High Speed Video Analysis Finding Forces on Bullets Using Difference Ratio Definitions of Derivatives

GROUP STRUCTURE:

You need to work as a group to accomplish this assignment. Your group is to be made up of two to four (2-3) students from your section. Select a *Group Leader* as soon as your group is formed. This person will be responsible for overseeing the project (schedule, tasks, etc.), leading the group, resolving issues, and interfacing with your instructor. Members will assist the leader (and the group) by following the leader's direction, offering counsel on the leader's decisions, and completing assigned tasks on schedule. Beware, without strong coordination, the "divide and conquer" method generally produces a substandard final product.

GRADING:

The group project point breakdown is as follows: 15 points for part 1, 15 points for part 2 and 50 points for part 3. Unless you are working with the Math Resource Center on a special project, your instructor will grade your project. In totaling your scores, your instructor will give weight to these areas: mathematical accuracy/adequacy; quality of writing; and formatting / presentation. Except for teams that have clearly had an uneven distribution of work, all members will receive the same grade for the project. Your instructor will decide what constitutes an uneven distribution of work on a case-by-case basis with input from all team members.

INSTRUCTIONS:

Project Part 1: Students should select project partners to form groups of two or three. Do not use the "divide and conquer" approach to solving the problem and preparing the analysis, figures, and report. Each member of the project group should be a full participant and make significant intellectual contributions to each facet of solving the problem and preparing the report. Since the analysis is greatly facilitated by using a spreadsheet, make sure at least one group member has some experience with spreadsheets.

Once your group is established inform your instructor. Your instructor will then give you a video to analyze. DO NOT analyze a video until you coordinate with your instructor. Videos are located in this link: *https://github.com/ke5urh/my_math_artifacts*

The challenge is to conduct an analysis of one (of four available) high speed videos of a bullet penetrating ballistic gelatin in order to use some elementary Physics formulas along with some Calculus concepts to determine the force (F) vs. penetration depth (x) curves by two different methods. Then you will confirm that the force-curves obtained with the different methods agree with each other. The needed background is:

> x = the bullet penetration depth in feet
>
> v = the bullet velocity in feet per second (fps or ft/s) which is also $v = dx/dt = x'(t)$
>
> a = the bullet acceleration in ft/s^2 which is also $a = d^2x/dt^2 = dv/dt = x''(t)$
>
> Newton's Second Law: Force = F = ma, where m is the mass in slugs. (A google search and some unit conversions might be necessary to convert the bullet weight in grains to the mass in slugs.)
>
> Kinetic energy = $E = \frac{1}{2} mv^2$.

By the Work-Energy Theorem, the force F can also be shown to be F = dE/dx.

That's right, the force is equal to the mass times the second derivative of the position with respect to time, but it is also equal to the rate of change of kinetic energy with respect to position!

The raw videos were obtained at a rate of 20,000 frames per second (also fps, so be careful with units and context). This frame rate and the small exposure times (microseconds) are fast enough to catch a bullet in flight and as it slows down due to the forces of interacting with ballistic gelatin. Ballistic gelatin is designed to closely simulate living muscle tissue. As every digital image is in pixels, one can use the pixel number of the leading edge of the bullet in each frame to determine pixel location vs. time. Then one can use scaling factors to convert pixels to inches and then compute the position of the bullet in feet as a function of time. This background work has already been done and the position of the bullet in feet has been edited into the videos.

Figure 1: Example image from video frame showing bullet at a position of x = 0.492 ft at a time of 1.112350 seconds.

The problem is best solved creating columns of data for each variable. A spreadsheet is particularly convenient for automating computations and organizing the data. The first three columns of data might be frame number, time, and bullet position. These can be generated by setting the video player on slow playback and using the start and pause features to determine the position vs. time data for the bullet in each high speed video frame.

The next task is to compute the velocity dx/dt. This derivative can be estimated as the change in x (dx) over the change in t (dt) as the change in time becomes very small. Hence the value of high speed video! The change in time in the video is 1/20000 seconds, so while not truly instantaneous, it is small enough for reasonable estimates. You may wish to compute separate dx and dt columns before computing the velocity difference ratio, dx/dt. Be sure to clearly label each column heading and to include units. You should also probably create a new time column with the time shifted so that the bullet impacts the gelatin at t = 0 s (when x = 0 ft). Subsequent analysis will be easier if the time refers to the impact time rather than the time the camera was started. You should now have columns of data for v (ft/s) and t (s). You should plot v vs. t and carefully consider the issues involved in computing the acceleration a = dv/dt.

Computing the second derivative directly from the data for x'(t) would be problematic. Due to discretization error, optical effects, rounding errors, and other experimental realities, the velocity is not monotonically decreasing, as we know it should. Think about this for a minute. A bullet penetrating ballistic gelatin can only slow down. It cannot speed up. It is like a car that has slammed on the brakes and is skidding to a stop. The velocity can only decease, it cannot increase. Therefore, the column of velocity data is imperfect. The task now is to create a mathematical model informed by the "noisy" velocity data but that has the expected behaviors: 1) Having the real impact velocity at $t = 0$ (the velocity listed in the file name was measured with a chronograph much more accurately than it can be determined from video). 2) Decreasing monotonically from the impact velocity 3) Going to zero at about the right time as one might reasonably infer from the graph and physical considerations. The model need not make sense for times prior to impact or for times well after the video, because it should only be used to estimate the velocity, acceleration, energy, and forces between the time of impact and the last frame in the video where the position is noted.

The trendline facility of Excel can be used to develop a suitable mathematical model for the velocity vs. time. (Alternatively, Graph.exe may be used for former Math 130 students who have the software and are capable of using it.) None of the trendline options are perfectly suitable, but the a polynomial model can be used if ample care is exercised. First, you want to be sure to display the trendline equation and R-squared on the graph. Second, consider whether you should set the intercept to a given value. Does the vertical intercept have a specific meaning so you know what it is before Excel computes the trendline? Third, since there is nothing forcing the trendline never to increase, you need to try different "order" polynomials until you find one with a pretty good R-squared (close to 1) but that does not ever increase. Fourth, you might consider adding some extra data points to the velocity vs. time data to represent some later time at which the velocity will be very close to zero. Getting the trendline right is a substantial sub-project in itself and will have a strong impact on downstream calculations of $x''(t)$, the force, and the energy, so take some care. Consider the physical meaning of the trendline that results and how to move forward.

Once you have an analytical model for $v(t)$, you could proceed either analytically (pencil and paper, taking derivatives of $v(t)$ and using equations and formulas) or you could continue pursuing a numerical approach in Excel. This is an important fork in the road in the project. Some of the skills to move forward analytically might not be covered until after Part 1 is due, so it might be better to keep things numerical for now and close the loop for Part 2 with an analytical explanation why $mx''(t) = dE/dx$. So, you'll need to create a column of values for the model values of $v(t)$ from the trendline. If you cut and paste the trendline equation into the cells for the $v(t)$ column, take care to reformat the trendline equation so you have at least 6 significant digits in the polynomial coefficients, so you don't propagate errors due to rounding.

Once you have a $v(t)$ column from the trendline model, you can easily compute a column of values for $a = x''(t)$ from the difference ratio formula for derivatives. You can then make another column for $F = mx''(t)$ (force in pounds) by multiplying the acceleration (in ft/s^2) by the bullet mass (in slugs).

Now, you need a column for the kinetic energy, $E = \frac{1}{2}mv^2$, at each point in time. Following this, you should compute a column for the force, $F = dE/dx$, using the difference ratio formula for derivatives.

The force can either be considered a function of time, $F(t)$ or a function of penetration depth $F(x)$. But since you have computed it two ways, both should agree. Graph them both (pick an input variable column and stick with it) and see if they agree. You might consider if you need to make a model $x(t)$ column by starting at zero and adding increments of $v(t)$ times the time interval $dt = 0.0005$ s to help smooth out the experimental errors in the original $x(t)$ column. In other words, it might make more sense to consider $dx = v(t)\,dt$, where $v(t)$ is the smooth model (trendline) velocity. This will probably bring the two force curves into much closer agreement compared with using dx as the difference in the originally measured x values.

Part 1 deliverables include a big table of data (from the Excel spreadsheet) with one row for every frame in the video with an x value as the bullet penetrates. There should be columns for: frame number, t (s), x (ft), dx (ft), dt (s), measured dx/dt (ft/s), model dx/dt (ft/s), a = d^2x/dt^2 (ft/s^2), E (ft lbs), model dx (ft), dE/dx (lbs), and F = mx'' (lbs). Your deliverables also include a graph of v(t) vs. t with the best fit trendline and a graph of the two different forces either plotted against time or against penetration depth. Labels should be legible, complete, and there should be a brief caption.

A detailed description of the method or discussion is not required for Part 1, but there should be complete documentation for sources of help and use of resources.

Project Part 2*:* Project Part 2 must be a complete intermediate draft, including abstract, introduction, methods, results, discussion, and reference sections, and incorporating any feedback from Part 1. The abstract is a one paragraph summary of the project's key findings. The introduction gives key background to establish the importance of the work and the context of prior work from a thorough literature search. Most sources from a "Google" search are not reliable, but searches on "Google Scholar" are usually better. "Wikipedia" articles are usually not reliable sources, but many Wikipedia articles contain impressive reference lists which contain reliable sources.

The method section should describe the materials, methods, and analysis process in sufficient detail that the results could be reproduced by an independent party with knowledge of Calculus and Physics, but no inside information about what was done in the project.

The results section should present the figures and tables and might make cursory observations such as the peak force, the interaction time, etc. There should be at least one paragraph for each figure and table, and each figure and table should have a caption. Also take care that formatting of tables and figures are consistent and that font sizes are legible when the figures are imported into a word document.

The discussion section should include analysis that interprets and explains the intriguing result that dE/dx = m d^2x/dt^2. We know they should be equal to each other since Newton's second law says F = ma, and the Work-Energy theorem says F = dE/dx. But you should explain why there is a mathematical tautology behind these physical results. Hint: Express E in terms of dx/dt and use the chain rule to compute dE/dx. Simplify.

Quality intermediate drafts are well-written, well-researched, and present well-considered analysis. All group members must carefully proof-read and have their feedback incorporated into the intermediate draft before it is submitted for a grade. Each group member must ensure that any works consulted are cited in the bibliography and all help received is properly acknowledged. Miscommunication is no excuse for failing to cite all sources, and each member is accountable for the whether the product that is turned in for a grade meets the DFMS documentation requirements. Failing to meet documentation requirements carries strict academic penalties and may be referred to the student honor system.

Project Part 3*:* Project Part 3 must be a complete final report, including abstract, introduction, method, results, discussion, and reference sections, and incorporating any feedback from Part 1 and Part 2. Quality final reports are well-written, well-researched, and present well-considered analysis. All group members must carefully proof-read and have their feedback incorporated into the final report before it is submitted for a grade. Each group member must ensure that any works consulted are cited in the bibliography and all help received is properly acknowledged. Miscommunication is no excuse for failing to cite all sources, and each member is accountable for the whether the product that is turned in for a grade meets the university's documentation requirements.