Response to pulse loading

- real cases
- pattern of all dynamic forces
- generic loading → next

Step loading

$F(t) = F\,H(t)$

$F(t) = \begin{cases} 0 & t < \tau = 0 \\ F & t > \tau = 0 \end{cases}$

amplitude

Heaviside function (unit step) $H(t)$

linear time invariant SDOF dynamical system

$m\ddot{u} + c\dot{u} + ku = F(t)$

$t = 0$ (instant where step loading takes place)

$\begin{cases} u_0 \\ \dot{u}_0 \end{cases}$ ics

$\zeta \ll 1$, $\omega_d \simeq \omega_1$

- Response

$u(t) = e^{-\zeta\omega_1 t}\left(A\sin\omega_d t + B\cos\omega_d t\right) + \dfrac{F}{k}$

particular integral $(t > 0)$

steady-state response, const, u_{st}

- By setting the ics:

$t = \tau = 0$: $\begin{cases} u_0 = B + \dfrac{F}{k} \Rightarrow B = u_0 - \dfrac{F}{k} \\ \dot{u}_0 = -\zeta\omega_1 B + \omega_d A \Rightarrow A = \dfrac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} - \dfrac{\zeta\omega_1}{\omega_d}\dfrac{F}{k} \end{cases}$

Final solution

$u(t) = \underbrace{e^{-\zeta\omega_1 t}\left(\dfrac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right)}_{\text{response to non-homogeneous ics}}$

$+ \underbrace{\dfrac{F}{k}\left(1 - e^{-\zeta\omega_1 t}\left(\dfrac{\zeta\omega_1}{\omega_d}\sin\omega_d t + \cos\omega_d t\right)\right)}_{\text{response to step loading for zero ics}: u_0(t)}$

$u_0(t) = F \cdot \boxed{A(t)}$ response to unit step $(H(t))$

where $A(t) = \dfrac{1}{k}\left(1 - e^{-\zeta\omega_1(t-\tau)}\left(\dfrac{\zeta\omega_1}{\omega_d}\sin\omega_d(t-\tau) + \cos\omega_d(t-\tau)\right)\right)$

$A(t) \to \dfrac{1}{k}$, $t \to \infty$ transient → 0, $t \to \infty$

If step occurs at τ

$t < \tau$, $u_0(t) = 0$

$u_0(t,\tau) = F \cdot A(t,\tau) = F \cdot A(t-\tau)$, $t > \tau$

Pulse loading

acting for short duration $\Delta\tau$ with constant magnitude F

$\Delta I = F\,\Delta\tau$ impulse

impacts, blasts, explosions

pulse loading

$\tau = 0$, $\Delta\tau \neq 0$

$t - \Delta\tau$

$u_0(t) = F \cdot A(t)$, $t > 0$

$u_0(t) = -F \cdot A(t - \Delta\tau)$, $t > \Delta\tau$

response $u_0(t) = F \cdot A(t) - F \cdot A(t - \Delta\tau)$

$= F \cdot \Delta\tau \dfrac{A(t) - A(t-\Delta\tau)}{\Delta\tau}$

limit process as $\Delta\tau \to 0$

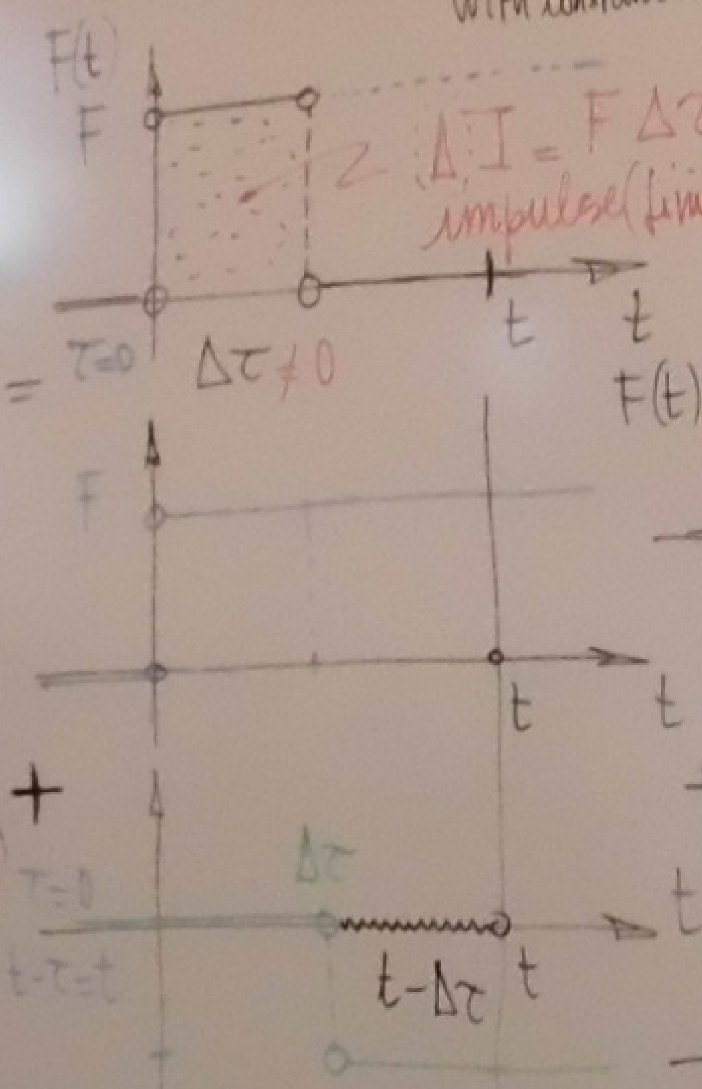
- By going to the limit $\Delta\tau \to 0$, at fixed impulse $\Delta I = I$

$u_0(t) = I\,\dot{A}(t) = I\,h(t)$ unit pulse response function

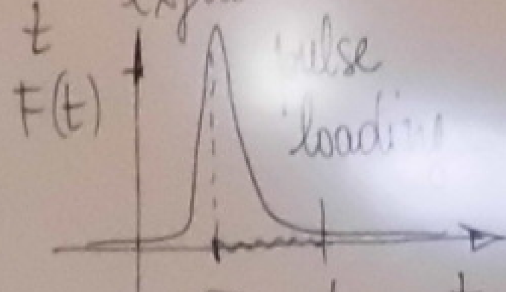
$h(t) = \dot{A}(t)$

Response to pulse loading
Step loading
Final solution
response to non-homogeneous ic
response to step load
where
Pulse loading
acting for short duration Δτ
with constant magnitude F
impacts,
blasts,
explosions
pulse loading
impulse finite
response
limit forces as Δτ→0
- By going to the limit Δτ→0, at fixed impulse ΔI→I
unit pulse response function
response to unit pulse
- Pulse acting in τ:

- Pulse loading acting for short duration $\Delta\tau$ with constant magnitude F



$\Delta I = F\,\Delta\tau$ impulse (finite) impacts, blasts, explosions



$u_0(t) = F \cdot A(t) \quad t>0$

$u_0(t) = -F\,A(t-\Delta\tau) \quad t>\Delta\tau$

response $\quad u_0(t) = F\cdot A(t) - F\cdot A(t-\Delta\tau)$

limit passes as $\Delta\tau \to 0$

$$= F\cdot\Delta\tau\,\frac{A(t)-A(t-\Delta\tau)}{\Delta\tau} = \Delta I\cdot\frac{\Delta A(t)}{\Delta\tau}$$

incremental ratio

- By going to the limit $\Delta\tau\to 0$, at fixed impulse $\Delta I \to I$

$$u_0(t) = I\cdot \dot{A}(t) = I\,h(t) \qquad h(t)\text{ unit pulse response function}$$

$$h(t) = \dot{A}(t) = \frac{1}{k}\left(\zeta\omega_1 e^{-\zeta\omega_1 t}\left(\zeta\frac{\omega_1}{\omega_d}\sin\omega_d t+\cos\omega_d t\right) - e^{-\zeta\omega_1 t}\,\omega_d\left(\zeta\frac{\omega_1}{\omega_d}\cos\omega_d t-\sin\omega_d t\right)\right)$$

$$= \frac{1}{k}\,e^{-\zeta\omega_1 t}\sin\omega_d t\left(\frac{(\zeta\omega_1)^2+\omega_d^2}{\omega_d}\right)$$

$$= \frac{m}{m}\,\frac{1}{k}\,e^{-\zeta\omega_1 t}\sin\omega_d t\,\frac{\omega_1^2}{\omega_d} \qquad \frac{1}{k} = \frac{1}{\omega_1^2}$$

$$\boxed{h(t) = \frac{1}{m\omega_d}\,e^{-\zeta\omega_1 t}\sin\omega_d t}$$

response to unit pulse in $\tau = 0$

$$\omega_d = \omega_1\sqrt{1-\zeta^2} \qquad \zeta \ll 1 \quad \omega_d \simeq \omega_1$$

- Pulse acting in τ:

$$u_0(t,\tau) = I\cdot h(t,\tau) = I\,h(t-\tau)$$

$$h(t-\tau) = \frac{1}{m\omega_d}\,e^{-\zeta\omega_1(t-\tau)}\sin\omega_d(t-\tau)$$

Interpretation in terms of initial velocity

Pulse theorem.

fundamental law of dynamics : $F = ma = m\dfrac{dv}{dt} \rightarrow \underbrace{F\,dt}_{\text{impulse } dI} = m\,dv = d(mv) = dq$

time-invariant

Pulse acting on $\Delta\tau \Rightarrow \Delta q = \Delta I = F_{tot} \cdot \Delta\tau$

$\Delta\tau \rightarrow 0$

$$m\,\dot{u}(\Delta\tau) - m\,\underbrace{\dot{u}(0)}_{\dot{u}_0 = 0} = \int_0^{\Delta\tau} \left(F(t) - (F_e + F_d)\right) dt$$

$$m\,\dot{u}_0^+ = I - \int_0^{\Delta\tau \to 0} \cancel{\underbrace{(F_e + F_d)}_{\text{bounded in time}}}\, dt$$

Thus, in the light of the pulse theorem, the effect of pulse loading on a system at rest is that of an initial velocity equal to the received impulse divided by the mass

initial velocity right after the application of the impulse

$$\boxed{\dot{u}_0^+ = \frac{I}{m}}$$

Response from (*) to $u_0 = 0$, $\dot{u}_0 = \dfrac{I}{m}$

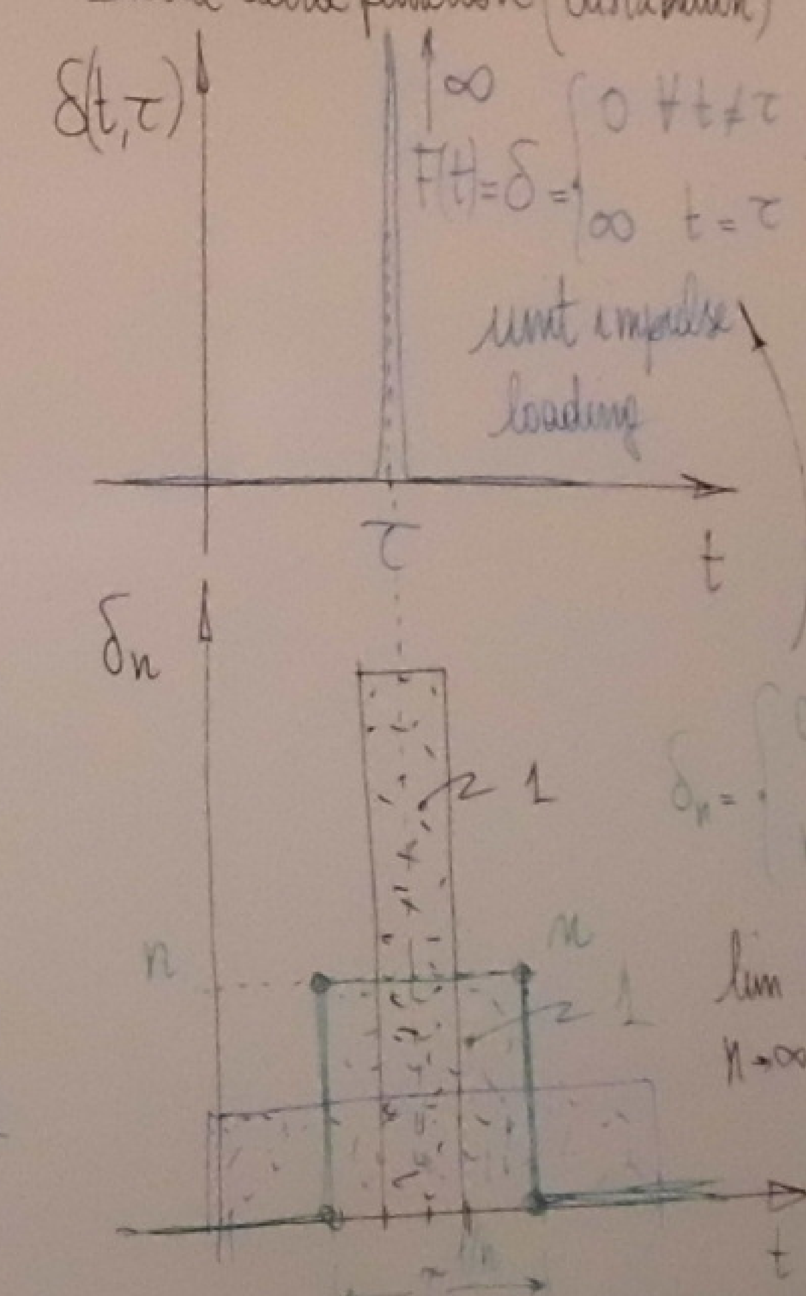
$$u_0(t) = e^{-\zeta\omega_1 t}\left(\frac{I}{m\omega_d}\sin\omega_d t\right)$$

$$= I \cdot \underbrace{\frac{1}{m\omega_d} e^{-\zeta\omega_1 t}\sin\omega_d t}_{h(t)}$$

response to unitary pulse

$\Delta\tau \rightarrow 0,\ F \rightarrow \infty \Rightarrow$ finite pulse I

Dirac delta function (distribution)



$F(t) = \delta = \begin{cases} 0 & \forall t \neq \tau \\ \infty & t = \tau \end{cases} \xrightarrow{\text{response}} h(t)$

$\delta_n = \begin{cases} 0, & |t-\tau| > \frac{1}{2n} \\ n, & |t-\tau| < \frac{1}{2n} \end{cases}$

$\lim_{n\to\infty} \delta_n = \delta$

INPUT OUTPUT

$\delta(t-\tau) \rightarrow$ SYSTEM $\rightarrow h(t-\tau)$

unit pulse in τ

Properties of $\delta(t,\tau) = \delta(t-\tau)$

- $\int_{-\infty}^{+\infty} \delta(t-\tau)\,dt = 1$
- $\int_{-\infty}^{+\infty} F(t)\,\delta(t-\tau)\,dt = F(\tau)$

Interpretation in terms of initial velocity

Pulse theorem

fundamental law of dynamics : $F = ma = m\frac{dv}{dt} \rightarrow \underbrace{F\,dt}_{\text{impulse } dI} = m\,dv = d(mv) = dq$

time constant

Pulse acting on $\Delta\tau \Rightarrow \Delta q = \Delta I = F_{tot} \cdot \Delta\tau$

$\Delta\tau \rightarrow 0$ $\quad m\dot{u}(\Delta\tau) - m\underbrace{\dot{u}(0)}_{\dot{u}_0 = 0} = \int_0^{\Delta\tau} \left(F(t) - (F_e + F_d)\right) dt$

$m\dot{u}_0^+ = I - \cancel{\int_0^{\Delta\tau \to 0} \underbrace{(F_e + F_d)}_{\text{bounded in time}} dt} \rightarrow 0$

$$\dot{u}_0^+ = \frac{I}{m}$$

initial velocity right after the application of the impulse

Thus, in the light of the pulse theorem, the effect of pulse loading on a system at rest is that of an initial velocity equal to the received impulse divided by the mass

Response from (*) to $u_0 = 0$, $\dot{u}_0 = \frac{I}{m}$

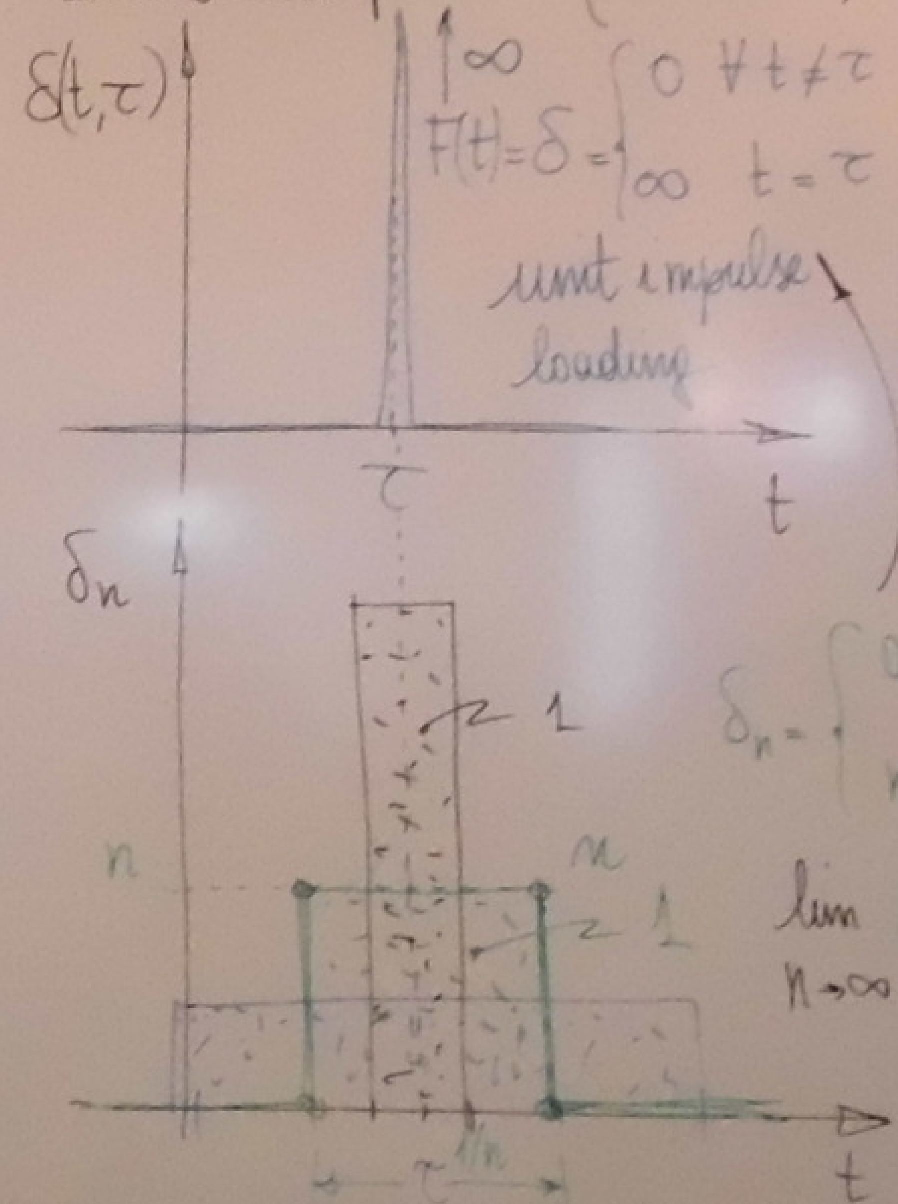
$$u_0(t) = e^{-\zeta\omega_n t}\left(\frac{I}{m\omega_d}\sin\omega_d t\right)$$

$$= I \cdot \underbrace{\frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t}_{h(t)}$$

response to unitary pulse

$\Delta\tau \rightarrow 0,\ F \rightarrow \infty \Rightarrow$ finite pulse I

Dirac delta function (distribution)



$F(t) = \delta = \begin{cases} 0 & \forall t \neq \tau \\ \infty & t = \tau \end{cases} \xrightarrow{\text{response}} h(t)$

unit impulse loading

$\delta_n = \begin{cases} 0, & |t-\tau| > \frac{1}{2n} \\ n, & |t-\tau| < \frac{1}{2n} \end{cases}$

$\lim_{n \to \infty} \delta_n = \delta$

INPUT $\quad$ OUTPUT

$\delta(t,\tau) \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h(t,\tau)$

Unit pulse in τ

Properties of $\delta(t,\tau) = \delta(t-\tau)$

- $\int_{-\infty}^{+\infty} \delta(t,\tau)\,dt = 1$
- $\int_{-\infty}^{+\infty} F(\tau)\,\delta(t,\tau)\,d\tau = F(t)$

leads us to represent any loading function as the superposition of infinite impulses $F(\tau)d\tau$ over time

$\rightsquigarrow$ response to generic loading

convolution integral

Response from (*) to $u_0 = 0$

$$u_0(t) = e^{-\xi\omega t}\left(\frac{I}{m\omega_d}\sin\omega_d t\right) = I\,\underbrace{\frac{1}{m\omega_d}e^{-\xi\omega t}\sin\omega_d t}_{h(t)}$$

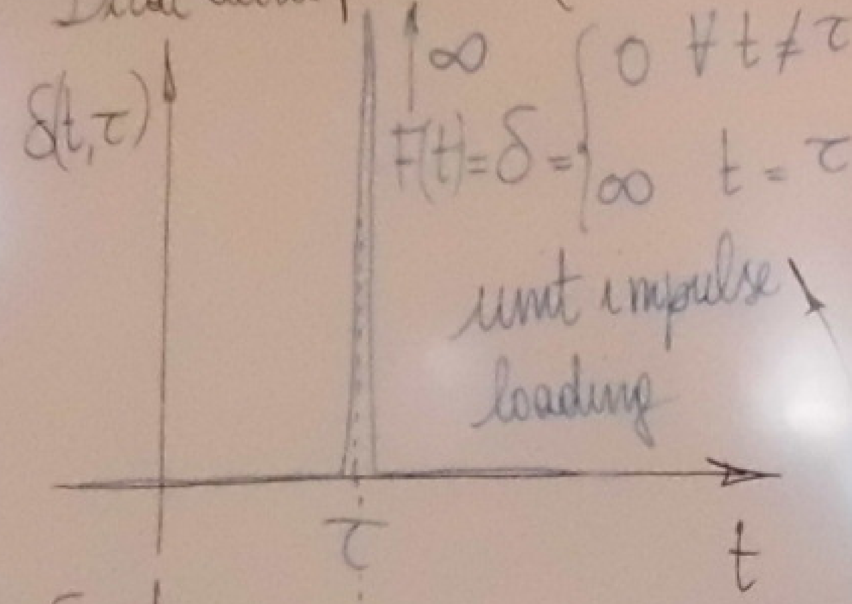
response to unit impulse

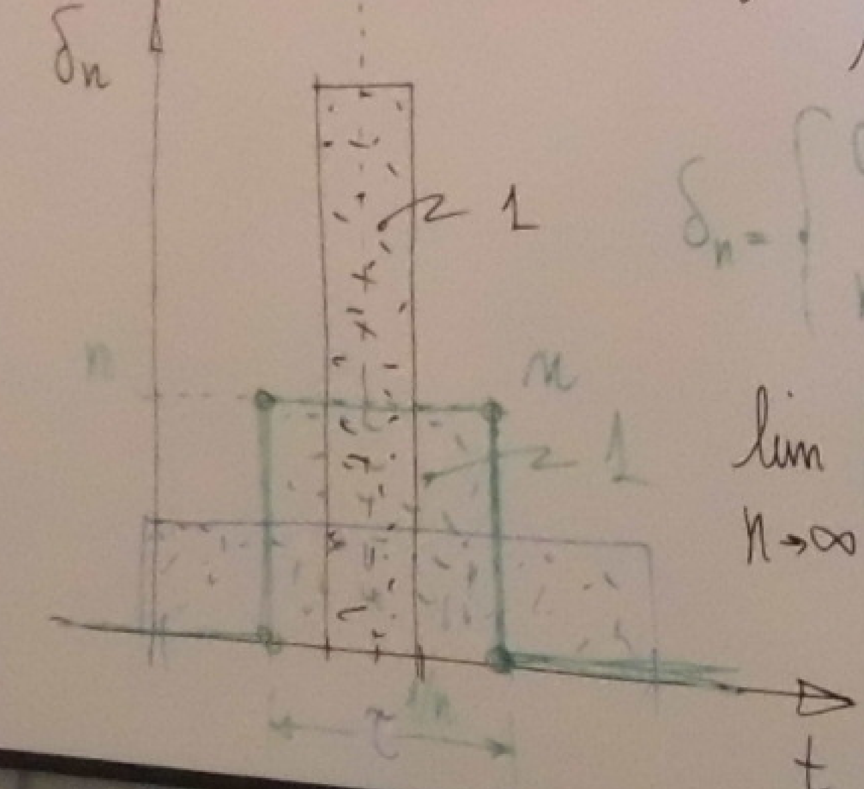
$\Delta\tau\to 0,\ F\to\infty \Rightarrow$ finite pulse I

Dirac delta function (distribution)

$$F(t)=\delta=\begin{cases}0 & \forall t\neq\tau\\ \infty & t=\tau\end{cases} \xrightarrow{\text{response}} h(t)$$

unit impulse loading





$$\delta_n=\begin{cases}0, & |t-\tau|>\frac{1}{2n}\\ n, & |t-\tau|<\frac{1}{2n}\end{cases}$$

$$\lim_{n\to\infty}\delta_n=\delta$$

INPUT $\delta(t,\tau) \rightarrow$ SYSTEM $\rightarrow h(t,\tau)$ OUTPUT

Unit pulse in τ

Properties of $\delta(t,\tau)=\delta(t-\tau)$

- $\displaystyle\int_{-\infty}^{+\infty}\delta(t,\tau)\,dt = 1$
- $\displaystyle\int_{-\infty}^{+\infty}F(\tau)\,\delta(t,\tau)\,d\tau = F(t)$

Leads us to represent any loading function as the superposition of infinite impulses $F(\tau)d\tau$ over time $\Rightarrow$ response to generic loading (convolution integral, Duhamel integral)