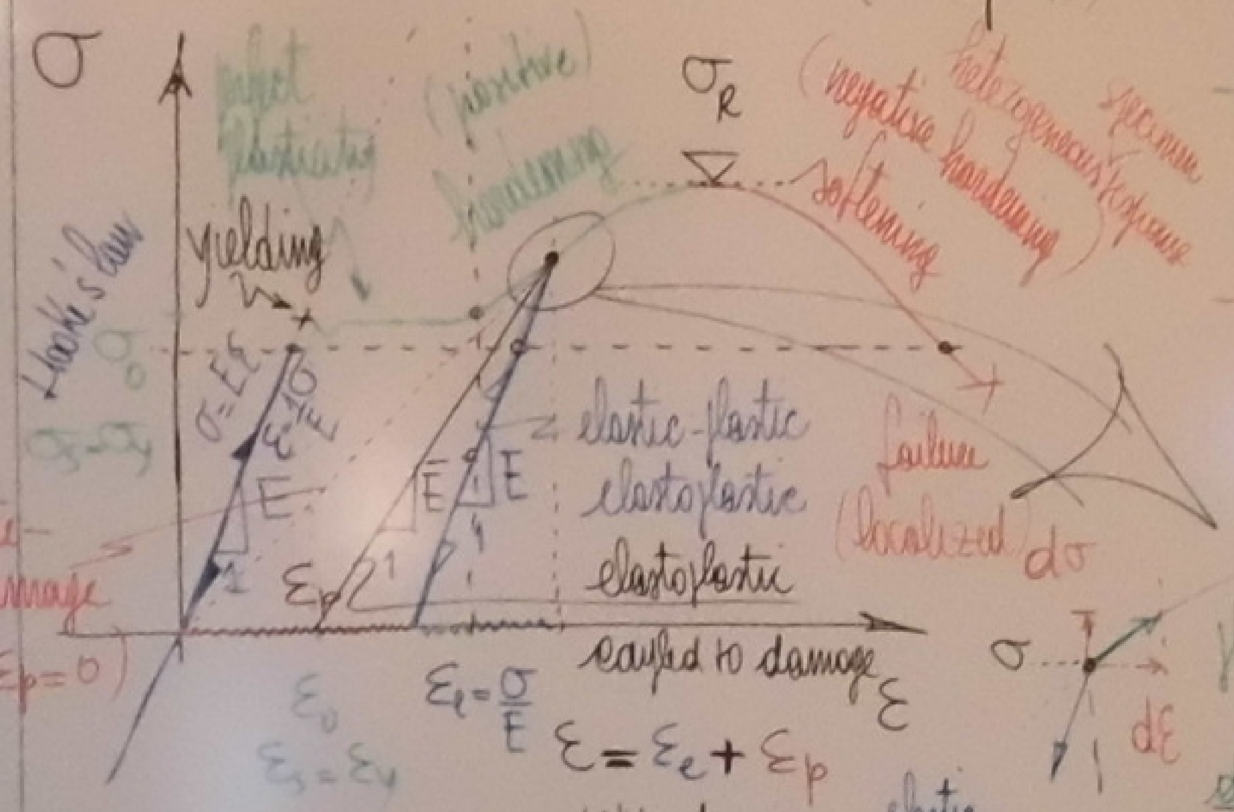
Part III - Anelasticity of Structures (DIAS)

Plasticity

- **Two main goals:**
  - Provide an introduction to the Theory of Plasticity at the continuum scale (Continuum Mechanics)
  - Outline the Limit Analysis of frames (alias Calcul à rupture => Calcolo a rottura)
- Elastoplastic material behaviour at small strains in a one-dimensional context (or scalar stress-strain law)
- Reference to ... materials at room temperature and ... of outlining some characteristic features (e.g. for metallic materials and engineering materials at small strains) => material non-linearity

- **Post-elastic response**

peak stress (limit point)



$\varepsilon_e = \frac{\sigma}{E}$

$\varepsilon = \varepsilon_e + \varepsilon_p$

additive strain decomposition at small strain

ε_p: plastic strain, irreversible strain, permanent strain

ε_e: elastic strain, reversible strain, recoverable strain

Damage Mechanics effects: stiffness degradation

$\bar{E} \leq E$

e.g. $\bar{E} = (1-D)E$ damage variable

σ, $d\sigma$, $d\varepsilon$, ε

elastic unloading

$d\sigma = E\, d\varepsilon$

elastic stiffness modulus

$\dot{\sigma} = E\dot{\varepsilon}$

plastic loading

$\frac{d\sigma}{dt} = E_t \frac{d\varepsilon}{dt}$

tangent stiffness modulus

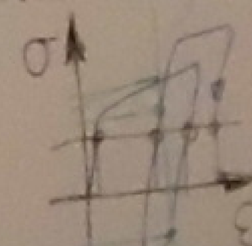
t time ordering variable marking the sequence of events (quasi static process)

Incremental law

$\dot{\sigma} = E_t\dot{\varepsilon}$

stress rate, strain rate

- Non-linear material behaviour with typical features:
  - Non-holonomic response leading to the need of describing the stress-strain law in incremental form
  - Plastic strain: phenomenological observation corresponding to internal rearrangements of the material microstructure (moving dislocations)
  - This reflects a form of energy dissipation occurring inside the material (heat generation) that cannot be recovered after unloading
  - This leads to a response that is "path-dependent", meaning that the current stress-strain response depends on the history undergone by the material



(history dependent)

- Incremental stress-strain law in direct form or ... form

$\dot{\sigma} = \dot{\sigma}(\dot{\varepsilon})$

(often used, FEM) ... stated in terms of the tangent modulus

$E_t > 0$ hardening

- Hardening typologies

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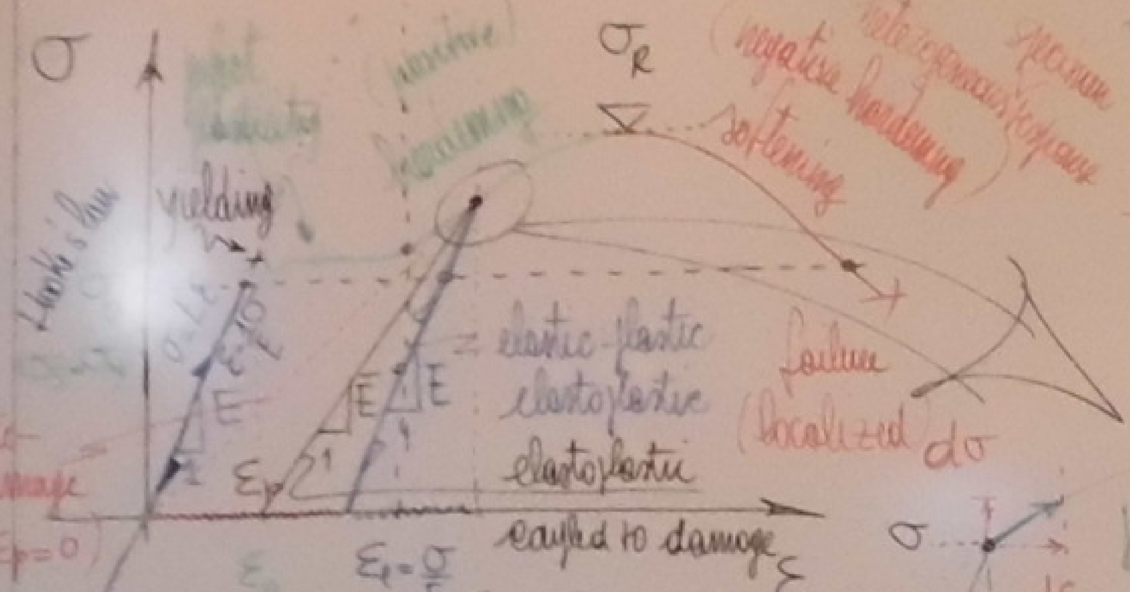
- Post-elastic response — peak stress (limit point)



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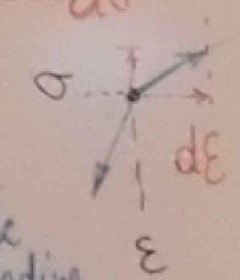

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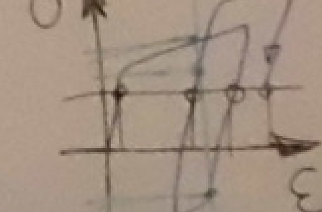


(history dependent) → history variables or internal variables

- Incremental stress-strain law in direct form or inverse form

$\dot{\sigma} = \dot{\sigma}(\dot{\varepsilon})$ — $\dot{\varepsilon} = \dot{\varepsilon}(\dot{\sigma})$

(often used, FEM) strain control — stress control (stress-based formulations)

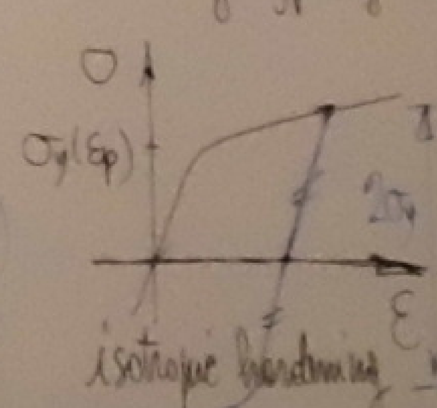
stated in terms of the tangent modulus

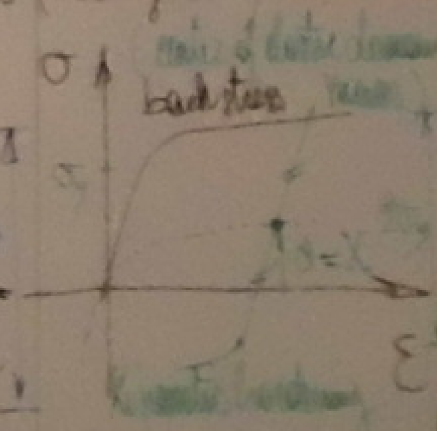
$E_t > 0$ hardening (positive) — unique for both forms

$E_t = 0$ perfect plasticity — non unique (unique just for the direct form)

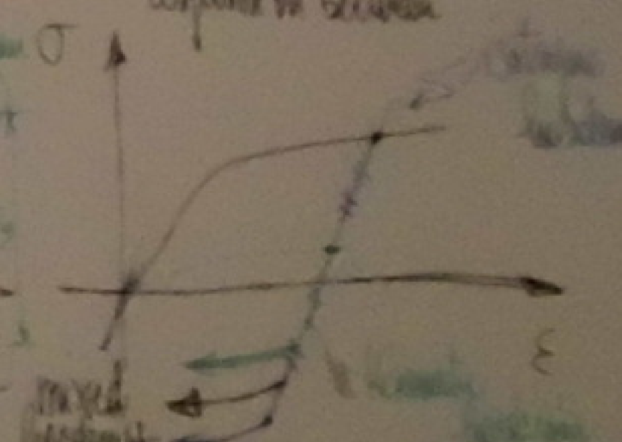
$E_t < 0$ softening (negative hardening)

- Hardening typologies (codified)





Bauschinger effect



- Non-linear material behaviour with typical features:
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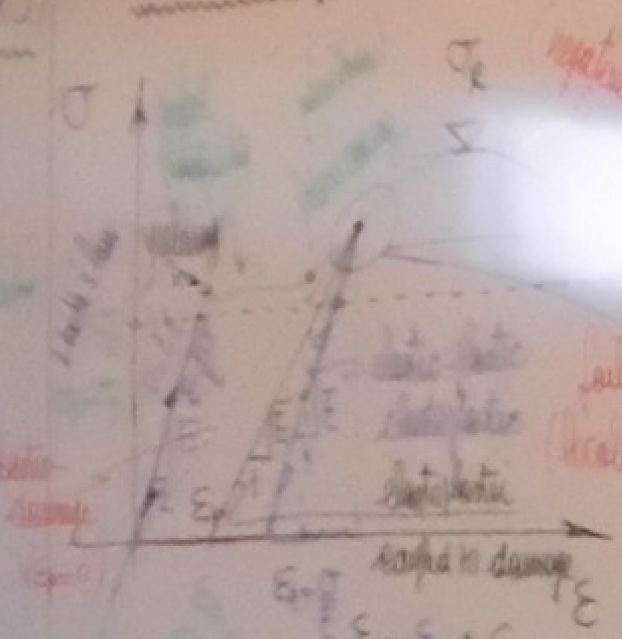


$\varepsilon = \varepsilon_e + \varepsilon_p$

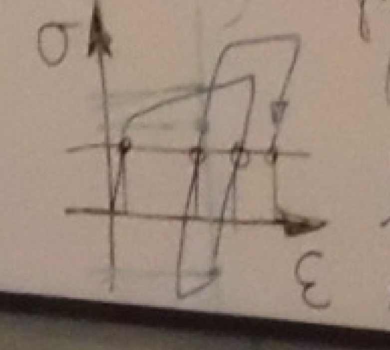
Incremental law

$\dot{\sigma} = E \dot{\varepsilon}$

$\dot{\sigma} = E_t \dot{\varepsilon}$

$\frac{d\sigma}{dt} = E_t \frac{d\varepsilon}{dt}$

t time ordering variable marking the sequence of events (quasi static process)



(history dependent)

history variables or internal variables

- Incremental stress-strain law in direct form or inverse form

$\dot{\sigma} = \dot{\sigma}(\dot{\varepsilon})$ (often used, FEM) strain control (stress-based formulations)

$\dot{\varepsilon} = \dot{\varepsilon}(\dot{\sigma})$ stress control (strain-based formulations)

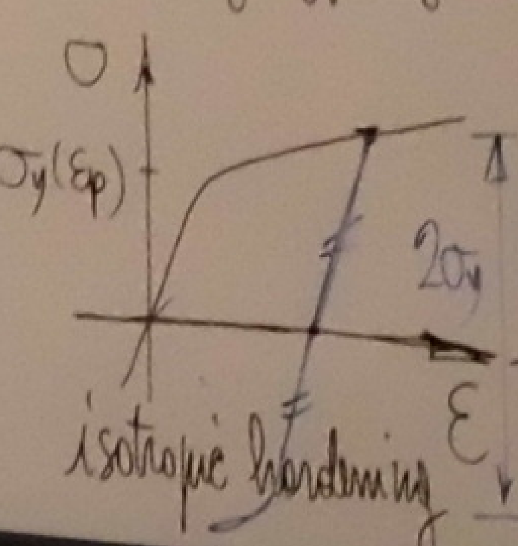
stated in terms of the tangent modulus:

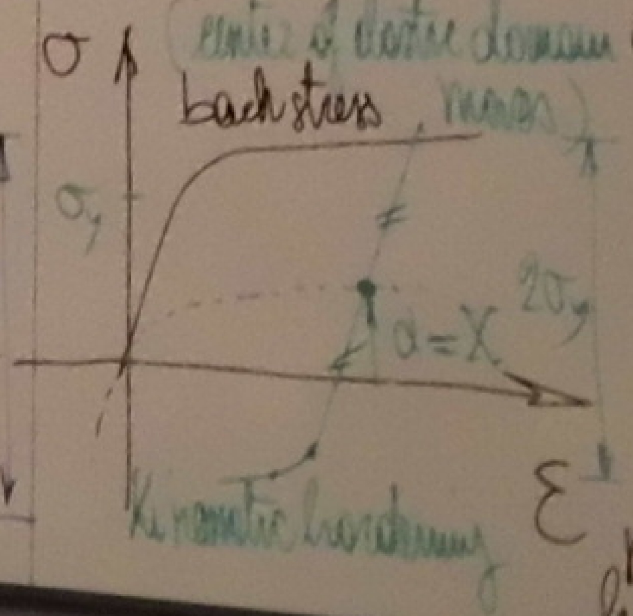
$E_t > 0$ hardening (positive) unique for both forms

$E_t = 0$ zero hardening (perfect plasticity) → constant flow stress — non unique (unique just for the direct form)

$E_t < 0$ softening (negative hardening)

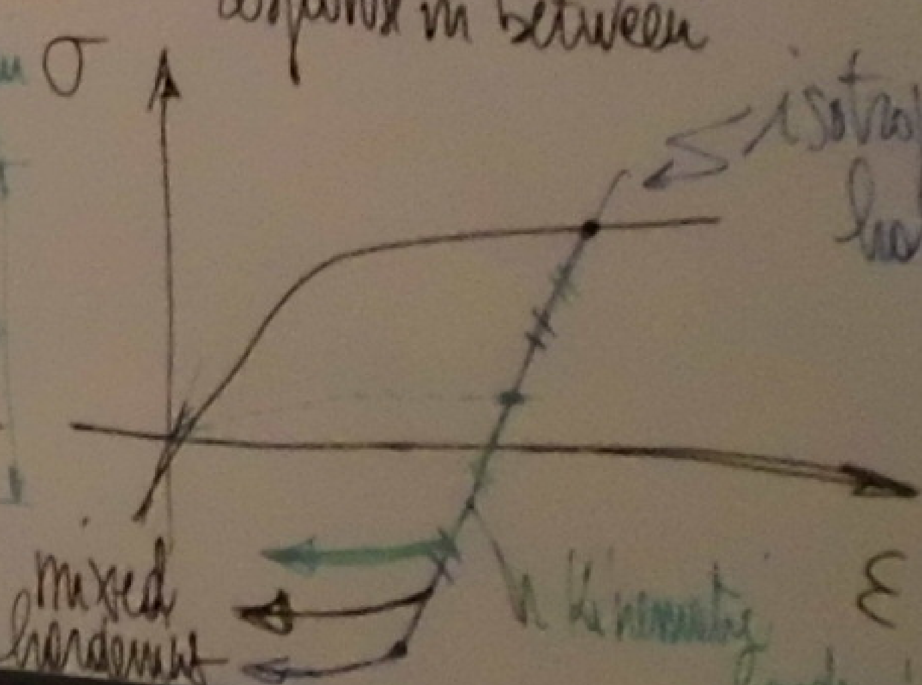
- Hardening typologies (codified) Modelization





Bauschinger effect

response in between



– Hardening (positive)

$(E_t > 0)$

direct
$\dot{\varepsilon} \geq 0 , \ \dot{\sigma} = E_t \dot{\varepsilon} \geq 0$
$\dot{\varepsilon} \leq 0 , \ \dot{\sigma} = E \dot{\varepsilon} \leq 0$

inverse
$\dot{\sigma} \geq 0 , \ \dot{\varepsilon} = \frac{1}{E_t} \dot{\sigma} \geq 0$
$\dot{\sigma} \leq 0 , \ \dot{\varepsilon} = \frac{1}{E} \dot{\sigma} \leq 0$

uniqueness for both laws

– Perfect-plasticity

constant flow stress

$\dot{\varepsilon} \geq 0 , \ \dot{\sigma} = 0 \ (E_t = 0)$
$\dot{\varepsilon} \leq 0 , \ \dot{\sigma} = E \dot{\varepsilon} \leq 0$

uniqueness of the direct form

$\dot{\sigma} > 0$ inadmissible

$\boxed{\dot{\sigma} = 0 , \ \varepsilon \text{ is undetermined}}$

$\dot{\sigma} < 0 , \ \dot{\varepsilon} = \frac{1}{E} \dot{\sigma} \leq 0$

loss of uniqueness for the inverse law

– Softening (negative hardening)

critical softening $E_t \to -\infty$ loosing strain control (limit case)

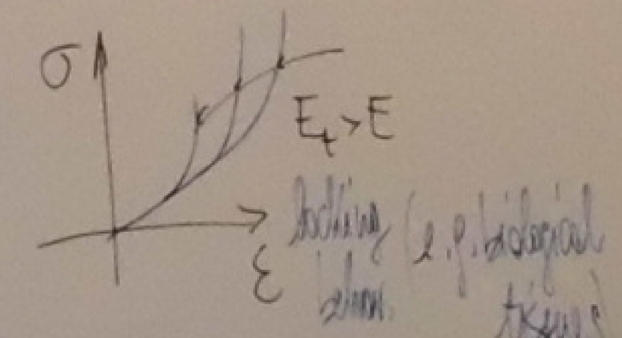
$(E_t < 0)$

$\dot{\varepsilon} \geq 0 , \ \dot{\sigma} = E_t \dot{\varepsilon} \leq 0$
$\dot{\varepsilon} \leq 0 , \ \dot{\sigma} = E \dot{\varepsilon} \leq 0$

$\dot{\sigma} > 0$ inadmissible

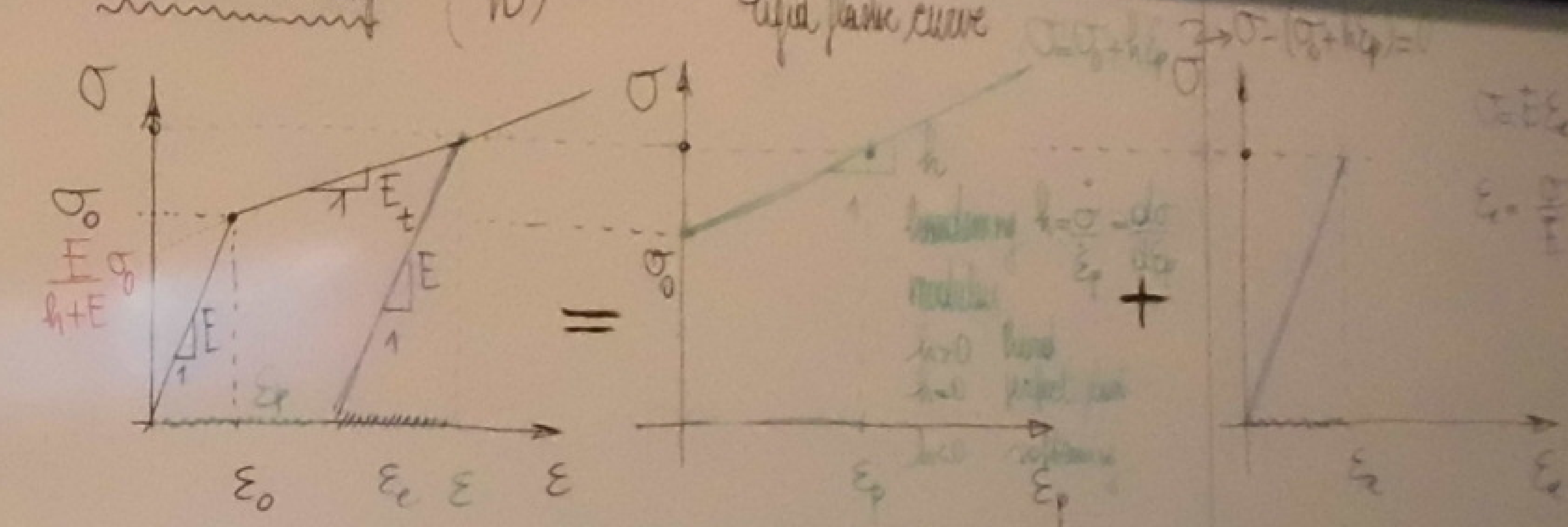
$\dot{\sigma} \leq 0 \begin{cases} \dot{\varepsilon} = \frac{1}{E_t} \dot{\sigma} \geq 0 \\ \dot{\varepsilon} = \frac{1}{E} \dot{\sigma} \leq 0 \end{cases}$

$-\infty < E_t < E$



Linear hardening (1D)

rigid plastic curve



$\varepsilon = \varepsilon_e + \varepsilon_p$

$= \frac{\sigma}{E} + \frac{\sigma}{h} - \frac{\sigma_0}{h}$

$= -\frac{\sigma_0}{h} + \left(\frac{1}{E} + \frac{1}{h}\right)\sigma \Rightarrow \sigma = E_t \varepsilon + \frac{E_t}{h}\sigma_0$

$= -\frac{\sigma_0}{h} + \frac{\sigma}{E_t} \quad \frac{h+E}{hE} = \frac{1}{E_t}$

$= \frac{E_t}{h}\sigma_0 + E_t \varepsilon$

$= \frac{E}{h+E}\sigma_0 + E_t \varepsilon$

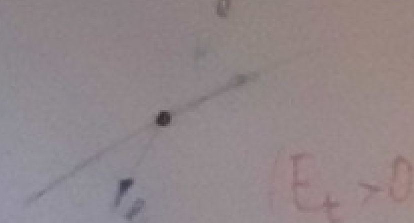
tangent modulus

$E_t = \frac{hE}{h+E}$

$\dot{\sigma} = E \dot{\varepsilon}_e$
$= E(\dot{\varepsilon} - \dot{\varepsilon}_p)$
$= E_t \dot{\varepsilon}$

$- \dot{\sigma} = \mathbf{E} : \dot{\varepsilon}$

- Hardening (positive)



$(E_t > 0)$

direct

$\dot{\varepsilon} \leq 0, \ \dot{\sigma} = E\dot{\varepsilon} \leq 0$

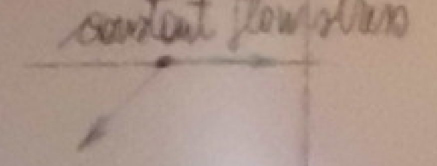
inverse

$\dot{\sigma} \leq 0, \ \dot{\varepsilon} = \frac{1}{E}\dot{\sigma} \leq 0$

uniqueness for both laws

Perfect plasticity

constant flow stress



$(E_t = 0)$

$\dot{\varepsilon} \leq 0, \ \dot{\sigma} = E\dot{\varepsilon} \leq 0$

uniqueness of the direct form

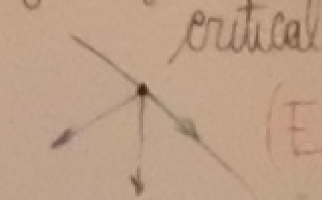
$\dot{\sigma} > 0$ inadmissible

$\dot{\sigma} < 0, \ \dot{\varepsilon} = \frac{1}{E}\dot{\sigma} \leq 0$

loss of uniqueness for the inverse law

- Softening (negative hardening)

critical softening $E_t \to -\infty$



$(E_t < 0)$

loosing strain control (limit case)

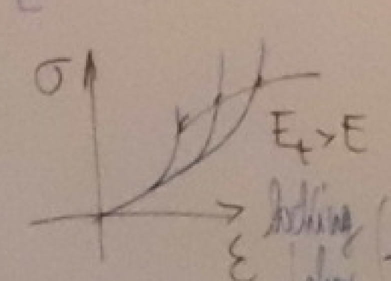
$\dot{\varepsilon} \geq 0, \ \dot{\sigma} = E_t\dot{\varepsilon} \leq 0$

$\dot{\varepsilon} \leq 0, \ \dot{\sigma} = E\dot{\varepsilon} \leq 0$

$\dot{\sigma} > 0$ inadmissible

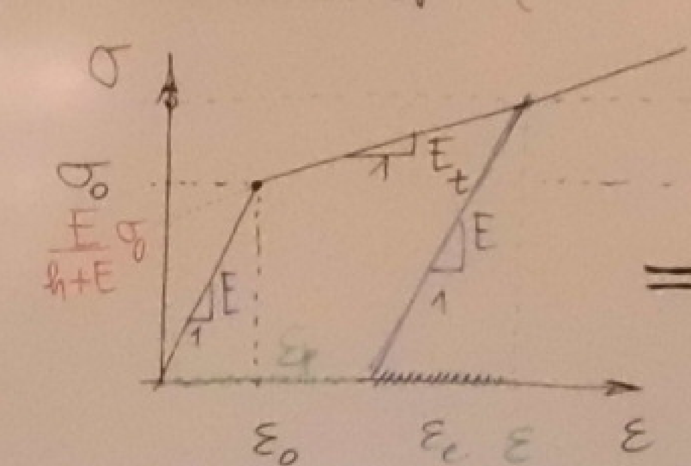
$\dot{\sigma} \leq 0$: $\dot{\varepsilon} = \frac{1}{E_t}\dot{\sigma} \geq 0$, $\dot{\varepsilon} = \frac{1}{E}\dot{\sigma} \leq 0$

$-\infty < E_t < E$



(e.g. biological tissues)

Linear hardening (1D)



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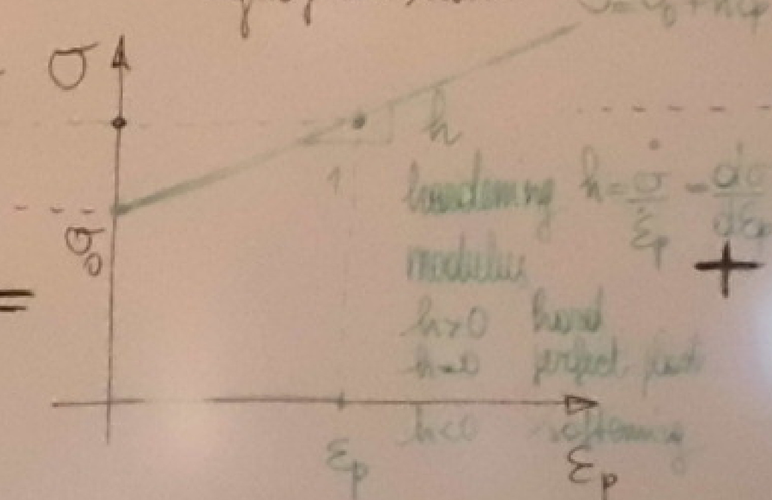
rigid plastic curve

$\sigma = \sigma_0 + h\varepsilon_p \to \sigma - (\sigma_0 + h\varepsilon_p) = 0$



hardening modulus $h = \frac{\dot{\sigma}}{\dot{\varepsilon}_p} = \frac{d\sigma}{d\varepsilon_p}$

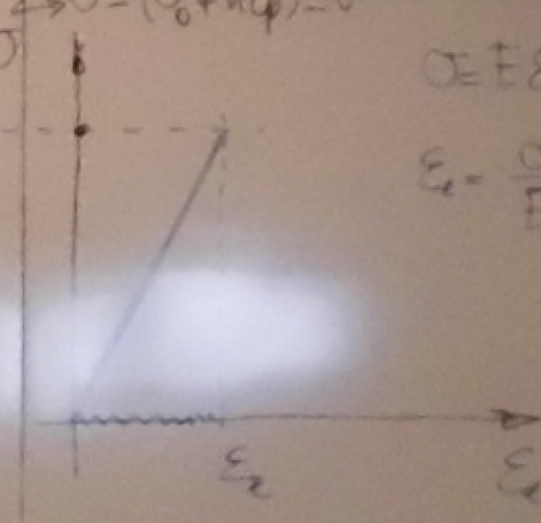
$h > 0$ hard.

$h = 0$ perfect plast.

$h < 0$ softening

$\frac{\sigma - \sigma_0}{h} = \varepsilon_p = -\frac{\sigma_0}{h} + \frac{\sigma}{h}$

+



$\sigma = E\varepsilon_e$

$\varepsilon_e = \frac{\sigma}{E}$

$$\varepsilon = \varepsilon_e + \varepsilon_p$$

$$= \frac{\sigma}{E} + \frac{\sigma}{h} - \frac{\sigma_0}{h}$$

$$= -\frac{\sigma_0}{h} + \left(\frac{1}{E} + \frac{1}{h}\right)\sigma \Rightarrow \sigma = E_t\varepsilon + \frac{E_t}{h}\sigma_0$$

$$= -\frac{\sigma_0}{h} + \frac{\sigma}{E_t} \qquad \frac{h+E}{hE} = \frac{1}{E_t}$$

$$= \frac{E_t}{h}\sigma_0 + E_t\varepsilon$$

$$= \frac{hE}{h+E}\frac{1}{h}\sigma_0 + E_t\varepsilon = \frac{E}{h+E}\sigma_0 + E_t\varepsilon$$

tangent modulus

$$\boxed{E_t = \frac{hE}{h+E}}$$

(Same could be obtained in rate form)

$$\dot{\sigma} = E\dot{\varepsilon}_e$$

$$= E(\dot{\varepsilon} - \dot{\varepsilon}_p)$$

$$= E_t\dot{\varepsilon}$$

Generalization to 3D

Yield Condition

- $F(\sigma) = f(\sigma) - r(h, \lambda) = 0$

  yield function (e.g. von Mises: $f(\sigma) = \sigma_{eq}^{vM} = \sqrt{3J_2}$)

- $\dot{\varepsilon}_p = \dot{\lambda}\, m$ — plastic multiplier, plastic flow direction

- $\dot{\sigma} = E_t : \dot{\varepsilon}$ — tangent stiffness matrix

- Hardening plastic

- Perfect plasticity

constant flow stress

- Softening (negative hardening)

critical softening $E_t \to -\infty$

losing strain control (limit case)

$(E_t < 0)$

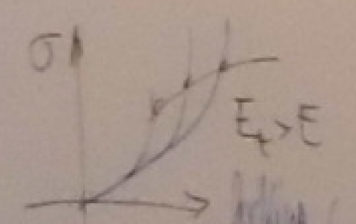
$\dot{\varepsilon} < 0,\ \dot{\sigma} \cdot E\dot{\varepsilon} \le 0$

uniqueness of the direct form

$\dot{\sigma} > 0$ inadmissible

$\dot{\sigma} \le 0$: $\dot{\varepsilon} = \frac{1}{E_t}\dot{\sigma} \ge 0$; $\dot{\varepsilon} = \frac{1}{E}\dot{\sigma} \le 0$

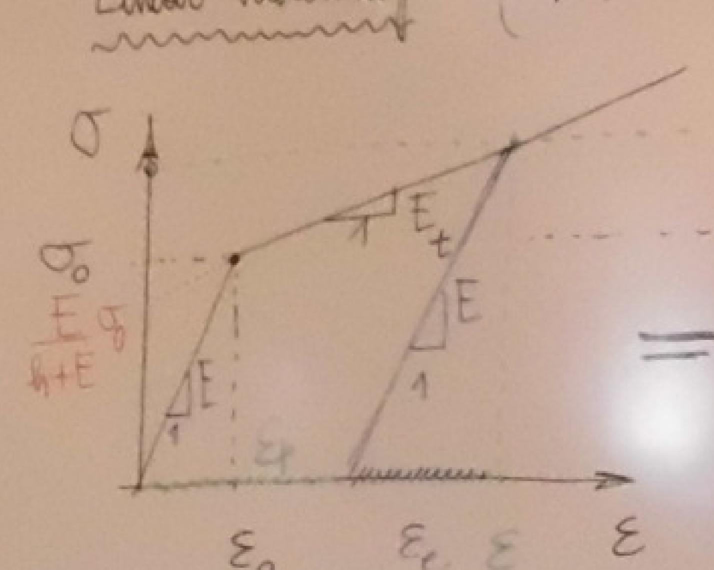
$-\infty < E_t < E$



$E_t > E$

locking (biological tissues)

Linear hardening (1D)



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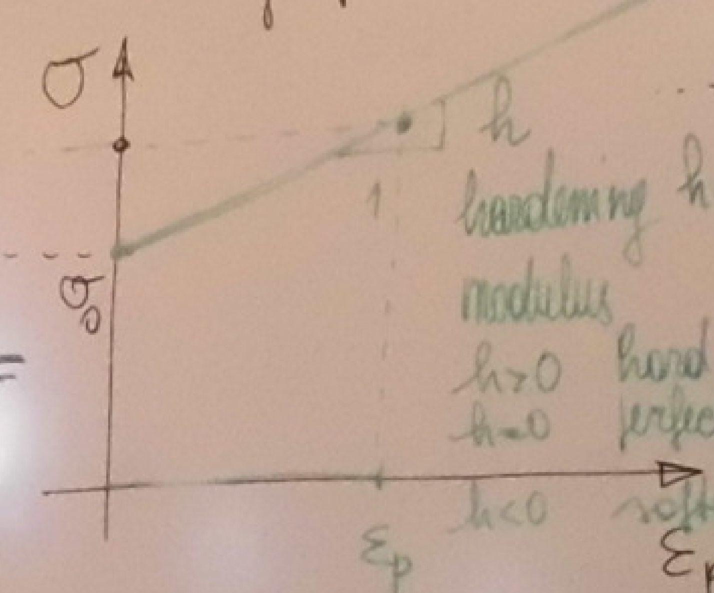

rigid plastic curve

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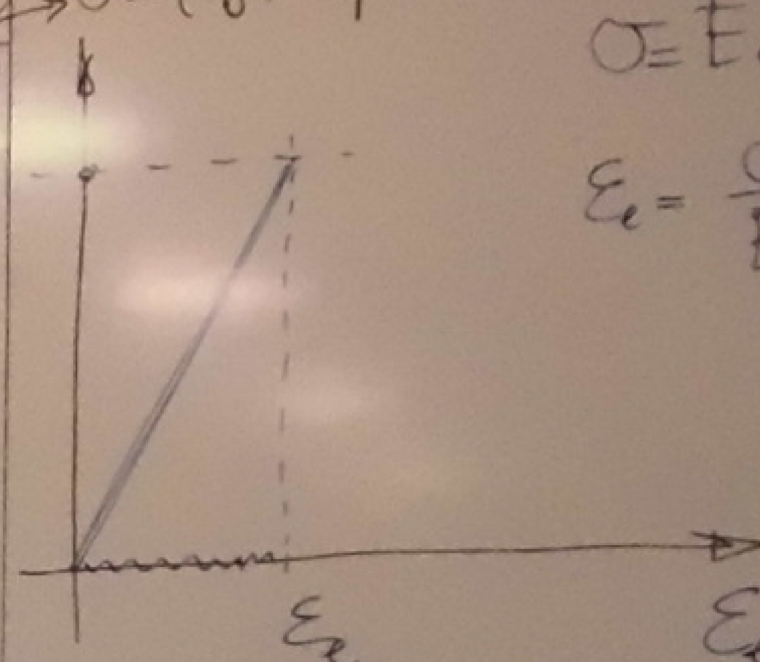
tangent modulus

$$\boxed{E_t = \frac{hE}{h+E}}$$

$\sigma - (\sigma_0 + h\varepsilon_p) = 0$



$\sigma = E\varepsilon_e$

$\varepsilon_e = \frac{\sigma}{E}$

(Same could be obtained in rate form)

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- $\dot{\varepsilon}_p = \dot{\lambda}\,m$
  plastic multiplier
  plastic flow direction
- $\dot{\sigma} = \mathbf{E}_t : \dot{\varepsilon}$
  tangent stiffness operator