Università degli studi di Bergamo

Scuola di Ingegneria (Dolmine)

CCS Ingegneria Edile

LM-24 Ingegneria delle Costruzioni Edili

Dinamica (, Instabilità) e Anelasticità delle Strutture

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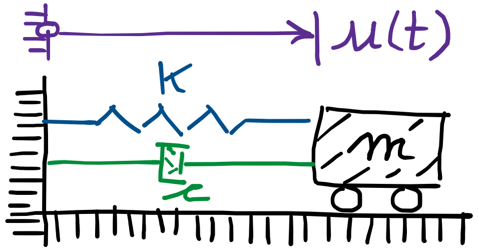
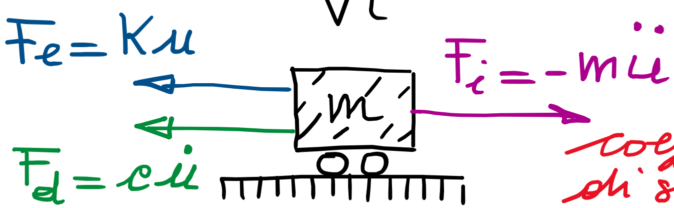
LEZIONE 03

Oscillazioni libere smorzate

$e.i.$ u_0 $\dot{u}_0$



$\forall t$



m, c, K cost (>0)
sistemi tempo-invarianti

Eq. del moto (equilibrio dinamico):

$$\cancel{m}\ddot{u}(t) + \underbrace{\frac{c}{m}}_{2\zeta\omega_1}\dot{u}(t) + \underbrace{\frac{K}{m}}_{\omega_1^2} u(t) = 0 \qquad (\zeta, \omega_1)$$

coefficiente di smorzamento

$\frac{c}{m} = 2\zeta\omega_1$ definizione

$\omega_1^2 \Rightarrow \omega_1 = \sqrt{\frac{K}{m}}$ pulsazione naturale del sistema non smorzato

$[\zeta] = [1]$ fattore di smorzamento (relativo al critico) ...... $\zeta = \frac{c}{2m\omega_1} = \frac{c}{2m\sqrt{\frac{K}{m}}}$

$$= \frac{c}{\underbrace{2\sqrt{Km}}_{c_{cr}}} = \frac{c}{c_{cr}} \qquad (c = c_{cr},\ \zeta = 1)$$

Per strutture civili:

$\zeta \simeq 1\%$ $(2\% - 7\%)$

tipic. $5\% = 0.05$

$\zeta << 1 \simeq 0.01$

eq. differenziale del 2° ordine a coeff. cost.

$$\boxed{\ddot{u} + 2\zeta\omega_1\dot{u} + \omega_1^2 u = 0} \quad (*)$$

eq. caratteristica (associata all'eq.ne differenziale di partenza)

$$\boxed{\lambda^2 + 2\zeta\omega_1\lambda + \omega_1^2\underbrace{\ }_{\lambda^0} = 0}$$

Si cercano soluz. nella forma:

$u(t) = e^{\lambda t}$
$\dot{u}(t) = \lambda e^{\lambda t}$
$\ddot{u}(t) = \lambda^2 e^{\lambda t}$

Sostituendo in $(*)$

$$\underbrace{(\lambda^2 + 2\zeta\omega_1\lambda + \omega_1^2)}_{=0}\cancel{e^{\lambda t}} = 0 \quad \forall t$$

algebrica, di 2° grado $\Rightarrow$ radici (due) $\lambda_{1,2}$ dell'eq.ne caratteristica (poli)

$$\lambda^2 + 2\zeta\omega_1\lambda + \omega_1^2 = 0 \Rightarrow \lambda_{1,2} = -\zeta\omega_1 \pm \sqrt{\zeta^2\omega_1^2 - \omega_1^2}$$

$$= -\zeta\omega_1 \pm \omega_1\sqrt{\zeta^2 - 1} = \omega_1\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

$$\zeta^2 - 1 = -(1-\zeta^2) = i^2(1-\zeta^2)$$

Casistica radici:

smorzamento

$\zeta < 1$ subcritico $\quad \lambda_{1,2} = -\zeta\omega_1 \pm i\,\omega_1\sqrt{1-\zeta^2}$ due radici complesse coniugate

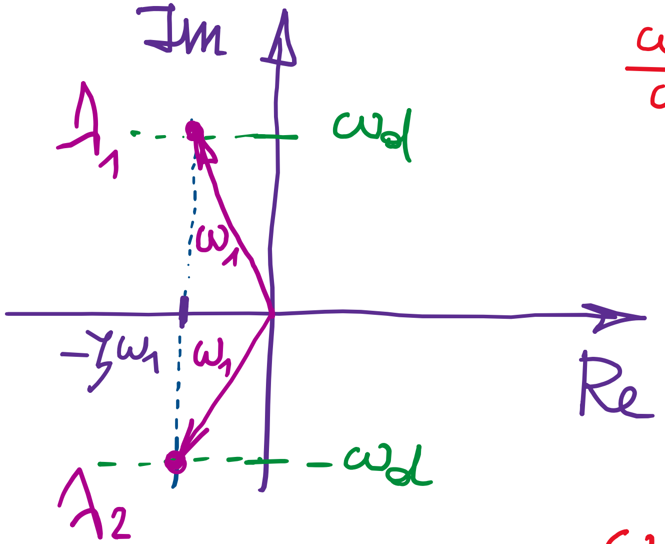
$\zeta = 1$ critico $\quad \lambda_{1,2} = -\omega_1$ due radici reali coincidenti (<0)

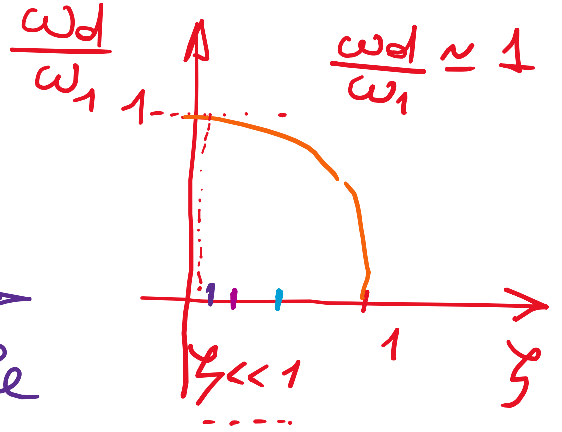
$\zeta > 1$ supercritico $\quad \lambda_{1,2} = -\zeta\omega_1 \pm \omega_1\sqrt{\zeta^2 - 1}$ due radici reale distinte (<0)

- Caso subcritico ($\zeta < 1$): $c < c_{cr}$

$$\lambda_{1,2} = \underbrace{-\zeta\omega_1}_{Re[\lambda_{1,2}] < 0} \pm \underbrace{i\,\omega_1\sqrt{1-\zeta^2}}_{Im[\lambda_{1,2}]}$$

$$|\lambda_{1,2}| = \omega_1$$





$$\omega_d = \omega_1\sqrt{1-\zeta^2} \leq \omega_1 \simeq \omega_1 \; (\zeta \ll 1)$$

pulsazione naturale del sistema smorzato

$$T_d \geq T_1 \simeq T_1$$

$$\frac{\omega_d}{\omega_1} = \sqrt{1-\zeta^2}; \quad \left(\frac{\omega_d}{\omega_1}\right)^2 = 1 - \zeta^2$$

$$\left(\frac{\omega_d}{\omega_1}\right)^2 + \zeta^2 = 1$$

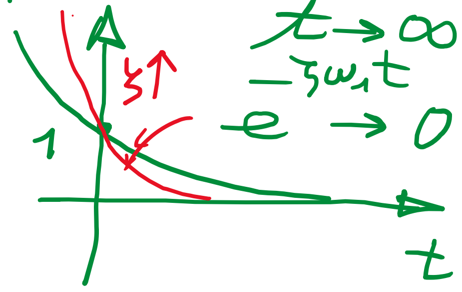
Pertanto: $\quad (\omega_d = \omega_1 \sqrt{1-\zeta^2})$

$$\lambda_{1,2} = -\zeta\omega_1 \pm i\,\omega_d$$

comp.te di moto armonico di pulsazione ω_d

Integrale $e^{\lambda_{1,2} t} = e^{(-\zeta\omega_1 \pm i\omega_d)t} = \underbrace{e^{-\zeta\omega_1 t}}_{\text{ampiezza esponenzialmente decadente in } t} \cdot \overbrace{e^{\pm i\omega_d t}}^{\sin\omega_d t,\ \cos\omega_d t}$

$e^{-\zeta\omega_1 t}$



$t \to \infty$ $\quad e^{-\zeta\omega_1 t} \to 0$

Integrale generale:

$$u(t) = e^{-\zeta\omega_1 t} \cdot (\underbrace{A}_{\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d}} \sin\omega_d t + \underbrace{B}_{u_0} \cos\omega_d t)$$

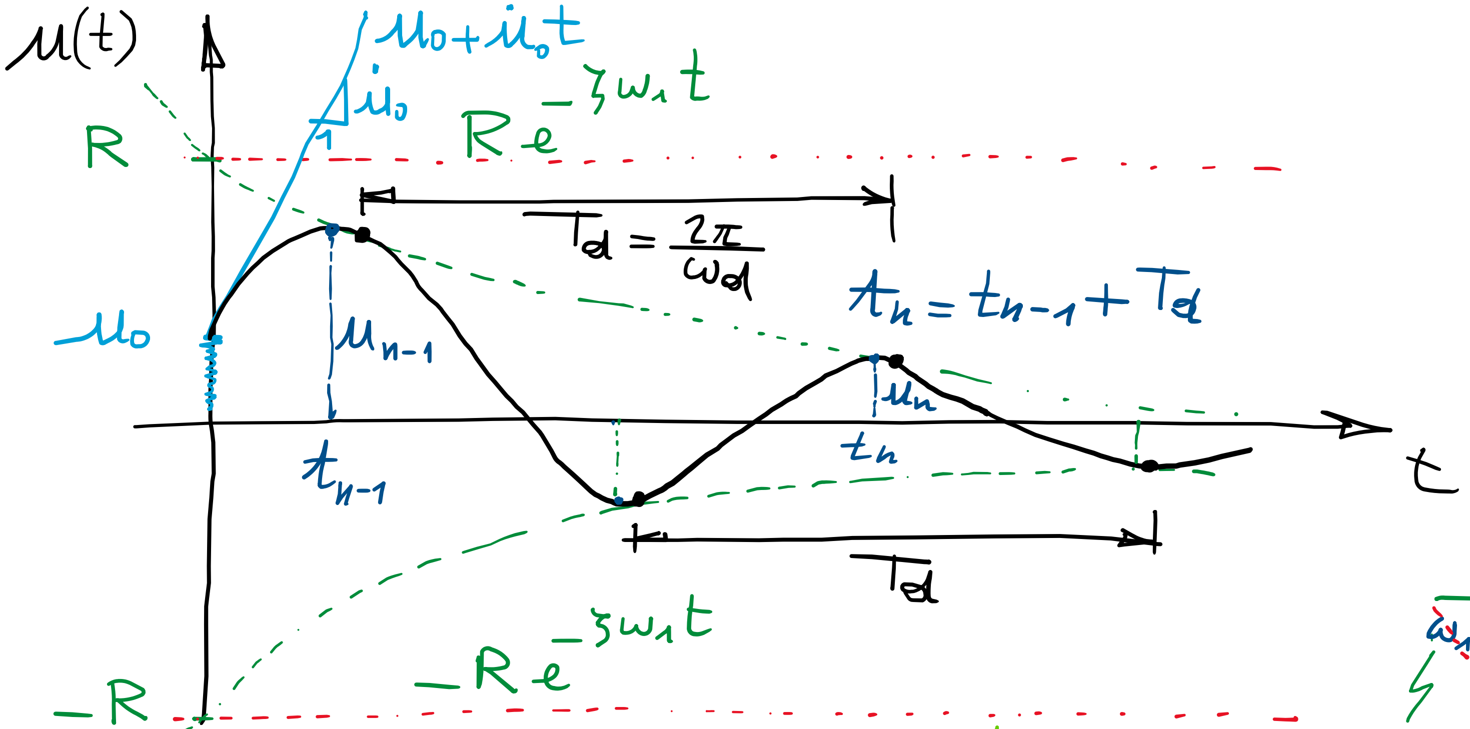
$$= R\, e^{-\zeta\omega_1 t} \cos(\omega_d t - \varphi) = R\, e^{-\zeta\omega_1 t} \sin(\omega_d t + \psi)$$

$$\Rightarrow \dot{u}(t) = -\zeta\omega_1 e^{-\zeta\omega_1 t} (A \sin\omega_d t + B\cos\omega_d t) + e^{-\zeta\omega_1 t}\omega_d (A\cos\omega_d t - B\sin\omega_d t)$$

Moto oscillatorio effetto di un moto armonico di periodo $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_1\sqrt{1-\zeta^2}} = \frac{T_1}{\sqrt{1-\zeta^2}}$ con ampiezza decadente in t (esponenzialmente) e rapidità legata a ζ.

Dalle c.i.:

$$\begin{cases} u_0 = u(0) = B \Rightarrow \boxed{B = u_0} \\ \dot{u}_0 = \dot{u}(0) = -\zeta\omega_1 B + \omega_d A \Rightarrow \boxed{A = \frac{\dot{u}_0 + \zeta\omega_1 B}{\omega_d} = \frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d}} \end{cases}$$



moto oscillatorio aperiodico con ampiezza decrescente e valori max/min che si riducono, ogni T_d, dello stesso rapporto $r = e^{\delta}$

$$T_d = \frac{2\pi}{\omega_1\sqrt{1-\zeta^2}}$$

rapporto tra ampiezze max/min successive

$$r = r_n = \frac{u_{n-1}}{u_n} = \frac{\cancel{R}\, e^{-\zeta\omega_1 t_{n-1}} \cancel{\cos}(t_{n-1})}{\cancel{R}\, e^{-\zeta\omega_1 (t_{n-1}+T_d)} \cancel{\cos}(t_{n-1}+T_d)} = e^{+\zeta\omega_1 T_d} = \underline{cost} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

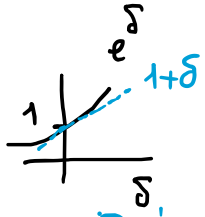
t_n

$$\delta = \ln r = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \underset{\zeta \ll 1}{\simeq} 2\pi\zeta \Rightarrow \boxed{\zeta = \frac{\delta}{2\pi}}$$

decremento logaritmico

$\delta \ll 1$

$$\hookrightarrow r = e^{\delta} \simeq 1 + \delta \Rightarrow \delta = r - 1 = \frac{u_{n-1}}{u_n} - 1 = \frac{u_{n-1} - u_n}{u_n} = \frac{\Delta u_n}{u_n}$$



approx per δ piccolo

stima pratica di δ per $\zeta \ll 1$ (e anche $\delta \ll 1$)

$$\delta^2(1-\zeta^2) = 4\pi^2\zeta^2$$

$$\delta^2 - \zeta^2\delta^2 = 4\pi^2\zeta^2$$

$$\delta^2 = (4\pi^2 + \delta^2)\zeta^2$$

stima di ζ (noto sperim. δ)

$$\boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{\delta}{2\pi\sqrt{\underbrace{1+(\delta/2\pi)^2}_{\simeq 1}}}}$$

NB: Il caso critico discrimina tra risposta oscillatoria e non

• Caso critico ($\zeta = 1$) $c = c_{cr}$

ulteriore integrale

$$\lambda_{1,2} = -\omega_1 \Rightarrow e^{-\omega_1 t} \; ; \quad t\, e^{-\omega_1 t} \text{ (verific.)}$$

$$u(t) = e^{-\omega_1 t} \cdot (A + B t)$$

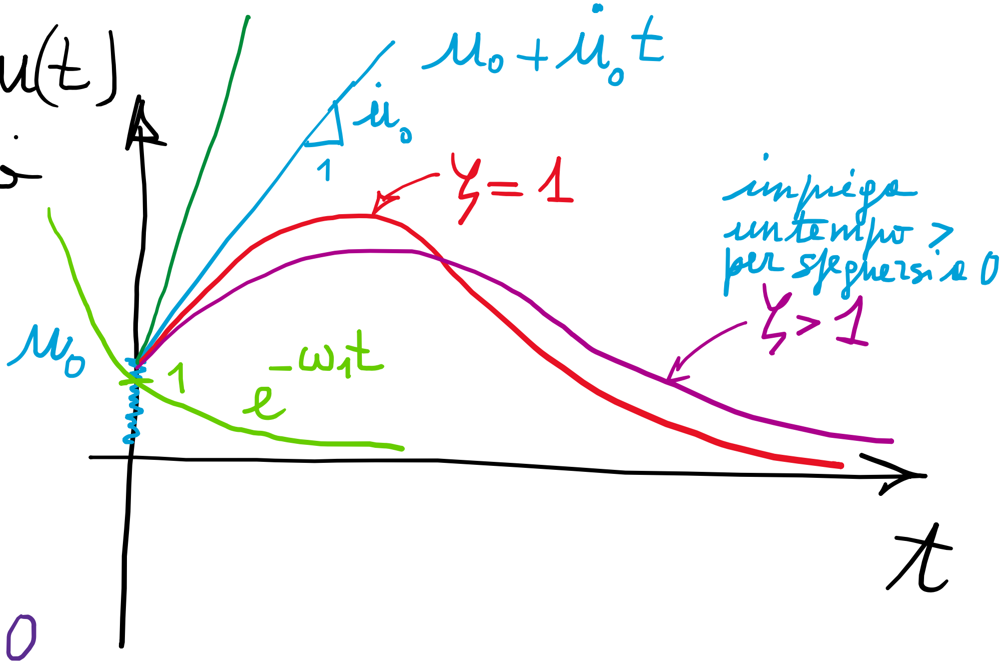
moto non oscillatorio decadente in t

$$\boxed{u_0 = A}$$

$$\dot{u}_0 = -\omega_1 A + B \Rightarrow \boxed{B = \dot{u}_0 + \omega_1 u_0}$$

$$= e^{-\omega_1 t} \cdot \left(u_0 + (\dot{u}_0 + \omega_1 u_0) t \right)$$

(risposte "resta della parte delle c.i.")



• Caso supercritico ($\zeta > 1$) $c > c_{cr}$

$$\lambda_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_1 < 0$$

$$\lambda_1 - \lambda_2 = 2\sqrt{\zeta^2 - 1}\,\omega_1 > 0$$

$$u(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$

$$\dot{u}(t) = \lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}$$

$$\begin{aligned} u_0 &= A + B \\ \dot{u}_0 &= \lambda_1 A + \lambda_2 B \end{aligned} \Rightarrow \begin{aligned} \lambda_2 u_0 - \dot{u}_0 &= -(\lambda_1 - \lambda_2) A \\ \lambda_1 u_0 - \dot{u}_0 &= (\lambda_1 - \lambda_2) B \end{aligned} \Rightarrow \begin{aligned} A &= -\frac{\lambda_2 u_0 - \dot{u}_0}{\lambda_1 - \lambda_2} \\ B &= \frac{\lambda_1 u_0 - \dot{u}_0}{\lambda_1 - \lambda_2} \end{aligned}$$

finale:

$$u(t) = -\frac{\lambda_2 u_0 - \dot{u}_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{\lambda_1 u_0 - \dot{u}_0}{\lambda_1 - \lambda_2} e^{\lambda_2 t}$$

resta moto non oscillatorio con tendenza asintotica

Concetti fondamentali:

$\omega_d = \omega_1\sqrt{1-\zeta^2}$ pulsaz. naturale sistema smorzato ← (tipicamente subcritico, $\zeta<1$ e $\ll 1$, $\zeta \sim 1\% = 0.01$)

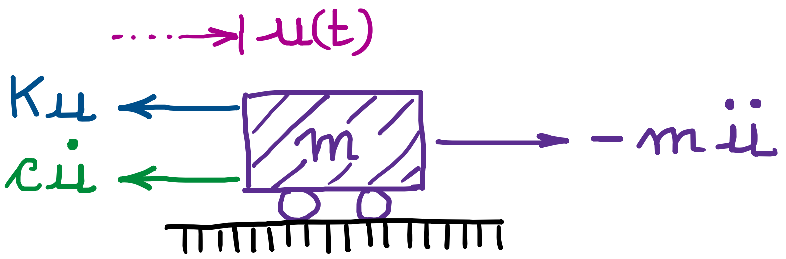
- Oscillazioni libere smorzate:

$$\zeta = \frac{c}{2\sqrt{Km}} = \frac{c}{c_{cr}}$$

fattore di smorzamento

$m, c, K = \text{cost}$

sistema tempo-invariante



$\frac{c}{m}$ $\frac{K}{m}$ $\omega_1 = \sqrt{\frac{K}{m}}$

con c.i. $\begin{cases} u_0 \\ \dot{u}_0 \end{cases}$ @ $t=t_0$

$$m\ddot{u}(t) + c\dot{u}(t) + Ku(t) = 0 \Rightarrow \ddot{u}(t) + \underbrace{2\zeta\omega_1}_{c/m}\dot{u}(t) + \underbrace{\omega_1^2}_{K/m}u(t) = 0$$

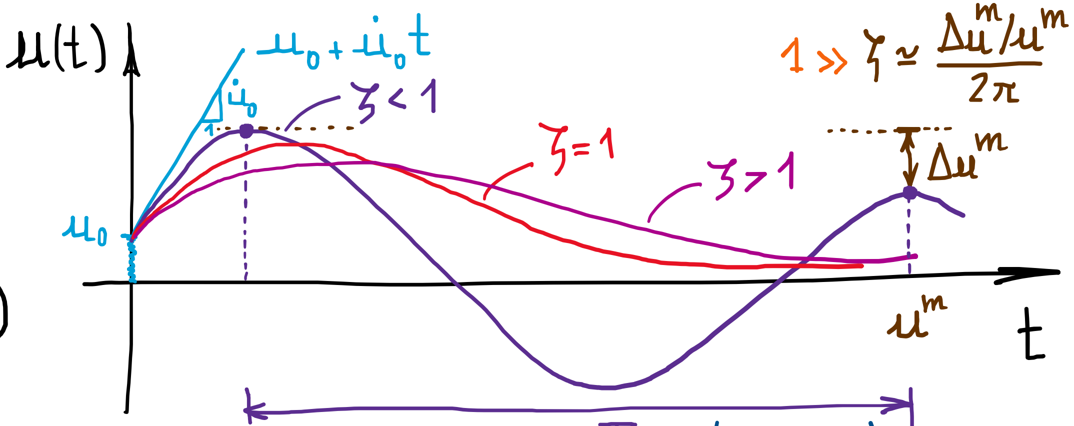
- Integrale generale (soluzione) ⇒ moto oscillatorio smorzato ad ampiezza variabile per $\zeta<1$,

moto non oscillatorio smorzato per $\zeta \geq 1$.

$\zeta<1$ $\quad u(t) = e^{-\zeta\omega_1 t}(A\sin\omega_d t + B\cos\omega_d t)$

$\zeta=1$ $\quad u(t) = e^{-\omega_1 t}(A + Bt)$

$\zeta>1$ $\quad u(t) = e^{-\zeta\omega_1 t}\left(A e^{\omega_1\sqrt{\zeta^2-1}\,t} + B e^{-\omega_1\sqrt{\zeta^2-1}\,t}\right)$



$1 \gg \zeta \simeq \frac{\Delta u^m / u^m}{2\pi}$

$T_d = \frac{T_1}{\sqrt{1-\zeta^2}}$ $\quad \left(T_1 = \frac{2\pi}{\omega_1}\right)$

SOMMARIO (Lez. 03)

- Oscillazioni libere smorzate (in risposta alle sole c.i.).
- Fattore di smorzamento (~ 1% per strutture civili).
- Radici dell'eq.ne caratteristica: poli.
- Casistica: subcritico → moto oscillatorio con ampiezza decadente.
  critico → moto aperiodico non oscillatorio.
  supercritico → idem, con ampiezza iniziale e picco inferiore.
- Decremento logaritmico e stima del fattore di smorzamento.
- Integrale generale e imposizione delle c.i.
- Next step: visto l'integrale generale dell'eq.ne omogenea con termine noto nullo. Da sovrapporsi ad integrale particolare dipendente dalla forzante → risposta forzata.