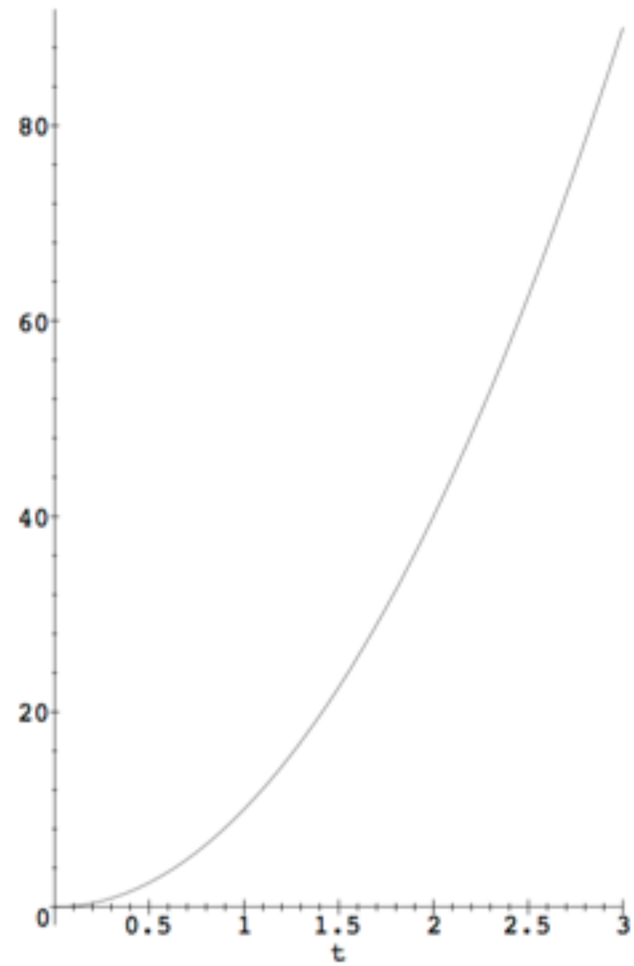
An Introduction to Derivatives

(Source: Most of this document was borrowed from Neal brand at Univ of North Texas)

The purpose of this project is to give you an introduction to the ideas that we will be using in calculus. Two of the main concepts we will be learning this semester are the concept of a limit and the the concept of the derivative. This project will give you an introduction to both.

Old Stuff A car at an intersection sits at a stop light. Once the light turns green, it accelerates away from the intersection. Assume that t seconds after the light turns green, the car is a distance of $f(t) = 10\,t^2$ meters away from the intersection. Then we can graph the distance from the intersection as a function of time. Refer to Figure 1 shown here.



How far is the car from the intersection after 2 seconds? After 3 seconds? 2.5 seconds? 2.1 seconds?

New Idea Let s now try to figure out how fast the car is moving after t seconds Notice that we can’t use the venerable distance rate time formula because the speed of the car is changing with time However, we can estimate the speed at t = 2 seconds by dividing the distance it travels between 2 and 3 seconds by the time elapsed between 2 and 3 seconds. Here’s the formula that we discussed in class:

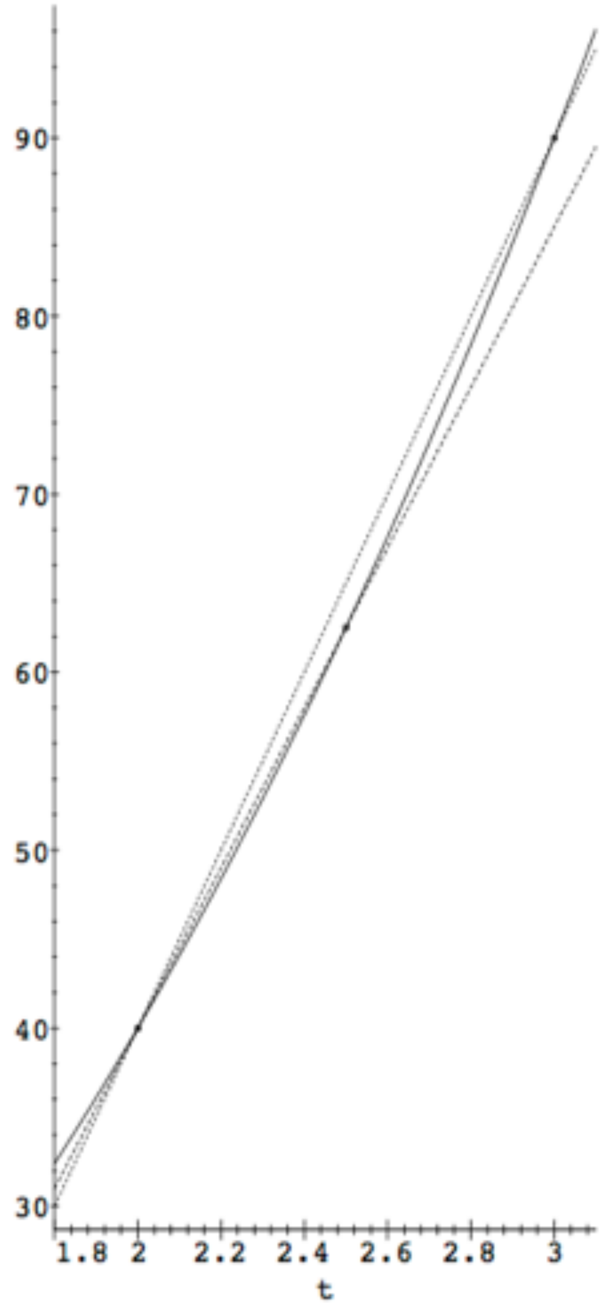
$$\begin{aligned}\text{Approx. Speed} &= \frac{f(3)-f(2)}{3\text{ s}-2\text{ s}}\\ &= \frac{90\text{ m}-40\text{ m}}{3\text{ s}-2\text{ s}}\\ &= 50\text{ m/s}.\end{aligned}$$

Geometrically, this ratio (50 m/s) corresponds to the slope of the line which conects the point (2,40) with the point (3,90). This line is called a secant of the curve. (This definition makes sense: recall from geometry that a secant of a circle is a line which intersect the circle at two points.) From the point-slope form of a line, this secant line has the equation:

$$y = 40 + 50\,(t - 2)$$

The approximate speed calculated in this manner is only an approimation, since the car does not maintain a constant speed between t = 2 seconds and t = 3 seconds. However, it is obviously clear that we will get a much better estimate of the speed by letting the time interval become smaller. The speed of the car will not change as much over a smaller interval. If we repeat the above calculations, but now we use the points (2.5,62.5) and (2,40), we get an approximate speed of 45 m/s. Corresponding secant line has the equation:

$$y = 40 + 45\,(t - 2)$$



Task #1. The second approximation (45 m/s) is better than the first (50 m/s) since the time interval is smaller (a half second compared to one second). Use the variable ***h*** to represent this time interval. Make a spreadsheet to create a table. Fill in the following table for smaller and smaller h to obtain better and better approximations for the speed of the car two seconds after the stoplight turns green. What is an approximate speed when h is only 0.0001 seconds?

Determine the equations for the four new secant lines represented by h = 0.2, 0.1, 0.01, 0.0001 seconds. Graph these secant lines and the graph of $f(t) = 10\,t^2$; i.e., draw one figure that contains the four new secant lines. Draw this figure using your favorite graphing software. Convince yourself that the slopes of these secant lines approach the slope of a line which is tangent to $y = 10\,t^2$ at (2,40).

Task #2. This part of your project is your responsibility. How do you know if your results from Task #1 are correct? Certainly, your secant lines are converging to some line; but which line are the secants converging to? How do you know exactly which line is the finally tangent line? What is the equation of this final tangent line?

<u>Task #3</u>. In Task #1, we calculated the speed of the car two seconds after the light turned green Now repeat Tasks 1 and 2 to find the speed of the car 1 second after the stoplight turns green. Repeat for t = 3 seconds. Repeat for t = 4 seconds. (You should have four sets of graphs.)

<u>Task #4</u>. Can you come up with a formula that gives you the exact speed of the car for any value of t? Prove that your formula works.

<u>Task #5</u>. Now let's see if our logic holds for a more complicated function: f(t) = sin (t). Repeat Tasks #1 and #2 for t = pi/3. Then complete Task #4 using this f(t) = sin (t).

Your final report should be professionally formatted. The graphs should all be constructed using graphing software; all figures should be labeled, and all equations whould be typed using a text editor. I will show you how to use these tools in class; pay attention!