Unpacking the combinatorics of modular operads

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Operads Pop-Up

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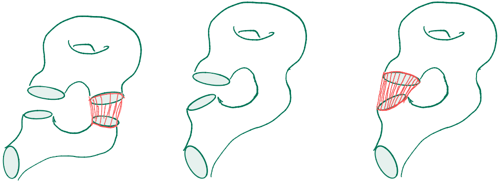
Outline

Modular operads

We develop a 'higher genus' analogue of operads ...in which graphs replace trees in the definition.

Abstract, Getzler-Kapranov 98



Getzler, E. and Kapranov, M. M.
Modular operads
Compositio Mathematica, 110(1):65–126, 1998.

Notation

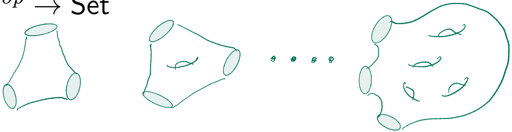
$\mathbb{P}$: groupoid of finite sets and bijections

$$\mathbf{n} = \{1, \ldots, n\}, \qquad \mathbf{0} = \emptyset.$$

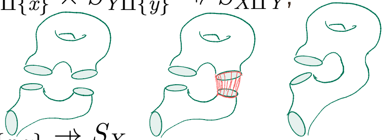
Definition 1

A *modular operad* is a

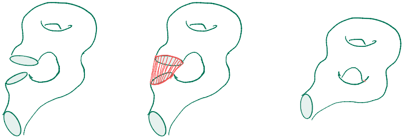
1. Functor $S : \mathbb{P}^{op} \to \mathsf{Set}$



2. together with a multiplication $\diamond : S_{X \amalg \{x\}} \times S_{Y \amalg \{y\}} \nrightarrow S_{X \amalg Y}$,



3. and a contraction operation $\zeta : S_{X \amalg \{x, y\}} \nrightarrow S_X$.



Modular operads

In this talk, definition modular operads will correspond to compact symmetric multicategories introduced by Joyal and Kock, 2011.

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Joyal, A. and Kock, J.,
Feynman Graphs, and Nerve Theorem for Compact Symmetric Multicategories (Extended Abstract)
Electronic Note in Theoretical Computer Science, 270(2):105–113, 2011.

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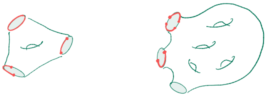
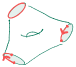
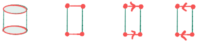
- coloured



- involutive colour set



- with multiplicative unit



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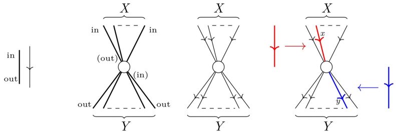
Graphical species (Joyal-Kock, 2011)

$\mathbb{P}^{\circlearrowleft}$	$\mathsf{GS} \overset{\text{def}}{=} \mathsf{PSh}(\mathbb{P}^{\circlearrowleft})$
	A graphical species S is described by:
$\mathbb{P}$ - groupoid of finite sets and bijections is full subcategory.	$\mathbb{P}$-presheaf $(S_X)_X$, a symmetric sequence or combinatorial species,
plus a distinguished object $\S$ with $\mathbb{P}^{\circlearrowleft}(\S, \S) = \{1, \tau\}, \quad \tau^2 = 1$	together with a pair $(\mathfrak{C}, \omega)$ of a set $\mathfrak{C} = S_{\S}$ and involution $\omega = S(\tau)$.
For each X, and $x \in X$, morphisms $ch_x, \; ch_x \circ \tau : \S \longrightarrow X.$	for all X, for all $x \in X$, a map $S(ch_x) : S_X \to \mathfrak{C}$.

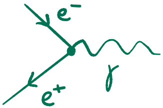
The boundary $\partial\phi$ of $\phi \in S_X$ is $(S(ch_x))_{x \in X}(\phi) \in \mathfrak{C}^X$.

Graphical species - examples

1. terminal species: $\S \mapsto \{*\}$, $X \mapsto \{*\}$ for all X.
2. directed species:
   Di is terminal species on $(\mathfrak{Di}, \sigma_{\mathfrak{Di}})$: $\mathfrak{Di} = \{\mathrm{in}, \mathrm{out}\}$, $\sigma_{\mathfrak{Di}} \neq 1$.



3. Feynman diagrams (particle interactions):

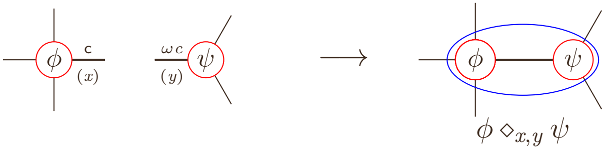


Multiplication.

Glue two elements along dual colours in boundaries:
Partial map

$$\diamond^{X,Y}_{x,y} : S_{X\amalg\{x\}} \times S_{Y\amalg\{y\}} \nrightarrow S_{X\amalg Y}.$$

commutative, equivariant with respect to $\mathbb{P}$ action

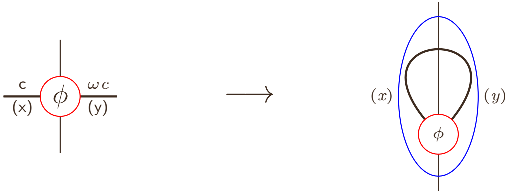


Contraction.

Self-gluing of one element along involutive pair of colours in its boundary:
Partial operation

$$\xi^X_{x,y} = \xi^X_{y,x} : S_{X \amalg \{x,y\}} \nrightarrow S_X,$$

equivariant with respect to $\mathbb{P}$ action.



$\zeta_{x,y}(\phi)$

Unit for $\diamond$.

Unit: injection $\epsilon : \mathfrak{C} = S_{\S} \rightarrowtail S_{\mathbf{2}}$:

$$\phi \diamond \epsilon(c) = \phi = \epsilon(c) \diamond \phi \quad \text{wherever defined,}$$
$$\epsilon \circ \omega = S(\sigma) \circ \epsilon, \text{ where } \sigma \in Aut(\mathbf{2}), \sigma \neq id$$

So $\partial(\epsilon(c)) = (c, \omega c)$.

A $(\mathfrak{C}, \omega)$-coloured modular operad $(S, \diamond, \zeta, \epsilon)$ is equipped with a contracted unit map

$$o : \mathfrak{C} \longrightarrow S_{\mathbf{0}}, \quad c \longmapsto \zeta\epsilon(c).$$

For all $c \in \mathfrak{C}$

$$o(c) = o(\omega c).$$

Unit for $\diamond$.

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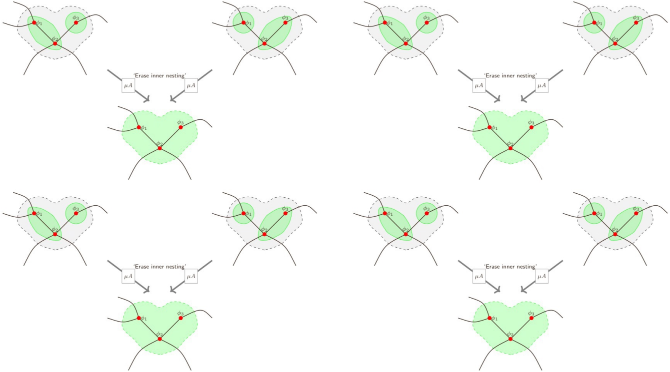
Category MO of modular operads

Objects: $(S, \diamond, \zeta, \epsilon)$ with 4 axioms that generalise associativity

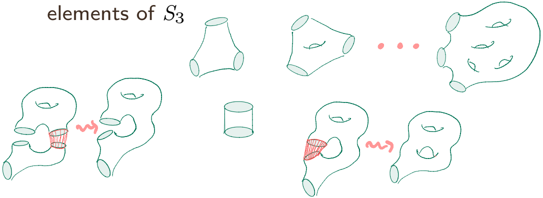


Morphisms in GS that preserve $(\diamond, \zeta, \epsilon)$.

Examples

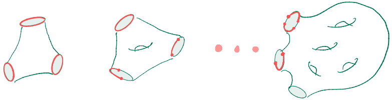
Oriented surfaces with closed boundary

elements of S_3



Oriented surfaces with open - closed boundary

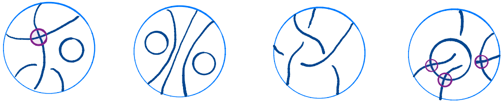
elements of OC_3



units

Undirected virtual tangles

elements of T_{2s}, $s = 3$



Compact closed categories

e.g. cobordism categories

Wheeled properads (Directed modular operads)

e.g. directed virtual tangles

Main theorems

Theorem (Joyal - Kock 2011, R. 2018/20, Hackney-Robertson-Yau 2020)

There is a category GS *of coloured collections – called graphical species – and a monad* $\mathbb{O}$ *on* GS *whose Eilenberg-Moore category of algebras* $\mathsf{GS}^{\mathbb{O}}$ *is canonically isomorphic to the category* MO *of modular operads.*

Theorem (Joyal - Kock 2011, R. 2018, Hackney-Robertson-Yau 2020)

There is a full, dense subcategory Ξ *of* MO *whose objects are graphs.*
The essential image of the induced fully faithful nerve $N : \mathsf{MO} \to \mathsf{PSh}(\Xi)$ *is characterised by Segal presheaves.*

A little context

- Stated by Joyal and Kock (2011), who consructed the category GS and an endofunctor on GS whose algebras are modular operads.
  However, this functor does not admit a monadic multiplication.
- Proof R. (2018).
- Hackney, Robertson and Yau (2020) have recently proved versions of these theorems by different methods, with explicit goal of defining ∞-modular operads.

The point of this talk is not these results, but to use their proof to understand more about the combinatorics.

The plan

Theorem (Joyal - Kock 2011, R. 2018/20, Hackney-Robertson-Yau 2020)

There is a category GS *of coloured collections – called graphical species – and a monad* $\mathbb{O}$ *on* GS *whose Eilenberg-Moore category of algebras* $\mathsf{GS}^{\mathbb{O}}$ *is canonically isomorphic to the category* MO *of modular operads.*

$$
\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}^{\mathbb{O}}"] & \mathsf{PSh}(\Xi) \arrow[d, "j^*"] \\
\mathbb{P}^{\circlearrowleft} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "\text{b.o.}"] & \mathsf{GS} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}^{\mathbb{O}}"] & \mathsf{PSh}(\mathsf{Gr})
\end{tikzcd}
$$

The plan

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\end{tikzcd}$$

Theorem (Joyal - Kock 2011, R. 2018, Hackney-Robertson-Yau 2020)

There is a full, dense subcategory Ξ of MO whose objects are graphs.
The essential image of the induced fully faithful nerve $N : \mathsf{MO} \to \mathsf{PSh}(\Xi)$ is characterised by Segal presheaves:
For all graphs $\mathcal{G}$

$$P(\mathcal{G}) \cong \lim_{(C,b)\in \mathbb{P}^{\circlearrowleft} \downarrow \mathcal{G}} P(C).$$

Abstract nerve theory

when is this fully faithful?

$$
\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}^{\mathbb{O}}"] & \mathsf{PSh}(\Xi) \arrow[d, "j^*"] \\
\mathbb{P}^{\circlearrowleft} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr} \arrow[u, "\text{b.o.}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}^{\mathbb{O}}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{PSh}(\mathsf{Gr})
\end{tikzcd}
$$

Weber, 2007:

If $\mathbb{O}$ has arities Gr, then N is fully faithful and it's essential image is characterised by Segal presheaves:

For all graphs $\mathcal{G}$

$$P(\mathcal{G}) \cong \lim_{(C,b)\in \mathbb{P}^{\circlearrowleft}\downarrow \mathcal{G}} P(C).$$

Example 1: Classical nerve theorem for categories

Segal condition for categorical nerve theorem:

$P : \Delta^{op} \to$ Set is the nerve of a category if and only if for all $n \geq 2$,

$$P_n \cong \underbrace{P_1 \times_{P_0} \cdots \times_{P_0} P_1}_{n \text{ times}}.$$

Weber picture:

$$\begin{tikzcd}
 & \Delta \arrow[r, hook] & \mathsf{Cat} \arrow[r, "N"] \arrow[d, shift left, "U^{Cat}"] & \mathsf{sSet} \arrow[d, "j^*"] \\
\mathcal{E} \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] & \Delta_0 \arrow[r, hook, "\text{dense}", "\text{f.f.}"'] \arrow[u, "j", "\text{b.o.}"'] & \mathsf{PSh}(\mathcal{E}) \arrow[r, hook] \arrow[u, shift left, "F^{Cat}"] & \mathsf{PSh}(\Delta_0).
\end{tikzcd}$$

Example 2: Dendroidal nerve theorem for operads

Σ^*: Objects: $n \in \mathbb{N}$, distinguished edge $\downarrow$
$\Sigma^*(\downarrow, n) \cong \{0, 1, \ldots, n\}$.

$$\begin{tikzcd}
 & \Omega \arrow[r, hook] & \mathsf{Op} \arrow[r, "N"] \arrow[d, shift left] & \mathsf{PSh}(\Omega) \arrow[d] \\
\Sigma^* \arrow[r, hook] & \Omega_0 \arrow[r, hook] \arrow[u, "j", "\text{b.o.}"'] & \mathsf{PSh}(\Sigma^*) \arrow[r, hook] \arrow[u, shift left] & \mathsf{PSh}(\Omega_0)
\end{tikzcd}$$

$P : \Omega^{op} \to \mathsf{Set}$ is the nerve of an operad if and only if,

$$P(T) \cong \mathsf{lim}_{(t,f) \in \Sigma^* \downarrow T} P(j(t)).$$

The key results: Distributive law

Theorem (R. 2020)

There are monads $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ *and* $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ *on* GS *and a distributive law* $\lambda : TD \Rightarrow DT$ *such that* $\mathbb{O} = \mathbb{DT}$ *on* GS.

The key results: Distributive law

Theorem (R. 2020)

There are monads $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ and $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ on GS *and a distributive law $\lambda : TD \Rightarrow DT$ such that $\mathbb{O} = \mathbb{D}\mathbb{T}$ on* GS.

Let GS_* be the category of $\mathbb{D}$-algebras.

$$
\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}"] & \mathsf{PSh}(\Xi) \arrow[d, "j^*"] \\
\mathbb{P}_*^{\circlearrowleft} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr}_* \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS}_* \arrow[u, shift left, "\text{free}"] \arrow[r, hook, "\text{f.f.}"'] \arrow[d, shift left, "\text{forget}"] & \mathsf{PSh}(\mathsf{Gr}_*) \arrow[d] \\
\mathbb{P}^{\circlearrowleft} \arrow[u, "\text{b.o.}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr} \arrow[u, "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{PSh}(\mathsf{Gr})
\end{tikzcd}
$$

The key results: Distributive law

$$
\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}"] & \mathsf{PSh}(\Xi) \arrow[d, "j^*"] \\
\mathbb{P}^{\circlearrowleft}_{*} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr}_* \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "j", "\text{b.o.}"'] & \mathsf{GS}_* \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{PSh}(\mathsf{Gr}_*) \arrow[d] \\
\mathbb{P}^{\circlearrowleft} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "\text{b.o.}"] & \mathsf{Gr} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "\text{b.o.}"'] & \mathsf{GS} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] & \mathsf{PSh}(\mathsf{Gr})
\end{tikzcd}
$$

The key results: Lift has arities

$$
\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}"] & \mathsf{PSh}(\Xi) \arrow[d, "j^*"] \\
\mathbb{P}_*^{\circlearrowleft} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr}_* \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "j", "\text{b.o.}"'] & \mathsf{GS}_* \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{PSh}(\mathsf{Gr}_*) \arrow[d] \\
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\end{tikzcd}
$$

Lemma (R. 2018)

There is a full, dense subcategory Gr_* *of* GS_* *such that the induced monad* $\mathbb{T}_*$ *on* GS_**, has arities* Gr_*.

Why is this interesting?

(i) Proof of Joyal and Kock's theorem using the originally intended methods: Weber nerve machinery and, in particular Berger-Mellies-Weber, 2012.

(ii) The proof exhibits the combinatorics of these structures explicitly. In particular it reveals where we need to take extra care.

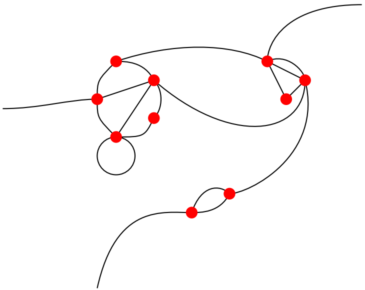
(iii) Abstract methods place structures in a wider context. Results from elsewhere may be generalised to modular operads.

(iv) Proof method suggests ways of building related constructions.

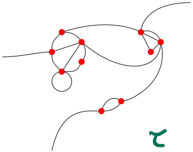
A monad for modular operads?

The configurations of formal composites are now general connected graphs, more precisely what we call Feynman graphs: they are (non-directed) graphs, allowed to have multiple edges and loops, as well as open edges.

Introduction, Joyal-Kock 2011



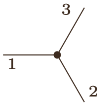
Graph category Gr - objects



Think of § are graph with two open ends
that can be permuted,
Think of X as corolla.

§ $\longmapsto$ 

$\{x\} \longmapsto$ 

$\mathbf{3} \longmapsto$ 

In general $\mathcal{G}$ has

- a finite set V of vertices,
- a finite set $\tilde{E}$ of edges
  (copies of §)

If

τ

$$E = \partial\tilde{E} \cong \tilde{E} \amalg \tilde{E},$$

$\mathcal{G}$ is described by a partial map

$$E \nrightarrow V$$

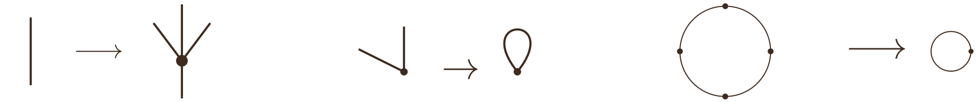
.

So $\mathcal{G}$ is a diagram

$$\begin{tikzcd} E \arrow[loop left, "\tau"] & H \arrow[l, tail, "s"'] \arrow[r, "t"] & V. \end{tikzcd}$$

Graph category Gr

Morphisms are local isomorphisms – they preserve vertex valency.



$$
\begin{tikzcd}
\mathcal{G} \arrow[d, "f"'] \\
\mathcal{G}'
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, leftrightarrow, "\tau"] \arrow[d, "f_E"'] & E \arrow[d, "f_E"'] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "f_H"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "f_V"] \\
E' \arrow[r, leftrightarrow, "\tau'"'] & E' & H' \arrow[l, "s'"] \arrow[r, "t'"'] & V'
\end{tikzcd}
$$

There are fully faithful dense embeddings

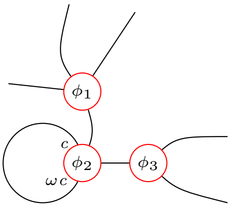
$$\mathbb{P}^{\circlearrowleft} \xrightarrow{\ \iota\ } \mathsf{Gr} \xrightarrow{\mathcal{G} \mapsto \mathsf{Gr}(\iota -, \mathcal{G})} \mathsf{GS}.$$

A monad for modular operads?

S is a graphical species. Build a species $T^\circ S$ of formal combinations of elements of S:

$T^\circ S(\S) = S_\S = (\mathfrak{C}, \omega)$.

$T^\circ S_X$: equivalence classes of graphs $\mathcal{G}$, with $\partial\mathcal{G} \cong X$, decorated by S:



$$S(\mathcal{G}) \overset{\text{def}}{=} \lim_{(Y,b)\in(\mathbb{P}^{\circlearrowleft}\downarrow\mathcal{G})} S_Y$$

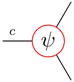
Colimit over graph isomorphisms that fix the bijections

$$X \cong \partial\mathcal{G}.$$

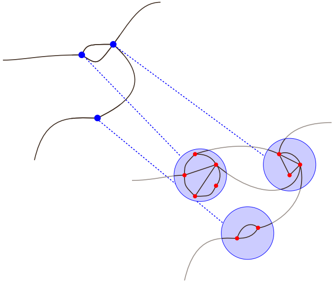
A monad for modular operads?

Monadic unit:

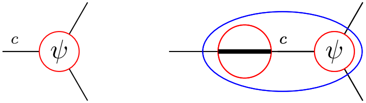
$\psi \longmapsto$



Monadic multiplication?



Units for operadic multiplication?



Take colimits of functors

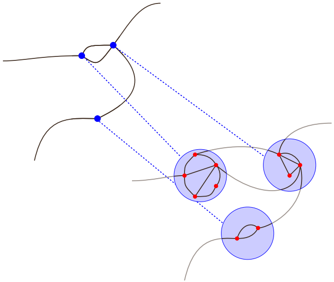
$$(\mathbb{P}^{\circlearrowleft} \downarrow \mathcal{G}) \to (\mathsf{Gr} \downarrow S),\ (X, f) \mapsto (\mathcal{G}, \alpha), \partial\mathcal{G} = X$$

that preserve boundaries and incidence.

A monad for modular operads?

Monadic unit:

Monadic multiplication?

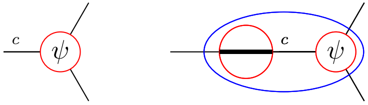


Take colimits of functors

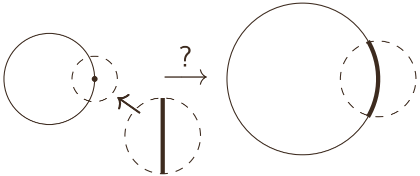
$$(\mathbb{P}^{\circlearrowleft} \downarrow \mathcal{G}) \to (\mathsf{Gr} \downarrow S),\ (X, f) \mapsto (\mathcal{G}, \alpha), \partial\mathcal{G} = X$$

that preserve boundaries and incidence.

Units for operadic multiplication?

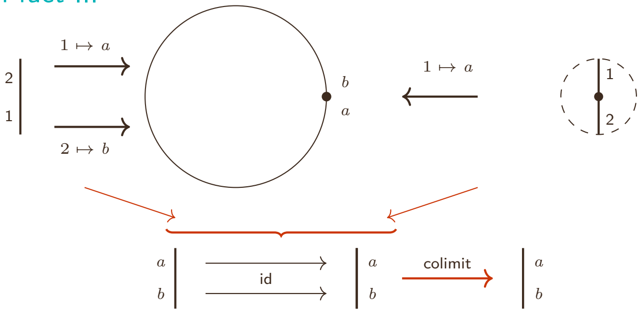


But...



Loops?

In fact ...



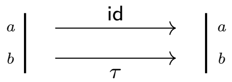
Can we add this object?

No!

We need

$$\zeta(\epsilon c) = \zeta(\epsilon(\omega c)), \forall c.$$

But we still need to take a colimit .



If the kids won't play nicely,

Separate them!

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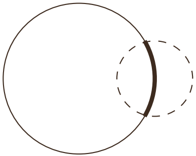
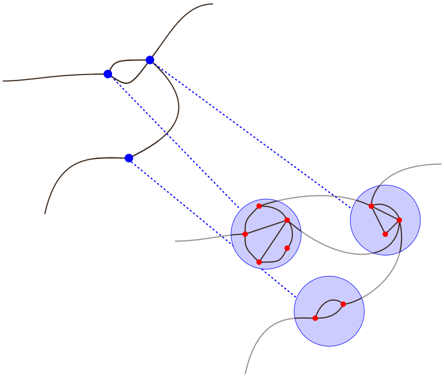
Separate them!

Non-unital monad

Besides this obstruction,



everything works fine.



Don't allow substitution by §.
Then there's a well defined monad $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ on GS that governs contraction and multiplication.
Just not multiplicative units.

Combinatorics of units 1: The monad

If S has a unital multiplication $\epsilon : \mathfrak{C} \to S_{\mathbf{2}}$, then it has distinguished elements $S_{\mathbf{2}}$:

$$\epsilon(c) \in S_{\mathbf{2}}, \quad \text{for all } c \in \mathfrak{C},$$

... but also in $S_{\mathbf{0}}$!

$$o(c) = \zeta\epsilon(c) = o(\omega c), \text{ for } c \in \mathfrak{C}.$$

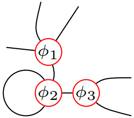
So, take endofunctor $D : \mathsf{GS} \to \mathsf{GS}$ that adjoins these elements:

- for each $c \in \mathfrak{C}$, add an extra element ϵ_c^+ to $S_{\mathbf{2}}$,
- for each orbit $\tilde{c}$ of ω in $\mathfrak{C}$, add an extra element $o_{\tilde{c}}^+$ to $S_{\mathbf{0}}$,

This extends to a monad $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ on GS.

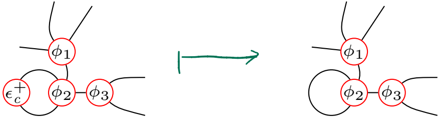
Distributive law

Natural transformation $\lambda : TD \Rightarrow DT$

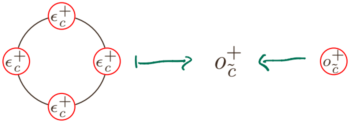


If all vertices are decorated by S, do nothing!

If $\mathcal{G}$ has vertices decorated by S, delete any vertices decorated by ϵ^+



Otherwise



A solution!

Theorem (R. 2020)

There are monads $\mathbb{T} = (T, \mu^{\mathbb{T}}, \eta^{\mathbb{T}})$ *and* $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ *on* GS *and a distributive law* $\lambda : TD \Rightarrow DT$ *such that* $\mathbb{O} = \mathbb{DT}$ *on* GS.

Let GS_* be the category of $\mathbb{D}$-algebras.

Theorem (R. 2018)

There is a full, dense subcategory Gr_* *of* GS_* *such that the induced monad* $\mathbb{T}_*$ *on* GS_*, *has arities** Gr_*.

A solution!

$$
\begin{tikzcd}
 & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}"] & \mathsf{PSh}(\Xi) \arrow[d, "j^*"] \\
\mathbb{P}_*^{\circlearrowleft} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr}_* \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "j", "\text{b.o.}"'] & \mathsf{GS}_* \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] \arrow[d, shift left, "\text{forget}"] & \mathsf{PSh}(\mathsf{Gr}_*) \arrow[d] \\
\mathbb{P}^{\circlearrowleft} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "\text{b.o.}"] & \mathsf{Gr} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "\text{b.o.}"'] & \mathsf{GS} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] & \mathsf{PSh}(\mathsf{Gr})
\end{tikzcd}
$$

Combinatorics of units 2: Graph morphisms

Endofunctor $D : \mathsf{GS} \to \mathsf{GS}$ that adjoins:

- for each $c \in \mathfrak{C}$, add an extra element ϵ_c^+ to $S_{\mathbf{2}}$,
- for each orbit $\tilde{c}$ of ω in $\mathfrak{C}$, add an extra element $o_{\tilde{c}}^+$ to $S_{\mathbf{0}}$,

What do the algebras look like?

Triples (S, ϵ, o)

- S is $(\mathfrak{C}, \omega)$ - graphical species
- $\epsilon : \mathfrak{C} \to S_{\mathbf{2}}$ is injective unit.
- $o : \mathfrak{C} \to S_{\mathbf{0}}$ factors through $\mathfrak{C}/\omega$.

Pointed graphical species

$\mathsf{GS}_* \stackrel{\text{def}}{=} Alg(\mathbb{D})$

A pointed graphical species (S, ϵ, o) is:

a $(\mathfrak{C}, \omega)$-graphical species S ,

$\epsilon : \mathfrak{C} \to S_{\mathbf{2}}$ is injective unit .

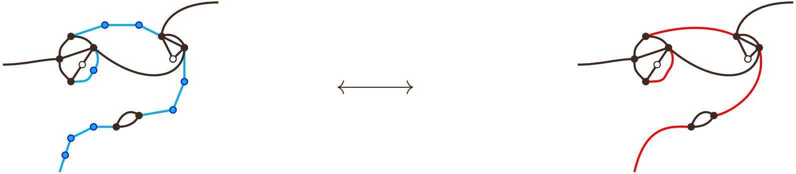
$o : \mathfrak{C} \to S_{\mathbf{0}}$ factors through $\mathfrak{C}/\omega$.

Pointed graphical species

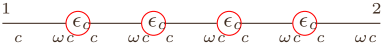
$\mathbb{P}^{\circlearrowleft}_*$	$\mathsf{GS}_* \overset{\text{def}}{=} \mathsf{PSh}(\mathbb{P}^{\circlearrowleft}_*)$
	A $\mathbb{P}^{\circlearrowleft}_*$-presheaf S_* is:
$\mathbb{P}^{\circlearrowleft}$	a $(\mathfrak{C}, \omega)$-graphical species S ,
with adjoined morphisms:	
$u : \mathbf{2} \to \S$ such that	
- $u \circ ch_1 = id_{\S}$ $\quad u \circ ch_2 = \tau$,	$\epsilon = S_*(u) : \mathfrak{C} \to S_{\mathbf{2}}$ is injective unit
- $\tau \circ u = u \circ \sigma_{\mathbf{2}} \in \mathbb{P}^{\circlearrowleft}(\mathbf{2}, \S)$,	
$z : \mathbf{0} \to \S$	
$z = \tau \circ z$	$o = S_*(z) : \mathfrak{C} \to S_{\mathbf{0}}$.

What should the monad $\mathbb{T}_*$ on GS_* do?

Ignore vertices decorated by units

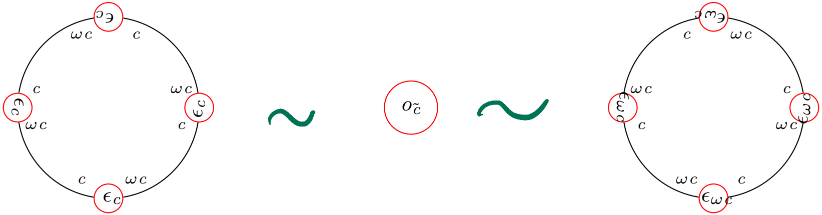


Units?



What should the monad $\mathbb{T}_*$ on GS_* do?

Contracted units? Identify $\mathbf{0}$-graphs decorated by (contracted) units



How does this work?

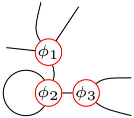
More graph morphisms

$$
\begin{tikzcd}
\mathbb{P}^{\circlearrowleft}{}_* \arrow[r, hook, "\mathsf{f.f.}"] & \mathsf{Gr}_* \arrow[r, hook, "\mathsf{f.f.}"] & \mathsf{GS}_* \arrow[r, hook] \arrow[d, shift left, "\mathsf{forget}^{\mathbb{T}_*}"] & \mathsf{PSh}(\mathsf{Gr}_*) \arrow[d] \\
\mathbb{P}^{\circlearrowleft} \arrow[u, "\mathsf{b.o.}"] \arrow[r, hook, "\mathsf{f.f.}"] \arrow[ur, dashed] & \mathsf{Gr} \arrow[u, "\mathsf{b.o.}"'] \arrow[r, hook, "\mathsf{f.f.}"] & \mathsf{GS} \arrow[u, shift left, "\mathsf{free}^{\mathbb{T}_*}"] \arrow[r, hook, "\mathsf{f.f.}"'] & \mathsf{PSh}(\mathsf{Gr}).
\end{tikzcd}
$$

Factorisation on Gr_* Right: Morphisms from Gr.
Left: Delete bivalent vertices as long as there is at least one remaining vertex (preserves graph boundary), and Special morphisms

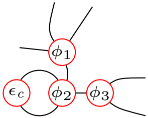
- $u : \mathbf{2} \to \S$,
- $z : \mathbf{0} \to \S$,
- $\kappa : \mathcal{W} \to \S$.

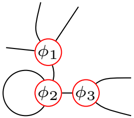
The lifted monad $\mathbb{T}_*$ on $\mathbb{D}$ algebras



If all vertices are decorated by S, do nothing!

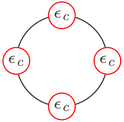
If $\mathcal{G}$ has vertices decorated by S, delete any vertices decorated by ϵ^+





Otherwise

$\rightsquigarrow \epsilon_c$



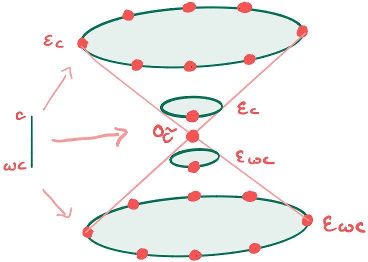
$\rightsquigarrow o^+_{\tilde{c}} \rightsquigarrow o_{\tilde{c}}$

A monad for modular operads?

Morphisms in left class of Gr_*-factorisation preserve boundary except at $z : \mathbf{0} \to \S$ and $\kappa : \mathcal{W}^m \to \S$.

$$T_* S_\S = S_\S = (\mathfrak{C}, \omega).$$

$T_* S_X$: equivalence classes of graphs $\mathcal{G}$, with $\partial \mathcal{G} \cong X$, decorated by S:



$$S(\mathcal{G}) \overset{\mathrm{def}}{=} \lim_{(Y,b) \in (\mathbb{P}^{\circlearrowleft} \downarrow \mathcal{G})} S_Y$$

Colimit over graph morphisms in the left class with fixed bijection

$$X \cong \partial \mathcal{G}$$

except at $(\S, c)$.

Remarks on construction

- No loop objects added to the construction.
- $z : \mathbf{0} \to \S$ comes directly from the definition
- $\kappa : \mathcal{W} \to \S$ gives contraction.
- Obtain multiplication for monad $\mathbb{T}_*$ by looking at only nice representatives.

Weak modular operads: Hackney, Robertson and Yau, 2020

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, hook] \arrow[rr, bend left, "\text{dense}"] & \Xi \arrow[r, hook, "\text{f.f.}"'] & \mathsf{MO} \arrow[r, "N_{\mathcal{U}}"] \arrow[d, shift left, "\text{forget}^{\mathbb{O}}"] & \mathsf{PSh}(\mathcal{U}) \arrow[d, "j^*"] \\
\mathbb{P}^{\circlearrowleft} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Gr} \arrow[u, "\text{b.o.}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}^{\mathbb{O}}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{PSh}(\mathsf{Gr})
\end{tikzcd}
$$

Theorem (Hackney, Robertson, Yau, 2020)

The graphical category $\mathbf{U} \subset \Xi$ *is dense in* MO.
Essential image of nerve satisfies the strict Segal condition.
There is a model category structure on $\mathsf{sSet}^{\mathbf{U}^{op}}$ *whose fibrant objects are those* S *satsifying the weak Segal condition:*
for all $\mathcal{G}$,

$$S(\mathcal{G}) \simeq \lim_{(C,b)\mathbb{P}^{\circlearrowleft}\downarrow\mathcal{G}} S(C).$$

Weak modular operads: Corollary and observation

It follows from the proof and Caviglia and Horel, 2016

Corollary (Raynor, 2020)

There is a model category structure on $\mathsf{sSet}^{\Xi^{op}}$ *whose fibrant objects are those* S *satsifying the weak Segal condition:*
for all $\mathcal{G}$*,*

$$S(\mathcal{G}) \simeq \lim_{(C,b)\mathbb{P}^{\circlearrowleft}\downarrow\mathcal{G}} S(C)$$

It remains to compare the versions of weak modular operads so obtained.

Extending the framework, concluding remarks.

Directions

- Circuit algebras/ modular operads with product and nerve theorem.
- Higher modular operads

Applications

- Extended cobordism categories.
- geometric applications from the cone..

To be continued...

THANK YOU!