Deflection Curve and Maximum Deflection of Simply Supported Beam Under Uniform Distributed Load

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Contents

1 Introduction

$$EI\frac{d^2y}{dx^2} = \frac{qLx}{2} - \frac{qx^2}{2}$$

This differential equation calculates the equation of the deflection curve and maximum deflection of a simply supported beam under an uniform distributed load. In structural engineering, understanding a beam's deformation under a load is crucial for ensuring the beam's strength and stability. It includes multiple important elements, such as the deflection curve, which displays the vertical displacement at every point along the beam, and the location of maximum deflection, which indicates the point where the greatest vertical displacement occurs. By solving this differential equation, engineers can predict the beam's behavior and ensure that the structure performs as intended under real-world conditions.

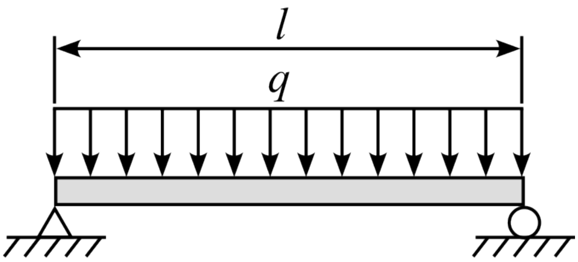

Figure 1: This image is a simply supported beam with a uniform distributed load. The left-hand side of the beam has a pinned support, and the right-hand side has a rolled support.

[2]

2 Definitions

Beam deflection is the movement of a beam from its original position due to the forces and loads being applied to the member. It is also known as displacement and can occur from externally applied loads, the weight of the structure itself, and the force of gravity [14].

Maximum deflection refers to the maximum displacement or bending of a structure or material under a load [9].

The **bending moment** is the internal resistance of a structure to bending. It is caused by the external forces acting on a structural member which creates tension on one side and compression on the other. It is calculated using the formula: force $\cdot$ distance [17].

The **deflection curve** is the curve illustrating the deflected shape of the neutral axis of a deflected beam [11].

A **simply supported beam** is a beam that spans a single distance between two supports. One end of the beam is supported by a pinned support, and the other end is supported by a roller support [8].

A **pinned support** is a common type of support and is most commonly compared to a hinge in civil engineering. A pinned support allows rotation to occur but no translation (i.e. it resists horizontal and vertical forces) [15].

A **rolled support** can resist a vertical force but not a horizontal force. A roller support or connection is free to move horizontally because there is no restriction [15].

A **uniform distributed load** is a force that is applied evenly over the distance of a support [3].

Young's Modulus, represented by E, is a constant characteristic that measures a material's capacity to deform elastically under tension or compression [6].

The **moment of inertia** is the scalar property of a body in rotational motion which opposes the change in its rotational motion due to external forces and is expressed in kgm^2. It is the sum of the product of the mass of each particle and the square of the distance from the rotational axis [10].

3 History

3.1 Leonardo da Vinci (1452 - 1519)

This differential equation's history can be traced back to Leonardo da Vinci's work "The Codex Madrid" published in 1493. The document contains concepts and illustrations regarding a beam's stress when bent, the influence of a beam's cross-sectional shape on its strength, and the deformation of beams under applied loads [7].

3.2 Galileo Galilei (1564 - 1642)

In 1638, Galileo attempted to elaborate upon Leonardo da Vinci's incomplete theory in his publication "Dialogues Concerning Two New Sciences". In the document, there is an illustration of a beam being supported by the wall. Since the beam rotated around its base at its form of support, Galileo assumes that the uniform tensile stress across the beam section was equivalent to the tensile strength of the material. Despite his hypothesis, Galileo's explanation was incorrect as his result was three times the correct value [7].

3.3 Robert Hooke (1635 - 1703)

In 1660, Robert Hooke, an English mathematician, discovered the law of elasticity or "Hooke's Law" which states that the displacement or size of the deformation was directly proportional to the deforming force or load. Once this load is removed, the object will return to its original shape and size. This deforming force is applied to solid structures by stretching, compressing, or bending. Furthermore, Hooke's findings describes the linear relationship between stress and strain. Stress is the force on unit areas within a material that develops as a result of the externally applied force. Strain is the relative deformation produced by stress [5].

3.4 Leonardo Euler (1707 - 1783) & Daniel Bernoulli (1700 - 1782)

A major breakthrough came in the mid-18th century with the Euler-Bernoulli Beam Theory. Prior to this, Leonardo Euler, a Swiss mathematician, and Daniel Bernoulli, a Swiss physicist, successfully derived equations for the vibration, deflection, and buckling load of beams [7]. The Euler-Bernoulli Theory is composed of two main assumptions. The first assumption is that any section of the beam that was a flat plane before deformation will remain a flat plane after. It presumes that any part of the beam will be perpendicular to the neutral axis before and after deformation. The second assumption is that the deformed angles or slopes are small. Based on these valid assumptions, the formula was created. E represents Young's Modulus and I is the moment of inertia. M is the moment of the beam at location x [16].

3.5 Augustin-Louis Cauchy (1789 - 1857)

In 1822, Augustin-Louis Cauchy, a French mathematician, laid the foundations of the mathematical theory of elasticity. His greatest contributions were written in his three greatest treatises: "Courses on Analysis from the École Royale Polytechnique" (1821), Résumé of Lessons on Infinitesimal Calculus" (1823), and "Lessons on the Applications of Infinitesimal Calculus to Geometry" (1826-1828) [4].

3.6 Saint Venant (1797 - 1886)

In the 19th century, Saint Venant, a French mathematician, made significant contributions to the study of elasticity and beam bending by discovering the Saint-Venant Principle which states that the stress, strain, and displacement fields caused by two different but statically equivalent force distributions on a body are approximately the same far away from the loading points [13].

3.7 Joseph Boussinesq (1842 - 1929)

In the 19th century, Joseph Boussinesq, a French mathematician, contributed to fluid mechanics. His first publication called *Comptes rendus de l'Academie des Sciences* elaborated on elastic theory and derived macroscopic partial differential equations and approximation strategies. He believed in a molecular foundation of mechanics and physics, and he developed his own vibrational theory of heat in Amperean tradition [12].

4 Initial Value Problem (IVP)

$$EI\frac{d^2y}{dx^2} = \frac{qLx}{2} - \frac{qx^2}{2}$$

Figure 2: This is the main differential equation

4.1 Initial Conditions

- The maximum deflection is located at $x = \frac{L}{2}$ where the simply supported beam deflects the most from its original position
- The slope of the deflection curve at the point of maximum deflection is 0
- The deflection is 0 at $x = 0$ and $x = L$ because the pinned support and roller support stabilize the simply supported beam

4.2 Main Differential Equation Definitions

On the left side of the differential equation:

- E represents Young's Modulus
- I represents the moment of inertia
- $\frac{d^2y}{dx^2}$ represents the curvature of the deflection curve of the simply supported beam

On the right side of the differential equation:

- q represents the uniform distributed load
- L represents the total length of the simply supported beam
- x represents a partial length of the simply supported beam

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Figure 3: This differential equation calculates the curvature of the beam at any point. It will be used to derive the main differential equation in the "Differential Equation Derivation" section

4.3 Secondary Differential Equation Definitions

- M represents the bending moment
- E represents Young's Modulus
- I represents the moment of inertia
- $\frac{d^2y}{dx^2}$ represents the curvature of the simply supported beam

5 Differential Equation Derivation

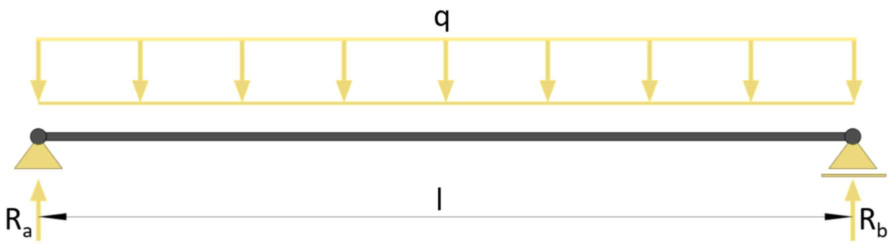

Figure 4: This image is a simply supported beam with a uniform distributed load. R_a represents Reaction A and R_b represents Reaction B.

[1]

This differential equation calculates the equation of the deflection curve and maximum deflection of a simply support beam under an uniform distributed load:

$$EI\frac{d^2y}{dx^2} = \frac{qLx}{2} - \frac{qx^2}{2}$$

To derive this differential equation, another equation must be incorporated:

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{1}$$

Since the sum of the vertical forces must equate to 0, therefore:

$$R_A + R_B = qL$$

Since the beam is symmetrical with an uniform distributed load, R_A and R_B both are $\frac{qL}{2}$

If a portion of the simply supported beam was analyzed with the partial length x, the bending moment equation is:

$$M = (\frac{qL}{2} \cdot x) - (qx \cdot \frac{x}{2})^1 \tag{2}$$

To simplify:

$$M = (\frac{qLx}{2}) - (\frac{qx^2}{2})$$

$$EI\frac{d^2y}{dx^2} = M \tag{3}$$

By using (2) and plugging it into (3), we have:

$$EI\frac{d^2y}{dx^2} = \frac{qLx}{2} - \frac{qx^2}{2}$$

[1] $\frac{qLx}{2}$ is positive because it opposes the bending moment, and $\frac{qx^2}{2}$ is negative because it goes in the same direction as the bending moment

6 Differential Equation Application

The function or the output is the equation of the deflection curve of a simply supported beam with a uniform distributed load. The first derivative is the slope of the deflection curve, and the second derivative is the curvature of the beam's deflection curve at a singular point.

Function	First Derivative	Second Derivative
Deflection or Equation of Deflection Curve	Slope of Deflection Curve	Curvature of Beam's Deflection Curve at a Point

Picking up from the "Differential Equation Derivation" section, this is the current differential equation that needs to be solved:

$$EI\frac{d^2y}{dx^2} = \frac{qLx}{2} - \frac{qx^2}{2}$$

By integrating each term, we have:

$$EI\int\frac{d^2y}{dx^2}dx = \int\frac{qLx}{2}dx - \int\frac{qx^2}{2}dx$$

or

$$EI\frac{dy}{dx} = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1$$

$\frac{dy}{dx}$ represents the slope of the deflection curve. It can be deduced that the slope of the deflection at the midpoint of the simply supported beam is 0 because it is the location of maximum deflection. Therefore $\frac{dy}{dx} = 0$ when $x = \frac{L}{2}$ and these values can be plugged into the equation:

$$EI \cdot 0 = \frac{qL}{4} \cdot (\frac{L}{2})^2 - \frac{qx}{6} \cdot (\frac{L}{2})^3 + C_1$$

Therefore, the value of C_1 is calculated:

$$C_1 = -\frac{qL^3}{24}$$

Therefore, the slope of the deflection curve becomes:

$$EI\frac{dy}{dx} = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24}$$

By integrating each term, we have:

$$EI\int\frac{dy}{dx} = \int\frac{qLx^2}{4}dx - \int\frac{qx^3}{6}dx - \int\frac{qL^3}{24}dx$$

or

$$EIy = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3x}{24} + C_2$$

y represents the deflection at any point along the simply supported beam and there is no deflection at the left-hand support. Therefore, $y = 0$ when $x = 0$ and these values can be plugged into the equation:

$$EI \cdot 0 = \frac{qL(0)^3}{12} - \frac{q(0)^4}{24} - \frac{qL^3(0)}{24} + C_2$$

Therefore, the value of C_2 is calculated:

$$C_2 = 0$$

Therefore, the equation of the deflection curve becomes:

$$EIy = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3x}{24}$$

or

$$v = -\frac{qx}{24EI} \cdot (L^3 - 2Lx^2 + x^3)$$

Since the maximum deflection occurs at the midpoint of the simply supported beam, setting $x = L/2$ would compute the equation for maximum deflection. This value can be plugged into the equation:

$$v \cdot (\frac{L}{2}) = \frac{q(\frac{L}{2})}{24EI} \cdot (L^3 - 2L \cdot (\frac{L}{2})^2 + \frac{L}{2}^3)$$

or

$$v \cdot (\frac{L}{2}) = -\frac{5qL^4}{384EI}$$

7 Conclusion

This differential equation calculates the equation of the deflection curve for a simply supported beam with a uniform distributed load on it, along with the maximum deflection. The equation of the deflection curve was calculated to be $EIy = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3x}{24}$ and the maximum deflection is computed to be $v \cdot (\frac{L}{2}) = -\frac{5qL^4}{384EI}$ under initial conditions. The solution demonstrates the importance of load distribution, beam geometry, and material properties, making it a critical tool for engineers in designing and analyzing beam structures.

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