Università degli studi di Bergamo
Scuola di Ingegneria (Dolmine)
CCS Ingegneria Edile
LM-24 Ingegneria delle Costruzioni Edili

Dinamica, Instabilità e Anelasticità delle Strutture

(ICAR/08 - SdC ; 6 CFU)

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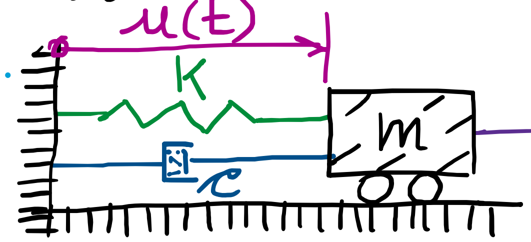
LEZIONE 05

Risposta smorzata $(\zeta \neq 0)$ a forzante armonica

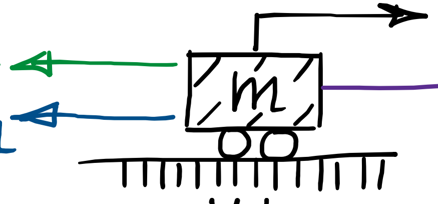
(forzante periodica di periodo $T = \frac{2\pi}{\omega} = \frac{1}{f}$
e pulsazione $\omega = 2\pi f$, frequenza ciclica f)

c.i. u_0, $\dot{u}_0$



$F(t) = F \sin \omega t$ — $f(t)$

da Principio di d'Alembert



$F_i = -m\ddot{u}$

$F_e = Ku$

$F_d = c\dot{u}$

$F \sin \omega t$

$\forall t$

Equil. "dinamico" $\Rightarrow$ eq.ne del moto

$$\cancel{m}\ddot{u}(t) + \frac{c}{m}\dot{u}(t) + \frac{K}{m}u(t) = \frac{K}{K}\frac{F}{m}\sin\omega t$$

pulsaz. nat. sist. non smorz.

coeffic. di smorz.

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{Km}} \Rightarrow 2\zeta\omega_1 \qquad \omega_1^2 = \frac{K}{m} \qquad \omega_1^2 \qquad \left(\omega_1 = \sqrt{\frac{K}{m}}\right)$$

fattore ζ di smorz. ("damping ratio")

$\zeta \ll 1 \sim 1\% = .01$

es. $5\% = .05$

spostamento "statico" $u_{st} = \frac{F}{K}$ (direttamente prop. a F)

$$\ddot{u} + 2\zeta\omega_1\dot{u} + \omega_1^2 u = \omega_1^2 u_{st} \sin\omega t \qquad (*)$$

Integrale particolare:

$$u_p(t) = N(\beta;\zeta)\, u_{st} \sin\left(\omega t - \xi(\beta;\zeta)\right), \qquad u_{st} = \frac{F}{K}$$

fase

fattore di amplificazione dinamica

$\beta = \frac{\omega}{\omega_1} = \frac{f}{f_1}$ rapporto di frequenze ("frequency ratio")

per $\zeta = 0$ (sist. non smorz.) $\Rightarrow N = \frac{1}{\sqrt{(1-\beta^2)^2}}$

condiz. di "risonanza" $(\rightarrow \infty,\ \beta \rightarrow 1)$

$$\xi = \begin{cases} 0 & \text{se } \beta < 1 \text{ in fase} \\ \pi & \text{se } \beta > 1 \text{ in opposiz. di fase} \end{cases}$$

$$\frac{d}{dt}\quad u_p = N u_{st}\sin(\omega t-\xi) = N u_{st}(\sin\omega t\cos\xi - \cos\omega t\sin\xi) = \underbrace{N u_{st}\cos\xi}_{Z_1}\sin\omega t - \underbrace{N u_{st}\sin\xi}_{Z_2}\cos\omega t$$

$$\frac{d}{dt}\quad \dot{u}_p = \omega N u_{st}\cos(\omega t-\xi)$$

$$\ddot{u}_p = -\omega^2 N u_{st}\sin(\omega t-\xi) = -\omega^2 u_p(t)$$

Quindi $Z_1 = N u_{st}\frac{1-\beta^2}{\sqrt{D}} = \frac{1-\beta^2}{D}u_{st}$ $\quad \sqrt{Z_1^2+Z_2^2} = N u_{st}$ $\quad Z_2 = N u_{st}\frac{2\zeta\beta}{\sqrt{D}} = \frac{2\zeta\beta}{D}u_{st}$

Sostituendo nell'eq.ne del moto (*) $\Rightarrow N, \xi$?

$$\left(\cancel{\omega_1^2} - \frac{\omega^2}{\underbrace{\omega_1^2}_{\beta^2}}\right) N \cancel{u_{st}}\sin(\omega t-\xi) + 2\zeta\frac{\cancel{\omega_1}\,\omega}{\underbrace{\cancel{\omega_1}\,\omega_1}_{\beta}} N\cancel{u_{st}}\cos(\omega t-\xi) = \frac{\cancel{\omega_1^2}}{\cancel{\omega_1^2}}\cancel{u_{st}}\sin\omega t$$

D termine a denominatore

$$(1-\beta^2)\underbrace{\sin(\omega t-\xi)}_{\cos\xi\sin\omega t-\sin\xi\cos\omega t} + 2\zeta\beta\underbrace{\cos(\omega t-\xi)}_{\cos\xi\cos\omega t+\sin\xi\sin\omega t} \overset{\forall t}{=} \frac{1}{N}\sin\omega t$$

$$D = (1-\beta^2)^2 + (2\zeta\beta)^2$$

- $\cos\xi = \frac{1}{\sqrt{1+\tan^2\xi}} = \frac{1-\beta^2}{\sqrt{\underbrace{(1-\beta^2)^2+(2\zeta\beta)^2}_{D}}} = \frac{1-\beta^2}{\sqrt{D}}$
- $\sin\xi = \tan\xi\cos\xi = \dots\dots = \frac{2\zeta\beta}{\sqrt{D}}$

$$\begin{cases} (1-\beta^2)\cos\xi + 2\zeta\beta\sin\xi \;\; \cancel{\sin\omega t} = \frac{1}{N}\cancel{\sin\omega t} \\ -(1-\beta^2)\sin\xi + 2\zeta\beta\cos\xi \;\; \cancel{\cos\omega t} = 0 \end{cases}$$

$$\Rightarrow \tan\xi = \frac{\sin\xi}{\cos\xi} = \frac{2\zeta\beta}{1-\beta^2}$$

Dalla prima eq.ne: $(1-\beta^2)\frac{1-\beta^2}{\sqrt{D}} + 2\zeta\beta\frac{2\zeta\beta}{\sqrt{D}} = \frac{1}{N}$

$$\frac{D}{\sqrt{D}} = \sqrt{D} = \frac{1}{N}$$

$$\Rightarrow \boxed{N(\beta;\zeta) = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}} \qquad \boxed{\xi(\beta;\zeta) = \operatorname{atan}\frac{2\zeta\beta}{1-\beta^2}}$$

N.B. per $\beta=1$, $N = 1/2\zeta$, $\xi = \pi/2$

fattore di amplificazione dinamica (di mst)

$$N = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}$$

Max. rel. (staz.)

$$\dot N = -\frac{\frac{1}{2\sqrt{D}}}{D}\,\dot D(\beta) = 0 \qquad \frac{d}{d\beta}$$

$$\cancel{2}(1-\beta^2)(-\cancel{2}\beta)+\cancel{2}(2\zeta\beta)2\zeta=0$$

$$-1+\beta^2+2\zeta^2=0 \Rightarrow \bar\beta^2 = 1-2\zeta^2$$

$$1-2\zeta^2>0,\ \zeta<\frac{1}{\sqrt{2}} \qquad \bar\beta=\sqrt{1-2\zeta^2}\le 1$$

$\simeq 1$ $\quad \zeta \ll 1$

@ $\beta=1$

$N = \frac{1}{2\zeta} = \frac{1/\zeta}{2}$

$\beta=0$

lì:

$$\bar D = (2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)_2$$

$$= 4\zeta^4+4\zeta^2-8\zeta^4 = 4\zeta^2-4\zeta^4 = 4\zeta^2(1-\zeta^2)$$

$$\bar N = \frac{1}{\sqrt{\bar D}} = \frac{1}{2\zeta}\,\frac{1}{\sqrt{1-\zeta^2}} \simeq \frac{1}{2\zeta} \quad \zeta\ll 1 \qquad N(\beta=1)=\frac{1/\zeta}{2} \sim N$$

ζ	0	1%	2%	5%
N	∞	50	25	10

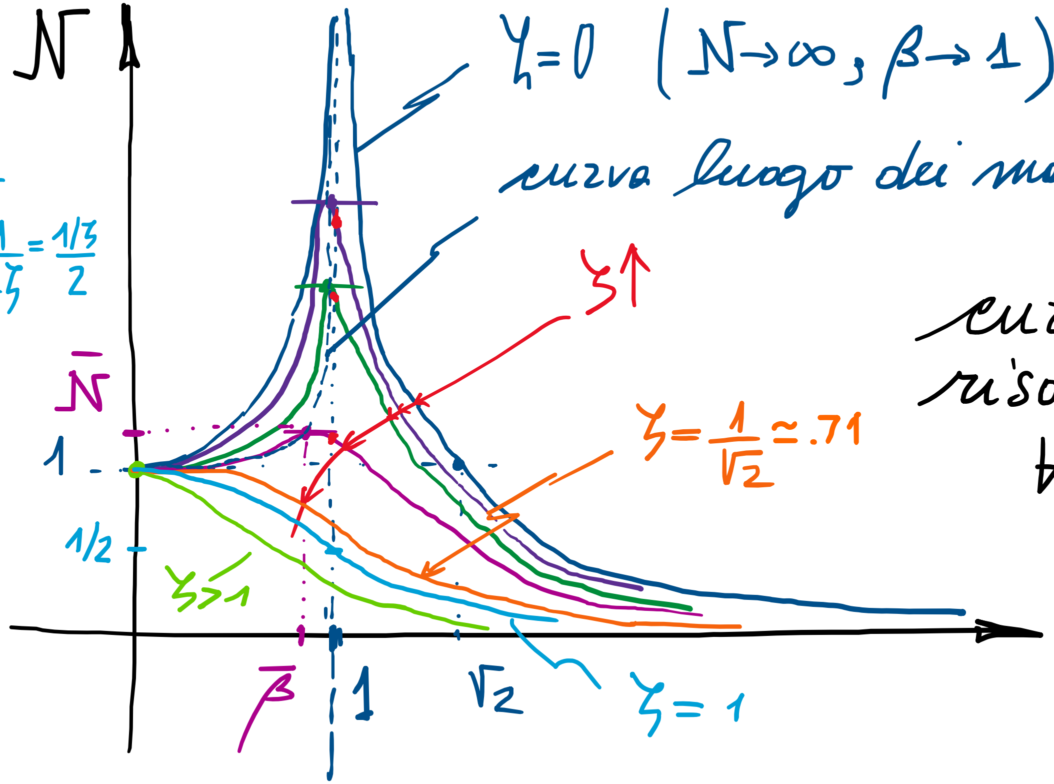
$\Rightarrow$ (può indurre uscita dal campo elastico)

$2\zeta^2 = 1-\bar\beta^2$

$$= \frac{1}{\cancel{2}\frac{\sqrt{1-\bar\beta^2}}{\sqrt{\cancel{2}}}}\,\frac{1}{\sqrt{\frac{\cancel{2}-1+\bar\beta^2}{\cancel{2}}}} = \frac{1}{\sqrt{1-\bar\beta^4}}$$

$$\bar N(\bar\beta) = \frac{1}{\sqrt{1-\bar\beta^4}}$$

traccia dei max (per tutti gli ζ)



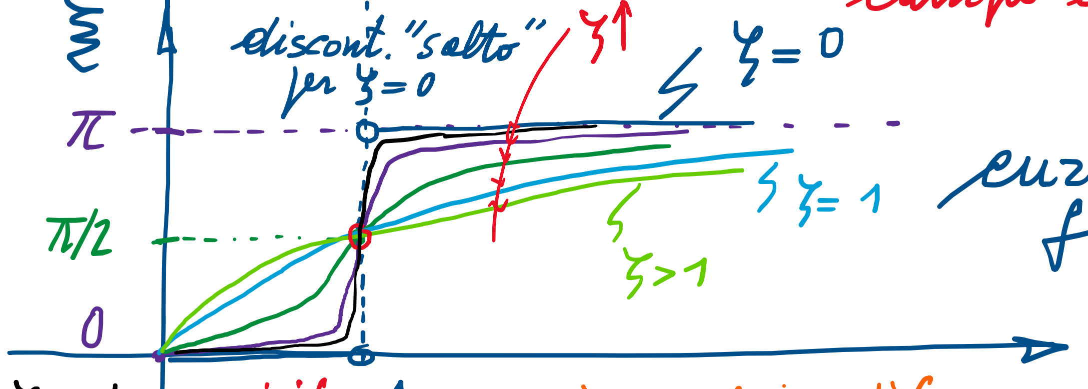
curva luogo dei max $\bar N = \frac{1}{\sqrt{1-\bar\beta^4}}$

curve di risonanza $\forall \zeta$

$\beta = \frac{\omega}{\omega_1}$

fase $\xi = \mathrm{atan}\,\frac{2\zeta\beta}{1-\beta^2}$

(sfasamento in ritardo di $u(t)$ rispetto a $F\sin\omega t$)



curve di fase $\forall \zeta$

$\beta = \frac{\omega}{\omega_1}$

per $\zeta \ll 1$: quasi in fase $\beta<1$ | 1 | quasi in opposizione di fase $\beta>1$

Integrale generale:

pulsazione naturale sistema smorzato

$$\omega_d = \omega_1\sqrt{1-\zeta^2} \underset{\zeta \ll 1}{\simeq} \omega_1$$

$$u(t) = u_{go}(t) + u_p(t)$$

$$= \underbrace{e^{-\zeta\omega_1 t}\left(A\sin\omega_d t + B\cos\omega_d t\right)}_{\text{risposta "transiente"}} + \underbrace{Z_1\sin\omega t - Z_2\cos\omega t}_{\text{risposta a regime ("steady state")}}$$

$$Z_1 = \frac{1-\beta^2}{D}u_{st}\,,\quad Z_2 = \frac{2\zeta\beta}{D}u_{st}\quad ;\quad u_{st} = \frac{F}{K}$$

$$\beta = \frac{\omega}{\omega_1}\quad ;\quad D = (1-\beta^2)^2 + (2\zeta\beta)^2$$

c.i.

$$\begin{cases} u_0 = B - Z_2 \Rightarrow B = \boxed{u_0 + Z_2} \\ \dot{u}_0 = -\zeta\omega_1 B + \omega_d A + \omega Z_1 \Rightarrow \boxed{A = \dfrac{\dot{u}_0 + \zeta\omega_1 B - \omega Z_1}{\omega_d} = \dfrac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d} + \underbrace{\dfrac{\zeta\omega_1 Z_2 - \omega Z_1}{\omega_d}}_{\frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}}}} \end{cases}$$

$$u(t) = e^{-\zeta\omega_1 t}\left(\frac{\dot{u}_0 + \zeta\omega_1 u_0}{\omega_d}\sin\omega_d t + u_0\cos\omega_d t\right)$$

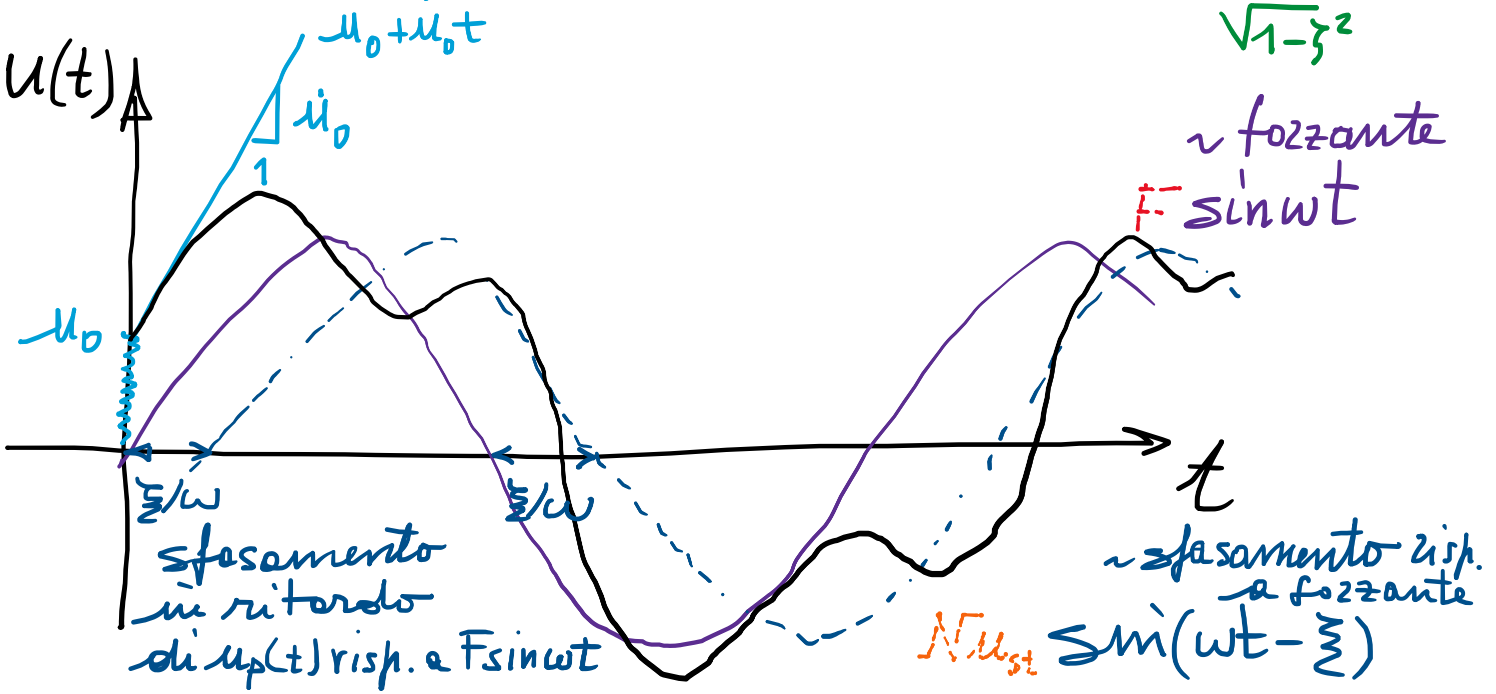
risposta alle c.i. per $F=0$ (spesso assente)

$$+ Z_1\sin\omega t - Z_2\cos\omega t$$

$$+ e^{-\zeta\omega_1 t}\left(\frac{\zeta Z_2 - \beta Z_1}{\sqrt{1-\zeta^2}}\sin\omega_d t + Z_2\cos\omega_d t\right)$$

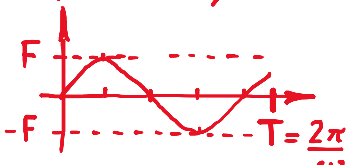
risposta a forzante armonica $F\sin\omega t$ per c.i. omogenee ($u_0 = 0$, $\dot{u}_0 = 0$)

ampiezze decadenti esponenzialmente in t (sino a far sopravvivere solo $u_p(t)$)



Concetti fondamentali :

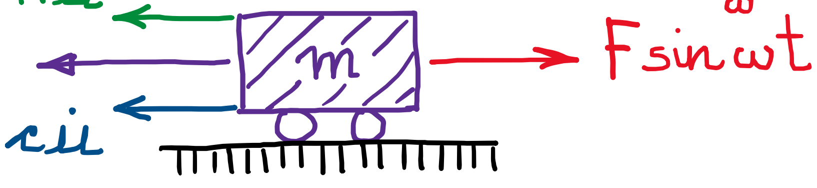
- Risposta smorzata $(c \neq 0, \zeta \neq 0)$ a forzante armonica $(F(t) = F \sin \omega t)$



$m, c, K = \text{cost}$

sistema tempo-invariante

con c.i. $\begin{cases} u_0 \\ \dot{u}_0 \end{cases}$ @ $t = t_0$



$$m\ddot{u}(t) + c\dot{u}(t) + Ku(t) = F \sin \omega t$$

$$\Downarrow$$

$$\ddot{u}(t) + \underbrace{2\zeta\omega_1}_{\frac{c}{m}} \dot{u}(t) + \underbrace{\omega_1^2}_{\frac{K}{m}} u(t) = \omega_1^2 \underbrace{u_{st}}_{\frac{F}{K}} \sin \omega t \qquad \omega_1 = \sqrt{\frac{K}{m}}$$

fattore di smorzamento

$$\zeta = \frac{c}{2\sqrt{Km}} = \frac{c}{c_{cr}}$$

(tipicamente subcritico, $\zeta < 1$ e $\ll 1$, $\zeta \sim 1\% = 0.01$)

risposta "a regime"

$$\underbrace{Z_1}_{N u_{st} \frac{1-\beta^2}{\sqrt{D}}} \sin \omega t - \underbrace{Z_2}_{N u_{st} \frac{2\zeta\beta}{\sqrt{D}}} \cos \omega t$$

$$\xi = \operatorname{atan} \frac{2\zeta\beta}{1-\beta^2}$$

- Integrale particolare:

$$u_p(t) = N(\beta;\zeta)\, u_{st} \sin(\omega t - \xi(\beta;\zeta))$$

fase (sfasamento in ritardo tra risposta e forzante)

fattore di amplificazione dinamica (rispetto a $u_{st} = F/K$)

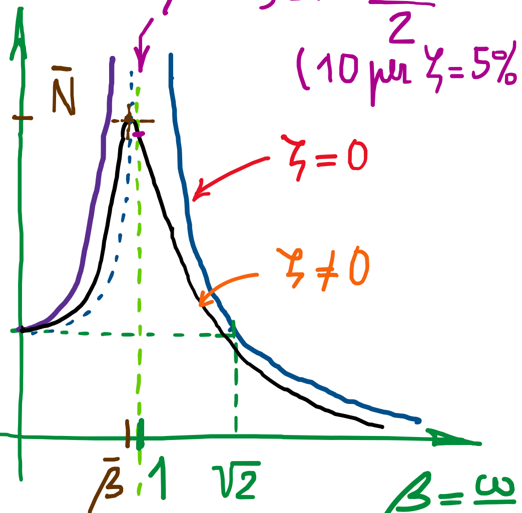
$$N = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$$

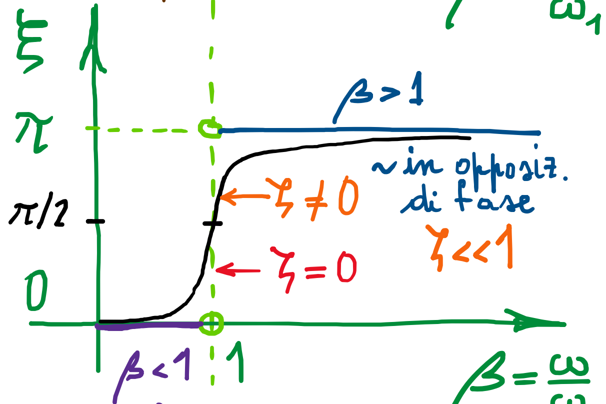
$$\bar{N}(\bar{\beta}) = \frac{1}{\sqrt{1-\bar{\beta}^4}} \quad \text{luogo dei max}$$

$$\bar{N} = \frac{1}{2\zeta} \frac{1}{\sqrt{1-\zeta^2}} \simeq \frac{1}{2\zeta}$$

$$\bar{\beta} = \sqrt{1-2\zeta^2} \simeq 1$$

$$N = \frac{1}{\sqrt{D}}$$





frequency ratio

SOMMARIO (Lez. 05)

- Risposta smorzata a forzante armonica.
- Effetto dello smorzamento su curve di risonanza e di fase.
- Picco finito di ampiezza in condizioni di risonanza; risposta in quadratura rispetto alla forzante.
- Risposta a regime in componenti $\sin\omega t$ e $\cos\omega t$.
- Integrale generale con risposta transiente e a regime.
- Next step: trattazione unificata in variabili complesse per risposta a $F\sin\omega t$ e/o $F\cos\omega t \rightsquigarrow Fe^{i\omega t}$.