Brauer diagrams, modular operads, & a graphical nerve theorem for circuit algebras

Sophie Raynor

Macquarie University

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I acknowledge the the traditional custodians of the land on which I work: the Wattamattageal clan of the Darug nation. I pay my respects to Elders past, present and future.

Aims

- Construct a monad for circuit algebras and prove a nerve theorem,
- describe relationships with other structures
- including a nice example of *'exceptional loops'*

This talk is based on my paper of the same name draft on www.sophieraynor.org, and my paper *Graphical combinatorics and a distributive law for modular operads* (arXiv:1911.05914) to appear *Adv Math.*.

Method

Use abstract nerve theory and (iterated) distributive laws

circuit algebras

we want!

$$
\begin{tikzcd}
 & \Xi^{\times} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{CA} \arrow[r, "N", "\text{f.f}"'] \arrow[d, shift left, "\text{forget}"] & \mathrm{psh}(\Xi^{\times}) \arrow[d, "j^{*}"] \\
\mathbf{B}^{\S} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{etGr} \arrow[u, "\text{b.o.}", "j"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}"] \arrow[r, hook, "\text{f.f.}"'] & \mathrm{psh}(\mathsf{etGr})
\end{tikzcd}
$$

This row has been described by Joyal-Kock '11.

fin sets & bijections + distinguished object

Graphs & vertex valency pres. morphisms

presheaves on $B^{\S}$

Philosophy

Monads encode algebraic stucture. Distributive laws tell us how structures interact.

1. Introduction
2. Brauer diagrams
3. Circuit algebras
4. The monad

Category BD of (monochrome) Brauer diagrams.

$(\underline{0} = \emptyset)$

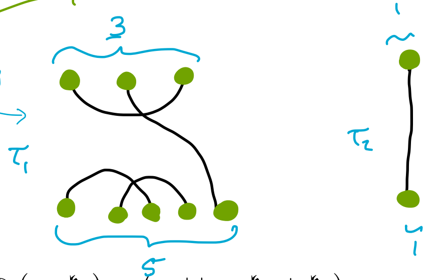
$\{1, \ldots, n\}$

Objects: $n \in \mathbb{N}$,

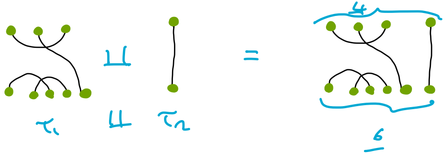
Morphisms: $f\colon m \to n$, $f = (\tau, \mathfrak{k})$ where

- τ is a perfect matching on $\mathbf{m} \amalg \mathbf{n}$
- $\mathfrak{k} \in \mathbb{N}$.

e.g



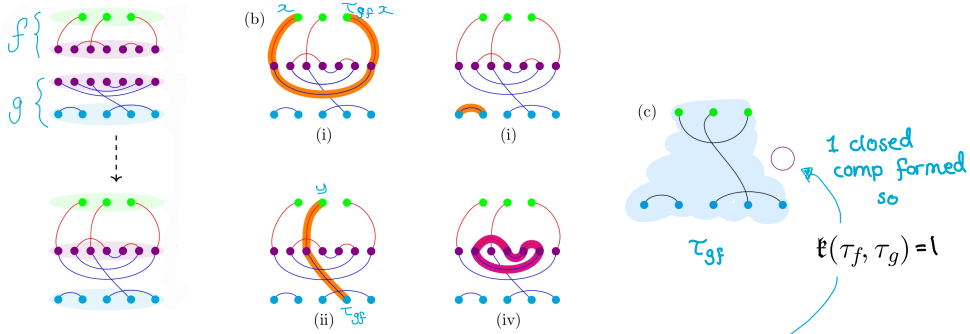
Horizontal (monoidal) composition: $(\tau_1, \mathfrak{k}_2) \oplus (\tau_2, \mathfrak{k}_2) = (\tau_1 \sqcup \tau_2, \mathfrak{k}_1 + \mathfrak{k}_2)$



$: 4 \to 6$

Composition in BD

Vertical (categorical) composition: $f = (\tau_f, \mathfrak{k}^f) \in \mathsf{BD}(k, m)$, $g = (\tau_g, \mathfrak{k}^g) \in \mathsf{BD}(m, n)$



$$gf = (\tau_{gf}, \mathfrak{k}^{gf}) \in \mathsf{BD}(k, n), \text{ where } \mathfrak{k}^{gf} = \mathfrak{k}^g + \mathfrak{k}^f + \mathfrak{k}(\tau_f, \tau_g).$$

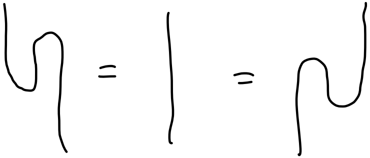
BD - duality

pair $(*, O)$ of the

The unique matching $*$ on $\mathbf{1} \sqcup \mathbf{1} \cong \mathbf{2}$ and $O \in \mathbb{N}$ describes morphisms

$$| = \mathrm{id} : 1 \to 1, \quad \cup : 2 \to 0, \quad \cap : 0 \to 2.$$

Together with $\times$, these generate BD under $\oplus$ and $\circ$. (Lehrer-Zheng '15)



So BD is a wheeled PROP.

Pushouts of cospans

Brauer diagrams are described by cospans

$$\mathbf{m} \xrightarrow{\tau|_{\mathbf{m}}} \mathbf{m} \amalg \mathbf{n} \xleftarrow{\tau|_{\mathbf{n}}} \mathbf{n}.$$

But composition in BD is not by pushouts of cospans.

e.g $f = \cap : 0 \to 2$
$g = \cup : 2 \to 0$

$$\begin{tikzcd}
\overset{\phi}{\cancel{k}} \arrow[dr, "\tau_f"'] & & \overset{2}{\cancel{\mathbf{m}}} \arrow[dl, "\tau_f", "="'] \arrow[dr, "\tau_g"', "="] & & \overset{\phi}{\cancel{\mathbf{n}}} \arrow[dl, "\tau_g"] \\
& \overset{2}{\cancel{k \amalg \mathbf{m}}} \arrow[dr, dashed, "="'] & & \overset{2}{\cancel{\mathbf{m} \amalg \mathbf{n}}} \arrow[dl, dashed, "="] & \\
& & \underset{2}{\cancel{?}} & &
\end{tikzcd}$$

$2 \neq \underline{0} \sqcup \underline{0}$

$= (-1, 1)$

Coloured Brauer diagrams

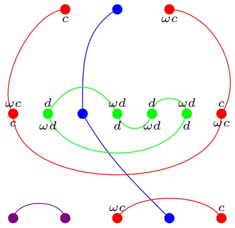
We can colour morphisms by involutive sets $(\mathfrak{C}, \omega)$:

$f = (\tau, k)$

$$\begin{tikzcd}
\mathbf{m} \amalg \mathbf{n} \arrow[r, "\mathring{\lambda}"] \arrow[d, "\tau"', "\cong"] & \mathfrak{C} \arrow[d, "\omega", "\cong"'] \\
\mathbf{m} \amalg \mathbf{n} \arrow[r, "\mathring{\lambda}"] \arrow[d] & \mathfrak{C} \arrow[d] \\
\pi_0(f) \arrow[r, "\widetilde{\lambda}"] & \widetilde{\mathfrak{C}}.
\end{tikzcd}$$

$k + \#\mathrm{orb}[\tau] =$ $\pi_0(f)$

$\widetilde{\mathfrak{C}}$ — orbits of ω in $\mathfrak{C}$



← some care with composition.

Example: $\mathfrak{C} = \{+, -\}$, $\omega : + \leftrightarrow -$ describes oriented Brauer diagrams.

With a little care, we get category $\mathsf{BD}^{(\mathfrak{C},\omega)}$ of $(\mathfrak{C}, \omega)$-coloured Brauer diagrams.

Operads of wiring diagrams

(list $\mathfrak{C}$)-coloured

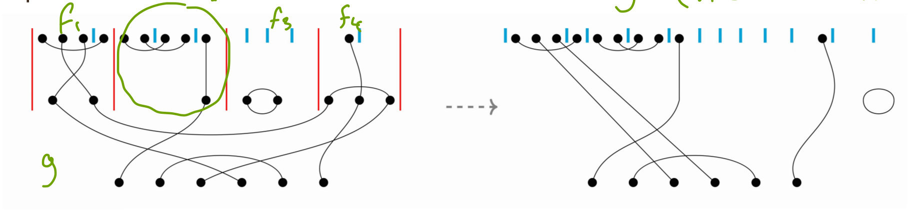
Operad $WD^{(\mathfrak{C},\omega)}$ of $(\mathfrak{C},\omega)$-coloured wiring diagrams is the operad underlying $\mathsf{BD}^{(\mathfrak{C},\omega)}$:

- for all lists of lists $(\boldsymbol{c}_1,\dots,\boldsymbol{c}_k;\boldsymbol{d})$, where $\underline{c_i} = (c_{i_1}, \dots, c_{m_i})$, $c_j \in \mathfrak{C}$ $\forall\, 1 \le j \le m_i$

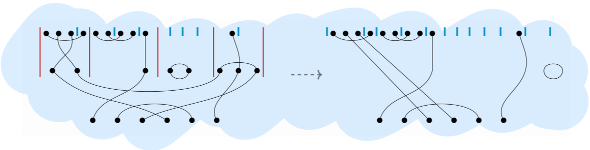
$$WD^{(\mathfrak{C},\omega)}(\boldsymbol{c}_1,\dots,\boldsymbol{c}_k;\boldsymbol{d}) = \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c}_1 \oplus \cdots \oplus \boldsymbol{c}_k, \boldsymbol{d}),$$

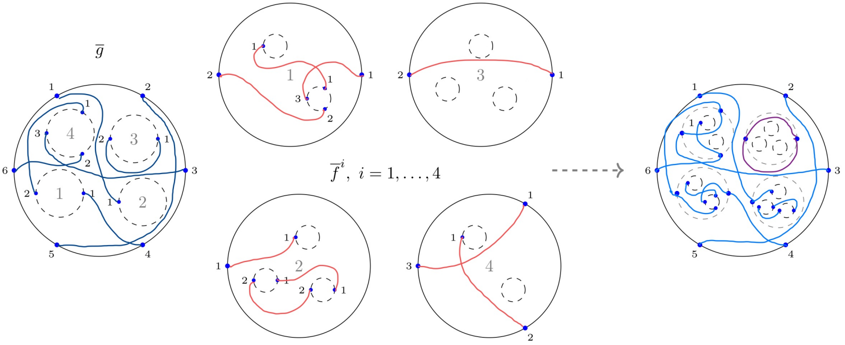
- composition:

$f_2 \in BD(2+2+1,1)$ $\qquad g \circ (f_1 \oplus \dots \oplus f_4) \overset{mor}{\in} BD^{(\mathfrak{C},\omega)}$



... but circuit algebras?





Circuit algebras

A $(\mathfrak{C}, \omega)$-coloured circuit algebra (in a symmetric monoidal category $(\mathsf{V}, \otimes, I)$) is an algebra for $WD^{(\mathfrak{C},\omega)}$:

- a coloured collection $\mathcal{A} = (\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c} \in \mathsf{list}\mathfrak{C}}$ in V,
- for all lists of lists $(\boldsymbol{c}_1, \dots, \boldsymbol{c}_k; \boldsymbol{d})$, and all $(f, \lambda) \in \mathsf{BD}^{(\mathfrak{C},\omega)}(\boldsymbol{c}_1 \oplus \dots \oplus \boldsymbol{c}_k; \boldsymbol{d})$,

$$\mathcal{A}(f, \lambda) \colon \mathcal{A}(\boldsymbol{c}_1) \otimes \mathcal{A}(\boldsymbol{c}_k) \to \mathcal{A}(\boldsymbol{d})$$

$$V(\mathcal{A}(c_1) \otimes \dots \otimes \mathcal{A}(c_k), \mathcal{A}(d))$$

$+$ axioms.

We'll assume that $V = \mathsf{Sets}$

$(\mathfrak{C}, \omega)$-coloured circuit algebras in $(\mathsf{V}, \otimes, I)$ are equivalent to lax symmetric monoidal functors $\mathsf{BD}^{(\mathfrak{C},\omega)} \to (\mathsf{V}, \otimes, I)$.

Some properties of circuit algebras

NOTE: In fact this slide is more complicated than necessary.

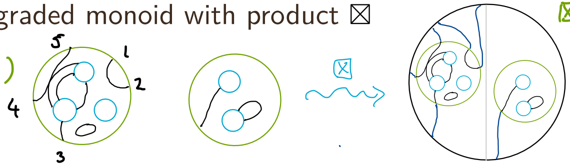
It would have been sufficient to draw the purple diagrams highlighted below

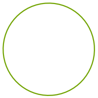
$\mathcal{A} = (\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}\in \text{list}\mathfrak{C}}$ is a circuit algebra. Then,

- $(\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}}$ is a graded monoid with product $\boxtimes$

$\boxtimes_{\underline{cd}} = \mathcal{A}(id_{\underline{cd}}) : \mathcal{A}(\underline{c}) \otimes \mathcal{A}(\underline{d}) \to \mathcal{A}(\underline{cd})$

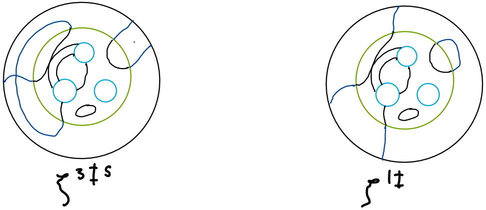
for $\underline{c}, \underline{d} \in$ list $\mathfrak{C}$, $id_{\underline{cd}} \in BD^{(\mathfrak{C},\omega)}(\underline{cd}, \underline{cd})$ describes element of $WD^{(\mathfrak{C},\omega)}(\underline{c},\underline{d}; \underline{cd})$

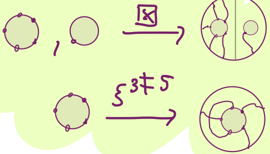




unit for $\boxtimes$ is empty diag.

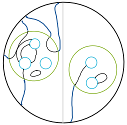
- $\mathcal{A}$ has a contraction operation ζ induced by $\cup \in \mathsf{BD}(2,0)$ (and permutations)

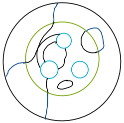




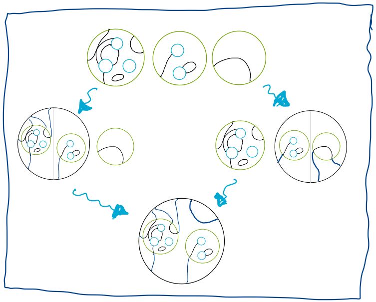
Some properties of circuit algebras

$\mathcal{A} = (\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c} \in \mathsf{list}\mathfrak{C}}$ is a circuit algebra.

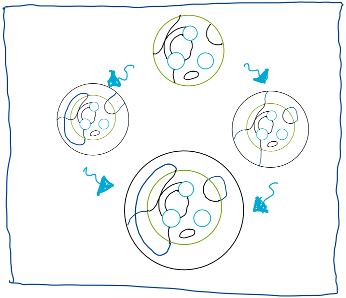




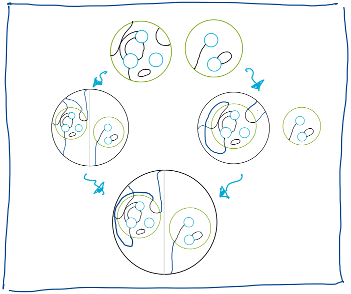
(c1) $\boxtimes$ is associative



(c2) ζ commutes

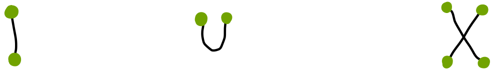


(c3) $\boxtimes$ and ζ commute



Downward Brauer diagrams

The category $\mathsf{BD}_\downarrow \subset \mathsf{BD}$ of downward Brauer diagrams is generated, under $\oplus$ and $\circ$, by



morphisms given by matchings alone

Algebras $\mathcal{D}$ for underlying operad also satisfy (c1)-(c3) and are called 'non-unital' circuit algebras.

- pushouts of cospans describe composition in $\mathsf{BD}_\downarrow$,
- Kock has described a monad for non-unital circuit algebras. (Kock, '18)
- symmetric strict monoidal functors $\mathsf{BD}_\downarrow \to \mathsf{Vect}_{\Bbbk}$ are equivalent to representations of $O(\infty)$. (Sam-Snowden, '15)

$\Bbbk$ has char. 0

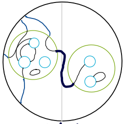
Modular operads

Again, diagram presented in the talk should have been simpler

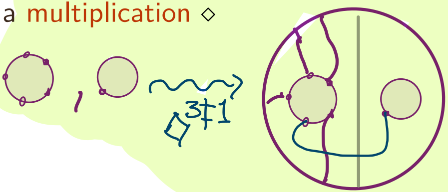
A circuit algebra $\mathcal{A} = (\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c} \in \mathsf{list}\mathfrak{C}}$ in $(\mathsf{V}, \otimes, I)$ has a multiplication $\diamond$

e.g. $\mathfrak{C} = \{*\}$

$\mathcal{A}(m) \otimes \mathcal{A}(n) \to \mathcal{A}(m+n-2)$



$\diamond^{2 \mp 1}$

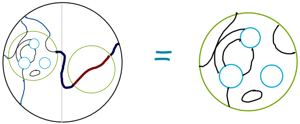


$\diamond^{3 \mp 1}$

such that $(\mathcal{A}, \diamond, \zeta)$ is a non-unital modular operad.

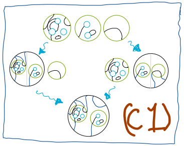
The multiplication $\diamond$ has a 2-sided unit induced by $\cap \in \mathsf{BD}(0,2)$:

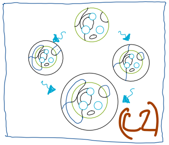
(e1) for all $c \in \mathfrak{C}$, the V-morphism $\epsilon_c \stackrel{\text{def}}{=} \mathcal{A}(\cap_c) \colon I \to (c, \omega c)$ satisfies

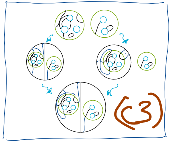


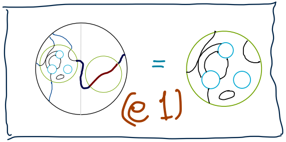
Taking stock: what we know so far

Circuit algebras are coloured collections $(\mathcal{A}(\boldsymbol{c}))_{\boldsymbol{c}\in\mathfrak{C}}$ with $\boxtimes$, ζ, and ϵ such that









$\boxtimes$, ζ, (c1)-(c3): non-unital circuit algebras.
(Kock, 18), monad $\mathbb{T}^\times$ on GS.

ζ, $\diamond = \zeta\boxtimes$, ϵ, (m1)-(m4), (e1): modular operads.
(R, 18), monad $\mathbb{D}\mathbb{T}$ on GS.

$\Xi^\times$ Ξ CA_ MO $\mathrm{psh}(\Xi^\times)$ $\mathrm{psh}(\Xi)$

$$
\begin{tikzcd}
 & \cancel{\Xi^\times} \arrow[r, hook, "\text{f.f.}"'] & \cancel{\mathsf{CA}} \arrow[r, "N", "\text{f.f.}"'] \arrow[d, shift left, "\text{forget}"] & \mathrm{psh}(\cancel{\Xi^\times}) \arrow[d, "j^*"] \\
\boldsymbol{B}^\S \arrow[r, hook, "\text{f.f.}"'] & \mathrm{etGr} \arrow[u, "j"', "\text{b.o.}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}"] \arrow[r, hook, "\text{f.f.}"'] & \mathrm{psh}(\mathrm{etGr})
\end{tikzcd}
$$

Graphical species (Joyal-Kock, '11)

Describe $(\mathfrak{C}, \omega)$-coloured collections $(S_c)_c$ as presheaves on a category $\boldsymbol{B}^{\S}$:

- groupoid $\boldsymbol{B}$ of finite sets and bijections is full subcategory,
- distinguished object $\S$ with $\boldsymbol{B}^{\S}(\S, \S) = \{id, \tau\}$, $\tau^2 = id$,
- For each X, and $x \in X$, morphisms ch_x, $ch_x \circ \tau : \S \longrightarrow X$ (ch_x 'chooses' x).

finite set

$\underset{\S}{2} \xrightarrow[1 \mapsto x]{ch_x}$ x

GS is category of presheaves or graphical species $S \colon \boldsymbol{B}^{\S\,\mathrm{op}} \to \mathsf{Set}$.

$S_{\S} = \mathfrak{C}$ $\qquad S(\tau) = \omega$

Graphs

Let $(\tau, 0) \in \mathsf{BD}(m_1 + \cdots + m_k, n)$:

$$\coprod_{i=1}^k \mathbf{m}_i \xrightarrow{\ \tau\ } (\coprod_{i=1}^k \mathbf{m}_i) \amalg \mathbf{n} \xleftarrow{\ \tau\ } \mathbf{n}.$$

So we can rewrite the corresponding wiring diagram in $WD(\mathbf{m}_1, \ldots, \mathbf{m}_k; \mathbf{n})$:

Edge ends   Half edges   V   vertices

$$\tau \circlearrowright (\coprod_{i=1}^k \mathbf{m}_i) \amalg \mathbf{n} \xleftarrow{\ f_{\mathrm{in}}\ } \coprod_{i=1}^k \mathbf{m}_i \xrightarrow{\ p\ } \mathbf{k}$$

free   involution   inclusion

(Joyal-Kock, '11.) A diagram of finite sets of this form is a graph.

$$\mathcal{G} = \tau \circlearrowright E \xleftarrow{\ s\ } H \xrightarrow{\ t\ } V$$

injective

free inv

Graphs and graphical species (Joyal-Kock,'11)

Objects of etGr are graphs.

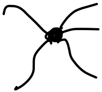
Morphisms:

pullback

$$
\begin{tikzcd}
\mathcal{G} \arrow[d] & E \arrow[r, leftrightarrow, "\tau"] \arrow[d] & E \arrow[d] & H \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d] \\
\mathcal{G}' & E' \arrow[r, leftrightarrow, "\tau'"'] & E' & H' \arrow[l, "s'"] \arrow[r, "t'"'] & V'.
\end{tikzcd}
$$

$\mathbf{B}^{\S} \hookrightarrow$ etGr

$\S \mapsto |$ given by $\{1,2\} \leftarrow \emptyset \longrightarrow \emptyset$

$X \mapsto$  given by $X \sqcup X \hookleftarrow X \longrightarrow *$

etGr $\hookrightarrow$ GS

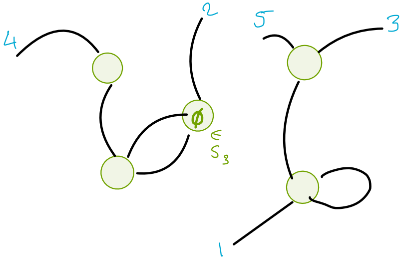
$\mathcal{G} \mapsto \mathrm{etGr}(-, \mathcal{G})$.

Kock's monad $\mathbb{T}^{\times}$ for non-unital circuit algebras

S is a graphical species. Build a species $T^{\times} S$ of formal combinations of elements of S:

$$T^{\times} S(\S) = S_{\S} = (\mathfrak{C}, \omega).$$

Take equivalence classes of graphs $\mathcal{G}$, with $\rho \colon \partial\mathcal{G} \xrightarrow{\cong} X$, decorated by S:



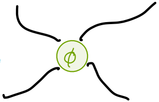
$\in T^{\times} S_5$

$T^{\times} S_X$ is colimit over graph isomorphisms that fix

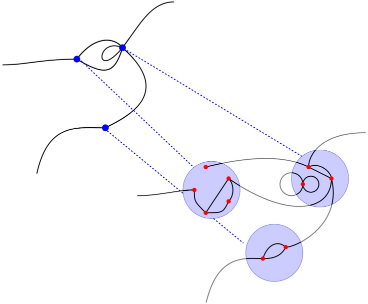
$$X \cong \partial\mathcal{G}.$$

Monad $\mathbb{T}^{\times}$ for (non-unital) circuit algebras (Kock, '18)

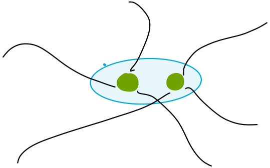
Monadic unit:

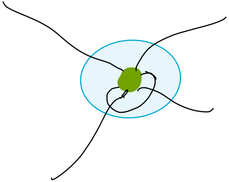
$S, \ni \phi \longmapsto$ 

Monadic multiplication:



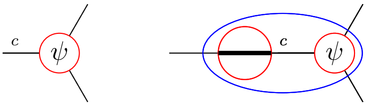
Algebras satisfy (c1)-(c3).



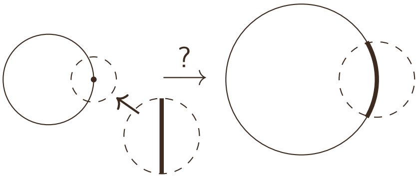


A monad for circuit algebras?

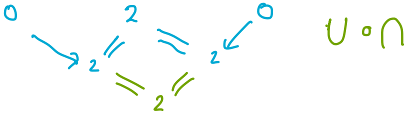
What about ϵ?



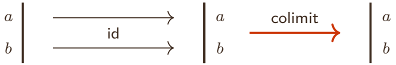
But...



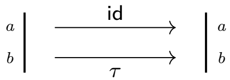
Recall the pesky pushout!



In fact we get



But we still need to take a colimit .



The monad for modular operads (R,'18, '21)

For modular operads, the problem can be solved by constructing the composite monad $\mathbb{DT}$ on GS, where

- $\mathbb{T}$ is the monad for contraction and multiplication defined exactly as $\mathbb{T}^{\times}$ but restricted to connected graphs,
- $\mathbb{D} = (D, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ is the monad that governs the multiplicative units ϵ.

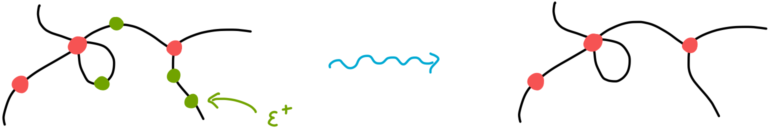
D : GS $\to$ GS adjoins elements

- ϵ_c^+ to $S_{\mathbf{2}}$, for all $c \in \mathfrak{C}$,
- $o_{\tilde{c}}^+$ to $S_{\mathbf{0}}$, for each orbit $\tilde{c}$ of ω in $\mathfrak{C}$.

Combining the monads: strategy I

Distributive law for modular operads: $\lambda_{\mathbb{DT}} : TD \Rightarrow DT$

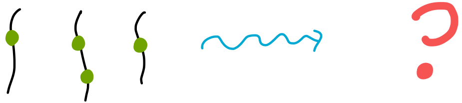
If $\mathcal{G}$ has vertices decorated by S, delete any vertices decorated by ϵ^+:



Otherwise

Strategy I: Try to modify $\lambda_{\mathbb{DT}}$ for $\mathbb{T}^\times$ and $\mathbb{D}$.

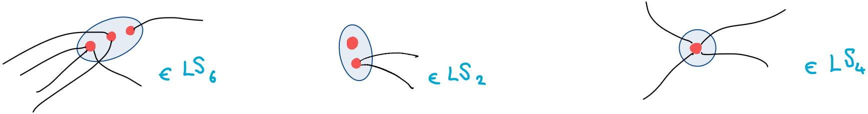
BUT



A solution: $\mathbb{T}^\times$ is also a composite!

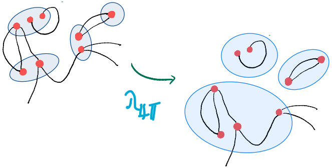
$$\mathbb{T}^\times = \mathbb{L}\mathbb{T}$$ non-unital mod op.

where $\mathbb{L} = (L, \mu^{\mathbb{L}}, \eta^{\mathbb{L}})$ is the free graded monoid monad on GS.



Distributive law

$\lambda_{\mathbb{LT}} \colon TL \Rightarrow LT$ takes a connected graph whose vertices are decorated by tuples of elements of S and returns a tuple of connected graphs decorated by S.



The monad $\mathbb{LDT}$ for circuit algebras.

By (Cheng, '11), composite $\mathbb{LDT}$ exists if there is a distributive law $\lambda_{\mathbb{LD}} : D\overset{L}{\cancel{B}} \Rightarrow \overset{L}{\cancel{B}}D$, and the Yang-Baxter equations hold:

$$
\begin{tikzcd}
 & DTL \arrow[r, "D\lambda_{\mathbb{TL}}"] & DLT \arrow[dr, "\lambda_{\mathbb{DL}}\,T"] & \\
TDL \arrow[ur, "\lambda_{\mathbb{TD}}\,L"] \arrow[dr, "T\lambda_{\mathbb{DL}}"'] & & & LDT \\
 & TLD \arrow[r, "\lambda_{\mathbb{TL}}\,D"'] & LTD \arrow[ur, "L\lambda_{\mathbb{TD}}"'] &
\end{tikzcd}
$$

Proposition (R, 2021)
There is, and they do!

The monad $\mathbb{LDT}$ for circuit algebras.

Proposition (R, '21)
Algebras for $\mathbb{LDT}$ satisfy (c1)-(c3) and (e1).

proof follows easily from what we already know.

Theorem (R, '21)
The category CA of Set valued circuit algebras is equivalent to the Eilenberg-Moore category of algebras for $\mathbb{LDT}$.

Need to check that wiring diagrams define suitable morphisms

- part coming from matching τ uses graph decr. of wiring diagrams.
- part coming from closed components uses dist. law for $\mathbb{D}$ & $\mathbb{T}$.

The nerve

Theorem (R,'21)
There is a full, dense subcategory $\Xi^\times$ of CA whose objects are graphs.
The essential image of the induced fully faithful nerve $N\colon \mathsf{CA} \to \mathsf{psh}(\Xi^\times)$ is characterised by Segal presheaves.

Proof sketch:
We want

$$\begin{tikzcd}
 & \Xi^\times \arrow[r, hook, "\text{f.f.}"'] & \mathsf{CA} \arrow[r, "N", "\text{f.f.}"'] \arrow[d, shift left, "\text{forget}"] & \mathsf{psh}(\Xi^\times) \arrow[d, "j^*"] \\
\mathbf{B}^\S \arrow[r, hook, "\text{f.f.}"'] & \mathsf{etGr} \arrow[u, "j", "\text{b.o.}"'] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{GS} \arrow[u, shift left, "\text{free}"] \arrow[r, hook, "\text{f.f.}"'] & \mathsf{psh}(\mathsf{etGr})
\end{tikzcd}$$

can't immediately apply abstract nerve

But $\mathbb{LDT}$ does not *'have aritites'*.

The nerve

Use the distributive law and abstract nerve theory (Berger-Mellies-Weber, '12)

$$
\begin{tikzcd}
 & \Xi^{\times} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{CA} \arrow[r, "N"] \arrow[d, shift left, "\text{forget}"] & \mathrm{psh}(\Xi^{\times}) \arrow[d, "j^*"] \\
\mathbf{B}^{\S}_{*} \arrow[r, hook, "\text{f.f.}"'] & \mathsf{etGr}_{*} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "j", "\text{b.o.}"'] & \mathsf{GS}_{*} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] \arrow[d, shift left, "\text{forget}"] & \mathrm{psh}(\mathsf{etGr}_{*}) \arrow[d] \\
\boldsymbol{B}^{\S} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "\text{b.o.}"] & \mathsf{etGr} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, "\text{b.o.}"'] & \mathsf{GS} \arrow[r, hook, "\text{f.f.}"'] \arrow[u, shift left, "\text{free}"] & \mathrm{psh}(\mathsf{etGr})
\end{tikzcd}
$$

$\widetilde{\mathbb{LT}}_*$

does have arities?

'lift' of $\mathbb{LT}$ to $\mathbb{D}$-algebras

✓

Algebras for $\mathbb{D}$

Comments and further work

- Another example of distributive laws working well in combination with abstract nerve machinery.
- Models of weak modular operads in (Hackney-Robertson-Yau, '20) and (R,'21) don't generalise to circuit algebras. However, there is a nice model (current work, together with Robertson).

- Finite dimensional representations of $O(\infty)$ and $GL(\infty)$ admit description as unital circuit algebras. Together with R. Street, we are working to understand these relationships more fully, and for general circuit algebras.
- Weakening the axioms e.g. braiding.

Thank you!