f \in \mathbf{Grpsf}(\mathcal{D})(\mathcal{H}, \mathcal{G})$, not necessarily étale, and any morphism $w: \mathcal{G}' \rightarrow \mathcal{G}$, the preimage $f^{-1}(\mathcal{G}') \rightarrow \mathcal{G}$ of \mathcal{G}' under f is defined as the pullback

$$\begin{array}{ccc} f^{-1}(\mathcal{G}') & \longrightarrow & \mathcal{H} \\ \downarrow & & \downarrow f \\ \mathcal{G}' & \xrightarrow{w} & \mathcal{G}. \end{array}$$

In particular, by Lemma 4.4, if $w: \mathcal{G}' \rightarrow \mathcal{G}$ is étale, then the preimage $f^{-1}(\mathcal{G}') \rightarrow \mathcal{H}$ is étale.

Epimorphisms in $\mathbf{Grpsf}(\mathcal{D})$ are pointwise surjections. However, the following example shows that monomorphisms in $\mathbf{Grpsf}(\mathcal{D})$, and in \mathbf{etGr} , are not necessarily pointwise injective. In particular, étale morphisms do not admit a unique epi-mono factorisation.

Example 4.6. Let \mathcal{W} be the wheel graph with vertex $v \in V(\mathcal{W})$. The essential morphism $\iota_v: \mathcal{C}_v \rightarrow \mathcal{W}$ is pointwise surjective, and easily seen to be a monomorphism. In fact, for all $k \geq 1$, the canonical morphism $\mathcal{L}^k \rightarrow \mathcal{W}^k$ (Example 3.31) is both an epimorphism and a monomorphism.

To characterise monomorphisms in $\mathbf{Grpsf}(\mathcal{D})$, observe that any (possibly empty) graph \mathcal{H} has the form $\mathcal{G}' \amalg \mathcal{S}$, where \mathcal{G}' is a graph without stick components and \mathcal{S} is a shrub.

Lemma 4.7. If $f \in \mathbf{Grpsf}(\mathcal{D})(\mathcal{H}, \mathcal{G})$ is a monomorphism, then

- (i) the images $f(\mathcal{G}')$ and $f(\mathcal{S})$ are disjoint in \mathcal{G} ;
- (ii) the restriction of f to \mathcal{S} is injective;
- (iii) f is injective on $V(\mathcal{H})$ and $H(\mathcal{H})$ (but not necessarily on $E(\mathcal{H})$).

Proof. Pairs of edges $e' \in E(\mathcal{G}')$ and $l \in E(\mathcal{S})$ of \mathcal{H} such that $f(e') = f(l) \in E(\mathcal{G})$ correspond to pairs of parallel morphisms $ch_{e'}, ch_l: (\mathbf{i}) \rightarrow \mathcal{H}$ such that $f \circ ch_l = f \circ ch_{e'}$. So monomorphisms in $\mathbf{Grpsf}(\mathcal{D})$ satisfy (i). A similar argument shows that monomorphisms also satisfy (ii) and (iii). \square

The converse also holds:

Proposition 4.8. Let $f \in \mathbf{Grpsf}(\mathcal{D})(\mathcal{H}, \mathcal{G})$ be a morphism. The following are equivalent:

- (1) f is a monomorphism;
- (2) f satisfies the conditions of Lemma 4.7;
- (3) if \mathcal{G}' has no inner edges, and $w: \mathcal{G}' \rightarrow \mathcal{G}$ satisfies the conditions of Lemma 4.7, then the induced morphism $f^{-1}(\mathcal{G}') \rightarrow \mathcal{G}'$ is pointwise injective.

Proof. The implication (1) \Rightarrow (2) follows from Lemma 4.7.

For (2) \Rightarrow (3), let \mathcal{G}' have no inner edges and let $w: \mathcal{G}' \rightarrow \mathcal{G}$ satisfy the conditions of Lemma 4.7. Then $f^{-1}(\mathcal{G}') \rightarrow \mathcal{G}'$ is pointwise injective if $f \in \mathbf{Grpsf}(\mathcal{D})(\mathcal{H}, \mathcal{G})$ satisfies the conditions of Lemma 4.7.

For (3) \Rightarrow (1), assume that (3) holds and let $f \in \mathbf{Grpsf}(\mathcal{D})(\mathcal{H}, \mathcal{G})$, and $w: \mathcal{G}' \rightarrow \mathcal{G}$ satisfy (3). So $f^{-1}(\mathcal{G}') \rightarrow \mathcal{G}'$ is a pointwise injection. If $g, h: \mathcal{H}' \rightrightarrows \mathcal{H}$ are parallel morphisms such that $f \circ g = f \circ h$, then, since $f^{-1}(\mathcal{G}')$ has no inner edges, neither does $(f \circ g)^{-1}(\mathcal{G}') = (f \circ h)^{-1}(\mathcal{G}')$ and the induced maps to $f^{-1}(\mathcal{G}')$ agree. By taking $w: \mathcal{G}' \rightarrow \mathcal{G}$ of the form $\mathcal{C}_0 \xrightarrow{v} \mathcal{G}$ for all $v \in V$, or $\iota_{\tilde{e}}: (\mathbf{i}_{\tilde{e}}) \rightarrow \mathcal{G}$ for $\tilde{e} \in \tilde{E}$, we see immediately that $g = h: \mathcal{H}' \rightarrow \mathcal{H}$. Hence f is a monomorphism in $\mathbf{Grpsf}(\mathcal{D})$. \square

By Lemma 3.14, any monomorphism $f: \mathcal{H} \rightarrow \mathcal{G}$ is either pointwise injective or, there exists a pair of ports $e_1, e_2 \in E_0(\mathcal{H})$ such that

- $\tau_{\mathcal{H}}e_1, \tau_{\mathcal{H}}e_2 \in s(H)$, and hence $e_2 \neq \tau_{\mathcal{H}}e_1$ (where $\tau_{\mathcal{H}}$ is the involution on $E(\mathcal{H})$),
- $\tau_{\mathcal{G}}f(e_2) = f(e_1) \in E_{\bullet}'(\mathcal{G})$ so $\{f(e_1), f(e_2)\}$ forms a $\tau_{\mathcal{G}}$ -orbit of inner edges of \mathcal{G} .

In this case we say that f *glues* pairs of ports of \mathcal{H} in \mathcal{G} .

Example 4.9. For all finite sets X and Y , the canonical étale morphisms $\mathcal{C}_{X \amalg \{x_0\}} \amalg \mathcal{C}_{Y \amalg \{y_0\}} \rightarrow \mathcal{M}_{x_0, y_0}^{X, Y}$ and $\mathcal{C}_{X \amalg \{x_0, y_0\}} \rightarrow \mathcal{N}_{x_0, y_0}^X$ are surjective and not pointwise injective. But they are monomorphisms.

4.2. Graph neighbourhoods and the essential category $\text{es}(\mathcal{G})$. A family of morphisms $\mathfrak{U} = \{f_i \in \text{etGr}(\mathcal{G}_i, \mathcal{G})\}_{i \in I}$ is *jointly surjective* on \mathcal{G} if $\mathcal{G} = \bigcup_{i \in I} \text{im}(f_i)$. By Lemma 4.4 etGr admits pullbacks, and jointly surjective families of morphisms $\{f_i \in \text{etGr}(\mathcal{G}_i, \mathcal{G})\}_{i \in I}$ define the covers at \mathcal{G} for a canonical *étale topology* J on etGr . Sheaves for this topology are those presheaves $P: \text{etGr}^{\text{op}} \rightarrow \text{Set}$ such that, for all graphs \mathcal{G} , and all covers $\mathfrak{U} = \{f_i \in \text{etGr}(\mathcal{G}_i, \mathcal{G})\}_{i \in I}$ at \mathcal{G} ,

$$(4.10) \quad P(\mathcal{G}) \cong \lim_{f_i \in \mathfrak{U}} P(\mathcal{G}_i).$$

In fact, as will be shown in Proposition 4.26, the category $\text{sh}(\text{etGr}, J)$ of sheaves for the étale site (etGr, J) is canonical to the category GS of graphical species (Definition 1.3).

In order to motivate this result, we will first establish more properties of étale morphisms. As usual, the analogy between étale morphisms of graphs and open maps of topological spaces provides some intuition.

Definition 4.11. A neighbourhood of a monomorphism $w: \mathcal{G}' \rightarrow \mathcal{G}$ is an étale monomorphism $u: \mathcal{U} \rightarrow \mathcal{G}$ such that $w = u \circ \tilde{w}: \mathcal{G}' \rightarrow \mathcal{U} \rightarrow \mathcal{G}$, for some monomorphism $\tilde{w}: \mathcal{G}' \rightarrow \mathcal{U}$.

A neighbourhood (\mathcal{U}, u) of $w: \mathcal{G}' \rightarrow \mathcal{G}$ is *minimal* if every other neighbourhood (\mathcal{U}', u') of $w: \mathcal{G}' \rightarrow \mathcal{G}$ is also a neighbourhood of (\mathcal{U}, u) .

Since vertices v of \mathcal{G} correspond to subgraphs $v: \mathcal{C}_0 \rightarrow \mathcal{G}$, and edges e of \mathcal{G} are in bijection with subgraphs $ch_e: (1) \hookrightarrow \mathcal{G}$, we may also refer to neighbourhoods of vertices and edges. Moreover, since $u: \mathcal{U} \rightarrow \mathcal{G}$ is a neighbourhood of $e \in E$ if and only if it is a neighbourhood of $\iota_{\tilde{e}}: (1_{\tilde{e}}) \rightarrow \mathcal{G}$, there is no loss of generality in referring to neighbourhoods of τ -orbits $\tilde{e} \in \tilde{E}$.

Let $\mathcal{S}(E_{\bullet}) = \coprod_{\tilde{e} \in \tilde{E}_{\bullet}} (1_{\tilde{e}})$ be the shrub on the inner edges of a graph \mathcal{G} . Given any subgraph $\mathcal{I} \hookrightarrow \mathcal{S}(E_{\bullet})$, there is a graph $\mathcal{G}_{\mathcal{I}}$ and a canonical surjective monomorphism $i_{\mathcal{I}}: \mathcal{G}_{\mathcal{I}} \rightarrow \mathcal{G}$ (as in Figure 12):

$$(4.12) \quad \begin{array}{ccccccc} \mathcal{G}_{\mathcal{I}} & & E \amalg (E(\mathcal{I}))^{\dagger} & \xleftarrow{\tau_{\mathcal{I}}} & E \amalg (E(\mathcal{I}))^{\dagger} & \xleftarrow{\hookrightarrow} & E & \xleftarrow{s} & H & \xrightarrow{t} & V \\ \downarrow i_{\mathcal{I}} & & \downarrow & & \downarrow & & & & \parallel & & \parallel \\ \mathcal{G} & & E & \xleftarrow{\tau} & E & \xleftarrow{s} & H & \xrightarrow{t} & H & \xrightarrow{t} & V, \end{array}$$

where

- $(E(\mathcal{I}))^{\dagger}$ is the formal involution $e \mapsto e^{\dagger}$ of the set $E(\mathcal{I})$ of edges of \mathcal{I} ,
- the involution $\tau_{\mathcal{I}}$ on $E \amalg (E(\mathcal{I}))^{\dagger}$ is defined by

$$e \mapsto \begin{cases} \tau e, & e \in E \setminus E(\mathcal{I}) \\ e^{\dagger}, & e \in E(\mathcal{I}), \end{cases}$$

- the surjection $E \amalg (E(\mathcal{I}))^{\dagger} \twoheadrightarrow E$ is the identity on E and $e^{\dagger} \mapsto \tau e$, $e \in E(\mathcal{I})$.

So, $\mathcal{G}_{\mathcal{I}}$ has inner edges $E_{\bullet}(\mathcal{G}_{\mathcal{I}}) = E_{\bullet} \setminus E(\mathcal{I})$, and boundary $E_0(\mathcal{G}_{\mathcal{I}}) = E_0 \amalg (E(\mathcal{I}))^{\dagger}$.

Informally, $\mathcal{G}_{\mathcal{I}}$ is the graph obtained from \mathcal{G} by ‘breaking the edges’ of \mathcal{I} , as in Figure 12.

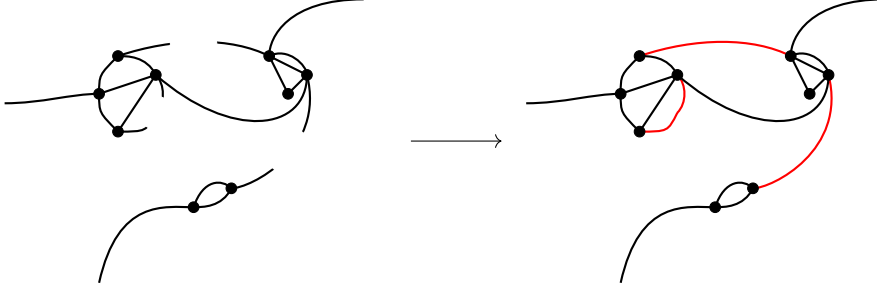


FIGURE 12. Graphs (left) $\mathcal{G}_{\hat{\mathcal{I}}}$ and, (right) \mathcal{G} , with subgraph \mathcal{I} indicated in red.

For $\tilde{e} \in \tilde{E}(\mathcal{I})$, the essential morphism $\iota_{\tilde{e}}: (\iota_{\tilde{e}}) \hookrightarrow \mathcal{G}$ (Example 3.28) factors in two ways through $\mathcal{G}_{\hat{\mathcal{I}}}$:

$$(4.13) \quad (\iota_{\tilde{e}}) \begin{array}{c} \xrightarrow{(e, \tau e) \mapsto (e, e^\dagger)} \\ \xrightarrow{(e, \tau e) \mapsto ((\tau e)^\dagger, \tau e)} \end{array} \mathcal{G}_{\hat{\mathcal{I}}} \xrightarrow{i_{\hat{\mathcal{I}}}} \mathcal{G}.$$

Hence there exist parallel morphisms $\mathcal{I} \rightrightarrows \mathcal{G}_{\hat{\mathcal{I}}}$, and a coequaliser diagram in $\mathbf{Grps}_f(\mathbf{D})$:

$$(4.14) \quad \mathcal{I} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{G}_{\hat{\mathcal{I}}} \xrightarrow{i_{\hat{\mathcal{I}}}} \mathcal{G}.$$

(The choice of morphisms $\mathcal{I} \rightrightarrows \mathcal{G}_{\hat{\mathcal{I}}}$ is not unique – there are $2^{|\tilde{E}(\mathcal{I})|}$ pairs – but it is unique up to isomorphism.)

For each $\mathcal{I} \subset \mathcal{S}(E_\bullet)$, the set of components of $\mathcal{I} \amalg \mathcal{G}_{\hat{\mathcal{I}}}$, together with the canonical monomorphisms to \mathcal{G} , defines an étale cover at \mathcal{G} .

The collection $\{(\mathcal{G}_{\hat{\mathcal{I}}}, i_{\hat{\mathcal{I}}})\}_{\mathcal{I} \subset \mathcal{S}(E_\bullet)} \subset \mathbf{etGr}/\mathcal{G}$ inherits a poset structure from the poset of subgraphs of $\mathcal{S}(E_\bullet)$, and the graph $\mathcal{G}_{\widehat{\mathcal{S}(E_\bullet)}}$ with no inner edges is initial in this poset. Moreover, any surjective monomorphism $\mathcal{G}' \rightarrow \mathcal{G}$ factors as $\mathcal{G}' \xrightarrow{\cong} \mathcal{G}_{\hat{\mathcal{I}}} \xrightarrow{i_{\hat{\mathcal{I}}}} \mathcal{G}$ for some unique $\mathcal{I} \subset \mathcal{S}(E_\bullet)$. Hence, we have proved the following:

Lemma 4.15. *A neighbourhood (\mathcal{U}, u) of a monomorphism $i \in \mathbf{Grps}_f(\mathbf{D})(\mathcal{G}', \mathcal{G})$ is minimal if and only if $E_\bullet(\mathcal{U}) = E_\bullet(\mathcal{G}')$ and the monomorphism $\mathcal{G}' \rightarrow \mathcal{U}$ induces a surjection on connected components.*

For all vertices $v \in V$, and all edges $e \in E$, the essential morphisms ι_v and $\iota_{\tilde{e}}$ describe minimal neighbourhoods of v and e respectively.

In case \mathcal{G} has no stick components, one readily checks that $\mathcal{G}_{\widehat{\mathcal{S}(E_\bullet)}} = \coprod_{v \in V} \mathcal{C}_v$. In particular, for any graph \mathcal{G} , there is a canonical choice of *essential cover* $\mathfrak{C}_{\mathcal{S}\mathcal{G}}$ at \mathcal{G} by the essential morphisms ι_v ($v \in V$), and $\iota_{\tilde{e}}$ ($\tilde{e} \in \tilde{E}$).

Definition 4.16. *Let \mathcal{G} be a graph. The essential category $\mathbf{es}(\mathcal{G})$ of \mathcal{G} is the full subcategory of $\mathbf{etGr}/\mathcal{G}$ on the essential monomorphisms $\iota_{\tilde{e}}: (\iota_{\tilde{e}}) \rightarrow \mathcal{G}$, $\tilde{e} \in \tilde{E}$, and $\iota_v: \mathcal{C}_v \rightarrow \mathcal{G}$, $v \in V$.*

Since $\mathbf{es}(\mathcal{G})$ has no non-trivial isomorphisms by definition, there is a canonical bijection $h = (e, v) \leftrightarrow (\delta_h: \iota_{\tilde{e}} \rightarrow \iota_v)$ between half-edges h of \mathcal{G} , and non-identity morphisms δ in $\mathbf{es}(\mathcal{G})$.

Lemma 4.17. *Each graph \mathcal{G} is canonically the colimit of the forgetful functor $\mathbf{es}(\mathcal{G}) \rightarrow \mathbf{etGr}$.*

A presheaf $P \in \mathbf{ps}(\mathbf{etGr})$ is a sheaf for the étale site (\mathbf{etGr}, J) if and only if for all \mathcal{G} ,

$$(4.18) \quad P(\mathcal{G}) \cong \lim_{(C, b) \in \mathbf{es}(\mathcal{G})} P(C).$$

Proof. If $e \in E_0$ is a port of \mathcal{G} , then there is, at most, one non-trivial morphism $\delta_h = \delta_{(\tau e, v)}$ with domain $\mathfrak{l}_{\bar{e}} \in \mathbf{es}(\mathcal{G})$. In this case, $\mathcal{C}_{\mathbf{v}}$ is the colimit of the diagram $\mathfrak{l}_{\bar{e}} \xrightarrow{\delta_h} \mathcal{C}_{\mathbf{v}}$. The first statement then follows from (4.13) and (4.14). The second statement is immediate since, by Lemma 4.15, the essential cover $\mathfrak{E}\mathfrak{s}_{\mathcal{G}}$ refines every étale cover \mathfrak{U} of \mathcal{G} . \square

4.3. Boundary-preserving étale morphisms. In general, morphisms $f \in \mathbf{etGr}(\mathcal{G}', \mathcal{G})$ do not satisfy $f(E'_0) \subset E_0$. Those that do are componentwise surjective *graphical covering morphisms* in the sense of Proposition 4.19 below. In particular, monomorphisms $f \in \mathbf{etGr}(\mathcal{G}', \mathcal{G})$ such that $f(E'_0) = E_0$ are componentwise isomorphisms.

Proposition 4.19. *For any étale morphism $f \in \mathbf{etGr}(\mathcal{G}', \mathcal{G})$, $f(E'_0) \subset E_0$ if and only if there exists an étale cover $\mathfrak{U} = \{\mathcal{U}_i, u_i\}_{i \in I}$ of \mathcal{G} , such that, for all i , $f^{-1}(\mathcal{U}_i)$ is isomorphic to a disjoint union of $k(\mathcal{U}_i, f) \in \mathbb{N}$ copies of \mathcal{U}_i .*

In this case, $k(\mathcal{U}_i, f) = k_f \in \mathbb{N}$ is constant on connected components of \mathcal{G} .

Proof. If $u: \mathcal{U} \rightarrow \mathcal{G}$ is an étale monomorphism for which there is a $k \in \mathbb{N}$ such that $f^{-1}(\mathcal{U}) \cong k(\mathcal{U})$, then also $f^{-1}(\mathcal{V}) \cong k\mathcal{V}$ for all monomorphisms $\mathcal{V} \rightarrow \mathcal{U}$. So, we may assume, without loss of generality, that $\mathfrak{U} = \mathfrak{E}\mathfrak{s}_{\mathcal{G}}$ is the essential cover of \mathcal{G} .

Observe first that, when $f(E'_0) \not\subset E_0$, there exists a vertex v of \mathcal{G} and a port e' of \mathcal{G}' such that $f(e') \in E/v$. Hence $(\mathfrak{l}_{\bar{e}'}) \not\cong \mathcal{C}_{\mathbf{v}}$ is a connected component of $f^{-1}(\mathcal{C}_{\mathbf{v}})$.

For the converse, let $(\mathcal{U}, u) = (\mathcal{C}_{\mathbf{v}}, \iota_v)$ for some $v \in V$. Since f is étale, $\mathcal{C}_{\mathbf{v}} \cong \mathcal{C}_{\mathbf{v}'}$ for all $v' \in V'$ such that $f(v') = v$. By the universal property of pullbacks, the canonical monomorphism $\coprod_{v': f(v')=v} \mathcal{C}_{\mathbf{v}'} \rightarrow \mathcal{G}'$ factors through $f^{-1}(\mathcal{C}_{\mathbf{v}}) \rightarrow \mathcal{G}'$. Hence,

$$f^{-1}(\mathcal{C}_{\mathbf{v}}) \cong \left(\coprod_{v': f(v')=v} \mathcal{C}_{\mathbf{v}'} \right) \amalg \mathcal{S}, \text{ for some shrub } \mathcal{S}.$$

The, by construction, a connected component of \mathcal{S} must be of the form $\iota_{\bar{e}'}: (\mathfrak{l}_{\bar{e}'}) \rightarrow \mathcal{G}'$ for some port e' of \mathcal{G}' satisfying $f(e') = e \in E/v$. But $f(E'_0) \subset E_0$ by assumption, so there is no such port. Hence $\mathcal{S} = \emptyset$ is the empty graph.

It is immediate that $f^{-1}(\mathfrak{l}_{\bar{e}}) \cong \coprod_{e' \in E', f(e')=e} (\mathfrak{l}_{\bar{e}'})$ for all $e \in E$, so $k((\mathfrak{l}_{\bar{e}}), f) = |f^{-1}(e)| \in \mathbb{N}$, and the first statement of the proposition is proved.

By condition (3) of Proposition 3.34, it is sufficient to verify the second part of the proposition componentwise on \mathcal{G} . Therefore we may assume, without loss of generality, that \mathcal{G} is connected.

Let $f \in \mathbf{etGr}(\mathcal{G}', \mathcal{G})$ satisfy $f(E'_0) \subset E_0$.

If $\mathcal{G} \cong (\mathfrak{l})$ is a stick, there is nothing to prove. Otherwise, the following diagram commutes for all vertices v , and all half edges $h = (e, v)$ of \mathcal{G} :

$$\begin{array}{ccccc} \coprod_{e' \in f^{-1}(e)} (\mathfrak{l}_{\bar{e}'}) & \xrightarrow{\coprod_{h' \in f^{-1}(h)} \delta_{h'}} & \coprod_{v' \in f^{-1}(v)} \mathcal{C}_{\mathbf{v}'} & \xrightarrow{\coprod_{v'} \iota_{v'}} & \mathcal{G}' \\ \downarrow & & \downarrow & & \downarrow f \\ (\mathfrak{l}_{\bar{e}}) & \xrightarrow{\delta_h} & \mathcal{C}_{\mathbf{v}} & \xrightarrow{\iota_v} & \mathcal{G}. \end{array}$$

The first part of the proof implies that both squares are pullbacks. In particular, if $f^{-1}(\mathcal{C}_{\mathbf{v}})$ is isomorphic to $k_v = k(\mathcal{C}_{\mathbf{v}}, \iota_v)$ copies of $\mathcal{C}_{\mathbf{v}}$ then $f^{-1}(\mathfrak{l}_{\bar{e}}) \cong k_v(\mathfrak{l}_{\bar{e}})$ for all $e \in E/v$. Hence $\mathcal{C}_{\mathbf{v}} \mapsto k_v$ extends to a functor $k_{\mathcal{G}}$ from $\mathbf{es}(\mathcal{G})$ to the discrete category \mathbb{N} . Since \mathcal{G} is connected, $k_{\mathcal{G}}$ must be constant. \square

Definition 4.20. A morphism $f \in \text{etGr}(\mathcal{G}', \mathcal{G})$ that restricts to an isomorphism $f_{E_0}: E_0 \xrightarrow{\cong} E'_0$ is called boundary-preserving.

The following is an immediate corollary of Proposition 4.19.

Corollary 4.21. If \mathcal{G} is connected, and \mathcal{G}' is non-empty, then any étale morphism $f \in \text{etGr}(\mathcal{G}', \mathcal{G})$ such that $f(E'_0) \subset E_0$, is surjective. If \mathcal{G}' is also connected and its boundary E'_0 is non-empty, then f is boundary-preserving if and only if it is an isomorphism.

Remark 4.22. The condition that E'_0 is non-empty is necessary in the statement of Corollary 4.21. For example, for any $m > 1$, each of the two étale morphisms $\mathcal{W}^m \rightarrow \mathcal{W}$ (see Example 4.3) is trivially boundary-preserving, but certainly not an isomorphism.

Recall from Example 3.30 that the line graph \mathcal{L}^k has totally ordered edge set $E(\mathcal{L}^k) = (l_j)_{j=0}^{2k+1}$ with ports $l_0 = 1_{\mathcal{L}^k}$ and $l_{2k+1} = 2_{\mathcal{L}^k}$. For each vertex $w_i \in V(\mathcal{L}^k)$, $E/w_i = \{l_{2i-1}, l_{2i}\}$.

Proposition 4.23. Let \mathcal{G} be a connected graph with only bivalent vertices. Then $\mathcal{G} = \mathcal{L}^k$ or $\mathcal{G} = \mathcal{W}^m$ for some $k \geq 0$ or $m \geq 1$.

Proof. Since \mathcal{G} is bivalent, every monomorphism $\mathcal{L}^k \rightarrow \mathcal{G}$ from a line graph is étale, for all $k \in \mathbb{N}$.

The result holds trivially if $\mathcal{G} \cong \mathcal{L}^0$ is a stick graph.

Otherwise, if $V = V_2$ is non-empty, then, for each $v \in V$, a choice of isomorphism $\mathcal{L}^1 \xrightarrow{\cong} \mathcal{C}_v$ describes a monomorphism $\mathcal{L}^1 \rightarrow \mathcal{G}$. Since \mathcal{G} is finite, there is a maximum $M \geq 1$ such that there exists a monomorphism $f: \mathcal{L}^M \rightarrow \mathcal{G}$.

Let $f \in \text{CetGr}(\mathcal{L}^M, \mathcal{G})$ be such a map. By Proposition 4.8, f is either injective on edges, or $f(2_{\mathcal{L}^M}) = \tau f(1_{\mathcal{L}^M}) \subset E_\bullet$. Let $e_1 = f(1_{\mathcal{L}^M})$, and $e_2 = f(2_{\mathcal{L}^M})$. Assume that e_j is not a port of \mathcal{G} for some $j = 1, 2$. Then $e_j \in E/v$ for some vertex $v \in V_2$.

If f is injective on edges then v is not in the image of f . But then, since v is bivalent, this means that f factors through monomorphisms $\mathcal{L}^M \rightarrow \mathcal{L}^{M+1} \rightarrow \mathcal{G}$, contradicting maximality of M . So $e_j \in E_0$ for $j = 1, 2$. Hence f is boundary-preserving, and $\mathcal{G} \cong \mathcal{L}^M$ by Corollary 4.21.

Otherwise, if f is not injective on edges, then, by Proposition 4.8, it must be the case that $f(l_0) = f(l_{2M})$. Therefore, f factors through a cycle $f: \mathcal{L}^M \rightarrow \mathcal{W}^M \rightarrow \mathcal{G}$. Since f is an étale monomorphism, the second map $\mathcal{W}^M \rightarrow \mathcal{G}$ is étale, monomorphic and boundary-preserving. Hence, $\mathcal{G} \cong \mathcal{W}^M$ by Corollary 4.21. \square

Étale morphisms of simply connected graphs are either subgraph inclusions or isomorphisms. (This is why the combinatorics of cyclic operads are much simpler than those of modular operads.)

Corollary 4.24. Let \mathcal{G} be simply connected. If $f \in \text{Grps}_f(\mathcal{D})(\mathcal{G}', \mathcal{G})$ is locally injective, then f is pointwise injective on connected components of \mathcal{G}' . Hence, if \mathcal{G}' is connected, it is simply connected.

It follows that any étale morphism of simply connected graphs is a pointwise injection,

Proof. We may assume, without loss of generality, that \mathcal{G} and \mathcal{G}' are connected. The corollary holds trivially if \mathcal{G}' , or \mathcal{G} is an isolated vertex. So, we may assume that \mathcal{G} and \mathcal{G}' have non-trivial edge sets.

Let $f: \mathcal{G}' \rightarrow \mathcal{G}$ be a local injection. Then so is $f \circ p: \mathcal{L}^k \rightarrow \mathcal{G}$, for any injective path $p: \mathcal{L}^k \rightarrow \mathcal{G}'$. Therefore, either $f \circ p$ is pointwise injective or it factors through a locally injective cycle $\mathcal{W}^m \rightarrow \mathcal{G}$. Hence, if \mathcal{G} is simply connected and \mathcal{G}' is connected, any local injection $f: \mathcal{G}' \rightarrow \mathcal{G}$ is a pointwise injection.

If \mathcal{G}' admits a locally injective cycle $c: \mathcal{W}^m \rightarrow \mathcal{G}'$, then $f \circ c: \mathcal{W}^m \rightarrow \mathcal{G}$ is locally injective cycle in \mathcal{G} . Hence, if \mathcal{G} is simply connected, then \mathcal{G}' is simply connected.

The final statement is immediate since étale morphisms are locally injective by definition. \square

Proposition 4.35 gives analogous results for directed acyclic graphs (Definition 4.34).

4.4. Étale sheaves on \mathbf{etGr} . Recall that graphical species are presheaves on the category \mathbf{B}^\odot , and that there is a full inclusion $\Phi: \mathbf{B}^\odot \hookrightarrow \mathbf{etGr}$. We prove that Φ induces an equivalence $\mathbf{GS} \simeq \mathbf{sh}(\mathbf{etGr}, J)$ between graphical species and sheaves for the étale topology on \mathbf{CetGr} .

Lemma 4.25. *The inclusion $\Phi: \mathbf{B}^\odot \hookrightarrow \mathbf{etGr}$ is dense.*

Proof. It is easy to check that any connected graph without inner edges is isomorphic to \S or \mathcal{C}_X for some finite set X , and therefore, the essential image $im^{es}(\Phi)$ of \mathbf{B}^\odot in \mathbf{etGr} is the full subcategory of connected graphs with no inner edges.

It is immediate from the definition of $\mathbf{es}(\mathcal{G})$ (Definition 4.16) that the canonical inclusion $\mathbf{es}(\mathcal{G}) \hookrightarrow im^{es}(\Phi)/\mathcal{G}$ is full and essentially surjective on objects, and hence an equivalence of categories.

Therefore $\mathbf{es}(\mathcal{G}) \simeq \mathbf{B}^\odot/\mathcal{G}$, and the lemma follows from Lemma 4.17. \square

In particular, \mathbf{etGr} is a full subcategory of \mathbf{GS} under the induced nerve functor $\Upsilon \stackrel{\text{def}}{=} N_{\mathbf{etGr}}: \mathbf{etGr} \rightarrow \mathbf{GS}$. embeds \mathbf{etGr} as a full subcategory of \mathbf{GS} .

By an application of the Yoneda lemma, the category $\mathbf{el}(\Upsilon\mathcal{G})$ of elements of $\Upsilon\mathcal{G}$ (see Definition 1.10) is canonically isomorphic to the slice category $\mathbf{el}(\mathcal{G}) \stackrel{\text{def}}{=} \mathbf{B}^\odot/\mathcal{G}$, and I will write $\mathbf{el}(\mathcal{G})$ for both. Since \mathbf{etGr} is a full subcategory of \mathbf{GS} under Υ , I will write \mathcal{G} rather than $\Upsilon\mathcal{G}$, where there is no risk of confusion.

Proposition 4.26. *There is a canonical equivalence of categories $\mathbf{sh}(\mathbf{etGr}, J) \simeq \mathbf{GS}$.*

Proof. This is straightforward from the definitions. Namely, the inclusion $\Phi: \mathbf{B}^\odot \rightarrow \mathbf{etGr}$ induces an *essential geometric morphism* between the presheaf categories $\mathbf{ps}(\mathbf{B}^\odot) = \mathbf{GS}$ and $\mathbf{ps}(\mathbf{etGr})$. The right adjoint to the pullback $\Phi^*: \mathbf{ps}(\mathbf{etGr}) \rightarrow \mathbf{GS}$ is given by

$$(4.27) \quad \Phi_*: \mathbf{GS} = \mathbf{ps}(\mathbf{B}^\odot) \rightarrow \mathbf{ps}(\mathbf{etGr}), \quad S \longmapsto (\mathcal{G} \mapsto \lim_{(\mathcal{C}, b) \in \mathbf{el}(\mathcal{G})} S(\mathcal{C})).$$

By Lemma 4.17, a presheaf P on \mathbf{etGr} is a sheaf for the canonical étale topology J on \mathbf{etGr} if and only if, for all graphs \mathcal{G} ,

$$(4.28) \quad P(\mathcal{G}) \cong \lim_{(\mathcal{C}, b) \in \mathbf{el}(\mathcal{G})} P(\mathcal{C}),$$

and therefore Φ_* factors through the inclusion $\mathbf{sh}(\mathbf{etGr}, J) \rightarrow \mathbf{ps}(\mathbf{etGr})$.

Moreover, since Φ is fully faithful, so is Φ_* (e.g. by [31, Section VII.2]), whereby Φ_* induces the equivalence $\mathbf{GS} \simeq \mathbf{sh}(\mathbf{etGr}, J)$. \square

I will use the same notation to denote a graphical species S and the corresponding sheaf on (\mathbf{etGr}, J) . So, for any graph \mathcal{G} ,

$$S(\mathcal{G}) \stackrel{\text{def}}{=} \lim_{(\mathcal{C}, b) \in \mathbf{el}(\mathcal{G})} S(\mathcal{C}).$$

The following is an immediate consequence of the proposition:

Corollary 4.29. *The restriction of the étale topology J to the category \mathbf{CetGr} of connected graphs and étale morphisms induces an equivalence $\mathbf{sh}(\mathbf{CetGr}, J) \simeq \mathbf{GS}$.*

Recall Definition 1.1 of elements of a presheaf.

Definition 4.30. Let S be a graphical species, viewed as a sheaf on \mathbf{etGr} . The category $\mathbf{etGr}(S)$ of S -structured graphs is defined as $\mathbf{etGr}(S) \stackrel{\text{def}}{=} \mathbf{El}_S(\mathbf{etGr})$.

Given a graph \mathcal{G} , $S(\mathcal{G})$ is called the set of S -structures on \mathcal{G} .

The full subcategory of $\mathbf{etGr}(S)$ with objects (\mathcal{G}, α) such that \mathcal{G} is connected is denoted by $\mathbf{CetGr}(S)$.

Remark 4.31. By the Yoneda lemma, $S(\mathcal{G}) \cong \mathbf{GS}(\Upsilon\mathcal{G}, S)$ canonically, so $S(\mathcal{G})$ is isomorphic to the set of objects of $\mathbf{etGr}(S)$ of the form (\mathcal{G}, α) . The same notation α (or (\mathcal{G}, α)) will be used for an element of $S(\mathcal{G})$ and the corresponding morphism in $\mathbf{GS}(\mathcal{G}, S)$.

A element $\alpha \in S(\mathcal{G})$ also describes a functor $\mathbf{el}(\mathcal{G}) \rightarrow \mathbf{el}(S)$ by $(\mathcal{C}_X, b) \mapsto (X, S(b)(\alpha))$. More generally, following Remark 1.9, a functor $\mathbf{el}(\mathcal{G}) \rightarrow \mathbf{C}$ may be described as a *decoration* of \mathcal{G} by a category \mathbf{C} .

4.5. Directed graphs. By way of example, and to provide extra context, this section ends with a discussion of directed graphs. In particular, we see that, if Di is the terminal directed graphical species defined in Example 1.8 (see also Example 1.11), then $\mathbf{etGr}(Di)$ defines a category of directed Feynman graphs and étale morphisms.

For any graph \mathcal{G} , a Di -structure $\xi \in Di(\mathcal{G})$ is precisely a partition $E = E_{\text{in}} \amalg E_{\text{out}}$, where $e \in E_{\text{in}}$ if and only if $\tau e \in E_{\text{out}}$. So τ induces bijections $E_{\text{in}} \cong \tilde{E} \cong E_{\text{out}}$.

So, an object (\mathcal{G}, ξ) of $\mathbf{CetGr}(Di)$ – called an *orientation on \mathcal{G}* – is given by a diagram of finite sets

$$(4.32) \quad \tilde{E} \xleftarrow{\widetilde{s}_{\text{in}}} H_{\text{in}} \xrightarrow{t_{\text{in}}} V \xleftarrow{t_{\text{out}}} H_{\text{out}} \xrightarrow{\widetilde{s}_{\text{out}}} \tilde{E},$$

where the maps $\widetilde{s}_{\text{in}}, \widetilde{s}_{\text{out}}$, and $t_{\text{in}}, t_{\text{out}}$ denote the appropriate (quotients of) restrictions of $s: H \rightarrow E$, respectively $t: H \rightarrow V$. Then morphisms in $\mathbf{etGr}(Di)$ are quadruples of finite set maps making the obvious diagrams commute, and such that the outer left and right squares are pullbacks. This is the definition of the category of directed graphs and étale morphisms in [28, Section 1.5].

Example 4.33. The line graphs \mathcal{L}^k with $E(\mathcal{L}^k) = \{l_i\}_{i=0}^{2k+1}$ admit a distinguished choice of orientation $\theta_{\mathcal{L}^k} \in Di(\mathcal{L}^k)$ given by

$$\theta_{\mathcal{L}^k}: E(\mathcal{L}^k) \rightarrow \{\text{in}, \text{out}\}, \quad l_{2i} \mapsto (\text{in}) \text{ and } l_{2i+1} \mapsto (\text{out}) \text{ for } 0 \leq i \leq k.$$

For $m \geq 1$, the canonical morphism $\mathcal{L}^m \rightarrow \mathcal{W}^m$ induces an orientation $\theta_{\mathcal{W}^m}$ (with $a_{2j} \mapsto (\text{in})$) on the wheel graph \mathcal{W}^m .

Definition 4.34. A directed path of length k in (\mathcal{G}, ξ) is a path $p: \mathcal{L}^k \rightarrow \mathcal{G}$ in \mathcal{G} such that, for all $l \in E(\mathcal{L}^k)$,

$$Di(ch_l)(\theta_{\mathcal{L}^k}) = Di(ch_{p(l)})(\xi) \in \{\text{in}, \text{out}\}.$$

A directed cycle of length m in (\mathcal{G}, ξ) is a cycle $c: \mathcal{W}^m \rightarrow \mathcal{G}$ in \mathcal{G} such that the induced morphism $\mathcal{L}^m \rightarrow \mathcal{W}^m \rightarrow \mathcal{G}$ is a directed path.

A directed acyclic graph (DAG) is a directed graph (\mathcal{G}, ξ) without directed cycles.

It is straightforward to verify that any directed path or cycle in a directed graph (\mathcal{G}, ξ) is locally injective. Hence, if (\mathcal{G}, ξ) admits a directed cycle, \mathcal{G} is not simply connected. The converse is not true.

There is a directed version of Corollary 4.24:

Proposition 4.35. For all morphisms $f \in \mathbf{etGr}(Di)((\mathcal{G}', \xi'), (\mathcal{G}, \xi))$ between connected DAGs, the underlying morphism $f \in \mathbf{etGr}(\mathcal{G}', \mathcal{G})$ is a monomorphism.

Moreover, if (\mathcal{G}, ξ) is a DAG and the set $\mathbf{etGr}(Di)((\mathcal{G}', \xi'), (\mathcal{G}, \xi))$ is non-empty, then (\mathcal{G}', ξ') is a DAG. Hence any morphism to a DAG in $\mathbf{etGr}(Di)$ is a pointwise injection.

Since Proposition 4.35 is not necessary for what follows, I leave its proof as an exercise to the interested reader.

A consequence of Proposition 4.35 is that the combinatorics of properads, which are governed by DAGs, are much simpler than those of wheeled properads or modular operads.

5. NON-UNITAL MODULAR OPERADS

The goal of the current section is to construct a monad $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ on \mathbf{GS} whose EM category of algebras $\mathbf{GS}^{\mathbb{T}}$ is isomorphic to the category \mathbf{MO}^- of non-unital modular operads (Remark 1.26).

To provide context for this section, consider the following example:

Example 5.1. Recall, from Example 1.19, the category \mathbf{B}^{\downarrow} , whose objects are either the directed exceptional edge (\downarrow), or rooted corollas t_X , with X a finite set.

The operad endofunctor M_{Op} on $\mathbf{ps}(\mathbf{B}^{\downarrow})$ from Example 2.10 is described in detail in [6, Section 3]. It takes a presheaf $P: \mathbf{B}^{\downarrow \text{op}} \rightarrow \mathbf{Set}$ to the presheaf $M_{\text{Op}}P$ on \mathbf{B}^{\downarrow} with $M_{\text{Op}}P(\downarrow) = P(\downarrow)$, and where $M_{\text{Op}}P(t_X)$ is the set of formal operadic compositions (i.e. root-to-leaf grafting as in Figure 7(b)) – with X inputs – of elements of P . So elements of $M_{\text{Op}}P(t_X)$ are represented by rooted trees $\mathfrak{T} \in \Omega$, whose leaves are bijectively labelled by X , together with a colouring of edges of \mathfrak{T} by $P(\downarrow)$, and a decoration of vertices v by elements of $P(t_{\partial v})$ (where $t_{\partial v}$ here denotes the minimal neighbourhood of v in \mathfrak{T}).

The monadic unit $\eta^{\mathbb{M}_{\text{Op}}}$ is induced by the inclusion of rooted corollas as trees with one vertex in Ω . So, $\eta^{\mathbb{M}_{\text{Op}}}(\phi) = (t_X, \phi)$ for all $\phi \in P(t_X)$ as in the left hand side of Figure 13. Applying the monad twice describes a nesting of trees, and the multiplication $\mu^{\mathbb{M}_{\text{Op}}}$ for M_{Op} is induced by erasing the inner nesting (the blue circles in the right hand side of Figure 13).

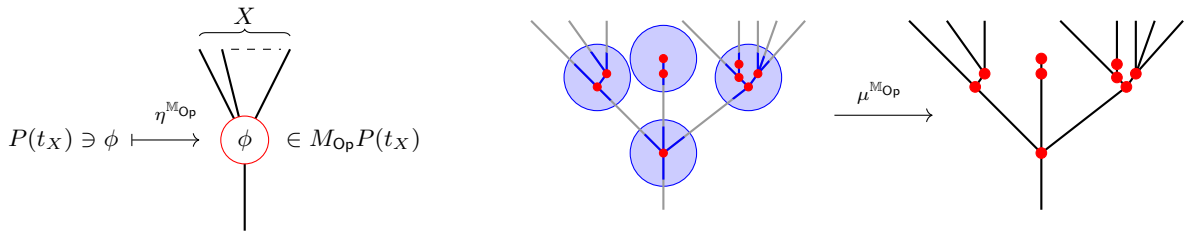


FIGURE 13. Visualising the unit and multiplication for the operad monad on rooted corollas.

If (P, h) is an algebra for \mathbb{M}_{Op} , then h describes a rule – that satisfies the axioms of operadic composition – for collapsing the inner edges of each P -decorated tree.

Just as the operad endofunctor takes a \mathbf{B}^{\downarrow} -presheaf O to trees decorated by O , the non-unital modular operad endofunctor T on \mathbf{GS} takes a graphical species S to the graphical species whose elements are formal multiplications and contractions in S , represented by S -structured connected graphs.

5.1. X -graphs and an endofunctor for non-unital modular operads. In order to define the endofunctor $T: \mathbf{GS} \rightarrow \mathbf{GS}$, we need to bijectively label graph boundaries by finite sets.

Definition 5.2. Let X be a finite set. An (admissible) X -graph is a pair $\mathcal{X} = (\mathcal{G}, \rho)$, where \mathcal{G} is a connected graph such that $V \neq \emptyset$ and $\rho: E_0 \xrightarrow{\cong} X$ is a bijection, or X -labelling for \mathcal{G} .

An X -isomorphism $\mathcal{X} \rightarrow \mathcal{X}'$ of X -graphs $\mathcal{X} = (\mathcal{G}, \rho)$ and $\mathcal{X}' = (\mathcal{G}', \rho')$, is an isomorphism $g \in \mathbf{CetGr}(\mathcal{G}, \mathcal{G}')$ that preserves the X -labelling: $\rho' \circ g_{E_0} = \rho: E_0 \rightarrow X$.

The groupoid of X -graphs and X -isomorphisms is denoted by $X\mathbf{CGr}_{\text{iso}}$.

Remark 5.3. It is convenient to use the same notation for labelled and unlabelled graphs. In particular, an X -graph $\mathcal{X} = (\mathcal{G}, \rho)$ is denoted simply by \mathcal{G} when the labelling ρ is trivial or completely canonical. For example, for all finite sets X , the corolla \mathcal{C}_X canonically defines an X -graph $\mathcal{C}_X = (\mathcal{C}_X, id)$.

Example 5.4. The line graph \mathcal{L}^k , with $E_0(\mathcal{L}^k) = \{1_{\mathcal{L}^k}, 2_{\mathcal{L}^k}\}$, $k \geq 0$ is labelled by $1_{\mathcal{L}^k} \mapsto 1 \in \mathbf{2}$ and therefore has the structure of a $\mathbf{2}$ -graph when $k \geq 1$. However, $\mathcal{L}^0 = (\mathbf{1})$ has empty vertex set and is therefore not a $\mathbf{2}$ -graph.

In particular, for all graphical species S and all finite sets X , S defines a presheaf on $X\mathbf{CGr}_{\text{iso}}$ with $S(\mathcal{X}) = S(\mathcal{G})$ for all X -graphs $\mathcal{X} = (\mathcal{G}, \rho)$. Recall Definition 1.1.

Definition 5.5. Let S be a graphical species and X a finite set. The groupoid $X\mathbf{CGr}_{\text{iso}}(S)$ of S -structured X -graphs is defined as the groupoid $\mathbf{El}_S(X\mathbf{CGr}_{\text{iso}})$ of elements of S on $X\mathbf{CGr}_{\text{iso}}$.

We can now define the non-unital modular operad endofunctor T on \mathbf{GS} , that takes a graphical species S to equivalence classes of graphs decorated by S .

For all graphical species S , let TS be defined on objects by

$$(5.6) \quad \begin{aligned} TS_{\mathbb{S}} &= S_{\mathbb{S}}, \\ TS_X &= \text{colim}_{\mathcal{X} \in X\mathbf{CGr}_{\text{iso}}} S(\mathcal{X}) \quad \text{for all finite sets } X. \end{aligned}$$

Let $\text{Aut}_X(\mathcal{X}) \stackrel{\text{def}}{=} X\mathbf{CGr}_{\text{iso}}(\mathcal{X}, \mathcal{X})$ be the automorphism group of an X -graph \mathcal{X} . If $g, g' \in X\mathbf{CGr}_{\text{iso}}(\mathcal{X}, \mathcal{X}')$ are parallel X -isomorphisms, then there are $\sigma \in \text{Aut}_X(\mathcal{X})$ and $\sigma' \in \text{Aut}_X(\mathcal{X}')$ such that $g' = \sigma'g\sigma$. Therefore, there is a completely canonical (independent of $g \in X\mathbf{CGr}_{\text{iso}}(\mathcal{X}, \mathcal{X}')$) choice of natural (in \mathcal{X}) isomorphism

$$(5.7) \quad \frac{S(\mathcal{X})}{\text{Aut}_X(\mathcal{X})} \xrightarrow{\cong} \frac{S(\mathcal{X}')}{\text{Aut}_X(\mathcal{X}')}, \quad [\alpha] \mapsto [g(\alpha)], \text{ for } \alpha \in S(\mathcal{X}).$$

It follows from (5.7), that

$$(5.8) \quad \begin{aligned} TS_X &= \coprod_{[\mathcal{X}] \in \pi_0(X\mathbf{CGr}_{\text{iso}})} \frac{S(\mathcal{X})}{\text{Aut}_X(\mathcal{X})} \\ &= \pi_0(X\mathbf{CGr}_{\text{iso}}(S)) \end{aligned}$$

where, for $\mathcal{X} = (\mathcal{G}, \rho)$, $[\mathcal{X}] \in \pi_0(X\mathbf{CGr}_{\text{iso}})$ is the connected component of \mathcal{X} in $X\mathbf{CGr}_{\text{iso}}$.

Hence, elements of TS_X may be viewed as isomorphism classes of S -structured X -graphs, and two X -labelled S -structured graphs (\mathcal{X}, α) and (\mathcal{X}', α') represent the same class $[\mathcal{X}, \alpha] \in TS_X$ precisely when there is an isomorphism $g \in X\mathbf{CGr}_{\text{iso}}(\mathcal{X}, \mathcal{X}')$ such that $S(g)(\alpha') = \alpha$.

Since bijections $f: X \xrightarrow{\cong} Y$ of finite sets induce isomorphisms $Y\mathbf{CGr}_{\text{iso}}(S) \xrightarrow{\cong} X\mathbf{CGr}_{\text{iso}}(S)$, the action of TS on isomorphisms in \mathbf{B}^{\odot} is the obvious one.

The projections $TS(ch_x): TS_X \rightarrow TS_{\mathbb{S}} = S_{\mathbb{S}}$ are induced by

$$(\mathcal{X}, \alpha) \mapsto S(ch_x^{\mathcal{X}})(\alpha),$$

for X -graph $\mathcal{X} = (\mathcal{G}, \rho)$ such that $ch_x^{\mathcal{X}} \in \mathbf{CetGr}(\mathbf{1}, \mathcal{G})$ is the map $ch_{\rho^{-1}(x)}$ defined by $1 \mapsto \rho^{-1}(x) \in E_0(\mathcal{G})$.

This is well-defined since, if (\mathcal{X}, α) and (\mathcal{X}', α') represent the same element of TS_X , then there is an X -isomorphism $g: \mathcal{X} \rightarrow \mathcal{X}'$ such that $S(g)(\alpha') = \alpha \in S_X$ and hence

$$S(ch_x^{\mathcal{X}'})(\alpha') = S(ch_x^{\mathcal{X}}) \circ S(g)(\alpha') = S(ch_x^{\mathcal{X}})(\alpha).$$

So TS describes a graphical species. Moreover, it is clear from the definition that the assignment $S \mapsto TS$ extends to an endofunctor T on \mathbf{GS} , with unit $\eta^{\mathbb{T}}: id_{\mathbf{GS}} \Rightarrow T$ given by the canonical maps $S_X \xrightarrow{\cong} S(\mathcal{C}_X) \rightarrow TS_X$.

5.2. Gluing constructions. A monadic multiplication $\mu^\mathbb{T}$ for the pointed endofunctor $(T, \eta^\mathbb{T})$ will be defined in terms of colimits of a certain class of diagrams in \mathbf{CetGr} . However, since \mathbf{CetGr} does not admit general colimits (see Examples 3.16 and 3.33), a small amount of preparation is necessary.

Let S be a graphical species and Y a finite set. Elements of TS_Y are represented by S -structured Y -graphs, and therefore, for all finite sets X , elements of T^2S_X are represented by X -graphs \mathcal{X} that are decorated by S -structured graphs.

In other words, each $[\mathcal{X}, \beta] \in T^2S_X$ is represented by a functor

$$el(\beta): el(\mathcal{X}) \rightarrow el(TS), \begin{cases} (\mathcal{C}_{X_b}, b) \mapsto (\mathcal{C}_{X_b}, S(b)(\beta)), & \text{where } S(b)(\beta) \in TS_{X_b} \\ (1, ch_e) \mapsto (1, c) & c \in S_\S \end{cases}$$

such that

$$el(\beta)(ch_{x_b})(S(b)(\beta)) = el(\beta)(ch_e) \in S(1).$$

for all morphisms in $el(\mathcal{X})$ of the form

$$\begin{array}{ccc} (1) & \xrightarrow{ch_{x_b}} & \mathcal{C}_{X_b} \\ & \searrow ch_e & \swarrow b \\ & \mathcal{X}. & \end{array}$$

Then, as in the operad case (Example 5.1, Figure 13), we would like to think of the monad multiplication as forgetting the vertices of the original graph \mathcal{X} to obtain an element of TS_X (Figure 14). Graphs of graphs are functors that encode this idea:

Definition 5.9. Let \mathcal{G} be a Feynman graph. A \mathcal{G} -shaped graph of graphs is a functor $\mathbf{\Gamma}: el(\mathcal{G}) \rightarrow \mathbf{etGr}$ (or $\mathbf{\Gamma}^\mathcal{G}$) such that

$$\begin{aligned} \mathbf{\Gamma}(ch_e) &= (1) & \text{for all } (1, ch_e) \in el(\mathcal{G}), \\ E_0(\mathbf{\Gamma}(b)) &= X_b & \text{for all } (\mathcal{C}_{X_b}, b) \in el(\mathcal{G}), \end{aligned}$$

and, for all $(\mathcal{C}_{X_b}, b) \in el(\mathcal{G})$ and all $x_b \in X_b$,

$$\mathbf{\Gamma}(ch_{x_b}) = ch_{x_b}^{\mathbf{\Gamma}(b)} \in \mathbf{etGr}(1, \mathbf{\Gamma}(b)).$$

A \mathcal{G} -shaped graph of graphs $\mathbf{\Gamma}: el(\mathcal{G}) \rightarrow \mathbf{etGr}$ is non-degenerate if, for all $v \in V$, $\mathbf{\Gamma}(\iota_v)$ has no stick components. Otherwise, $\mathbf{\Gamma}$ is degenerate.

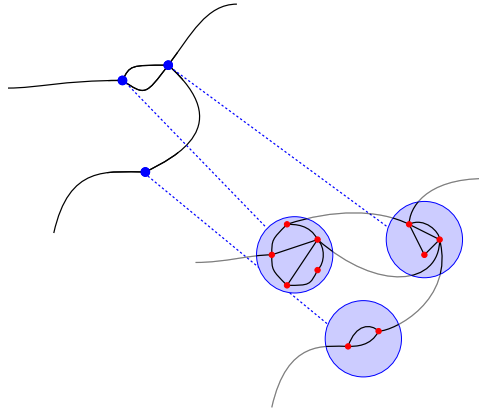


FIGURE 14. A \mathcal{G} -shaped graph of graphs $\mathbf{\Gamma}$ describes *graph substitution*: each vertex v of \mathcal{G} is replaced by a graph \mathcal{G}_v according to a bijection $E_0(\mathcal{G}_v) \xrightarrow{\cong} (E/v)^\dagger$. When $\mathbf{\Gamma}$ is non-degenerate, taking its colimit in \mathbf{etGr} corresponds to erasing the inner (blue) nesting.

Informally, a non-degenerate \mathcal{G} -shaped graph of graphs is a rule for substituting graphs into vertices of \mathcal{G} as in Figure 14. However, this intuitive description of a graph of graphs in terms of graph insertion does not always apply in the degenerate case (see Sections 6 and 7).

By Lemma 4.17, every graph \mathcal{G} is the colimit of the (non-degenerate) *identity \mathcal{G} -shaped graph of graphs* given by the forgetful functor $\text{el}(\mathcal{G}) \rightarrow \text{etGr}$, $(\mathcal{C}, b) \mapsto \mathcal{C}$. It follows from the discussion around (4.14) that, if \mathcal{G} has no stick components, this is equivalent to the statement that \mathcal{G} is the coequaliser of the canonical diagram

$$(5.10) \quad \mathcal{S}(E_\bullet) \rightrightarrows \coprod_{v \in V} \mathcal{C}_v \xrightarrow{\Pi(\iota_v)} \mathcal{G}.$$

To prove that all non-degenerate graphs of graphs admit a colimit in etGr , we generalise this observation using a modification of the *gluing data* (for directed graphs) from [28, Section 1.5.1].

Definition 5.11. *Let $\mathcal{S} = \coprod_{i \in I} (\iota_i)$ be a shrub, and let \mathcal{G} be a (not necessarily connected) graph without stick components. A pair of parallel morphisms $\delta_1, \delta_2: \mathcal{S} \rightrightarrows \mathcal{G}$ such that*

- δ_1, δ_2 are injective and have disjoint images in \mathcal{G} ; and
- for all $i \in I$, $\delta_1(1_i)$ and $\delta_2(2_i)$ are ports of \mathcal{G} ,

is called a gluing datum in etGr .

Lemma 5.12. *Gluing data admit coequalisers in etGr .*

Proof. Let \mathcal{G} be a graph without stick components and $\delta_1, \delta_2: \mathcal{S} = \coprod_{i \in I} (\iota_i) \rightrightarrows \mathcal{G}$ a gluing datum with coequaliser $\bar{p}: \mathcal{G} \rightarrow \bar{\mathcal{G}} = (\bar{E}, \bar{H}, \bar{V}, \bar{s}, \bar{t}, \bar{\tau})$ in the category $\text{psf}(\mathcal{D})$ of graph-like diagrams.

If $\bar{\mathcal{G}}$ is a graph, then it is the coequaliser of δ_1, δ_2 in $\text{Grpsf}(\mathcal{D})$.

Half-edges h and h' of \mathcal{G} are identified in \bar{H} if and only if there is an edge $l \in E(\mathcal{S})$ such that $\delta_1(l) = s(h)$ and $\delta_2(l) = s(h')$. This contradicts the conditions of Definition 5.11, whereby $\bar{H} = H$. Moreover $e, e' \in E(\mathcal{G})$ are identified in \bar{E} if and only if there is an $l \in E(\mathcal{S})$ such that $\delta_1(l) = e$ and $\delta_2(l) = e'$ (or vice versa). So, since \mathcal{G} has no stick components, and δ_1, δ_2 have disjoint images, we may assume that e is a port and $e' \in s(H)$. In particular, $\bar{s}: \bar{H} \rightarrow \bar{E}$ is injective, and since δ_1 and δ_2 have disjoint images, $\bar{\tau}: \bar{E} \rightarrow \bar{E}$ is a fixed-point free involution. Hence $\bar{\mathcal{G}}$ is a graph.

In particular, $\bar{V} = V$ since \mathcal{S} is a shrub, and $\bar{H} = H$ by construction, so $\bar{p}: \mathcal{G} \rightarrow \bar{\mathcal{G}}$ is an étale monomorphism. \square

Example 5.13. In Example 3.22, the graphs $\mathcal{M}_{x,y}^{X,Y}$ and $\mathcal{N}_{x,y}^X$ (Examples 3.8, 3.9) were constructed as coequalisers of gluing data (3.23), (3.24):

$$(ch_x, ch_y \circ \tau: (\iota) \rightrightarrows (\mathcal{C}_{X \amalg \{x\}} \amalg \mathcal{C}_{Y \amalg \{y\}})) \longrightarrow \mathcal{M}_{x,y}^{X,Y}, \quad (ch_x, ch_y \circ \tau: (\iota) \rightrightarrows \mathcal{C}_{X \amalg \{x,y\}}) \longrightarrow \mathcal{N}_{x,y}^X.$$

This is visualised in Figure 15.

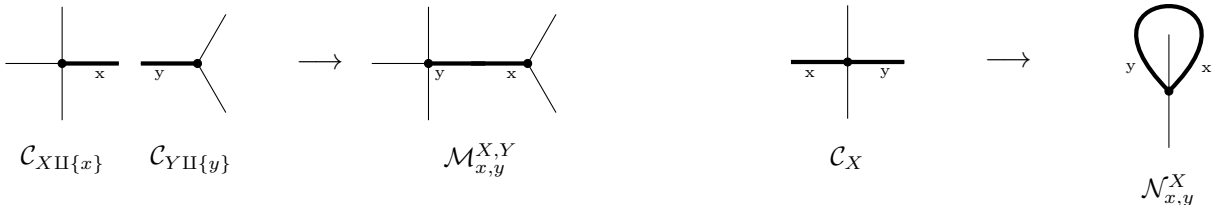


FIGURE 15. Construction of $\mathcal{M}_{x,y}^{X,Y}$ and $\mathcal{N}_{x,y}^X$ as coequalisers of gluing data.

Proposition 5.14. *A non-degenerate \mathcal{G} -shaped graph of graphs $\Gamma: \text{el}(\mathcal{G}) \rightarrow \text{etGr}$ admits a colimit $\Gamma(\mathcal{G})$ in etGr .*

Proof of Proposition 5.14. For all graphs \mathcal{G} , $\text{el}(\mathcal{G})$ is a connected category if and only if \mathcal{G} is a connected graph. So, the colimit $\Gamma(\mathcal{G})$ of a \mathcal{G} -shaped graph of graphs $\Gamma: \text{el}(\mathcal{G}) \rightarrow \text{etGr}$, if it exists, may be constructed componentwise.

A non-degenerate (\mathfrak{i}) -shaped graph of graphs is just an isomorphism $(\mathfrak{i}) \xrightarrow{\cong} (\mathfrak{i})$.

Assume therefore, that \mathcal{G} is a graph without stick components. Since Γ preserves objectwise boundaries and incidence data, we may apply Γ to all components of (5.10) to obtain a diagram

$$(5.15) \quad \coprod_{\tilde{e} \in \widetilde{E}_\bullet} (\iota_{\tilde{e}}) \xrightarrow{\quad} \coprod_{v \in V} \Gamma(\mathcal{C}_v, \iota_v)$$

in etGr . This is a gluing datum since Γ is non-degenerate, and therefore (5.15) has a colimit $\overline{\mathcal{G}}$ in etGr by Lemma 5.12. Moreover, the diagram (5.15) forms a cocone in $\text{psf}(\mathcal{D})$ over the colimit $\Gamma(\mathcal{G})$ of Γ in $\text{psf}(\mathcal{D})$. Hence,

Conversely, Γ has a colimit in the category $\text{psf}(\mathcal{D})$ of graph-shaped diagrams and (5.15) forms a cocone in $\text{psf}(\mathcal{D})$ over this colimit. Hence, by the universal properties of colimits $\Gamma(\mathcal{G}) = \overline{\mathcal{G}}$, if there is a functor from $\text{es}(\mathcal{G})$ to the diagram (5.15), and hence a cocone of Γ over $\overline{\mathcal{G}}$.

For vertices $v' \in V$, and inner edges $e' \in E_\bullet$, there are canonical inclusions

$$(5.16) \quad \Gamma(\iota_{v'}) \hookrightarrow \coprod_{v \in V} \Gamma(\iota_v), \quad \text{and} \quad \Gamma(\iota_{\tilde{e}'}) \hookrightarrow \coprod_{\tilde{e} \in \widetilde{E}_\bullet} \Gamma(\iota_{\tilde{e}}) \xrightarrow{\cong} \coprod_{\tilde{e} \in \widetilde{E}_\bullet} (\iota_{\tilde{e}}).$$

By assumption, \mathcal{G} has no stick components. Therefore, for any port $e \in E_0(\mathcal{G})$, there is a unique half-edge $(\tau e, w) \in H(\mathcal{G})$ and the morphism $\Gamma(\delta_{(\tau e, w)}): \Gamma(\iota_{\tilde{e}}) \hookrightarrow \Gamma(\iota_w)$ induces an inclusion

$$(5.17) \quad \Gamma(\iota_{\tilde{e}}) \hookrightarrow \Gamma(\iota_w) \hookrightarrow \coprod_{v \in V} \Gamma(\iota_v).$$

The inclusions (5.16) and (5.17) describe the desired functor $\text{es}(\mathcal{G}) \rightarrow \overline{\mathcal{G}}$.

So, $\Gamma(\mathcal{G}) = \overline{\mathcal{G}}$ is the colimit of Γ in $\text{psf}(\mathcal{D})$, and hence, since $\overline{\mathcal{G}}$ is a graph, Γ has a colimit in etGr . \square

Remark 5.18. In fact, as will follow from Proposition 7.14, all graphs of graphs admit a colimit in CetGr . However, the non-degeneracy condition simplifies the proof of Proposition 5.14, and is all that is needed for now.

Corollary 5.19. *If \mathcal{G} is a graph, and Γ is a non-degenerate \mathcal{G} -shaped graph of graphs with colimit $\Gamma(\mathcal{G})$, then the induced map $E(\mathcal{G}) \rightarrow E(\Gamma)$ is injective and restricts to the identity $E_0(\mathcal{G}) \xrightarrow{=} E_0(\Gamma(\mathcal{G}))$ on ports. For each $(\mathcal{C}, b) \in \text{el}(\mathcal{G})$, the universal map $\Gamma(b) \rightarrow \Gamma(\mathcal{G})$ is an étale monomorphism. In particular,*

$$E(\Gamma(\mathcal{G})) \cong E(\mathcal{G}) \amalg \coprod_{v \in V} E_\bullet(\Gamma(\iota_v)).$$

Proof. The final statement follows directly from the first two.

By the proof of Proposition 5.14, only the inner edges of \mathcal{G} , and, for all $(\mathcal{C}, b) \in \text{el}(\mathcal{G})$, the τ -orbits of ports of $\Gamma(b)$ are involved in forming the colimit $\Gamma(\mathcal{G})$ of Γ . Hence Γ induces a strict inclusion

$$\coprod_{\tilde{e} \in \widetilde{E}} (\iota_{\tilde{e}}) \xrightarrow{\cong} \coprod_{\tilde{e} \in \widetilde{E}} \Gamma(\iota_{\tilde{e}}) \hookrightarrow \Gamma(\mathcal{G})$$

that restricts to an identity $E_0(\mathcal{G}) = E_0(\Gamma(\mathcal{G}))$ of ports. The second part is immediate. \square

Corollary 5.20. *(Corresponds to [28, Lemma 1.5.12].) If Γ is non-degenerate and, for each $(\mathcal{C}, b) \in \text{el}(\mathcal{G})$, the graph $\Gamma(\mathcal{C}, b)$ is connected, then $\Gamma(\mathcal{G}) = \text{colim}(\Gamma)$ is a connected graph if and only if \mathcal{G} is.*

Proof. Since a (\mathbb{I}) -shaped graph of graphs is isomorphic to the identity functor $(\mathbb{I}) \mapsto (\mathbb{I})$ and hence has colimit (\mathbb{I}) , we assume that \mathcal{G} has no stick components.

Let $\Gamma: \text{el}(\mathcal{G}) \rightarrow \text{CetGr}$ be a non-degenerate \mathcal{G} -shaped graph of graphs with colimit $\Gamma(\mathcal{G})$.

A morphism $\gamma \in \text{psf}(\text{D})(\Gamma(\mathcal{G}), \star \amalg \star)$ is equivalently described by a diagram in $\text{psf}(\text{D})$:

$$(5.21) \quad \mathcal{S}(E_\bullet) \xrightarrow{\quad \quad \quad} \coprod_{v \in V(\mathcal{G})} \Gamma(\iota_v) \xrightarrow{\quad \quad \quad} \star \amalg \star.$$

If $\Gamma(\iota_v)$ is connected for each $v \in V$, then all maps $\Gamma(\iota_v) \rightarrow \star \amalg \star$ are constant. Hence morphisms $\coprod_{v \in V} \Gamma(\iota_v) \rightarrow \star \amalg \star$ are in bijection with morphisms $\coprod_{v \in V} \mathcal{C}_v \rightarrow \star \amalg \star$.

So, $\text{psf}(\text{D})(\mathcal{G}, \star \amalg \star) \cong \text{psf}(\text{D})(\Gamma(\mathcal{G}), \star \amalg \star)$, and it follows from Proposition 3.34 that $\Gamma(\mathcal{G})$ is connected if and only if \mathcal{G} is connected. \square

Let \mathcal{G} be a graph and S any graphical species. As usual, let K be the terminal graphical species.

Definition 5.22. A (non-degenerate) \mathcal{G} -shaped graph of S -structured graphs is a functor $\Gamma_S: \text{el}(\mathcal{G}) \rightarrow \text{CetGr}(S)$ such that the forgetful functor induced by the unique morphism $S \rightarrow K$:

$$(5.23) \quad \Gamma: \text{el}(\mathcal{G}) \xrightarrow{\Gamma_S} \text{etGr}(S) \xrightarrow{\quad \quad \quad} \text{etGr}(K) = \text{etGr}$$

is a (non-degenerate) graph of graphs

The category $\text{CetGr}^{(\mathcal{G})}(S)$ of non-degenerate \mathcal{G} -shaped graphs of S -structured graphs is the category whose objects are non-degenerate \mathcal{G} -shaped graphs of S -structured graphs $\Gamma_S: \text{el}(\mathcal{G}) \rightarrow \text{CetGr}(S)$ and whose morphisms are natural transformations.

Lemma 5.24. For $\mathcal{G} \not\cong \mathcal{C}_0$, two \mathcal{G} -shaped graphs of S -structured graphs Γ_S^1, Γ_S^2 are in the same connected component of $\text{CetGr}^{(\mathcal{G})}(S)$ if and only if, for all $(\mathcal{C}_{X_b}, b) \in \text{el}(\mathcal{G})$, $\Gamma_S^1(b)$ and $\Gamma_S^2(b)$ are in the same connected component of $X_b \text{CGr}_{\text{iso}}(S)$.

In particular, if Γ_S^1 and Γ_S^2 are in the same connected component of $\text{CetGr}^{(\mathcal{G})}(S)$, then Γ_S^1 and Γ_S^2 have isomorphic colimits in $\text{CetGr}(S)$.

Proof. Let $\phi: \Gamma_S^1 \Rightarrow \Gamma_S^2$ be a morphism in $\text{CetGr}^{(\mathcal{G})}(S)$. For each element (\mathcal{C}_{X_b}, b) of \mathcal{G} , the component $\phi_{(b)}$ of ϕ at b , is, by definition a boundary-preserving morphism in $\text{CetGr}(S)$ between X_b -graphs with non-empty vertex sets. Since $\mathcal{G} \not\cong \mathcal{C}_0$, if $(\mathcal{C}_{X_b}, b) \in \text{el}(\mathcal{G})$, then X_b is non-empty. So, by Corollary 4.21, ϕ_b is an isomorphism in $\text{CetGr}(S)$ of S -structured X_b -graphs.

The converse is immediate, as is the final statement. \square

5.3. Multiplication for the monad T . The aim of this section is to describe the multiplication $\mu^T: T^2 \Rightarrow T$ in terms of colimits of graphs of graphs.

From now on, all graphs will be connected, unless explicitly stated otherwise.

Let X be a finite set and $\mathcal{X} = (\mathcal{G}, \rho)$ an X -graph. If $\Gamma = \Gamma^{\mathcal{X}}: \text{el}(\mathcal{X}) \rightarrow \text{CetGr}$ is a non-degenerate \mathcal{X} -shaped graph of graphs, then the colimit $\Gamma(\mathcal{X}) = \text{colim}_{\text{el}(\mathcal{X})} \Gamma$ exists by Proposition 5.14 and it inherits the X -labelling ρ of \mathcal{X} , by Corollary 5.19.

Given a graphical species S and finite set X , elements $[\mathcal{X}, \beta]$ of $T^2 S_X$ are represented X -graphs \mathcal{X} and \mathcal{X} -shaped graphs of S -structured graphs Γ_S . Let $(\Gamma(\mathcal{X}), \alpha)$ be the colimit of Γ_S in $\text{CetGr}(S)$, where $\alpha \in S(\Gamma(\mathcal{X}))$ and $\Gamma(\mathcal{X})$ is the colimit of the \mathcal{X} -shaped graph of graphs defined as in (5.23). The assignment

$$[\mathcal{X}, \beta] \mapsto [\Gamma(\mathcal{X}), \alpha]$$

is independent of the choice of representative of $[\mathcal{X}, \beta] \in T^2 S_X$: Namely, if, for $j = 1, 2$, $(\mathcal{X}^j, \mathbf{\Gamma}_S^j) : \text{el}(\mathcal{X}^j) \rightarrow \text{CetGr}(S)$ both represent the same element $[\mathcal{X}, \beta]$ of $T^2 S_X$, then $\mathcal{X}^1 \cong \mathcal{X}^2$ in $X\text{CGr}_{\text{iso}}$ by definition of T , and, by Lemma 5.24,

$$(5.25) \quad \text{colim}_{\text{el}(\mathcal{X}^1)} \mathbf{\Gamma}_S^1 \cong \text{colim}_{\text{el}(\mathcal{X}^2)} \mathbf{\Gamma}_S^2 \in X\text{CGr}_{\text{iso}}(S).$$

To see that this extends to a morphism $\mu^{\mathbb{T}} S : T^2 S \rightarrow TS$ of graphical species, let $[\mathcal{X}, \beta] \in T^2 S_X$ be represented by an \mathcal{X} -shaped graph of S -structured graphs $\mathbf{\Gamma}_{\mathcal{X}} : \text{el}(\mathcal{X}) \rightarrow \text{CetGr}(S)$ with colimit $\mathbf{\Gamma}_S(\mathcal{X}) = (\mathbf{\Gamma}(\mathcal{X}), \alpha) \in X\text{CGr}_{\text{iso}}(S)$.

By Corollary 5.19, there is a canonical inclusion $E(\mathcal{X}) \hookrightarrow E(\mathbf{\Gamma}(\mathcal{X}))$ and, for each $e \in E(\mathcal{X})$,

$$S(ch_e^{\mathbf{\Gamma}(\mathcal{X})})(\alpha) = S(ch_e^{\mathcal{X}})(\beta) \in S(\mathbf{i}).$$

Hence, for all $x \in X$, there is a commutative diagram of sets

$$\begin{array}{ccc} T^2 S_X & \xrightarrow{\mu^{\mathbb{T}} S_X} & TS_X \\ & \searrow T^2 S(ch_x) \quad \swarrow TS(ch_x) & \\ & T^2 S(\mathbf{i}) = TS(\mathbf{i}). & \end{array}$$

Naturality of $\mu^{\mathbb{T}} S$ in S is immediate from the definition and, by a straightforward modification of [28, Section 2.2], it may be shown that $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ satisfies the two axioms for a monad.

Remark 5.26. For all graphical species S , $\mu^{\mathbb{T}} S$ and $\eta^{\mathbb{T}} S$ are palette-preserving morphisms in GS . So \mathbb{T} restricts to a monad $\mathbb{T}^{(\mathfrak{C}, \omega)}$ on $\text{GS}^{(\mathfrak{C}, \omega)}$, for all (\mathfrak{C}, ω) . If A is a (\mathfrak{C}, ω) -coloured graphical species and $h \in \text{GS}(TA, A)$, then (A, h) is a \mathbb{T} -algebra if and only if it is a $\mathbb{T}^{(\mathfrak{C}, \omega)}$ -algebra.

Example 5.27. Since K is the terminal graphical species, $\text{CetGr}(K) \cong \text{CetGr}$ and hence elements of TK are boundary-preserving isomorphism classes of graphs in CetGr . The unique morphism $! \in \text{GS}(TK, K)$ makes K into an algebra for \mathbb{T} . Likewise, for any palette (\mathfrak{C}, ω) , the unique palette-preserving morphism $!(\mathfrak{C}, \omega) \in \text{GS}^{(\mathfrak{C}, \omega)}(TK^{(\mathfrak{C}, \omega)}, K^{(\mathfrak{C}, \omega)})$ makes the terminal (\mathfrak{C}, ω) -coloured graphical species $K^{(\mathfrak{C}, \omega)}$ into an algebra for \mathbb{T} .

5.4. \mathbb{T} -algebras are non-unital modular operads. Having constructed the monad \mathbb{T} , it just remains to prove that \mathbb{T} -algebras are non-unital modular operads.

Lemma 5.28. *A \mathbb{T} -algebra (A, h) admits a multiplication ${}_h \diamond$ and contraction ${}_h \zeta$. The induced operations are preserved by morphisms in $\text{GS}^{\mathbb{T}}$.*

Proof. Let X and Y be finite sets and let $\mathcal{M}_{x,y}^{X,Y}$ be the $X \amalg Y$ -graph described in Examples 3.8 and 5.13 obtained by gluing the corollas $\mathcal{C}_{X \amalg \{x\}}$ and $\mathcal{C}_{Y \amalg \{y\}}$ along ports x and y .

Given a (\mathfrak{C}, ω) -coloured graphical species S , $\underline{c} \in \mathfrak{C}^X$, $\underline{d} \in \mathfrak{C}^Y$ and $c \in \mathfrak{C}$, ordered pairs $(\phi, \psi) \in S_{(\underline{c}, c)} \times S_{(\underline{d}, \omega c)}$ determine elements $\mathcal{M}_c(\phi, \psi)$ of $S(\mathcal{M}_{x,y}^{X,Y})$.

The canonical map $S(\mathcal{M}_{x,y}^{X,Y}) \rightarrow TS_{X \amalg Y}$ is injective unless $X = Y = \emptyset$, in which case $[\mathcal{M}_c(\phi_1, \psi_1)] = [\mathcal{M}_c(\phi_2, \psi_2)]$ if and only if $(\phi_2, \psi_2) = (\psi_1, \phi_1)$.

If (A, h) is a (\mathfrak{C}, ω) -coloured \mathbb{T} -algebra, then the family of maps given by the compositions

$${}_h \diamond : S_{(\underline{c}, c)} \times S_{(\underline{d}, \omega c)} \xrightarrow{[\mathcal{M}(\cdot, \cdot)]} TS_{\underline{cd}} \xrightarrow{h} S_{\underline{cd}}$$

defines a multiplication on A (see Figure 16).

Similarly, for a finite set X , let $\mathcal{N}_{x,y}^X$ be the X -graph described in Examples 3.9, 5.13, obtained by gluing the ports of $\mathcal{C}_{X \amalg \{x,y\}}$ labelled by x and y .

For $\underline{c} \in \mathfrak{C}^X$ and $c \in \mathfrak{C}$, each $\phi \in S_{(\underline{c}, c, \omega c)} \subset S_{X \amalg \{x, y\}}$ determines an element $\mathcal{N}_c^S(\phi) \in S(\mathcal{N}_{x, y}^X)$. The only non-trivial boundary-preserving automorphism of $\mathcal{N}_{x, y}^X$ is the permutation $\sigma_{x, y} \in \text{Aut}(X \amalg \{x, y\})$ that switches x and y and leaves the other elements unchanged, so $[\mathcal{N}_c^S(\phi)] = [\mathcal{N}_c^S(\psi)]$ in TS_X if and only if $S(\sigma_{x, y})(\phi) = \psi$.

If (A, h) is a (\mathfrak{C}, ω) -coloured algebra for T , the family of maps given by the compositions

$${}_h\zeta: A_{(\underline{c}, c, \omega c)} \xrightarrow{[\mathcal{N}^A(\cdot)]} TA_{\underline{c}} \xrightarrow{h} A_{\underline{c}}$$

defines an equivariant contraction on A (see Figure 16). \square

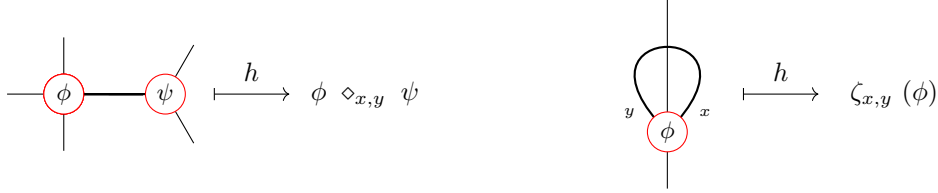


FIGURE 16. If (A, h) is a \mathbb{T} -algebra, h induces a multiplication and contraction on A .

We are now able to describe the algebras for the monad \mathbb{T} on \mathbf{GS} as precisely non-unital modular operads.

Proposition 5.29. *There is a canonical isomorphism of categories $\mathbf{GS}^{\mathbb{T}} \cong \mathbf{MO}^-$.*

Proof. A \mathbb{T} -algebra structure $h: TA \rightarrow A$ equips a graphical species A with a multiplication $\diamond = {}_h\Diamond$, and contraction $\zeta = {}_h\zeta$ as in Lemma 5.28. We must show that (A, \diamond, ζ) satisfies conditions (C1)-(C4) of Definition 1.25.

The key observation for this result is that, up to choice of boundary E_0 , any connected graph with two inner edge orbits has one of the forms illustrated in Figures 17-20.

Condition (C1) is illustrated in Figure 17. Precisely, let $\phi_1 \in A_{(\underline{b}, c)}$, $\phi_2 \in A_{(\underline{c}, \omega c, d)}$, $\phi_3 \in A_{(\underline{d}, \omega d)}$. By Lemma 5.28, and the monad algebra axioms,

$$\begin{aligned} (\phi_1 \diamond_c \phi_2) \diamond_d \phi_3 &= h [\mathcal{M}_d^A ((\phi_1 \diamond_c \phi_2), \phi_3)] \\ &= h [\mathcal{M}_d^A (h[\mathcal{M}_c(\phi_1, \phi_2)], h\eta^\mathbb{T} \phi_3)] \\ &= h\mu^\mathbb{T} [\mathcal{M}_d^{TA} ([\mathcal{M}_c(\phi_1, \phi_2)], \eta^\mathbb{T} \phi_3)], \end{aligned}$$

and, likewise

$$\phi_1 \diamond_c (\phi_2 \diamond_d \phi_3) = h\mu^\mathbb{T} [\mathcal{M}_c^{TA} (\eta^\mathbb{T} A\phi_1, [\mathcal{M}_d^A(\phi_2, \phi_3)])].$$

So, it suffices to show that, for all ϕ_1, ϕ_2, ϕ_3 as above,

$$\mu^\mathbb{T} [\mathcal{M}_d^{TA} ([\mathcal{M}_c(\phi_1, \phi_2)], \eta^\mathbb{T} \phi_3)] = \mu^\mathbb{T} [\mathcal{M}_c^{TA} (\eta^\mathbb{T} A\phi_1, [\mathcal{M}_d^A(\phi_2, \phi_3)])].$$

By Example 5.13, since colimits commute,

$$\text{coeq}_{\mathbf{CetGr}} \left(ch_y, ch_z \circ \tau: (i) \rightrightarrows \mathcal{M}_{w, x}^{X_1, (X_2 \amalg \{y\})} \amalg \mathcal{C}_{X_3 \amalg \{z\}} \right) = \text{coeq}_{\mathbf{CetGr}} \left(ch_w, ch_x \circ \tau: (i) \rightrightarrows \mathcal{C}_{X_1 \amalg \{w\}} \amalg \mathcal{M}_{y, z}^{(X_2 \amalg \{x\}), X_3} \right).$$

The coherence conditions (C2)-(C4) all follow in the same way from the defining axioms of monad algebras. Figures 18-20 illustrate each condition.

The assignment $(A, h) \mapsto (A, \diamond, \zeta)$ clearly extends to a functor $\mathbf{GS}^{\mathbb{T}} \rightarrow \mathbf{MO}^-$.

The proof of the converse closely resembles that of [16, Theorem 3.7]. Namely, let (S, \diamond, ζ) satisfy the conditions in the statement of the proposition. We construct a morphism $h \in \mathbf{GS}(TS, S)$ by successively

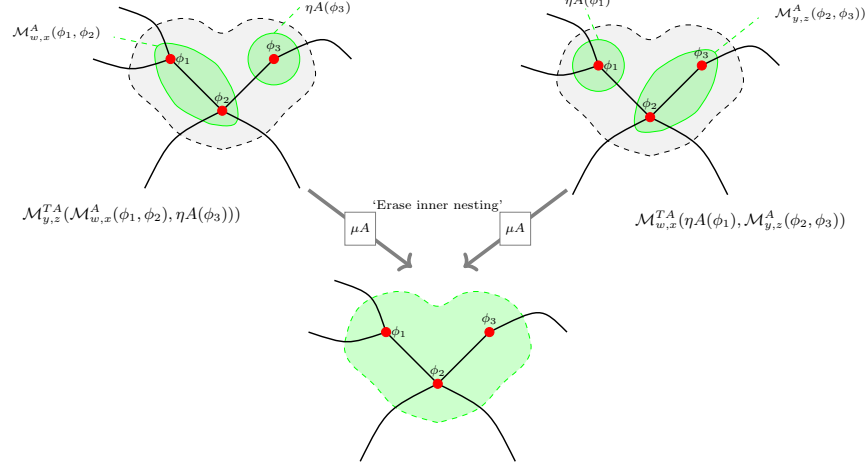


FIGURE 17. Coherence condition (C1) Applying $\mu^{\mathbb{T}}A: T^2A \rightarrow A$ amounts to *erasing inner nesting*.

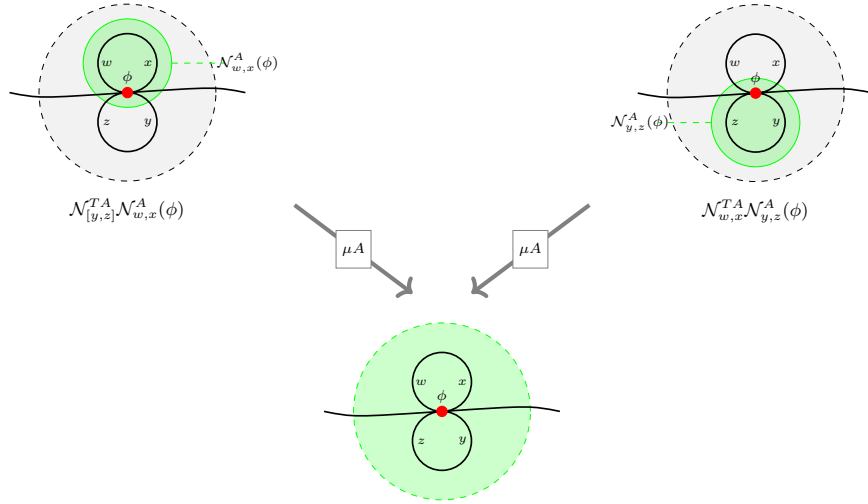


FIGURE 18. Coherence condition (C2)

using \diamond and ζ to *collapse* inner edge orbits of S -structured X -graphs (\mathcal{X}, α) . This results in a finite sequence of S -structured X -graphs that terminates in an S -structured corolla (\mathcal{C}_X, ϕ) .

As usual, let X be a finite set and let (\mathcal{X}, α) be a representative of $[\mathcal{X}, \alpha] \in TS_X$.

If \mathcal{X} has no inner edges, then $\mathcal{X} = \mathcal{C}_X$, and so $[\mathcal{X}, \alpha] = \eta^{\mathbb{T}}S(\phi)$ for some $\phi \in S_X$. In this case, define

$$(5.30) \quad h[\mathcal{X}, \alpha] \stackrel{\text{def}}{=} \phi \in S_X.$$

Otherwise, let $\tilde{e} \in \widetilde{E}_{\bullet}$ be the orbit of a pair $e, \tau e$ of inner edges of \mathcal{X} . Write $t(e) \stackrel{\text{def}}{=} ts^{-1}(e)$ for the vertex v with $e \in E/v$.

There are two possibilities: either $t(e) = t(\tau e)$ or $t(e) \neq t(\tau e)$.

CASE 1: $t(e) = v_1$ and $t(\tau e) = v_2$ are distinct vertices in V .

Let $\mathcal{X}_{/\tilde{e}}$ be the graph obtained from \mathcal{X} by removing the τ -orbit $\{e, \tau e\}$ and identifying v_1 and v_2 to a vertex $\bar{v} \in V/(v_1 \sim v_2)$:

$$\mathcal{X}_{/\tilde{e}} \stackrel{\text{def}}{=} \tau \bigcirc_{\leftarrow} (E \setminus \{e, \tau e\}) \xleftarrow{s} (H \setminus s^{-1}\{e, \tau e\}) \xrightarrow{\bar{t}} V/(v_1 \sim v_2),$$

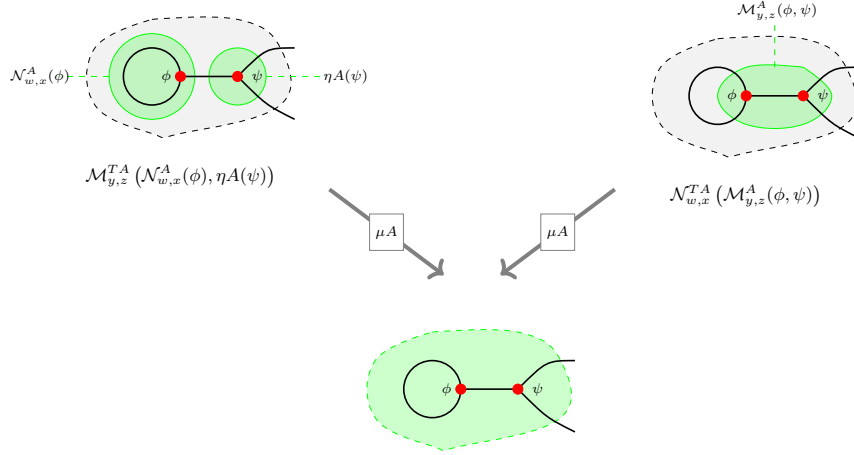


FIGURE 19. Coherence condition (C3).

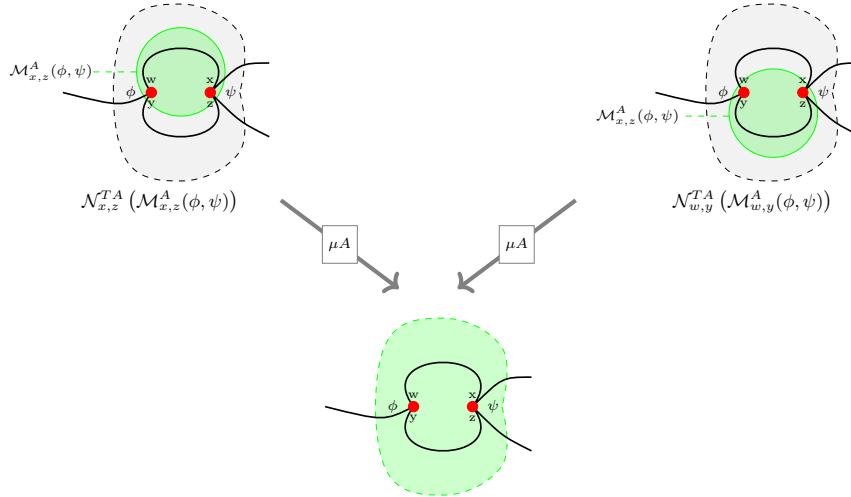


FIGURE 20. Coherence condition (C4).

where \bar{t} is the composition of $t: H \rightarrow V$ with the quotient $V \twoheadrightarrow V/(v_1 \sim v_2)$.

Let $(\mathcal{C}_{X_1 \amalg \{x_1\}}, b_1) \in \text{el}(\mathcal{X})$ be a neighbourhood of v_1 in \mathcal{X} such that $b_1(x_1) = \tau e$, and let $(\mathcal{C}_{X_2 \amalg \{x_2\}}, b_2) \in \text{el}(\mathcal{X})$ be a neighbourhood of v_2 such that $b_2(x_2) = e$. Then $b_{x_1, x_2}: \mathcal{M}_{x_1, x_2}^{X_1, X_2} \rightarrow \mathcal{X}$ is a minimal neighbourhood containing v_1, v_2 , and \bar{e} in \mathcal{X} .

Hence, the monomorphism $\bar{b}: (\mathcal{M}_{x_1, x_2}^{X_1, X_2})_{/\bar{e}} \rightarrow \mathcal{X}_{/\bar{e}}$, obtained by collapsing \bar{e} in $\mathcal{M}_{x_1, x_2}^{X_1, X_2}$ describes a minimal neighbourhood of \bar{v} , and therefore $(\mathcal{M}_{x_1, x_2}^{X_1, X_2})_{/\bar{e}} \cong \mathcal{C}_{X_1 \amalg X_2} \cong \mathcal{C}_{\mathbf{v}'}$ by Lemma 4.15.

For $i = 1, 2$, let $\phi_i \stackrel{\text{def}}{=} S(b_1)(\alpha) \in S(\mathcal{C}_{X_i \amalg \{x_i\}})$. There is an S -structure $\alpha_{/\bar{e}}$ on $\mathcal{X}_{/\bar{e}}$ given by

$$\begin{aligned} S(\bar{b})(\alpha_{/\bar{e}}) &= \phi_1 \diamond_{x_1, x_2} \phi_2, \\ S(b)(\alpha_{/\bar{e}}) &= S(b)(\alpha) \quad \text{whenever } (\mathcal{C}_Y, b) \text{ is a neighbourhood of } v' \in V \setminus \{v_1, v_2\}. \end{aligned}$$

CASE 2: $t(e) = t(\tau e) = v \in V$.

In this case, the graph $\mathcal{X}_{/\bar{e}}$ obtained from \mathcal{X} by collapsing $\{e, \tau e\}$ has the form

$$\mathcal{X}_{/\bar{e}} \stackrel{\text{def}}{=} \tau \bigcirc_{\leftarrow} (E \setminus \{e, \tau e\}) \xleftarrow{s} (H \setminus s^{-1}\{e, \tau e\}) \xrightarrow{\bar{t}} V.$$

If $(\mathcal{C}_{X_v \amalg \{x,y\}}, b) \in \text{el}(\mathcal{X})$ is a neighbourhood of v such that $b(x) = \tau e$ and $b(y) = e$, then the minimal neighbourhood containing v and \tilde{e} in \mathcal{X} is of the form $b_{x,y}: \mathcal{N}_{x,y}^{X_v} \rightarrow \mathcal{X}$.

The canonical monomorphism $\bar{b}: (\mathcal{N}_{x,y}^{X_v})_{/\bar{e}} \rightarrow \mathcal{X}_{/\bar{e}}$ obtained by collapsing \bar{e} is a minimal neighbourhood of v in $\mathcal{X}_{/\bar{e}}$. Hence, $(\mathcal{N}_{x,y}^{X_v})_{/\bar{e}} \cong \mathcal{C}_{X_v}$ by Lemma 4.15.

Let $\phi \stackrel{\text{def}}{=} S(b)(\alpha) \in S_{X_v \amalg \{x,y\}}$. So, there is an S -structure $\alpha_{/\bar{e}}$ on $\mathcal{X}_{/\bar{e}}$ defined by

$$\begin{aligned} S(\bar{b})(\alpha_{/\bar{e}}) &= \zeta_{x,y}(\phi), \\ S(b)(\alpha_{/\bar{e}}) &= S(b)(\alpha) \quad \text{whenever } (\mathcal{C}_Y, b) \text{ is a neighbourhood of } w \in V \setminus \{v\}. \end{aligned}$$

It follows that an ordering $(\tilde{e}_1, \dots, \tilde{e}_N)$ of the set \widetilde{E}_\bullet of inner τ -orbits of \mathcal{X} defines a terminating sequence of S -structured X -graphs

$$(\mathcal{X}, \alpha) \mapsto (\mathcal{X}_{/\tilde{e}_1}, \alpha_{/\tilde{e}_1}) \mapsto ((\mathcal{X}_{/\tilde{e}_1})_{/\tilde{e}_2}, (\alpha_{/\tilde{e}_1})_{/\tilde{e}_2}) \mapsto \dots \mapsto (((\mathcal{X}_{/\tilde{e}_1}) \dots)_{/\tilde{e}_N}, (\alpha_{/\tilde{e}_1}) \dots)_{/\tilde{e}_N}.$$

Since $((\mathcal{X}_{/\tilde{e}_1}) \dots)_{/\tilde{e}_N} = \mathcal{C}_X$ has no inner edges, there is a $\phi_{(\mathcal{X}, \alpha)} \in S_X$ such that

$$(\alpha_{/\tilde{e}_1} \dots)_{/\tilde{e}_N} = \eta^\mathbb{T} S(\phi_{(\mathcal{X}, \alpha)}) \in TS_X.$$

The coherence conditions (C1)-(C4) are equivalent to the statement that $\phi_{(\mathcal{X}, \alpha)} \in S_X$ so obtained is unchanged if the order of collapse of consecutive pairs $\tilde{e}_j, \tilde{e}_{j+1} \in \widetilde{E}_\bullet$ of inner τ -orbits is switched. In other words, $\phi_{(\mathcal{X}, \alpha)}$ is independent of the choice of ordering of \widetilde{E}_\bullet . Moreover, it is independent of the choice of representative of (\mathcal{X}, α) by the definition of TS .

To complete the proof of the proposition, it remains to establish that h satisfies the monad algebra axioms for \mathbb{T} . Compatibility of h with $\eta^\mathbb{T}$ is immediate from equation (5.30). Compatibility of h with $\mu^\mathbb{T}$ follows since the coherence conditions (C1)-(C4) ensure that $h[\mathcal{X}, \alpha]$ is independent of the order of collapse of the inner edges of \mathcal{X} .

So (S, \diamond, ζ) defines a \mathbb{T} -algebra (S, h) , and this assignment extends in the obvious way to a functor $\text{MO}^- \rightarrow \text{GS}^\mathbb{T}$ that, by construction, is inverse to the functor $\text{GS}^\mathbb{T} \rightarrow \text{MO}^-$ defined above. \square

6. THE PROBLEM OF LOOPS

Before constructing the monad \mathbb{D} that encodes the combinatorics of units in Section 7.1, let us first pause to discuss the obstruction to obtaining a monadic multiplication for the modular operad endofunctor in the construction outlined in [23].

Example 6.1. In Example 5.1, I sketched the idea behind the construction of the operad monad \mathbb{M}_{Op} on $\text{ps}(\mathbf{B}^\downarrow)$, whose underlying endofunctor takes a \mathfrak{D} -coloured presheaf P to the \mathfrak{D} -coloured presheaf of formal operadic compositions in P , encoded as P -decorated rooted trees. However, I did not describe how the units for the operadic composition are obtained.

Unlike T on GS , the definition of M_{Op} on $\text{ps}(\mathbf{B}^\downarrow)$ allows *degenerate substitution* of the exceptional directed tree (\downarrow) into the vertex of the corolla t_1 with one leaf (Figure 21). Grafting an exceptional directed edge (\downarrow) onto the leaf or root of any tree \mathfrak{T} leaves \mathfrak{T} unchanged (see Figure 7(b)). So, if (P, θ) is a \mathfrak{D} -coloured algebra for \mathbb{M}_{Op} , and hence a \mathfrak{D} coloured operad, the elements $\theta(\downarrow, d) \in P(t_1)$ (with $d \in \mathfrak{D}$) provide units for the operadic composition.

The modular operad endofunctor $T^{\text{ds}}: \text{GS} \rightarrow \text{GS}$ defined in [23, Section 5], whose algebras are modular operads, is obtained by a slight modification of the non-unital modular operad endofunctor T , to allow degenerate substitutions as in Example 6.1.

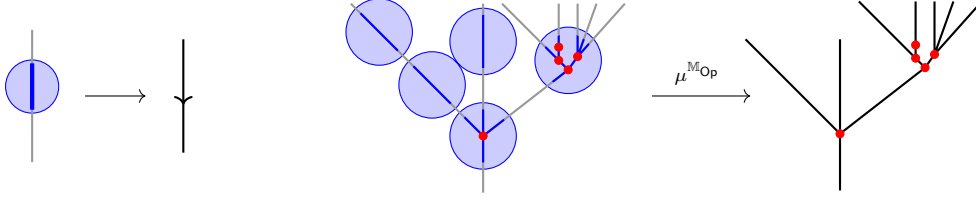


FIGURE 21. The combinatorics of the operadic unit are represented graphically by the degenerate substitution of the exceptional tree into t_1 . Applying the monad multiplication $\mu^{M_{Op}}P$ to nested trees in M_{Op}^2P deletes vertices decorated by elements of $P(\downarrow)$. (See also Figure 13.)

If S is a graphical species and X a finite set, then let $XCetGr_{iso}^{ds}(S)$ be the groupoid obtained from $XCGr_{iso}$ by dropping the condition that X -graphs must have non-empty vertex set. So,

$$XCetGr_{iso}^{ds} = XCGr_{iso} \text{ for } X \not\cong \mathbf{2} \text{ and } \mathbf{2}CetGr_{iso}^{ds} \cong \mathbf{2}CGr_{iso} \amalg \{(1, c)\}_{c \in S_{\S}},$$

(where the boundary of (1) has the identity $\mathbf{2}$ -labelling).

The modular operad endofunctor $T^{ds} : GS \rightarrow GS$ is defined pointwise by

$$\begin{aligned} T^{ds}S_{\S} &\stackrel{\text{def}}{=} S_{\S}, \\ T^{ds}S_X &\stackrel{\text{def}}{=} \text{colim}_{\mathcal{X} \in XCetGr_{iso}^{ds}} S(\mathcal{X}), \quad \text{for all finite sets } X, \end{aligned}$$

together with the obvious extension of T on morphisms in \mathbf{B}° .

Clearly $\eta^{\mathbb{T}}$, together with the inclusion $TS \subset T^{ds}S$ (for all S), induces a unit η^{ds} for T^{ds} . (See Definition 2.13.)

Proposition 6.2. *Algebras for the pointed endofunctor (T^{ds}, η^{ds}) on GS are modular operads.*

Proof. Since $T \subset T^{ds}$, algebras for T^{ds} have the structure of unpointed modular operads by Proposition 5.29. If (A, h) is an algebra for T^{ds} , then, for each $c \in A_{\S}$, $h(1, c) \in A_{\mathbf{2}}$ provides a c -coloured unit for the induced multiplication. \square

For all graphical species S , an element of $T^{ds^2}S_X$ is represented by an X -graph \mathcal{X} and a (possibly degenerate) \mathcal{X} -shaped graph of S -structured connected graphs $\Gamma_S : \text{el}(\mathcal{X}) \rightarrow \text{CetGr}(S)$.

In particular, if T^{ds} admits a monad multiplication $\mu^{ds} : T^{ds^2} \Rightarrow T^{ds}$, then $\mu^{ds}S = \mu^{\mathbb{T}}S$ on T^2S for all graphical species S . But this creates problems, as the following example shows:

Example 6.3. As usual, let \mathcal{W} be the wheel graph with one vertex v and edges $\{a, \tau a\}$. Its category of elements $\text{el}(\mathcal{W})$ has skeletal subcategory

$$(6.4) \quad \begin{array}{ccc} (1) & \xrightleftharpoons[ch_1^{c_2 \circ \tau}]{ch_1^{c_2}} & \mathcal{C}_2 \\ & \searrow ch_a \quad \swarrow 1_{\mathcal{C}_2} \mapsto a & \\ & \mathcal{W} & \end{array}$$

If S is a (\mathfrak{C}, ω) -coloured graphical species and $c \in \mathfrak{C}$, then there is a \mathcal{W} -shaped graph of S -structured graphs $\mathbf{\Lambda}_{S,c}$ given by

$$(6.5) \quad \begin{array}{ccccc} & \text{el}(\mathcal{W}) & & (1) & \xrightarrow{\quad ch_1^{\mathcal{C}_2} \quad} \mathcal{C}_2 \\ & \downarrow \mathbf{\Lambda}_{S,c} & & \downarrow \mathbf{\Lambda}_{S,c}(ch_a) & \downarrow \mathbf{\Lambda}_{S,c}(1_{\mathcal{C}_2} \mapsto a) \\ & \text{CetGr}(S) & & (1, c) & \xrightarrow[\mathbf{\Lambda}_{S,c}(ch_1^{\mathcal{C}_2} \circ \tau)]{\mathbf{\Lambda}_{S,c}(ch_1^{\mathcal{C}_2})} (1, c). \end{array}$$

In particular, $\mathbf{\Lambda}_{S,c}(ch_1^{\mathcal{C}_2}) = id_{(1,c)} = \mathbf{\Lambda}_{S,c}(ch_1^{\mathcal{C}_2} \circ \tau)$.

Hence $\mathbf{\Lambda}_{S,c}$ has colimit $id_c: (1, c) \rightarrow (1, c)$ in $\text{CetGr}(S)$.

In the first place, this is a little surprising since $E_0(1) \neq E_0(\mathcal{W})$ so Corollary 5.19 does not hold for $\mathbf{\Lambda}_{S,c}$.

Moreover, \mathcal{W} admits a unique non-trivial – but trivially boundary fixing – automorphism $\tau_{\mathcal{W}}: \mathcal{W} \rightarrow \mathcal{W}$. So, if $\mathbf{\Lambda}_{S,\omega c}$ is the \mathcal{W} -shaped graph of S -structured graphs

$$ch_a \mapsto (1, \omega c) \text{ and } (1_{\mathcal{C}_2} \mapsto a) \mapsto (1, \omega c),$$

then $\mathbf{\Lambda}_{S,c}$ and $\mathbf{\Lambda}_{S,\omega c}$ represent the same element of $T^{\text{ds}^2}S_0$ since

$$(6.6) \quad \mathbf{\Lambda}_{S,\omega c}(\mathcal{C}, b) = \mathbf{\Lambda}_{S,c}(\mathcal{C}, \tau_{\mathcal{W}} \circ b) \text{ for all } (\mathcal{C}, b) \in \text{el}(\mathcal{W}),$$

so $\tau_{\mathcal{W}} \in \mathbf{0CGr}_{\text{iso}}(\mathcal{W}, \mathcal{W})$.

But $\mathbf{\Lambda}_{S,c}$ has colimit $(1, c)$ in $\text{CetGr}(S)$ while $\mathbf{\Lambda}_{S,\omega c}$ has colimit $(1, \omega c) \in \text{CetGr}(S)$, and these are distinct if $c \neq \omega c$.

So X -isomorphisms do not respect colimits of graphs of graphs on the nose.

As Example 6.3 shows, taking colimits in $X\text{CetGr}_{\text{iso}}^{\text{ds}}$ of degenerate graphs of S -structured graphs does not always lead to a well-defined class of S -structured graphs, let alone one in the correct arity.

The issue is that the parallel morphisms $id_{(1)}, \tau: (1) \rightrightarrows (1)$ do not admit a coequaliser in etGr . In $\text{psf}(\mathbf{D})$ their coequaliser is the exceptional loop \bigcirc (Example 3.16). Therefore, an obvious first attempt at resolving this problem in order to extend $\mu^{\text{T}}S$ to a well-defined morphism $\mu^{\text{ds}}S: T^{\text{ds}^2}S \rightarrow T^{\text{ds}}S$, is therefore to enlarge CetGr to include \bigcirc .

Remark 6.7. In the formalism of [16, 7] described in Example 3.1 (as well as in, for example [18, 32, 43]), where graph ports are defined to be the fixed points of edge involution, the graph substitution is not defined in terms of a functorial construction, but by ‘removing neighbourhoods of vertices and gluing in graphs’. Therefore, the exceptional loop arises from substitution as in Figure 22.

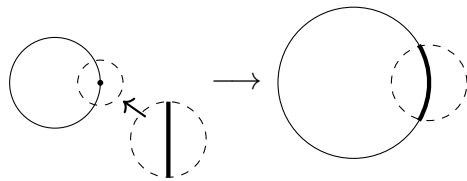


FIGURE 22. Constructing an exceptional loop by removing a vertex and substituting the stick graph.

Since the goal of this work is to unpick and understand this issue of loops, let us dig a little deeper.

Let \mathbf{CetGr}° be the category of *fully generalised Feynman graphs* and étale morphisms, obtained from \mathbf{CetGr} by adding the object \circ and a unique morphism $(\mathfrak{i}) \rightarrow \circ$. In other words, objects of \mathbf{CetGr}° are \mathbf{Set}_τ -diagrams $\tau \begin{smallmatrix} \curvearrowright \\ \hookrightarrow \end{smallmatrix} E \xleftarrow{s} H \xrightarrow{t} V$, such that $s: H \rightarrow E$ is injective, and the involution τ on E is allowed to have fixed points in $E_0 = E \setminus \text{im}(s)$ but not in $\text{im}(s)$.

By definition, \circ is the coequaliser of $\text{id}, \tau: (\mathfrak{i}) \rightrightarrows (\mathfrak{i})$, so we can define the category of elements $\text{el}(\circ) \stackrel{\text{def}}{=} \mathbf{B}^\circ / \circ$, and thereby extend any graphical species $S: \mathbf{B}^{\circ\text{op}} \rightarrow \mathbf{Set}$ to a presheaf on \mathbf{CetGr}° according to $\mathcal{G} \mapsto \lim_{\text{el}(\mathcal{G})} S$. But then $\text{el}(\circ) \cong \text{el}(\mathfrak{i})$, and hence $S(\circ) \cong S(\mathfrak{i})$ for all graphical species S .

It follows that \mathbf{CetGr}° does not embed in \mathbf{GS} . (See also [18, 19] for a discussion of some of these issues.) In particular, there is no monad \mathbb{M} on \mathbf{GS} with arities \mathbf{CetGr}° (see Section 2).

Let $T^\circ: \mathbf{GS} \rightarrow \mathbf{GS}$ be the endofunctor

$$\begin{aligned} T^\circ S_{\mathfrak{s}} &\stackrel{\text{def}}{=} S_{\mathfrak{s}}, \\ T^\circ S_X &\stackrel{\text{def}}{=} \text{colim}_{(\mathcal{G}, \rho) \in X \mathbf{CetGr}_{\text{iso}}^\circ} S(\mathcal{G}). \end{aligned}$$

So, the graph of graph functors $\mathbf{\Lambda}_{S,c}$ and $\mathbf{\Lambda}_{S,\omega c}$ described in Example 6.3 represent the same element $[\mathcal{W}, \beta] \in T^{\circ^2} S_0$. Then since $S(\circ) \cong S(\mathfrak{i}) = S_{\mathfrak{s}}$,

$$[\circ, c] \neq [\circ, \omega c] \in T^\circ S_0 \text{ whenever } c \neq \omega c \in S_{\mathfrak{s}}$$

It follows that μ^\top cannot be extended to a multiplication $\mu^\circ: T^{\circ^2} \Rightarrow T^\circ$.

Indeed, this is not surprising: for each $c \in S_{\mathfrak{s}}$, the contraction of $\phi \in S_{c,\omega c} \subset S_2$ factors through the quotient \bar{S}_2 of S_2 under the action of $\text{Aut}(\mathbf{2})$. Hence, $\zeta(\phi)$ loses data relative to ϕ . The map $(\mathfrak{i}) \rightarrow \circ$ in \mathbf{CetGr}° would seem to be *in the wrong direction*!

The combinatorics of contracted units are examined more closely in the next section. The *problem of loops* discussed in this section will be resolved by adjoining a map that acts as a formal *equaliser*, rather than a coequaliser, of $\text{id}, \tau: (\mathfrak{i}) \rightrightarrows (\mathfrak{i})$.

Remark 6.8. Hackney, Robertson and Yau [21, Definition 1.1] are able to construct the modular operad monad on \mathbf{GS} , within the framework of Feynman graphs, by including extra *boundary* data in their definition of graphs. For them, a graph is a pair $(\mathcal{G}, \bar{\mathfrak{d}}(\mathcal{G}))$ of a Feynman graph \mathcal{G} , and subset $\bar{\mathfrak{d}}(\mathcal{G}) \subset E_0$ of ports, that satisfies certain conditions. So, in their formalism, $(\mathfrak{i}, \mathbf{2})$ and $(\mathfrak{i}, \mathbf{0})$ are different graphs, and $(\mathfrak{i}, \mathbf{0})$ plays the role of the exceptional loop \circ .

However, this approach does not result in a suitable dense functor from the graph category to \mathbf{GS} . Therefore, though they construct a fully faithful nerve for modular operads in terms of a dense subcategory $U \hookrightarrow \mathbf{MO}$ of graphs, the inclusion is not fully faithful, and so U does not fully describe the graphical combinatorics of modular operads.

Remark 6.9. To my knowledge, the construction that I present in Section 7 is unique among graphical descriptions of unital modular operads (or wheeled prop(erad)s), in that all others include some version of the exceptional loop as a graph. (See e.g. [32, 33, 18, 43, 2].)

7. MODULAR OPERADS WITH UNIT

Proposition 5.29 identifies the category of non-unital modular operads with the EM category of algebras for the monad \mathbb{T} on \mathbf{GS} . The goal of this section is to modify this in order to obtain (unital) modular operads. Some potential obstacles have been discussed in Section 6, where it was also explained (in Example 6.1) why the ‘obvious’ modification of the operad monad (Example 5.1) does not work for unital modular operads.

This section begins by returning to the definition of modular operads in Definition 1.25 and looking in more detail at the combinatorics of (contracted) units. This combinatorial information can be encoded in a monad $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ on \mathbf{GS} .

Once \mathbb{D} is defined, it is a small step to obtaining the distributive law $\lambda: TD \Rightarrow DT$, whose construction provides us with an explicit description of the modular operad monad \mathbb{DT} in terms of equivalence classes of graphs structured by graphical species. Moreover, as discussed in Section 7.5, the construction of \mathbb{DT} is such that it is always possible to work with nice (non-degenerate and with well-behaved colimits) representatives of these classes, thereby avoiding the problem of loops (Section 6).

7.1. Pointed graphical species. By definition, if $(S, \diamond, \zeta, \epsilon)$ is a modular operad, then the unit $\epsilon: S_{\S} \rightarrow S_{\mathbf{2}}$ is an injective map such that

$$(7.1) \quad \epsilon \circ S_{\tau} = S(\sigma_{\mathbf{2}})(\epsilon).$$

The key point is that the combination of a unit and a contraction implies that, as well as the structure in arity $\mathbf{2}$ provided by $\epsilon: S_{\S} \rightarrow S_{\mathbf{2}}$, modular operads are equipped with a map $S_{\S} \rightarrow S_{\mathbf{0}}$, providing structure in arity $\mathbf{0}$. Namely, as in (1.24), the contracted unit map

$$(7.2) \quad o = \zeta \circ \epsilon: S_{\S} \rightarrow S_{\mathbf{0}}$$

is well-defined and invariant under $S_{\tau}: S_{\S} \rightarrow S_{\S}$.

Definition 7.3. *Objects of the category \mathbf{GS}_* of pointed graphical species are triples $S_* = (S, \epsilon, o)$ (or (S, ϵ^S, o^S)) where S is a graphical species and $\epsilon: S_{\S} \rightarrow S_{\mathbf{2}}$, and $o: S_{\S} \rightarrow S_{\mathbf{0}}$ are maps satisfying conditions (7.1) and (7.2) above. Morphisms in \mathbf{GS}_* are morphisms in \mathbf{GS} that preserve the additional structure.*

Example 7.4. Given a palette (\mathfrak{C}, ω) , terminal (\mathfrak{C}, ω) -coloured graphical species $K^{(\mathfrak{C}, \omega)}$ is trivially pointed and hence terminal in the category of (\mathfrak{C}, ω) -coloured pointed graphical species and palette-preserving morphisms.

The category \mathbf{GS}_* is also a presheaf category: Namely, we can obtain the category \mathbf{B}_*^{\odot} from \mathbf{B}^{\odot} by formally adjoining morphisms $u: \mathbf{2} \rightarrow \S$ and $z: \mathbf{0} \rightarrow \S$, subject to the relations

- $u \circ ch_1 = id \in \mathbf{B}^{\odot}(\S, \S)$ and $u \circ ch_2 = \tau \in \mathbf{B}^{\odot}(\S, \S)$,
- $\tau \circ u = u \circ \sigma_{\mathbf{2}} \in \mathbf{B}^{\odot}(\mathbf{2}, \S)$, and
- $z = \tau \circ z \in \mathbf{B}^{\odot}(\mathbf{0}, \S)$.

Lemma 7.5. *The following are equivalent:*

- (1) S_* is a presheaf on \mathbf{B}_*^{\odot} that restricts to a graphical species S on \mathbf{B}^{\odot} ;
- (2) (S, ϵ, o) , with $\epsilon = S_*(u)$ and $o = S_*(z)$ is a pointed graphical species.

Proof. It is easy to check directly that \mathbf{B}_*^{\odot} is completely described by

- $\mathbf{B}_*^{\odot}(\S, \S) = \mathbf{B}^{\odot}(\S, \S)$ and $\mathbf{B}_*^{\odot}(Y, X) = \mathbf{B}^{\odot}(Y, X)$ whenever $Y \not\cong \mathbf{0}$ and $Y \not\cong \mathbf{2}$,
- $\mathbf{B}_*^{\odot}(\mathbf{0}, \S) = \{z\}$, and $\mathbf{B}_*^{\odot}(\mathbf{0}, X) = \mathbf{B}^{\odot}(\mathbf{0}, X) \amalg \{ch_x \circ z\}_{x \in X}$,
- $\mathbf{B}_*^{\odot}(\mathbf{2}, \S) = \{u, \tau \circ u\}$, and $\mathbf{B}_*^{\odot}(\mathbf{2}, X) = \mathbf{B}^{\odot}(\mathbf{0}, X) \amalg \{ch_x \circ u, ch_x \circ \tau \circ u\}_{x \in X}$,

and the lemma follows immediately. \square

As a consequence of Lemma 7.5, the notation S_* and (S, ϵ, o) will be used interchangeably to denote the same pointed graphical species.

The forgetful functor $\mathbf{GS}_* \rightarrow \mathbf{GS}$ has a left adjoint $(\cdot)^+$ that takes a graphical species S to its left Kan extension S^+ along the inclusion $(\mathbf{B}^\odot)^{\text{op}} \hookrightarrow (\mathbf{B}_*^\odot)^{\text{op}}$. This does nothing more than formally adjoin elements $\{\epsilon_c^+\}_{c \in S_2}$ to S_2 and $\{o_{\bar{c}}\}_{\bar{c} \in \widetilde{S}_2}$ to S_0 according to the combinatorics of contracted units:

Lemma 7.6. *The forgetful functor $\mathbf{GS}_* \rightarrow \mathbf{GS}$ is strictly monadic. In other words, \mathbf{GS}_* is the EM category of algebras for the monad $\mathbb{D} = (D, \mu^\mathbb{D}, \eta^\mathbb{D})$ on \mathbf{GS} induced by the adjunction.*

The monadic unit $\eta^\mathbb{D}$ is provided by the inclusion $S \hookrightarrow DS$, and the multiplication $\mu^\mathbb{D}$ is induced by the canonical projections $D^2S_2 \rightarrow DS_2$. So S^+ is described by $(DS, \epsilon^+, o^+) = (DS, \epsilon^{DS}, o^{DS})$.

7.2. Pointed graphs. Let \mathbf{CetGr}_* be the category obtained in the bo-ff factorisation of $(\Upsilon -)^+ : \mathbf{CetGr} \hookrightarrow \mathbf{GS} \rightarrow \mathbf{GS}_*$, making the following diagram commute:

$$(7.7) \quad \begin{array}{ccccc} & & \text{f.f.} & & \\ & \text{B}_*^\odot & \xrightarrow[\text{f.f.}]{\text{dense}} & \mathbf{CetGr}_* & \xrightarrow[\text{f.f.}]{\Upsilon_*} \mathbf{GS}_* \\ \text{b.o.} \uparrow & & & \uparrow \text{b.o.} & \\ \mathbf{B}^\odot & \hookrightarrow & \mathbf{CetGr} & \xrightarrow{\Upsilon} & \mathbf{GS} \\ & & & & \downarrow (\cdot)_* \text{ forget} \end{array}$$

The inclusion $\mathbf{B}_*^\odot \rightarrow \mathbf{CetGr}_*$ is fully faithful (by uniqueness of bo-ff factorisation), and also dense, since the induced nerve $\Upsilon_* : \mathbf{CetGr}_* \rightarrow \mathbf{GS}_*$ is fully faithful by construction.

Let $\mathcal{G} \in \mathbf{CetGr}$ be a graph. For each edge $e \in E$, the ch_e -coloured unit for $\Upsilon_*\mathcal{G}$ is defined, by Lemma 7.5, as $\epsilon_e^\mathcal{G} \stackrel{\text{def}}{=} ch_e \circ u \in \mathbf{CetGr}_*(\mathcal{C}_2, \mathcal{G})$, and the corresponding contracted unit is given by $o_e^\mathcal{G} = ch_e \circ z \in \mathbf{CetGr}_*(\mathcal{C}_0, \mathcal{G})$.

Definition 7.8. *Let S_* be a pointed graphical species. The category of pointed elements of S_* is given by $\text{el}_*(S) \stackrel{\text{def}}{=} \text{El}_{\mathbf{B}_*^\odot}(S_*)$. (See Definition 1.1.)*

By an application of the Yoneda lemma $\text{el}_*(\Upsilon_*\mathcal{G}) \cong \mathbf{B}_*^\odot/\mathcal{G}$ for all graphs \mathcal{G} . Therefore, we usually identify these and write simply $\text{el}_*(\mathcal{G})$ for the *category of pointed elements of a graph \mathcal{G}* .

The forgetful functor $\mathbf{GS}_* \rightarrow \mathbf{GS}$ induces injective-on-objects inclusions $\text{el}(S) \rightarrow \text{el}_*(S_*)$ for all pointed graphical species S_* . In particular $\text{el}(\mathcal{G}) \hookrightarrow \text{el}_*(\mathcal{G})$.

Recall [30, Section IX.3] that a functor $\Psi : \mathbf{C} \rightarrow \mathbf{D}$ is called *final* if the slice category $d/\Psi \stackrel{\text{def}}{=} \Psi^{\text{op}}/d$ is non-empty and connected for all $d \in \mathbf{D}$, and that $\Psi : \mathbf{C} \rightarrow \mathbf{D}$ is final if and only if for any functor $\Phi : \mathbf{D} \rightarrow \mathbf{E}$ such that $\text{colim}_{\mathbf{C}}(\Phi \circ \Psi)$ exists in \mathbf{E} , $\text{colim}_{\mathbf{D}}\Phi$ also exists in \mathbf{E} and the two colimits agree.

Lemma 7.9. *For all graphs \mathcal{G} , the inclusion $\text{el}(\mathcal{G}) \hookrightarrow \text{el}_*(\mathcal{G})$ is final. Hence, \mathcal{G} is the colimit of the forgetful functor $\text{el}_*(\mathcal{G}) \rightarrow \mathbf{CetGr}_*$, $(\mathcal{C}, b) \rightarrow \mathcal{C}$ and, for all pointed graphical species $S_* = (S, \epsilon, o)$*

$$\lim_{(\mathcal{C}', b') \in \text{el}_*(\mathcal{G})} S_*(\mathcal{C}') = \lim_{(\mathcal{C}, b) \in \text{el}(\mathcal{G})} S(\mathcal{C}) = S(\mathcal{G}).$$

Proof. By definition, $\text{el}_*(\mathcal{G})$ is obtained from $\text{el}(\mathcal{G})$ by adjoining, for each $e \in E$, the objects $(\mathbf{2}, ch_e \circ u)$ and $(\mathbf{0}, ch_e \circ u) = (\mathbf{0}, ch_{\tau_e} \circ u)$. Hence, for all $(\mathcal{C}, b) \in \text{el}_*(\mathcal{G})$, the slice category $b/\text{el}(\mathcal{G})$ is connected and non-empty. \square

So by Lemma 7.9, a morphism $f \in \text{CetGr}_*(\mathcal{G}, \mathcal{G}')$ is described by a functor $\text{el}(\mathcal{G}) \rightarrow \text{el}_*(\mathcal{G}')$ such that for each $(\mathcal{C}, b) \in \text{el}(\mathcal{G})$, $(\mathcal{C}, b) \mapsto (\mathcal{C}, f \circ b)$, and there is a commuting diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{g} & \mathcal{C}' \\ & \searrow f \circ b & \swarrow b' \\ & \mathcal{G}' & \end{array}$$

such that $g \in \mathbf{B}_*^\odot(\mathcal{C}, \mathcal{C}')$ and $b' \in \text{CetGr}(\mathcal{C}', \mathcal{G}')$, so (\mathcal{C}', b') is an (unpointed) element of \mathcal{G}' .

Example 7.10. (Compare Example 6.3.) A surprising consequence of the definitions is that the morphism set $\text{CetGr}_*(\mathcal{W}, \mathfrak{l})$ is non-empty. There are two morphisms $\kappa, \tau \circ \kappa \in \text{CetGr}_*(\mathcal{W}, \mathfrak{l})$:

(7.11)

$$\begin{array}{ccc} & \mathcal{W} & \\ \nearrow ch_a & & \nwarrow 1_{\mathcal{C}_2} \mapsto a \\ (\mathfrak{l}) & \xrightleftharpoons{ch_1} & \mathcal{C}_2 \\ & \searrow ch_2 \circ \tau & \\ & (\mathfrak{l}) & \end{array}$$

(7.12)

$$\begin{array}{ccc} & \mathcal{W} & \\ \nearrow ch_a & & \nwarrow 1_{\mathcal{C}_2} \mapsto a \\ (\mathfrak{l}) & \xrightleftharpoons{ch_1} & \mathcal{C}_2 \\ & \searrow ch_2 \circ \tau & \downarrow \sigma_2 \\ (\mathfrak{l}) & \xrightleftharpoons{ch_2} & \mathcal{C}_2 \\ & \searrow ch_1 \circ \tau & \\ & (\mathfrak{l}) & \end{array}$$

Hence, $\text{CetGr}_*(\mathcal{W}, \mathfrak{l}) \cong \text{CetGr}_*(\mathfrak{l}, \mathfrak{l}) \cong \text{CetGr}_*(\mathcal{W}, \mathcal{W})$. In particular, for all graphs $\mathcal{G} \not\cong \mathcal{W}$,

$$\text{CetGr}_*(\mathcal{W}, \mathcal{G}) \cong E(\mathcal{G}) \quad \text{by } ch_e \circ \kappa \mapsto e.$$

These morphisms play a crucial role in the proof of the nerve theorem, Theorem 8.1.

Now, let $W \subset V_2$ be a subset of bivalent vertices of a connected graph \mathcal{G} .

Definition 7.13. A vertex deletion functor (for W) is a \mathcal{G} -shaped graph of graphs $\Lambda_{\setminus W}^{\mathcal{G}}: \text{el}(\mathcal{G}) \rightarrow \text{CetGr}_*$ such that for $(\mathcal{C}_X, b) \in \text{el}(\mathcal{G})$,

$$\Lambda_{\setminus W}^{\mathcal{G}}(b) = \begin{cases} (\mathfrak{l}) & \text{if } (\mathcal{C}_X, b) \text{ is a neighbourhood of } v \in W, \\ \mathcal{C}_X & \text{otherwise.} \end{cases}$$

If $\Lambda_{\setminus W}^{\mathcal{G}}$ admits a colimit $\mathcal{G}_{\setminus W}$ in CetGr_* , then the induced morphism $\text{del}_{\setminus W} \in \text{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ is called the vertex deletion morphism corresponding to W .

Note, in particular, that a vertex deletion functor $\Lambda_{\setminus W}^{\mathcal{G}}$ is non-degenerate if and only if $W = \emptyset$ in which case $\Lambda_{\setminus W}^{\mathcal{G}}$ is the identity graph of graphs $(\mathcal{C}, b) \mapsto \mathcal{C}$.

It follows immediately that, if $W = W_1 \amalg W_2$ and $\text{del}_{\setminus W}: \mathcal{G} \rightarrow \mathcal{G}_{\setminus W}$ exists in CetGr_* , then

$$\text{del}_{\setminus W_1}: \mathcal{G} \rightarrow \mathcal{G}_{\setminus W_1} \quad \text{and} \quad \text{del}_{\setminus W_2}^{\mathcal{G}_{\setminus W_1}}: \mathcal{G}_{\setminus W_1} \rightarrow (\mathcal{G}_{\setminus W_1})_{\setminus W_2} = \mathcal{G}_{\setminus W}$$

also exist in CetGr_* and $\text{del}_{\setminus W} = \text{del}_{\setminus W_2}^{\mathcal{G}_{\setminus W_1}} \circ \text{del}_{\setminus W_1}$.

Proposition 7.14. For all graphs \mathcal{G} and all $W \subset V_2$, the colimit $\mathcal{G}_{\setminus W}$ of $\Lambda_{\setminus W}^{\mathcal{G}}$ exists in CetGr_* .

Moreover, $E_0(\mathcal{G}) = E_0(\mathcal{G}_{\setminus W})$ unless $\mathcal{G} = \mathcal{W}^m$ and $W = V$ for some $m \geq 1$.

Example 7.15. For $\mathcal{G} = \mathcal{C}_2$ and $W = V = \{*\}$, $\mathbf{\Lambda}_{\setminus W}^{\mathcal{G}}$ is the constant functor induced by the cocone of $\text{el}(\mathcal{C}_2)$ over (\mathfrak{l}) in CetGr_* :

$$(7.16) \quad \begin{array}{ccccc} (\mathfrak{l}) & \xrightarrow{ch_1} & \mathcal{C}_2 & \xleftarrow{ch_2 \circ \tau} & (\mathfrak{l}) \\ & \searrow id_{(\mathfrak{l})} & \downarrow u & \swarrow id_{(\mathfrak{l})} & \\ & & (\mathfrak{l}) & & \end{array} .$$

So, trivially, $\mathbf{\Lambda}_{\setminus W}^{\mathcal{G}}$ has colimit (\mathfrak{l}) in CetGr_* and $\text{del}_{\setminus W} = u \in \text{CetGr}_*(\mathcal{C}_2, \mathfrak{l})$.

More generally, if $\mathcal{G} = \mathcal{L}^k$, and $W = V$, then $\mathbf{\Lambda}_{\setminus W}^{\mathcal{G}}$ is also the constant functor to $\mathcal{G}_{\setminus W} = (\mathfrak{l})$, and $u^k \stackrel{\text{def}}{=} \text{del}_{\setminus W}: \mathcal{L}^k \rightarrow \text{CetGr}_*$ is induced by the CetGr_* -cocone under $\text{el}_*(\mathcal{G})$:

$$\begin{array}{ccccccc} (\mathfrak{l}) & \xrightarrow{ch_1} & \mathcal{C}_2 & \xleftarrow{ch_2 \circ \tau} & \dots & \xrightarrow{ch_1} & \mathcal{C}_2 & \xleftarrow{ch_2 \circ \tau} & (\mathfrak{l}) \\ & \searrow id_{(\mathfrak{l})} & \downarrow u & & & & \downarrow u & \swarrow id_{(\mathfrak{l})} & \\ & & (\mathfrak{l}) & \equiv & \dots & \equiv & (\mathfrak{l}) & & \end{array}$$

(So $u^1 = u: \mathcal{C}_2 \rightarrow (\mathfrak{l})$ and u^0 is just the identity on (\mathfrak{l}) .)

For any graph \mathcal{G} , a pointwise étale injection $\iota \in \text{CetGr}(\mathcal{L}^k, \mathcal{G})$ includes $W = V(\mathcal{L}^k)$ as a subset $W \subset V_2(\mathcal{G})$ of bivalent vertices of \mathcal{G} . Hence, $\text{del}_{\setminus W} \in \text{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ exists in CetGr_* and there is a commuting diagram

$$(7.17) \quad \begin{array}{ccc} \mathcal{L}^k & \xrightarrow{\iota} & \mathcal{G} \\ u^k \downarrow & & \downarrow \text{del}_{\setminus W} \\ (\mathfrak{l}) & \xrightarrow{ch_{e_{\setminus W}}} & \mathcal{G}_{\setminus W}, \end{array}$$

where $e_{\setminus W} = \text{del}_{\setminus W}(\iota(1_{\mathcal{L}^k}))$.

Example 7.18. By Example 7.10, $\mathcal{W}_{\setminus \{v\}} = \text{colim}_{\text{el}(\mathcal{W})} \mathbf{\Lambda}_{\setminus \{v\}}^{\mathcal{W}}$ exists and is isomorphic to (\mathfrak{l}) in CetGr . (See also Section 6.) The induced morphism $\text{del}_{\setminus \{v\}}$ is precisely $\kappa: \mathcal{W} \rightarrow (\mathfrak{l})$.

Let $(v_i)_{i=1}^m$ be the (cyclically) ordered vertices of the wheel graph \mathcal{W}^m , and let $\iota \in \text{CetGr}(\mathcal{L}^{m-1}, \mathcal{W}^m)$ be a pointwise étale inclusion that respects the (cyclic) ordering of vertices. So, if W is the image of $V(\mathcal{L}^{m-1})$ in $V(\mathcal{W}^m)$, then $V(\mathcal{W}^m) \cong W \amalg \{*\}$. By (7.17), ι defines a vertex deletion morphism $\text{del}_{\setminus W} \in \text{CetGr}_*(\mathcal{W}^m, \mathcal{W})$. Therefore $\kappa^m \stackrel{\text{def}}{=} \text{del}_{\setminus V(\mathcal{W}^m)}$ is given by the composite $\kappa \circ \text{del}_{\setminus W}: \mathcal{W}^m \rightarrow (\mathfrak{l})$.

In particular, for all $m \geq 1$, there are two distinct pointed morphisms κ^m , and $\tau \circ \kappa^m$ in $\text{CetGr}_*(\mathcal{W}^m, \mathfrak{l})$. Hence, for all \mathcal{G} ,

$$\text{CetGr}_*(\mathcal{W}^m, \mathcal{G}) = \text{CetGr}(\mathcal{W}^m, \mathcal{G}) \amalg \{ch_e \circ \kappa^m\}_{e \in E(\mathcal{G})} \cong \text{CetGr}_*(\mathcal{W}, \mathcal{G}).$$

Proof of Proposition 7.14. If W is empty, then $\mathcal{G}_{\setminus W} = \mathcal{G}$ and $\text{del}_{\setminus W}$ is the identity on \mathcal{G} . On the other hand, if $W = V$ then, by Proposition 4.23, $\mathcal{G} = \mathcal{L}^k$ or $\mathcal{G} = \mathcal{W}^m$ for some $k \geq 0$, $m \geq 1$. In these cases, it follows from Examples 7.15 and 7.18 that $\mathcal{G}_{\setminus W} = (\mathfrak{l})$ in CetGr_* . For $\mathcal{G} = \mathcal{L}^k$, the vertex deletion morphism $u^k: \mathcal{L}^k \rightarrow (\mathfrak{l})$ induces a bijection on boundaries. So the proposition is proved when $W = V$ or $W = \emptyset$.

So, assume that $\emptyset \neq W \subsetneq V$ is a proper, non-empty subset of bivalent vertices of \mathcal{G} .

Let $\mathcal{G}^W \subset \mathcal{G}$ be the subgraph with vertices $V(\mathcal{G}^W) = W$, half-edges $H(\mathcal{G}^W) = \coprod_{v \in W} H/v$ and whose edge set $E(\mathcal{G}^W)$ is the τ -closure of $\coprod_{v \in W} E/v$. (See Figure 23.)

By Proposition 4.23, $\mathcal{G}^W = \coprod_{i=1}^m \mathcal{L}^{k_i}$ is a disjoint union of line graphs, where $k_i \geq 1$ for all i . So,

$$(7.19) \quad E_0(\mathcal{G}^W) = \coprod_{i=1}^m \{1_{\mathcal{L}^{k_i}}, 2_{\mathcal{L}^{k_i}}\}, \quad \text{and} \quad \left(\coprod_{i=1}^m \{1_{\mathcal{L}^{k_i}}\} \right) \cap \left(\coprod_{i=1}^m \{2_{\mathcal{L}^{k_i}}\} \right) = \emptyset \text{ in } E(\mathcal{G}).$$

For $1 \leq i \leq m$, let $e_i \in E(\mathcal{G})$ be the edge corresponding to $1_{\mathcal{L}^{k_i}}$.

The graph $\mathcal{G}_{\setminus W}$ is obtained by applying $u^{k_i}: \mathcal{L}^{k_i} \rightarrow (1)$ on each component \mathcal{L}^{k_i} of $\mathcal{G}^W \subset \mathcal{G}$. Since these are disjoint in \mathcal{G} and $W \neq V$, there is a commuting diagram

$$\begin{array}{ccc} \mathcal{G}^W = \coprod_{i=1}^m \mathcal{L}^{k_i} & \xrightarrow{\iota} & \mathcal{G} \\ \coprod_{i=1}^m u^{k_i} \downarrow & & \downarrow \text{del}_{\setminus W} \\ \coprod_{i=1}^m (1) & \xrightarrow{\coprod_i \text{che}_{e_{\setminus W_i}}} & \mathcal{G}_{\setminus W}, \end{array}$$

and $\text{del}_{\setminus W}: \mathcal{G} \rightarrow \mathcal{G}_{\setminus W}$ exists in CetGr_* .

The graph $\mathcal{G}_{\setminus W}$ is described explicitly (when $W \neq V$) by:

$$\mathcal{G}_{\setminus W} = \tau_W \circ \left(E_{\setminus W} \xleftarrow{s_{\setminus W}} H_{\setminus W} \xrightarrow{t_{\setminus W}} V_{\setminus W} \right),$$

where

$$\begin{aligned} V_{\setminus W} &= V \setminus W, \\ H_{\setminus W} &= H \setminus H(\mathcal{G}^W) = H \setminus \coprod_{v \in W} H/v, \\ E_{\setminus W} &= E \setminus \coprod_{v \in W} E/v, \end{aligned}$$

$s_{\setminus W}, t_{\setminus W}$ are just the restrictions of s and t , and the involution $\tau_W: E_{\setminus W} \rightarrow E_{\setminus W}$ is given by

$$\begin{aligned} \tau_W(e) &= \tau e & \text{for } e \in E \setminus E(\mathcal{G}^W), \\ \tau_W(1_{\mathcal{L}^{k_i}}) &= 2_{\mathcal{L}^{k_i}} & \text{for } 1 \leq i \leq m. \end{aligned}$$

This is fixed point free by (7.19). See Figure 23.

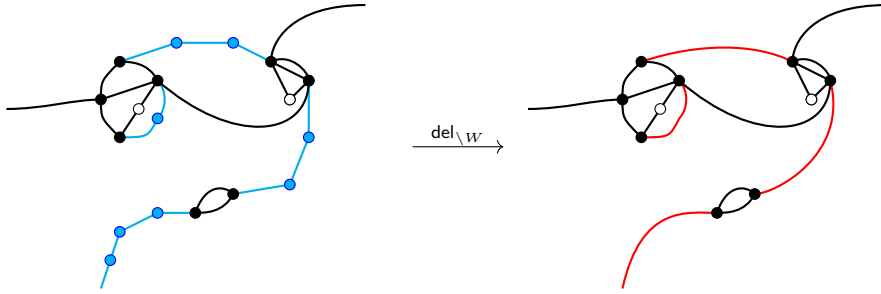


FIGURE 23. Vertex deletion $\text{del}_{\setminus W}: \mathcal{G} \rightarrow \mathcal{G}_{\setminus W}$, with $\mathcal{G}^W \subset \mathcal{G}$ and $W \subset V_2$, and $\coprod_{i=1}^3 u^{k_i}(\mathcal{G}^W) \subset \mathcal{G}_{\setminus W}$.

The final statement, that $E_0(\mathcal{G}) = E_0(\mathcal{G}_{\setminus W})$ except when $\mathcal{G} = \mathcal{W}^m$ and $W = V$ for some $m \geq 1$, follows since $\text{del}_{\setminus W} \in \text{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ restricts to an identity on boundaries when $W \neq V$ by (7.19). \square

Definition 7.20. The similarity category $\text{Gr}_{\text{sim}_*} \hookrightarrow \text{CetGr}_*$ is the identity-on-objects subcategory of CetGr_* whose morphisms are given by $z: \mathcal{C}_0 \rightarrow (1)$, the vertex deletion morphisms, and graph isomorphisms.

Morphisms in Gr_{sim_*} are called similarity morphisms, and connected components of Gr_{sim_*} are similarity classes. Graphs in the same connected component of Gr_{sim_*} are similar.

Example 7.21. In \mathbf{CetGr}_* , any morphism with codomain (\mathbb{I}) is, up to isomorphism, of the form $z \in \mathbf{CetGr}_*(\mathcal{C}_0, \mathbb{I})$, $\kappa^m \in \mathbf{CetGr}_*(\mathcal{W}^m, \mathbb{I})$ for $m \geq 1$, or $u^k \in \mathbf{CetGr}_*(\mathcal{L}^k, \mathbb{I})$ for $k \geq 0$ and hence a similarity morphism.

For any graph \mathcal{G} without isolated vertices, let $\mathcal{P}(V_2)$ be poset of subsets $W \subset V_2$ of bivalent vertices of \mathcal{G} , ordered by inclusion. Any pointed graphical species $S_* = (S, \epsilon, o)$ in \mathcal{E} defines a functor $\mathcal{P}(V_2) \rightarrow \mathcal{E}$, with colimit $S(\mathcal{G})$, by $W \mapsto S(\mathcal{G}_{\setminus W})$ and $(W_1 \subset W_2) \mapsto \left(S(\text{del}_{\setminus(W_2 \setminus W_1)}^{\mathcal{G}_{\setminus W_1}}) : S(\mathcal{G}_{\setminus W_2}) \rightarrow S(\mathcal{G}_{\setminus W_1}) \right)$.

In particular, for all graphical species S in \mathcal{E} and all graphs \mathcal{G} with no isolated vertices,

$$(7.22) \quad DS(\mathcal{G}) = \coprod_{W \in \mathcal{P}(V_2)} S(\mathcal{G}_{\setminus W}).$$

Example 7.23. For all graphs \mathcal{G} and all $k \in \mathbb{N}$,

$$\mathbf{CetGr}_*(\mathcal{L}^k, \mathcal{G}) \cong \coprod_{j=0}^k \binom{k}{j} \mathbf{CetGr}(\mathcal{L}^j, \mathcal{G}).$$

Corollary 7.24 (Corollary to Proposition 7.14). *The pair $(\mathbf{Gr}_{\text{sim}_*}, \mathbf{CetGr})$ of subcategories of \mathbf{CetGr}_* defines an orthogonal factorisation system on \mathbf{CetGr}_* .*

In particular, assume that $E_0(\mathcal{G}) \neq \emptyset$ (hence $\mathcal{G} \not\cong \mathcal{C}_0$), and let $f \in \mathbf{CetGr}_(\mathcal{G}, \mathcal{G}')$ factor as $(\text{del}_{\setminus W_f}, f_{\setminus W_f})$. If f is boundary-preserving, then $f_{\setminus W_f} \in \mathbf{CetGr}(\mathcal{G}_{\setminus W_f}, \mathcal{G}')$ is an isomorphism.*

Proof. Let $\mathcal{G} \not\cong \mathcal{C}_0$ be a connected graph, and let $f \in \mathbf{CetGr}_*(\mathcal{G}, \mathcal{G}')$. Let W_f be the set of bivalent vertices w of \mathcal{G} such that, if (\mathcal{C}_2, b) is a minimal neighbourhood of w , then there is an edge e' of \mathcal{G}' such that

$$f \circ b = ch_{e'} \circ u : \mathcal{C}_2 \rightarrow \mathcal{G}'.$$

Then, by Proposition 7.14, f factors uniquely as a vertex deletion morphism $\text{del}_{\setminus W_f} \in \mathbf{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W_f})$ followed by a morphism $f_{\setminus W_f} \in \mathbf{CetGr}(\mathcal{G}_{\setminus W_f}, \mathcal{G}')$.

The only morphisms in \mathbf{CetGr}_* with domain \mathcal{C}_0 are of the form $id_{\mathcal{C}_0}$ or $ch_e \circ z : \mathcal{C}_0 \rightarrow \mathcal{G}$ for some graph \mathcal{G} and edge e of \mathcal{G} . Hence the corollary follows immediately from Proposition 7.14 and Corollary 4.21. \square

The factorisation system described by $(\mathbf{Gr}_{\text{sim}_*}, \mathbf{CetGr}_*)$ is called the *generic-free* factorisation system on \mathbf{CetGr}_* (see [42]).

By Corollary 7.24, a morphism $f \in \mathbf{CetGr}_*(\mathcal{G}, \mathcal{G}')$ is uniquely characterised by a commuting diagram of the form (7.25), and such that $\mathfrak{f}_V^{-1}(\widetilde{E'}) \subset V_0 \amalg V_2$ is either a single isolated vertex or a (possibly empty) subset of bivalent vertices, and the induced square 7.26 is a pullback.

$$(7.25) \quad \begin{array}{ccccc} E & \xleftarrow{\tau} & E & \xleftarrow{s} & H & \xrightarrow{t} & V \\ \mathfrak{f}_E \downarrow & & \mathfrak{f}_E \downarrow & & \mathfrak{f}_H \downarrow & & \mathfrak{f}_V \downarrow \\ E' & \xleftarrow{\tau'} & E' & \xleftarrow{s' \amalg id'} & H' \amalg E' & \xrightarrow{t' \amalg q'} & V' \amalg \widetilde{E'} \end{array} \quad (7.26) \quad \begin{array}{ccc} H & \xrightarrow{t} & (V \setminus V_0) \\ \mathfrak{f}_H \downarrow & & \mathfrak{f}_V \downarrow \\ H' \amalg E' & \xrightarrow{t' \amalg q'} & (V \setminus V_0) \amalg \widetilde{E'} \end{array}$$

If $\mathcal{G} \not\cong \mathcal{C}_0$, and $f = f_{\setminus W_f} \circ \text{del}_{\setminus W_f}$ is the generic-free factorisation of f , then $W_f = \mathfrak{f}_V^{-1}(\widetilde{E'}) \subset V_2$.

Example 7.27. Diagram (7.28) describes $z : \mathbf{0} \rightarrow \S$, and (7.29) describes $u : \mathbf{2} \rightarrow \S$ in $\mathbf{B}_*^\odot \subset \mathbf{CetGr}_*$:

$$(7.28) \quad \begin{array}{ccccc} \emptyset & \xleftarrow{\quad} & \emptyset & \xrightarrow{\quad} & \{v\} \\ \downarrow & & \downarrow & & \downarrow \\ \tau \curvearrowright \{1, 2\} & \xleftarrow{id} & \{1, 2\} & \xrightarrow{q_1} & \{\tilde{1}\} \end{array} \quad (7.29) \quad \begin{array}{ccccc} \tau_2 \curvearrowright \left\{ \begin{array}{l} e_1, \\ \tau_2 e_1, \end{array} \right\} & \xleftarrow{\quad} & \{(e_1, v), (e_2, v)\} & \xrightarrow{\quad} & \{v\} \\ \downarrow & & \downarrow & & \downarrow \\ \tau \curvearrowright \{1, 2\} & \xleftarrow{id} & \{1, 2\} & \xrightarrow{q_1} & \{\tilde{1}\}. \end{array}$$

Lemma 3.14 admits the following extension to CetGr_* , that says that most morphisms in CetGr_* are completely determined by their action on edges:

Lemma 7.30. *If $\mathcal{G} \not\cong \mathcal{C}_0$ and $\mathcal{G}' \not\cong \mathcal{W}$, then \mathfrak{f}_E is sufficient to define $f \in \text{CetGr}_*(\mathcal{G}, \mathcal{G}')$.*

Proof. Assume that \mathcal{G} is not an isolated vertex, hence $E \neq \emptyset$. If $\mathcal{G}' \not\cong \mathcal{W}$, then

$$\tau' e'_1 \neq e'_2 \text{ for all } e'_1, e'_2 \in E' \text{ such that } \{e'_1, e'_2\} = E'/v', v' \in V'_2.$$

So, $v \in V_2$ is a bivalent vertex of \mathcal{G} with $E/v = \{e_1, e_2\} \subset E_2$, and

$$\mathfrak{f}_E(e_1) = \mathfrak{f}_E(\tau e_2) \text{ if and only if } \mathfrak{f}_V(v) = q'(\mathfrak{f}_E(e_1)) = q'(\mathfrak{f}_E(\tau e_2)) \in \widetilde{E'}.$$

Otherwise, if $\mathfrak{f}_E(e_1) \neq \mathfrak{f}_E(\tau e_2)$, then $\mathfrak{f}_V(v) = t' s'^{-1}(\mathfrak{f}_E(e_1)) \in V'$. □

Example 7.31. Lemma 7.30 does not hold if $\mathcal{G}' = \mathcal{W}$. For example, there are only two maps $E(\mathcal{W}^2) \rightarrow E(\mathcal{W})$ that are compatible with the involution, and these correspond to the two maps in $\text{CetGr}(\mathcal{W}^2, \mathcal{W})$. However, $\text{CetGr}_*(\mathcal{W}^2, \mathcal{W})$ has six distinct elements.

7.3. S_* -structured graphs. The étale topology on CetGr extends to a topology on CetGr_* whose covers at \mathcal{G} are jointly surjective collections $\mathfrak{U} \subset \text{CetGr}_*/\mathcal{G}$.

So, $P_* \in \text{ps}(\text{CetGr}_*)$ is a sheaf for this topology if and only if, for all graphs \mathcal{G} ,

$$P_*(\mathcal{G}) \cong \lim_{(\mathcal{C}, b) \in \text{el}(\mathcal{G})} P(\mathcal{C}), \text{ where } P: \text{CetGr}^{\text{op}} \rightarrow \mathcal{G} \text{ is the restriction to } \text{CetGr}.$$

In particular, there is a canonical equivalence $\text{sh}(\text{CetGr}_*, J_*) \simeq \text{GS}_*$.

Let $S_* = (S, \epsilon, o)$ be a pointed graphical species.

Definition 7.32. (Compare Definition 4.30.) *The element category $\text{El}_{S_*}(\text{CetGr}_*)$ is called the category of S_* -structured graphs and denoted by $\text{CetGr}_*(S_*)$.*

An S_ -structured graph (\mathcal{G}, α) is called admissible if $\mathcal{G} \not\cong (1)$ is not a stick graph.*

Example 7.33. For $k \geq 0$ and $m \geq 1$, the vertex deletion morphisms $u^k: \mathcal{L}^k \rightarrow (1)$ and $\kappa^m: \mathcal{W}^m \rightarrow (1)$ in CetGr_* induce injective maps in $\text{CetGr}_*(S_*)$:

$$(7.34) \quad \epsilon^k \stackrel{\text{def}}{=} S_*(u^k): S_{\S} \rightarrow S(\mathcal{L}^k), \text{ and } o^m \stackrel{\text{def}}{=} S_*(\kappa^m): S_{\S} \rightarrow S(\mathcal{W}^m),$$

where $\epsilon^1 = \epsilon$, and ϵ^k, o^m factor through $\epsilon: S_{\S} \rightarrow S_2$ for all $k \geq 0, m \geq 1$.

For each $c \in S_{\S}$, let

$$\mathcal{L}^k(\epsilon_c) \stackrel{\text{def}}{=} \epsilon^k(c) \in S_*(\mathcal{L}^k) \text{ and } \mathcal{W}^m(\epsilon_c) \stackrel{\text{def}}{=} o^m(c) \in S_*(\mathcal{W}^m)$$

be the c -coloured unit structures on \mathcal{L}^k and \mathcal{W}^m , as pictured in Figure 24.

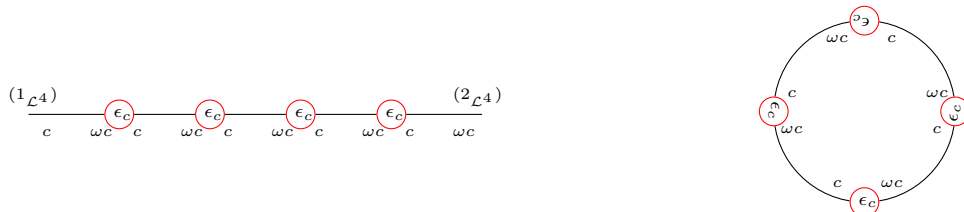


FIGURE 24. The c -coloured unit structures $\mathcal{L}^4(\epsilon_c)$ and $\mathcal{W}^4(\epsilon_c)$.

For any subset $W \subset V_2$ of bivalent vertices of a graph \mathcal{G} , and any S_* -structure $\alpha_{\setminus W} \in S(\mathcal{G}_{\setminus W})$, there is a unique S_* -structure $\alpha \in S_*(\mathcal{G})$ such that $\text{del}_{\setminus W} \in \text{CetGr}_*(S_*)(\alpha, \alpha_{\setminus W})$: Let $(\mathcal{C}_2, b) \in \text{el}(\mathcal{G})$ be a minimal neighbourhood of $v \in W$. Then there is some $e \in E(\mathcal{G}_{\setminus W})$, such that $\text{del}_{\setminus W} \circ b = ch_e \circ u$, and hence

$$S_*(b)(\alpha) = S_*(\text{del}_{\setminus W} \circ b)(\alpha_{\setminus W}) = S_*(u)S(ch_e)(\alpha_{\setminus W}) = \epsilon(S(ch_e)(\alpha_{\setminus W})).$$

Definition 7.35. Let (\mathcal{G}, α) be an S_* -structured graph with $\mathcal{G} \not\cong \mathcal{C}_0$. The set

$$W_\alpha = \{v \mid \text{there is a neighbourhood } (\mathcal{C}, b) \text{ of } v \text{ such that } S(b)(\alpha) \in \text{im}(\epsilon) \cup \text{im}(o)\} \subset V_0 \amalg V_2,$$

is the subset of vertices α -decorated by (contracted) units.

For all \mathcal{G} , an S_* -structure (\mathcal{G}, α) is called *reduced* if $W_\alpha = \emptyset$.

For all $\mathcal{G} \not\cong \mathcal{C}_0$, and all subsets $W \subset V_2$ of bivalent vertices of \mathcal{G} , there is an S_* -structure α_W such that $\text{del}_{\setminus W} \in \text{CetGr}_*(\mathcal{G}, \mathcal{G}_{\setminus W})$ describes a morphism in $\text{CetGr}_*(S_*)((\mathcal{G}, \alpha), (\mathcal{G}_{\setminus W}, \alpha_{\setminus W}))$ if and only if $W \subset W_\alpha$.

For all $(\mathcal{G}, \alpha) \in \text{CetGr}_*(S_*)$ we can define the *reduced similar S_* -structure* $(\mathcal{G}_\alpha^\perp, \alpha^\perp)$ corresponding to (\mathcal{G}, α) : If $(\mathcal{G}, \alpha) \cong (\mathcal{C}_0, o_{\bar{c}})$ – and hence $\mathcal{G} \cong \mathcal{C}_0$ and $W_\alpha = \{*\}$ – then $(\mathcal{G}_\alpha^\perp, \alpha^\perp)$ is unique up to unique isomorphism. It is induced by $z: \mathcal{C}_0 \rightarrow (1)$ and has the form $(1, c)$ or $(1, \omega c)$. Otherwise, $(\mathcal{G}_\alpha^\perp, \alpha^\perp)$ is induced by $\text{del}_{\setminus W_\alpha}$ and $(\mathcal{G}_\alpha^\perp, \alpha^\perp) = (\mathcal{G}_{\setminus W_\alpha}, \alpha_{\setminus W_\alpha})$ is unique on the nose.

7.4. Similar structures. The issues that can arise from trying to incorporate degenerate substitution by the stick graph into the definition of the modular operad monad in order to obtain multiplicative units have been outlined in Section 6.

Degenerate substitutions – and exceptional loops – can be avoided if there is a suitable notion of equivalence of S_* -structured graphs, for which all constructions can be obtained in terms of admissible representatives.

This principle will inform the construction of the distributive law $\lambda: TD \Rightarrow DT$.

For any finite set X , the category $X\text{Gr}_{\text{sim}_*}$ is obtained from the category $X\text{CGr}_{\text{iso}}$ of (admissible) X -graphs and port-preserving isomorphisms, by including the vertex deletion morphisms $\text{del}_{\setminus W}$, and $z: \mathcal{C}_0 \rightarrow (1)$, whenever the domain is an (admissible) X -graph. In this way, for any morphism f in $X\text{Gr}_{\text{sim}_*}$, whose domain is an X -graph \mathcal{X} , the codomain $f(\mathcal{X})$ inherits a X -labelling from \mathcal{X} :

- If $X \not\cong 0$, $X \not\cong 2$, then $X\text{Gr}_{\text{sim}_*}$ is the category whose objects are connected X -graphs and whose morphisms are similarity morphisms that preserve the labelling of the ports.
- For $X = 2$, $2\text{Gr}_{\text{sim}_*}$ contains the morphisms $\text{del}_{\setminus V}: \mathcal{L}^k \rightarrow (1)$, and hence the labelled stick graphs $(1, id)$ and $(1, \tau)$. There are no non-trivial morphisms out of these special graphs, and \mathcal{X} is in the same connected component as $(1, id)$ if and only if $\mathcal{X} = \mathcal{L}^k$ (with the identity labelling) for some $k \in \mathbb{N}$.
- Finally, when $X = 0$, the morphisms $\text{del}_{\setminus V}: \mathcal{W}^m \rightarrow (1)$, and $z: \mathcal{C}_0 \rightarrow (1)$ are not boundary-preserving, and, in particular, do not equip (1) with any labelling of its ports. So, the objects of $0\text{Gr}_{\text{sim}_*}$ are the admissible 0 -graphs, and (1) .

In particular, $\mathcal{W}^m, \mathcal{C}_0$ and (1) are in the same connected component of $0\text{Gr}_{\text{sim}_*}$, and there are no non-trivial morphisms in $0\text{Gr}_{\text{sim}_*}$ with (1) as domain.

In what follows, to simplify notation, we'll write $\mathcal{C}_{0 \setminus V} \stackrel{\text{def}}{=} (1)$ and $\text{del}_{\setminus V} = z: \mathcal{C}_0 \rightarrow (1)$.

Definition 7.36. Let S_* be a pointed graphical species. For each finite set X , let $X\text{Gr}_{\text{sim}_*}(S_*) \stackrel{\text{def}}{=} \text{El}_{S_*}(X\text{Gr}_{\text{sim}_*})$ be the category of similar S_* -structured X -graphs.

Admissible S_* -structured X -graphs $(\mathcal{X}^1, \alpha^1), (\mathcal{X}^2, \alpha^2)$ are called *similar*, written $(\mathcal{X}^1, \alpha^1) \sim (\mathcal{X}^2, \alpha^2)$ (or just $\alpha^1 \sim \alpha^2$), if they are in the same connected component of $X\text{Gr}_{\text{sim}_*}(S_*)$.

Example 7.37. (Compare Section 6.) If $\mathcal{X} \not\cong \mathcal{C}_0$ and $W_\alpha = V(\mathcal{G})$, then $(\mathcal{X}, \alpha) \in X\mathbf{Gr}_{\text{sim}_*}(S_*)$ has the form $\mathcal{L}^k(\epsilon_c)$, or $\mathcal{W}^m(\epsilon_c)$ for some $c \in S_\S$. So, the reduced similar structure $(\mathcal{X}_\alpha^\perp, \alpha^\perp) = (1, c)$ is not admissible.

Since $x: \mathcal{C}_0 \rightarrow (1)$ induces morphisms $\mathbf{CetGr}_*(S_*)((\mathcal{C}_0, o_{\bar{c}}), (1, c))$ and $z \in \mathbf{CetGr}_*(S_*)((\mathcal{C}_0, o_{\bar{c}}), (1, \omega c))$ there are two reduced structures similar to $(\mathcal{C}_0, o_{\bar{c}})$ in $\mathbf{0Gr}_{\text{sim}_*}$.

For all $c \in C$ and all $k, n, l, m \geq 1$,

$$\mathcal{L}^k(\epsilon_c) \sim \mathcal{L}^n(\epsilon_c) \in \mathbf{2Gr}_{\text{sim}_*}(S) \text{ and } \mathcal{W}^l(\epsilon_c) \sim \mathcal{W}^m(\epsilon_c) \sim \mathcal{W}^m(\epsilon_{\omega c}) \in \mathbf{0Gr}_{\text{sim}_*}(S).$$

But also, the similarity maps $\kappa: \mathcal{W} \rightarrow (1) \leftarrow \mathcal{C}_0: z$ in \mathbf{CetGr}_* induce morphisms

$$(7.38) \quad \mathcal{W}(\epsilon_c) \longrightarrow (1, c) \longleftarrow (\mathcal{C}_0, o_{\bar{c}}) \longrightarrow (1, \omega c) \longleftarrow \mathcal{W}(\epsilon_{\omega c}).$$

So, $\mathcal{W}(\epsilon_c) \sim (\mathcal{C}_0, o_{\bar{c}})$ in $\mathbf{0Gr}_{\text{sim}_*}(S_*)$. (This was illustrated in the Introduction, Figure 2.)

In particular, for all $c \in S_\S$, $(1, c)$ and $(1, \omega c)$ are in the same connected component of $\mathbf{0Gr}_{\text{sim}_*}(S_*)$ but are in disjoint components of $\mathbf{2Gr}_{\text{sim}_*}(S_*)$ whenever $c \neq \omega c$.

Lemma 7.39. *For all pointed graphical species S_* and all finite sets X , there is a canonical bijection*

$$(7.40) \quad \text{colim}_{(\mathcal{G}, \rho) \in X\mathbf{Gr}_{\text{sim}_*}} S_*(\mathcal{G}) \cong \pi_0(X\mathbf{Gr}_{\text{sim}_*}(S_*)).$$

Proof. Since $X\mathbf{CGr}_{\text{iso}}(S) \subset X\mathbf{Gr}_{\text{sim}_*}(S_*)$, there is a surjection of connected components $\pi_0(X\mathbf{CGr}_{\text{iso}}(S)) \rightarrow \pi_0(X\mathbf{Gr}_{\text{sim}_*}(S_*))$. So every component of $X\mathbf{Gr}_{\text{sim}_*}(S_*)$ has a representative in $X\mathbf{CGr}_{\text{iso}}(S)$, and the result follows from the definition and equation (5.8). \square

7.5. A distributive law for modular operads. Let S be a graphical species and X a finite set. An element of TDS_X is represented by an X -graph \mathcal{X} , with a decoration $\alpha \in DS(\mathcal{X}) = S^+(\mathcal{X})$. The idea is to construct a natural transformation $\lambda: TD \Rightarrow DT$ so that λS is invariant under morphisms in $X\mathbf{Gr}_{\text{sim}_*}(S_*)$. In other words, λ ignores vertices decorated by units.

Proposition 7.41. *There is a distributive law $\lambda: TD \Rightarrow DT$ such that for all graphical species S and finite sets X , and all $[\mathcal{X}, \alpha]$ and $[\mathcal{X}', \alpha']$ in TDS_X ,*

$$\lambda[\mathcal{X}, \alpha] = \lambda[\mathcal{X}', \alpha'] \text{ in } DTS_X \text{ if and only if } [\mathcal{X}, \alpha] \sim [\mathcal{X}', \alpha'] \in X\mathbf{Gr}_{\text{sim}_*}(S^+).$$

Proof. Since the endofunctor D just adjoins elements, there are canonical inclusions $TS \hookrightarrow DTS$ and $TS \hookrightarrow TDS$. The natural transformation $\lambda: TD \Rightarrow DT$ will restrict to the identity on TS .

For a finite set X , elements of TDS_X are represented by DS -structured X -graphs (\mathcal{X}, α) , whereas elements of DTS_X are either of the form $\epsilon_c^{DTS}, o_{\bar{c}}^{DTS}$ for $c \in S_\S$, or are represented by S -structured X -graphs (\mathcal{X}', α') . Observe also that an object $(\mathcal{X}, \alpha) \in X\mathbf{Gr}_{\text{sim}_*}(S^+)$ is reduced and admissible if and only if $[\mathcal{X}, \alpha] \in TS_X$.

Let $(\mathcal{X}, \alpha) \in \mathbf{CGr}_{\text{iso}}(DS)$. If $X = \mathbf{2}$, and (\mathcal{X}, α) has the form $\mathcal{L}^k(\epsilon_c)$, set

$$\lambda[\mathcal{X}, \alpha] = \epsilon_c^{DS} \in DTS_{\mathbf{2}}.$$

And, if $X = \mathbf{0}$, and $(\mathcal{X}, \alpha) = \mathcal{W}^m(\epsilon_c)$ or $(\mathcal{X}, \alpha) = (\mathcal{C}_0, o_{\bar{c}})$, set

$$\lambda[\mathcal{X}, \alpha] = o_{\bar{c}}^{DTS} \in DTS_{\mathbf{0}}.$$

Otherwise, the component of (\mathcal{X}, α) in $X\mathbf{Gr}_{\text{sim}_*}(S^+)$ has an admissible and reduced (hence terminal) object $(\mathcal{X}_\alpha^\perp, \alpha^\perp)$, so we can set

$$\lambda[\mathcal{X}, \alpha] = [\mathcal{X}_\alpha^\perp, \alpha^\perp] \in TS_X \subset DTS_X.$$

The assignment so defined clearly extends to a natural transformation $\lambda: TD \Rightarrow DT$.

The verification that λ satisfies the four axioms [3, Section 1] for a distributive law is straightforward but tedious, so I prove just one here, that the following diagram of natural transformations commutes:

$$(7.42) \quad \begin{array}{ccccc} TD^2 & \xRightarrow{\lambda D} & DTD & \xRightarrow{D\lambda} & D^2T \\ T(\mu^{\mathbb{D}}) \Downarrow & & & & \Downarrow (\mu^{\mathbb{D}})T \\ TD & \xRightarrow{\lambda} & & & DT \end{array}$$

So, let $[\mathcal{X}, \alpha] \in TD^2S_X$. The result is immediate when $\mathcal{X} = \mathcal{C}_0$. Moreover, all the maps in (7.42) restrict to the identity on $TS \subset TD^2S$.

So we may assume that $[\mathcal{X}, \alpha] \notin TS_X$ and that $\mathcal{X} \not\cong \mathcal{C}_0$. For $j = 1, 2$, define the sets W^j , of vertices decorated by distinguished elements adjoined in the j^{th} application of D .

$$W^j \stackrel{\text{def}}{=} \{v | v \text{ has a minimal neighbourhood } (\mathcal{C}_X, b) \text{ with } D^2S(b)(\alpha) \in \text{im}(\epsilon^{DS})\}.$$

Then $[\mathcal{X}, \mu^{\mathbb{D}}\alpha] = T\mu^{\mathbb{D}}S[\mathcal{X}, \alpha] \in TDS_X$ is described by

$$DS(b)(\mu^{\mathbb{D}}\alpha) = \begin{cases} \epsilon_c^{DS} & \text{if } (\mathcal{C}, b) \text{ is a minimal neighbourhood of } v \in W^1 \amalg W^2, \\ D^2S(b)(\alpha) \in S(\mathcal{C}) & \text{otherwise.} \end{cases}$$

If $W^1 \cup W^2 \neq V$, the diagram gives

$$\begin{array}{ccccc} [\mathcal{X}, \alpha] & \xrightarrow{\lambda D} & [\mathcal{X}_{\setminus W^2}, \alpha_{\setminus W^2}] & \xrightarrow{D\lambda} & [(\mathcal{X}_{\setminus W^2})_{\setminus W^1}, (\alpha_{\setminus W^2})_{\setminus W^1}] \\ & & & & \parallel \\ [\mathcal{X}, \alpha] & \xrightarrow{T(\mu^{\mathbb{D}})} & [\mathcal{X}, \mu^{\mathbb{D}}\alpha] & \xrightarrow{\lambda} & [\mathcal{X}_{\setminus (W^1 \amalg W^2)}, \alpha_{\setminus (W^1 \amalg W^2)}] \in TS_X \end{array}$$

If $W^1 \amalg W^2 = V$, then $T(\mu^{\mathbb{D}})[\mathcal{X}, \alpha]$ has the form $\mathcal{L}^k(\epsilon_c^{DS})$ or $\mathcal{W}^m(\epsilon_c^{DS})$ and both paths map to the corresponding (contracted) unit in $DT S$. \square

So, λ induces a composite monad $\mathbb{D}\mathbb{T}$ on \mathbf{GS} .

Remark 7.43. There is also a distributive law for composing \mathbb{D} and \mathbb{T} in the other direction $DT \Rightarrow TD$. Algebras for the composite monad $\mathbb{T}\mathbb{D}$ are just the cofibred coproducts of algebras for \mathbb{D} and \mathbb{T} . There is no further relationship between the two structures. (See also Example 2.15.)

By [3, Section 3], $\lambda: TD \Rightarrow DT$ induces a lift \mathbb{T}_* of $\mathbb{D}\mathbb{T}$ to \mathbf{GS}_* , such that the EM categories $\mathbf{GS}^{\mathbb{D}\mathbb{T}}$ and $\mathbf{GS}_*^{\mathbb{T}_*}$ are canonically isomorphic. (See also Section 2.2.)

Corollary 7.44. *The monad $\mathbb{T}_* = (T_*, \mu^{\mathbb{T}_*}, \eta^{\mathbb{T}_*})$ on \mathbf{GS}_* is given by*

$$T_*S_{\S} = S_{\S}, \text{ and } T_*S_X = \text{colim}_{(\mathcal{G}, \rho) \in X\mathbf{Gr}_{\text{sim}_*}} S_*(\mathcal{G}).$$

The unit $\eta^{\mathbb{T}_}: 1_{\mathbf{GS}_*} \Rightarrow T_*$ and multiplication $\mu^{\mathbb{T}_*}: T_*^2 \Rightarrow T_*$ are induced by the unit $\eta^{\mathbb{T}}$ and multiplication $\mu^{\mathbb{T}}$ for \mathbb{T} . In other words, if \mathcal{X} is an X -graph and $\alpha \in S(\mathcal{X})$ and $[\mathcal{X}, \alpha]_*$ denotes the class of $[\mathcal{X}, \alpha] \in TS_X$ in T_*S_X , then*

$$\eta^{\mathbb{T}_*}(\phi) = [\eta^{\mathbb{T}}\phi]_* \text{ and } \mu^{\mathbb{T}_*}[\mathcal{X}, \beta] = [\mu^{\mathbb{T}}[\mathcal{X}, \beta]]_*.$$

Proof. Let S be a graphical species with \mathbb{D} -algebra structure $h_{\mathbb{D}}: DS \rightarrow S$ so that $S_* = (S, \epsilon, o)$, with $\epsilon = h_{\mathbb{D}}(\epsilon^+)$ and $o = h_{\mathbb{D}}(o^+)$, is a pointed graphical species.

Observe first that

$$(\mu^{\mathbb{D}}T) \circ (D\lambda) = (\mu^{\mathbb{D}}\mu^{\mathbb{T}}) \circ (D\lambda T) \circ (DT D\eta^{\mathbb{T}}): DTD \Rightarrow DT,$$

so, by [3, Section 3], $T_*(S_*)$ is described by the coequaliser

$$(7.45) \quad \begin{array}{ccccc} DTDS & \xrightarrow{DTh_{\mathbb{D}}} & DTS & \xrightarrow{\pi} & T_*S_* \\ & \searrow D\lambda S & \nearrow \mu^{\mathbb{D}}TS & & \\ & & D^2TS & & \end{array}$$

On $TS \subset DTDS$, both maps in (7.45) restrict to the inclusion $TS \hookrightarrow DTS$. By the application of $h_{\mathbb{D}}$, the upper path, $\pi: DTS \rightarrow T_*S_*$ identifies all occurrences of ϵ and ϵ^{DS} in elements of $DTDS$ and, by the application of $\mu^{\mathbb{D}}$ in the lower path, it identifies all occurrences of ϵ^{DS} and ϵ^{DTDS} . The occurrence of λ means that similar elements of $TDS \hookrightarrow DTDS$ are identified by the lower path. Hence, similar elements of $TS \subset DTS$ are identified by the quotient π and $T_*S_X = \text{colim}_{(\mathcal{G}, \rho) \in X\text{Gr}_{\text{sim}_*}} S_*(\mathcal{G})$ as required.

The unit for T_*S_* is provided by the map $[\mathcal{L}^k(\epsilon)]_*: S_{\S} \rightarrow T_*S_2$ that takes c to the class of all ϵ_c decorated line graphs \mathcal{L}^k , and the contracted unit by $[\mathcal{W}^m(\epsilon)]_* = [\mathcal{C}_0, o]_*: S_{\S} \rightarrow T_*S_0$ that takes c to the class of all ϵ_c decorated wheel graphs, and the isolated vertex decorated by $\zeta(\epsilon_c)$.

Let $[\mathcal{X}, \beta]$ represent an element of T^2S_X . If $\text{del}_{\setminus W} \in X\text{Gr}_{\text{sim}_*}((\mathcal{X}, \beta), (\mathcal{X}_{\setminus W}, \beta_{\setminus W}))$, then $(\mathcal{C}_2, b) \in \text{el}(\mathcal{X})$ is a neighbourhood of $v \in W$ if and only if there is a $k \geq 1$, and a $c \in S_{\S}$, such that $S(b)(\beta) = [\mathcal{L}^k(\epsilon_c)] \in TS_2$. It follows that $\mu^{\mathbb{T}*} = [\mu^{\mathbb{T}}(-)]_*$ is well-defined and provides the multiplication for \mathbb{T}_* .

Clearly $\eta^{\mathbb{T}*} = [\eta^{\mathbb{T}}(-)]_*$, whereby the result follows immediately. \square

So, elements of T_*S_X may be viewed as similarity classes of S_* -structured X -graphs. Two X -labelled S_* -structured graphs (\mathcal{X}, α) and (\mathcal{X}', α') represent the same class $[\mathcal{X}, \alpha] \in TS_X$ precisely when there is a similarity morphism $g \in X\text{Gr}_{\text{sim}_*}(\mathcal{X}, \mathcal{X}')$ such that $S(g)(\alpha') = \alpha$.

In particular, to compute the image of $[\mathcal{X}, \beta] \in T^2S_*$ under the multiplication $\mu^{\mathbb{T}*}(S_*): T^2S_* \rightarrow T_*S_*$, it is sufficient to chose a non-degenerate graph of graphs and quotient by similarity at the end.

7.6. \mathbb{T}_* -algebras are modular operads. At last we are ready to prove the first main theorem – that modular operads are \mathbb{DT} -algebras in GS or, equivalently, \mathbb{T}_* -algebras in GS_* .

Observe first that, if $(S, \diamond, \zeta, \epsilon)$ is a modular operad, then $(S, \epsilon, \zeta\epsilon)$ is a pointed graphical species, and (S, \diamond, ζ) is an unpointed modular operad. Therefore, by Proposition 5.29, S is equipped with a \mathbb{T} -algebra structure $p_{\mathbb{T}} = p_{\mathbb{T}}^{\diamond, \epsilon}: TS \rightarrow S$.

Lemma 7.46. *Let $(S, \diamond, \zeta, \epsilon)$ be a modular operad, and let $p_{\mathbb{T}}: TS \rightarrow S$ denote the corresponding \mathbb{T} -algebra structure on S . Objects in the same connected component of $X\text{Gr}_{\text{sim}_*}(S, \epsilon, \zeta\epsilon)$ have the same image under the map*

$$X\text{CGr}_{\text{iso}}(S) \twoheadrightarrow TS_X \xrightarrow{p_{\mathbb{T}}} S_X.$$

Proof. Since $(S, \diamond, \zeta, \epsilon)$ is a modular operad, $p_{\mathbb{T}}$ satisfies

$$p_{\mathbb{T}}[\mathcal{M}_c(\phi, \epsilon_c)] = \phi \diamond_c \epsilon_c = \phi = p_{\mathbb{T}}(\eta^{\mathbb{T}}\phi) \text{ for all } \phi \in S_{(\underline{c}, c)},$$

and, $p_{\mathbb{T}}[\mathcal{N}_c(\psi)] = \zeta(\psi)$ for all $\psi \in S_{(\underline{c}, c, \omega_c)}$. So, in particular

$$p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}(\epsilon_c)] = \zeta(\epsilon_c).$$

Hence, by the monad algebra axioms, $p_{\mathbb{T}}[\mathcal{W}^m(\epsilon)] = p_{\mathbb{T}}[\mathcal{C}_0, \zeta\epsilon]: S_{\S} \rightarrow S_0$. Otherwise, by Proposition 5.29, $p_{\mathbb{T}}[\mathcal{X}, \alpha] = p_{\mathbb{T}}[\mathcal{X}_0, \alpha_0]$ where, either $(\mathcal{X}_0, \alpha_0) = (\mathcal{X}^{\perp}_{\alpha}, \alpha^{\perp})$ if the latter is admissible, or $(\mathcal{X}_0, \alpha_0) = (\mathcal{C}_2, \epsilon_c)$ if $(\mathcal{X}, \alpha) \sim \mathcal{L}^k(\epsilon)$. \square

It is now straightforward to prove that \mathbb{DT} is the desired modular operad monad on GS .

Theorem 7.47. *The EM category $\text{GS}^{\mathbb{DT}}$ of algebras for \mathbb{DT} is canonically isomorphic to MO .*

Proof. A $\mathbb{D}\mathbb{T}$ -algebra (A, h) induces \mathbb{D} - and \mathbb{T} -structure morphisms

$$h_{\mathbb{D}} \stackrel{\text{def}}{=} h \circ D\eta^{\mathbb{T}} A: DA \rightarrow A, \quad \text{and} \quad h_{\mathbb{T}} \stackrel{\text{def}}{=} h \circ D\eta^{\mathbb{D}} A: TA \rightarrow A$$

by [3, Section 2], where, since $\eta^{\mathbb{D}}$ is just an inclusion, $h_{\mathbb{T}} = h|_{TA}: TA \rightarrow A$ is the restriction to $TA \subset DTA$. By Proposition 5.29, A is equipped with a multiplication $\diamond = h \circ [\mathcal{M}(-, -)]$ and contraction $\zeta = h \circ [\mathcal{N}(-)]$, so that (A, \diamond, ζ) is a non-unital modular operad.

It remains to show that ϵ provides a unit for the multiplication \diamond . By the monad algebra axioms, there are commuting diagrams

$$(7.48) \quad \begin{array}{ccc} A & \xrightarrow{\eta^{\mathbb{D}} \eta^{\mathbb{T}} A} & DTA \\ & \searrow & \downarrow h \\ & & A, \end{array}$$

$$(7.49) \quad \begin{array}{ccc} (DT)^2 A & \xrightarrow{D\lambda T A} & D^2 T^2 A \xrightarrow{\mu^{\mathbb{D}} \mu^{\mathbb{T}} A} DTA \\ \downarrow DTh & & \downarrow h \\ DTA & \xrightarrow{h} & A. \end{array}$$

For all $\underline{c} \in S_{\S}^X$, and all $\phi \in A_{\underline{c}, c}$, the image of the element $[\mathcal{M}_c(\eta^{\mathbb{T}}(\phi), \epsilon_c^{DTA})] \in (TD)^2 A_X$ under the top-right path in (7.49) is just $\phi \in A_X$, since the application of $D\lambda T A$ deletes the unit ϵ_c^{DTA} .

Now $DTh[\mathcal{M}_c(\eta^{\mathbb{T}}(\phi), \epsilon_c^{DTA})] = [\mathcal{M}_c(\phi, \epsilon_c^{DTA})] \in DTA_X$ by construction, and, since (7.49) commutes,

$$\phi \diamond_c \epsilon_c = h[\mathcal{M}_c(\phi, \epsilon_c)] = \phi.$$

Hence, ϵ is a unit for \diamond .

Conversely, a modular operad $(S, \diamond, \zeta, \epsilon)$ induces a pointed graphical species $(S, \epsilon, \zeta \epsilon)$. And, by Proposition 5.29, since (S, \diamond, ζ) is an unpointed modular operad, it is a \mathbb{T} -algebra with structure map $p_{\mathbb{T}}: TS \rightarrow S$ satisfying

$$\diamond = p_{\mathbb{T}} \circ [\mathcal{M}(\cdot, \cdot)] \quad \text{and} \quad \zeta = p_{\mathbb{T}} \circ [\mathcal{N}(\cdot)].$$

By Lemma 7.46, since ϵ is a unit for \diamond , for all $c \in S_{\S}$ and all $m \geq 1$,

$$p_{\mathbb{T}}[\mathcal{N}(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)].$$

So, $o_{\bar{c}} \stackrel{\text{def}}{=} \zeta \epsilon_c = p_{\mathbb{T}}[\mathcal{W}^m(\epsilon_c)]$.

Then (7.48) commutes when $h = p: DTS \rightarrow S$ is defined by the restriction $p|_{TS_X} = p_{\mathbb{T}}: TS \rightarrow S$, together with $p(\epsilon^{DTS}) = \epsilon: S_{\S} \rightarrow S_2$, and $p(o^{DTS}) = \zeta \epsilon: S_{\S} \rightarrow S_0$.

It remains to check that (7.49) commutes for S and $h = p$. This is clear for the adjointed (contracted) units $\epsilon^{(TDTS)^+}$, and $o^{(TDTS)^+}$ for $(DT)^2 S$. So we must check that the restriction

$$(7.50) \quad \begin{array}{ccccc} TDTS & \xrightarrow{\lambda TS} & DT^2 S & \xrightarrow{D\mu^{\mathbb{T}} S} & DTS \\ \downarrow Tp & & & & \downarrow p \\ TS & \xrightarrow{p} & & & S \end{array}$$

of (7.49) commutes.

To this end, let $[\mathcal{X}, \beta] \in TDTS_X$. Exactly one of the following four conditions holds:

- (i) $X = \mathbf{0}$ and $[\mathcal{X}, \beta] = [\mathcal{C}_0, o_{\bar{c}}^{DTS}]$, in which case its image under both paths in (7.50) is $o_{\bar{c}}$,
- (ii) $X = \mathbf{0}$ and $[\mathcal{X}, \beta] = [\mathcal{W}^m(\epsilon_c^{DTS})]$ for some $m \geq 1$, and $c \in S_{\S}$.

Then the application of λTS in the top-right path means that this path takes $[\mathcal{W}^m(\epsilon_c^{DTS})]$ to $o_{\bar{c}} \in S_0$, whereas the bottom left path takes $[\mathcal{W}^m(\epsilon_c^{DTS})]$ first to $[\mathcal{W}^m(\epsilon_c)] \in TS_0$ by applying p inside, and then to

$$p[\mathcal{W}^m(\epsilon_c)] = p[\mathcal{W}(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{N}(\epsilon_c)] = \zeta \epsilon_c = o_{\bar{c}}$$

by an application of Lemma 7.46 and the definition of the contraction o .

- (iii) $X = \mathbf{2}$ and $[\mathcal{X}, \beta] = [\mathcal{L}^k(\epsilon_c^{DTS})]$ for some $k \geq 1$, and $c \in S_\S$.

Then, once again, the application of λTS in the top-right path means that this path takes $[\mathcal{L}^k(\epsilon_c^{DTS})]$ to $\epsilon_c \in S_2$, whereas the bottom left path takes $[\mathcal{L}^k(\epsilon_c^{DTS})]$ first to $[\mathcal{L}^k(\epsilon_c)] \in TS_2$ by applying p inside, and then, by an application of Lemma 7.46,

$$p[\mathcal{L}^k(\epsilon_c)] = p_{\mathbb{T}}[\mathcal{L}^k(\epsilon_c)] = \epsilon_c.$$

- (iv) Otherwise, the reduced structure $(\mathcal{X}_\beta^\perp, \beta^\perp)$ similar to (\mathcal{X}, β) is admissible in $DTS(\mathcal{X})$. Hence $[\mathcal{X}_\beta^\perp, \beta^\perp] \in T^2 S_X$, and therefore $Tp[\mathcal{X}, \beta] = Tp_{\mathbb{T}}[\mathcal{X}^\perp, \beta^\perp]$ by Lemma 7.46. Then (7.50) commutes, since $(S, p_{\mathbb{T}})$ is an algebra for \mathbb{T} .

Therefore, (S, p) naturally admits the structure of a \mathbb{DT} -algebra. It is straightforward to verify that the functors $\mathbf{MO} \hookrightarrow \mathbf{GS}^{\mathbb{DT}}$ so defined are each others' inverses. \square

Remark 7.51. In particular, \mathbb{DT} is the algebraically free monad [26] on the endofunctor T^{ds} in Section 6.

8. A NERVE THEOREM FOR MODULAR OPERADS

The goal of this section is to prove the following nerve theorem for modular operads using the abstract machinery described in Section 2.

Theorem 8.1. *The functor $N: \mathbf{MO} \rightarrow \mathbf{ps}(\Xi)$ is full and faithful. Its essential image consists of precisely those presheaves P on Ξ whose restriction to $\mathbf{ps}(\Xi)$ are graphical species. In other words,*

$$(8.2) \quad P(\mathcal{G}) = \lim_{(C, b) \in \text{el}(\mathcal{G})} P(C) \text{ for all graphs } \mathcal{G}.$$

Remark 8.3. A version of this theorem was stated in [23], and another version was proved, by different methods in [20, Theorem 3.8]. In [35], I proved this theorem by essentially the same methods, but without the use of the distributive law. In all these versions, the statement of the Segal condition is the same.

By Section 7, there is a diagram of functors

$$(8.4) \quad \begin{array}{ccccccc} & & \Xi & \hookrightarrow & \mathbf{MO} & \xrightarrow{N} & \mathbf{ps}(\Xi) \\ & & \uparrow j & \text{b.o.} & \uparrow \text{free}^{\mathbb{T}*} & \downarrow \text{forget}^{\mathbb{T}*} & \downarrow j^* \\ \mathbf{B}^\odot & \xrightarrow{\text{dense}} & \mathbf{CetGr}_* & \xrightarrow{\text{dense}} & \mathbf{GS}_* & \xrightarrow{\text{f.f.}} & \mathbf{ps}(\mathbf{CetGr}_*) \\ \uparrow \text{b.o.} & \nearrow \text{dense} & \uparrow \text{b.o.} & \text{f.f.} & \uparrow \text{free}^{\mathbb{D}} & \downarrow \text{forget}^{\mathbb{D}} & \downarrow \\ \mathbf{B}^\odot & \xrightarrow{\text{dense}} & \mathbf{CetGr} & \xrightarrow{\text{dense}} & \mathbf{GS} & \xrightarrow{\text{f.f.}} & \mathbf{ps}(\mathbf{CetGr}). \end{array}$$

where Ξ is the category obtained in the bo-ff factorisation of $\mathbf{CetGr} \rightarrow \mathbf{GS} \rightarrow \mathbf{MO}$, and also in the bo-ff factorisation of $\mathbf{CetGr}_* \rightarrow \mathbf{GS}_* \rightarrow \mathbf{MO}$.

As discussed in Section 2, Theorem 8.1 follows immediately if the monad \mathbb{DT} on \mathbf{GS} has arities \mathbf{CetGr} (see [5, Definition 1.8]). Unfortunately, this is not the case. The obstruction, unsurprisingly, relates to the contracted units. (See Remark 8.13.)

The remainder of this work is devoted to showing that \mathbb{T}_* has arities \mathbf{CetGr}_* . In this case, the nerve $N: \mathbf{MO} \rightarrow \mathbf{ps}(\Xi)$ is fully faithful. Moreover, because \mathbf{B}^\odot is dense in \mathbf{CetGr}_* , the essential image of N is characterised as the Ξ -presheaves P that satisfy the Segal condition (8.2).

The first step is to study the graphical category Ξ in more detail.

8.1. **The category Ξ .** By construction, $\Xi \subset \mathbf{MO}$ is the full subcategory on the modular operads $\Xi(\mathcal{G})$ free on connected graphs $\mathcal{G} \in \mathbf{CetGr}$.

Example 8.5 (The free modular operad on a graph). Let $\mathcal{H} = (E, H, V, s, t, \tau)$ be a graph. To streamline the notation, let $T\mathcal{H} \stackrel{\text{def}}{=} T\Upsilon\mathcal{H}$ denote the free non-unital modular operad on \mathcal{H} , and $T_*\mathcal{H} \stackrel{\text{def}}{=} T_*\Upsilon_*\mathcal{H}$ the corresponding free unital modular operad on \mathcal{H} .

Of course, $T_*\mathcal{H}(\mathbf{1}) = \{ch_e\}_{e \in E} = \mathbf{CetGr}_*(\mathbf{1}, \mathcal{H})$. Recall that the unit for $\Upsilon_*\mathcal{H}$ is given by $ch_e \mapsto \epsilon_e^{\mathcal{H}} \stackrel{\text{def}}{=} ch_e \circ u \in \mathbf{CetGr}_*(\mathcal{C}_2, \mathcal{H})$, and the contracted unit for $\Upsilon_*\mathcal{H}$ is given by $ch_e \mapsto o_e^{\mathcal{H}} \stackrel{\text{def}}{=} ch_e \circ z \in \mathbf{CetGr}_*(\mathcal{C}_0, \mathcal{H})$.

So, $T_*\mathcal{H}$ has units

$$ch_e \mapsto \epsilon_e^{T_*\mathcal{H}} \stackrel{\text{def}}{=} [\eta^{\mathbb{T}} \epsilon_e^{\mathcal{H}}]_* = [ch_e \circ u^k], \quad k \geq 1.$$

and contracted units

$$ch_e \mapsto o_e^{T_*\mathcal{H}} \stackrel{\text{def}}{=} [\eta^{\mathbb{T}} o_e^{\mathcal{H}}]_* = [ch_e \circ z] = [ch_e \circ \kappa^m], \quad m \geq 1.$$

By Corollary 7.44, for all finite sets X , elements of $T_*\mathcal{H}_X$ are represented by pairs (\mathcal{X}, f) where $\mathcal{X} = (\mathcal{G}, \rho)$ is an admissible X -graph and $f \in \mathbf{CetGr}_*(\mathcal{G}, \mathcal{H})$, and pairs (\mathcal{X}^1, f^1) and (\mathcal{X}^1, f^2) represent the same element $[\mathcal{X}, f]_* \in T_*\mathcal{H}_X$ if and only if there is a commuting diagram

$$(8.6) \quad \begin{array}{ccccc} \mathcal{X}^1 & \xrightarrow{g^1} & \mathcal{X}^\perp & \xleftarrow{g^2} & \mathcal{X}^2 \\ & \searrow f^1 & \downarrow f^\perp & \swarrow f^2 & \\ & & \mathcal{H} & & \end{array}$$

in \mathbf{CetGr}_* such that, for $j = 1, 2$, g^j is a morphism in $X\mathbf{Gr}_{\text{sim}*}$, and \mathcal{X}^\perp is reduced, so $f^\perp: \mathcal{X}^\perp \rightarrow \mathcal{H}$ is an (unpointed) étale morphism in \mathbf{CetGr} .

Outside the (contracted) units, \mathcal{X}^\perp is admissible. Otherwise f^\perp is of the form $ch_e \in \mathbf{CetGr}(\mathbf{1}, \mathcal{H})$.

Observe, in particular, that, for all $e \in E$, the following special case of (8.6) commutes in \mathbf{CetGr}_* :

$$(8.7) \quad \begin{array}{ccccc} \mathcal{C}_0 & \xrightarrow{z} & (\mathbf{1}) & \xleftarrow{\kappa} & \mathcal{W}^m \\ & \searrow ch_e \circ z & \downarrow ch_e & \swarrow ch_e \circ \kappa & \\ & & \mathcal{H} & & \end{array}$$

This will be essential in the proof of Theorem 8.1.

By definition, Ξ is the restriction to \mathbf{CetGr}_* of the Kleisli category of \mathbb{T}_* . So, for all pairs $(\mathcal{G}, \mathcal{H})$ of graphs

$$\Xi(\mathcal{G}, \mathcal{H}) = \mathbf{GS}_*(\mathcal{G}, T_*\mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G}).$$

In particular, for $\mathcal{G} \cong \mathcal{C}_X$ or $\mathcal{G} \cong (\mathbf{1})$, $\Xi(\mathcal{G}, \mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G})$ has been described in Example 8.5.

For the general case, it follows from Example 8.5 that, $\Xi(\mathcal{G}, \mathcal{H}) \cong T_*\mathcal{H}(\mathcal{G})$ is obtained as a quotient of $T\mathcal{H}(\mathcal{G})$. So, a morphism $\gamma \in \Xi(\mathcal{G}, \mathcal{H})$ is represented by a non-degenerate \mathcal{G} -shaped graph of graphs $\mathbf{\Gamma}$ with colimit $\mathbf{\Gamma}(\mathcal{G})$, together with a morphism $f \in \mathbf{CetGr}_*(\mathbf{\Gamma}(\mathcal{G}), \mathcal{H})$.

Since every graph \mathcal{G} is trivially the colimit of the identity \mathcal{G} -shaped graph of graphs $(\mathcal{C}, b) \mapsto \mathcal{C}$, the assignment $f \mapsto (id, f) \in \Xi(\mathcal{G}, \mathcal{H})$ induces an inclusion $\mathbf{CetGr}_* \hookrightarrow \Xi$. So, there is weak factorisation system on Ξ where the left class consists of (boundary preserving) morphisms $\mathcal{G} \rightarrow \mathbf{\Gamma}(\mathcal{G})$ induced by non-degenerate graphs of graphs, and the right class is given by morphisms in the image of \mathbf{CetGr}_* .

Recall (Corollary 7.24) that the pair $(\mathbf{Gr}_{\text{sim}*}, \mathbf{CetGr})$ defines a factorisation system on \mathbf{CetGr}_* . Hence, the composite structure of the monad \mathbb{DT} induces a ternary factorisation system on Ξ .

Lemma 8.8. *Let $\mathcal{G} \not\cong \mathcal{C}_0$ and \mathcal{H} be graphs and, for $i = 1, 2$, let Γ^i be a non-degenerate \mathcal{G} -shaped graph of graphs with colimit $\Gamma^i(\mathcal{G})$, and $f^i \in \text{CetGr}_*(\Gamma^i(\mathcal{G}), \mathcal{H})$. Then $(\Gamma^1, f^1), (\Gamma^2, f^2)$ represent the same element α of $\Xi(\mathcal{G}, \mathcal{H})$ if and only if there is a representative (Γ, f) of α , and a commuting diagram in CetGr_* of the following form, where the morphisms in the top row are vertex deletion morphisms:*

$$(8.9) \quad \begin{array}{ccccc} \Gamma^1(\mathcal{G}) & \xrightarrow{\quad} & \Gamma(\mathcal{G}) & \xleftarrow{\quad} & \Gamma^2(\mathcal{G}) \\ & \searrow f^1 & \downarrow f & \swarrow f^2 & \\ & & \mathcal{H} & & \end{array}$$

Proof. By definition, if (Γ^1, f^1) and (Γ^2, f^2) represent the same element, then, for all $(\mathcal{C}_{X_b}, b) \in \text{el}(\mathcal{G})$, $(\Gamma^1(b), f^1 \circ b)$ and $(\Gamma^2(b), f^2 \circ b)$ are similar in $X_b \text{Gr}_{\text{sim}_*}(\Upsilon_* \mathcal{H}) \subset \text{CetGr}_*/\mathcal{H}$. In which case, since $\mathcal{G} \not\cong \mathcal{C}_0$, by Example 8.5, there is an admissible graph $\Gamma(b)$ and a morphism $f_b \in \text{CetGr}_*(\Gamma(b), \mathcal{H})$, such that the following diagram – in which the horizontal morphisms are vertex deletion morphism – commutes in CetGr_* :

$$\begin{array}{ccccc} \Gamma^1(b) & \xrightarrow{\quad} & \Gamma(b) & \xleftarrow{\quad} & \Gamma^2(b) \\ & \searrow f^1 \circ b & \downarrow f_b & \swarrow f^2 \circ b & \\ & & \mathcal{H} & & \end{array}$$

So, the \mathcal{G} -shaped graph of $T_*\mathcal{H}$ -structured graphs $\Gamma_{T_*\mathcal{H}}: \text{el}(\mathcal{G}) \rightarrow \text{CetGr}(T_*\mathcal{H}), b \mapsto (\Gamma(b), f_b)$ so obtained is non-degenerate and hence admits a colimit $(\Gamma(\mathcal{G}), f)$ in $\text{CetGr}(T_*\mathcal{H})$ such that (8.9) commutes.

The converse follows immediately from the definitions. \square

8.2. Factorisation categories. The graphical category Ξ is the full subcategory on graphs of the Kleisli category $\text{GS}_{*\mathbb{T}_*}$ of T_* with $\text{GS}_{*\mathbb{T}_*}(S_*, S'_*) = \text{GS}_*(S_*, T_*S'_*)$ for all $S_*, S'_* \in \text{GS}_*$.

Since elements of T_*S_X correspond to similarity classes of S_* -structured X -graphs (\mathcal{X}, α) , each element $\beta \in \text{GS}_*(\Upsilon_*\mathcal{G}, T_*S_*)$ is represented by a non-degenerate \mathcal{X} -shaped graph of S_* -structured graphs Γ_S . The colimit of Γ_S describes an S_* structured X -graph $(\Gamma(\mathcal{X}), \alpha)$, where $\Gamma(\mathcal{X})$ is the colimit of the underlying \mathcal{X} -shaped graph of graphs $\Gamma: \text{el}(\mathcal{X}) \rightarrow \text{CetGr}(S_*) \rightarrow \text{CetGr}$.

By [5, Proposition 2.5], the monad \mathbb{T}_* has arities CetGr_* if certain categories associated to factorisations of morphisms in $\text{GS}_{*\mathbb{T}_*}$ ([5, Section 2.4]) are connected.

Let S_* be a pointed graphical species, \mathcal{G} be a graph, and let $\beta \in \text{GS}_*(\Upsilon_*\mathcal{G}, T_*S_*)$.

Definition 8.10. *The factorisation category $\text{fact}_*(\beta)$ of β is the category whose objects are pairs (Γ, α) , where Γ is a non-degenerate \mathcal{G} -shaped graph of graphs with colimit $\Gamma(\mathcal{G})$ and $\alpha \in S(\Gamma(\mathcal{G}))$ is such that β is equal to the composition:*

$$\Upsilon_*\mathcal{G} \xrightarrow{\quad \Gamma \quad} \Upsilon_*\Gamma(\mathcal{G}) \xrightarrow{\quad \alpha \quad} S_*$$

in GS_ . Morphisms in $\text{fact}_*(\beta)((\Gamma^1, \alpha^1), (\Gamma^2, \alpha^2))$ are commuting diagrams in $\text{GS}_{*\mathbb{T}_*}$*

$$(8.11) \quad \begin{array}{ccccc} & & \Gamma^1(\mathcal{G}) & & \\ & \nearrow & \downarrow g & \searrow \alpha^1 & \\ \mathcal{G} & & & & S_* \\ & \searrow & \downarrow g & \swarrow \alpha^2 & \\ & & \Gamma^2(\mathcal{G}) & & \end{array}$$

such that g is a morphism in $\text{CetGr}_ \hookrightarrow \text{GS}_{*\mathbb{T}_*}$.*

Lemma 8.12. *For all graphs \mathcal{G} and all $\beta \in \mathbf{GS}_*(\mathcal{G}, T_*S)$, the category $\mathbf{fact}_*(\beta)$ is connected.*

Proof. This follows directly from the discussion above, and in particular Example 8.5.

Let S_* be a pointed graphical species. For X a finite set, S_* -structured X -graphs $(\mathcal{X}^1, \alpha^1), (\mathcal{X}^2, \alpha^2)$ represent the same element of T_*S_X if and only if they are similar in $X\mathbf{Gr}_{\mathbf{sim}_*}(S_*) \cong \mathbf{GS}_*(\mathcal{C}_X, T_*S_*)$. So the lemma holds for elementary graphs.

Now let $\mathcal{G} \not\cong \mathcal{C}_0$ be any connected graph. Elements of $\mathbf{GS}_*(\Upsilon_*\mathcal{G}, T_*S) \cong T_*S(\mathcal{G})$ are represented by non-degenerate \mathcal{G} -shaped graphs of S_* -structured graphs. Since there is no object of the form (\mathcal{C}_0, b) in $\mathbf{el}(\mathcal{G})$, two such S -structured graphs of graphs, $\mathbf{\Gamma}_{S_*}^1, \mathbf{\Gamma}_{S_*}^2$ represent the same element of $T_*S(\mathcal{G})$ if and only if

$$\mathbf{\Gamma}_{S_*}^1(\mathcal{C}_X, b) \sim \mathbf{\Gamma}_{S_*}^2(\mathcal{C}_X, b) \in X\mathbf{Gr}_{\mathbf{sim}_*}(S)$$

for all $(\mathcal{C}_X, b) \in \mathbf{el}(\mathcal{G})$ and hence their colimits are also similar. In other words, $\mathbf{fact}_*(\beta)$ is connected. \square

Theorem 8.1 now follows from [5, Sections 1 & 2].

Proof of Theorem 8.1. The category \mathbf{CetGr}_* is dense in \mathbf{GS}_* . By [5, Proposition 2.5], the statement of Lemma 8.12 is equivalent to the statement that the monad \mathbb{T}_* has arities \mathbf{CetGr}_* . Therefore, the monad \mathbb{T}_* on \mathbf{GS}_* has arities \mathbf{CetGr}_* .

Hence the induced nerve functor $N: \mathbf{MO} \rightarrow \mathbf{ps}(\Xi)$ is fully faithful by [5, Propositions 1.5 & 1.9]. Moreover, by [5, Theorem 1.10] its essential image is the subcategory of those presheaves on Ξ whose restriction to \mathbf{CetGr}_* are in the image of the fully faithful embedding $\mathbf{GS}_* \hookrightarrow \mathbf{ps}(\mathbf{CetGr}_*)$.

So a presheaf P on Ξ is in the essential image of N if and only if

$$P(\mathcal{G}) = \lim_{(\mathcal{C}, b) \in \mathbf{el}_*(\mathcal{G})} P(\mathcal{C}),$$

and, by finality of $\mathbf{el}(\mathcal{G}) \subset \mathbf{el}_*(\mathcal{G})$, this is the case precisely if

$$P(\mathcal{G}) = \lim_{(\mathcal{C}, b) \in \mathbf{el}(\mathcal{G})} P(\mathcal{C}).$$

\square

Remark 8.13. Using the method of [5, Section 2], we can construct the corresponding *unpointed* factorisation categories for the monad \mathbb{DT} on \mathbf{GS} .

For any graphical species S and graph \mathcal{G} , $\mathbf{GS}_{\mathbb{DT}}(\Upsilon\mathcal{G}, S) \cong \mathbf{GS}_*(\Upsilon_*\mathcal{G}, T_*S^+) \cong S^+(\mathcal{G})$ canonically.

So, a morphism $\beta: \Upsilon\mathcal{G} \rightarrow S$ in the Kleisli category $\mathbf{GS}_{\mathbb{DT}}$ is represented by a \mathcal{G} -shaped graph of graphs $\mathbf{\Gamma}$ with colimit $\mathbf{\Gamma}(\mathcal{G})$, and a DS -structure $\alpha \in DS(\mathbf{\Gamma}(\mathcal{G}))$.

Such pairs $(\mathbf{\Gamma}, \alpha)$ are the objects of the unpointed factorisation category $\mathbf{fact}(\beta)$. Morphisms in $\mathbf{fact}(\beta)((\mathbf{\Gamma}, \alpha), (\mathbf{\Gamma}', \alpha'))$ are morphisms in $\mathbf{CetGr}(\mathbf{\Gamma}(\mathcal{G}), \mathbf{\Gamma}'(\mathcal{G}'))$ making the diagram (8.11) commute.

By [5, Proposition 2.5], \mathbb{DT} has arities \mathbf{CetGr} if and only if $\mathbf{fact}(\beta)$ is connected for all S, \mathcal{G} , and β .

To see that this is not the case, let $S = \Upsilon(i)$, let $\mathcal{G} = (i)$ and let $\beta = \sigma^+ = z: \mathcal{C}_0 \rightarrow (i)$. Then $TS \cong S$ and the diagrams $\mathcal{C}_0 \rightarrow \mathcal{C}_0 \xrightarrow{z} (i)$, and $\mathcal{C}_0 \rightarrow \mathcal{W} \xrightarrow{\kappa} (i)$ describe objects in $\mathbf{fact}(\beta)$. Since there are no non-trivial morphisms in \mathbf{CetGr} with domain or codomain \mathcal{C}_0 , these objects are in disjoint components of $\mathbf{fact}(\beta)$. Therefore, \mathbb{DT} does not have arities \mathbf{CetGr} .

8.3. Weak modular operads. In [20, 21], Hackney, Robertson and Yau have proved a version of Theorem 8.1 in terms of a bijective-on-objects subcategory U of Ξ that was constructed precisely so as to have a generalised Reedy structure. The category U does not contain those morphisms in $\mathbf{CetGr}_* \hookrightarrow \Xi$ that factor through $z: \mathcal{C}_0 \rightarrow (i)$ or $\kappa^m: \rightarrow (i)$, $m \geq 1$, nor does it contain any morphisms of \mathbf{CetGr} that

are not monomorphism. So $U \hookrightarrow \mathbf{MO}$ is not fully faithful. However, by [21, Theorem 3.6], it is dense and hence induces a fully faithful nerve.

In [20, Theorem 3.8], they have shown that the simplicial presheaf category $\mathbf{ps}_{\mathbf{sSet}}(U)$ admits a cofibrantly generated model structure, obtained by localising the Reedy model structure at the *Segal maps*

$$\lim_{(\mathcal{C}, b) \in \mathbf{el}(\mathcal{G})} P(\mathcal{C}) \longrightarrow P(\mathcal{G}),$$

and that the fibrant objects for this model structure are those simplicial presheaves on U that satisfy the *weak Segal condition*

$$(8.14) \quad P(\mathcal{G}) \simeq \lim_{(\mathcal{C}, b) \in \mathbf{el}(\mathcal{G})} P(\mathcal{C}) \quad \text{for all graphs } \mathcal{G} \in U.$$

In [9], Caviglia and Horel describe a general class of rigidification results whereby, given a dense inclusion $\mathbf{D} \hookrightarrow \mathbf{C}$ of categories satisfying certain conditions, an equivalence is established between \mathbf{sSet} -valued presheaves on \mathbf{D} that satisfy a weak Segal condition, and \mathbf{C} objects internal to \mathbf{sSet} that are Segal on the nose. As an application, they apply their result to a certain class of monads with arities. Their result can be applied directly to obtain the following corollary to Theorem 8.1.

Corollary 8.15. *There is a model category structure on the category $\mathbf{ps}_{\mathbf{sSet}}(\Xi)$ of simplicial presheaves $P: \Xi^{\text{op}} \rightarrow \mathbf{sSet}$ whose fibrant objects are presheaves P satisfying the weak Segal condition*

$$(8.16) \quad P(\mathcal{G}) \simeq \lim_{(\mathcal{C}, b) \in \mathbf{el}(\mathcal{G})} P(\mathcal{C}) \quad \text{for all graphs } \mathcal{G} \in \mathbf{CetGr}.$$

Proof. The monad \mathbb{T}_* has arities \mathbf{CetGr}_* and $\mathbf{B}^\circ/\mathcal{G}$ is small and connected for all connected graphs \mathcal{G} . Therefore the assumptions of [9, Assumptions 7.9] are satisfied. By [9, Section 7.5], \mathbf{MO} is equivalent to the category of models in \mathbf{Set} of the limit sketch $L = (\mathbf{CetGr}_*, \{(\mathcal{G}/\mathbf{B}^{\circ\text{op}})_{\mathcal{G} \in \mathbf{CetGr}}\})$ and there is a Segal model structure on the category of \mathbf{sSet} valued models for L .

By [9, Proposition 7.1], this can be transferred along a Quillen equivalence to a model structure on $\mathbf{ps}_{\mathbf{sSet}}(\Xi)$, whose fibrant objects are those presheaves that satisfy the weak Segal condition. \square

In current work with M. Robertson, we are comparing the existing models for weak modular operads. We expect that there is a direct Quillen equivalence between the model structure on $\mathbf{ps}_{\mathbf{sSet}}(\Xi)$ of Corollary 8.15 and the model structure on $\mathbf{ps}_{\mathbf{sSet}}(U)$ of [20].

Remark 8.17. The graphical category \overline{Gr} whose morphisms are described in [23, Section 6] is the bijective-on-objects subcategory of Ξ that does not contain the morphisms $z: \mathcal{C}_0 \rightarrow (i)$ or any morphisms of the form $ch_e \circ \kappa^m \in \mathbf{CetGr}_*(\mathcal{W}^m, \mathcal{G})$. Therefore, \overline{Gr} does not embed fully faithfully in \mathbf{MO} .

There are inclusions $U \subset \overline{Gr} \subset \Xi$. So, since Ξ and U are both dense in \mathbf{MO} , \overline{Gr} is also dense in \mathbf{MO} , and yields a fully faithful nerve functor whose essential image satisfies the same Segal condition (8.2); whence we may deduce the main statement of [23]. (See also [21, Theorem 3.6 & Section 4] for details.)

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