Adam Gluck CSE 2321 Exam 1

Prepositional Logic

- Symbols
  - $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- English Prepositions
  - I wear a hat if it's sunny
    - $sunny \Rightarrow hat$
  - I wear a hat only if it's sunny
    - $hat \Rightarrow sunny$
- P and Q are logically equivalent if $P \Leftrightarrow Q$ is a tautology
- Double Negation
  - $\big(\neg(\neg P)\big) \equiv P$
- Commutative Laws
  - $(P \vee Q) \equiv (Q \vee P)$
  - $(P \wedge Q) \equiv (Q \wedge P)$
- Associative Laws
  - $\big((P \wedge Q) \vee R\big) \equiv \big(P \vee (Q \vee R)\big)$
- Distributive Laws
  - $\big(P \vee (Q \wedge R)\big) \equiv \big((P \vee Q) \wedge (P \vee R)\big)$
  - $\big(P \wedge (Q \vee R)\big) \equiv \big((P \wedge Q) \vee (P \wedge R)\big)$
- DeMorgans Law
  - $\big(\neg(P \vee Q)\big) \equiv \big((\neg P) \wedge (\neg Q)\big)$
  - $\big(\neg(P \wedge Q)\big) \equiv \big((\neg P) \vee (\neg Q)\big)$
- The conditional and the contrapositive
  - $(P \Rightarrow Q) \equiv \big((\neg Q) \Rightarrow (\neg P)\big)$
- The inverse and the converse
  - $\big((\neg P) \Rightarrow (\neg Q)\big) \equiv (Q \Rightarrow P)$
- The conditional and a not/or expression
  - $(P \Rightarrow Q) \equiv \big((\neg P) \vee Q\big)$
- Tautology: Always True
- Contradiction: Always False
- Contingency: Not tautology or contradiction

Predicate Logic

- Notation
  - $\forall R(x), P(x) \equiv \forall x(R(x) \Rightarrow P(x)$
  - $\exists R(x), P(x) \equiv \exists x(R(x) \wedge P(x)$
- Interactions with $\neg$
  - $\neg\forall x \in S, P(x) \equiv \exists x \in S, \neg P(x)$
  - $\neg\exists x \in S, P(x) \equiv \forall x \in S, \neg P(x)$
- There is at most one $P(X)$ in S
  - $\exists x \in S, \forall y \in S, \Big(\neg(x \neq y) \Rightarrow \big(P(x) \wedge \neg P(y)\big)\Big)$
- There is exactly one $P(X)$ in S
  - $\exists x \in S, \Big(P(x) \wedge \big(\forall y \in S, P(y) \Rightarrow (x = y)\big)\Big)$
- Is a prime number
  - $P(x) = \forall y, z \in \mathbb{N}, (x = yz) \Rightarrow \neg(y = 1 \Leftrightarrow z = 1)$

Sets

- Notation
  - $S_1 = \{x : P(x)\}$
  - $S_2 = \{x \in S : P(x)\}$
- Schema of Separation
  - If ϕ is a predicate, then for any set X there exists a set $Y = \{x \in X : \phi(x)\}$
- X equals Y
  - $\forall u(u \in X \Leftrightarrow u \in Y)$
- $\emptyset$ is a set with no elements
- $|X|$ is cardinality of X
  - Number of elements
- X is a subset of Y
  - $X \subseteq Y \equiv \forall x(x \in X \Rightarrow x \in Y)$
- X is a proper subset of Y
  - $(X \subseteq Y) \wedge (X \neq Y)$
- Union of X and Y
  - $Z = X \cup Y \equiv \forall u(u \in Z \Leftrightarrow u \in X \vee U \in Y)$
- Intersection of X and Y
  - $X \cap Y \equiv \{u \in X : u \in Y\}$
- Difference of X and Y
  - $X \backslash Y \equiv \{u \in X : u \notin Y\}$
- Cartesian Product
  - $\{(u, v) : u \in X \wedge v \in Y\}$
- Power Set
  - $Pow(X) \equiv \{u : u \subseteq X\}$
  - $|Pow(x)| = 2^{|x|}$

Graphs

- A directed graph is an order pair (V, E) of sets
  - $E \subseteq \{(v, w) : v, w \in V \wedge v \neq w\}$
- A undirected graph is an order pair (V, E) of sets
  - $E \subseteq \{\{v, w\} : v, w \in V \wedge v \neq w\}$
- Maximum number of edges in an undirected graph with n vertices
  - $\frac{(|v|)(|v|-1)}{2}$

- Maximum number of edges in an directed graph with n vertices
  - $(|v|)(|v|-1)$
- Degree of a vertex (numbers of edges connected)
  - undirected graph
    - $deg(v) = |\{w: \{v, w\} \in E\}|$
  - directed graph
    - $indeg(v) = |(w: (w, v) \in E\}|$

    - $outdeg(v) = |(w: (v, w) \in E\}|$
    - $deg(v) = indeg(v) + outdeg(v)$
- Path in a graph
  - Directed
    - $(v_i, v_{i+1}) \in E$
  - Undirected
    - $\{v_i, v_{i+1}\} \in E$
  - The length of a path is the number of edges
    - if no vertices are repeated it is a simple path
- A cycle is a path where its length is its number of vertices, and $v_0 = v_k$
  - if the path defined by the cycle is a simple path, it is a simple cycle
- Hamiltonian
  - A path is Hamiltonian if it includes every vertex exactly once
  - A cycle is Hamiltonian if it includes every vertex exactly once
- Eulerian
  - A path is Eulerian if it includes every edge exactly once
  - A cycle is Eulerian if it includes every edge exactly once
- An undirected graph $G = (V, E)$ is connected if for all $x, y \in V$ there is a path in G from x to y
- Let $G = (V, E)$ be a connected undirected graph. If every vertex in V has even degree then G has an Eulerian cycle.

Proofs

- Two Essential Features
  - It must be finitely long so you can give it to someone else
  - It must be possible for someone else to check the proof for correctness without needing brilliant flashes of insight
- Direct Proof
  - State the statement you want to prove.
  - Proof.
    - Each line should be a statement or declaration.
    - Each statement should be true.
    - The truth of each statement should be clear from a definition, axiom, lemma, theorem, the previous line, or some combination.
    - Provide a short and precise explanation/justification when needed.
    - Conclusion should mirror the statement we are proving.
- Proof By Induction
  - State the statement you want to prove, typically a statement like ∀n ∈ N, f(n).
  - Proof by Induction.
    - *Base Case*:
      - Compute f(n) directly for the base case, the smallest value of n, typically n = 0 or n = 1.
    - Induction Step:
      - Assume the statement is true for an arbitrary value n.
      - Prove the statement is true for n + 1. In other words, we prove here that f(n) ⇒ f(n + 1).
      - Conclusion: State what you have proven