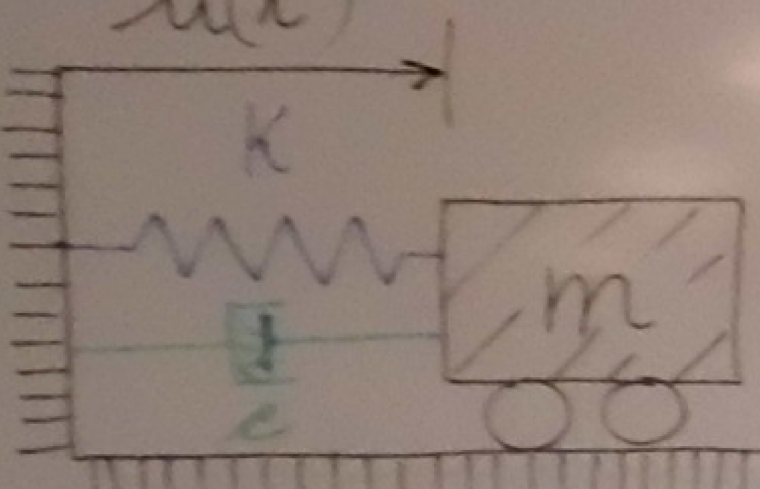
Damped free vibrations

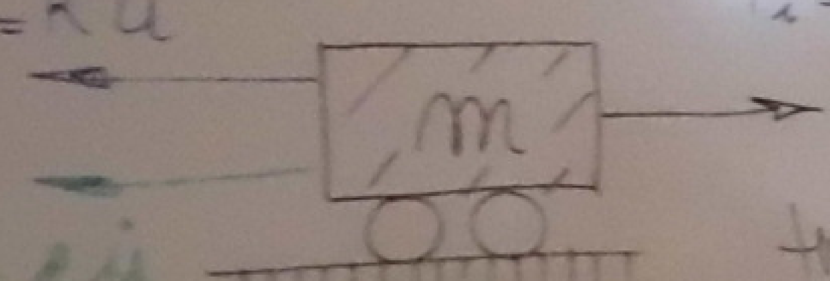
$u(t)$



I.Cs

$$\begin{cases} u(0) = u_0 \\ \dot{u}(0) = \dot{u}_0 \end{cases}$$

$F_e = Ku$ $\qquad F_i = -m\ddot{u}$



$F_d = c\dot{u}$

damping coeff.

$$\frac{m\ddot{u} + c\dot{u} + Ku = 0}{m}$$

$\underbrace{\frac{c}{m}}_{2\zeta\omega_1} \qquad \underbrace{\frac{K}{m}}_{\omega_1^2}$

$\frac{c}{m} = 2\zeta\omega_1$

$\zeta = \frac{c}{2m\omega_1} = \frac{c}{2\sqrt{Km}} = \frac{c}{c_{cr}}$ damping ratio

$\omega_1^2 = \frac{K}{m} \Rightarrow \omega_1 = \sqrt{\frac{K}{m}}$ undamped natural angular frequency

Eq. of motion becomes:

$$\ddot{u} + 2\zeta\omega_1\dot{u} + \omega_1^2 u = 0$$

Seek solution in the form:

$$u(t) = e^{\lambda t};\ \dot{u} = \lambda e^{\lambda t};\ \ddot{u} = \lambda^2 e^{\lambda t}$$

By substituting

$$\lambda^2 + 2\zeta\omega_1\lambda + \omega_1^2 = 0$$

characteristic equation

$$\lambda_{1,2} = -\zeta\omega_1 \pm \sqrt{\zeta^2\omega_1^2 - \omega_1^2}$$
$$= -\zeta\omega_1 \pm \omega_1\sqrt{\zeta^2 - 1}$$
$$= \omega_1\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \qquad \zeta^2 - 1 > 0$$

two roots of the characteristic eqn.

Cases

1) $-\ \zeta > 1,\ c > c_{cr}$ supercritical, two real roots $\lambda_{1,2} < 0$

2) $-\ \zeta = 1,\ c = c_{cr}$ critical, two coincident roots $\lambda_{1,2} = -\omega_1$

3) $\zeta < 1,\ c < c_{cr}$ subcritical, two complex conjugate roots

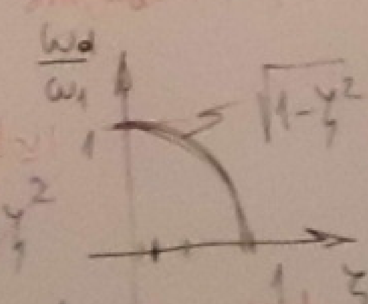
- Subcritical damping ($\zeta < 1$;

$$e^{\lambda_{1,2}t} = e^{(-\zeta\omega_1 \pm i\omega_d)t} = \underbrace{e^{-\zeta\omega_1 t}}_{\text{amplitude decay}} \cdot \underbrace{e^{\pm i\omega_d t}}_{\text{harmonic motion}}$$

General integral

$\sin\omega_d t, \cos\omega_d t$

$$u(t) = e^{-\zeta\omega_1 t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$
$$= Re^{-\zeta\omega_1 t}\cos(\omega_d t - \varphi)$$
$$= Re^{-\zeta\omega_1 t}\sin(\omega_d t + \psi)$$

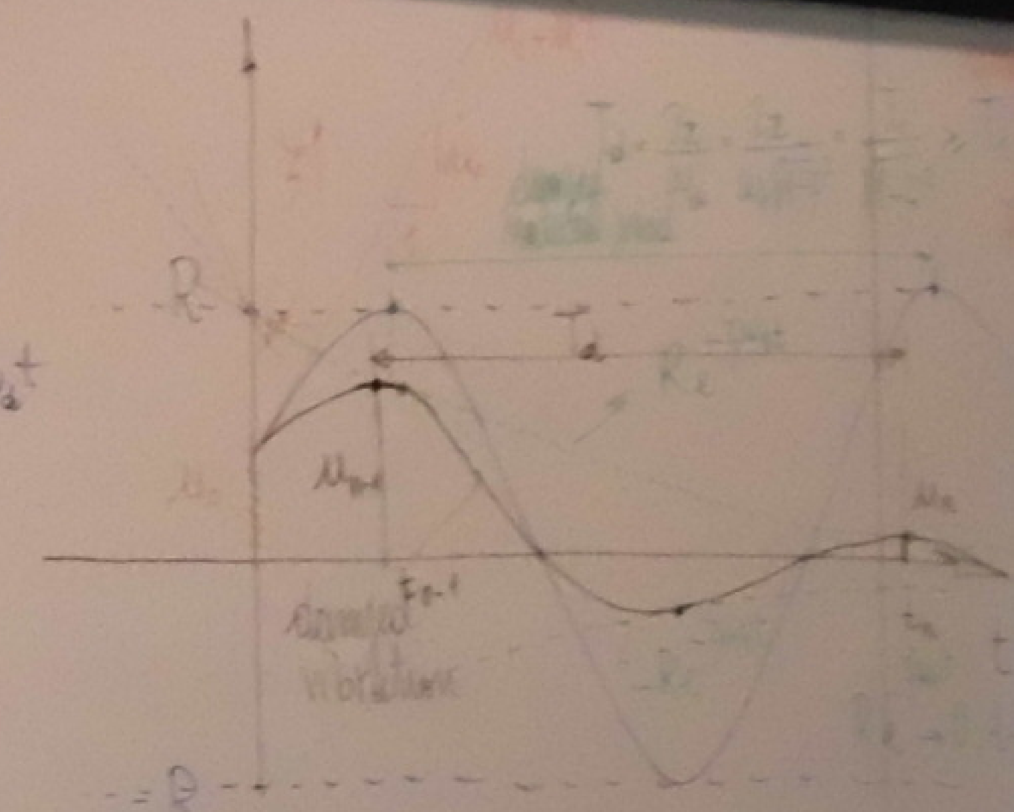


$\frac{\omega_d^2}{\omega_1^2} = 1 - \zeta^2$

$\omega_d = \omega_1\sqrt{1-\zeta^2} \leq \omega_1$

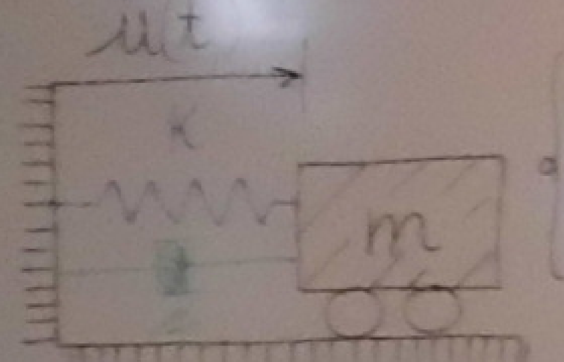
damped angular frequency

$0 > \zeta^2 - 1 = -(1-\zeta^2) = i^2(1-\zeta^2)$



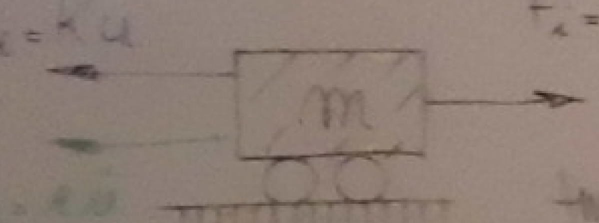
motion: harmonic motion with period T_d but with amplitude decaying in time proportional to $e^{-\zeta\omega_1 t}$

Damped free vibrations



I.C.s
$$\begin{cases} u(0) = u_0 \\ \dot{u}(0) = \dot{u}_0 \end{cases}$$

$F_e = k u$ $\quad F_i = -m\ddot{u}$



$$m\ddot{u} + c\dot{u} + k u = 0$$

$\frac{c}{m} = 2\zeta\omega_1$

$\zeta = \frac{c}{2m\omega_1} = \frac{c}{2\sqrt{km}} = \frac{c}{c_{cr}}$

$\underbrace{c/m}_{2\zeta\omega_1}$ $\quad \underbrace{k/m}_{\omega_1^2}$

$\omega_1^2 = \frac{k}{m} \to \omega_1 = \sqrt{\frac{k}{m}}$ undamped natural angular frequency

Eq of motion becomes:

$$\ddot{u} + 2\zeta\omega_1\dot{u} + \omega_1^2 u = 0$$

Seek solution in the form:

$u(t) = e^{\lambda t}$; $\dot{u} = \lambda e^{\lambda t}$; $\ddot{u} = \lambda^2 e^{\lambda t}$

substituting

$\lambda^2 + 2\zeta\omega_1\lambda + \omega_1^2 = 0$

characteristic equation

$$\lambda_{1,2} = -\zeta\omega_1 \pm \sqrt{\zeta^2\omega_1^2 - \omega_1^2}$$
$$= -\zeta\omega_1 \pm \omega_1\sqrt{\zeta^2 - 1}$$
$$= \omega_1\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

two roots of the characteristic eqn.

$\zeta^2 - 1 > 0$

Cases

1) $\zeta > 1$, $c > c_{cr}$ supercritical, two real roots $\lambda_{1,2} < 0$

2) $\zeta = 1$, $c = c_{cr}$ critical, two coincident roots $\lambda_{1,2} = -\omega_1$

3)

- Subcritical damping ($\zeta < 1$; $\zeta << 1 \sim 1\%$)

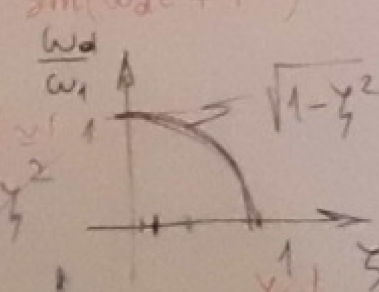
$$e^{\lambda_{1,2}t} = e^{(-\zeta\omega_1 \pm i\omega_d)t} = \underbrace{e^{-\zeta\omega_1 t}}_{\text{amplitude decay}} \cdot \underbrace{e^{\pm i\omega_d t}}_{\text{harmonic motion}}$$

General integral

$$u(t) = e^{-\zeta\omega_1 t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$

$\sin\omega_d t$, $\cos\omega_d t$

$$= R e^{-\zeta\omega_1 t}\cos(\omega_d t - \varphi)$$
$$= R e^{-\zeta\omega_1 t}\sin(\omega_d t + \psi)$$



$\zeta^2 + \frac{\omega_d^2}{\omega_1^2} = 1 - \zeta^2$

$\omega_d = \omega_1\sqrt{1-\zeta^2} \simeq \omega_1 \leq \omega_1$

damped angular frequency

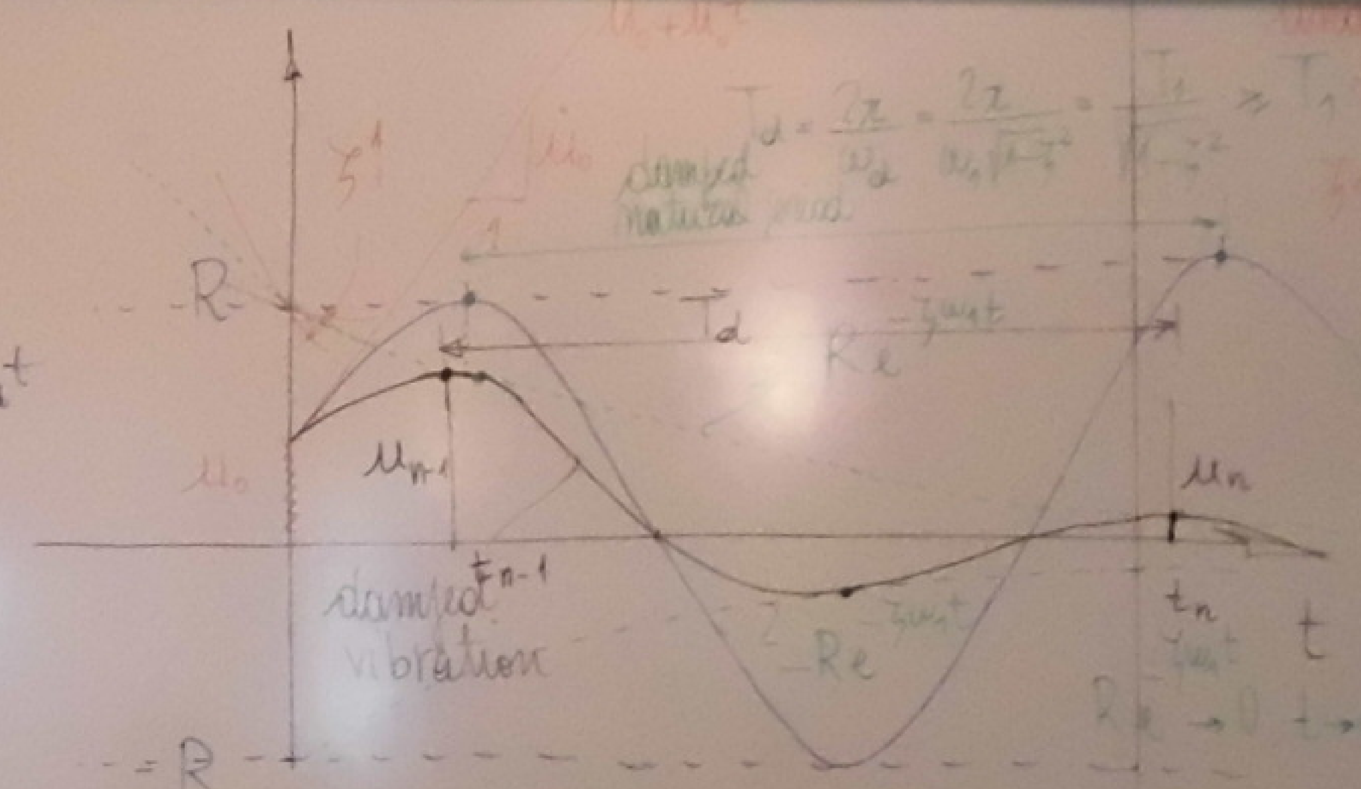
$0 > \zeta^2 - 1 = -(1-\zeta^2) = i^2(1-\zeta^2)$

ω_d

Civil engng $\zeta \sim 1\% = 0.01$

$2\% - 7\%$

$5\% = 0.05$



$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_1\sqrt{1-\zeta^2}}$

damped natural period

motion: harmonic motion with period T_d but with amplitude decaying in time according to $e^{-\zeta\omega_1 t}$ (not a harmonic motion)

oscillatory regime → free vibration (damped)

Amplitude ratio

$r_n = \frac{u_{n-1}}{u_n}$

$\delta = \ln r = \zeta\omega_1 T_d$

logarithmic decrement $= \zeta\omega_1\frac{2\pi}{\omega_1}\frac{1}{\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

$r = e^{\delta}$

$\delta^2(1-\zeta^2) = 4\pi^2\zeta^2$

$\delta^2 = (\delta^2 + 4\pi^2)\zeta^2$

$$\boxed{\zeta = \frac{\delta}{\sqrt{\delta^2 + 4\pi^2}}}$$

$$\boxed{\zeta \simeq \frac{\delta}{2\pi}}$$

Damped free vibrations
Subcritical damping
general integral
harmonic motion
$u(t) = e^{-\zeta\omega_1 t}(A\sin\omega_d t + B\cos\omega_d t)$
$\sin\omega_d t, \cos\omega_d t$
$\omega_d = \omega_1\sqrt{1-\zeta^2}$
damped angular frequency
$\sqrt{1-\zeta^2}$
Civil engng $\zeta \sim 1\% = 0.01$
$2\% - 7\%$
$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_1\sqrt{1-\zeta^2}} = \frac{T_1}{\sqrt{1-\zeta^2}}$
damped natural period
undamped natural period
$Re^{-\zeta\omega_1 t}$
$-Re^{-\zeta\omega_1 t}$
u_{n-1}
u_n
t_{n-1}
t_n
damped vibration
$Re^{-\zeta\omega_1 t} \to 0,\ t \to \infty$
motion: harmonic motion with period T_d but with amplitude decaying in time according to $e^{-\zeta\omega_1 t}$ (not a harmonic motion)
(damped)
Amplitude ratio
$r_n = \frac{u_{n-1}}{u_n} = \frac{e^{-\zeta\omega_1(t_n - T_d)} R\cos(\ )}{e^{-\zeta\omega_1 t_n} R\cos(\ )}$
$= e^{\zeta\omega_1 T_d} = \text{cost} = r$
$\delta = \ln r = \zeta\omega_1 T_d$
logarithmic decrement
$= \zeta\omega_1 \frac{2\pi}{\omega_1}\frac{1}{\sqrt{1-\zeta^2}}$
$= \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$
$r = e^{\delta} \simeq 1 + \delta$
$\delta^2(1-\zeta^2) = 4\pi^2\zeta^2$
$\delta^2 = (\delta^2 + 4\pi^2)\zeta^2$
$\boxed{\zeta = \frac{\delta}{\sqrt{\delta^2 + 4\pi^2}}}$
$\boxed{\zeta \simeq \frac{\delta}{2\pi}}$
$\delta \ll 1$

$$u(t) = e^{-\zeta\omega_1 t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$

$$\dot{u}(t) = -\zeta\omega_1 e^{-\zeta\omega_1 t}(A\sin\omega_d t + B\cos\omega_d t) + e^{-\zeta\omega_1 t}\omega_d\left(A\cos\omega_d t - B\sin\omega_d t\right)$$

By imposing the i.c.s

- $u(0) = \boxed{B = u_0}$
- $\dot{u}(0) = -\zeta\omega_1 B + \omega_d A = \dot{u}_0$

$$A = \frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} \leftarrow \omega_d = \omega_1\sqrt{1-\zeta^2}$$

Final response:

$$u(t) = e^{-\zeta\omega_1 t}\left(\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right)$$

$A, B \leftrightarrow R, \varphi$

$, \psi$

Same as for free vibrations (undamped)

$$= \omega_1\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \qquad \zeta^2 - 1 > 0$$

Cases

1) – $\zeta > 1$, $c > c_{cr}$ supercritical, two real roots $\lambda_{1,2} < 0$

2) – $\zeta = 1$, $c = c_{cr}$ <u>critical</u>, two coincident roots $\lambda_{1,2} = -\omega_1$

3) – $\zeta < 1$, $c < c_{cr}$ ...

2) Critical damping $\zeta = 1$ $(c = c_{cr} = 2\sqrt{km})$

$$\lambda_{1,2} = -\omega_1 \rightarrow e^{-\omega_1 t};\ t e^{-\omega_1 t}$$

$$u(t) = e^{-\omega_1 t}(\underset{u_0}{A} + Bt)$$

I.c.s

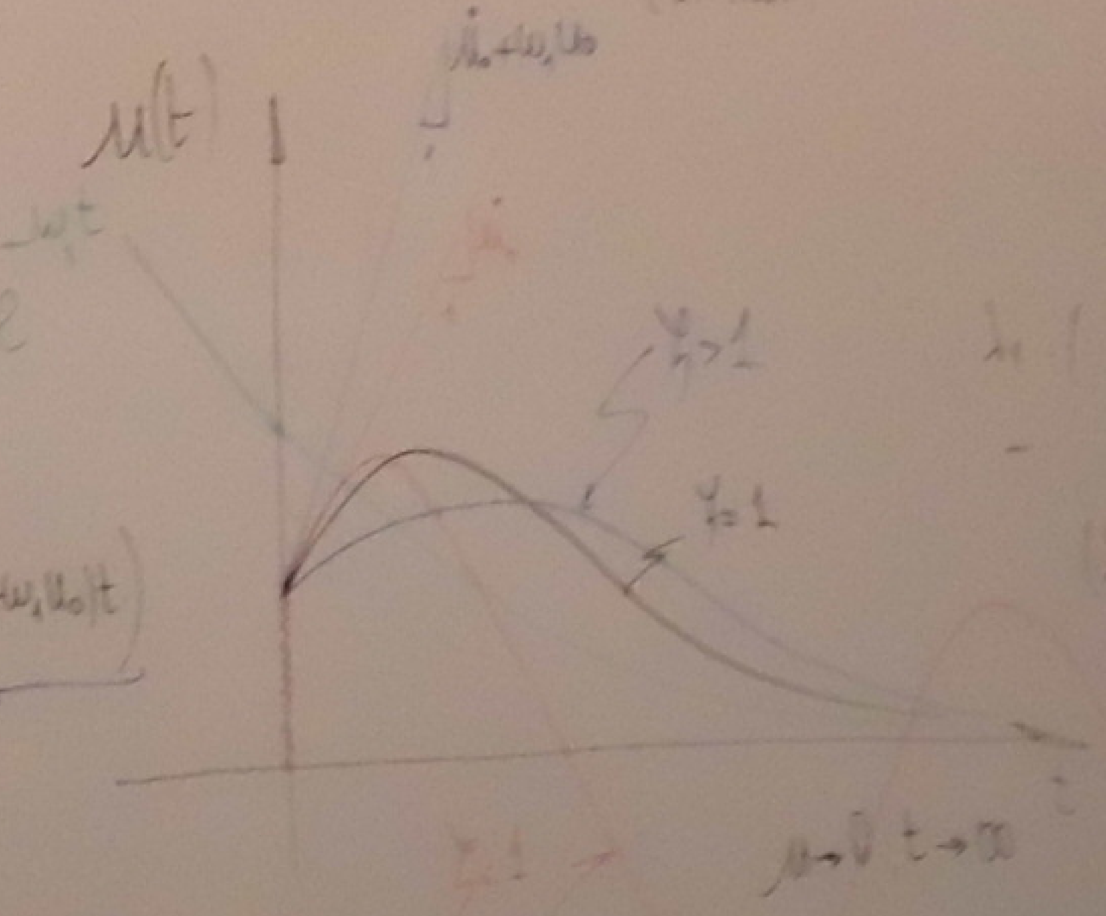
$\boxed{u_0 = \ldots}$

$\dot{u}_0 = -\omega_1 u_0 + B$

$\hookrightarrow B = \dot{u}_0 + \omega_1 u_0$

Find: $u(t) = e^{-\omega_1 t}\left(u_0 + (\dot{u}_0 + \omega_1 u_0)t\right)$

bounds the limit of oscillatory response (critical)



1) Supercritical damping ...

$u(t) = e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t)$

$+e^{-\zeta\omega_n t}\,\omega_d\,(A\cos\omega_d t - B\sin\omega_d t)$

By imposing the i.cs

- $u(0) = \boxed{B = u_0}$
- $\dot u(0) = -\zeta\omega_n B + \omega_d A = \dot u_0$

$$A = \frac{\dot u_0 + \zeta\omega_n u_0}{\omega_d} \quad \leftarrow \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Final response

$$u(t) = e^{-\zeta\omega_n t}\left(\frac{\dot u_0 + \zeta\omega_n u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right)$$

$A, B \leftrightarrow R, \varphi, \psi$

Same as for free vibrations (undamped)

$= \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \qquad \zeta^2 - 1 > 0$

cases

1) $\zeta > 1$, $c > c_{cr}$ supercritical, two real roots $\lambda_{1,2} < 0$

2) $\zeta = 1$, $c = c_{cr}$ critical, two coincident roots $\lambda_{1,2} = -\omega$

3)

2) Critical damping $\zeta = 1$ ($c = c_{cr} = 2\sqrt{km}$)

$\lambda_{1,2} = -\omega_1 \rightarrow e^{-\omega_1 t};\ t e^{-\omega_1 t}$

$u(t) = e^{-\omega_1 t}(A + Bt)$

I.cs

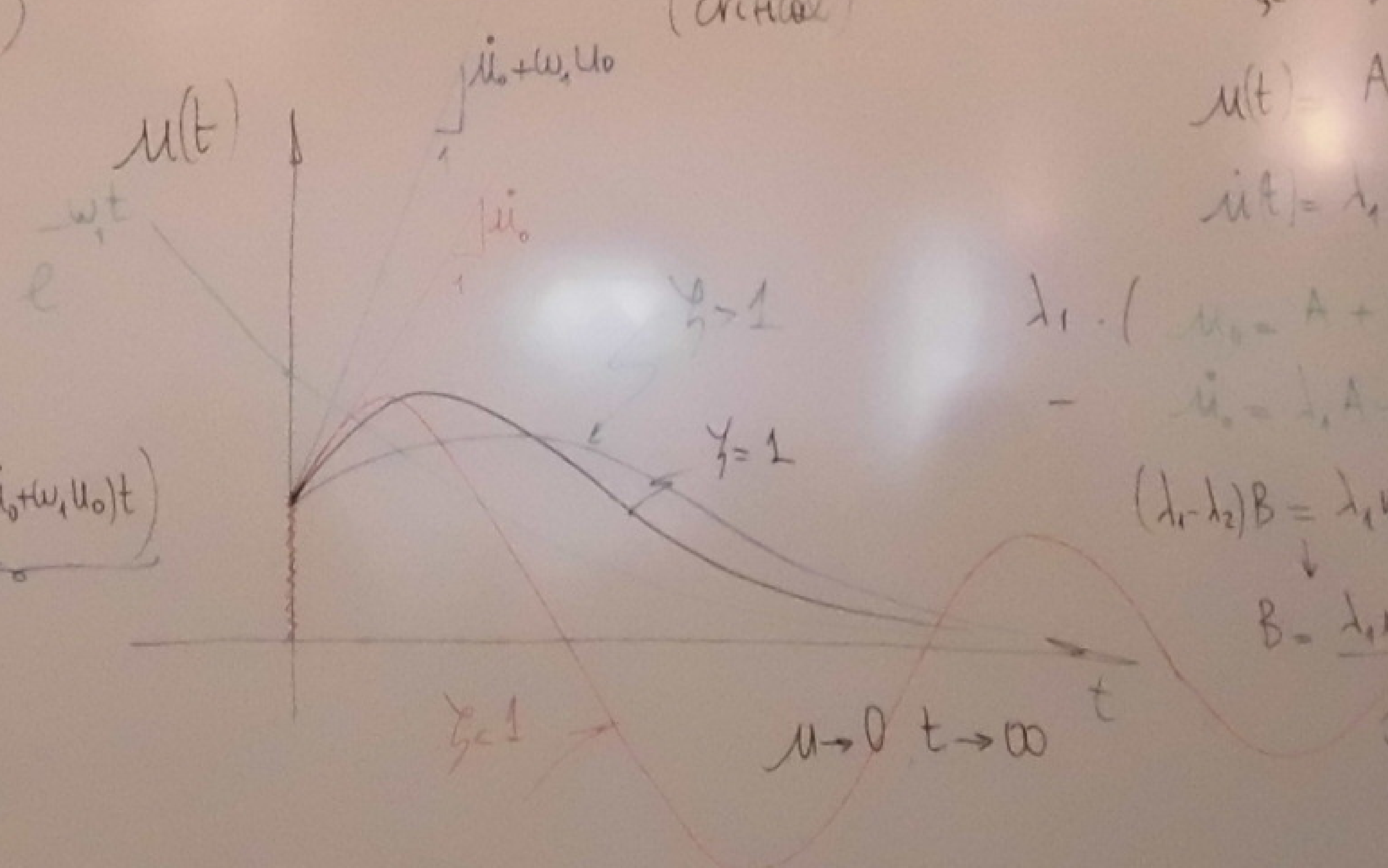
$u_0 = A$

$\dot u_0 = -\omega_1 u_0 + B$

$\rightarrow B = \dot u_0 + \omega_1 u_0$

Final: $u(t) = e^{-\omega_1 t}\left(u_0 + (\dot u_0 + \omega_1 u_0)t\right)$

bounds the limit of oscillatory response (critical)



$u \to 0 \quad t \to \infty$

1) Supercritical damping $\zeta > 1$ ($c > c_{cr} = 2\sqrt{km}$)

$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} < 0$

$u(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$

$$\dot u(t) = \lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t} = -\frac{\lambda_2 u_0 - \dot u_0}{\lambda_1 - \lambda_2}e^{\lambda_1 t} + \frac{\lambda_1 u_0 - \dot u_0}{\lambda_1 - \lambda_2}e^{\lambda_2 t}$$

$\lambda_1 \cdot (\ u_0 = A + B\ ) \cdot \lambda_2$

$-\quad \dot u_0 = \lambda_1 A + \lambda_2 B$

$(\lambda_1 - \lambda_2)B = \lambda_1 u_0 - \dot u_0$

$$B = \frac{\lambda_1 u_0 - \dot u_0}{\underbrace{\lambda_1 - \lambda_2}_{2\sqrt{\zeta^2-1}}}$$

$(\lambda_1 - \lambda_2)A = \lambda_2 u_0 - \dot u_0$

$$A = -\frac{\lambda_2 u_0 - \dot u_0}{\lambda_1 - \lambda_2}$$

2) Critical damping $\zeta = 1$ ($c = c_{cr} = 2\sqrt{km}$)

$\lambda_{1,2} = -\omega_1 \rightarrow e^{-\omega_1 t} ; t e^{-\omega_1 t}$

$u(t) = e^{-\omega_1 t}(A + Bt)$

I.C.

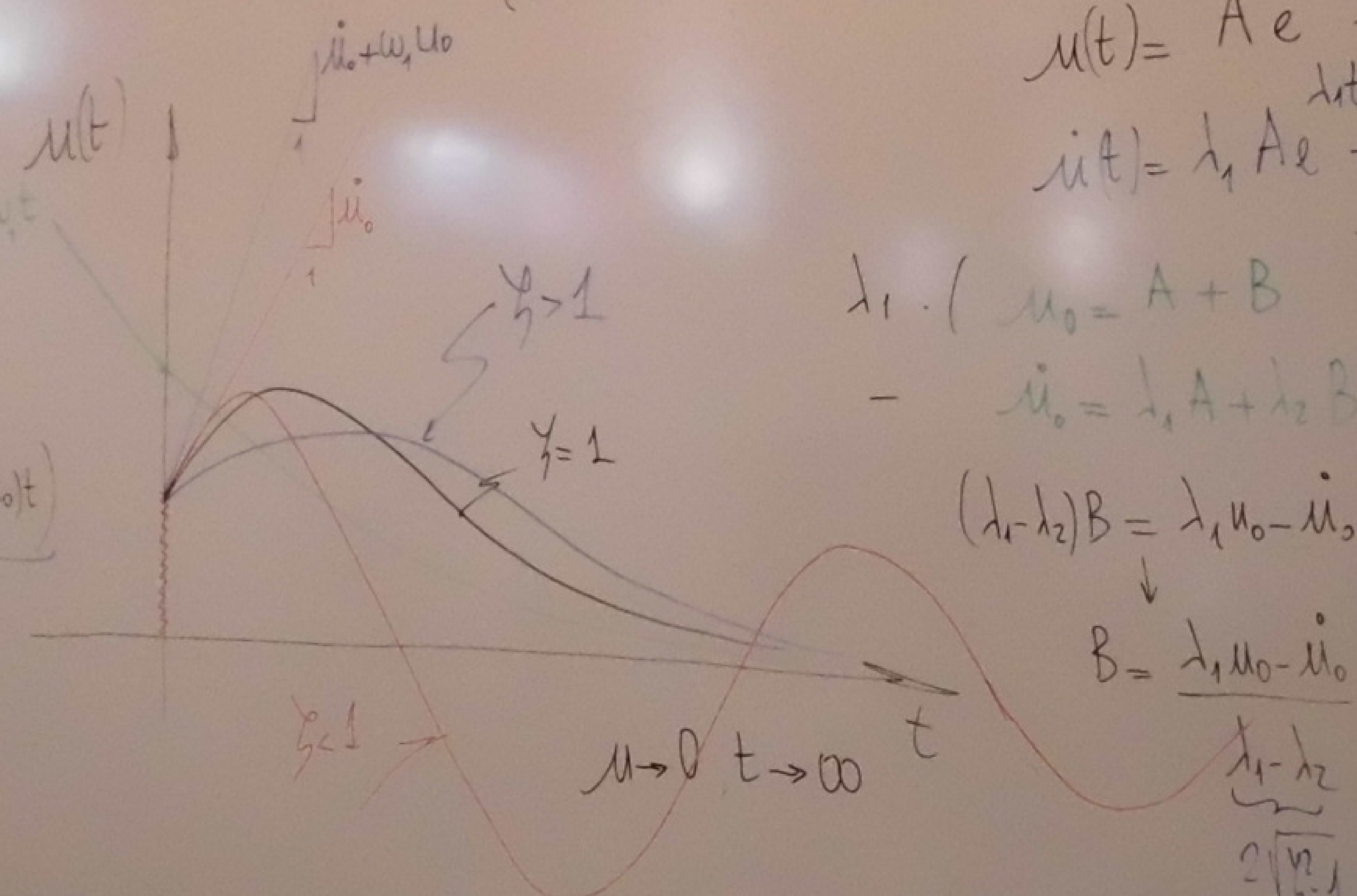
$u_0 = A$

$\dot{u}_0 = -\omega_1 u_0 + B$

$\to B = \dot{u}_0 + \omega_1 u_0$

Final $u(t) = e^{-\omega_1 t}\left(u_0 + (\dot{u}_0 + \omega_1 u_0)t\right)$

bounds the limit of oscillatory response (critical)



$u \to 0 \quad t \to \infty$

1) Supercritical damping $\zeta > 1$ ($c > c_{cr} = 2\sqrt{km}$)

$\lambda_{1,2} = -\zeta\omega_1 \pm \omega_1\sqrt{\zeta^2 - 1} < 0$

$u(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$

$\dot{u}(t) = \lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t} = -\frac{\lambda_2 u_0 - \dot{u}_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{\lambda_1 u_0 - \dot{u}_0}{\lambda_1 - \lambda_2} e^{\lambda_2 t}$

$\lambda_1 \cdot (u_0 = A + B) \cdot \lambda_2$

$- \quad \dot{u}_0 = \lambda_1 A + \lambda_2 B \quad -$

$(\lambda_1 - \lambda_2)B = \lambda_1 u_0 - \dot{u}_0$

$B = \frac{\lambda_1 u_0 - \dot{u}_0}{\underbrace{\lambda_1 - \lambda_2}_{2\sqrt{\zeta^2 - 1}}}$

$(\lambda_2 - \lambda_1)A = \lambda_2 u_0 - \dot{u}_0$

$A = -\frac{\lambda_2 u_0 - \dot{u}_0}{\lambda_1 - \lambda_2}$