

# Instability of discrete structural systems

## nd 2-order theory - Formalization

- Analyze systems that may be conveniently described by a finite number of def.

- Analysis under large displacements (there are such to influence the equilibrium config.) but within

"geometrically small" displacements  $\Rightarrow$  allows for:

- linearization of the equil. equations
- quadratic representation of the total potential energy  
(useful to access stability conditions by the SC provided by Dirichlet's theorem)

- Total potential energy:

$$V(q) \approx V(q) = V(q_0) + \frac{\partial V}{\partial q} \Big|_{q_0} \delta q + \frac{1}{2} \delta q^T \frac{\partial^2 V}{\partial q^2} \Big|_{q_0} \delta q + \dots$$

lagrangian variables  $V$

reference configuration (often  $q_0 = 0$ )

undisplaced config.  $\delta q = q$

$\Delta V$  first-order variation

$\frac{1}{2} \delta q^T \frac{\partial^2 V}{\partial q^2} \Big|_{q_0} \delta q$  second-order variation

$$\Delta V = \delta V + \frac{1}{2} \delta^2 V$$

- Equilibrium equations by the stationary condition:

$$\frac{\partial V}{\partial q} = 0 \Rightarrow -P + K \delta q = 0 \Rightarrow K \delta q = P \text{ linearized equil. eqns.}$$

Stability condition:

$$\frac{\partial^2 V}{\partial q^2} \delta q^T K \delta q \geq 0 \quad [\text{loss of positive definiteness of matrix } K]$$

( $P = 0$  if  $q_0$  is an equil. cond.)

[the algebraic properties of Kronecker delta]

- Eulerian stability problems (determination of princi. loads)

• Hypotheses:

- Equil. config. ruled linearly by the load

load vector  $P = p P_0$

load multiplier  $p$

- Pr. critical linearity

$$q = p \bar{q}_0$$

- Second-order variation must be linear in  $p$

$$\frac{1}{2} \delta q^T K \delta q = \frac{1}{2} \delta V = \delta w - p \delta w \text{ linear in } p$$

$\delta w = \frac{1}{2} \delta q^T K \delta q$  static def.

$K = K_E - p K_d$  the def. of  $K$

$\delta w = \frac{1}{2} \delta q^T K_E \delta q$  the critical load  $p$

local stability

stationary

stable

unstable

Instability of discrete structural systems: - Total potential energy:

### 2<sup>nd</sup>-order theory - Formalization

Lagrangian system that now is conveniently associated with a finite number of displacements

Analyse within large displacements (there are such to influence the equilibrium state) but within

'particularly small' displacements  $\Rightarrow$  allows for

- linearization of the equl. equations

- quadratic representation of the total potential energy (useful to assess stability conditions by the SC provided by Dirichlet's theorem)

Lagrangian variables  $V$  reference configuration (often  $q_0 = 0$ )

$$V(q) \approx V(q) = V(q_0) + \frac{\partial V}{\partial q} \Big|_{q_0} \delta q + \frac{1}{2} \delta q^T \frac{\partial^2 V}{\partial q \partial q} \Big|_{q_0} \delta q + \dots$$

$\Delta V$  undistorted config.  $\delta q = q - q_0$

$\Delta V = \delta V + \frac{1}{2} \delta^2 V$

first-order variation  $\delta V$

$$P = \frac{\partial V}{\partial q}$$

second-order variation  $\frac{1}{2} \delta q^T K \delta q$

load vector  $P = P_0$  base loads

$$K = \frac{\partial^2 V}{\partial q \partial q}$$

load multiplier

external load vector of equiv. forces

total stiffness matrix

Equilibrium equations by the stationary condition:

$\frac{\partial V}{\partial q} = 0 \Rightarrow -P + K \delta q = 0 \Rightarrow K \delta q = P$  linearized equl. eqns.

( $P = 0$  if  $q_0$  is an equiv. conf.)

$\delta q^T K \delta q \geq 0$  (lossy feedback elements of matrix  $K$ )

[the algebraic properties of  $K$  rule stability]

- Eulerian stability problems (determination of critical loads)

### • Hypotheses

- Equil. config. ruled linearly by the load

load vector  $P = p P_0$

load multiplier  $p = p^E$

The critical condition  $\frac{1}{2} \delta^2 V = 0$

$\det K = 0$

$K \delta q = p K_0 \delta q$  generalized equl. pb

$\delta q = p \bar{q}_0$  eigenvalues

$\frac{1}{2} \delta^2 V = \sum p \bar{q}_0^2$  eigenvectors

$\sum p \bar{q}_0^2 = \frac{1}{2} \delta q^T K \delta q$  static energy  $K$  stiffness

$K = K_E - p K_0$   $\delta q = \frac{1}{2} \delta q^T K \delta q$  1<sup>st</sup> order external work;  $K_0$  contribution

loss of stability  $\frac{1}{2} \delta^2 V > 0 \Rightarrow \sum p \bar{q}_0^2 > 0$

$p = \frac{\sum p \bar{q}_0^2}{\sum \bar{q}_0^2}$

Roughness ratio provides an upper bound of  $p^F$  (starting from estimate of  $\bar{q}_0$ )

- Eulerian critical load

$$p_{cr}^E = \min \{ p^F \} \rightarrow P_{cr}^E = p_{cr}^E P_0$$

$$\frac{1}{2} \delta^2 V = \sum p \bar{q}_0^2 - p_{cr}^E \sum \bar{q}_0^2 \geq 0$$

$$\text{st } p_{cr}^E \leq \sum p \bar{q}_0^2 / \sum \bar{q}_0^2$$

$$\text{Roughness ratio provides an upper bound of } p^F \text{ (starting from estimate of } \bar{q}_0 \text{)}$$

$$\frac{1}{2} \delta^2 V = \sum p \bar{q}_0^2$$

- Eulerian stability problems (determination of critical loads)

• Hypotheses:

- Equil. config. ruled linearly by the load

load vector  $P = p \frac{P_0}{2} \begin{matrix} \text{base loads} \\ \text{load multiplier} \end{matrix}$

- The critical condition

$$\frac{1}{2} \delta^2 V = 0$$

$$\det K = 0$$

- Pre-critical linearity

$$q = p \bar{q}$$

- Second-order variation may be written as

$$\delta q^T K \delta q = \frac{1}{2} \delta^2 V = \delta w - p \delta w \quad \text{linear in } p$$

$$K \delta q = p K_{\text{cr}} \delta q$$

generalized eigenv prob.

$p$ : eigenvalues

$\delta q^+$ : eigenvectors

$$\left\{ \begin{array}{l} \delta w = \frac{1}{2} \delta q^T K_{\text{cr}} \delta q \text{ elastic energy} \\ \delta w = \frac{1}{2} \delta q^T K_{\text{cr}} \delta q \text{ 2nd order external work; } K_{\text{cr}} \text{ geometric stiffness} \end{array} \right.$$

- Loss of stability

$$\frac{1}{2} \delta^2 V = 0 \Rightarrow \delta w - p \delta w = 0 \Rightarrow$$

$$p_i^* = \frac{\delta w(\delta q^+)}{\delta w(\delta q)} = \frac{\frac{1}{2} \delta q^+ K_{\text{cr}} \delta q^+}{\frac{1}{2} \delta q^+ K_{\text{cr}} \delta q^+}$$

Rayleigh ratio (instability problems)  $p_i^*$   
critical load multipliers;  $\delta q^+$  critical configuration  $= R(\delta q^+)$

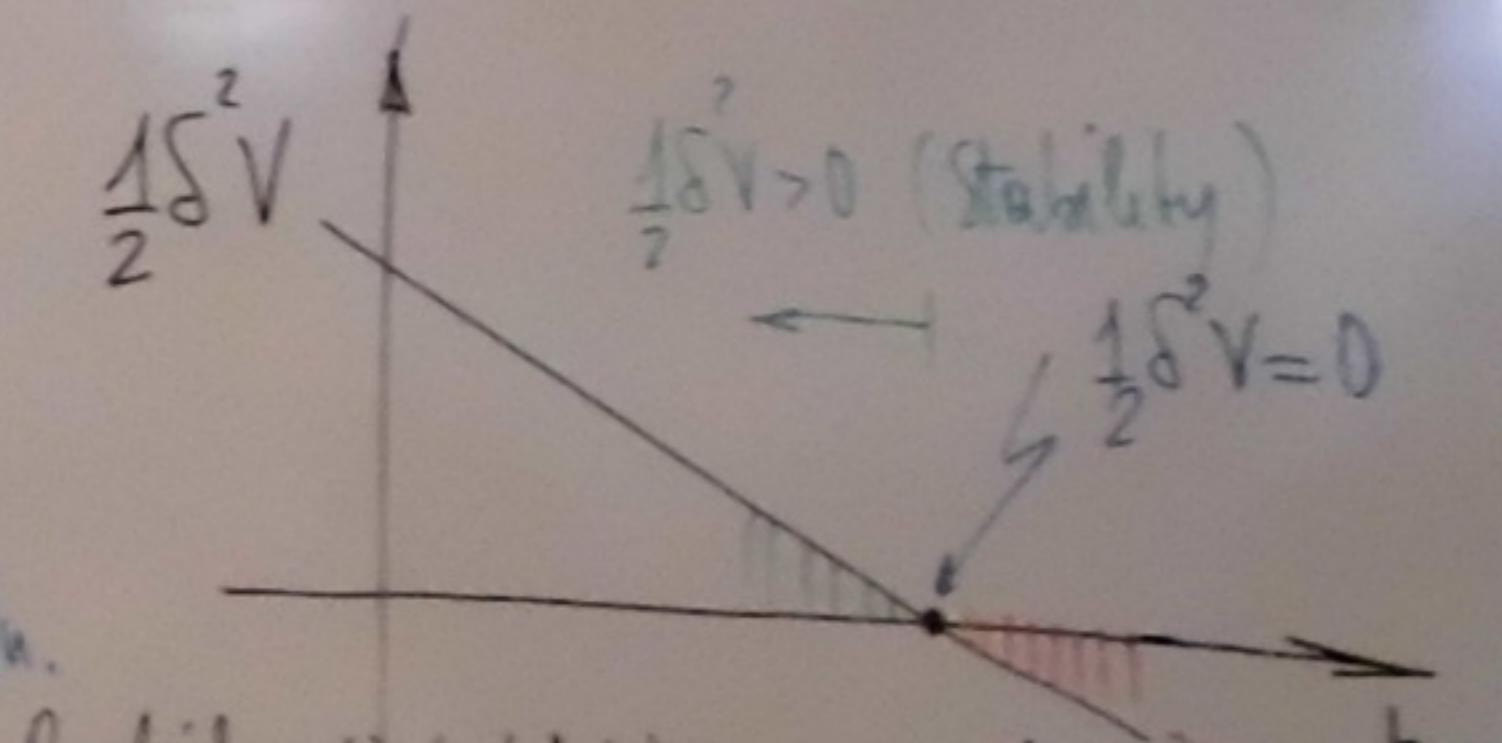
- Eulerian critical load

$$p_{\text{cr}}^E = \min \{ p_i^* \} \rightarrow P_{\text{cr}}^E = p_{\text{cr}}^E P_0^E$$

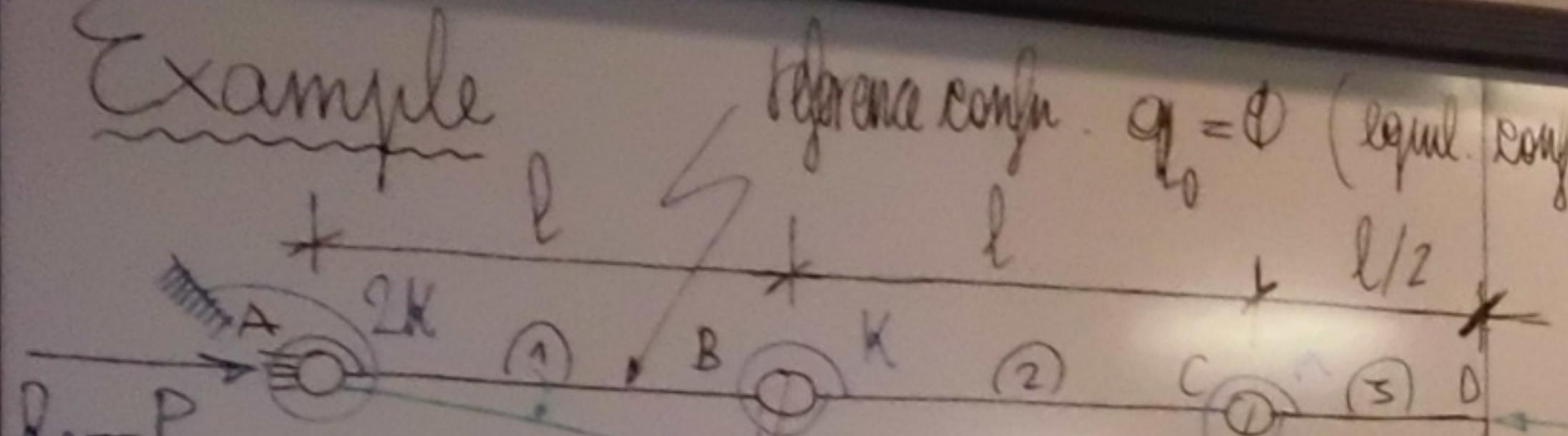
$$\frac{1}{2} \delta^2 V = \delta w(\delta q) - p_{\text{cr}}^E \delta w(\delta q) \geq 0$$

$$\text{at } p = p_{\text{cr}}^E \quad \delta q = p_{\text{cr}}^E \leq \frac{\delta w(\delta q)}{\delta w(\delta q)} = R(\delta q)$$

Rayleigh ratio provides an upper bound of  $p_{\text{cr}}^E$  (starting from estimates of  $\delta q^*$ )



Example



vacuum  
or deformed  
config  
 $\bar{q} = [\theta_1 \quad \theta_2 \quad \theta_3]$

$$u = \bar{AD} - (\bar{AD})_0$$

$$= l + l + \frac{l}{2} - (l \cos \theta_1 + l \cos \theta_2 + \frac{l}{2} \cos \theta_3)$$

$$= \frac{5l}{2} - l(\cos \theta_1 + \cos \theta_2 + \frac{1}{2} \cos \theta_3)$$

large displacements (rotations)  $-P = \frac{\partial V}{\partial q_i} = 0$

$$W_i \ll 1, \cos \theta_i \approx 1 - \frac{\theta_i^2}{2}, \text{ is an equil. config}$$

reference config.  $q_0 = \emptyset$  (equil. config.)

$$q = \bar{q}$$

$$V(q) = \frac{1}{2} (2K(\theta_1)^2 + K(\theta_2 - \theta_1)^2 + K(\theta_3 - \theta_2)^2) - Pu$$

$$\text{total potential energy } V_E = \frac{1}{2} K(2\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 + \theta_3^2 + \theta_2^2 - 2\theta_2\theta_3) V_F - Pu$$

$$\bar{V}(q) = \frac{V(q)}{K} = \frac{1}{2} (3\theta_1^2 + 2\theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3) + p = \frac{Pl}{K} = \frac{P}{K} - \frac{P}{K} \bar{u}$$

$$+ \left( \frac{5}{2} - \cos \theta_1 - \cos \theta_2 - \frac{1}{2} \cos \theta_3 \right)$$

2nd-order theory

- 2nd-order derivatives  $\frac{\partial^2 V}{\partial q_i^2}$

$$\begin{cases} \bar{V}_{\theta_1 \theta_1} = 3 - p \cos \theta_1 & ; \bar{V}_{\theta_1 \theta_2} = -1 & ; \bar{V}_{\theta_1 \theta_3} = 0 \\ \bar{V}_{\theta_2 \theta_2} = -1 & ; \bar{V}_{\theta_2 \theta_1} = 2 - p \cos \theta_1; \bar{V}_{\theta_2 \theta_3} = -1 & \\ \bar{V}_{\theta_3 \theta_3} = 0 & ; \bar{V}_{\theta_3 \theta_1} = -1 & ; \bar{V}_{\theta_3 \theta_2} = -\frac{1}{2} p \cos \theta_3 \end{cases}$$

$\cos \theta_i \approx 1$  (in the first iteration)  $p = 3$

- Equilibrium config by the stationary condition:  $\bar{V} \approx \bar{V}_2(q) = \bar{V}_E(q) - p \left( \frac{5}{2} - \frac{1}{2} \theta_1^2 + \frac{1}{2} \theta_2^2 + \frac{1}{2} \theta_3^2 \right)$

$$\frac{\partial \bar{V}}{\partial \theta_1} = 3\theta_1 - \theta_2 - p \sin \theta_1 = 0$$

$$\frac{\partial \bar{V}}{\partial \theta_2} = 2\theta_2 - \theta_1 - \theta_3 - p \sin \theta_2 = 0$$

$$\frac{\partial \bar{V}}{\partial \theta_3} = \theta_3 - \theta_2 - p \sin \theta_3 = 0$$

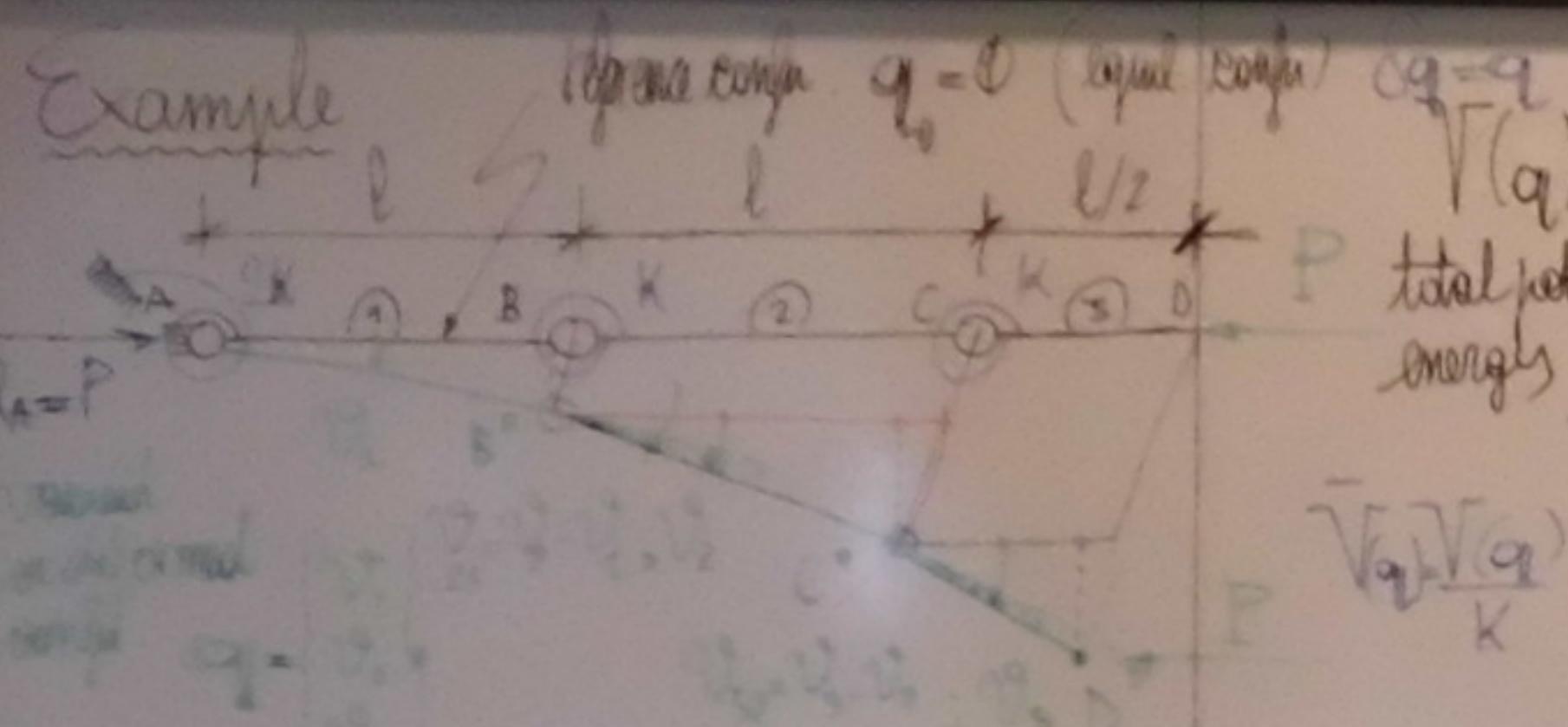
$$+ p \left( \theta_1^2 + \theta_2^2 + \frac{1}{2} \theta_3^2 \right) \downarrow$$

$$(non-linear) \rightarrow \frac{\partial \bar{V}_2}{\partial \theta_i} = 0 \text{ (linearized equil. eqns.)} \quad \begin{bmatrix} 3-p & -1 & 0 \\ -1 & 2-p & -1 \\ 0 & -1 & 1-p \end{bmatrix} \quad \begin{bmatrix} 3-p & 0 & 0 \\ -1 & 2-p & 0 \\ 0 & -1 & 1-p \end{bmatrix}$$

$$\downarrow \frac{\partial \bar{V}_2}{\partial \theta_i} = 0 \Rightarrow K_{ij} = 0 \sin \theta_i \text{ and} \quad \begin{bmatrix} 3-p & 0 & 0 \\ -1 & 2-p & 0 \\ 0 & -1 & 1-p \end{bmatrix} \quad \begin{bmatrix} 3-p & 0 & 0 \\ -1 & 2-p & 0 \\ 0 & -1 & 1-p \end{bmatrix}$$

$$\downarrow K_{ij} = p K_{ij} \quad \begin{bmatrix} 3-p & 0 & 0 \\ -1 & 2-p & 0 \\ 0 & -1 & 1-p \end{bmatrix} \quad \begin{bmatrix} 3-p & 0 & 0 \\ -1 & 2-p & 0 \\ 0 & -1 & 1-p \end{bmatrix}$$

$$\downarrow K = K_E - p K_E \quad \begin{bmatrix} 3-p & 0 & 0 \\ -1 & 2-p & 0 \\ 0 & -1 & 1-p \end{bmatrix} \quad \begin{bmatrix} 3-p & 0 & 0 \\ -1 & 2-p & 0 \\ 0 & -1 & 1-p \end{bmatrix}$$



$$u = \bar{v}_1 + \bar{v}_2$$

$$= l + l - l \cos \theta_1 + l \cos \theta_2 + \frac{1}{2} \cos \theta_3$$

$$= \frac{5}{2}l - l(\cos \theta_1 + \cos \theta_2 + \frac{1}{2} \cos \theta_3)$$

large displacements (rotations)  $\rightarrow P = \frac{\partial V}{\partial q} = 0$

$$W_1: \text{act } \cos \theta_1 = 1 - \frac{\bar{v}_1^2}{2} + \dots \quad \text{isom: } \bar{v}_1 = 0$$

$$V(q) = \frac{1}{2} \left( 2K(\bar{v}_1^2) + K(\bar{v}_2^2) + K(\bar{v}_3^2) \right) - P u$$

$$\text{total potential energy } V_E = \frac{1}{2} K \left( 2\bar{v}_1^2 + \bar{v}_2^2 + \bar{v}_3^2 - 2\bar{v}_1\bar{v}_2 + \bar{v}_3^2 - 2\bar{v}_2\bar{v}_3 \right)$$

$$V(q) = \frac{1}{2} \left( 3\bar{v}_1^2 + 2\bar{v}_2^2 + \bar{v}_3^2 - 2\bar{v}_1\bar{v}_2 - 2\bar{v}_2\bar{v}_3 \right) + P = \frac{Pl}{K} = \frac{P}{K} + \frac{P}{F_0}$$

$\rightarrow$  2nd-order theory

$$\text{Equilibrium config by the stationary condition: } \bar{V} \approx \bar{V}_2(q) = \bar{V}_E(q) - \frac{1}{2} \left( \bar{v}_1^2 + \bar{v}_2^2 + \bar{v}_3^2 \right) \quad \frac{\delta^2 V}{2} = \delta u - p \delta w$$

$$\frac{\partial \bar{V}}{\partial \bar{v}_1} = 3\bar{v}_1 - \bar{v}_2 - p \frac{\sin \theta_1}{\bar{v}_1} = 0$$

$$\frac{\partial \bar{V}}{\partial \bar{v}_2} = 2\bar{v}_2 - \bar{v}_1 - \bar{v}_3 - p \frac{\sin \theta_2}{\bar{v}_2} = 0 \quad (\text{non-linear}) \rightarrow \frac{\partial \bar{V}_2}{\partial \bar{v}_2} = 0 \quad (\text{linearized equil. eqns.})$$

$$\frac{\partial \bar{V}}{\partial \bar{v}_3} = \bar{v}_3 - \bar{v}_2 - p \frac{\sin \theta_3}{\bar{v}_3} = 0$$

$$\frac{\partial^2 \bar{V}_2}{\partial \bar{v}_2^2} = \frac{1}{2} \quad \frac{\delta^2 V}{2} = K_{Gq} = 0 \quad \text{Simplifying by } \frac{1}{2} \quad K_{Gq} = p K_G$$

- 2nd-order derivatives:  $\frac{\partial^2 V}{\partial q_i \partial q_j}$

$$\begin{cases} \bar{V}_{\bar{v}_1 \bar{v}_1} = 3 - p \cos \theta_1 & ; \bar{V}_{\bar{v}_1 \bar{v}_2} = -1 & ; \bar{V}_{\bar{v}_1 \bar{v}_3} = 0 \\ \bar{V}_{\bar{v}_2 \bar{v}_1} = -1 & ; \bar{V}_{\bar{v}_2 \bar{v}_2} = 2 - p \cos \theta_2 & ; \bar{V}_{\bar{v}_2 \bar{v}_3} = -1 \\ \bar{V}_{\bar{v}_3 \bar{v}_1} = 0 & ; \bar{V}_{\bar{v}_3 \bar{v}_2} = -1 & ; \bar{V}_{\bar{v}_3 \bar{v}_3} = 1 - p \cos \theta_3 \end{cases}$$

- loss of stability

$$0 = \det K = -1$$

$$-(3-p)(-p)(1-p) + 1 - p^2 = 0$$

characteristic eqn  $\sim p^3$

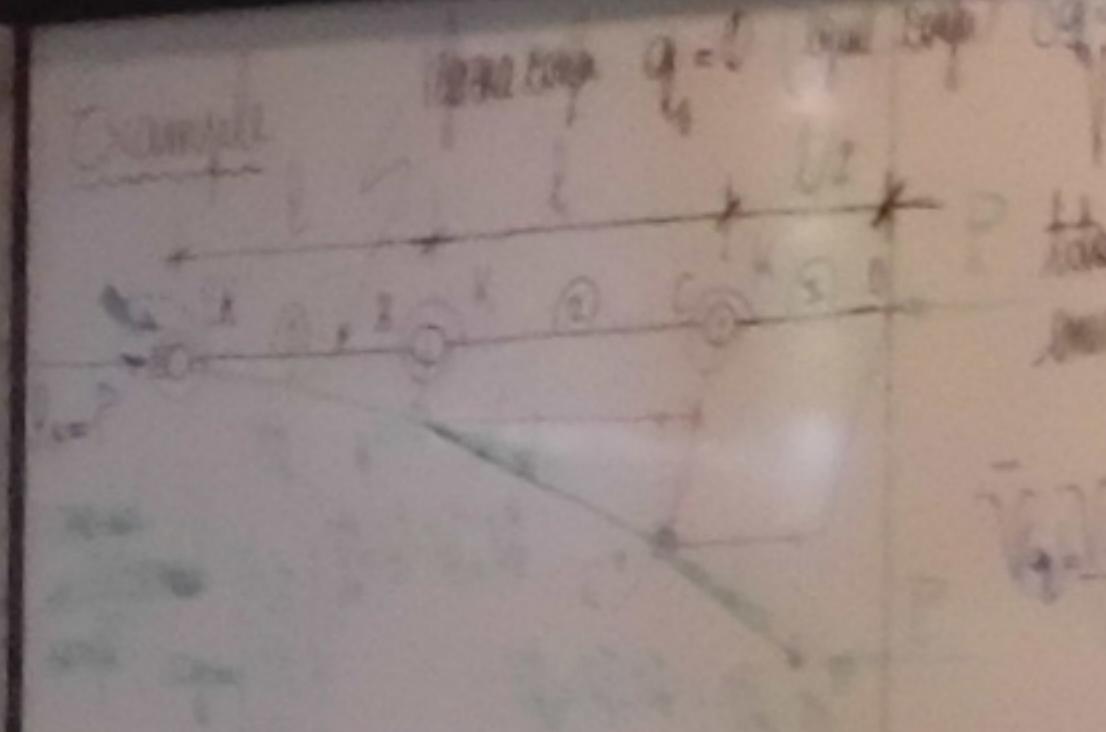
$$(p-4)(p^2-3p+1) = 0$$

$$p_{c,2} = \frac{3 + \sqrt{9 - 4p^2}}{2} \approx 2.658$$

$$p_c = 4$$

critical value of the load multiplier  $P_{cr} = p_c \cdot \frac{K}{l}$

$$P_{cr}^E = 320, P_{cr}^F = 380 \frac{K}{l}$$



$$V(q) = \frac{1}{2} \left[ K_{11}q_1^2 + K_{12}q_1q_2 + K_{13}q_1q_3 + K_{22}q_2^2 + K_{23}q_2q_3 + K_{33}q_3^2 \right] - Pq_2$$

total potential energy

$$= \frac{1}{2} \left[ 3\left(\frac{P}{K}\right)^2 + 2q_1^2 + q_2^2 - 2q_1q_2 - 2q_1q_3 - 2q_2q_3 \right] - Pq_2$$

$\rightarrow$  2<sup>nd</sup> order theory

- 2<sup>nd</sup> order derivatives:  $\frac{\partial^2 V}{\partial q_i \partial q_j}$

$$\begin{cases} \bar{V}_{1,1,1} = 3-p\cos\theta_1 & ; \bar{V}_{1,1,2} = -1 & ; \bar{V}_{1,1,3} = 0 \\ \bar{V}_{1,2,1} = -1 & ; \bar{V}_{1,2,2} = 2-p\cos\theta_2 & ; \bar{V}_{1,2,3} = -1 \\ \bar{V}_{1,3,1} = 0 & ; \bar{V}_{1,3,2} = -1 & ; \bar{V}_{1,3,3} = 1-\frac{p}{2}\cos\theta_3 \end{cases}$$

$\cos\theta_i \approx 1$  (in the 2<sup>nd</sup> order derivatives)

- loss of stability

$$0 = \det K = -1$$

$$= (3-p)(2-p)(1-\frac{p}{2}) + 1(-1)(1-\frac{p}{2}) = 0$$

characteristic eqn.  $\sim p$

$$(p-4)(p^2-3p+1) = 0$$

$$\begin{cases} p_{1,2} = \frac{3 \pm \sqrt{5}}{2} \\ p_3 = 4 \end{cases}$$

critical values of the load multiplier

$$P_{cr,i} = p_i \cdot \frac{K}{l}$$

$$P_{cr}^E = 3820, \quad P_{cr}^E = 3820 \cdot \frac{K}{l}$$

- Equilibrium angle by the stationary condition:  $\bar{V} \approx \bar{V}_2(q) = \bar{V}_E(q) - p \left( \frac{1}{2} \left( \frac{q_1^2}{2} + \frac{q_2^2}{2} + \frac{q_3^2}{2} \right) \right) \quad \frac{1}{2} \delta V = \delta w - p \delta w$

$$\frac{\partial V}{\partial q_1} = 3\theta_1 - \theta_2 - p \sin\theta_2 = 0$$

$$\frac{\partial V}{\partial q_2} = 2\theta_2 - \theta_1 - \theta_3 - p \sin\theta_2 = 0$$

$$\frac{\partial V}{\partial q_3} = \theta_3 - \theta_2 - \frac{p}{2} \sin\theta_3 = 0$$

$$\frac{\partial V_2}{\partial q_1} = 0 \quad (\text{linearized equil. eqns.})$$

$$\frac{\partial V_2}{\partial q_2} = 0 \quad (\text{linearized equil. eqns.})$$

$$\frac{\partial V_2}{\partial q_3} = 0 \quad (\text{linearized equil. eqns.})$$

$$0 = \begin{bmatrix} 3-p & -1 & 0 \\ -1 & 2-p & -1 \\ 0 & -1 & 1-p \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$K_E$$

$$K_G$$