Wave propagation in random media with fluctuating turbulent parameters

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Wave propagation in a random medium with the fluctuating index-of-refraction structure constant C_n^2 is considered. Under certain conditions, these fluctuations can essentially influence the results of field-moment measurements. The randomized probability-distribution function of the index of refraction is used to derive equations for arbitrary field moments in such a medium. With respect to the mean field and the two-point coherence function, it is shown that C_n^2 fluctuations lead to an increase in these moments and also to the change in their analytical expressions as compared with the case of the unperturbed C_n^2. For certain field moments the small-angle approximation for the radiative-transport equation and that of the equation for the radiant intensity autocorrelation function are derived. The relations among various spatial scales, such as the propagation distance, the local and global variation scales, and the coherence radius, are shown to play an important role in the outcome of the field-moment measurements.

1. INTRODUCTION

To describe wave propagation in random media one has to use statistical characteristics of the media and a combination of simplifying assumptions regarding propagation conditions. These assumptions relate, on the one hand, to the mechanism of wave scattering and, on the other, to the properties of the statistical medium. For instance, in the theory of wave propagation in a turbulent medium, the model of the statistically homogeneous and isotropic medium with a Gaussian probability distribution of the index of refraction is most often used. On the basis of this model the most important, classical results have been obtained. The radiative-transport theory is also based on the model of a medium with homogeneous or regularly inhomogeneous scattering and absorption coefficients, which essentially are also statistical characteristics of the medium. At the same time, such models are only rough approximations to the real-world media because the statistical characteristics of such media are, in their turn, subject to random changes in both space and time. This variability can be associated with different mechanisms and can touch on various parameters of the statistics of the media. For instance, for the turbulent medium such mechanisms can be the variability of the meteorological conditions on the propagation path, the inhomogeneous character of the underlying surface, or such an inner mechanism as the intermittency of turbulence.[1,2] The index-of-refraction structure constant C_n^2, the inner and outer scales of turbulence, the spectral exponent, etc. can undergo random variations. For turbid scattering media the situation is similar: It is well known that the optical parameters of such natural formations as clouds, fog, and sea water can randomly vary in space and time. Incidentally, rather recently, in solving the radiation-transport equation in the regularly inhomogeneous absorbing medium the effect of the translucence of such a medium as compared with the homogeneous medium possessing the equivalent optical thickness was discovered.[3] Several papers subsequently studied the radiative transport in media with fluctuating optical characteristics in which a similar effect in various aspects was discussed.[4-9] Such a formulation of the problem for turbulent media is rare in theoretical papers. As an example of the closed formulation of the problem, we may cite Ref. 10, which considers the effect of large-scale index-of-refraction fluctuations on the very-high-frequency component beyond the horizon, and Ref. 11, in which the effect of the intermittence of turbulence on the wave scattering is studied.

Let us now give the initial formulation of the problem, whose solution is suggested below. Let us assume that wave propagation occurs in a medium with a random field of the index of refraction (which is complex in the general case) $n(\mathbf{r})$. It is described by the $P(n)$ probability-distribution function. Let us assume that this probability-distribution function can be represented in the form

$$P(n) = \int p(n,\{A\})f(\{A\})\mathrm{d}\{A\}, \tag{1}$$

where $p(n,\{A\})$ is a certain probability-density function of the value n that depends on the set of parameters $\{A\}$. These parameters are regarded here as random values. $f(\{A\})$ is the probability-density function of these values. The process described by formula (1) is called the randomization of the probability-density function $p(n,\{A\})$, and $P(n)$ is called randomized distribution.[12]

Let us consider now a procedure for measuring the statistical moments of the field in such a medium. We divide this procedure into two stages. First, during the measuring process the temporal or spatial averaging is carried out over the interval τ or the region of space σ, respectively. Let us assume that τ and σ are, on the one hand, large compared with the characteristic time (or scales) of the variation of the random field of the index of refraction $n(\mathbf{r}, t)$ and that, on the other hand, they are small compared with the characteristic time (or scales) of the change in the statistical parameters $\{A\}$ of the random field n. We shall denote this averag-

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ing a small-scale averaging. After it, the measured wavefield moments will randomly vary because of the large-scale fluctuations in parameters $\{A\}$. These moments or their combinations are then finally averaged over a sufficiently large series of measurements. Let us call this procedure a large-scale averaging.

It is clear that by using a similar scheme it is possible to calculate the averages during the derivation of equations for appropriate field moments. The terms that contain the random value n are at first subject to small-scale averaging, that is, they are integrated with the density $p(n, \{A\})$ and then are integrated with the density $f(\{A\})$, which corresponds to the large-scale averaging.

2. PROPAGATION IN THE RANDOM INHOMOGENEOUS MEDIUM: LAYERED VARIATION APPROXIMATION

Let us consider the case of a nonabsorbing turbulent medium with the parameter C_n^2 fluctuating in space. C_n^2 is the index-of-refraction structure constant. The spatial scale of these C_n^2 fluctuations is such that, while describing wave propagation to a small-angle approximation ($\lambda \ll l_0$; l_0 is the microscale of turbulence), C_n^2 variations can be ignored in the plane transverse to the propagation direction, and only the one-dimensional dependence $C_n^2(\xi)$ need be taken into account. Let us call this approximation the layered variation approximation. Then the results obtained in Refs. 13 and 14 for the parabolic equation approximation and for the Markov approximation can be used. The averaging that figures in the above approximations will act in our case as a small-scale averaging since the problem will boil down to the calculation of the additional large-scale averaging with respect to variations of $C_n^2(\xi)$. As is well known from Ref. 13, for the plane wave passing through the turbulent layer ($0 < \xi < z$) characterized by the random distribution of $C_n^2(\xi)$ it is possible to write approximate analytical expressions for the mean field and the two-point coherence function as follows:

The mean field is given by

$$\langle v(z)\rangle = v_0 \exp\left[-(k^2/8)\int_0^z A(\xi; 0)\mathrm{d}\xi\right], \tag{2}$$

where

$$A(\xi; \boldsymbol{\rho}) = 8\pi \iint \Phi_n(\xi; 0, \boldsymbol{\kappa}) \exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\rho})\mathrm{d}^2\kappa, \tag{3}$$

where

$$\Phi_n(\xi; \kappa_\parallel, \boldsymbol{\kappa}_\perp) = C_n^2(\xi)\tilde{\Phi}(\kappa_\parallel, \boldsymbol{\kappa}_\perp) \tag{4}$$

is the wave-number spectrum of the index-of-refraction fluctuations n. Strictly speaking, it is necessary to impose a limitation on the scale of the change in C_n^2 with respect to ξ in order to determine correctly the wave-number spectrum Φ_n as a function of ξ. We shall assume below that all such restrictions are fulfilled.

The two-point coherence function is defined by

$$\Gamma_{1,1}(z; \mathbf{R}, \boldsymbol{\rho}) \equiv \langle v(z; \mathbf{R}-\boldsymbol{\rho}/2)v^*(z; \mathbf{R}+\boldsymbol{\rho}/2)\rangle = |v_0|^2 \exp\left[-(k^2/4)\int_0^z D(\xi; \boldsymbol{\rho})\mathrm{d}\xi\right], \tag{5}$$

where

$$D(\xi; \boldsymbol{\rho}) \equiv C_n^2(\xi)\tilde{D}(\boldsymbol{\rho}) = 8\pi C_n^2(\xi) \iint [1-\cos(\boldsymbol{\kappa}\cdot\boldsymbol{\rho})]\Phi(0, \boldsymbol{\kappa})\mathrm{d}^2\kappa. \tag{6}$$

If the angle brackets in formulas (2) and (5) denote the small-scale averaging, $\langle v\rangle$ and $\Gamma_{1,1}$ will become random quantities, as they depend on the random parameter C_n^2.

Equations (2) and (5) can be further averaged with respect to C_n^2 fluctuations, or moments of higher orders composed of them can be calculated. Here we shall limit ourselves to the type of probability distribution indicated in the exponents of these formulas. First, some useful conclusions can be drawn from the general form of functions (2) and (5). Since they have the form of convex functions, in this case $\exp(-x)$, the Jensen inequality, is valid[12]:

$$\langle \exp(-\kappa)\rangle_{\mathrm{ls}} \geqslant \exp(-\langle x\rangle_{\mathrm{ls}}), \tag{7}$$

where the subscript ls means large-scale averaging. Thus the presence of the variations of parameters in the exponentials of formulas (2) and (5) will lead to increases in full mean values compared with the case in which these variations are absent. Therefore we have an amplification of the mean field and an increase in the coherence radius of the plane wave. At first sight it seems paradoxical that the presence of fluctuations should lead to an increase in the degree of coherence of the wave. But everything is explained by the convexity of the function $\exp(-x)$ and hence by the nonuniform contribution (during the averaging) for $x < \langle x\rangle$ and $x > \langle x\rangle$.

Now let us investigate in more detail what the large-scale averaging of expressions (2) and (5) leads to. Equations (2) and (5) can be written in the general form

$$E \equiv \langle \exp\{-Nx\}\rangle_{\mathrm{ls}}, \tag{8}$$

where N is introduced so that equation $\langle x\rangle_{\mathrm{ls}} = 1$ is fulfilled for the random value x. In our problem it is essential that $x \geqslant 0$. Hence we take the gamma density as a model density of probability:

$$\mathcal{F}(x) = \begin{cases}\beta^\alpha \exp(-\beta x)/\Gamma(\alpha) & \text{when } x \geqslant 0\\ 0 & \text{when } x < 0\end{cases}. \tag{9}$$

Since we have posed the restriction that $\langle x\rangle_{\mathrm{ls}} = 1$, it must be true in Eq. (9) that $\beta = \alpha$. Then, if $x \geqslant 0$,

$$\mathcal{F}(x) = \alpha^\alpha \exp(-\alpha x)/\Gamma(\alpha). \tag{10}$$

If Eq. (8) is calculated with the density of Eq. (10), we obtain

$$E = (1 + N/\alpha)^{-\alpha}. \tag{11}$$

The parameter α is evaluated by

$$b_x \equiv \langle x^2\rangle - \langle x\rangle^2 = \alpha^{-1}. \tag{12}$$

Comparing the definition of E in Eq. (8) with expressions (2) and (5), we see that x is proportional to the integration over ξ from a random function $y(\xi)$. Taking into account the normalization of $\langle x\rangle_{\mathrm{ls}} = 1$, the value x can be represented as

$$x = S/\langle S\rangle_{\mathrm{ls}}, \tag{13}$$

where

$$S(z) = \int_0^z y(\xi)\mathrm{d}\xi. \tag{14}$$

To deal with the statistical parameters of the medium measured in practice, it is convenient to pass over from the variance b_x for the integral value of x to local statistical parameters of the process $y(\xi)$. For this purpose let us transfer the averaging procedure under the integral sign and use the statistical homogeneity of $y(\xi)$. Then the following expression can be written for b_x:

$$b_x = z^{-2}\int_0^z d\xi_1 \int_0^z d\xi_2 B_y(\xi_1 - \xi_2), \tag{15}$$

where

$$B_y(\xi_1 - \xi_2) = [\langle y(\xi_1)y(\xi_2)\rangle - \langle y\rangle^2]/\langle y\rangle^2$$

is a function of the longitudinal covariance of $y(\xi)$, which is characterized by the limited longitudinal scale of L_y. Let us now consider two limiting cases. Assuming that $z \ll L_y$, we have that

$$b_x \approx B_y(0), \tag{16}$$

and, accordingly, the value $B_y^{-1}(0)$ must be ascribed to the parameter α in Eq. (10). If, however, $z \gg L_y$, then it can be assumed that y is δ correlated over ξ:

$$B_y(\xi_1 - \xi_2) = R_y^0\delta(\xi_1 - \xi_2), \tag{17}$$

where

$$R_y^0 = \int_{-\infty}^{\infty} B_y(\xi')d\xi' \approx B_y(0)L_y, \tag{18}$$

and therefore

$$b_x = R_y^0/z \approx B_y(0)L_y/z = \alpha^{-1}. \tag{19}$$

Comparing Eqs. (16) and (19), we conclude that in the second case the variance of the integral value x is L_y/z times smaller than that of the first case. This result is the consequence of the averaging of the fluctuations of y over the ray.

Thus the expressions for E in two limiting cases can be derived as

$z \ll L_y$:

$$E = [1 + NB_y(0)]^{-B_y^{-1}(0)}, \tag{20}$$

$z \gg L_y$:

$$E = [1 + NB_y(0)L_y/z]^{-z/B_y(0)L_y}. \tag{21}$$

To pass over from general formulas (20) and (21) to specific expressions for the mean field and the coherence function,[15] it is necessary to determine for each case the types of parameters N and $B_y(0)$ by comparing Eqs. (2) and (5) with formula (8). For the mean field we have

$$N = k^2 z\langle C_n^2\rangle\tilde{A}(0)/8, \qquad \tilde{A}(0) = 8\pi\iint \Phi(0, \kappa)d^2\kappa,$$

$$B_y(0) = [\langle (C_n^2)^2\rangle - \langle C_n^2\rangle^2]/\langle C_n^2\rangle^2 \equiv B_{C_n^2}(0).$$

Substituting these expressions into formula (21), we obtain the mean field when $z \gg L_y$:

$$\langle v(z)\rangle = [1 + k^2 L_y\langle C_n^2\rangle\tilde{A}(0)B_{C_n^2}(0)/8]^{-z/B_{C_n^2}(0)L_y}. \tag{22}$$

The parameters N and $B_y(0)$ for the two-point coherence function will assume the following form:

$$N = k^2 z\langle C_n^2\rangle\tilde{D}(\rho), \qquad B_y(0) = B_{C_n^2}(0).$$

Hence, when $z \gg L_y$,

$$\Gamma_{1,1}(z;\boldsymbol{\rho}) = [1 + k^2 L_y\langle C_n^2\rangle\tilde{D}(\boldsymbol{\rho})B_{C_n^2}(0)/8]^{-z/B_{C_n^2}(0)L_y}. \tag{23}$$

For $z \ll L_y$, expressions for $\langle v(z)\rangle$ and $\Gamma_{1,1}(z;\boldsymbol{\rho})$ are obtained from Eqs. (22) and (23) by formally replacing L_y with z. This is evident from a comparison between Eqs. (20) and (21). As for the example of expression (11), it can be rewritten as

$$E = \exp[-\alpha \ln(1 + N/\alpha)]. \tag{24}$$

Let α^{-1} tend to zero, which corresponds to the decrease in the fluctuations of the integral $S(z) = \int_0^z y(\xi)d\xi$. The logarithm in Eq. (24) can then be expanded into a power series in terms of α^{-1}, and we shall limit ourselves to the first two terms:

$$E \approx \exp[-N(1 - N/2\alpha)]. \tag{25}$$

Such a form for E is obtained when $F(x)$ is assumed to be a Gaussian distribution. Hence this distribution can be used only for sufficiently weak fluctuations of $S(x)$, which is also evident from formula (25). When S fluctuations are absent, then $\alpha^{-1} \equiv 0$ and

$$E = \exp\{-N\}, \tag{26}$$

which corresponds to the well-known formulas for the unperturbed C_n^2 similar to Eqs. (2) and (5).

Expression (25) demonstrates the effects of fluctuations of C_n^2. Since the additional term containing α^{-1} has a plus in the exponent, it leads to an increase in E as compared with that of the unperturbed case. For the mean field, it is equivalent to an increase in the effective length of the field attenuation and, for the two-point coherence function, to an increase in the effective coherence radius.

The simple examples shown above have demonstrated the effect of the fluctuations of such a statistical parameter of the turbulent medium C_n^2 on the values of fluctuations under measurement. In a similar way, more-complex situations can be discussed, for instance, the turbulent spread of the limited wave beam in a medium with the fluctuating $C_n^2(\xi)$. Calculations show that the presence of $C_n^2(\xi)$ fluctuations will lead to a decrease in the beam spread as compared with the unperturbed case. It is easy to show that this effect is reduced to the above-mentioned increase of the coherence radius. It is possible also to consider moments composed of quantities such as those of Eqs. (2) and (5) averaged over the small scale and subjected additionally to the large-scale averaging; for instance,

$$\langle\langle v(z)\rangle_{ss}\langle v(z)\rangle_{ss}\rangle_{ls} \equiv \tilde{\Gamma}_{2,0}, \tag{27}$$

$$\langle\Gamma_{1,1}^{ss}(z;\mathbf{R}_1,\boldsymbol{\rho}_1)\Gamma_{1,1}^{ss}(z;\mathbf{R}_2,\boldsymbol{\rho}_2)\rangle_{ls} \equiv \tilde{\Gamma}_{2,2}. \tag{28}$$

Here ss means the small scale. Let us pay attention to the autocorrelation of the two-point coherence function $\tilde{\Gamma}_{2,2}$ determined by formula (28). It does not lead to the conventional fourth moment of the field, which should be written as

$$\langle\langle v(z,\mathbf{R}_1 - \boldsymbol{\rho}_1/2)v^*(z;\mathbf{R}_1 + \boldsymbol{\rho}_1/2) \times v(z,\mathbf{R}_2 - \boldsymbol{\rho}_2/2)v^*(z;\mathbf{R}_2 + \boldsymbol{\rho}_2/2)\rangle_{ss}\rangle_{ls} \equiv \Gamma_{2,2}. \tag{29}$$

As can be easily seen, Eq. (29) differs from Eq. (28) in the way in which the averaging for the small scale was taken: In Eq. (28) this is carried out at the level of second moments, while in Eq. (20) it is at the level of the fourth moments. As is shown below, definition (28) can be used for obtaining the correlation of radiant intensities (brightness) used in the radiative-transport theory. This correlation was first introduced in Ref. 9 to describe how radiation becomes stochastic during its propagation in a medium with the three-dimensional fluctuations of the extinction coefficient.

In this section the case of a layered medium with statistical parameters fluctuating in only one dimension has been discussed. In Section 3 we consider more-general cases of three-dimensional variations of the statistical parameters.

3. CASES OF THREE-DIMENSIONAL VARIATIONS

If the variability of statistical parameters proves to be essential not only in the longitudinal direction but also in transverse directions, it is necessary to take this fact into account from the beginning, that is, to try to obtain appropriate differential equations for fully averaged field moments.

We shall consider a lossy medium characterized by a complex dielectric permittivity. As it will be seen, taking absorption into account does not necessarily lead to a complicated mathematical model but does offer us a more complete comparison with the results obtained by the radiative-transport equation.

The wave equation in an inhomogeneous absorbing medium has the form

$$\Delta u(\mathbf{r}) + k_0^2\epsilon(\mathbf{r})u(\mathbf{r}) = 0, \tag{30}$$

where $u(\mathbf{r})$ is the scalar-wave field, $\epsilon(\mathbf{r})$ is the complex relative permittivity, and $k_0 = \omega/c$ is the wave number in vacuum.

Let us represent $\epsilon(\mathbf{r})$ in the form

$$\epsilon(\mathbf{r}) = \bar{\epsilon}_1 + \tilde{\epsilon}_1(\mathbf{r}) + i\epsilon_2(\mathbf{r}), \tag{31}$$

where $\bar{\epsilon}_1$ is a positive constant, $\epsilon_2(\mathbf{r}) \geqslant 0$ and $\tilde{\epsilon}_1(\mathbf{r})$ are real, and $|\tilde{\epsilon}_1| \ll \bar{\epsilon}_1$. If the length of the attenuation of the wave's amplitude that is due to ϵ_2 and the spatial scale of the variations $\epsilon(\mathbf{r})$ are both much greater than the λ, we can derive a parabolic equation by substituting

$$u(\mathbf{r}) = v(z; \rho)\exp(ik_0\sqrt{\bar{\epsilon}_1}z) \tag{32}$$

into Eq. (30) and discarding the term $\partial^2 v/\partial z^2$:

$$2ik\partial v/\partial z + \Delta_\perp v + k^2\mu(z; \rho)v = 0, \tag{33}$$

where $k = k_0\sqrt{\bar{\epsilon}_1}$, $\mu = \eta + i\chi$, $\eta = \tilde{\epsilon}_1/\bar{\epsilon}_1$, and $\chi = \epsilon_2/\bar{\epsilon}_1$. Equation (33) has the same form as that of the parabolic equation for the complex amplitude of the field v widely used in wave-propagation theory.[13] The only difference is that the parameter μ in Eq. (33) is complex. However, this does not hinder us from using Eq. (33) for solving problems similar to those associated with nonabsorbing media.

Let us assume that $\mu(z; \rho)$ is a fluctuating quantity and proceed to derive the equation for the arbitrary statistical field moment. Following the procedure set forth in Refs. 13 and 14, one can obtain from Eq. (33) the following equation:

$$2ik\partial\gamma/\partial z + \hat{L}\gamma + k^2\nu(z, \{\rho\})\gamma = 0 \tag{34}$$

for the quantity

$$\gamma = v(z, \rho_1')v(z, \rho_2') \times \ldots v(z, \rho_n')v^*(z, \rho_1'')v^*(z, \rho_2'') \ldots v^*(z, \rho_m''), \tag{35}$$

where

$$\hat{L} = \Delta_{\perp 1}' + \Delta_{\perp 2}' + \ldots + \Delta_{\perp n}' - \Delta_{\perp 1}'' - \Delta_{\perp 2}'' - \ldots - \Delta_{\perp m}'',$$

$$\nu(z, \{\rho\}) = \mu(z, \rho_1' + \ldots + \mu(z, \rho_n') - \mu^*(z, \rho_1'') - \ldots - \mu^*(z, \rho_m'').$$

The averaging of Eq. (34) over all possible fluctuations of the parameter ν can be more easily done by shifting from Eq. (34) to the integrodifferential equation for $\Gamma_{n,m} = \langle\gamma(z; \rho_1', \ldots, \rho_m'')\rangle$ (Refs. 13 and 14), that is,

$$2ik\Gamma_{n,m}(z; \rho_\alpha', \rho_\beta'') - 2ik\left\langle \exp\left(\frac{ik}{2}\right) \times \int_0^z \nu(\xi; \rho_\alpha', \rho_\beta'')d\xi\right\rangle\Gamma_{n,m}(0; \rho_\alpha', \rho_\beta'') = -\int_0^z dz'\left\langle\exp\left(\frac{ik}{2}\right)\int_{z'}^z \nu(\xi; \rho_\alpha', \rho_\beta'')d\xi\right\rangle\hat{L}\Gamma_{n,m}(z'; \rho_\alpha', \rho_\beta'). \tag{36}$$

If the largest scale of the variation in the fluctuating parameter ν over ξ is much smaller than the length of the path z, it can be assumed that the random field ν is delta correlated over ξ. The statistical independence of any values of ν at different ξ's stems from this assumption. Using this property and employing a procedure similar to that of Ref. 14, we arrive at a differential equation

$$\partial\Gamma_{n,m}/\partial z = (i/2k)\hat{L}\Gamma_{n,m} + \Gamma_{n,m}(\partial/\partial z)\ \ln W(z; \{\rho\}), \tag{37}$$

where

$$W(z; \{\rho\}) = \left\langle \exp\left[(ik/2)\int_0^z \nu(\xi, \{\rho\})d\xi\right]\right\rangle \cdot \tag{38}$$

During the derivation of Eq. (37) only the δ-correlation property is used; thus Eq. (37) is valid for any ν distribution.

Equation (37) is similar to the well-known equation for $\Gamma_{n,m}$ in the nonabsorbing random medium. The difference lies only in the specific forms of the function $W(z; \{\rho\})$. We now consider the averaging procedure. For this purpose let us first write the function $\nu(\xi; \{\rho\})$ in the form

$$\nu = \nu_1 + i\nu_2,$$

where [see Eq. (33)]

$$\nu_1 = \sum_{\alpha=1}^{n} \eta(\rho_\alpha') - \sum_{\beta=1}^{m} \eta(\rho_\beta''), \tag{39}$$

$$\nu_2 = \sum_{\alpha=1}^{n} \chi(\rho_\alpha') + \sum_{\beta=1}^{m} \chi(\rho_\beta''). \tag{40}$$

Since the probability-distribution function of the index of refraction $n(\mathbf{r})$ and hence ν are presented in the form of a mixture [see Eq. (1)], the calculation of the average in Eq. (38) consists of the calculation of the average on the small scale and then on the large scale. The characters of the fluctuations of values ν_1 and ν_2 differ because ν_1 describes random refraction, whereas ν_2 describes random adsorption. The value ν_1 is alternating, whereas ν_2 is nonnegative. These can easily be seen from expressions (39) and (40).

Fluctuations of ν_1 can usually be described by a Gaussian distribution, which is not suitable for describing fluctuations of ν_2. We shall apply Gaussian distribution only to the part of W that depends on ν_1 in the small-scale averaging and then carry out the averaging of the entire expression for the large scale using the distribution in Eq. (9), which describes both fluctuations of ν_2 and fluctuations of statistical parameters of ν_1 that appear in W after the small-scale averaging is carried out. Let us represent W in the form

$$W(z, \{\rho\}) = \langle \Psi_1 \Psi_2 \rangle_{\text{ls}}, \tag{41}$$

where

$$\Psi_1 = \left\langle \exp\left[(ik/2) \int_0^z \nu_1(\xi, \{\rho\}) \mathrm{d}\xi \right] \right\rangle_{\text{ss}}, \tag{42}$$

$$\Psi_2 = \exp\left[- (k/2) \int_0^z \nu_2(\xi, \{\rho\}) \mathrm{d}\xi \right] \cdot \tag{43}$$

If the field $\eta(\xi, \boldsymbol{\rho})$ is Gaussian and is delta correlated over ξ, the averaging in Eq. (42) will yield

$$\Psi_1 = \exp\left[- (k^2/8) \int_0^z Q_{n,m}(\xi; \boldsymbol{\rho}_\alpha', \boldsymbol{\rho}_\beta'') \mathrm{d}\xi \right], \tag{44}$$

where

$$\begin{aligned} Q_{n,m} = & \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} A[\xi; \boldsymbol{\rho}_\alpha' - \boldsymbol{\rho}_\beta', (\boldsymbol{\rho}_\alpha' + \boldsymbol{\rho}_\beta')/2] \\ & - 2 \sum_{\alpha=1}^{n} \sum_{\beta=1}^{m} A[\xi; \boldsymbol{\rho}_\alpha' - \boldsymbol{\rho}_\beta'', (\boldsymbol{\rho}_\alpha' + \boldsymbol{\rho}_\beta'')/2] \\ & + \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} A[\xi; \boldsymbol{\rho}_\alpha'' - \boldsymbol{\rho}_\beta'', (\boldsymbol{\rho}_\alpha'' + \boldsymbol{\rho}_\beta'')/2], \end{aligned} \tag{45}$$

with

$$\begin{aligned} & A[\xi; \boldsymbol{\rho}_\alpha - \boldsymbol{\rho}_\beta, (\boldsymbol{\rho}_\alpha + \boldsymbol{\rho}_\beta)/2] \\ & = \int_{-\infty}^{\infty} \langle \eta(\xi - \xi'/2, \boldsymbol{\rho}_\alpha) \eta(\xi + \xi'/2, \boldsymbol{\rho}_\beta) \rangle_{\text{ss}} \mathrm{d}\xi. \end{aligned} \tag{46}$$

In deriving formula (44) it was taken into account that the field $\eta(\xi, \boldsymbol{\rho})$ is not statistically homogeneous since the statistical characteristics of A depend on the coordinate of the center of gravity and on the longitudinal coordinate ξ. This dependence is of a random nature.

Before calculating the average over the large scale, let us consider another method of averaging on the small scale related to the use of radiant intensity. The relationship between the two-point coherence function and radiant intensity has been mentioned previously. Bearing in mind the moments of radiant intensity and assuming that $m = n$ is true everywhere, we single out pairs of arguments $\boldsymbol{\rho}_\alpha'$ and $\boldsymbol{\rho}_\beta'$ that are parts of the combination of the fields of the type vv^* in expression (34). If such pairs are averaged over the small scale, then instead of $\Gamma_{n,m}(z; \{\rho\})$ we shall have $\tilde{\Gamma}_{n,n}(z; \{\rho\})$, defined as

$$\tilde{\Gamma}_{n,n} = \left\langle \prod_{\alpha=1}^{n} \langle v(z; \boldsymbol{\rho}_\alpha') v^*(z; \boldsymbol{\rho}_\alpha'') \rangle_{\text{ss}} \right\rangle_{\text{ls}} \cdot \tag{47}$$

It is easy to see that the statistical moment defined this way satisfies an equation similar to Eq. (37):

$$\partial \tilde{\Gamma}_{n,n}/\partial z = (i/2k) \hat{L} \tilde{\Gamma}_{n,n} + (\partial/\partial z) \ln \tilde{W}(z; \boldsymbol{\rho}_\alpha', \boldsymbol{\rho}_\alpha''), \tag{48}$$

where

$$\tilde{W} = \langle \tilde{\Psi}_1 \tilde{\Psi}_2 \rangle_{\text{ls}}, \tag{49}$$

$$\tilde{\Psi}_1 = \prod_{\alpha=1}^{n} \left\langle \exp\{(ik/2) \int_0^z [\eta(\xi, \boldsymbol{\rho}_\alpha') - \eta(\xi, \boldsymbol{\rho}_\alpha'')] \mathrm{d}\xi\} \right\rangle_{\text{ss}}, \tag{50}$$

$$\tilde{\Psi}_2 = \exp\left\{ - (k/2) \int_0^z \mathrm{d}\xi \sum_{\alpha=1}^{n} [\chi(\xi; \boldsymbol{\rho}_\alpha') + \chi(\xi; \boldsymbol{\rho}_\alpha'')] \right\} \cdot \tag{51}$$

Under the same assumptions made for Eq. (42), namely, the Gaussian distribution and the delta correlation, the averaging in Eq. (50) yields

$$\begin{aligned} \tilde{\Psi}_1 = & \exp[- (k^2/4) \int_0^z d\xi \sum_{\alpha=1}^{n} \\ & \times D(\xi, \boldsymbol{\rho}_\alpha' - \boldsymbol{\rho}_\alpha'', (\boldsymbol{\rho}_\alpha' + \boldsymbol{\rho}_\alpha'')/2)], \end{aligned} \tag{52}$$

where

$$\begin{aligned} D[\xi; \boldsymbol{\rho}' - \boldsymbol{\rho}'', (\boldsymbol{\rho}' + \boldsymbol{\rho}'')/2] = & A(\xi; 0\, \boldsymbol{\rho}')/2 + A(\xi; 0, \boldsymbol{\rho}'')/2 \\ & - A[\xi; \boldsymbol{\rho}' - \boldsymbol{\rho}'', (\boldsymbol{\rho}' + \boldsymbol{\rho}'')/2]. \end{aligned} \tag{53}$$

The calculation of average values in formulas (41) and (49) for arbitrary n and m is a laborious task. Therefore, we shall write out equations only for the most widely used moments $\Gamma_{n,m}$ and $\tilde{\Gamma}_{n,n}$ and then carry out the final computation of functions W and $\tilde{W}$ for these cases.

The equation for the mean field, $\Gamma_{1,0} = \langle v \rangle$, is obtained from Eq. (37) when $n = 1$ and $m = 0$. It satisfies

$$\partial \langle v \rangle/\partial z = (i/2k) \Delta_\perp \langle v \rangle + \langle v \rangle (\partial/\partial z) \ln W_0, \tag{54}$$

where

$$W_0 = \left\langle \exp\left\{ - (k^2/8) \int_0^z [A(\xi, 0, \boldsymbol{\rho}) + 4\chi(\xi, \boldsymbol{\rho})/k] \mathrm{d}\xi_{\text{ls}} \right\} \right\rangle \cdot \tag{55}$$

The equation for the two-point coherence function $\Gamma_{1,1}$ can be obtained either from Eq. (37) or from Eq. (48), assuming that $n = m = 1$:

$$\partial \Gamma_{1,1}/\partial z = (i/2k)(\Delta_\perp' - \Delta_\perp'') \Gamma_{1,1} + \Gamma_{1,1} \partial/\partial z\, (\ln W_1), \tag{56}$$

where

$$\begin{aligned} W_1 = \Big\langle \exp\Big(& - (k^2/4) \int_0^z \{D[\xi; \boldsymbol{\rho}_1' - \boldsymbol{\rho}_1'', (\boldsymbol{\rho}_1' + \boldsymbol{\rho}_1'')/2] \\ & + 2\chi(\xi, \boldsymbol{\rho}_1')/k + 2\chi(\xi, \boldsymbol{\rho}_1'')/k\} \mathrm{d}\xi \Big) \Big\rangle_{\text{ls}} \cdot \end{aligned} \tag{57}$$

The equation for the correlation function $\tilde{\Gamma}_{2,2}$ is obtained from Eq. (48) when $n = 2$:

$$\begin{aligned} \partial \tilde{\Gamma}_{2,2}/\partial z = & (i/2k)(\Delta_{\perp 1}' + \Delta_{\perp 2}' - \Delta_{\perp 1} - \Delta_{\perp 2}'') \\ & \times \tilde{\Gamma}_{2,2} + \tilde{\Gamma}_{2,2} (\partial/\partial z) \ln \tilde{W}_2, \end{aligned} \tag{58}$$

where

$$\begin{aligned} \tilde{W}_2 = \Big\langle \exp\Big(& -(k^2/4) \int_0^z \{D[\xi; \boldsymbol{\rho}_1' - \boldsymbol{\rho}_1'', (\boldsymbol{\rho}_1' + \boldsymbol{\rho}_1'')/2] \\ & + D[\xi; \boldsymbol{\rho}_2' - \boldsymbol{\rho}_2'', (\boldsymbol{\rho}_2' + \boldsymbol{\rho}_2'')/2] + 2[\chi(\xi, \boldsymbol{\rho}_1') \\ & + \chi(\xi, \boldsymbol{\rho}_2') + \chi(\xi, \boldsymbol{\rho}_1'' + \chi(\xi, \boldsymbol{\rho}_2'')]/k\} d\xi \Big) \Big\rangle_{\text{ls}} \cdot \end{aligned} \tag{59}$$

The small-angle approximation for the radiative-transport equation can be derived from Eq. (56) if the variables $\rho = \rho_1' - \rho_1''$ and $\mathbf{R} = (\rho_1' + \rho_1'')/2$ and the Wigner representation are used[16]:

$$\Gamma_{1,1}(z; \mathbf{R}, \rho) = \iint \bar{J}(\xi; \mathbf{R}, \kappa)\exp(i\kappa \cdot \rho)\mathrm{d}^2\kappa, \qquad (60)$$

where $\bar{J} = \langle J\rangle_{ls}$ is the radiant intensity averaged over the large scale. The nonaveraged radiant intensity J is related to the two-point coherence function $\Gamma_{1,1}{}^{ss} = \langle v(z; \mathbf{R} - \rho/2)v^*(z; \mathbf{R} + \rho/2)\rangle$ similarly to Eq. (60):

$$\Gamma_{1,1}{}^{ss}(z; \mathbf{R}, \rho) = \iint \bar{J}(\xi; \mathbf{R}, \kappa)\exp(i\kappa\rho)d^2\kappa, \qquad (61)$$

that is, Eq. (60) is obtained through the averaging of Eq. (61) on a large scale. The equation for $\bar{J}$ has the following form:

$$\partial\bar{J}/\partial z + (\kappa/k)\partial\bar{J}/\partial\mathbf{R} = G_1(z; \mathbf{R}, \kappa), \qquad (62)$$

where

$$G_1 = (2\pi)^{-2}\iint d^2\rho[(\partial/\partial z)\ln W_1] \times \iint d^2\kappa'\bar{J}(z; \kappa', \mathbf{R})\exp[i(\kappa' - \kappa)\rho]. \qquad (63)$$

Let us assume that the function $(\partial/\partial z)\ln W_1$ has the form

$$(\partial/\partial z)\ln W_1 = H(\rho) - H_e, \qquad (64)$$

where H_e = constant, $H(\rho)$ is a positive, finite function, and the transform

$$H(\rho) = 2\pi\iint \varphi(\kappa)\exp(i\kappa\rho)d^2\kappa \qquad (65)$$

exists. Then the standard transport equation can be derived from Eq. (62):

$$[\partial/\partial z + \mathbf{n}_\perp\nabla_\mathbf{R} + H_e]\bar{J}(z; \mathbf{R}, n_\perp = H_s\iint \bar{J}(z; \mathbf{R}, \mathbf{n}_\perp')X(|\mathbf{n}_\perp - \mathbf{n}_\perp'|)\mathrm{d}\mathbf{n}_\perp', \qquad (66)$$

where $\mathbf{n}_\perp = \kappa/k$;.$H_e$ represents the effective extinction coefficient. If it is written as a sum of the effective scattering coefficient H_s and absorption ratio H_a, $H_e = H_s + H_a$, the effective scattering indicator $X(\gamma)$ will be

$$X(\gamma) = 2\pi k^2\varphi(k\gamma)/H_s. \qquad (67)$$

For the equation for the autocorrelation function of J, similar to the derivation of Eq. (62) or (66) from Eq. (56), we can obtain from Eq. (58) the equation for the quantity

$$B_J(z; \mathbf{R}, \mathbf{R}', \mathbf{n}_\perp, \mathbf{n}_\perp') = \langle J(z; \mathbf{R}, \mathbf{n}_\perp)J(\mathbf{z}; \mathbf{R}', \mathbf{n}_\perp')\rangle \qquad (68)$$

introduced in Ref. 9.

For this purpose it is necessary to shift to difference ρ, ρ' and summation $\mathbf{R}$, $\mathbf{R}'$ variables and to use Wigner's representation for the function $\bar{\Gamma}_{2,2}$:

$$\bar{\Gamma}_{2,2}[z; \mathbf{R}, \mathbf{R}', \rho, \rho') = k^4\iint \mathrm{d}^2n_\perp \times \iint \mathrm{d}^2n_\perp'B_J(z; \mathbf{R}, \mathbf{R}', \mathbf{n}_\perp, \mathbf{n}_\perp')\exp(ik[\mathbf{n}_\perp\rho + \mathbf{n}_\perp' \cdot \rho'), \qquad (69)$$

which easily can be obtained by using definitions (47) and (61). The result is

$$\partial B_J/\partial z + \mathbf{n}_\perp\partial B_J/\partial\mathbf{R} + \mathbf{n}_\perp'\partial B_J/\partial\mathbf{R}' = G_2(z; \mathbf{R}, \mathbf{R}', \mathbf{n}_\perp, \mathbf{n}_\perp'), \qquad (70)$$

where

$$G_2 = (2\pi)^{-4}k^4\iint \mathrm{d}^2\rho\iint \mathrm{d}^2\rho'(\partial/\partial z)\ln W_2 \times \iint \mathrm{d}^2m_\perp\iint \mathrm{d}^2m_\perp'B_J(z; \mathbf{R}, \mathbf{R}', \mathbf{m}_\perp, \mathbf{m}_\perp') \times \exp\{ik[\mathbf{m}_\perp - \mathbf{n}_\perp)\cdot\rho + (\mathbf{m}_\perp' - \mathbf{n}_\perp')\cdot\rho']\}. \qquad (71)$$

To solve the equations thus obtained it is necessary first to determine the functions W_0, W_1, and W_2 given by formulas (55), (57), and (59), respectively. This can be done in a manner similar to the examples in Section 2. By using the gamma-distribution model for the integral values in the exponents of expressions (55), (57), and (59) and taking into account that all these formulas have a form similar to that of Eq. (8), formula (11) can be conveniently employed for the evaluation of W_i's. Equations (12)–(19) can still be used to calculate N and α if the function $y(\xi)$ is redefined as a function of both ξ and ρ_α. It should also be noted that, in the three-dimensional problem, we can always assume that $z \gg L_y$. To avoid cumbersome expressions, we shall not give the detailed derivation of W_i's but only the final results:

$$W_0 = (1 + N_0/\alpha_0)^{-\alpha_0}, \qquad (72)$$

where

$$N_0 = k^2z[\langle C_n{}^2\rangle\tilde{A}(0) + 4\langle\chi\rangle/k]/8, \qquad (73)$$

$$\alpha_0 = z(\langle C_n{}^2\rangle\tilde{A}(0) + 4\langle\chi\rangle/k)^2[\tilde{A}^2(0)R_{cc}(0) + 8\tilde{A}(0)R_{\chi c}(0)/k + 16R_{\chi\chi}(0)/k^2]^{-1}, \qquad (74)$$

$$\tilde{\bar{A}}(\rho) = 8\pi\iint \tilde{\Phi}(0, \kappa)\exp(i\kappa\cdot\rho)\mathrm{d}^2\kappa,\ R_{pq}(\rho_1 - \rho_2) = \int_{-\infty}^{\infty}[\langle p(0, \rho_1)q(\xi, \rho_2\rangle - \langle p(0, \rho_1)\rangle\langle q(\xi, \rho_2)\rangle]\mathrm{d}\xi. \qquad (75)$$

In Eq. (74) the symbol c is used instead of $C_n{}^2$ for the sake of simplicity, and we maintain this convention in what follows.

$$W_1 = (1 + N_1/\alpha_1)^{-\alpha_1}, \qquad (76)$$

where

$$N_1 = k^2z[\langle C_n{}^2\rangle\tilde{D}(\rho_1' - \rho_1'') + 4\langle\chi\rangle/k]/4, \qquad (77)$$

$$\alpha_1 = 2z[\langle C_n{}^2\rangle\tilde{D}(\rho_1' - \rho_1'') + 4\langle\chi\rangle/k]^2 \times \Bigg(\tilde{A}(0)[R_{cc}(0) + R_{cc}(\rho_1' - \rho_1'')] - 4\tilde{A}(0)\tilde{A}(\rho_1' - \rho_1'')R_{cc} \times \left(\frac{\rho_1' - \rho_1''}{2}\right) + 2\tilde{A}^2(\rho_1' - \rho_1'')R_{cc}(0) + \frac{8}{k}\left\{\tilde{A}(0)[R_{\chi c}(0) + R_{\chi c}(\rho_1' - \rho_1'')] - 2\tilde{A}(\rho_1' - \rho_1'')R_{\chi c}\left(\frac{\rho_1 - \rho_1'}{2}\right)\right\} + \frac{16}{k^2}[R_{\chi\chi}(0) + R_{\chi\chi}(\rho_1' - \rho_1'')]\Bigg)^{-1} \qquad (78)$$

It is evident from formulas (73) and (74) that W_0, and hence the mean field, are given in terms of the one-point moments $R_{cc}(0)$, $R_{\chi c}(0)$, and $R_{\chi\chi}(0)$, the same parameters that appear in the solution for the layered medium. Actually, for statistically homogeneous three-dimensional variations of $C_n{}^2$ and χ, the solution of equation (54) for the mean field of a plane wave is $\langle v\rangle \equiv W_0$. If one assumes that $\chi = 0$, the expression for $\langle v\rangle$ will completely coincide with formula (12), obtained in Section 2 for layered (one-dimensional) variations of $C_n{}^2$, if all other conditions remain the same.

In the expression for W_1 in Eq. (56) for the two-point coherence function $\Gamma_{1,1}$ and in the radiative-transport equation (66), there are dependences of transverse coordinates in covariations R_{cc}, $R_{\chi c}$, and $R_{\chi\chi}$. It is clearly seen from formulas (76)–(78) under which conditions the transition to the approximation of layered variations is possible.

$$W_2 = (1 + N_2/\alpha_2)^{-\alpha_2}. \tag{79}$$

In this case the exact expression for α_2 proves to be so cumbersome and inconvenient to use that it is pointless to give it here. We shall therefore present only the expressions for N_2 and α_2 for the case when distances between points ρ_1' and ρ_1'' and, correspondingly, those between ρ_2' and ρ_2'' do not exceed L_R, the scale of covariations $R_{cc}(\rho)$, $R_{\chi c}(\rho)$, and $R_{\chi\chi}(\rho)$. However, the distance between these pairs can be arbitrary. This limitation is not serious because $\tilde{\Gamma}_{2,2}$ converges to zero rapidly for

$$|\rho_1' - \rho_1''| > \rho_0, \qquad |\rho_2' - \rho_2''| > \rho_0,$$

where $\rho_0 \ll L_R$ is the coherence radius or the scale of the function $\Gamma_{1,1}$. Hence in the region of interest, where essential changes of the function $\tilde{\Gamma}_{2,2}$ take place, the above limitation is not severe. Those expressions are

$$N_2 = k^2 z\{\langle C_n^2\rangle[\tilde{D}(\rho_1' - \rho_1'') + \tilde{D}(\rho_2' - \rho_2'')] + 8\langle\chi\rangle/k\}, \tag{80}$$

$$\begin{aligned}\alpha_2 = z\{&\langle C_n^2\rangle[\tilde{D}(\rho_1' - \rho_1'') + \tilde{D}(\rho_2' - \rho_2'')] + 8\langle\chi\rangle/k\}^2 \\ &\times \{R_{cc}(0)[\tilde{D}^2(\rho_1' - \rho_1'') + \tilde{D}^2(\rho_2' - \rho_2'')] \\ &+ 2R_{cc}(\rho_1 - \rho_2')\tilde{D}(\rho_1' - \rho_1'')\tilde{D}(\rho_2' - \rho_2'') \\ &+ 8[R_{\chi c}(0) + R_{\chi c}(\rho_1' - \rho_2')] \\ &\times [\tilde{D}(\rho_1' - \rho_1'') + \tilde{D}(\rho_2' - \rho_2'')]/k \\ &+ 32[R_{\chi\chi}(0) + R_{\chi\chi}(\rho_1' - \rho_2')]/k^2\}^{-1}.\end{aligned} \tag{81}$$

4. CONCLUSIONS

We have studied a random inhomogeneous medium in which, along with ordinary, sufficiently small-scale index-of-refraction fluctuations, large-scale fluctuations of such statistical parameters as the turbulence structure constant C_n^2 or the absorption ratio are also present. If the entire time span during which the field moments are measured is much larger than the characteristic time of the change in small-scale fluctuations yet still much smaller than the characteristic time of large-scale fluctuations, all conclusions of the conventional, classic theory are valid,[13,14] since the statistical characteristics of the medium can be considered stationary.

The situation changes when the measuring-time span becomes much longer than the characteristic time of the large-scale fluctuations of the statistical characteristics of the medium. In this paper the randomized probability distribution function of the index of refraction has been used to describe such a medium. Equations for arbitrary field moments recorded according to the new scheme have been derived. In the special case of a nonabsorbing turbulent medium analytical expressions for the mean field and for the two-point coherence function were obtained. It was shown that the presence of C_n^2 fluctuations leads to an increase in the field moments compared with those for the unperturbed case. For the mean field this is equivalent to an increase in the effective coherence radius. Furthermore, the form of the dependence of the field moments on the parameters varies compared with the unperturbed case. For instance, on short propagation paths the dependence of the mean field and the coherence function on the distance assumes the form of a power series, and for paths longer than the characteristic scale of C_n^2 fluctuations this dependence becomes exponential. In the limit of large distances the exponent tends to the value corresponding to the unperturbed C_n^2, that is, the effect of C_n^2 fluctuations vanishes. This is a result of the averaging of fluctuations on the wave-propagation path.

The small-angle approximations of the radiative-transport equation and of the equation for the autocorrelation function of radiant intensity B_J [see Eq. (68)] were obtained from the equations for the corresponding field moments. In Refs. 4, 5, and 9, during the derivation of such equations the transport equation for radiant intensity not yet averaged over the large scale is given as the initial equation. The fluctuations of the extinction coefficient—a positive value—are assumed to be Gaussian. This assumption limits the applicable region of the results. Our approach avoids this shortcoming because, during the randomization of the initial distribution, model density was used, which satisfies the conditions of positivity for the corresponding random value. The expansion with respect to a small parameter gives the transition to the case of Gaussian density. This model probability-distribution function was chosen also for the simplification of calculations. Naturally other distributions could be more appropriate in certain situations.

As is evident from the expressions obtained, the three-dimensional character of the fluctuations of the statistical parameters of the medium is reflected to the greatest degree on the fourth field moment of the $\tilde{\Gamma}_{2,2}$ type or on the radiant intensity autocorrelation function B_J. In Ref. 9 the asymptotic behavior of B_J over large distances in a medium with Gaussian fluctuations of the extinction coefficient was studied. The use of such statistics considerably facilitates computations but, in principle, may greatly distort the real behavior of B_J. In that sense, equations obtained in this paper are more nearly correct, although they are also more complicated. We hope that they will offer a realistic means to predict the measured results of the correlation functions $\tilde{\Gamma}_{2,2}$ and B_J.

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