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NUMERICAL SIMULATION OF STATIONARY AND NON-STATIONARY GAUSSIAN RANDOM PROCESSES*

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1. Introduction. If r is a random variable, ranging over some measure space, the collection of functions of time $x(t) = x_r(t)$ is called a *random process*. The process is called *Gaussian* if, for every finite collection of times $t_1 < \cdots < t_n$, the random variables $x(t_1), \cdots, x(t_n)$ have a multivariate Gaussian distribution. The process is called *stationary* if, for any fixed increment Δt, the random variables $x(t_i + \Delta t)$ have the same joint distribution as the random variables $x(t_i)$.

A linear transformation of a Gaussian random process is another Gaussian random process. The theory of linear transformations of Gaussian random processes is very well developed; see, for example, Doob [1]. But there is not an equally good theory for nonlinear transformations. For example, if $x(t)$ is a Gaussian random process with mean zero and given autocorrelation, as defined in §2, there is no known formula for the distribution or for the autocorrelation of the random process $y(t)$ defined by an equation $N[y(t)] = x(t)$, where N is a given nonlinear differential operator. The best available way to study the process $y(t)$ is to compute samples of it. First we need a method for computing numerical samples of the Gaussian process $x(t)$ with given autocorrelation. If each computed sample of $x(t)$ is bounded and smooth, as it should be if its given autocorrelation is bounded, then samples of $y(t)$ may be computed from the differential equation $N[y(t)] = x(t)$ by any standard numerical technique, such as Adams' method or the Runge-Kutta method; see Hildebrand [2].

The purpose of this paper is to present numerical methods for the computation of samples of a Gaussian random process $x(t)$ with prescribed autocorrelation or power spectral density.

2. Gaussian processes and linear filters. In this section we shall summarize the results which we shall need from the theory of linear transformations of Gaussian processes. Throughout the paper we shall assume that $Ex(t) = 0$, where E denotes the expected value operator.

The *autocorrelation* function is defined as

$$R(t_1, t_2) = E(x(t_1)x(t_2)). \tag{2.1}$$

If $x(t)$ is a stationary process, we write

$$R(\tau) = E(x(t)x(t - \tau)). \tag{2.2}$$

For stationary processes, we also define the *power spectral density*:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\, d\tau. \tag{2.3}$$

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In texts such as Davenport and Root [3] it is shown that $S(\omega)\,d\omega$ may be thought of as the average contribution of the frequency interval $d\omega$ to the power of a sample function $x(t)$. Since, in general, the Fourier transform (2.3) can be inverted, a knowledge of the power spectral density $S(\omega)$ is equivalent to a knowledge of the autocorrelation $R(\tau)$. As it is stressed by Blackman and Tukey [4], it is usually more natural to work with $S(\omega)$.

A linear processor with time-invariant elements may be described mathematically as a linear transformation

$$x(t) = \int_{-\infty}^{t} g(t - s)w(s)\,ds. \tag{2.4}$$

This transformation produces an output $x(t)$ depending on the input process $w(s)$ for times $s \leqq t$. Setting w equal to the Dirac delta function, we see that $g(t)$ is the response of the filter to a delta function input. Thus, $g(t) = 0$ for $t < 0$. As it is shown in [3], the power spectral densities $S_w(\omega)$ and $S_x(\omega)$ of the input and output are related by the equation

$$S_x(\omega) = |\, G(\omega)\,|^2 S_w(\omega), \tag{2.5}$$

where $e^{i\omega t}G(\omega)$ is the response of the filter to the input $e^{i\omega t}$; equivalently, $G(\omega)$ is the Fourier transform of $g(t)$.

Formula (2.5) suggests a method for creating a signal x with prescribed power spectral density $S(\omega) = S_x(\omega)$. We pick a transfer-function $G(\omega)$ satisfying $|\, G(\omega)\,|^2 = S(\omega)$, and we choose the random signal w to have spectral density $S_w(\omega) \equiv 1$.

The condition $S_w(\omega) \equiv 1$ defines an idealized random process $w(t)$ known as *white noise*. From (2.3) we see that $R_w(\tau) = \delta(\tau)$. In particular, the variance $Ew^2(t)$ is $\delta(0) = +\infty$ for all time. Although white noise is convenient for theoretical discussion, it is not suitable for direct input in digital computation.

If we assume that $S(\omega)$ may be represented with sufficient accuracy by a rational function, i.e., a quotient of two polynomials in ω, then it is possible to find a transform which satisfies the condition $|\, G(\omega)\,|^2 = S(\omega)$. As it is shown in [3], the conditions

$$0 \leqq S(\omega) < \infty, \qquad S(\omega) = S(-\omega), \qquad S(\omega) \to 0 \text{ as } \omega \to \pm\infty \tag{2.6}$$

insure that the rational function $S(\omega)$ has a representation

$$S(\omega) = \left| \frac{P(i\omega)}{Q(i\omega)} \right|^2, \qquad \omega \text{ real.} \tag{2.7}$$

where $P(z)$ and $Q(z)$ are polynomials in z with real coefficients, where the degree of P is less than that of Q, and where the zeros of $Q(z)$ lie in the halfplane $\operatorname{Re} z < 0$. Thus we may choose

$$G(\omega) = \frac{P(i\omega)}{Q(i\omega)}$$

or, equivalently, if $D = d/dt$ = the differential operator,

$$x(t) = \frac{P(D)}{Q(D)} w(t). \tag{2.8}$$

The last equation is to be understood in the sense that we first solve the differential equation

$$Q(D)\phi(t) = w(t), \qquad -\infty < t < \infty, \tag{2.9}$$

for the steady-state solution $\phi(t)$, and then obtain the required signal

$$x(t) = P(D)\phi(t) \tag{2.10}$$

as a linear combination of derivatives of $\phi(t)$ of order lower than the degree of Q.

Further analysis is required for numerical computation because of the numerically unrealizable white noise $w(t)$ in the differential equation (2.9). We shall present in §6 a numerical method for solving the stochastic differential equation (2.9). The intervening sections summarize the necessary techniques and discuss the more general numerical simulation of nonstationary processes.

3. The Crout factorization. Let $P = (p_{ij})$, $i, j = 1, \cdots, n$, be a positive definite, real, symmetric matrix. It is shown in [2, Appendix A], or in Gantmacher [5], that P has a factorization

$$P = TT^*, \tag{3.1}$$

where T is lower-triangular, with positive elements on the main diagonal. If the existence of the factorization (3.1) is given, it is easy to see how to compute the components t_{ij} of T in the order

$$ij = 11, 21, \cdots, n1; 22, 32, \cdots, n2; \cdots; nn.$$

Since $t_{ij} = 0$ for $j > i$, (3.1) states that

$$p_{ij} = \sum_{k=1}^{j} t_{ik} t_{jk}\,. \tag{3.2}$$

First we compute

$$t_{11} = p_{11}^{1/2}. \tag{3.3}$$

The other elements in the first column are

$$t_{i1} = t_{11}^{-1} p_{i1}\,, \qquad i = 2, \cdots, n. \tag{3.4}$$

If the preceding columns $k < j$ have been computed, we compute the diagonal element

$$t_{jj} = \left(p_{jj} - \sum_{k=1}^{j-1} t_{jk}^2\right)^{1/2}. \tag{3.5}$$

If $j < n$, the elements below the diagonal are computed from the formula

$$t_{ij} = t_{jj}^{-1}\left(p_{ij} - \sum_{k=1}^{j-1} t_{ik}\, t_{jk}\right), \qquad i = j + 1, \cdots, n. \tag{3.6}$$

4. Completely equidistributed sequences. Suppose that we wish to simulate a

random sequence $x_1, x_2, \cdots$ of independent samples from the uniform distribution on the interval $0 \leqq x < 1$. Various methods of simulating such a sequence are discussed or referenced in [6]. For the purposes of this paper it is recommended that some transcendental number $\theta > 1$ be chosen, and that the fractional parts

$$x_n = \{\theta^n\}, \qquad n = 1, 2, \cdots, \tag{4.1}$$

be computed as accurately as possible. By fractional part we mean the part beyond the decimal, e.g., $\{3.14\} = .14$.

We have chosen (in binary notation)

$$\theta = 11.001001000011111101101010100010001. \tag{4.2}$$

This number represents $\theta = \pi$ to 35 bits. The powers $\theta^n (n = 1, 2, \cdots, 20000)$ were computed with no round-off error, i.e., θ^n was computed to $35n$ bits. Since this was a long computation, which would be expensive to repeat for individual applications, the numbers $x_n = \{\theta^n\}$, rounded to 35 bits, $n = 1, \cdots, 20000$, have been stored permanently on magnetic tape. By an abbreviated method of programming suggested by Mr. Robert Schneider, who wrote our present program, we plan later to compute and store 350,000 of these numbers. When several different sequences, $x_n^{(1)}, \cdots, x_n^{(s)} (n = 0, 1, 2, \cdots)$ are required for a single application, we use the interlaced sequences

$$x_n^{(1)} = x_{ns+1}, \qquad x_n^{(2)} = x_{ns+2}, \cdots, \qquad x_n^{(s)} = x_{ns+s}, \qquad n = 0, 1, \cdots.$$

The reasons for preferring this time-consuming method of generating the sequence x_n are discussed rigorously in [6]. As it is discussed in [6], almost every sequence of the form (4.1) has the property of *complete equidistribution*, which is defined as follows. A sequence x_n is *equidistributed* if, for any interval $a \leqq x < b$, where $0 \leqq a < b \leqq 1$, the number of $x_1, \cdots x_N$ lying in the interval, divided by N, approaches $(b - a)/1 = b - a$ as $N \to \infty$, i.e.,

$$\frac{1}{N} \sum_{\substack{a \leqq x_n < b \\ 1 \leqq n \leqq N}} 1 \to b - a \quad \text{as} \quad N \to \infty . \tag{4.3}$$

Although we should expect a truly random sequence to be equidistributed, not every equidistributed sequence looks like a random sequence. For example, as it follows from a general result of H. Weyl [7], the numbers $\{n\sqrt{2}/1000\}$ are equidistributed; but they do not look random because for more than 99% of all n (taken sequentially) we have $x_{n+1} > x_n$, whereas $x_{n+1} > x_n$ should occur 50% of the time. Therefore, we require a stronger condition than simple equidistribution (4.3).

Let k be any positive integer. We shall say that a sequence x_n is *equidistributed by k's* if for every set of k intervals (a_i, b_i), $i = 1, \cdots, k$, where $0 \leqq a_i < b_i \leqq 1$, we have

$$\frac{1}{N} \sum_{\substack{a_i \leqq x_{n+i} < b_i \\ (i=1,\cdots,k) \\ n=1,\cdots,N}} 1 \to \prod_{i=1}^{k} (b_i - a_i) \quad \text{as} \quad N \to \infty . \tag{4.4}$$

In other words, considering the sequence of successive groups of k numbers $x_{n+1}, x_{n+2}, \cdots, x_{n+k} (n = 1, 2, \cdots)$ to be successive points P_n in the space of k dimensions, we require that the points P_n be equidistributed in the k-dimensional unit hypercube. This requirement has many consequences. For example, if a sequence is equidistributed by twos ($k = 2$), then it is simply equidistributed (4.3) and $x_{n+1} > x_n$ 50% of the time. A sequence equidistributed by k's for all k is called *completely equidistributed.*

Let $x_n = \{\theta^n\}$. It is shown in [6] that, for almost all $\theta > 1$, for every pair of positive integers k and r the successive groups of r numbers $x_{nk}, x_{nk+1}, \cdots, x_{nk+r-1} (n = 1, 2, \cdots)$ are equidistributed in the unit hypercube of r dimensions. In particular, since we may set $k = r$ in the preceding sentence, almost all sequences $x_n = \{\theta^n\}$ with $\theta > 1$ are completely equidistributed. It is apparent that these sequences x_n have all of the properties commonly attributed to random sequences. In particular, for the purpose of spectral synthesis it is essential to know that completely equidistributed sequences x_n are *white sequences*, i.e., sequences for which

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(x_n - \frac{1}{2}\right)\left(x_{n+\tau} - \frac{1}{2}\right) = 0 \quad \text{if} \quad \tau > 0. \tag{4.5}$$

Given a sequence x_n of independent samples from the uniform distribution on $0 \leqq x < 1$, we can construct a sequence w_n of independent samples from the Gaussian distribution with mean 0 and variance 1. We use the method of Box and Muller. The sequence $w_1, w_2, w_3, \cdots$ is obtained from the formulas

$$w_{2n-1} = (-2 \ln x_{2n-1})^{1/2} \cos 2\pi x_{2n}, \qquad n = 1, 2, \cdots, \tag{4.6}$$

$$w_{2n} = (-2 \ln x_{2n-1})^{1/2} \sin 2\pi x_{2n}, \qquad n = 1, 2, \cdots.$$

In their paper [9] Box and Muller prove that the independence of the x_n implies the independence of the w_n.

Suppose that we wish to simulate a sequence of n-dimensional vectors $z^{(0)}, z^{(1)}, z^{(2)}, \cdots$, which are independent samples from the n-dimensional multivariate Gaussian distribution with positive definite moment matrix M. We can do so by means of a one-dimensional sequence $w_1, w_2, \cdots$ of the type discussed in the preceding paragraph. We first define vectors

$$w^{(0)} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \quad w^{(1)} = \begin{bmatrix} w_{n+1} \\ \vdots \\ w_{2n} \end{bmatrix}, \quad w^{(3)} = \begin{bmatrix} w_{2n+1} \\ \vdots \\ w_{3n} \end{bmatrix}, \cdots. \tag{4.7}$$

These vectors simulate samples from the n-dimensional Gaussian distribution with the identity moment matrix I. Let M be factored by the Crout factorization $M = TT^*$ discussed in §3. Then we define

$$z^{(i)} = Tw^{(i)}, \qquad i = 0, 1, 2, \cdots. \tag{4.8}$$

These vectors are the required vectors because $z = Tw$ comes from the multi-

variate Gaussian distribution with mean zero and moment matrix

$$[E(z_\alpha z_\beta)] = E(zz^*) = E(Tww^*T^*) = TT^* = M. \tag{4.9}$$

5. Simulation of nonstationary Gaussian random processes. Let $x(t)$ be a random process which is Gaussian with mean zero but which is not necessarily stationary. For all sampling times $t_1 < t_2 < \cdots < t_n$ the samples $x(t_\nu)$, $\nu = 1, \cdots, n$, come from the n-dimensional Gaussian distribution. The moment matrix of this distribution is

$$M = [E(x(t_\alpha)x(t_\beta))] = [R(t_\alpha, t_\beta)], \tag{5.1}$$

where $R(s, t)$ is the nonstationary autocorrelation.

We suppose that the function $R(s, t)$ is given. Let a finite collection of times $t_1 < t_2 < \cdots < t_n$ be given. We suppose that the matrix $[R(t_\alpha, t_\beta)]$ is symmetric and positive definite, i.e., we suppose that

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} R(t_\alpha, t_\beta)\lambda_\alpha \lambda_\beta = E\left(\sum_{\alpha=1}^{n} \lambda_\alpha x(t_\alpha)\right)^2 > 0 \tag{5.2}$$

unless all $\lambda_\alpha = 0$. The moment matrix (5.1) is now factored by the Crout factorization $M = TT^*$. Numbers $w_1, w_2, \cdots, w_n$ are generated simulating independent samples from the Gaussian distribution with mean 0 and variance 1; the generation of these numbers was discussed in §4. Then the required samples $x(t_\alpha)$ are given by the formula $x = Tw$ in an obvious vector notation. Explicitly,

$$\begin{aligned} x(t_1) &= t_{11}w_1, \\ x(t_2) &= t_{21}w_1 + t_{22}w_2, \\ &\cdots \\ x(t_n) &= t_{n1}w_1 + t_{n2}w_2 + \cdots + t_{nn}w_n. \end{aligned} \tag{5.3}$$

Some discussion of the assumption (5.2) is in order. Suppose this assumption fails. Then for some $\lambda_1, \cdots, \lambda_n$ not all 0, we have

$$E\left(\sum_{\alpha=1}^{n} \lambda_\alpha x(t_\alpha)\right)^2 = 0. \tag{5.4}$$

Without loss of generality, suppose $\lambda_n \neq 0$. Then

$$E\left(x(t_n) - \sum_{\alpha=1}^{n-1} \lambda_\alpha' x(t_\alpha)\right)^2 = 0 \tag{5.5}$$

with $\lambda_\alpha' = -\lambda_\alpha/\lambda_n$. In other words, $x(t_n)$ is redundant; $x(t_n)$ should not be sampled, but should be *calculated* as a linear combination $\sum \lambda_\alpha' x(t_\alpha)$ of the other samples. In general, if

$$\text{rank } [R(t_\alpha, t_\beta)] = r < n \tag{5.6}$$

and if columns $\beta_1, \cdots, \beta_r$ of the matrix $[R(t_\alpha, t_\beta)]$ are independent, then the random variables $x(t_\beta)$ for $\beta = \beta_1, \cdots, \beta_r$ have a positive definite moment

matrix. The variables $x(t_\beta)$ for $\beta \neq \beta_1, \cdots, \beta_r$ can be calculated as linear combinations of the $x(t_\beta)$ for $\beta = \beta_1, \cdots, \beta_r$.

6. Simulation of stationary Gaussian random processes. A stationary process is, of course, a particular type of nonstationary process. Therefore, we could apply the general method described in the last section to stationary processes. But a stationary process can be simulated in a more efficient manner, which does not become increasingly cumbersome as the number of samples to be computed increases.

In engineering and scientific problems we are usually given, not the autocorrelation $R(\tau)$, but the power spectral density $S(\omega)$, which is a given function of ω with the properties

$$S(\omega) \geqq 0, \qquad S(\omega) = S(-\omega), \qquad S(\omega) \to 0 \text{ as } \omega \to \infty. \tag{6.1}$$

We are also given a positive time increment Δt, which may or may not be small. If $x(t)$ is the corresponding stationary Gaussian random process, we wish to compute indefinitely many samples of $x(t)$ for $t = 0, \Delta t, 2\Delta t, \cdots$.

Assuming that $S(\omega)$ may be represented with sufficient accuracy by a rational function, we now use (2.7) through (2.10). Let (2.9) have the form

$$\phi^{(n)}(t) + a_1\phi^{(n-1)}(t) + \cdots + a_n\phi(t) = w(t), \qquad -\infty < t < \infty, \tag{6.2}$$

where $w(t)$ is white noise. In order to compute $x(t) = P(D)\phi(t)$, we shall require samples of the state-vector

$$z(t) = \begin{bmatrix} \phi(t) \\ \phi'(t) \\ \vdots \\ \phi^{n-1}(t) \end{bmatrix}, \qquad t = 0, \Delta t, 2\Delta t, \cdots. \tag{6.3}$$

The vector z satisfies the stochastic differential equation

$$\frac{dz(t)}{dt} = Az(t) + f(t), \qquad -\infty < t < \infty, \tag{6.4}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \qquad f(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ w(t) \end{bmatrix}. \tag{6.5}$$

First we shall compute $z(0)$. This vector is a Gaussian random vector with positive definite moment matrix

$$\begin{aligned} M &= E(z(0)z^*(0)) = E(z(t)z^*(t)), && \text{for any time } t, \\ &= [E(z_iz_j)] = [E\phi^{(i-1)}\phi^{(j-1)}], && i, j = 1, \cdots, n. \end{aligned} \tag{6.6}$$

The moment matrix M can be computed without a knowledge of the eigen-

values of A. A proof of the following two formulas appears in [8, Theorem 2]. The matrix M has components of the form

$$m_{ij} = \begin{cases} 0, & i + j \text{ odd} \\ (-1)^{(j-i)/2} m_{(j+i)/2}, & i + j \text{ even}, \end{cases} \tag{6.7}$$

where the numbers $m_0, m_1, \cdots, m_{n-1}$ can be computed by solving the n linear equations

$$(-1)^k \sum_{k/2 \leqq q \leqq (n+k)/2} (-1)^q a_{n-2q+k} m_q = \begin{cases} 0, & k = 0, \cdots, n-2, \\ \frac{1}{2}, & k = n-1, \end{cases} \tag{6.8}$$

where we define $a_0 = 1$.

Example. For $n = 6$, (6.7) and (6.8) become

$$[m_{ij}] = \begin{bmatrix} m_0 & 0 & -m_1 & 0 & m_2 & 0 \\ 0 & m_1 & 0 & -m_2 & 0 & m_3 \\ -m_1 & 0 & m_2 & 0 & -m_3 & 0 \\ 0 & -m_2 & 0 & m_3 & 0 & -m_4 \\ m_2 & 0 & -m_3 & 0 & m_4 & 0 \\ 0 & m_3 & 0 & -m_4 & 0 & m_5 \end{bmatrix},$$

where $m_0, \cdots, m_5$ are found by solving

$$\begin{bmatrix} a_6 & -a_4 & a_2 & -1 & 0 & 0 \\ 0 & a_5 & -a_3 & a_1 & 0 & 0 \\ 0 & -a_6 & a_4 & -a_2 & 1 & 0 \\ 0 & 0 & -a_5 & a_3 & -a_1 & 0 \\ 0 & 0 & a_6 & -a_4 & a_2 & -1 \\ 0 & 0 & 0 & a_5 & -a_3 & a_1 \end{bmatrix} \cdot \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

When M has been computed, it can be factored by the Crout factorization, $M = TT^*$. The required initial vector $z(0) = z^{(0)}$ can now be computed from the formula

$$z(0) = Tw^{(0)}, \tag{6.9}$$

which is found by setting $i = 0$ in (4.8).

Suppose $z(t)$ has been computed for any $t \geqq 0$. We shall now show how to compute the succeeding state-vector, $z(t + \Delta t)$. Let

$$X(t) = e^{At} = \exp At \tag{6.10}$$

be the solution of the initial-value problem

$$\begin{aligned} \frac{dX(t)}{dt} &= AX(t), & t > 0, \\ X(0) &= I. \end{aligned} \tag{6.11}$$

We shall need to know the value of the matrix $\exp At$ for the single value of

time $t = \Delta t$. If the eigenvalues of A are unknown, the constant matrix $\exp(A\Delta t)$ can be computed directly from the initial-value problem (6.11) by any standard technique such as the Runge-Kutta method.

Since z satisfies the inhomogeneous differential equation (6.4), we have

$$z(t + \Delta t) = e^{A\Delta t}z(t) + r, \tag{6.12}$$

where $r = r(t)$ is the Gaussian random vector

$$r = \int_0^{\Delta t} e^{(\Delta t - s)A} f(t + s)\, ds. \tag{6.13}$$

This vector is uncorrelated with the random vector

$$z(t) = \int_{-\infty}^{t} e^{A(t-\tau)} f(\tau)\, d\tau, \tag{6.14}$$

since

$$\begin{aligned} E(f(t + s)f^*(\tau)) &= C\delta(t + s - \tau) \\ &= 0 \text{ for } s > 0 \text{ and } \tau < t, \end{aligned} \tag{6.15}$$

where, by (6.5), C is the positive semidefinite matrix

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{6.16}$$

Since r is uncorrelated with $z(t)$, a sample of r may be computed independently of $z(t)$ in order to compute $z(t + \Delta t)$ from (6.4).

To compute a sample of r, we need to know the moment matrix

$$\begin{aligned} M_r &= E(rr^*) \\ &= E\int_0^{\Delta t}\int_0^{\Delta t} e^{(\Delta t - s_1)A} f(t + s_1) f^*(t + s_2) e^{(\Delta t - s_2)A^*}\, ds_1\, ds_2 \\ &= \int_0^{\Delta t} e^{sA} C e^{sA^*}\, ds\,. \end{aligned} \tag{6.17}$$

Let the integrand in (6.17) be called $J(s)$. Then

$$\frac{d}{ds} J(s) = AJ(s) + J(s)A^*. \tag{6.18}$$

Integration of (6.18) for $0 < s < \Delta t$ gives

$$e^{A\Delta t}C[e^{A\Delta t}]^* - C = AM_r + M_rA^*. \tag{6.19}$$

The eigenvalues of A are the zeros of the polynomial

$$Q(\zeta) = \zeta^n + a_1\zeta^{n-1} + a_2\zeta^{n-2} + \cdots + a_n\,. \tag{6.20}$$

By the choice of $Q(\zeta)$ in the representation (2.7), the zeros of $Q(\zeta)$ lie in the half-plane Re $\zeta < 0$. Therefore, the Ljapunov equation (6.19) has a unique solution M_r ; for a proof of the existence and uniqueness of the solution M_r see [5] or [8].

Note that the matrix M_r depends on Δt but not on t. Written out, (6.19) becomes n^2 linear inhomogeneous equations for the n^2 unknown components of M_r . If we define

$$\exp A\Delta t = [e_{ij}], \qquad M_r = [\mu_{ij}], \qquad i, j = 1, \cdots, n, \tag{6.21}$$

(6.19) becomes

$$\sum_{k=1}^{n} (a_{ik}\mu_{kj} + a_{jk}\mu_{ik}) = \begin{cases} e_{in}e_{jn} & \text{if } i < n \text{ or } j < n. \\ e_{nn}^2 - 1 & \text{if } i = n \text{ and } j = n. \end{cases} \tag{6.22}$$

Since the moment matrix M_r is symmetric, the number of equations can be reduced to $\frac{1}{2}n(n + 1)$. Let the right-hand side of (6.22) be called b_{ij} . Since $\mu_{ij} = \mu_{ji}$ and $b_{ij} = b_{ji}$, we may rewrite (6.22) in the form

$$\sum_{k=1}^{n} (a_{ik}\mu_{jk} + a_{jk}\mu_{ik}) = b_{ij}, \qquad i, j = 1, \cdots, n. \tag{6.23}$$

This equation is unchanged if i and j are interchanged. Therefore, it is sufficient to use (6.23) for $i \geqq j$. We then obtain $\frac{1}{2}n(n + 1)$ equations in the same number of unknowns:

$$\sum_{k \leqq j} a_{ik}\mu_{jk} + \sum_{k>j} a_{ik}\mu_{kj} + \sum_{k \leqq i} a_{jk}\mu_{ik} + \sum_{k>i} a_{jk}\mu_{ki} = b_{ij}, \qquad 1 \leqq j \leqq i \leqq n. \tag{6.24}$$

Having computed the matrix M_r from (6.24), we factor it by the Crout factorization $M_r = T_rT_r^*$. We are now ready to compute a sample for r. Let $t + \Delta t = \nu\Delta t$. Let $w^{(\nu)}$ be one of the vectors defined in (4.7). We then compute

$$r = T_r w^{(\nu)}, \qquad z(t + \Delta t) = e^{A\Delta t}z(t) + r. \tag{6.25}$$

This process may be repeated indefinitely for $t = 0, \Delta t, 2\Delta t, \cdots$. We recall that the initial vector $z(0)$ was computed in (6.9).

In the representation (2.7) let the polynomial P have the form

$$P(\zeta) = b_0\zeta^m + b_1\zeta^{m-1} + \cdots + b_m, \qquad b_0 \neq 0, \quad m < n. \tag{6.26}$$

The required scalar function $x(t)$, with power spectral density $S(\omega)$, is computed as the linear combination

$$x(t) = b_0 z_{m+1}(t) + b_1 z_m(t) + \cdots + b_m z_1(t), \qquad t = 0, \Delta t, 2\Delta t, \cdots, \tag{6.27}$$

where $z_1(t), \cdots, z_n(t)$ are the components of the vector $z(t)$; (6.27) is equivalent to the relation $x(t) = P(D)\phi(t)$ applied to the definition (6.3) of $z(t)$.

7. A numerical example. Let the spectral density

$$S(\omega) = \frac{9\omega^2 + 1}{(\omega^2 - 5)^2 + 4\omega^2} \tag{7.1}$$

be prescribed. Let $x(t)$ be the corresponding Gaussian process. We wish to compute samples of $x(t)$ for $t = 0.0, 0.1, 0.2, 0.3, \cdots$.

We know that for real ω, $S(\omega)$ has a factorization

$$S(\omega) = \left| \frac{b_0(i\omega) + b_1}{(i\omega)^2 + a_1(i\omega) + a_2} \right|^2, \tag{7.2}$$

where the a's and b's are real, and where the zeros of $\zeta^2 + a_1\zeta + a_2$ lie in the halfplane Re $\zeta < 0$. Factorization of the numerator and of the denominator in (7.1) yields

$$b_0 = 3, \quad b_1 = 1; \qquad a_1 = 2, \quad a_2 = 5. \tag{7.3}$$

We are thus led to the differential equation

$$\ddot{\phi} + 2\dot{\phi} + 5\phi = w(t). \tag{7.4}$$

We require samples of the vector

$$z = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} \tag{7.5}$$

for $t = k(0.1)$. We shall then compute the samples $x(t)$ as linear combinations:

$$x(t) = 3\dot{\phi}(t) + \phi(t), \qquad t = k(0\,1). \tag{7.6}$$

First let us compute the moment matrix

$$M = E[z(0)z^*(0)]. \tag{7.7}$$

According to §6 we have

$$M = \begin{bmatrix} m_0 & 0 \\ 0 & m_1 \end{bmatrix},$$

where

$$\begin{bmatrix} a_2 & -1 \\ 0 & a_1 \end{bmatrix} \cdot \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}.$$

Therefore, since $a_1 = 2$ and $a_2 = 5$,

$$M = \begin{bmatrix} \frac{1}{20} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}. \tag{7.8}$$

The Crout factorization of M is $M = TT^*$, where

$$T = \begin{bmatrix} \frac{1}{10}\sqrt{5} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{7.9}$$

Then the initial vector $z(0)$ is computed as

$$z(0) = Tw^{(0)} = \begin{bmatrix} \frac{1}{10}\sqrt{5}\, w_1 \\ \frac{1}{2}w_2 \end{bmatrix}, \tag{7.10}$$

where $w_1, w_2, \cdots$ simulate independent samples from the Gaussian distribution with mean 0 and variance 1, as discussed in §4.

We now require the basic solution exp (At) for $t = \Delta t = 0.1$. We have

$$e^{At} = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) \end{bmatrix} = X(t), \tag{7.11}$$

where

$$\ddot{\phi}_j + 2\dot{\phi}_j + 5\phi_j = 0, \qquad j = 1, 2,$$
$$\phi_1(0) = 1, \quad \dot{\phi}_1(0) = 0; \qquad \phi_2(0) = 0, \quad \dot{\phi}_2(0) = 1.$$

In fact, these equations are equivalent to the initial-value problem

$$\dot{X} = AX = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \cdot X, \qquad X(0) = I. \tag{7.12}$$

The solution of (7.11) is

$$e^{At} = \begin{bmatrix} e^{-t}\cos 2t & \frac{1}{2}e^{-t}\sin 2t \\ -e^{-t}(\cos 2t + 2\sin 2t) & \frac{1}{2}e^{-t}(2\cos 2t - \sin 2t) \end{bmatrix}. \tag{7.13}$$

For $t = 0.1$ computation gives

$$e^{A\Delta t} = \begin{bmatrix} 0.886804 & 0.089882 \\ -1.24633 & 0.796922 \end{bmatrix} = [e_{ij}]. \tag{7.14}$$

Then

$$[e_{i2}\, e_{j2}] - C = [e_{i2}\, e_{j2}] - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} .00808 & .07163 \\ .07163 & -.3649 \end{bmatrix}. \tag{7.15}$$

Let

$$M_n = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{7.16}$$

be the moment matrix defined by (6.17). Then (6.19) becomes

$$\begin{bmatrix} .00808 & .07163 \\ .07163 & -.3649 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix} \cdot \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix}. \tag{7.17}$$

This system yields three equations in three unknowns:

$$\begin{aligned} 2b &= .00808, \\ -5a - 2b + c &= .0716, \\ -10b - 4c &= -.3649. \end{aligned} \tag{7.18}$$

The solution of these equations is

$$a = .0003, \qquad b = .0041, \qquad c = .0811. \tag{7.19}$$

We then find the Crout factorization

$$M_r = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = T_r T_r^* = \begin{bmatrix} t_{11} & 0 \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} t_{11} & t_{21} \\ 0 & t_{22} \end{bmatrix}, \tag{7.20}$$

where

$$t_{11} = .017, \qquad t_{21} = .24, \qquad t_{22} = .16. \tag{7.21}$$

(Of course, automatic digital computation would conserve more accuracy). Formula (6.25) now yields

$$z(t_{k+1}) = e^{A\Delta t} z(t_k) + T_r w^{(k+1)}, \qquad k = 0, 1, \cdots, \tag{7.22}$$

or, by (7.14) and (7.21),

$$z(t_{k+1}) = \begin{bmatrix} .89 & .09 \\ -1.25 & .80 \end{bmatrix} z(t_k) + \begin{bmatrix} .017 & 0 \\ .24 & .16 \end{bmatrix} \begin{bmatrix} w_{2k+3} \\ w_{2k+4} \end{bmatrix}, \tag{7.23}$$

where $z(0) = z(t_0)$ is given by (7.10). It should be emphasized that, apart from round-off error, (7.23) is exact; there is no truncation error. Finally, we have the required samples

$$x(t_k) = z_1(t_k) + 2z_2(t_k), \qquad k = 0, 1, \cdots. \tag{7.24}$$

Except for the generation of the basic, independent, artificial Gaussian numbers w_k, the whole process is so simple that it could be carried out on a desk-calculator. The temptation would be to do so, where one would obtain the numbers w_k from any available table of random numbers. This course would be risky because the validity of the calculations depends upon the assumption that the w_k form a white sequence. If a table of w_k is used, the table should be constructed according to the principles discussed in [6].

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