Critical deformed shapes – Example – $\mathbb{K}\,\Phi_i^{Dyn} = \omega_i^2\,\mathbb{M}\,\Phi_i$

$$-\frac{1}{2}\delta^2 V_2 = \frac{1}{2}\,\delta q^T \underbrace{(\mathbb{K}_E - p\,\mathbb{K}_G)}_{\mathbb{K}}\,\delta q = 0 \Rightarrow \mathbb{K}_E\,\delta q_i^* = p_i^*\,\mathbb{K}_G\,\delta q_i^*$$

from 2^{nd}-order theory — loss of stability

("geometrically-small" displacements) $|\vartheta_i| \ll 1$

Φ (equil. config.)

$$\delta q = q - q_0 = q$$

$$\begin{cases} (3-p)\,\vartheta_1 - \vartheta_2 = 0 \\ -\vartheta_1 + (2-p)\,\vartheta_2 - \vartheta_3 = 0 \\ -\vartheta_2 + \left(1-\frac{p}{2}\right)\vartheta_3 = 0 \end{cases} \Rightarrow \boxed{\mathbb{K}\,\delta q = \Phi}$$

$\mathbb{K}$ singular

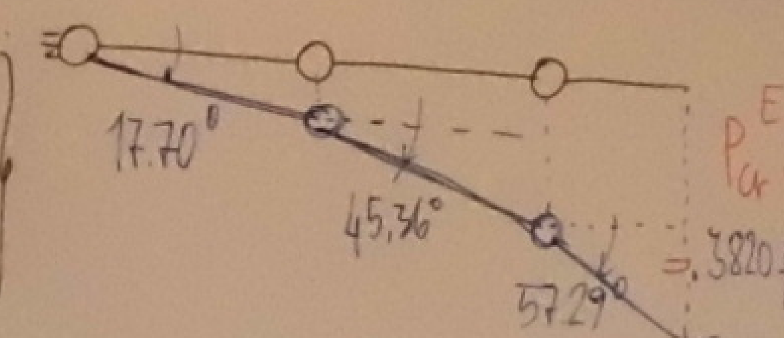
+ $\det \mathbb{K} = 0 \Rightarrow p_1^* = .3820,\ p_2^* = 2.618,\ p_4^* = 4$ critical load multipliers

– from 1^{st}: $\vartheta_1 = \dfrac{\vartheta_2}{3-p} = \dfrac{1}{2}\,\dfrac{2-p}{3-p}\,\vartheta_3$

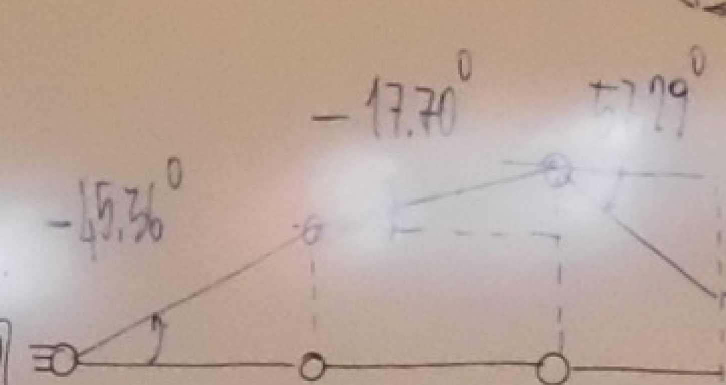
– from 3^{rd}: $\vartheta_2 = \left(1-\frac{p}{2}\right)\vartheta_3$; $\vartheta_3 = 1\,\text{rad} = 57.29°$

Critical deformed shapes
$$\delta q_i^* = \begin{Bmatrix} \frac{1}{2}\,\frac{2-p}{3-p} \\ \frac{2-p}{2} \\ 1 \end{Bmatrix}$$
"instability modes"

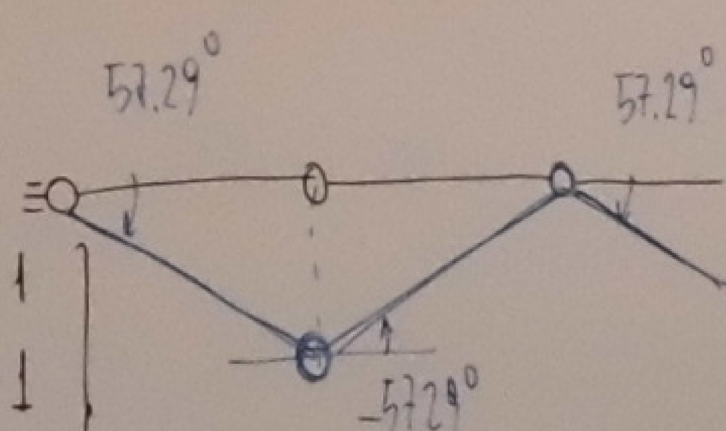
1^{st}
$$\begin{Bmatrix} .3090 \\ .8090 \\ 1 \end{Bmatrix}$$



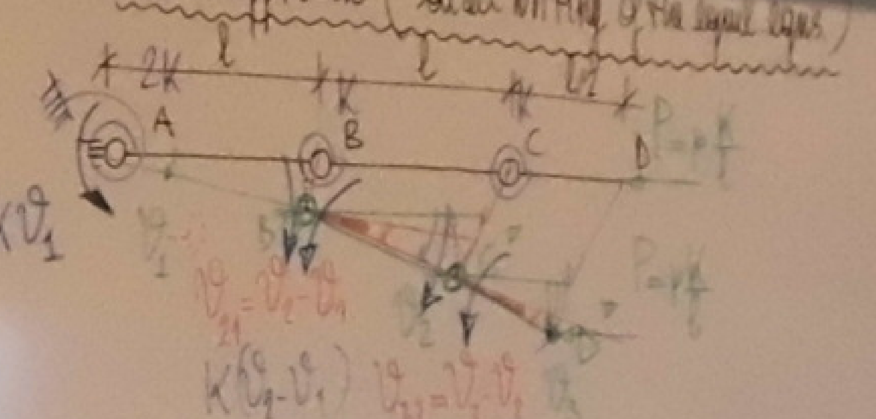
2^{nd}
$$\begin{Bmatrix} -.8090 \\ -.3090 \\ 1 \end{Bmatrix}$$



3^{rd}
$$\begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$



Static approach (direct writing of the equil. eqns.)



non-linear

– Equilibrium eqns. (large displacements) in the deformed config.

$$\begin{cases} \sum M_{C'}^{CD} = 0 \Rightarrow K(\vartheta_3 - \vartheta_2) - p\frac{K}{\ell}\,\frac{\ell}{2}\sin\vartheta_3 = 0 \\ \sum M_{B'}^{BCD} = 0 \Rightarrow K(\vartheta_2 - \vartheta_1) - p\frac{K}{\ell}\left(\frac{\ell}{2}\sin\vartheta_3 + \ell\sin\vartheta_2\right) = 0 \\ \sum M_{A'\equiv A}^{ABCD} = 0 \Rightarrow 2K\vartheta_1 - p\frac{K}{\ell}\left(\frac{\ell}{2}\sin\vartheta_3 + \ell\sin\vartheta_2 + \ell\sin\vartheta_1\right) = 0 \end{cases}$$

– "Geometrically-small" displacements $|\vartheta_i| \ll 1$; $\sin\vartheta_i \simeq \vartheta_i$ to the first order

$\to$ linearized equil. eqns.

$$\tilde{\mathbb{K}}\,\delta q = (\tilde{\mathbb{K}}_E - p\,\tilde{\mathbb{K}}_G)\,\delta q = 0;\quad \tilde{\mathbb{K}} = \begin{bmatrix} 0 & -1 & 1-\frac{p}{2} \\ -1 & 1-p & -\frac{p}{2} \\ 2-p & -p & -\frac{p}{2} \end{bmatrix}$$

linearized equil. eqns.:

$$-\vartheta_2 + \left(1-\frac{p}{2}\right)\vartheta_3 = 0$$
$$-\vartheta_1 + (1-p)\,\vartheta_2 - \frac{p}{2}\,\vartheta_3 = 0$$
$$(2-p)\,\vartheta_1 - p\,\vartheta_2 - \frac{p}{2}\,\vartheta_3 = 0$$

$$\tilde{\mathbb{K}}_E = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix};\ \tilde{\mathbb{K}}_G = \begin{bmatrix} 0 & 0 & [illegible] \\ 0 & 1 & [illegible] \\ 1 & 1 & [illegible] \end{bmatrix}$$

– Critical loads

$\det\tilde{\mathbb{K}} = [illegible] = 0$

$\to \frac{1}{2}p^3 - \frac{3}{2}p^2 + [illegible] = 0 \Rightarrow p_i^* [illegible]$

Critical deformed shapes - Example -

$\mathbb{K}\,\Phi_i = \omega_i^2\, \mathbb{M}\,\Phi_i$

$-\frac{1}{2}\delta^2 V_t = \frac{1}{2}\delta q^T \underbrace{(\mathbb{K}_E - p\,\mathbb{K}_G)}_{\mathbb{K}}\delta q = 0 \Rightarrow \boxed{\mathbb{K}_E\,\delta q_i^* = p_i^*\,\mathbb{K}_G\,\delta q_i^*}$

from 2^{nd}-order theory — loss of stability

"geometrically-small" displacements $|\vartheta_i| \ll 1$

$\delta q = q - q_0 = q$ — Φ (equil. config.)

$$\begin{cases} (3-p)\,\vartheta_1 - \vartheta_2 = 0 \\ -\vartheta_1 + (2-p)\,\vartheta_2 - \vartheta_3 = 0 \Rightarrow \boxed{\mathbb{K}\,\delta q = 0} \\ -\vartheta_2 + \left(1-\frac{p}{2}\right)\vartheta_3 = 0 \end{cases}$$

$\mathbb{K}$ singular

- $\det \mathbb{K} = 0 \Rightarrow p_1^* = .3820,\ p_2^* = 2.618,\ p_4^* = 4$ critical load multipliers

- from 1^{st}: $\vartheta_1 = \frac{\vartheta_2}{3-p} = \frac{1}{2}\frac{2-p}{3-p}\vartheta_3$
- from 3^{rd}: $\vartheta_2 = \left(1-\frac{p}{2}\right)\vartheta_3$

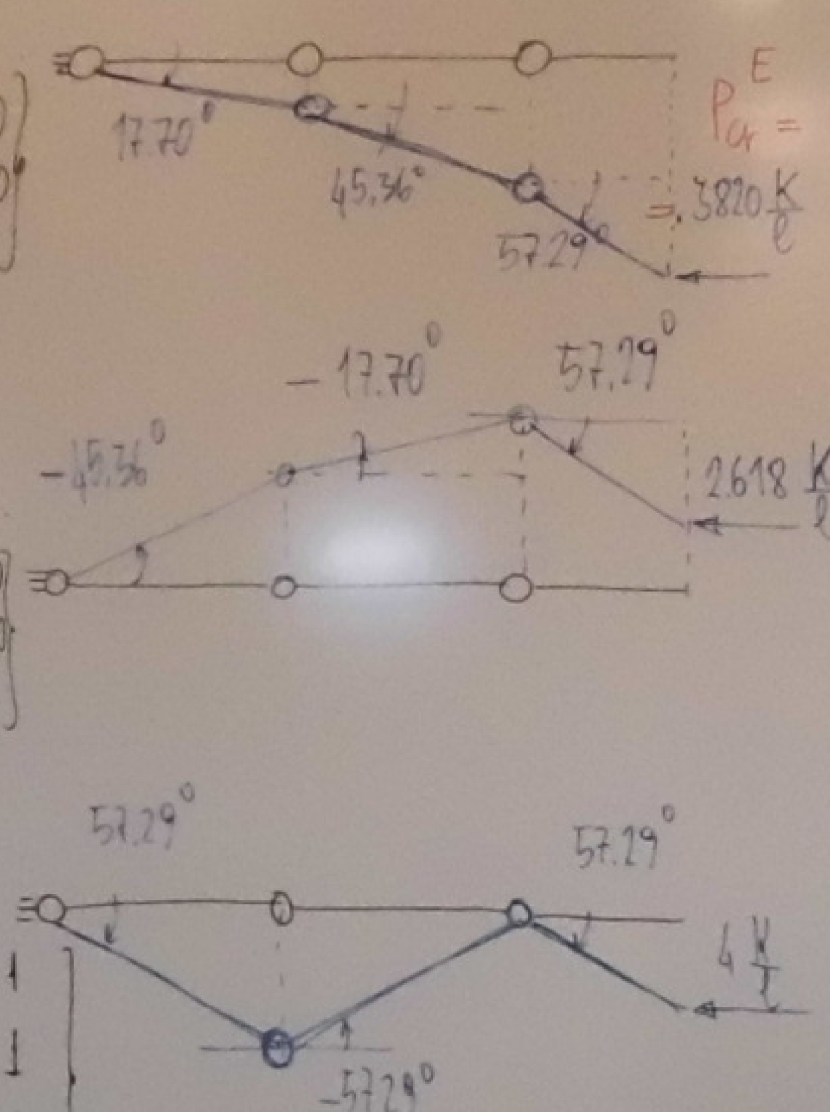
$\vartheta_3 = 1\text{ rad} = 57.29^0$

Critical deformed shapes $\delta q_i^* = \begin{Bmatrix} \frac{1}{2}\frac{2-p}{3-p} \\ \frac{2-p}{2} \\ 1 \end{Bmatrix}$ "instability modes"

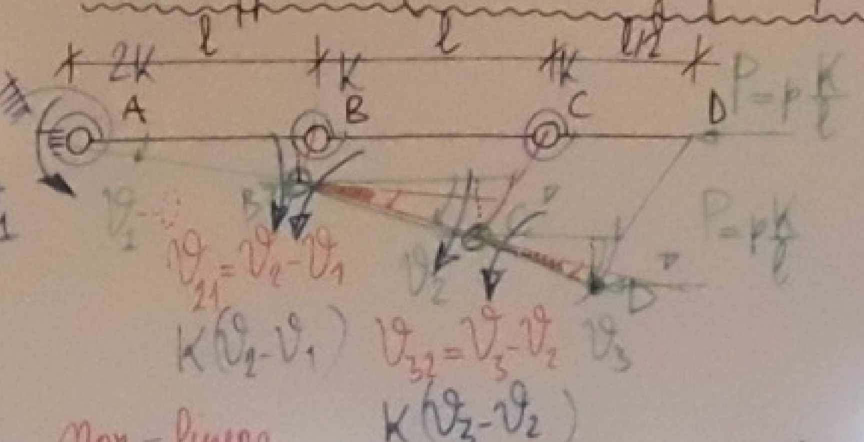
1^{st} $\begin{Bmatrix} .3090 \\ .8090 \\ 1 \end{Bmatrix}$

2^{nd} $\begin{Bmatrix} -.8090 \\ -.3090 \\ 1 \end{Bmatrix}$

3^{rd} $\begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$



Static approach (direct writing of the equil. equs.)



non-linear

- Equilibrium equs. (large displacements) in the deformed config. ($\vartheta_1 = \vartheta_2 = \vartheta_3$ equil. config.)

$$\begin{cases} \sum M_{C}^{CD} = 0 \Rightarrow K(\vartheta_3 - \vartheta_2) - p\frac{K}{\ell}\frac{\ell}{2}\sin\vartheta_3 = 0 \\ \sum M_{B}^{BCD} = 0 \Rightarrow K(\vartheta_2 - \vartheta_1) - p\frac{K}{\ell}\left(\frac{\ell}{2}\sin\vartheta_3 + \ell\sin\vartheta_2\right) = 0 \\ \sum M_{A \equiv A}^{ABCD} = 0 \Rightarrow 2K\vartheta_1 - p\frac{K}{\ell}\left(\frac{\ell}{2}\sin\vartheta_3 + \ell\sin\vartheta_2 + \ell\sin\vartheta_1\right) = 0 \end{cases}$$

- "Geometrically-small" displacements $|\vartheta_i| \ll 1$: $\sin\vartheta_i \simeq \vartheta_i$ to the first-order

$\to$ linearized equil. equs.

$\tilde{\mathbb{K}}\,\delta q = (\tilde{\mathbb{K}}_E - p\,\tilde{\mathbb{K}}_G)\,\delta q = 0;\quad \tilde{\mathbb{K}} = \begin{bmatrix} 0 & -1 & 1-\frac{p}{2} \\ -1 & 1-p & -\frac{p}{2} \\ 2-p & -p & -\frac{p}{2} \end{bmatrix}$

Linearized equil. equs:

$$\begin{cases} -\vartheta_2 + \left(1-\frac{p}{2}\right)\vartheta_3 = 0 \\ -\vartheta_1 + (1-p)\,\vartheta_2 - \frac{p}{2}\,\vartheta_3 = 0 \\ (2-p)\,\vartheta_1 - p\,\vartheta_2 - \frac{p}{2}\,\vartheta_3 = 0 \end{cases}$$

… combinations of the equs. coming from the energy approach (2^{nd}-order theory)

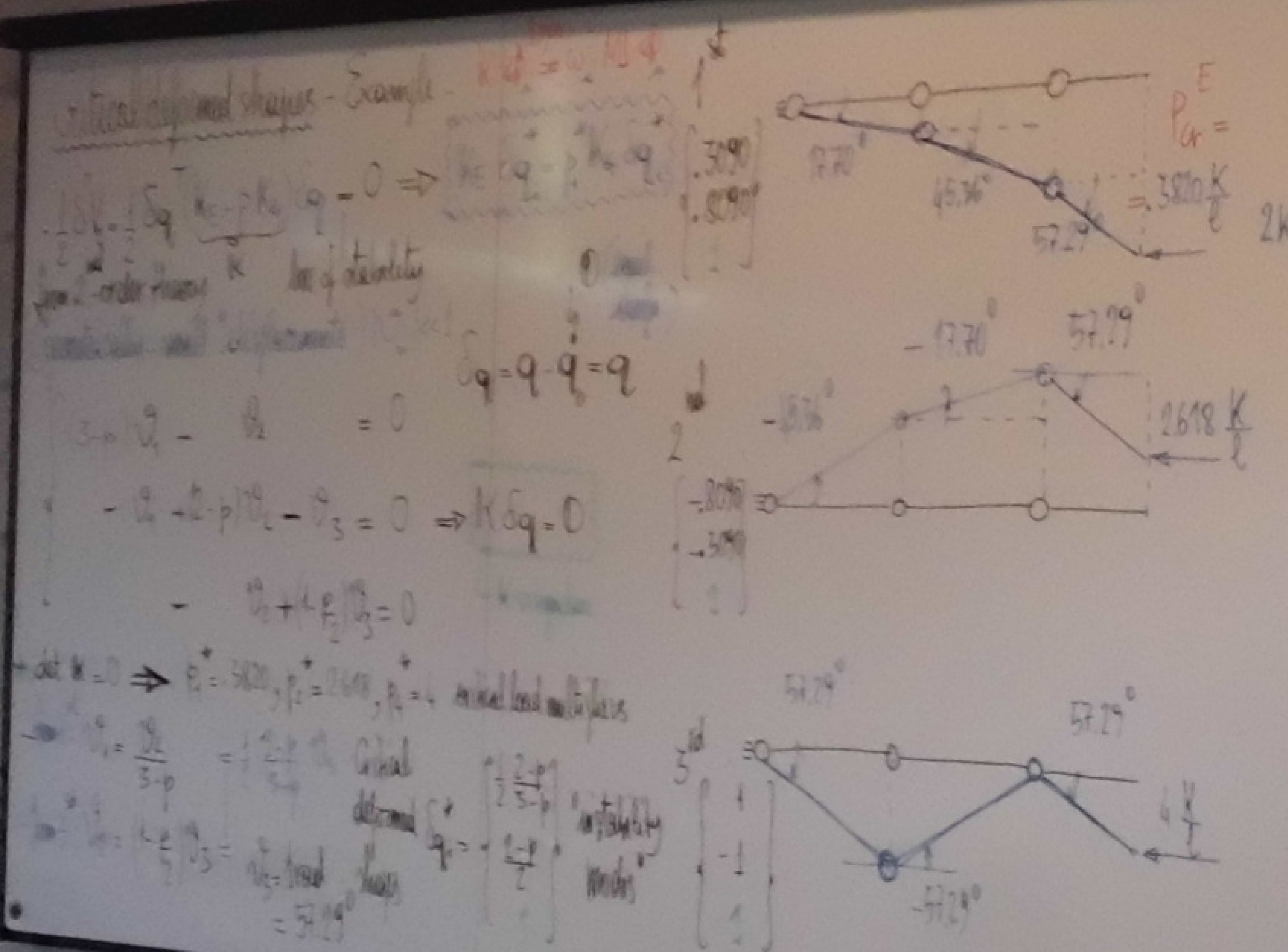
$\tilde{\mathbb{K}}_E = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix};\quad \tilde{\mathbb{K}}_G = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \end{bmatrix}$

in general, non-symmetric matrices

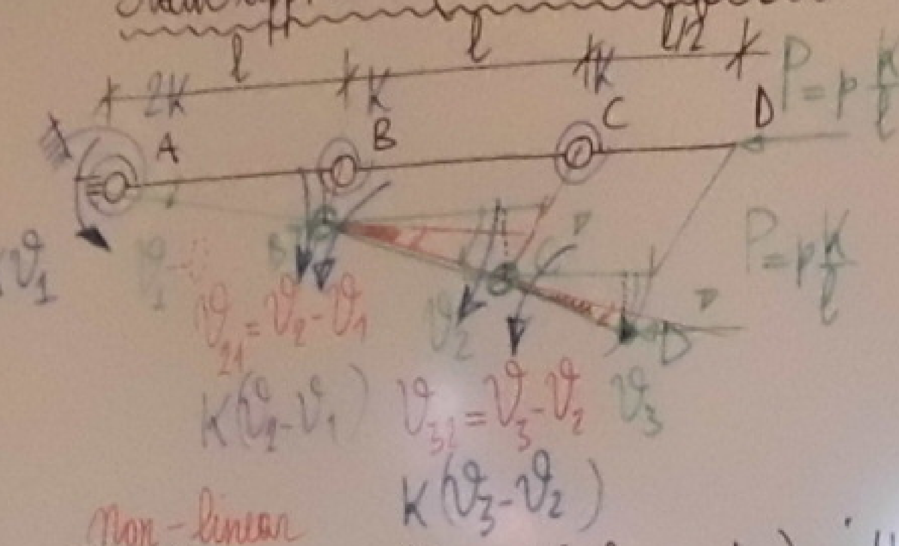
- Critical loads:

$\det \tilde{\mathbb{K}} = 1\left[\frac{p}{2} + \frac{p}{2}(2-p)\right] + \left(1-\frac{p}{2}\right)\left(p - (1-p)(2-p)\right) = 0$

$\to \frac{1}{2}p^3 - \frac{7}{2}p^2 + \frac{13}{2}p - 2 = 0 \Rightarrow p_i^* \ldots$



Static approach (direct writing of the equil. equs.)



Non-linear

– Equilibrium equs. (large displacements) in the deformed confu. ($\vartheta_1 = \vartheta_2 = \vartheta_3$ equil. confu.)

$$\begin{cases} \Sigma M_C^{CD} = 0 \Rightarrow K(\vartheta_3 - \vartheta_2) - p\frac{K}{\ell}\frac{\ell}{2}\sin\vartheta_3 = 0 \\ \Sigma M_B^{BCD} = 0 \Rightarrow K(\vartheta_2 - \vartheta_1) - p\frac{K}{\ell}\left(\frac{\ell}{2}\sin\vartheta_3 + \ell\sin\vartheta_2\right) = 0 \\ \Sigma M_{A\equiv A}^{ABCD} = 0 \Rightarrow 2K\vartheta_1 - p\frac{K}{\ell}\left(\frac{\ell}{2}\sin\vartheta_3 + \ell\sin\vartheta_2 + \ell\sin\vartheta_1\right) = 0 \end{cases}$$

– "Geometrically-small" displacements $|\vartheta_i| \ll 1$; $\sin\vartheta_i \simeq \vartheta_i$ | to the first-order

2> linearized equil. equs,

$$\tilde{\mathbb{K}}\,\delta q = (\tilde{\mathbb{K}}_E - p\tilde{\mathbb{K}}_G)\,\delta q = 0; \quad \tilde{\mathbb{K}} = \begin{bmatrix} 0 & -1 & 1-\frac{p}{2} \\ -1 & 1-p & -\frac{p}{2} \\ 2-p & -p & -\frac{p}{2} \end{bmatrix}$$

Linearized equil. equs.:

$$\begin{cases} -\vartheta_2 + \left(1 - \frac{p}{2}\right)\vartheta_3 = 0 \\ -\vartheta_1 + (1-p)\vartheta_2 - \frac{p}{2}\vartheta_3 = 0 \\ (2-p)\vartheta_1 - p\vartheta_2 - \frac{p}{2}\vartheta_3 = 0 \end{cases}$$

linear combinations of the equs. coming from the energy approach (2nd-order theory)

$$\tilde{\mathbb{K}}_E = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}; \quad \tilde{\mathbb{K}}_G = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \end{bmatrix}$$

in general, non symmetric matrices.

– Critical loads:

$$\det\tilde{\mathbb{K}} = 1\left[\frac{p}{2} + \frac{p}{2}(2-p)\right] + \left(1 - \frac{p}{2}\right)\left(p - (1-p)(2-p)\right) = 0$$

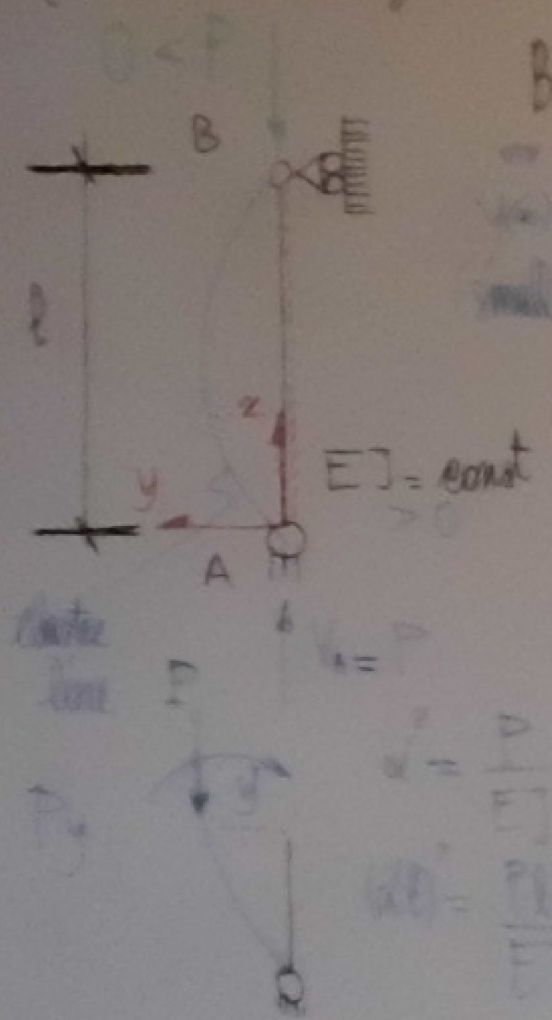
$$2 \Rightarrow \frac{1}{2}p^3 - \frac{7}{2}p^2 + \frac{13}{2}p - 2 = 0 \Rightarrow p_i^*, \delta q_i^*$$

same as by energy

Euler beam (~1750) → continuous systems (static approach)
(Elastic bending instability of compressed beams
0 < P
B
A
l
x
y
EJ = const > 0
elastic line
P
Py
V_A = P
B-E-N theory (bending deflection of beams)
y''(x) ≃ χ = M / EJ
(small rotations)
deformed config.
EJ y''(x) = M(x) = -P y(x)
EJ y''(x) + P y(x) = 0
y''(x) + α² y(x) = 0 (harmonic motion eqn.)
α² = P / EJ > 0
(αl)² = Pl² / EJ ; P = (αl)² EJ / l²
boundary conditions
y(0) = 0
y(l) = 0
solution: (A, B arbitrary constants)
y(x) = A sin αx + B cos αx
y'(x) = αA cos αx - αB sin αx
y''(x) = -α² (A sin αx + B cos αx) = -α² y
By setting the b.c.s:
y(0) = B = 0
y(l) = A sin αl + B cos αl = 0 → A sin αl = 0
y=0 (ref conf.) equil. conf. ∀P
A = 0, sin αl arb.
sin αl = 0, A arb.
system of
0 1
sin αl cos αl
A
B
0
0
D · X = 0 → det D = -sin αl = 0
αl = nπ, n = 1, 2, 3 ...
(αl)² = n² π²
∞ bifurcated solutions
P_{cr,n} = n² π² EJ / l²
y_{cr,n}(x) = A sin nπx / l
appear @ P = P_{cr,n}
P^E_{cr} = P_{cr,n=1} = π² EJ / l²
flex points
y'' = 0 ⇒ M = 0

Euler beam (~1750) → continuous systems (static approach)

(Elastic bending instability of compressed beams



B-E-N theory (bending deflection of beams)

$\chi \simeq \frac{M}{EJ}$

small rotations

deformed config.

$EJ\, y''(x) = M(x) = -P\, y(x)$

$EJ\, y''(x) + P\, y(x) = 0$

$y''(x) + \alpha^2 y(x) = 0$ (harmonic motion eqn.) $\rightarrow y(x) = A\sin\alpha x + B\cos\alpha x$

$\alpha^2 = \frac{P}{EJ} > 0$

boundary conditions

$\begin{cases} y(0) = 0 \\ y(\ell) = 0 \end{cases}$

solution (A, B arbitrary constants)

$y'(x) = \alpha A\cos\alpha x - \alpha B\sin\alpha x$

$y''(x) = -\alpha^2(A\sin\alpha x + B\cos\alpha x) = -\alpha^2 y$ ✓

By setting the b.cs:

$\begin{cases} y(0) = \boxed{B = 0} \\ y(\ell) = A\sin\alpha\ell + B\cos\alpha\ell = 0 \end{cases} \rightarrow A\sin\alpha\ell = 0$

$A = 0$, $\sin\alpha\ell$ arb.

$\boxed{\sin\alpha\ell = 0,\ A \text{ arb.}}$

$y \equiv 0$ (ref. conf.) equil. conf. $\forall P$

system of b.cs

$\begin{bmatrix} 0 & 1 \\ \sin\alpha\ell & \cos\alpha\ell \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$\mathbb{D}\, X = 0 \rightarrow \det \mathbb{D} = \boxed{-\sin\alpha\ell = 0}$

$\alpha\ell = n\pi$, $n = 1, 2, 3 \ldots$

$(\alpha\ell)^2 = n^2\pi^2$

∞ bifurcated solutions $\quad P_{cr,n} = n^2\pi^2 \frac{EJ}{\ell^2} = \pi^2 \frac{EJ}{\underbrace{\left(\frac{\ell}{n}\right)^2}_{\ell_{0,n}}} = \pi^2 \frac{EJ}{\ell_{0,n}^2}$

$y_{cr,n}(x) = A\sin\frac{n\pi x}{\ell}$

appear @ $P = P_{cr,n}$

$\ell_{0,n} = \frac{\ell}{n}$ free inflection length

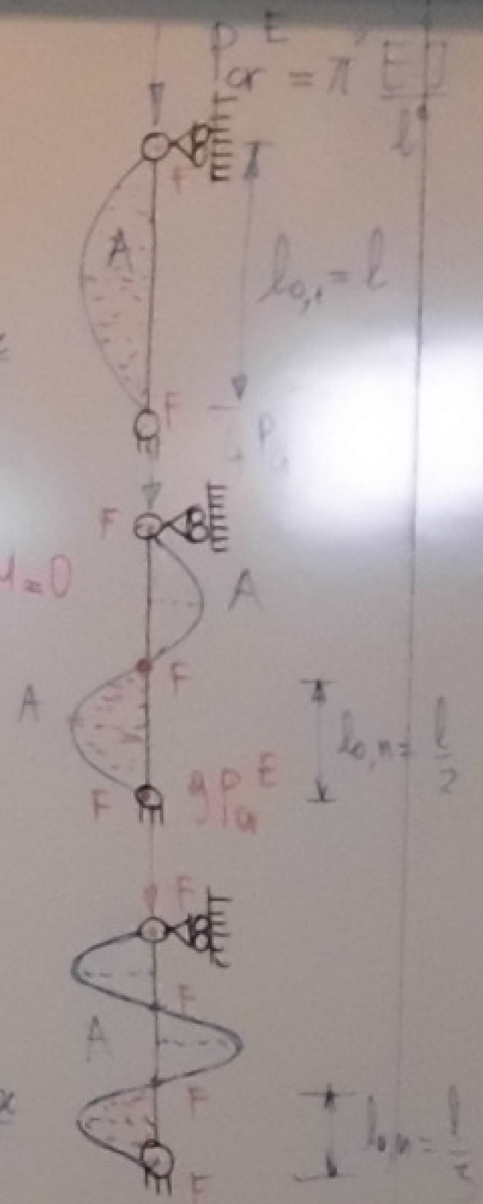
$P^E_{cr} = P_{cr,n=1} = \pi^2 \frac{EJ}{\ell^2}$

flex points $y'' = 0 \Rightarrow M = 0$

$A\sin\frac{\pi x}{\ell}$

$A\sin\frac{2\pi x}{\ell}$

$A\sin\frac{3\pi x}{\ell}$



- The critical loads are ruled by the free inflection length (the distance between two consecutive flex points of the critical deformed shape) as

$$P_{cr,n} = \pi^2 \frac{EJ}{\ell_{0,n}^2}$$

- The critical load is directly proportional to the bending elastic stiffness EJ
- It is inversely proportional to the square of free inflection length $\ell_{0,n} = \frac{\ell}{n}$

By setting the b.c.s

$y(0) = \boxed{B = 0}$

$y(l) = A\sin\alpha l + B\cos\alpha l = 0 \rightarrow A\sin\alpha l = 0$

$y \equiv 0$ (ref. conf.) equil. conf. $\forall P$

$A = 0$, $\sin\alpha l$ arb.

$\boxed{\sin\alpha l = 0,\ A \text{ arb.}}$

$\alpha l = n\pi \quad , n = 1, 2, 3 \ldots$

$(\alpha l)^2 = n^2\pi^2$

$\begin{bmatrix} 0 & 1 \\ \sin\alpha l & \cos\alpha l \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$D \cdot X = 0 \rightarrow \det D = -\boxed{\sin\alpha l = 0}$

solution (A, B arbitrary constants)

$\rightarrow y(x) = A\sin\alpha x + B\cos\alpha x$

∞ bifurcated solutions $\quad P_{cr,n} = n^2\pi^2 \dfrac{EJ}{l^2}$

$y_{cr,n}(x) = A\sin\dfrac{n\pi x}{l} \quad = \pi^2 \dfrac{EJ}{\underbrace{\left(\frac{l}{n}\right)^2}_{l_{0,n}}} = \pi^2 \dfrac{EJ}{l_{0,n}^2}$

appear @ $P = P_{cr,n}$

$P_{cr}^E = P_{cr,n=1} = \pi^2 \dfrac{EJ}{l^2}$ $\quad l_{0,n} = \dfrac{l}{n}$

free inflection length

$A\sin\dfrac{\pi x}{l}$

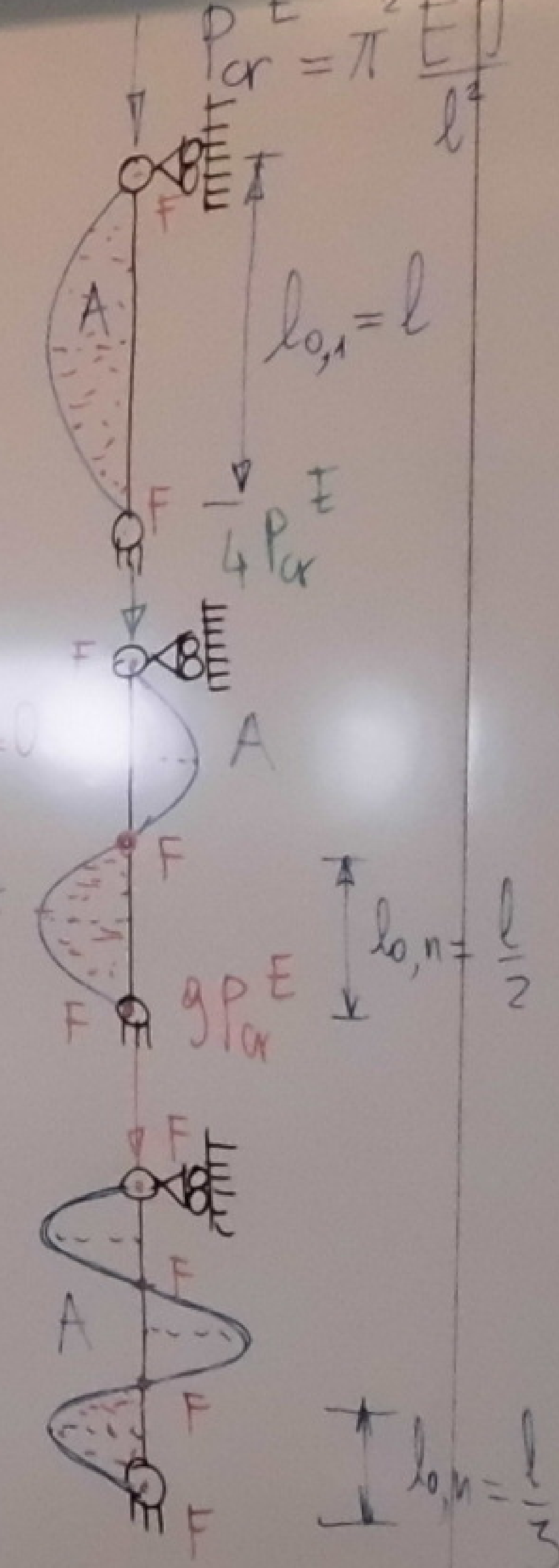
Flex points

$y'' = 0 \Leftrightarrow M = 0$

$A\sin\dfrac{2\pi x}{l}$

$A\sin\dfrac{3\pi x}{l}$

$P_{cr}^E = \pi^2 \dfrac{EJ}{l^2}$



- The critical load(s) are ruled by the free inflection length (the distance between two consecutive flex points of the critical deformed shape) as

$$\boxed{P_{cr,n} = \pi^2 \frac{EJ}{l_{0,n}^2}}$$

- The critical load is directly proportional to the bending elastic stiffness EJ
- It is indirectly proportional to the square of free inflection length $l_{0,n} = \dfrac{l}{n}$