Extruding Kolmogorov-type phase screen ribbons

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A phase screen ribbon extrusion process is presented that allows a phase screen ribbon of any specified width to be extruded, one column at a time, producing a ribbon of any desired length, with Kolmogorov statistics (i.e., having a five-thirds power-law-dependent structure function) for all separations up to some selected upper limit—which upper limit can be as large as desired. The method is an adaptation of the method described by [Assémat *et al.* Opt. Express **14**, 988 (2006)]. © 2008 Optical Society of America

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1. INTRODUCTION

A method of extruding a phase screen ribbon—i.e., a method by which one successively adds columns of values to an array of values—so as to allow production of an indefinitely long ribbon of values, with the values being randomly selected in such a manner as to have very nearly stationary statistics with a specified covariance between elements of the ribbon has been described by Assémat *et al.* [1]. They showed how this method could be used to produce a phase screen ribbon with statistics corresponding to turbulence with Von Kármán-type statistics.

Because of the divergent nature of the covariance for Kolmogorov-type turbulence the method presented by Assémat *et al.* does not seem to be directly applicable when it is such turbulence that is to be simulated. Neither does their method seem to be able to incorporate information about the value of the covariance for separations that are greater than the width of the ribbon. Moreover, the method seems to produce results with statistics that are noticeably different for separations along the length of the ribbon than they are for separations along the width of the ribbon when the separations are greater than about one third the width of the ribbon.

We have found a way to circumvent these limitations and discrepancies, a way that allows phase screen ribbons of any specified width to be extruded to whatever length is desired, with the structure function very closely conforming to the five-thirds power law of Kolmogorov-type turbulence for all separations up to some prespecified distance. Our method uses the same statistical formulation for the extrusion process as that used by Assémat *et al.*, but applies this extrusion tool differently. The variant formulation of this phase screen extrusion process we have developed is based on recognition of the fact that a statistical property that is *not imposed* in the generation of random values is *not to be expected* in the random values that are produced.

2. EXTRUSION PROCESS

Consider two sets of random variables, $\{x_1,x_2,x_3,\ldots,x_N\}$ appearing as the elements of the column vector $\mathbf{x}$ and $\{z_1,z_2,z_3,\ldots,z_M\}$ appearing as the elements of the column vector $\mathbf{z}$, with the covariance between the various elements of these two column vectors being the elements of the covariance matrices

$$\mathbf{P_{xx}} = \langle \mathbf{x}\mathbf{x}^{\mathrm{T}} \rangle, \quad \mathbf{P_{zz}} = \langle \mathbf{z}\mathbf{z}^{\mathrm{T}} \rangle, \quad \mathbf{P_{xz}} = \langle \mathbf{x}\mathbf{z}^{\mathrm{T}} \rangle, \quad \mathbf{P_{zx}} = \langle \mathbf{z}\mathbf{x}^{\mathrm{T}} \rangle. \tag{1}$$

Assémat *et al.* develop the equation

$$\mathbf{x} = \mathbf{Az} + \mathbf{Bg}, \tag{2}$$

where $\mathbf{g}$ is a column vector of length M with statistically independent elements each having zero mean and unity variance, and where

$$\mathbf{A} = \mathbf{P_{xz}}\mathbf{P_{zz}}^{-1}, \quad \mathbf{B} = \sqrt{\mathbf{P_{xx}} - \mathbf{P_{xz}}\mathbf{P_{zz}}^{-1}\mathbf{P_{zx}}}, \tag{3}$$

as their method of developing randomly selected values for $\mathbf{x}$, values that will be statistically appropriate given a previously (and statistically suitably) selected set of values for $\mathbf{z}$. This result is apparently fairly well known in the field of controls systems [2].

In interpreting the square root notation appearing in the expression for $\mathbf{B}$ given in Eq. (3) it should first be noted that the matrix with value $\mathbf{P_{xx}} - \mathbf{P_{xz}}\mathbf{P_{zz}}^{-1}\mathbf{P_{zx}}$ is a covariance matrix [3]. (It is the covariance matrix for the random variable $\mathbf{x}$ conditioned on the random variable $\mathbf{z}$ having some particular set of values.) Being a covariance matrix, this matrix is positive semidefinite [4]. From this it follows that all of its eigenvalues are nonnegative [5]. Now consider a matrix formed as the product of the unitary matrix whose columns are the eigenvectors of this covariance matrix times a diagonal matrix whose diagonal elements are the square roots of the eigenvalues of this covariance matrix. Because all of the eigenvalues are nonnegative, all of the elements of this product matrix are real. It is easy to see that this real product matrix times its transpose is equal to the covariance matrix. Accordingly, this real product matrix can be considered to be the square root of the covariance matrix. It is this real product matrix that the square root notation in Eq. (3) is meant to indicate. (The generation of this square root matrix is easily accomplished by using singular value decomposition.)

Given a phase screen ribbon of some length, Assémat *et al.* consider the values of the last two (or the last one, or the last four) columns at the leading edge of the ribbon as constituting the elements of $\mathbf{z}$ and proceed to apply Eq. (2) for the calculation of the values for the vector $\mathbf{x}$—the values of $\mathbf{P_{xx}}$, $\mathbf{P_{zx}}$, and $\mathbf{P_{xz}}$ having been set to be such that these calculated values of $\mathbf{x}$ are statistically appropriate for the elements of the next column beyond the leading edge of the ribbon. Appending this column of $\mathbf{x}$ values to the leading edge of the phase screen ribbon and considering the leading edge of the ribbon to be thus advanced, they then repeat the process, successively adding as many columns as desired. For convenience in describing our modification of this extrusion process we introduce the term "stencil" to describe the process of selecting the last two (or one, or four) columns of the ribbon to make up the elements of $\mathbf{z}$. The stencil is a pattern that is overlaid on the existing portion of the phase screen ribbon, aligned to the ribbon's leading edge, and having openings appearing above the ribbon elements that are to be selected—i.e., having openings that will appear over every one of the elements of the last two (or one, or four) columns of the phase screen ribbon when the stencil is placed on and aligned to the leading edge of the ribbon.

3. MODIFIED EXTRUSION PROCESS

There are two basic changes that we have made in the approach described by Assémat *et al.* One change concerns the nature of the stencil used to select the elements of the existing phase screen ribbon that will go into $\mathbf{z}$. The other has to do with accommodating the fact that the covariance for a Kolmogorov phase screen pattern is divergent. We treat these two matters in the following two subsections.

A. Stencil Pattern

In place of the type of stencil used by Assémat *et al.* we use a stencil of the type shown in Fig. 1. The sample stencil pattern shown in Fig. 1 is for a phase screen ribbon with a width of $2^4+1=17$ sample points. (Generalization of this sample stencil pattern to one that is applicable for the case of a ribbon with a width of 2^N+1, for any value of N, should be obvious.) The fine dots in Fig. 1 define the location of the elements, up to the leading edge (which is shown here as being on the right), of an existing phase screen ribbon—a ribbon with a width of 17. The small plus signs define the location, just beyond the leading edge of the ribbon, of the elements of the column that are to be added by one step or cycle of the extrusion process—which elements will be equal to the elements of $\mathbf{x}$ produced by Eq. (2). The small open circles about some of those fine dots define the location of the openings of the stencil. These opening will specify or select the elements of the existing phase screen ribbon that will be the elements of $\mathbf{z}$ used in that step or cycle of the extrusion process, i.e., that will be used in Eq. (2). As each step or cycle of the extrusion process is completed, the values of $\mathbf{x}$ are put in place at the positions indicated by the plus signs, and thus the leading edge of the phase screen ribbon is advanced one column. In preparation for the next step or cycle of the extrusion process, the stencil is moved one column farther to the right, so as to be registered with the new position of the leading edge of the phase screen ribbon.

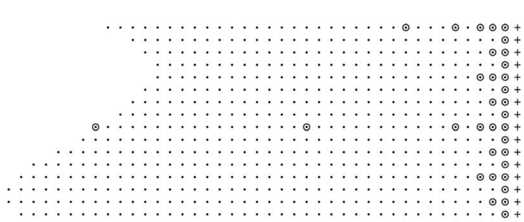

Fig. 1. Sample stencil pattern.

To facilitate our description of the form or nature of the stencil pattern, we find it convenient to separate the stencil pattern into two parts, which we call the "body" and the "tail." When the stencil is registered with the leading edge of the phase screen ribbon, the body of the stencil covers the first to the 2^Nth columns of the phase screen ribbon. The tail of the stencil covers every column of the phase screen ribbon from the (2^N+1)th column to as far back as may be desired.

The pattern of openings in the tail of the stencil is quite simple. There are openings only in the $k(2^N+1)$th columns—where $k=\{1,2,3,\ldots\}$. In those columns there is only a single opening, and that opening is at the center of the column, i.e., at the $(2^{N-1}+1)$th position down the length of the column. (For the 17-wide phase screen ribbon shown in Fig. 1, for which $N=4$, these are the 17th, $2\times 17=34$th, $3\times 17=51$st, $4\times 17=68$th,... columns. The openings are at the ninth position down the lengths of these columns.)

The pattern of openings in the body of the stencil is as follows. For the first column there is an opening in the stencil for every position. There are N additional columns in the body of the stencil for which there are openings. These are $2^{n-1}+1$ columns, where $n=\{1,2,3,\ldots,N\}$. (For the stencil pattern shown in Fig. 1, for which $N=4$, these are the second, third, fifth, and ninth columns.) These columns each have $2^{N-n}+1$ uniformly spaced openings, the first of these openings being at the first position along the length of the column. For the stencil pattern shown in Fig. 1.

For $n=1$, i.e., for the $(2^{1-1}+1)$th=second column, there are $2^{4-1}+1=9$ openings,

For $n=2$, i.e., for the $(2^{2-1}+1)$th=third column, there are $2^{4-2}+1=5$ openings,

For $n=3$, i.e., for the $(2^{3-1}+1)$th=fifth column, there are $2^{4-3}+1=3$ openings,

For $n=4$, i.e., for the $(2^{4-1}+1)$th=ninth column, there are $2^{4-4}+1=2$ openings.

B. Accommodating the Divergence of the Covariance of a Kolmogorov Phase Screen

Because of the five-thirds power-law form and the consequent divergence of the covariance of a Kolmogorov phase screen, we have to make do with information about the value of the phase structure function, i.e., values of the mean of the square of the difference between phase screen values. To let us get around the problem of having to work with structure functions rather than covariance functions, we use the stencil to also specify one additional location. This has to be a location different from any of those specified by the openings of the stencil. There is no other constraint with regard to where it is placed on the stencil—but to minimize possible round-off errors in the computations it is desirable to select a position near the leading edge of the stencil. We call this the "reference" position and denote the value from the existing phase screen ribbon that it selects by the notation z_{Ref}. We can then write in place of Eq. (2) that

$$\mathbf{x} = [\mathbf{A}(\mathbf{z} - z_{\mathrm{Ref}}) + \mathbf{Bg}] + z_{\mathrm{Ref}}, \tag{4}$$

which, from consideration of Eq. (2), can be seen to be the equivalent of the development of the set of random variables $\boldsymbol{\xi}=\mathbf{x}-z_{\mathrm{Ref}}$ given the set of random variables $\boldsymbol{\zeta}=\mathbf{z}-z_{\mathrm{Ref}}$. The value of the covariance between elements of $\boldsymbol{\xi}$ and/or $\boldsymbol{\zeta}$ can be developed from values of the mean square difference between values of $\mathbf{x}$ and/or $\mathbf{z}$, using the easily proven fact that for any three random variables, say α, β, and γ, having zero mean and equal variances

$$\langle(\alpha-\gamma)(\beta-\gamma)\rangle = -\tfrac{1}{2}\langle(\alpha-\beta)^2\rangle + \tfrac{1}{2}\langle(\alpha-\gamma)^2\rangle + \tfrac{1}{2}\langle(\beta-\gamma)^2\rangle. \tag{5}$$

The terms on the right-hand side of Eq. (5) correspond to structure functions relating elements of $\mathbf{x}$ and/or $\mathbf{z}$, while the term on the left-hand side corresponds to covariances relating terms of $\boldsymbol{\xi}$ and/or $\boldsymbol{\zeta}$—which means that the extrusion process can be implemented by using only Kolmogorov phase screen, five-thirds power-law, structure function information.

4. SAMPLE RESULTS

To start the process of phase screen ribbon extrusion we first need an initial section of ribbon (where by the term "section of the ribbon" we are denoting a segment of the ribbon with length equal to the length of the stencil). We have found that when attempting to match Kolmogorov theory in implementing the phase screen ribbon extrusion process we can start with an initial section of phase screen ribbon filled entirely with zeros. We have found that by the time we have extruded an additional ten sections of ribbon the statistics of that tenth section (and presumably of all subsequent sections) conform to the Kolmogorov phase screen, five-thirds power-law form—at least out to separations equal to the length of the stencil.

We have also found if we start with an initial section of ribbon populated by using a variation of the method of Lane *et al.* [6] we get a good match to Kolmogorov theory by the time we have extruded only four sections. The statistics of that fourth section conform to the Kolmogorov phase screen, five-thirds power-law form.

In Fig. 2 we show the structure function for the tenth extruded section of what we intend to be a 257-element-wide phase screen ribbon representing Kolmogorov turbulence, when the length of the stencil was 1,028 elements and the initial section of the ribbon was filled with all zeros—and for the fourth extruded section when the initial section of the ribbon was filled by use of our version of the Lane *et al.* algorithm. As can be seen, the match with the five-thirds power-law dependence associated with Kolmogorov theory is very good. The difficulty in identifying the dotted line in each plot of Fig. 2 indicates the closeness of the ribbon's structure function to Kolmogorov theory—out to the full length of the stencil!

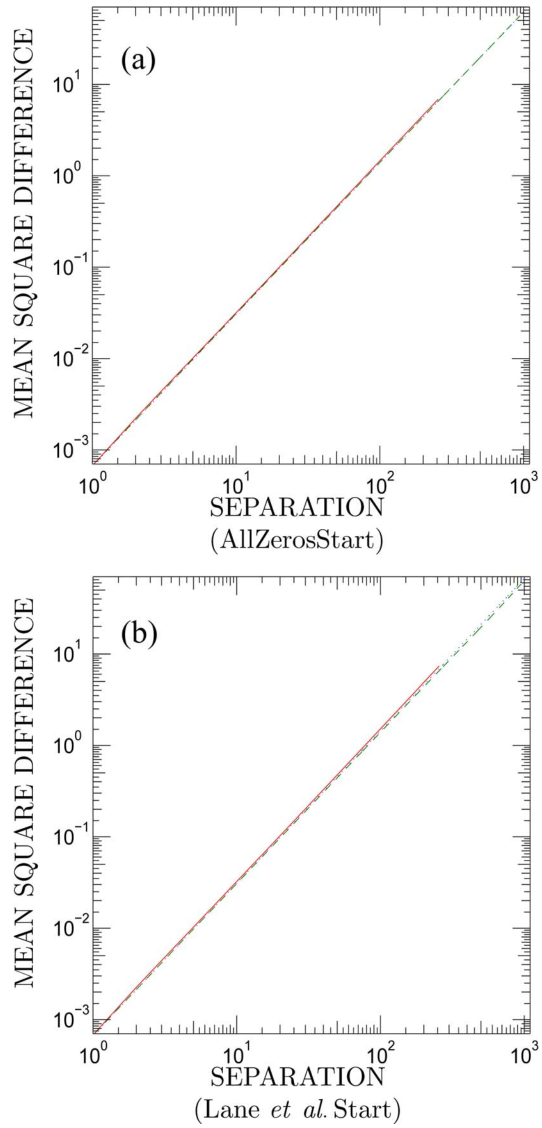

Fig. 2. (Color online) Phase screen ribbon structure function results for a 257-element wide ribbon. (a) Tenth extruded section, starting with an all-zeros initial section. (b) Fourth extruded section, starting with an initial section filled using our version of the Lane *et al.* algorithm. Each plot shows three curves. In each the (barely visible) dotted line shows the five-thirds, power-law, Kolmogorov theory values. The dashed line and the solid line show mean square difference values for separations along the length and for separations along the width of the phase screen ribbon, respectively. These mean values are each calculated from 1,000 independent realizations of a 257×1028 section of the phase screen ribbon. The stencil pattern used in developing these results had a total length equal to four times the width of the phase screen ribbon.

In conversations with colleagues questions have been raised regarding isotropy and stationarity of phase screen ribbon results produced by this process—considering the anisotropy and the nonstationarity of the stencil. We understand this as not being a question of exact conformity to stationarity and isotropy constraints (since there is no basis for expecting such conformance when nothing in the phase screen extrusion process imposes such a exacting constraint) but rather of being a question about whether there is any *significant* deviation from isotropy and stationarity.

We feel that the fact that the structure function results for separations along the length of the ribbon direction and for separations along the width of the ribbon direction can be seen in Fig. 2 to be so nearly equal shows that there is no significant anisotropy in the statistics of the phase screen ribbon produced by this process.

To address the question of stationarity of the statistics of the phase screen ribbon we have focused our attention on the fact that the stencil pattern suggests that

if:	there is any significant nonstationarity in the statistics of the phase screen ribbon that is produced,
then:	this would manifest itself as a row-to-row dependence in the mean square difference for separations along the length of the ribbon

—a row-to-row dependence as this is the most pronounced pattern in the stencil, and for a separation along the length of the ribbon as any other orientation of the separation might smear the row-to-row dependence. If there is a significant nonstationarity of this form then for pairs of positions along a given row of the phase screen ribbon—for pairs of positions with a given separation—the mean of the square of the difference of the phase at such a pair of positions would be significantly different for different rows.

In Fig. 3 we show results developed from the same set of 1,000 statistically independent sections (of size 257×1028) of phase screen ribbons as were used to produce the results shown in Fig. 2(a). For the results shown in Fig. 3 mean square difference values were calculated only for separations along the length of the ribbon, and were separately calculated for each row. In Fig. 3 we show the row dependence of the mean square difference for separations of 2, 4, 8,…, 512 and 1024 units. As can be seen from consideration of this figure, not only are the results apparently in good agreement with Kolmogorov theory, but more significantly there is no significant row dependence.

Actually, there is a slight row dependence, apparently picking up the every-other-row pattern of the second column of the stencil. This is illustrated in Fig. 4, where it can be seen that this row dependence, which appears as a row-to-row sawtooth pattern, is rather slight. The peak-to-peak variation of this sawtooth pattern is about 0.1×10^{-3}, while the mean value is about 2.2×10^{-3}, so the amplitude of this variation corresponds to only about $\frac{1}{2}(0.1 \times 10^{-3}/2.2 \times 10^{-3}) = 0.023 \equiv 2.3\%$. For larger separations the row-dependent sawtooth pattern has about the same absolute amplitude and thus significantly less as a percentage. We believe that for most purposes this magnitude of row dependence is inconsequential.

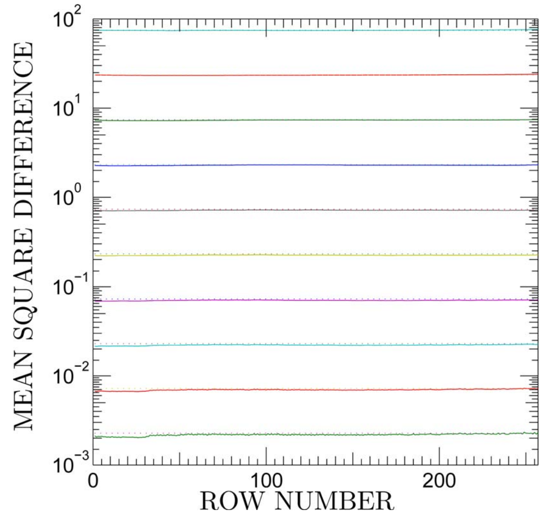

Fig. 3. (Color online) Row dependence for various separations. The same set of phase screen ribbon sections as was used to produce the results shown in Fig. 2 have been processed, treating each row separately to produce the results shown here. The ten solid curves show the mean square difference values for separations of 2, 4, 8,…, and 1,024 along the direction of the length of the ribbon. Accompanying each such curve there is a (just barely visible) dotted horizontal line showing the Kolmogorov theory value.

As a sort of *tour de force* demonstration of the capability of this phase screen ribbon extrusion process we have used this process to generate phase screen ribbon sections of size $257 \times 25{,}700$, a section for which the length is 100 times the width of the ribbon. This was carried out by using a stencil of that same size—so the mean square difference of phase values should follow the Kolmogorov theory five-thirds power-law form for all separations up to the full length of one of these sections. We generated

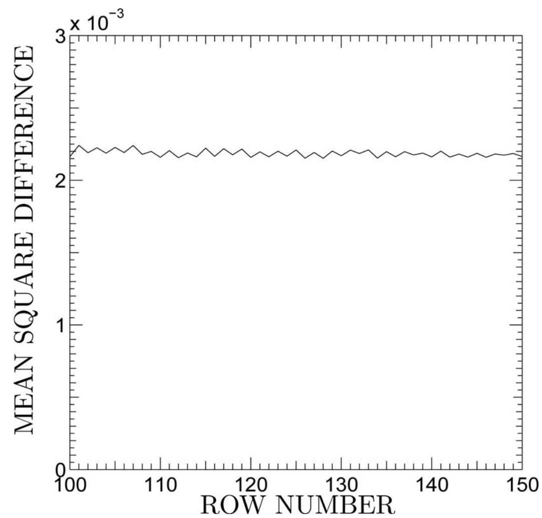

Fig. 4. Row dependence for a separation of two units. Shown here is a limited portion of the lowest of the solid line curves shown in Fig. 3. The expanded (and linear) scale of the ordinate allows a row-dependent variation in the mean square difference to be clearly seen. The row-to-row sawtooth variation appears to have an amplitude of no more than about 2.3%. (Note: what appears as a sawtooth pattern could just as well be represented as a square-wave pattern. That it appears as a sawtooth is due to there being no data points to be plotted between the successive max and min values.)

2,630 statistically independent random realizations of such sections. For each such realization we started with a set of zero values and extruded a total of 257,000 columns of phase screen ribbon, taking the last 25,700 columns as making up our phase screen ribbon section.

In Fig. 5 we show the mean square difference values we obtained when averaging over these 2,630 realizations, for separations aligned with the length of the phase screen ribbon section. For each separation we considered all possible pairs of positions on the section that had that separation. Each of the mean square difference values shown in Fig. 5 represents an average over all $\mathcal{N}_{\mathrm{RR}}=2{,}630$ random realizations and over all $\mathcal{N}_{\mathrm{Pairs}}$ pairs of positions applicable for that separation. Except for the largest separations the number of pairs with a particular separation, $\mathcal{N}_{\mathrm{Pairs}}$, is a rather large number.

Also shown in Fig. 5 is a straight line showing the Kolmogorov theory five-thirds power-law values. The difficulty in distinguishing between the two lines indicates the goodness of the conformity of the statistics of the phase screen ribbon random values to what is intended.

To make the conformity shown in Fig. 5 between the mean square values and the Kolmogorov five-thirds power-law more explicit, in Fig. 6 we show the normalized deviation of the mean square values curve relative to the theory values—the normalized deviation being equal to the difference between the calculated mean square values and the theoretical (five-thirds power-law) values, divided by the theoretical values. We consider the smallness of the normalized deviation—a fraction of 1% for the smaller separations and no more that 3% for the largest separation—to be a clear demonstration of the efficacy of the phase screen ribbon generation process.

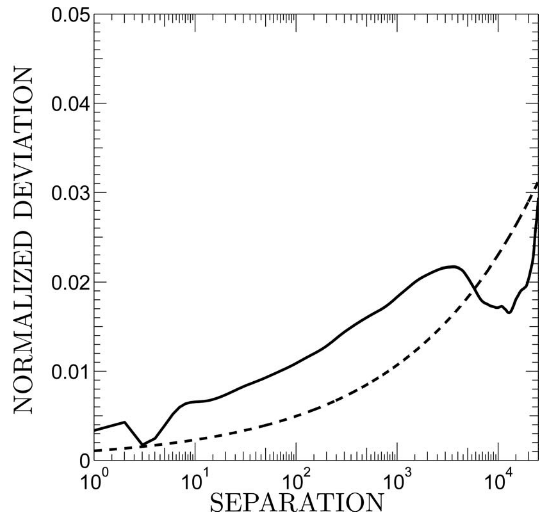

Fig. 6. Normalized deviation of long phase screen ribbon sections. The solid curve shows the normalized deviation of the mean square difference results shown in Fig. 5, i.e., the difference between the mean square values and the five-thirds power-law values, divided by the five-thirds power-law values. The dashed curve shows the rms value that is to be associated with the normalized deviation. These results are for $\mathcal{N}_{\mathrm{RR}}=2{,}630$ random realizations of a phase screen ribbon section of size $257\times 25{,}700$. The separations are all parallel to the length of the ribbon.

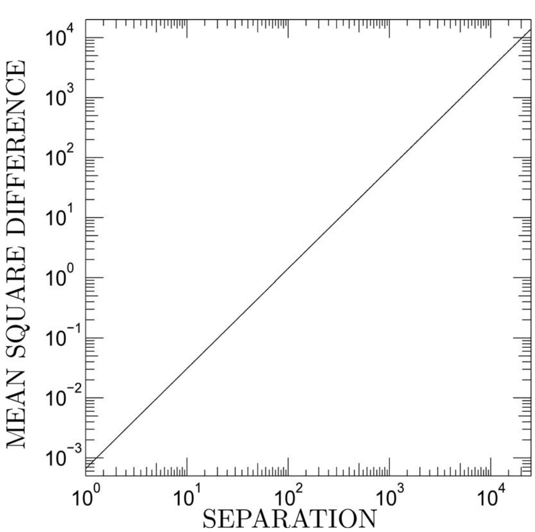

Fig. 5. Long phase screen ribbon mean square difference results. The solid curve represents the mean of the square of the difference of phase screen ribbon values, the average being formed over $\mathcal{N}_{\mathrm{RR}}=2{,}630$ independent random realizations of the phase screen ribbon section. The separations considered are aligned with the length of the ribbon, and for each separation the averaging process utilizes every pair of positions with that separation. The ribbon width is 257, and the length of the section is $100\times 257=25{,}700$. Also shown here is a (barely discernable) dotted line indicating the five-thirds power-law dependence the mean square difference should follow.

To fully assess the significance of this normalized deviation we have developed an estimate of the rms value that is to be associated with this deviation. We have been able to show that for a separation (p,q) with phase screen sections of size (P,Q) the calculated value of the normalized deviation should be expected to have a standard deviation, σ, with a value given by the equation

$$\sigma^2=\frac{2}{\mathcal{N}_{\mathrm{RR}}\mathcal{N}_{\mathrm{Pairs}}}+\left(\frac{1}{2\mathcal{N}_{\mathrm{RR}}}-\frac{1}{2\mathcal{N}_{\mathrm{RR}}\mathcal{N}_{\mathrm{Pairs}}}\right)\tilde{V}\left(\frac{p}{P},\frac{q}{Q}\right), \quad (6)$$

where

$$\tilde{V}(\mu,\nu)=[\mu^2+\nu^2]^{-5/3}\int_{-(1-|\mu|)}^{+(1-|\mu|)}\mathrm{d}x\int_{-(\zeta-|\nu|)}^{+(\zeta-|\nu|)}\mathrm{d}y\left[\frac{1-|\mu|-|x|}{(1-|\mu|)^2}\right]\times\left[\frac{\zeta-|\nu|-|y|}{(\zeta-|\nu|)^2}\right]([(x+\mu)^2+(y+\nu)^2]^{5/6}+[(x-\mu)^2+(y-\nu)^2]^{5/6}-2[x^2+y^2]^{5/6})^2. \quad (7)$$

The dashed curve shown in Fig. 6 shows the value of this standard deviation, calculated with $\mathcal{N}_{\mathrm{RR}}=2{,}630$ and $\mathcal{N}_{\mathrm{Pairs}}=\infty$. As can be seen, not only is the normalized deviation very small (varying from a fraction of a percent to a few percent), but it also is quite comparable with the standard deviation that is to be associated with this normalized deviation. (Presumably then, if we had used 100 times more phase screen realizations, the normalized deviation we would be showing in Fig. 6 would be about ten times smaller.)

To make clear the role played by the tail of the stencil in forcing the statistics of the phase screen ribbon to conform to Kolmogorov statistics for large separations—and to make manifest the import of the previously offered assertion that "a statistical property that is *not imposed* in

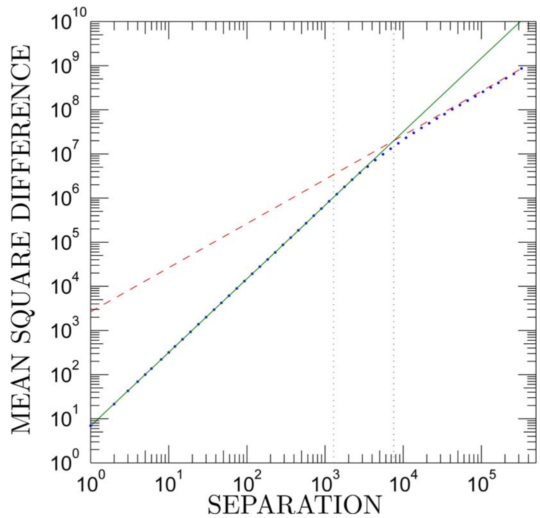

Fig. 7. (Color online) Mean square difference results for separations much larger than the stencil length. The small filled circles show the calculated average of the squared difference of phase screen values. A 6.88 $(\text{separation})^{5/3}$ dependence, i.e., a Kolmogorov dependence, is shown by the solid line. The dashed line shows a linear dependence on separation. The left of the two faint vertical dotted lines is at a separation of 1,281—which is the length of the stencil used in generating the phase screen ribbons. It is to be noted that the structure function results are very close to Kolmogorov predictions for separations up to about 4,000. The right of the two vertical dotted lines shows where the Kolmogorov and the linear dependence lines intersect, which is at a separation of 7,549.

the generation of random values is *not to be expected* in the random values that are produced"—we have also generated a set of results for a very long ribbon and calculated the structure function results for separations much longer than the length of the tail of the stencil. The ribbons we generated had a width of 257 and a length of 512,400. The stencil, including its tail, had an over all length of 1,281. Structure function results were calculated for separations as large as 325,631. These results are shown in Fig. 7.

Consideration of the results shown in this figure makes it clear that the structure function follows the five-thirds power law associated with Kolmogorov statistics for all separations up to and even somewhat beyond the stencil length, but for separations much greater than the stencil length the structure function follows a first power law. This is what one would expect for a random-walk process. That the mean square difference manifests random-walk-type behavior may be taken as meaning that over distances much greater than the stencil tail length the development of the extruded phase screen ribbon is basically unconstrained.

5. COMMENTS

In considering the utility of the phase screen ribbon extrusion process note must be taken of the facts that it is hard to compete with the speed of the fast Fourier transform (FFT) algorithm and that matrix multiplication is inherently slower (at least on a general purpose computer). Generating a pair of $2^{\nu}\times 2^{\nu}$ phase screens involves a single FFT operation, with one half of the required time being charged to each of the two phase screens generated.

Extruding a phase screen ribbon of width $2^{N}+1$ involves a stencil with about $2^{N+1}+1$ elements. The **A** matrix has a $(2^{N}+1)\times(2^{N+1}+1)$ size. For each extruded column of the phase screen ribbon this matrix will multiply a $2^{N+1}+1$ long column vector. The **B** matrix has a $(2^{N}+1)\times(2^{N}+1)$ size. For each extruded column of the ribbon this matrix will multiply a $2^{N}+1$ length column vector. A comparison, using MATLAB, of the speed for these two processes for the $N=8$ case (and for the $N=10$ case) indicates that the extrusion process can produce a single column of the phase screen ribbon in 0.0929 (and 0.0574) of the time it takes the FFT process to produce an entire screen. Accordingly, if one were simply interested in producing a set of square phase screens, the extrusion process would be 23.8 (and 58.8) times slower than the FFT process.

Clearly, except for a slight advantage in conforming to Kolmogorov statistics, the phase screen ribbon extrusion process is not to be preferred to the FFT phase screen generation process if a single moderate size phase screen is all that is required. But for some studies very large phase screens are required—particularly for time-dependence studies. In such a case one might start with an FFT-generated phase screen and, as time progresses and the phase screen has to be shifted to simulate the effects of wind and/or of slewing of the line of sight, columns or rows could be dropped from one edge of the phase screen and new columns or rows added to the opposite edge by using the extrusion process. In this sort of application the time required for the extrusion process would be inconsequential.

ACKNOWLEDGMENT

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