Università degli studi di Bergamo

Scuola di Ingegneria (Dolmine)

CCS Ingegneria Edile

LM-24 Ingegneria delle Costruzioni Edili

# Dinamica (, Instabilità) e Anelasticità delle Strutture

(ICAR/08 - SdC ; 6 CFU)

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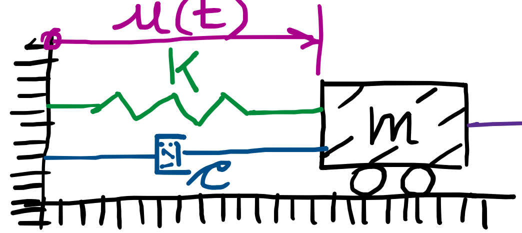
## LEZIONE 05

# Risposta smorzata ($\zeta \neq 0$) a forzante armonica

(forzante periodica di periodo $T = \frac{2\pi}{\omega} = \frac{1}{f}$ e pulsazione $\omega = 2\pi f$, frequenza ciclica $f$)

c.i. $u_0$, $\dot{u}_0$




$F(t) = F \sin\omega t$ ($f(t)$)

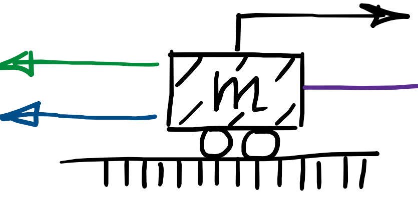
da Principio di d'Alembert

Equil. "dinamico" ⇒ eq.ne del moto

$F_i = -m\ddot{u}$

$F_e = Ku$

$F_d = c\dot{u}$




$F \sin\omega t$

$\forall t$

$$\cancel{m}\ddot{u}(t) + \frac{c}{m}\dot{u}(t) + \frac{K}{m} u(t) = \frac{K F}{K \cancel{m}} \sin\omega t$$

pulsaz. nat. sist. non smorz.

coeffic. di smorz.

$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{Km}}$ ⇒ $2\zeta\omega_1$

fattore $\zeta$ di smorz. ("damping ratio")

$\zeta \ll 1 \sim 1\% = .01$

es. $5\% = .05$

$\omega_1^2 = \frac{K}{m}$ $\left(\omega_1 = \sqrt{\frac{K}{m}}\right)$

spostamento "statico" $u_{st} = \frac{F}{K}$ (di rett. prop. a $F$)

$$\boxed{\ddot{u} + 2\zeta\omega_1 \dot{u} + \omega_1^2 u = \omega_1^2 u_{st} \sin\omega t} \quad (*)$$

Integrale particolare:

$$u_p(t) = N(\beta;\zeta)\, u_{st} \sin(\omega t - \xi(\beta;\zeta)),$$

$u_{st} = \frac{F}{K}$; fase

$N$: fattore di amplificazione dinamica

$\beta = \frac{\omega}{\omega_1} = \frac{f}{f_1}$ rapporto di frequenze ("frequency ratio")

per $\zeta = 0$ (sist. non smorz.) ⇒ $N = \frac{1}{\sqrt{(1-\beta^2)^2}}$

condiz. di "risonanza" ($\to\infty$, $\beta \to 1$)

; $\xi = \begin{cases} 0 & \text{se } \beta < 1 \text{ in fase} \\ \pi & \text{se } \beta > 1 \text{ in opposiz. di fase} \end{cases}$

$$u_p = N u_{st}\sin(\omega t - \xi) = N u_{st}(\sin\omega t\cos\xi - \cos\omega t\sin\xi) = \underbrace{N u_{st}\cos\xi}_{Z_1}\sin\omega t - \underbrace{N u_{st}\sin\xi}_{Z_2}\cos\omega t$$

$\frac{d}{dt}$

$$\dot u_p = \omega N u_{st}\cos(\omega t - \xi)$$

$\frac{d}{dt}$

$$\ddot u_p = -\omega^2 N u_{st}\sin(\omega t - \xi) = -\omega^2 u_p(t)$$

Quindi $Z_1 = N u_{st}\frac{1-\beta^2}{\sqrt{D}} = \frac{1-\beta^2}{D}u_{st}$ $\quad \sqrt{Z_1^2+Z_2^2} = N u_{st}$ $\quad Z_2 = N u_{st}\frac{2\zeta\beta}{\sqrt{D}} = \frac{2\zeta\beta}{D}u_{st}$

Sostituendo nell'eq.ne del moto (*) $\Rightarrow N, \xi$ ?

$$(\cancel{\omega_1^2} - \omega^2) N \cancel{u_{st}}\sin(\omega t - \xi) + 2\zeta\cancel{\omega_1}\,\omega N \cancel{u_{st}}\cos(\omega t - \xi) = \cancel{\omega_1^2}\cancel{u_{st}}\sin\omega t$$

$$\underbrace{\frac{\omega^2}{\omega_1^2}}_{\beta^2} \qquad \underbrace{\frac{\omega}{\omega_1}}_{\beta}$$

D termine a denominatore

$$D = (1-\beta^2)^2 + (2\zeta\beta)^2$$

$$(1-\beta^2)\sin(\omega t - \xi) + 2\zeta\beta\cos(\omega t - \xi) = \frac{1}{N}\sin\omega t \qquad \forall t$$

$$\cos\xi\sin\omega t - \sin\xi\cos\omega t \qquad \cos\xi\cos\omega t + \sin\xi\sin\omega t$$

- $\cos\xi = \frac{1}{\sqrt{1+\tan^2\xi}} = \frac{1-\beta^2}{\underbrace{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}_{D}} = \frac{1-\beta^2}{\sqrt{D}}$
- $\sin\xi = \tan\xi\cos\xi = \dots = \frac{2\zeta\beta}{\sqrt{D}}$

$$\begin{cases} (1-\beta^2)\cos\xi + 2\zeta\beta\sin\xi\ \cancel{\sin\omega t} = \frac{1}{N}\cancel{\sin\omega t} \\ -(1-\beta^2)\sin\xi + 2\zeta\beta\cos\xi\ \cancel{\cos\omega t} = 0 \end{cases} \Rightarrow \tan\xi = \frac{\sin\xi}{\cos\xi} = \frac{2\zeta\beta}{1-\beta^2}$$

Dalla prima eq.ne: $(1-\beta^2)\frac{1-\beta^2}{\sqrt{D}} + 2\zeta\beta\frac{2\zeta\beta}{\sqrt{D}} = \frac{1}{N}$

$$\frac{D}{\sqrt{D}} = \sqrt{D} = \frac{1}{N}$$

$$\Rightarrow \boxed{N(\beta;\zeta) = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}} \qquad \boxed{\xi(\beta;\zeta) = \operatorname{atan}\frac{2\zeta\beta}{1-\beta^2}}$$

N.B. per $\beta = 1$, $N = 1/2\zeta$, $\xi = \pi/2$

fattore di amplificazione dinamica (di ust) $N = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}$

Max. rel. (staz.)

$$N' = -\frac{1}{2\sqrt{D}}\frac{D'}{D} \qquad D'(\beta) = 0 \qquad \frac{d}{d\beta}$$

$\beta = 0$

$$\cancel{2}(1-\beta^2)(-\cancel{2}\cancel{\beta}) + \cancel{2}(2\zeta\cancel{\beta})\cancel{2}\zeta = 0$$

$$-1+\beta^2+2\zeta^2 = 0 \Rightarrow \bar\beta^2 = 1-2\zeta^2$$

$1-2\zeta^2>0, \quad \zeta < \frac{1}{\sqrt{2}}$

$$\bar\beta = \sqrt{1-2\zeta^2} \leq 1$$

$\simeq 1 \quad \zeta \ll 1$

lì:

$$\bar D = (2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)$$
$$= 4\zeta^4+4\zeta^2-8\zeta^4 = 4\zeta^2-4\zeta^4 = 4\zeta^2(1-\zeta^2)$$

@ $\beta = 1$ $\quad N = \frac{1}{2\zeta} = \frac{1/\zeta}{2}$

$$\bar N = \frac{1}{\sqrt{\bar D}} = \frac{1}{2\zeta}\frac{1}{\sqrt{1-\zeta^2}} \simeq \frac{1}{2\zeta} \quad \zeta \ll 1 \quad N(\beta=1) = \frac{1/\zeta}{2} \sim$$

| $\zeta$ | 0 | 1% | 2% | 5% |
|---|---|---|---|---|
| $N$ | $\infty$ | 50 | 25 | 10 |

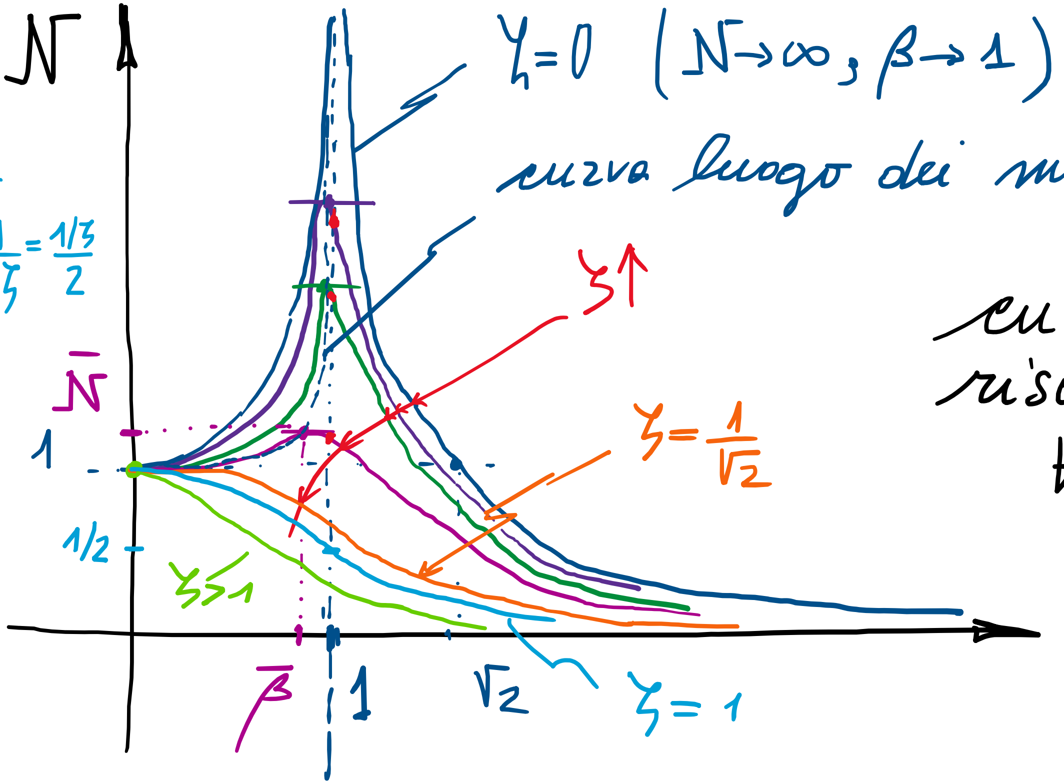
$\Rightarrow$ (può indurre uscita dal campo elastico)

$2\zeta^2 = 1-\bar\beta^2$

$$= \frac{1}{\cancel{2}\frac{\sqrt{1-\bar\beta^2}}{\sqrt{\cancel{2}}}}\;\frac{1}{\sqrt{\frac{2-1+\bar\beta^2}{\cancel{2}}}}$$

$$\bar N(\bar\beta) = \frac{1}{\sqrt{1-\bar\beta^4}}$$

traccia dei max (per tutti gli $\zeta$)

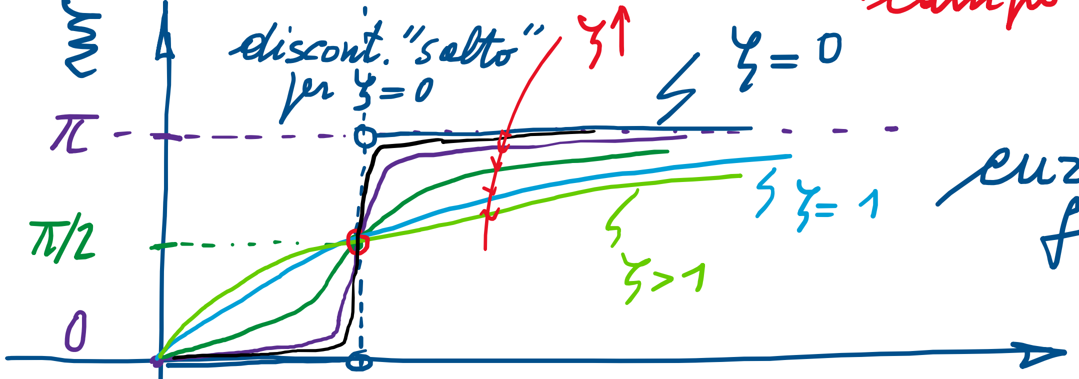



curve di risonanza $\forall\zeta$

$\beta = \frac{\omega}{\omega_1}$

fase $\xi = \text{atan}\frac{2\zeta\beta}{1-\beta^2}$

(sfasamento in ritardo di $u(t)$ rispetto a $F\sin\omega t$)




curve di fase $\forall\zeta$

$\beta = \frac{\omega}{\omega_1}$

per $\zeta \ll 1$: quasi in fase $\beta<1$ | 1 | quasi in opposizione di fase $\beta>1$

Integrale generale:

pulsazione naturale sistema smorzato

$$\omega_d = \omega_1\sqrt{1-\zeta^2} \underset{\zeta \ll 1}{\simeq} \omega_1$$

$$Z_1 = \frac{1-\beta^2}{D} u_{st}, \quad Z_2 = \frac{2\zeta\beta}{D} u_{st} \quad ; \quad u_{st} = \frac{F}{K}$$

$$\beta = \frac{\omega}{\omega_1} \quad ; \quad D = (1-\beta^2)^2 + (2\zeta\beta)^2$$

$$u(t) = u_{go}(t) + u_p(t)$$

$$= \underbrace{e^{-\zeta\omega_1 t}\left(A\sin\omega_d t + B\cos\omega_d t\right)}_{\text{risposta "transiente"}} + \underbrace{Z_1\sin\omega t - Z_2\cos\omega t}_{\text{risposta a regime ("steady state")}}$$

c.i.

$$\begin{cases} u_0 = B - Z_2 \Rightarrow B = \boxed{u_0 + Z_2} \\ \dot{u}_0 = -\zeta\omega_1 B + \omega_d A + \omega Z_1 \Rightarrow \boxed{A = \dfrac{\dot{u}_0 + \zeta\omega_1 B - \omega Z_1}{\omega_d} = \dfrac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \underbrace{\dfrac{\zeta\omega_1 Z_2 - \omega Z_1}{\omega_d}}_{\frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}}}} \end{cases}$$

$$u(t) = e^{-\zeta\omega_1 t}\left(\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right)$$

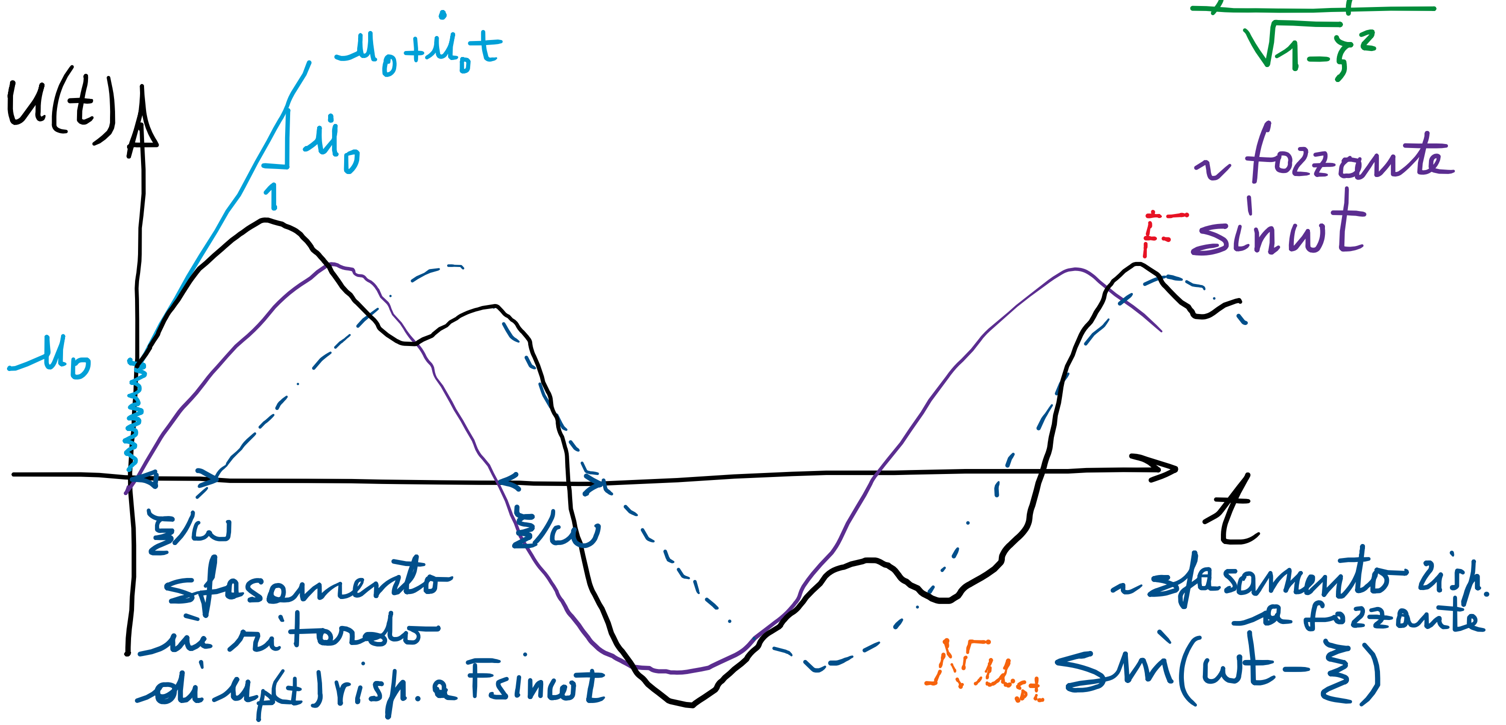
risposta alle c.i. per $F=0$ (spesso assente)

$$+ Z_1\sin\omega t - Z_2\cos\omega t$$

$$+ e^{-\zeta\omega_1 t}\left(\frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}}\sin\omega_d t + Z_2\cos\omega_d t\right)$$

risposta a forzante armonica $F\sin\omega t$ per c.i. omogenee ($u_0 = 0$, $\dot{u}_0 = 0$)

ampiezze decadenti esponenzialmente in $t$ (sino a far sopravvivere solo $u_p(t)$)



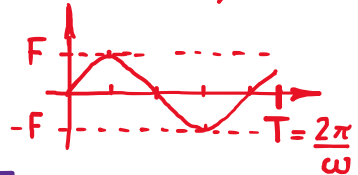
# Concetti fondamentali:

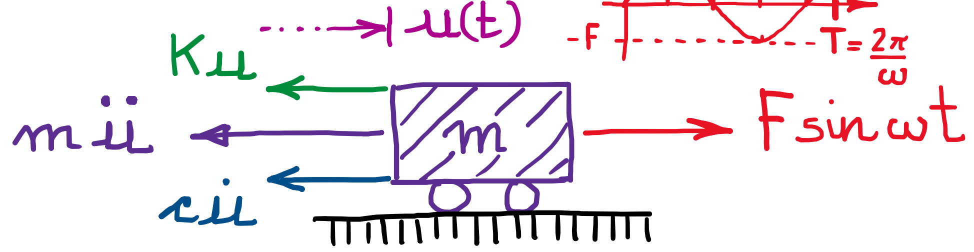
- Risposta smorzata ($c \neq 0, \zeta \neq 0$) a forzante armonica ($F(t) = F \sin \omega t$)

$m, c, K = \text{cost}$

sistema tempo-invariante

con c.i. $\begin{cases} u_0 \\ \dot{u}_0 \end{cases}$ @ $t = t_0$




$$m\ddot{u}(t) + c\dot{u}(t) + Ku(t) = F \sin \omega t$$

$$\Downarrow$$

$$\ddot{u}(t) + \underbrace{2\zeta\omega_1}_{\frac{c}{m}} \dot{u}(t) + \underbrace{\omega_1^2}_{\frac{K}{m}} u(t) = \omega_1^2 \underbrace{u_{st}}_{\frac{F}{K}} \sin \omega t \qquad \omega_1 = \sqrt{\frac{K}{m}}$$

fattore di smorzamento

$$\zeta = \frac{c}{2\sqrt{Km}} = \frac{c}{c_{cr}}$$

(tipicamente subcritico, $\zeta < 1$ e $\ll 1$, $\zeta \sim 1\% = 0.01$)

- Integrale particolare: risposta "a regime" $\underbrace{Z_1}_{N u_{st} \frac{1-\beta^2}{\sqrt{D}}} \sin \omega t - \underbrace{Z_2}_{N u_{st} \frac{2\zeta\beta}{\sqrt{D}}} \cos \omega t$

$$\xi = \operatorname{atan} \frac{2\zeta\beta}{1-\beta^2}$$

$$u_p(t) = N(\beta;\zeta)\, u_{st} \sin(\omega t - \xi(\beta;\zeta))$$

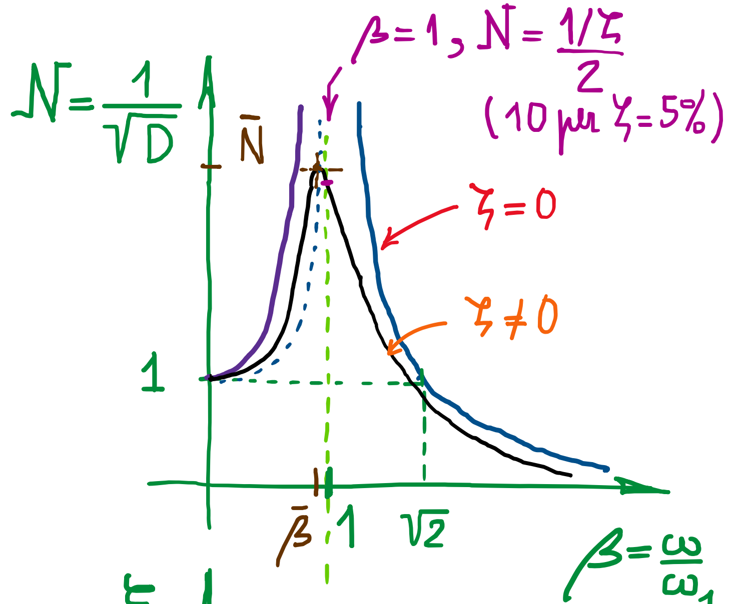
$\xi$: fase (sfasamento in ritardo tra risposta e forzante)

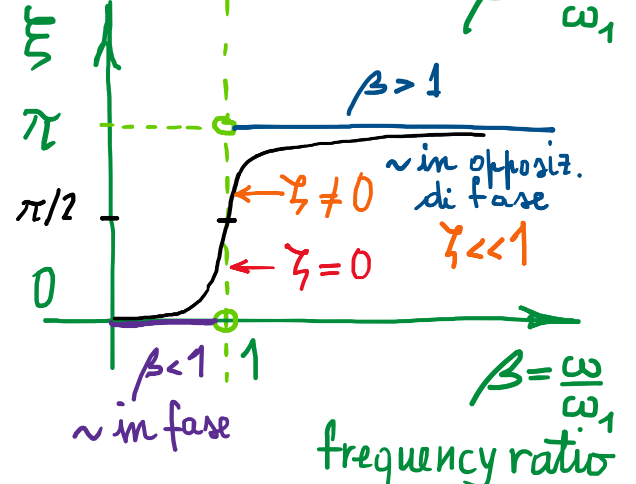
$N$: fattore di amplificazione dinamica (rispetto a $u_{st} = F/K$) $\quad N = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$

$\bar{N}(\bar{\beta}) = \frac{1}{\sqrt{1-\bar{\beta}^4}}$ luogo dei max

$\bar{N} = \frac{1}{2\zeta} \frac{1}{\sqrt{1-\zeta^2}} \simeq \frac{1}{2\zeta}$

$\bar{\beta} = \sqrt{1-2\zeta^2} \simeq 1$



# SOMMARIO (Lez. 05)

- Risposta smorzata a forzante armonica.
- Effetto dello smorzamento su curve di risonanza e di fase.
- Picco finito di ampiezza in condizioni di risonanza; risposta in quadratura rispetto alla forzante.
- Risposta a regime in componenti $\sin\omega t$ e $\cos\omega t$.
- Integrale generale con risposta transiente e a regime.
- Next step: trattazione unificata in variabili complesse per risposta a $F\sin\omega t$ e/o $F\cos\omega t \rightsquigarrow F e^{i\omega t}$.