JOSA COMMUNICATIONS

Communications are short papers. Appropriate material for this section includes reports of incidental research results, comments on papers previously published, and short descriptions of theoretical and experimental techniques. Communications are handled much the same as regular papers. Galley proofs are provided.

Phase variance and Strehl ratio in adaptive optics

Jan Herrmann

Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, Massachusetts 02173

Received February 20, 1991; revised manuscript received August 4, 1992; accepted August 10, 1992

Some correction schemes in adaptive optics are performed with the use of the phase reconstructed from measured gradients. For these cases it can be shown that the minimum phase-variance estimator leads to the maximum Strehl ratio, that is, to the maximum peak intensity in the far field.

INTRODUCTION

It has been possible for some time to design adaptive-optics systems that use phase conjugation to compensate for the effect of atmospheric turbulence.[1] The implementation of such systems is based on phase-slope measurements and a phase reconstruction from these measurements. Wallner[2] and Welsh and Gardner[3] analyze models of adaptive-optics systems that include a great variety of effects. All proposed systems use a control law that optimizes the mean-square residual phase error.

One of the prime objectives of adaptive optics is to maximize the Strehl ratio, and a well-known relation between the variance of the phase error σ^2 and the Strehl ratio S is given by the Maréchal approximation[4]

$$S \approx 1 - \sigma^2,$$

which is valid only for small variances. But in general the Strehl ratio is a function not only of the phase variance but also of the spatial correlation of the phase. The principal point of this Communication is to show that the phase with the minimum variance automatically leads to the maximum Strehl ratio.

THE STREHL RATIO

For our purpose it is convenient to define the Strehl ratio in the following discrete form:

$$S = \left\langle \left| \frac{1}{N} \sum_n e^{ie_n} \right|^2 \right\rangle,$$

where the sum extends over all points of the transmitter aperture and e_n is the phase error at the nth point. We assume a uniform amplitude of the corrected beam. The Strehl ratio can be written as

$$S = \frac{1}{N^2} \left\langle \sum_{n,m} e^{i(e_n - e_m)} \right\rangle = \frac{1}{N^2} \sum_{n,m} e^{-D_{nm}/2},$$

where D_{nm} is the structure function of the phase error. The last step uses the relation[5]

$$\langle e^{ir} \rangle = e^{-\langle (r - \bar{r})^2 \rangle /2},$$

which is valid for a Gaussian random variable r.

The structure function of the error can be written in terms of the covariance function of the error $C_{nm} = \langle (e_n e_m) \rangle$ by expansion of the structure function:

$$D_{nm} = \langle (e_n - e_m)^2 \rangle = {\sigma_n}^2 + {\sigma_m}^2 - 2C_{nm}.$$

We can now prove that the optimal phase estimator leads to the maximum Strehl ratio. From estimation theory[6–8] we need only a fundamental property of the error's covariance matrix. If $\hat{x}$ is the linear, unbiased, minimum variance estimator with the error's covariance matrix

$$C_e = \langle (x - \hat{x})(x - \hat{x})^T \rangle,$$

then any other linear, unbiased estimator x' will give a covariance matrix C_e', so that the difference

$$\Delta = C_e' - C_e$$

is positive definite. This property is sufficient for our proof.

The Strehl ratio for the estimator x' can be written in terms of the estimator $\hat{x}$:

$$\begin{aligned} S' &= \frac{1}{N^2} \sum_{n,m} \exp\left[+ \frac{1}{2} (C_{nn}' + C_{mm}' - 2C_{nm}') \right] \\ &= \frac{1}{N^2} \sum_{n,m} \exp\left[- \frac{1}{2} (C_{nn} + C_{mm} - 2C_{nm}) \right] \\ &\quad \times \exp\left[- \frac{1}{2} (\Delta_{nn} + \Delta_{mn} - 2\Delta_{nm}) \right]. \end{aligned}$$

0740-3232/92/122257-02$05.00

The quantity inside the additional second exponential is always positive for every n and m,

$$\Delta_{nn} + \Delta_{mm} - 2\Delta_{nm} \geq 0,$$

because the difference matrix Δ is positive definite. Therefore it follows that the minimum variance estimator $\hat{x}$ also gives the maximum Strehl ratio.

CONCLUSIONS

The preceding conclusions apply to adaptive-optics systems that use a phase-conjugate correction. The phase correction is derived from measured gradients by the use of a wave-front sensor, followed by a phase reconstruction that is usually performed with a least-squares estimation. The primary interest for most applications is to maximize the Strehl ratio, and we have shown in this Communication that this happens automatically.

ACKNOWLEDGMENT

The author thanks Edward J. Kelly of the Massachusetts Institute of Technology Lincoln Laboratory, who provided the basic ideas presented in this Communication.

This study was sponsored by the U.S. Army Space Defense Command under Air Force contract F19628-90-C-0002.

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