1. Proposed thermal blooming term $\Delta n_{thermal}$

$$\frac{\partial A_\ell(r)}{\partial z} = \frac{i}{2k^\ell}\Delta_\perp A_\ell(r) + ik_0^\ell \delta n_{\text{turb}}(r) A_\ell(r) - \frac{\alpha_{\text{loss}}^\ell}{2} A_\ell(r)$$

$$\downarrow$$

$$\frac{\partial A_\ell(r)}{\partial z} = \frac{i}{2k^\ell}\Delta_\perp A_\ell(r) + ik_0^\ell (\delta n_{\text{turb}}(r) + \Delta n_{\text{thermal}}) A_\ell(r) - \frac{\alpha_{\text{loss}}^\ell}{2} A_\ell(r)$$

$$\Delta n_{thermal} = \text{index}(T)$$

$$\text{index}(T) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial^k n}{\partial T^k}\right)_{T_0} (T - T_0)^k$$

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + S(r,t)$$

$$S(r,t) = \frac{\alpha I(r,t)}{\rho c_\rho}$$

$$I(r,t) = I_0 \exp\left(-\frac{2r^2}{w^2} - \frac{(t-t_0)^2}{\tau^2}\right)$$

t : time

t_0 : starting time

r : radial distance from the beam's center

n : refractivity

T : temperature

T_0 : Reference temperature of the atmosphere at the ground level

$\frac{\partial^k n}{\partial T^k}$: kth derivative of the refractive index with respect to temperature

$\frac{\partial T}{\partial t}$: Temperature change per unit time

α : absorptivity (absorption per unit length)

κ : thermal Diffusivity of the medium

I : laser irradiance

$\alpha I(r,t)$: power absorbed by the medium per unit volume

$S(r,t)$: heat source term, rate at which the laser's energy is absorbed and converted into heat

ρ : mass density of the medium

c_ρ : specific heat at constant pressure

w : laser beam waist radius

τ : pulse duration

I_0 : peak intensity at the beam's center

This proposed model incorporates a thermal blooming effect, and a "Feedback" mechanism the models how changes in temperature overtime worsens the index of refraction.

2. Simplified SOE

$$\frac{\partial A_\ell(r)}{\partial z} = \frac{i}{2k^\ell}\Delta_\perp A_\ell(r) + ik_0^\ell\left(\delta n_{\text{turb}}(r) + \left(\frac{\partial n}{\partial T}\right)_{T_0}\Delta T\right)A_\ell(r) - \frac{\alpha_{\text{loss}}^\ell}{2}A_\ell(r)$$

Let $A_\ell(r) = a(r)\exp\left(-\frac{\alpha_{\text{loss}}^\ell}{2}z\right)$, then $\frac{\partial A_\ell(r)}{\partial z} = \frac{\partial a(r)}{\partial z}\exp\left(-\frac{\alpha_{\text{loss}}^\ell}{2}z\right) - \frac{\alpha_{\text{loss}}^\ell}{2}a(r)\exp\left(-\frac{\alpha_{\text{loss}}^\ell}{2}z\right)$

$$\frac{\partial a(r)}{\partial z} = \frac{i}{2k^\ell}\Delta_\perp a(r) + ik_0^\ell\left(\delta n_{\text{turb}}(r) + \left(\frac{\partial n}{\partial T}\right)_{T_0}\Delta T\right)a(r)$$

$$\frac{\partial T}{\partial t} = \kappa\nabla^2 T + \frac{\alpha}{\rho c_\rho}I(r,t)$$

*** Arguments to justify that $\nabla^2 T = T_{xx} + T_{yy} + T_{zz} \approx \nabla_\perp^2 T = T_{xx} + T_{yy}$
*** $S + T_0 = T$ into equation 2 we get

$$\begin{aligned}
\frac{\partial a(r)}{\partial z} &= \frac{i}{2k^\ell}\Delta_\perp a(r) + ik_0^\ell\left(\delta n_{\text{turb}}(r) + \left(\frac{\partial n}{\partial T}\right)_{T_0}S\right)a(r) \\
\frac{\partial S}{\partial t} &= \kappa\nabla_\perp^2 S + \frac{\alpha}{\rho c_\rho}|a(r)|^2\exp\left(-\alpha_{\text{loss}}^\ell z\right)
\end{aligned}$$

The equation for S is only parametrically dependent on z. Taking the represenation for $a(r) = a(r_\perp, z)$ as

$$a\big(\mathbf{r}_\perp, \mathbf{p}(z)\big) = I\big(\mathbf{p}(z)\big)e^{-\left(\Theta\left(\mathbf{r}_\perp,\mathbf{p}(z)\right) + i\Phi\left(\mathbf{r}_\perp,\mathbf{p}(z)\right)\right)},$$

with

$$\begin{aligned}
I\big(\mathbf{p}(z)\big) &= \frac{C(z)\sqrt{W_x(z)W_y(z)}}{\sqrt{\pi}}, \\
\Theta\big(\mathbf{r}_\perp, \mathbf{p}(z)\big) &= \frac{1}{2}\left(W_x^2(z)\big(x - X(z)\big)^2 + W_y^2(z)\big(y - Y(z)\big)^2\right), \\
\Phi\big(\mathbf{r}_\perp, \mathbf{p}(z)\big) &= P(z) + T_x(z)\big(x - X(z)\big) + T_y(z)\big(y - Y(z)\big) + \\
&\quad F_x(z)\big(x - X(z)\big)^2 + F_y(z)\big(y - Y(z)\big)^2, \text{ and} \\
\mathbf{p}(z) &= \big[C(z)\ W_x(z)\ W_y(z)\ T_x(z)\ T_y(z)\ X(z)\ Y(z) \\
&\quad F_x(z)\ F_y(z)\ P(z)\big]^{\mathrm{T}}\ .
\end{aligned}$$

We can use the Lagrangian approach sub the ODES for the PDE in a.

$$\frac{\partial S}{\partial t} = \kappa\nabla_\perp^2 S + \frac{\alpha}{\rho c_\rho}C_0^2\frac{W_x(z)W_y(z)}{\pi}e^{-\left(W_x^2(z)\left(x - X(z)\right)^2 + W_y^2(z)\left(y - Y(z)\right)^2\right)}\exp\left(-\alpha_{\text{loss}}^\ell z\right)$$

$$
\begin{aligned}
C(z) &= C(0)\\
\frac{dW_x}{dz}(z) &= \frac{2}{n\xi}F_x(z)W_x(z)\\
\frac{dW_y}{dz}(z) &= \frac{2}{n\xi}F_y(z)W_y(z)\\
\frac{dT_x}{dz}(z) &= -n\gamma^2\left\langle \delta n_{\text{turb}}(r) + \left(\frac{\partial n}{\partial T}\right)_{T_0} S, M_{T_x}(\mathbf{r})\right\rangle\\
\frac{dT_y}{dz}(z) &= -n\gamma^2\left\langle \delta n_{\text{turb}}(r) + \left(\frac{\partial n}{\partial T}\right)_{T_0} S, M_{T_y}(\mathbf{r})\right\rangle\\
\frac{dX}{dz}(z) &= -2T_x(z)\\
\frac{dY}{dz}(z) &= -2T_y(z)\\
\frac{dF_x}{dz}(z) &= -\frac{W_x^4(z)}{2n\xi} + \frac{2F_x^2(z)}{n\xi} + \frac{\gamma^2}{\xi}\left\langle \delta n_{\text{turb}}(r) + \left(\frac{\partial n}{\partial T}\right)_{T_0} S, M_{F_x}(\mathbf{r})\right\rangle\\
\frac{dF_y}{dz}(z) &= -\frac{W_y^4(z)}{2n\xi} + \frac{2F_y^2(z)}{n\xi} + \frac{\gamma^2}{\xi}\left\langle \delta n_{\text{turb}}(r) + \left(\frac{\partial n}{\partial T}\right)_{T_0} S, M_{F_y}(\mathbf{r})\right\rangle\\
\frac{dP}{dz}(z) &= \frac{\left(W_x^2(z) + W_y^2(z)\right)}{2n\xi} - \frac{\gamma^2}{\xi}\left\langle \delta n_{\text{turb}}(r) + \left(\frac{\partial n}{\partial T}\right)_{T_0} S, M_P(\mathbf{r})\right\rangle
\end{aligned}
$$

Generlizing the equation for S:

$$
\begin{aligned}
S_t &= \kappa\nabla_\perp^2 S + F(r_\perp; z)\\
S(r_\perp, t) &= 0 \quad \text{for} \quad |r_\perp| >> l_0\\
F(r_\perp; z) &= \frac{\alpha}{\rho c_\rho} C_0^2 \frac{W_x(z)W_y(z)}{\pi} e^{-\left(W_x^2(z)\left(x - X(z)\right)^2 + W_y^2(z)\left(y - Y(z)\right)^2\right)} \exp\left(-\alpha_{\text{loss}}^{\ell} z\right)
\end{aligned}
$$

Assume that S has reached a steady state, i.e. $S_t = 0$ than

$$
\begin{aligned}
\kappa\nabla_\perp^2 S &= -F(r_\perp; z)\\
S(r_\perp, t) &= 0 \quad \text{for} \quad |r_\perp| >> l_0
\end{aligned}
$$

The solution to this Poission Equation is given by the 2 D Green's function

$$
S(r_\perp; z) = \frac{1}{\kappa}\iint \frac{1}{2\pi}\ln\left(|r_\perp - r'_\perp|\right) F(r'_\perp; z) dr'_\perp
$$