Università degli studi di Bergamo

Scuola di Ingegneria (Dolmine)

CCS Ingegneria Edile

LM-24 Ingegneria delle Costruzioni Edili

Dinamica, Instabilità e Anelasticità delle Strutture

(ICAR/08 - SdC; 6 CFU)

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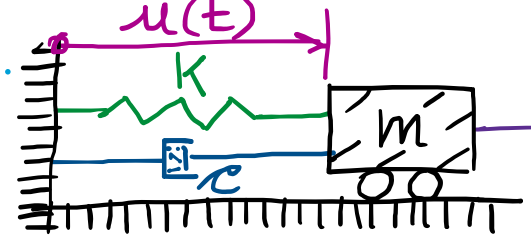
LEZIONE 05

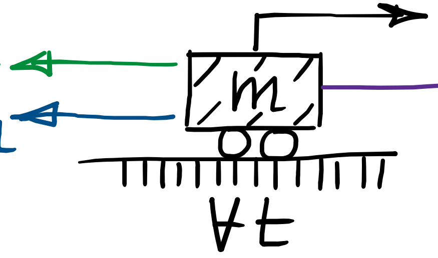
Risposta smorzata $(\zeta \neq 0)$ a forzante armonica

(forzante periodica di periodo $T = \frac{2\pi}{\omega} = \frac{1}{f}$ e pulsazione $\omega = 2\pi f$, frequenza ciclica f)

c.i. u_0, $\dot u_0$



$\overbrace{F(t)}^{f(t)} = F \sin\omega t$



$F_i = -m\ddot u$

$F_e = K u$

$F_d = c\dot u$

$F \sin\omega t$

da Principio di d'Alembert

Equil. "dinamico" ⇒ eq.ne del moto

$$\cancel{m}\ddot u(t) + \frac{c}{m}\dot u(t) + \frac{K}{m} u(t) = \frac{K}{m}\frac{F}{K}\sin\omega t$$

$\frac{c}{m} = 2\zeta\omega_1$; $\omega_1^2 = \frac{K}{m}$

pulsaz. nat. sist. non smorz. $\left(\omega_1 = \sqrt{\frac{K}{m}}\right)$

spostamento "statico" $u_{st} = \frac{F}{K}$ (direttamente prop. a F)

coeffic. di smorz.

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{Km}} \Rightarrow 2\zeta\omega_1$$

fattore di smorz. ("damping ratio")

$\zeta \ll 1 \sim 1\% = .01$

es. $5\% = .05$

$$\boxed{\ddot u + 2\zeta\omega_1\dot u + \omega_1^2 u = \omega_1^2 u_{st}\sin\omega t} \quad (*)$$

Integrale particolare:

$$u_p(t) = N(\beta;\zeta)\, u_{st} \sin(\omega t - \xi(\beta;\zeta)),$$

$u_{st} = \frac{F}{K}$; ξ fase

N fattore di amplificazione dinamica

$\beta = \frac{\omega}{\omega_1} = \frac{f}{f_1}$ rapporto di frequenze ("frequency ratio")

per $\zeta = 0$ (sist. non smorz.) $\Rightarrow N = \frac{1}{\sqrt{(1-\beta^2)^2}}$

condiz. di "risonanza" $(\to\infty,\ \beta\to 1)$;

$$\xi = \begin{cases} 0 & \text{se } \beta < 1 \text{ in fase} \\ \pi & \text{se } \beta > 1 \text{ in opposiz. di fase} \end{cases}$$

$$u_p = N u_{st}\sin(\omega t-\xi) = N u_{st}(\sin\omega t\cos\xi - \cos\omega t\sin\xi) = \underbrace{N u_{st}\cos\xi}_{Z_1}\sin\omega t - \underbrace{N u_{st}\sin\xi}_{Z_2}\cos\omega t$$

$$\frac{d}{dt}\downarrow \qquad \dot u_p = \omega N u_{st}\cos(\omega t-\xi)$$

$$\frac{d}{dt}\downarrow \qquad \ddot u_p = -\omega^2 N u_{st}\sin(\omega t-\xi) = -\omega^2 u_p(t)$$

Quindi $Z_1 = N u_{st}\dfrac{1-\beta^2}{\sqrt D} = \dfrac{1-\beta^2}{D}u_{st}$ $\qquad \sqrt{Z_1^2+Z_2^2} = N u_{st}$ $\qquad Z_2 = N u_{st}\dfrac{2\zeta\beta}{\sqrt D} = \dfrac{2\zeta\beta}{D}u_{st}$

Sostituendo nell'eq.ne del moto (*) $\Rightarrow N, \xi$?

$$(\cancel{\omega_1^2} - \underbrace{\frac{\omega^2}{\omega_1^2}}_{\beta^2}) N \cancel{u_{st}}\sin(\omega t-\xi) + 2\zeta\underbrace{\frac{\omega_1}{\omega_1}}\underbrace{\frac{\omega}{\omega_1}}_{\beta} N\cancel{u_{st}}\cos(\omega t-\xi) = \frac{\cancel{\omega_1^2}}{\cancel{\omega_1^2}}\cancel{u_{st}}\sin\omega t$$

D termine a denominatore

$$(1-\beta^2)\sin(\omega t-\xi) + 2\zeta\beta\cos(\omega t-\xi) \overset{\forall t}{=} \frac{1}{N}\sin\omega t \qquad D = (1-\beta^2)^2+(2\zeta\beta)^2$$

$$\cos\xi\sin\omega t - \sin\xi\cos\omega t \qquad \cos\xi\cos\omega t + \sin\xi\sin\omega t$$

- $\cos\xi = \dfrac{1}{\sqrt{1+\tan^2\xi}} = \dfrac{1-\beta^2}{\sqrt{\underbrace{(1-\beta^2)^2+(2\zeta\beta)^2}_{D}}} = \dfrac{1-\beta^2}{\sqrt D}$
- $\sin\xi = \tan\xi\cos\xi = \dots = \dfrac{2\zeta\beta}{\sqrt D}$

$$\begin{cases} \left((1-\beta^2)\cos\xi + 2\zeta\beta\sin\xi\right)\cancel{\sin\omega t} = \dfrac{1}{N}\cancel{\sin\omega t} \\ \left(-(1-\beta^2)\sin\xi + 2\zeta\beta\cos\xi\right)\cancel{\cos\omega t} = 0 \end{cases}$$

$$\Rightarrow \tan\xi = \frac{\sin\xi}{\cos\xi} = \frac{2\zeta\beta}{1-\beta^2}$$

Dalla prima eq.ne: $(1-\beta^2)\dfrac{1-\beta^2}{\sqrt D} + 2\zeta\beta\dfrac{2\zeta\beta}{\sqrt D} = \dfrac{1}{N}$

$$\frac{D}{\sqrt D} = \sqrt D = \frac{1}{N}$$

$$\Rightarrow \boxed{N(\beta;\zeta) = \frac{1}{\sqrt D} = \frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}} \qquad \boxed{\xi(\beta;\zeta) = \operatorname{atan}\frac{2\zeta\beta}{1-\beta^2}}$$

N.B. per $\beta = 1$, $N = 1/(2\zeta)$, $\xi = \pi/2$

fattore di amplificazione dinamica (di ust) $N = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}$

Max. rel. (staz.)

$\frac{d}{d\beta}$ $N' = -\frac{\frac{1}{2\sqrt{D}}}{D}$ $D'(\beta) = 0$

$\beta = 0$

$\cancel{2}(1-\beta^2)(-\cancel{2}\cancel{\beta})+\cancel{2}(2\zeta\cancel{\beta})2\zeta=0$

$-1+\beta^2+2\zeta^2=0 \Rightarrow \bar\beta^2 = 1-2\zeta^2$

$1-2\zeta^2>0,\ \zeta<\frac{1}{\sqrt{2}}$ $\bar\beta=\sqrt{1-2\zeta^2}\le 1$

$\simeq 1 \quad \zeta \ll 1$

@ $\beta=1$ $N = \frac{1}{2\zeta} = \frac{1/\zeta}{2}$

Lì:

$\bar D = (2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)$

$= 4\zeta^4+4\zeta^2-8\zeta^4 = 4\zeta^2-4\zeta^4 = 4\zeta^2(1-\zeta^2)$

$\bar N = \frac{1}{\sqrt{\bar D}} = \frac{1}{2\zeta}\frac{1}{\sqrt{1-\zeta^2}} \simeq \frac{1}{2\zeta}$ $\zeta \ll 1$ $N(\beta=1)=\frac{1/\zeta}{2} \sim N$

$2\zeta^2 = 1-\bar\beta^2$

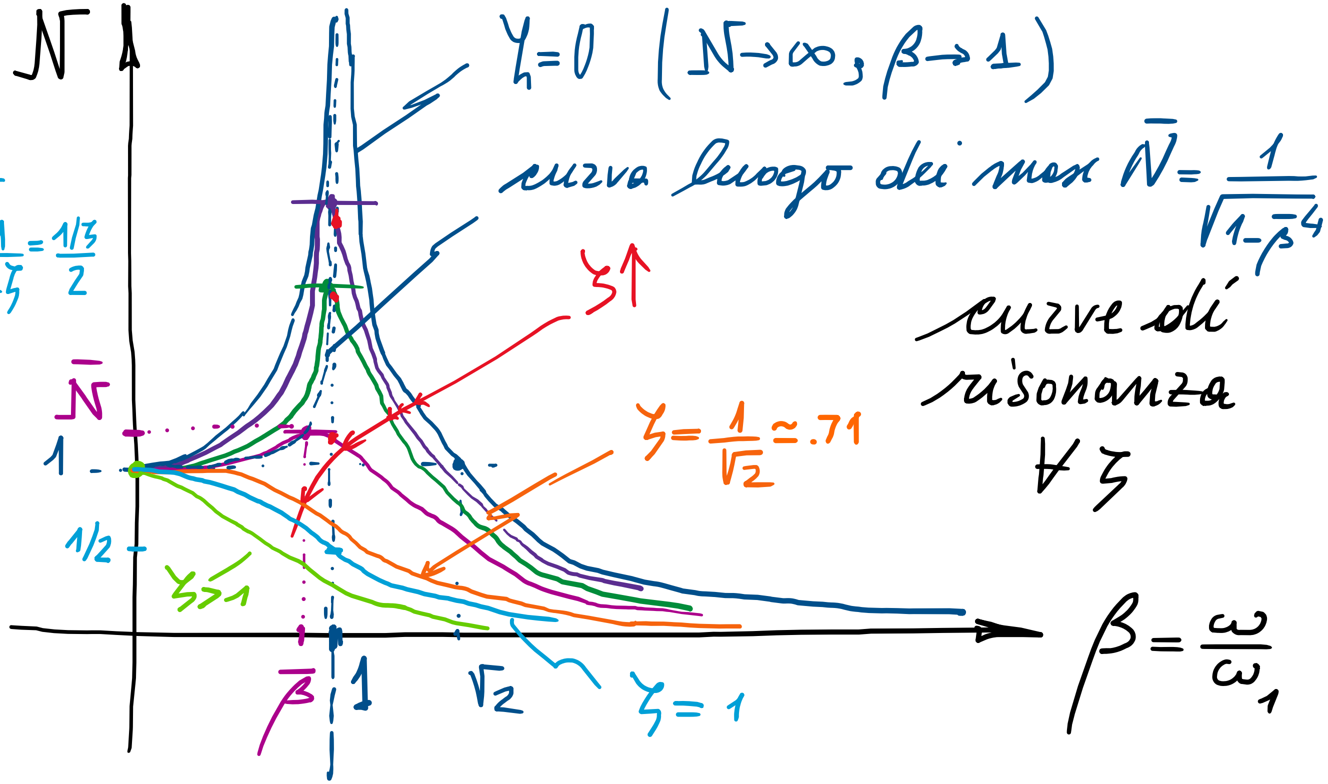
$= \frac{1}{\frac{\sqrt{1-\bar\beta^2}}{\sqrt{2}}}\frac{1}{\sqrt{\frac{2-1+\bar\beta^2}{2}}} = \frac{1}{\sqrt{1-\bar\beta^4}}$

$\bar N(\bar\beta) = \frac{1}{\sqrt{1-\bar\beta^4}}$

traccia dei max (per tutti gli ζ)

fase $\xi = \text{atan}\,\frac{2\zeta\beta}{1-\beta^2}$

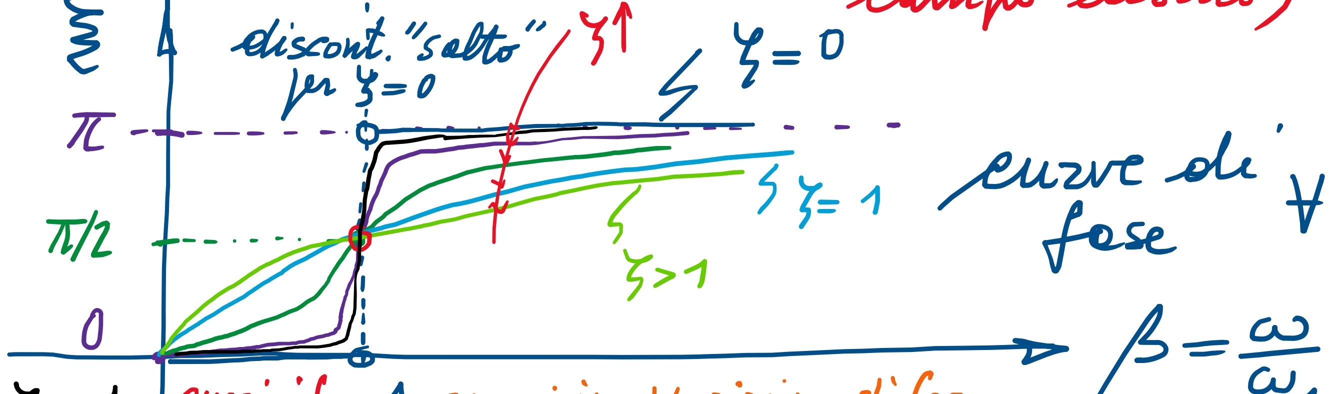
(sfasamento in ritardo di $u(t)$ rispetto a $F\sin\omega t$)



curve di risonanza $\forall\zeta$

ζ	0	1%	2%	5%
N	∞	50	25	10

$\Rightarrow$ (può indurre uscita dal campo elastico)



curve di fase $\forall\zeta$

per $\zeta\ll 1$: quasi in fase $\beta<1$ | 1 | quasi in opposizione di fase $\beta>1$

Integrale generale:

pulsazione naturale sistema smorzato

$$\omega_d = \omega_1\sqrt{1-\zeta^2} \simeq \omega_1 \quad (\zeta \ll 1)$$

$$Z_1 = \frac{1-\beta^2}{D} u_{st}, \quad Z_2 = \frac{2\zeta\beta}{D} u_{st} \quad ; \quad u_{st} = \frac{F}{K}$$

$$\beta = \frac{\omega}{\omega_1} \quad ; \quad D = (1-\beta^2)^2 + (2\zeta\beta)^2$$

$$u(t) = u_{go}(t) + u_p(t)$$

$$= \underbrace{e^{-\zeta\omega_1 t}\left(A\sin\omega_d t + B\cos\omega_d t\right)}_{\text{risposta "transiente"}} + \underbrace{Z_1\sin\omega t - Z_2\cos\omega t}_{\text{risposta a regime ("steady state")}}$$

c.i.

$$\begin{cases} u_0 = B - Z_2 \Rightarrow B = \boxed{u_0 + Z_2} \\ \dot{u}_0 = -\zeta\omega_1 B + \omega_d A + \omega Z_1 \Rightarrow A = \dfrac{\dot{u}_0 + \zeta\omega_1 B - \omega Z_1}{\omega_d} = \dfrac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \underbrace{\dfrac{\zeta\omega_1 Z_2 - \omega Z_1}{\omega_d}}_{\frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}}} \end{cases}$$

$$u(t) = e^{-\zeta\omega_1 t}\left(\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right)$$

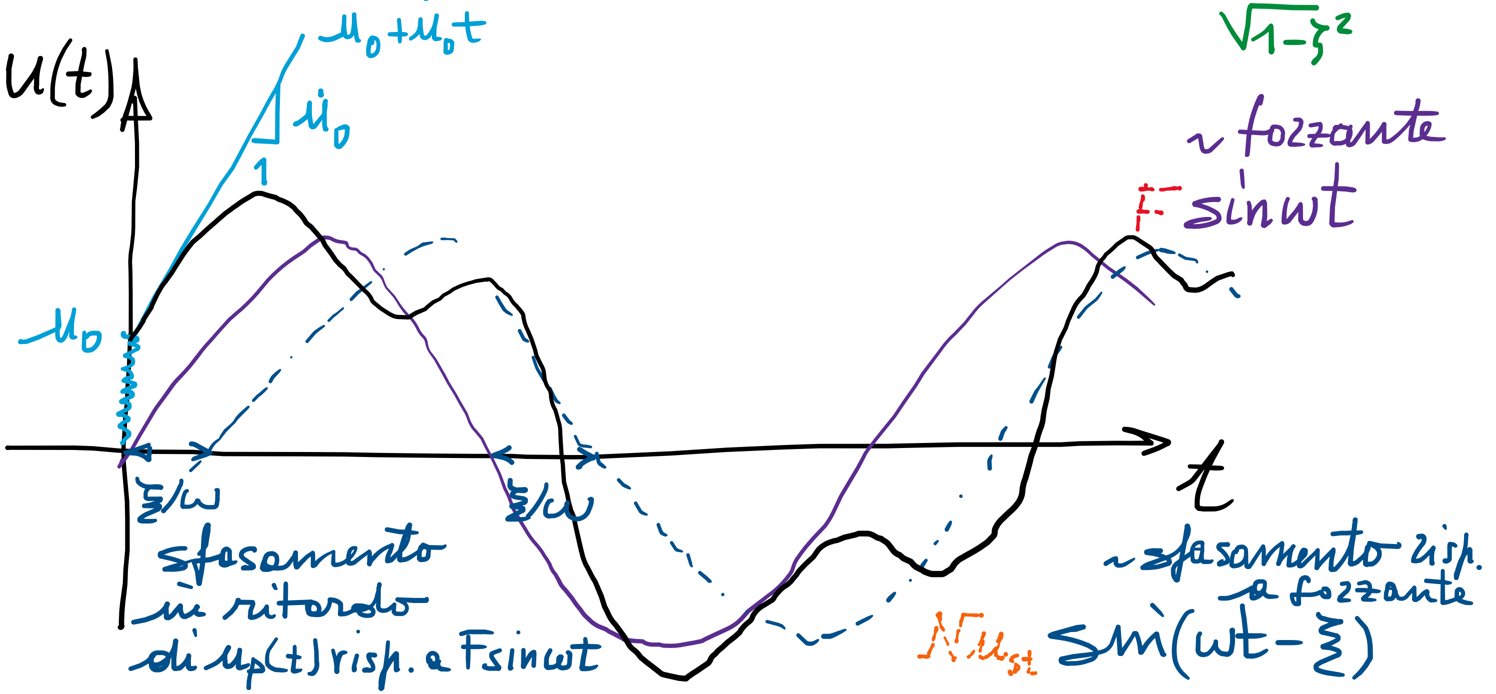
risposta alle c.i. per $F=0$ (spesso assente)

$$+ Z_1\sin\omega t - Z_2\cos\omega t$$

$$+ e^{-\zeta\omega_1 t}\left(\frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}}\sin\omega_d t + Z_2\cos\omega_d t\right)$$

risposta a forzante armonica $F\sin\omega t$ per c.i. omogenee ($u_0 = 0$, $\dot{u}_0 = 0$)

ampiezze decadenti esponenzialmente in t (sino a far sopravvivere solo $u_p(t)$)



Concetti fondamentali:

- Risposta smorzata a forzante armonica ($F(t) = F\sin\omega t$)
  ($c \neq 0, \zeta \neq 0$)

$\bar N(\bar\beta) = \frac{1}{\sqrt{1-\bar\beta^4}}$ luogo dei max

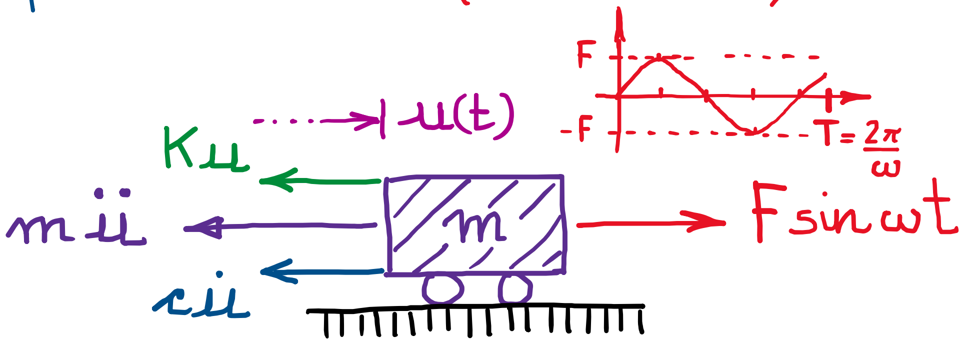
$\bar N = \frac{1}{2\zeta}\frac{1}{\sqrt{1-\zeta^2}} \simeq \frac{1}{2\zeta}$

$\bar\beta = \sqrt{1-2\zeta^2} \simeq 1$

$m, c, K = \text{cost}$

sistema tempo-invariante

con c.i. $\begin{cases} u_0 \\ \dot u_0 \end{cases}$ @ $t = t_0$



$$m\ddot u(t) + c\,\dot u(t) + K u(t) = F\sin\omega t$$

$$\Downarrow$$

$$\ddot u(t) + \underbrace{2\zeta\omega_1}_{\frac{c}{m}}\dot u(t) + \underbrace{\omega_1^2}_{\frac{K}{m}} u(t) = \omega_1^2 \underbrace{u_{st}}_{\frac{F}{K}} \sin\omega t \qquad \omega_1 = \sqrt{\frac{K}{m}}$$

fattore di smorzamento

$\zeta = \frac{c}{2\sqrt{Km}} = \frac{c}{c_{cr}}$

(tipicamente subcritico, $\zeta < 1$
e $\ll 1$, $\zeta \sim 1\% = 0.01$)

risposta "a regime"

$\underbrace{Z_1}_{N u_{st}\frac{1-\beta^2}{\sqrt D}} \sin\omega t - \underbrace{Z_2}_{N u_{st}\frac{2\zeta\beta}{\sqrt D}} \cos\omega t$

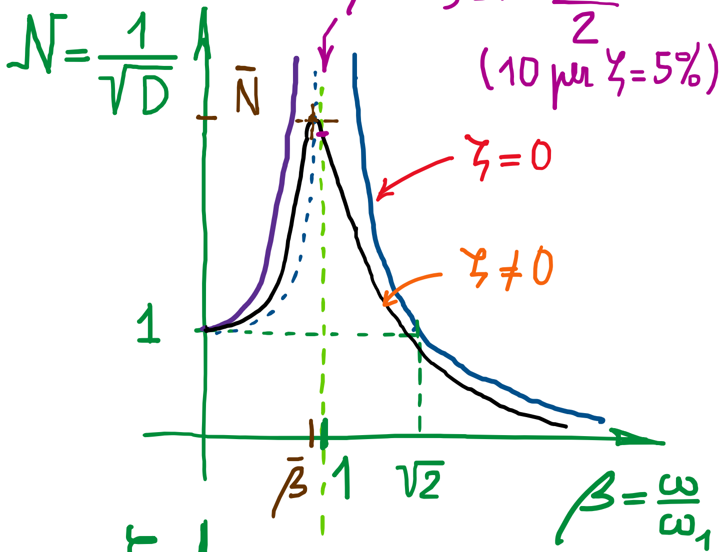
- Integrale particolare:

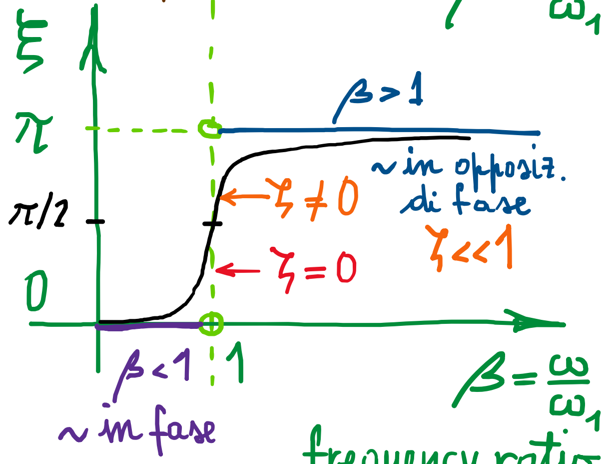
$\xi = \text{atan}\,\frac{2\zeta\beta}{1-\beta^2}$

$$u_p(t) = N(\beta;\zeta)\, u_{st}\, \sin(\omega t - \xi(\beta;\zeta))$$

fase (sfasamento in ritardo tra risposta e forzante)

fattore di amplificazione dinamica (rispetto a $u_{st} = F/K$) $\quad N = \frac{1}{\sqrt D} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$





SOMMARIO (Lez. 05)

- Risposta smorzata a forzante armonica.
- Effetto dello smorzamento su curve di risonanza e di fase.
- Picco finito di ampiezza in condizioni di risonanza; risposta in quadratura rispetto alla forzante.
- Risposta a regime in componenti $\sin\omega t$ e $\cos\omega t$.
- Integrale generale con risposta transiente e a regime.
- Next step: trattazione unificata in variabili complesse per risposta a $F\sin\omega t$ e/o $F\cos\omega t \rightsquigarrow Fe^{i\omega t}$.