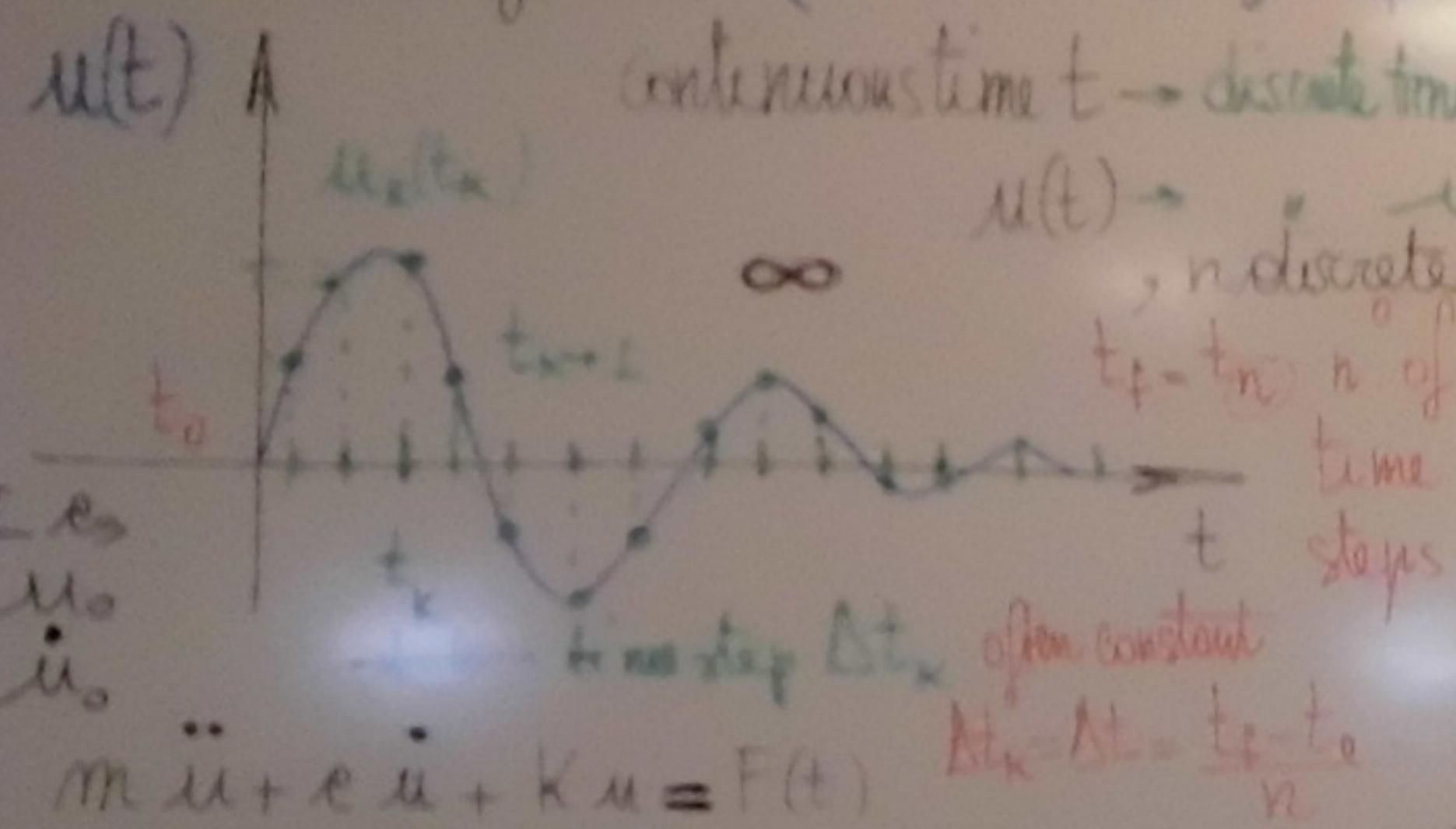


Time integration (direct, step-by-step)



Known the solution at $t=t_k$

$$m\ddot{u}_k + c\dot{u}_k + Ku_k = F_k(t_k)$$

Advance to the solution at $t=t_k + \Delta t = t_{k+1}$

$$m\ddot{u}_{k+1} + c\dot{u}_{k+1} + Ku_{k+1} = F_{k+1}(t_{k+1})$$

$$m\ddot{u}_k + c\dot{u}_k + Ku_k + \Delta t(m\ddot{u}_k + c\dot{u}_k + Ku_k) = F_{k+1}(t_{k+1})$$

$$m\ddot{u}_k + c\dot{u}_k + Ku_k + \Delta t(m\ddot{u}_k + c\dot{u}_k + Ku_k) = F_{k+1}(t_{k+1})$$

$$\Delta t = \frac{t_{k+1} - t_k}{n}$$

Finite differences

$$\frac{df(x)}{dx} \sim \frac{\Delta f}{\Delta x}$$

Taylor expansion

$$\Delta f_x = f(x+\Delta x) - f(x) = f(x) \Delta x + \frac{1}{2} f'(x) \Delta x^2 + \frac{1}{3} f''(x) \Delta x^3 + \dots$$

Method of linear acceleration ($2\beta = \frac{1}{3}; \gamma = \frac{1}{2}$)

$$\begin{aligned} \ddot{u} &= \text{acc} & \ddot{u}_k &= \ddot{u}(t) \\ \dot{u} &= \text{velocity} & \dot{u}_k &= \dot{u}(t) \\ u &= \text{displacement} & u_k &= u(t) \end{aligned}$$

Δt

t_k, t_{k+1}, t

Δt

t_k, t_k, t_{k+1}, t

Δt

t_k, t_k, t_{k+1}, t

Δt

Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing in sec

$$\begin{cases} \Delta u = u \Delta t + \frac{1}{2} \Delta t^2 (\ddot{u} + 2\beta \Delta \ddot{u}) = u \Delta t + \frac{1}{2} \Delta t^2 + \beta \Delta t^2 \quad (1) \\ \Delta \ddot{u} = \Delta t (\ddot{u} + \gamma \Delta \ddot{u}) = \ddot{u} \Delta t + \gamma \Delta \ddot{u} \Delta t \quad (2) \end{cases}$$

$(1-\gamma)\ddot{u} + \gamma \ddot{u}_{k+1}$ weighted average (β, γ)

Mean acceleration ($2\beta = 1; \gamma = \frac{1}{2}$)

$u \text{ const.} \cdot \dot{u}_k, \beta = \frac{1}{2}$

$\ddot{u}_k = \frac{\dot{u}_k - \dot{u}_{k-1}}{2}$

Some relations

$$\ddot{u} = \frac{\Delta u}{\Delta t^2} - \frac{\Delta \ddot{u}}{\Delta t}$$

$$\Delta \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

Substitute (1) into (2)

$$\Delta u - \beta \Delta \ddot{u} + \beta \Delta t \cdot \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

$$\Delta t^2 + \beta \Delta t \cdot \ddot{u} - \beta \Delta t \cdot \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

$$\Delta t^2 + \beta \Delta t \cdot \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

$$\Delta t^2 + \beta \Delta t \cdot \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

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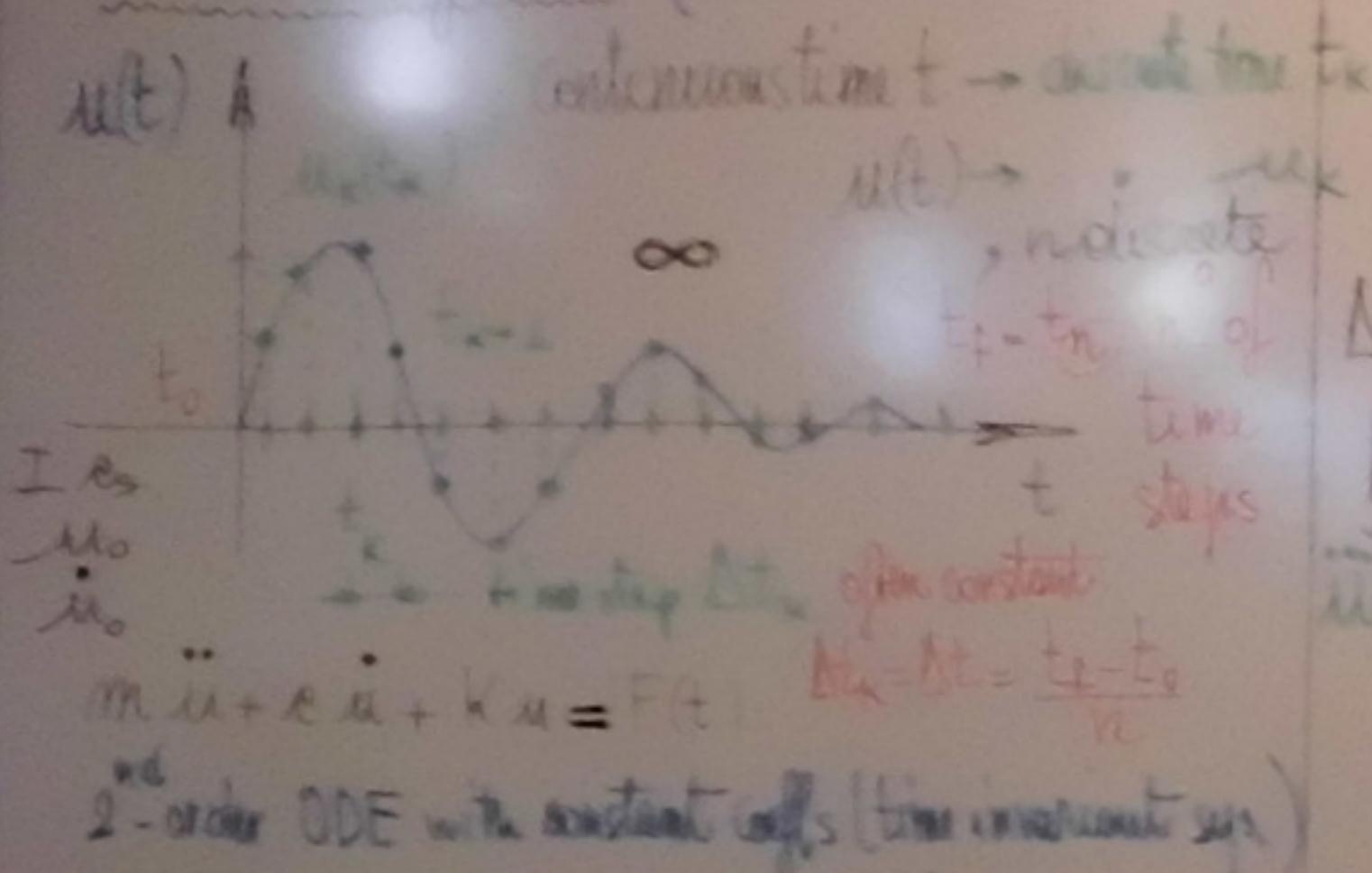
$$\Delta t^2 + \beta \Delta t \cdot \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

$$\Delta t^2 + \beta \Delta t \cdot \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

$$\Delta t^2 + \beta \Delta t \cdot \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

$$\Delta t^2 + \beta \Delta t \cdot \ddot{u} = \frac{\Delta u}{\Delta t} - \frac{\Delta u}{\Delta t}$$

Time integration (direct, step-by-step)



know the solution at $t=t_k$

$$m\ddot{u}_k + c\dot{u}_k + Ku_k = F(t_k)$$

known to the solution at $t=t_k+\Delta t = t_{k+1}$

$$m\ddot{u}_{k+1} + c\dot{u}_{k+1} + Ku_{k+1} = F(t_{k+1})$$

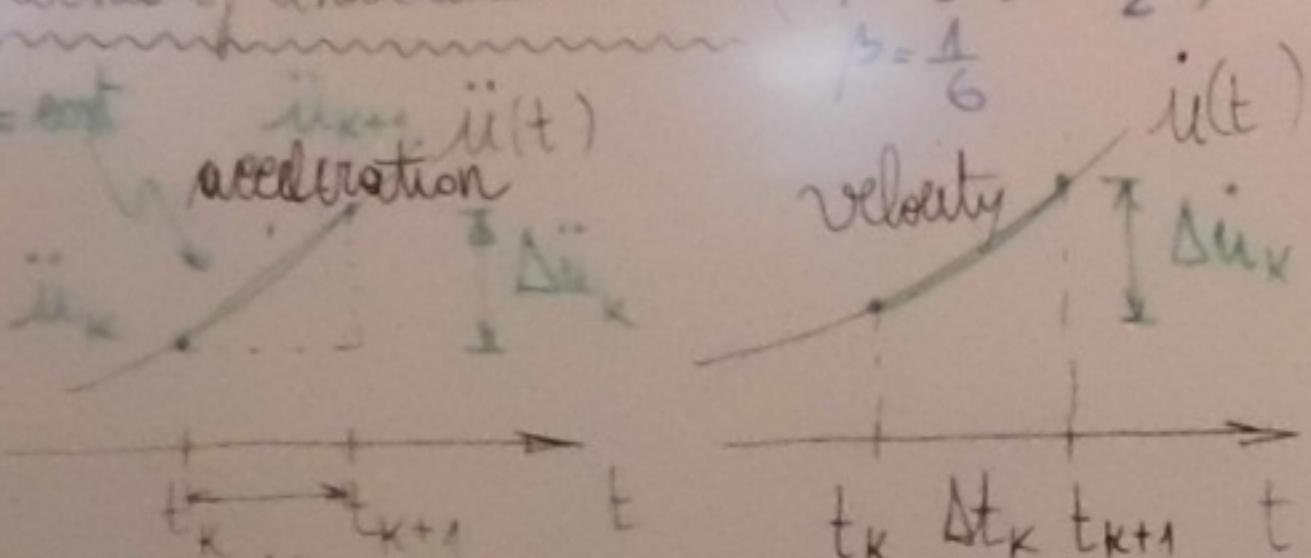
Finite differences

$$\frac{df(x)}{dx} \approx \frac{\Delta f}{\Delta x}$$

Taylor expansion

$$\Delta f_x = f(x+\Delta x) - f(x) = f(x) + \frac{1}{2}f'(x)\Delta x + \frac{1}{6}f''(x)\Delta x^2 + \dots$$

Method of linear acceleration ($2\beta = \frac{1}{3}; \gamma = \frac{1}{2}$)

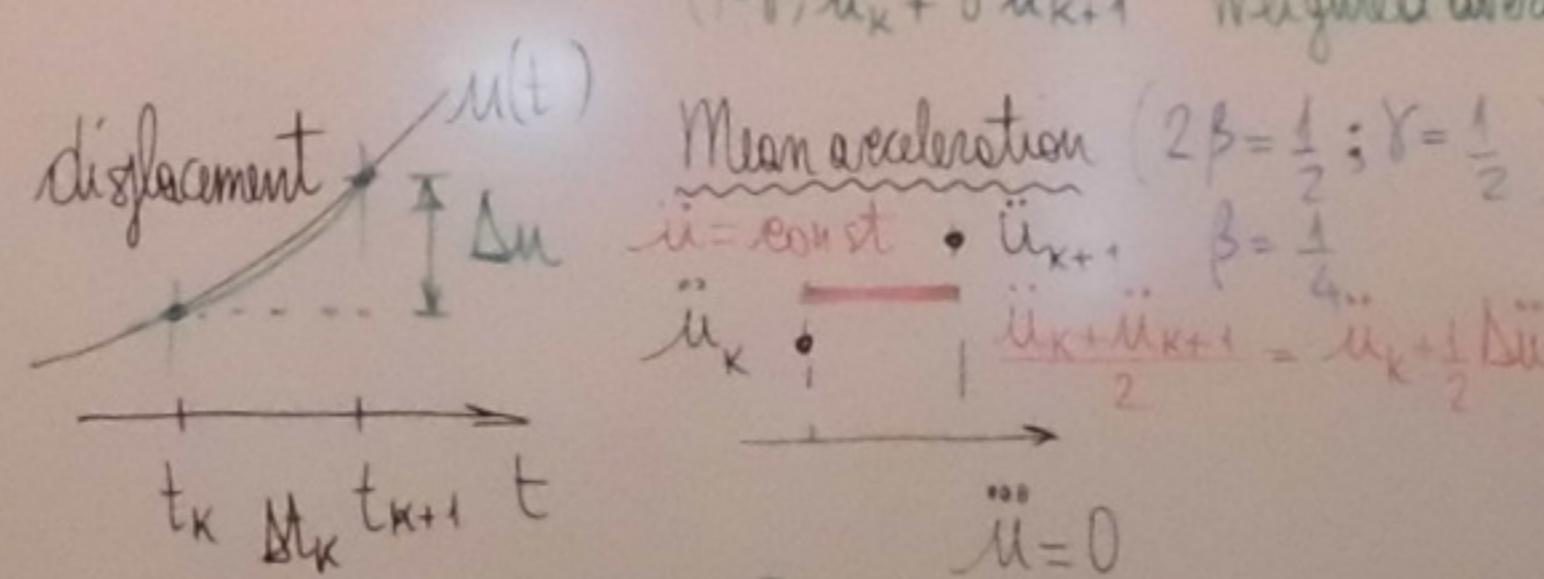


Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing lin sec.

$$\begin{cases} \Delta u = \ddot{u}\Delta t + \frac{1}{2}\Delta t^2(\ddot{u} + 2\beta\Delta \ddot{u}) = \ddot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \beta\Delta \ddot{u}\Delta t^2 & (1) \\ \Delta \ddot{u} = \Delta t(\ddot{u} + \gamma\Delta \ddot{u}) = \ddot{u}\Delta t + \gamma\Delta \ddot{u}\Delta t & (2) \end{cases}$$

$(1-\gamma)\ddot{u}_k + \gamma\ddot{u}_{k+1}$ weighted average (β, γ)



Solve (1) wrt $\Delta \ddot{u}$

$$\Delta \ddot{u} = \frac{\Delta u - \ddot{u}\Delta t - \frac{1}{2}\ddot{u}\Delta t^2}{\beta\Delta t^2}$$

$$\Delta \ddot{u}(\Delta u) = \frac{\Delta u - u}{\beta\Delta t^2} - \frac{u}{2\beta} \quad (3)$$

Substitute (3) into (2):

$$\Delta u = \ddot{u}\Delta t + \gamma\Delta \ddot{u} = \frac{\Delta u}{\beta\Delta t^2} - \frac{u}{\beta\Delta t} - \frac{u}{2\beta}$$

$$= \ddot{u}\Delta t + \frac{\gamma}{\beta}\frac{\Delta u}{\Delta t} - \frac{u}{\beta} - \frac{u}{2\beta}$$

$$\text{By substituting (3) and (1) into the differential eqn of } u(t) \quad \left\{ \begin{array}{l} \Delta u(\Delta u) = \frac{1}{\beta}\frac{\Delta u}{\Delta t} - \frac{u}{\beta} + (1-\frac{1}{2}\beta)\ddot{u}\Delta t \\ \ddot{u} = 0 \end{array} \right\} \quad (4)$$

$$\Delta u = \ddot{u}\Delta t = (\ddot{u} + \frac{1}{2}\Delta \ddot{u})\Delta t \quad \left\{ \begin{array}{l} m(\Delta u - \ddot{u} - \frac{1}{2}\Delta \ddot{u}) + c\left(\frac{1}{\beta}\frac{\Delta u}{\Delta t} - \frac{u}{\beta} + (1-\frac{1}{2}\beta)\ddot{u}\Delta t\right) + Ku = F \\ \Delta u = \ddot{u}\Delta t + \frac{1}{2}\Delta \ddot{u}\Delta t^2 \end{array} \right. \\ = \ddot{u}\Delta t + \frac{1}{2}\Delta \ddot{u}\Delta t^2 \quad \left\{ \begin{array}{l} m(\Delta u - \ddot{u} - \frac{1}{2}\Delta \ddot{u}) + c\left(\frac{1}{\beta}\frac{\Delta u}{\Delta t} - \frac{u}{\beta} + (1-\frac{1}{2}\beta)\ddot{u}\Delta t\right) + Ku = F \\ \Delta u = \ddot{u}\Delta t + \frac{1}{2}\left(\ddot{u} + \frac{1}{2}\Delta \ddot{u}\right)\Delta t^2 \end{array} \right. \\ = \ddot{u}\Delta t + \frac{1}{2}\left(\ddot{u} + \frac{1}{2}\Delta \ddot{u}\right)\Delta t^2 \quad \left\{ \begin{array}{l} m(\Delta u - \ddot{u} - \frac{1}{2}\Delta \ddot{u}) + c\left(\frac{1}{\beta}\frac{\Delta u}{\Delta t} - \frac{u}{\beta} + (1-\frac{1}{2}\beta)\ddot{u}\Delta t\right) + Ku = F \\ \Delta u = \ddot{u}\Delta t + \frac{1}{2}\left(\ddot{u} + \frac{1}{2}\Delta \ddot{u}\right)\Delta t^2 \end{array} \right. \\ \Rightarrow \Delta u = R \cdot \Delta F \end{math>$$

Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing lin acc.

$$\Delta u = \ddot{u} \Delta t + \frac{1}{2} \Delta t^2 (\ddot{\ddot{u}} + 2\beta \Delta \ddot{u}) = \ddot{u} \Delta t + \frac{1}{2} \ddot{u} \Delta t^2 + \beta \Delta \ddot{u} \Delta t^2 \quad (1)$$

$$\Delta \ddot{u} = \Delta t (\ddot{u} + \gamma \Delta \ddot{u}) \quad (1-\gamma)\ddot{u}_k + 2\beta \ddot{u}_{k+1} \\ = \ddot{u} \Delta t + \gamma \Delta \ddot{u} \Delta t \quad (2)$$

$(1-\gamma)\ddot{u}_k + \ddot{u}_{k+1}$ weighted average ($2\beta, \gamma$)

displacement $u(t)$
velocity $\dot{u}(t)$
acceleration $\ddot{u}(t)$

Mean acceleration ($2\beta = \frac{1}{2}; \gamma = \frac{1}{2}$)

$$\ddot{u} = \text{const} \cdot \ddot{u}_{k+1}, \quad \beta = \frac{1}{4}$$

$$\ddot{u}_k + \frac{\ddot{u}_k + \ddot{u}_{k+1}}{2} = \ddot{u}_k + \frac{1}{2} \Delta \ddot{u}_k$$

$$\ddot{u} = 0$$

By substituting (3) and (1)
into incremental eqn. of $u(t)$

$$\Delta u = \ddot{u} \Delta t - \left(\ddot{u} + \frac{1}{2} \Delta \ddot{u} \right) \Delta t$$

$$\Delta u = \ddot{u} \Delta t + \frac{1}{2} \ddot{u} \Delta t^2$$

$$= \ddot{u} \Delta t + \frac{1}{2} \left(\ddot{u} + \frac{1}{2} \Delta \ddot{u} \right) \Delta t^2$$

Solve (1) wrt $\Delta \ddot{u}$:

$$\Delta \ddot{u} = \frac{\Delta u}{\beta \Delta t^2} - \frac{\ddot{u} \Delta t}{\beta \Delta t^2} - \frac{1}{2} \frac{\ddot{u} \Delta t^2}{\beta \Delta t^2} \quad (3)$$

$$\Delta \ddot{u}(\Delta u) = \frac{\Delta u}{\beta \Delta t^2} - \frac{\ddot{u}}{\beta \Delta t} - \frac{\ddot{u}}{2\beta} \quad (3)$$

Substitute (3) into (2):

$$\Delta \ddot{u} = \Delta t \ddot{u} + \gamma \Delta t \left(\frac{\Delta u}{\beta \Delta t^2} - \frac{\ddot{u}}{\beta \Delta t} - \frac{\ddot{u}}{2\beta} \right)$$

$$= \Delta t \ddot{u} + \frac{\gamma \Delta u}{\beta \Delta t} - \frac{\gamma}{\beta} \ddot{u} - \frac{\gamma}{2\beta} \ddot{u} \Delta t$$

$$\Delta \ddot{u}(\Delta u) = \frac{\gamma}{\beta} \frac{\Delta u}{\Delta t} - \frac{\gamma}{\beta} \ddot{u} + \left(1 - \frac{\gamma}{2\beta} \right) \ddot{u} \Delta t \quad (4)$$

$$\Delta u = \frac{\gamma}{\beta} \frac{\Delta u}{\Delta t} - \frac{\gamma}{\beta} \ddot{u} + \left(1 - \frac{\gamma}{2\beta} \right) \ddot{u} \Delta t + K \Delta u = \Delta F \quad \text{elective stiffness}$$

$$\Delta u = \frac{m}{\beta} \left(\frac{\ddot{u}}{\Delta t} - \frac{\ddot{u}}{2} \right) + \left(\frac{\gamma}{\beta} \ddot{u} - \left(1 - \frac{\gamma}{2\beta} \right) \ddot{u} \Delta t \right) + K \Delta u = \Delta F$$

$$\Delta u = \frac{m}{\beta} \left(\frac{\ddot{u}}{\Delta t} - \frac{\ddot{u}}{2} \right) + \left(\frac{\gamma}{\beta} \ddot{u} - \left(1 - \frac{\gamma}{2\beta} \right) \ddot{u} \Delta t \right) + K \Delta u = \Delta F$$

$$\Delta u = \tilde{K} \Delta u = \tilde{K} \Delta F$$

$$\Delta u = \tilde{K} \Delta F$$

Iterative loop (Start. $u_0, \dot{u}_0; \ddot{u}_0$ from eq. of motion)

- ① Initial state $u_k, \dot{u}_k, \ddot{u}_k; \ddot{F}_k \leftarrow m\ddot{u}_k + c\dot{u}_k + Ku_k = F_k$
 - ② Set constants β, γ and evaluate (at given $\Delta t = \text{const}$)
 $\tilde{K} = \frac{m}{\beta \Delta t^2} + \frac{\gamma}{\beta} \frac{c}{\Delta t} + K$ once for all ($\Delta t_k = \Delta t$)
 $\tilde{F} = F + \frac{m}{\beta} \left(\frac{\dot{u}_k + \ddot{u}_k}{2} \right) + c \left(\frac{1}{\beta} \dot{u}_k - \frac{1}{2\beta} \ddot{u}_k \right) \Delta t$
 - ③ Solve for increment Δu
 $\Delta u = \tilde{K}^{-1} \cdot \tilde{F}$
 - ④ Update variables:
 $u = u_k + \Delta u$
 $\dot{u} = \dot{u}_k + \Delta \dot{u} \quad \leftarrow (4$
 $\ddot{u} = \ddot{u}_k + \Delta \ddot{u} \quad \leftarrow (5)$
- loop over the time steps
 $k=1, \dots, N-1$

Monitoring of the progress of the numerical solution

Numerical considerations

- In general N.M. is implicit (meaning that the solution at time t_{k+1} is not automatically known from values at t_k (one has to calculate $\Delta u = \tilde{K}^{-1} \cdot \tilde{F}$))
- Explicit iff $\beta = 0$ (forward), consistently with estimate of Δu from $\tilde{K} \Delta u = \tilde{F}$ for $\beta \rightarrow 0$
- Accuracy of the 2nd order iff $\gamma = 1/2$ $\Delta u \sim \Delta t^2$ ($\Delta t \approx T_1/10$)
- Numerical stability (solution shall be bounded in time at variable Δt)
 - unconditionally stable if $2\beta \geq \gamma \geq 1/2$ (any Δt fine)
 - conditionally stable if $2\beta < \gamma > 1/2$ (depending on Δt)

$$\Delta t \leq \Delta t_{cr} = \frac{\Omega_{cr}}{\omega_1} = \frac{\Omega_{cr} T_1}{2\pi} \text{ where } \Omega_{cr} = \frac{3(\gamma - 1/2) + \sqrt{9(\gamma - 1/2)^2 + 4\beta}}{4\beta}$$

Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing for use

$$\begin{cases} \Delta u = \dot{u}\Delta t + \frac{1}{2}\Delta t^2 (\ddot{u} + 2\beta\Delta \dot{u}) = \dot{u}\Delta t + \frac{1}{2}\Delta t^2 + \beta\Delta t^2 \\ \Delta \dot{u} = \Delta t (\ddot{u} + \gamma \Delta \ddot{u}) = \ddot{u}\Delta t + \gamma \Delta \ddot{u} \Delta t \end{cases}$$

Method	β	γ	Type	Stability
Heun method	$\frac{1}{4}$	$\frac{1}{2}$	implicit	second stable
Linear acceleration	$\frac{1}{6}$	$\frac{1}{2}$	consistent	$\Delta t \leq T_1/6$
Central difference	0	$\frac{1}{2}$	implicit	$\Delta t \leq T_1/3$

Iterative loop

- ① Initial state $u_k, \dot{u}_k, \ddot{u}_k \leftarrow \ddot{m}\ddot{u} + c\dot{u} + Ku = F_k$
 - ② Set constants β, γ and evaluate (at given $\Delta t = \text{const}$)
 $R = \frac{m}{\beta \Delta t^2} + \frac{c}{\beta \Delta t} + K$ (use for all $t_{k+1} = t_k + \Delta t$)
 $\tilde{\Delta F} = \Delta F + \frac{1}{2} \left(\frac{c}{\beta \Delta t} - \frac{1}{2} R \right) \Delta t$
 - ③ Solve for movement Δu
 $\Delta u = \tilde{\Delta F}$
 - ④ Update variables:
 $u = u_k + \Delta u$
 $\dot{u} = \dot{u}_k + \Delta \dot{u} \quad \leftarrow (4)$
 $\ddot{u} = \ddot{u}_k + \Delta \ddot{u} \quad \leftarrow (5)$
- between the time step, the next of the numerical solution*

Numerical considerations

In general N.M. is implicit (meaning that the solution at time t_{k+1} is not automatically known from values at t_k one has to calculate $\Delta u = \tilde{K}^{-1} \tilde{\Delta F}$)

- Explicit iff $\beta=0$ (forward), consistently with estimate of Δu from $K \Delta u = \tilde{\Delta F}$ for $\beta \rightarrow 0$
- Accuracy of the 2nd order iff $\gamma=1/2$ $\Delta u \sim \Delta t$ ($\Delta t \approx \frac{T_1}{10}$)
- Numerical stability (solution shall be bounded in time at variable Δt)
 - unconditionally stable if $2\beta \geq \gamma \geq 1/2$ (any Δt fine)
 - conditionally stable if $2\beta < \gamma > 1/2$ (damping ratio)

$$\Delta t \leq \Delta t_{cr} = \frac{\Omega_{cr}}{\omega_i} = \frac{\Omega_{cr} T_1}{2\pi} \text{ where } \Omega_{cr} = \sqrt{\frac{3(1-\frac{1}{2}) + \frac{3}{4}(\frac{1}{2}-\frac{1}{2}) + \frac{1}{2}\beta}{\frac{1}{2}\beta}}$$

Newmark method (≈ 1959) (γ, β method)

Family of integration methods, generalizing like see

$$\begin{cases} \Delta u = \dot{u} \Delta t + \frac{1}{2} \Delta t^2 (\ddot{u} + 2\beta \Delta \dot{u}) = \dot{u} \Delta t + \frac{1}{2} \dot{u} \Delta t^2 + \beta \Delta \dot{u} \Delta t^2 & (1) \\ \Delta \dot{u} = \Delta t (\ddot{u} + \gamma \Delta \ddot{u}) & (2) \end{cases}$$

Method	β	γ	Type	Stability
Mean acceleration	$\frac{1}{4}$	$\frac{1}{2}$	implicit	Uncond. stable
Linear acceleration	$\frac{1}{6}$	$\frac{1}{2}$	implicit	Conditionality
Central differences	0	$\frac{1}{2}$	explicit	$\Delta t_{cr} = \frac{T_1}{\pi}$

$$\Omega_{cr} = \sqrt{\frac{3}{\pi}} T_1$$

Iteration steps

- Initial state $\mathbf{u}_0, \dot{\mathbf{u}}_0, \ddot{\mathbf{u}}_0$
- \mathbf{F}_0
- Iteration loop
 - Update state $\mathbf{u}_k, \dot{\mathbf{u}}_k, \ddot{\mathbf{u}}_k$
 - Update \mathbf{F}_k
 - Update \mathbf{K} and calculate Δt_{k+1} (at given Δt_k -const)
 - $\mathbf{u}_{k+1} = \mathbf{u}_k + \mathbf{K} \cdot \Delta t_k$ one for all ($\Delta t_k = \Delta t$)
 - $\Delta t_{k+1} = \frac{\Delta t_k}{\beta}$
- Stop for moment Δt_k

3. Solve for moment Δt_k

4. Update solution

Handwritten notes:

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