Instability of discrete structural systems:

2nd-order theory – Formalization

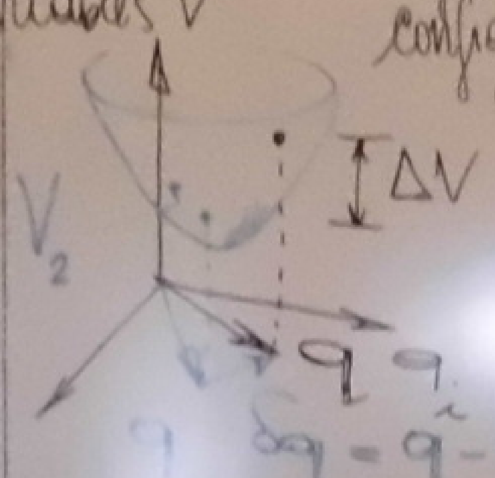
- Analyze systems that may be conveniently described by a finite number of dof
- Analysis under large displacements (these are such to influence the equilibrium config.) but within "geometrically small" displacements => allows for:
  - linearization of the equil. equations
  - quadratic representation of the total potential energy (useful to access stability conditions by the SC provided by Dirichlet's theorem)

– Total potential energy:

$$V(q) \simeq V_2(q) = V(q_0) + \underbrace{\frac{\partial V}{\partial q}\Big|_{q_0}^{T} \delta q}_{\delta V} + \underbrace{\frac{1}{2}\,\delta q^T \frac{\partial^2 V}{\partial q\,\partial q}\Big|_{q_0} \delta q}_{\frac{1}{2}\delta^2 V} + \dots$$

lagrangian variables V

reference configuration (often $q_0 = 0$) undeformed config.: $\delta q = q$



$\delta q = \tilde{q} - q_0$

$\Delta V = \delta V + \frac{1}{2}\delta^2 V$

δV: first-order variation: $-\mathbb{P}^T \delta q$

$\frac{1}{2}\delta^2 V$: second-order variation: $\frac{1}{2}\delta q^T \mathbb{K}\, \delta q$

$-\mathbb{P} = \frac{\partial V}{\partial q}\Big|_{q_0}$ external load vector of conserv. forces

$\mathbb{K} = \frac{\partial^2 V}{\partial q\,\partial q}\Big|_{q_0}$ total stiffness matrix

– Equilibrium equations by the stationary condition:

$$\frac{\partial V_2}{\partial q} = 0 \Rightarrow -\mathbb{P} + \mathbb{K}\,\delta q = 0 \Rightarrow \mathbb{K}\,\delta q = \mathbb{P} \quad \text{linearized equil. eqns.}$$

($\mathbb{P} = 0$ if q_0 is an equil. conf.)

Stability condition:

$$\frac{\partial^2 V_2}{\partial q\,\partial q} \Longleftrightarrow \frac{1}{2}\delta^2 V = \frac{1}{2}\delta q^T \mathbb{K}\,\delta q \geq 0 \;\; \forall \delta q \neq 0$$

loss of positive definiteness of matrix $\mathbb{K}$

[The algebraic properties of $\mathbb{K}$ rule stability]

– Eulerian stability problems (determination of critical loads)

- Hypotheses
  – Equil. config. ruled linearly by the load

load vector $\mathbb{P} = p\,\mathbb{P}_0$ — $\mathbb{P}_0$: base loads, p: load multiplier

  – Pre-critical linearity

$$q = p\,\bar{q}_0$$

  – Second-order variation may be written as

$$\frac{1}{2}\delta q^T \mathbb{K}\,\delta q = \frac{1}{2}\delta^2 V = \delta\omega - p\,\delta W \quad \text{linear in } p$$

$$\boxed{\mathbb{K} = \mathbb{K}_E - p\,\mathbb{K}_G}$$

$\delta\omega = \frac{1}{2}\delta q^T \mathbb{K}_E\, \delta q$ elastic energy — $\mathbb{K}_E$ elastic stiffness

$\delta W = \frac{1}{2}\delta q^T \mathbb{K}_G\, \delta q$ 2nd order external work; $\mathbb{K}_G$ geometric stiffness

– The critical condition

$\frac{1}{2}\delta^2 V = 0$

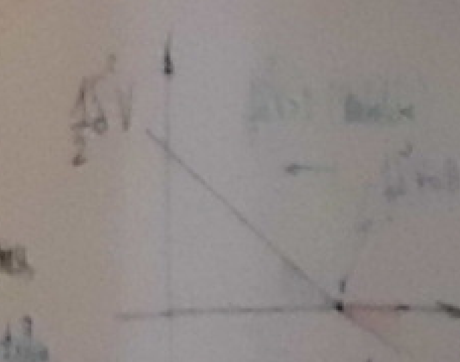
$\det \mathbb{K} = 0$

$\mathbb{K}_E\, \delta q^* = p\,\mathbb{K}_G\, \delta q^*$ generalized eigen. pb.

p^* eigenvalues

δq^* eigenvectors

– Eulerian critical load

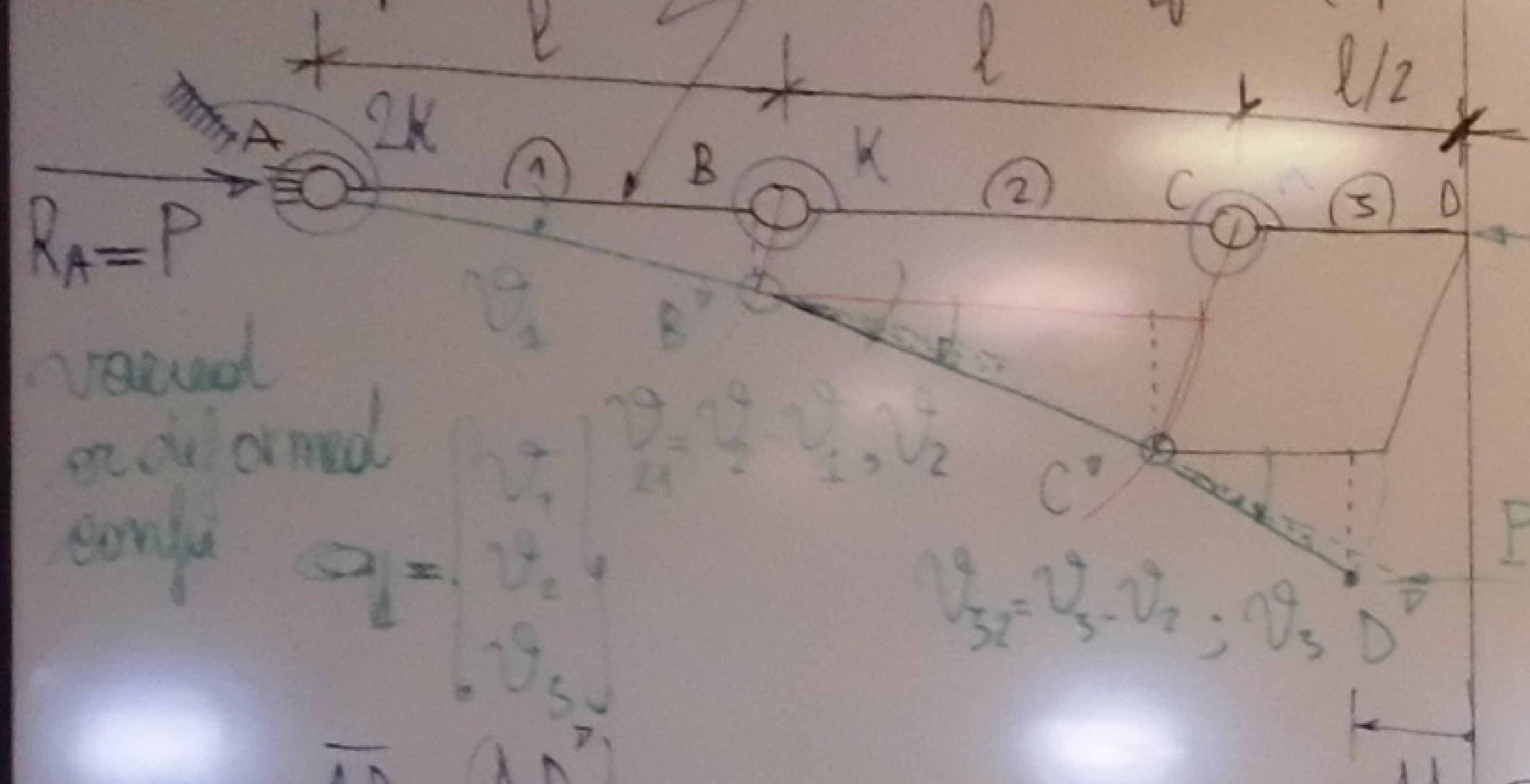


Instability of discrete structural systems: – Total potential energy
2nd-order theory – Formalization
allows for
– linearization of the equil. equations
lagrangian variables V
reference configuration
often q0 = 0
undeformed config.
first-order variation
second-order variation
external load vector of conserv. forces
total stiffness matrix
– Equilibrium equations by the stationary condition:
linearized equil. eqns.
[The algebraic properties of K rule stability]
– Eulerian stability problems (determination of critical loads)
Hypotheses
– Equil. config. ruled linearly by the load
load vector
base loads
load multiplier
– Pre-critical linearity
– Second-order variation may be written as
linear in p
– The critical condition
det K = 0
generalized eigv. pb
eigenvalues
eigenvectors
elastic stiffness
loss of stability
– Eulerian critical load
Rayleigh ratio provides an upper bound of
(starting from estimates of

– Eulerian stability problems (determination of critical loads)
Hypotheses:
– Equil. confn ruled linearly by the load
load vector
P = p P0 base loads
load multiplier
– Pre-critical linearity
q = p q̄0
– Second-order variation may be written as
½δ²V = δω − p δw linear in p
K = KE − p KG
δω = ½ δqᵀ KE δq elastic energy
δw = ½ δqᵀ KG δq 2nd order external work; KG geometric stiffn.
; KE elastic stiffness
loss of stability
– The critical condition
½δ²V = 0
det K = 0
KE δq = p KG δq
generalized eigv. pb
pi eigenvalues
δqi* eigenvectors
critical load multipliers ; δqi critical configurations
– Eulerian critical load
pcr^E = min{pi*} → Pcr^E = pcr^E P0
– ½δ²V = δω(δq) − pcr^E δw(δq) ≥ 0
at p = pcr^E ∀δq   pcr^E ≤ δω(δq)/δw(δq) = R(δq)
Rayleigh ratio provides an upper bound of pcr^E (starting from estimates of δq*)
½δ²V
½δ²V > 0 (stability)
½δ²V = 0
½δ²V < 0
p
pi*
Rayleigh ratio (instability problems)
= R(δqi*)

Example

reference confn. $q_0 = 0$ (equil. confn)



varied or deformed confn $\quad q = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix}$

$$u = \overline{AD} - (AD^*)_x$$
$$= l + l + \frac{l}{2} - \left(l\cos\vartheta_1 + l\cos\vartheta_2 + \frac{l}{2}\cos\vartheta_3\right)$$
$$= \left(\frac{5}{2} - \left(\cos\vartheta_1 + \cos\vartheta_2 + \frac{1}{2}\cos\vartheta_3\right)\right) l$$

large displacements (rotations) $\quad -\mathbb{P} = \left.\frac{\partial V}{\partial q}\right| = 0$

$|\vartheta_i| \ll 1$, $\cos\vartheta_i \simeq 1 - \frac{\vartheta_i^2}{2} + \dots$ $\quad$ is an equil. confn $q_0 = 0$

$\sin\vartheta_i \simeq \vartheta_i$

$q = q$

$$V(q) = \frac{1}{2}\Big(\underbrace{2k\vartheta_1^2 + k(\vartheta_2 - \vartheta_1)^2 + k(\vartheta_3 - \vartheta_2)^2}_{\text{elastic energy } V_E}\Big) - \underbrace{Pu}_{\text{external work } L_F}$$

total potential energy

$$= \frac{1}{2}k\left(2\vartheta_1^2 + \vartheta_2^2 + \vartheta_1^2 - 2\vartheta_1\vartheta_2 + \vartheta_3^2 + \vartheta_2^2 - 2\vartheta_2\vartheta_3\right) V_F - Pu$$

$$\bar{V}(q) = \frac{V(q)}{k} = \frac{1}{2}\left(3\vartheta_1^2 + 2\vartheta_2^2 + \vartheta_3^2 - 2\vartheta_1\vartheta_2 - 2\vartheta_2\vartheta_3\right) + \quad p = \frac{Pl}{k} = \frac{P}{k/l} = \frac{P}{P_0}$$

$$-p\left(\frac{5}{2} - \cos\vartheta_1 - \cos\vartheta_2 - \frac{1}{2}\cos\vartheta_3\right)$$

2^{nd}-order theory

– Equilibrium confn by the stationary condition: $\quad \bar{V} \simeq \bar{V}_2(q) = \bar{V}_E(q) - p\left(\frac{5}{2} - 1 + \frac{\vartheta_1^2}{2} - 1 + \frac{\vartheta_2^2}{2} - \frac{1}{2} + \frac{1}{2}\frac{\vartheta_3^2}{2}\right)$

$$-p\frac{1}{2}\left(\vartheta_1^2 + \vartheta_2^2 + \frac{1}{2}\vartheta_3^2\right)$$

$$\frac{\partial \bar{V}}{\partial \vartheta_1} = 3\vartheta_1 - \vartheta_2 - p\underbrace{\sin\vartheta_1}_{\vartheta_1} = 0$$

$$\frac{\partial \bar{V}}{\partial \vartheta_2} = 2\vartheta_2 - \vartheta_1 - \vartheta_3 - p\underbrace{\sin\vartheta_2}_{\vartheta_2} = 0$$

$$\frac{\partial \bar{V}}{\partial \vartheta_3} = \vartheta_3 - \vartheta_2 - \frac{p}{2}\underbrace{\sin\vartheta_3}_{\vartheta_3} = 0$$

(non-linear equil. eqns.) $\xrightarrow{|\vartheta_i| \ll 1}$ $\frac{\partial V_2}{\partial q} = 0$ (linearized equil. eqns.)

$$0 = \begin{bmatrix} 3-p & -1 & 0 \\ -1 & 2-p & -1 \\ 0 & -1 & 1-\frac{p}{2} \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\frac{\partial^2 V_2}{\partial q \partial q}$$

$\frac{1}{2}\delta^2 V = 0 \Rightarrow \mathbb{K}\delta q = 0$

$\boxed{\mathbb{K}_E\,\delta q = p\,\mathbb{K}_G\,\delta q}$

$\boxed{\mathbb{K} = \mathbb{K}_E - p\mathbb{K}_G}$

– 2^{nd}-order derivatives $\frac{\partial \bar{V}}{\partial q_i \partial q_j}$

$$\begin{bmatrix} \bar{V}_{,\vartheta_1\vartheta_1} = 3 - p\cos\vartheta_1 & ; \bar{V}_{,\vartheta_1\vartheta_2} = -1 & ; \bar{V}_{,\vartheta_1\vartheta_3} = 0 \\ \bar{V}_{,\vartheta_2\vartheta_1} = -1 & ; \bar{V}_{,\vartheta_2\vartheta_2} = 2 - p\cos\vartheta_2 & ; \bar{V}_{,\vartheta_2\vartheta_3} = -1 \\ \bar{V}_{,\vartheta_3\vartheta_1} = 0 & ; \bar{V}_{,\vartheta_3\vartheta_2} = -1 & ; \bar{V}_{,\vartheta_3\vartheta_3} = 1 - \frac{p}{2}\cos\vartheta_3 \end{bmatrix}$$

$\cos\vartheta_i \simeq 1$ (in the 2^{nd}-order derivatives)


Example
total potential energy
elastic energy V_E
external work L_F
2nd-order theory
- Equilibrium confu by the stationary condition:
$\frac{\partial \bar V}{\partial \vartheta_1} = 3\vartheta_1 - \vartheta_2 - p \sin\vartheta_1 = 0$
$\frac{\partial \bar V}{\partial \vartheta_2} = 2\vartheta_2 - \vartheta_1 - \vartheta_3 - p \sin\vartheta_2 = 0$
$\frac{\partial \bar V}{\partial \vartheta_3} = \vartheta_3 - \vartheta_2 - \frac{p}{2} \sin\vartheta_3 = 0$
(non-linear equil. equs.)
$\frac{\partial V_2}{\partial q} = 0$ (linearized equil. equs.)
- 2nd-order derivatives $\frac{\partial^2 V}{\partial q_i \partial q_j}$
$\bar V_{,\vartheta_1\vartheta_1} = 3 - p\cos\vartheta_1$; $\bar V_{,\vartheta_1\vartheta_2} = -1$; $\bar V_{,\vartheta_1\vartheta_3} = 0$
$\bar V_{,\vartheta_2\vartheta_1} = -1$; $\bar V_{,\vartheta_2\vartheta_2} = 2 - p\cos\vartheta_2$; $\bar V_{,\vartheta_2\vartheta_3} = -1$
$\bar V_{,\vartheta_3\vartheta_1} = 0$; $\bar V_{,\vartheta_3\vartheta_2} = -1$; $\bar V_{,\vartheta_3\vartheta_3} = 1 - \frac{p}{2}\cos\vartheta_3$
$\mathbb{K} = \mathbb{K}_E - p\,\mathbb{K}_G$
- Loss of stability
$0 = \det \mathbb{K}$
characteristic equ.
$(p-4)(p^2-3p+1) = 0$
$p_3 = 4$
critical value of the load multiplier

Example
total potential energy
elastic energy V_E
external work L_F
2nd-order theory
Equilibrium confs by the stationary condition:
(linearized equil. eqns.)
generalized eig. pb.
K = K_E − p K_G
2nd-order derivatives:
V,ϑ1ϑ1 = 3 − p cos ϑ1 ; V,ϑ1ϑ2 = −1 ; V,ϑ1ϑ3 = 0
V,ϑ2ϑ1 = −1 ; V,ϑ2ϑ2 = 2 − p cos ϑ2 ; V,ϑ2ϑ3 = −1
V,ϑ3ϑ1 = 0 ; V,ϑ3ϑ2 = −1 ; V,ϑ3ϑ3 = 1 − p/2 cos ϑ3
cos ϑi ≃ 1 (in the 2nd-order derivatives)
K_E
K_G
− loss of stability
0 = det K =
characteristic eqn ~ p
(p − 4)(p² − 3p + 1) = 0
p1,2 = (3 ∓ √5)/2 = .3820, 2.618
p3 = 4
critical values of the load multiplier
p_cr,i = p_i K/l
p_cr^E = .3820 , P_cr^E = .3820 K/l