Fundamental Theorems of Limit Analysis (of frames – Calcul à rupture)

- Static quantities
  Bending moments $M(x)$
  that are admissible
  (statically and plastically)

– statically admissible, namely equilibrated,
with "live" loads affected by load multiplier $\bar{\lambda}$

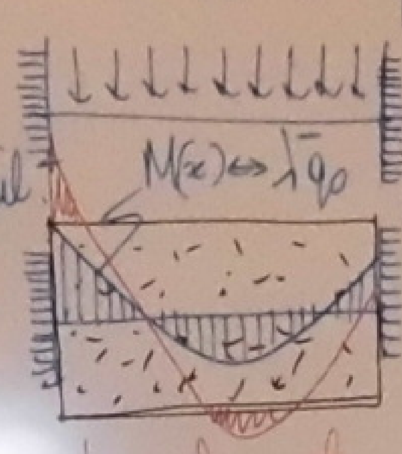
$$M(x) \leftrightarrow \bar{\lambda} P_{0i}, \bar{\lambda} q_0$$

– plastically admissible, namely conformal

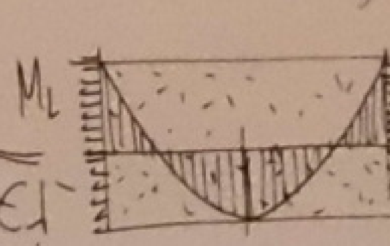
$$M_L^- \leq M(x) \leq M_L^+$$

$\bar{\lambda}$, to which $M(x)$ is in equil.
sets the family of possible λ_L
(collapse load multiplier λ_L is a $\bar{\lambda}$)

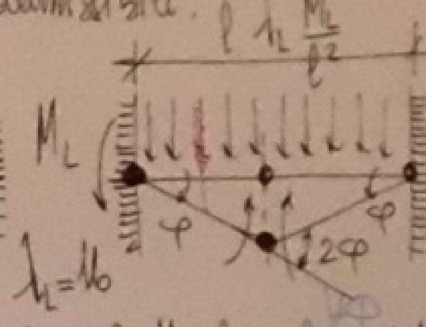
$$q = \bar{\lambda} q_0 = \bar{\lambda} \frac{M_L}{\ell^2}$$



$\lambda_L \in \bar{\lambda}$ set



– As said, the true collapse internal actions
and collapse mechanism are both statically
and kinematically admissible.



$\lambda_L \in \lambda^+ \text{set}^{M_L}$

$$\mathcal{L}_e = \ell \lambda_L \frac{M_L}{\ell^2} \frac{\ell}{2} \varphi \frac{\ell}{4} = 2 M_L \varphi + M_L 2\varphi = 4 M_L \varphi = \mathcal{L}_{ip} = D$$

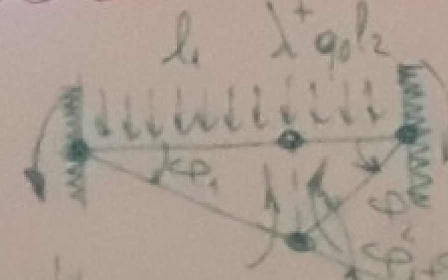
- Kinematic quantities:
  Kinematic mechanism, displacements $v(x)$
  and rotation φ_i that are admissible
  (kinematically and plastically)

– Kinematically admissible, namely compatible
with a (potential) collapse mechanism at associated
load multiplier λ^+ such that to produce positive external work

$$v(x), \varphi \leftrightarrow \lambda^+ P_{0i}, \lambda^+ q_0$$

– plastically admissible, namely such that to provide positive plastic dissip.
plastic rotations $\varphi_i > 0$ when $M = M_L^+$, $\varphi_i < 0$ when $M = M_L^-$

λ^+, to which a (potential) collapse mechanism is
associated sets the family of possible λ_L
(collapse load multiplier λ_L is a λ^+)



Here, the associated λ is
calculated by energy balance as if this would be
the collapse mode

λ^+ associated to such a mechanism, can be evaluated as

$$\lambda^+ = \frac{D = \mathcal{L}_{ip} > 0}{\mathcal{L}_{e0} > 0} = \frac{\sum_i M_{L_i}^+ \varphi_i + M_{L_i}^- \varphi_i \left[- \int g(x) y(x) dx \right]}{\sum_i P_{0i} y_i + \int q_0(x) y(x) dx}$$

$$\mathcal{L}_e = \sum_i \underbrace{\lambda^+ P_{0i} y_i + \int_{S_\ell} \lambda^+ q_0(x) y(x) dx}_{\text{live loads}} + \underbrace{\int_{S_{\ell}} g(x) y(x) dx}_{\text{permanent loads}} = \sum_i M_{L_i}^+ \varphi_i + M_{L_i}^- \varphi_i = D = \mathcal{L}_{ip} > 0$$

– Static theorem

– Kinematic theorem

– Mixed theorem

Fundamental Theorems of Limit Analysis (of Frames – Calcul à rupture)

- Static quantities: bending moments $M(x)$ that are admissible (statically and plastically)
  - statically admissible, namely equilibrated, with "live" loads affected by load multiplier λ^-

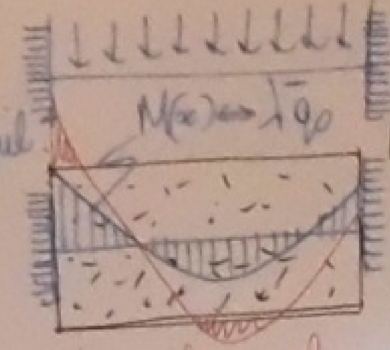
$$M(x) \leftrightarrow \lambda^- P_{0i}, \lambda^- q_0$$

  - plastically admissible, namely conformal

$$M_L^- \le M(x) \le M_L^+$$

λ^-, to which $M(x)$ is an equil., sets the family of possible λ_L (collapse load multiplier λ_L is a λ^-)

$$q = \lambda^- q_0 = \lambda^- \frac{M_L}{\ell^2}$$



equil.

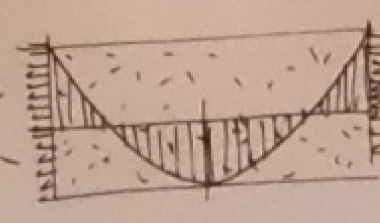
$M(x) \leftrightarrow \lambda^- q_0$

$M_L^- = -M_L$

$M_L^+ = M_L$

not conformal

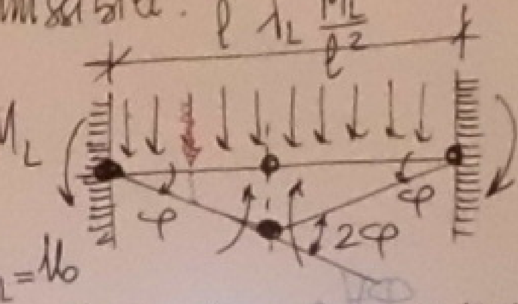
$\lambda_L \in \lambda^-$ set



M_L

- As said, the true collapse internal actions and collapse mechanism are both statically and kinematically admissible.

$\ell \quad \lambda_L \frac{M_L}{\ell^2}$



M_L

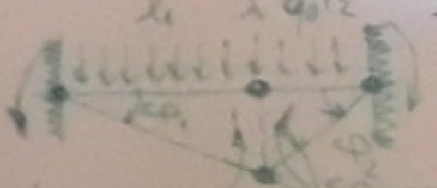
$\lambda_L = 16$

$$\mathcal{L}_e = 2\lambda_L \frac{M_L}{\ell^2}\frac{\ell}{2}\varphi\frac{\ell}{4} = 2M_L\varphi + M_L 2\varphi = 4M_L\varphi = \mathcal{L}_{ip} = D$$

$$\lambda_L \in \lambda^+ \text{set}$$

- Kinematic quantities: ...

λ^+, to which a (potential) collapse mechanism is associated, sets the family of possible λ_L (collapse load multiplier λ_L is a λ^+)



$\ell_1 \quad \lambda^+ q_0 \ell_2$

Here, the associated λ^+ is calculated by energy balance as if this would be the collapse mode

λ^+ associated to such a mechanism, can be evaluated as

$$\lambda^+ = \frac{D = \mathcal{L}_{ip} > 0}{\mathcal{L}_{e0} > 0} = \frac{\sum_i M_{Li}^+\varphi_i + M_{Li}^-\varphi_i \left[- \int g(x)y(x)\,dx\right]}{\sum_i P_{0i}\,y_i + \int q_0(x)\,y(x)\,dx}$$

equilibrium at incipient collapse

$$\mathcal{L}_e = \underbrace{\sum_i \lambda^+ P_{0i}\,y_i + \int_{St} \lambda^+ q_0(x)y(x)\,dx}_{\text{live loads}} + \underbrace{\int_{St} g(x)y(x)\,dx}_{\text{permanent loads}} = \sum_i M_{Li}^+\varphi_i + M_{Li}^-\varphi_i = \underset{>0}{D} = \mathcal{L}_{ip}$$

- Static theorem (lower bound theorem) – "Safe theorem"

The collapse load multiplier is the max of all λ^-

$$S = \lambda_L = \max\{\lambda^-\}$$

$\lambda^- \le \lambda_L = S$

λ^- lower-bound (conservative) estimate of λ_L

- Kinematic theorem (upper-bound theorem)

... load multiplier is the min of all λ^+:

$$S = \lambda_L = \min\{\lambda^+\}$$

$S = \lambda_L \le \lambda^+$

upper-bound (non conservative) estimate of λ_L

- Mixed theorem

Any statically admissible distribution leading to the same $\lambda^- = \lambda^+$ of a kinematically admissible mechanism provides the true collapse multiplier. $\lambda^- = \lambda^+ = \lambda_L = S$

In fact, ...


"Safe theorem"
- Static theorem (lower-bound theorem)
The collapse load multiplier is the max of all λ⁻
S = λ_L = max {λ⁻}
0 λ⁻ ≤ λ_L = S λ
lower-bound (conservative) estimate of λ_L
- Kinematic theorem (upper-bound theorem)
The collapse load multiplier is the min. of all λ⁺:
S = λ_L = min {λ⁺}
0 S = λ_L ≤ λ⁺ λ
upper-bound (non conservative) estimate of λ_L
- Mixed theorem
Any statically admissible distribution leading to the same λ⁻ = λ⁺ of a kinematically admissible mechanism provides the true collapse multiplier: λ⁻ = λ⁺ = λ_L = S
- As said, the true collapse internal actions and collapse mechanism are both statically and kinematically admissible:
not conformal
λ⁺ associated to such a mechanism, can be evaluated as
equilibrium at incipient collapse
D = L_ip = 4 M_L φ

Proofs

Le   PVW $\mathcal{L}_{ip} = D$

S. Th. — Kinematic quantities at collapse

① static: collapse $\sum_i \lambda_L P_{0i} y_i = \sum_i M^+_{Li} \varphi^+_i + M^-_{Li} \varphi^-_i$

This is the equation that one would use to get λ_L if collapse mode would be known

② static: statically admissible $\sum_i \lambda^- P_{0i} y_i = \sum_i M_i \varphi^+_i + M_i \varphi^-_i$

$$\boxed{\lambda^- \leq \lambda_L} \Leftarrow \lambda_L - \lambda^- \geq 0 \Leftarrow (\lambda_L - \lambda^-) \underbrace{\sum_i P_{0i} y_i}_{\mathcal{L}_{e0} > 0} = \sum_i \underbrace{(M^+_{Li} - M_i)}_{\geq 0} \varphi^+_i + \underbrace{(M^-_{Li} - M_i)}_{\leq 0} \underbrace{\varphi^-_i}_{< 0} \geq 0$$

and since λ_L is a λ^-, λ_L is the max of all λ^-

K. Th. — Kinematic quantities at kinematically admissible collapse

① static: kinematically admissible $\sum_i \lambda^+ P_{0i} y_i = \sum_i M^+_{Li} \varphi^+_i + M^-_{Li} \varphi^-_i$

This is the equation that one would use to get λ^+ for the given admissible mechanism

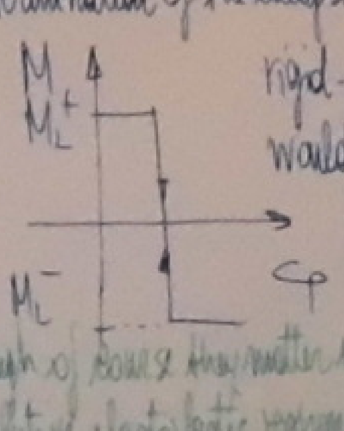
② static: collapse mode $\sum_i \lambda_L P_{0i} y_i = \sum_i M_i \varphi^+_i + M_i \varphi^-_i$

$$\boxed{\lambda^+ \geq \lambda_L} \Leftarrow \lambda^+ - \lambda_L \geq 0 \Leftarrow (\lambda^+ - \lambda_L) \underbrace{\sum P_{0i} y_i}_{\mathcal{L}_{e0} > 0} = \sum_i \underbrace{(M^+_i - M_i)}_{\geq 0} \underbrace{\varphi^+_i}_{> 0} + \underbrace{(M^-_i - M_i)}_{\leq 0} \underbrace{\varphi^-_i}_{< 0} \geq 0$$

and since λ_L is a λ^+, λ_L is the min of all λ^+

Corollaries:

- Elastic structural properties are irrelevant (for the determination of the collapse load)



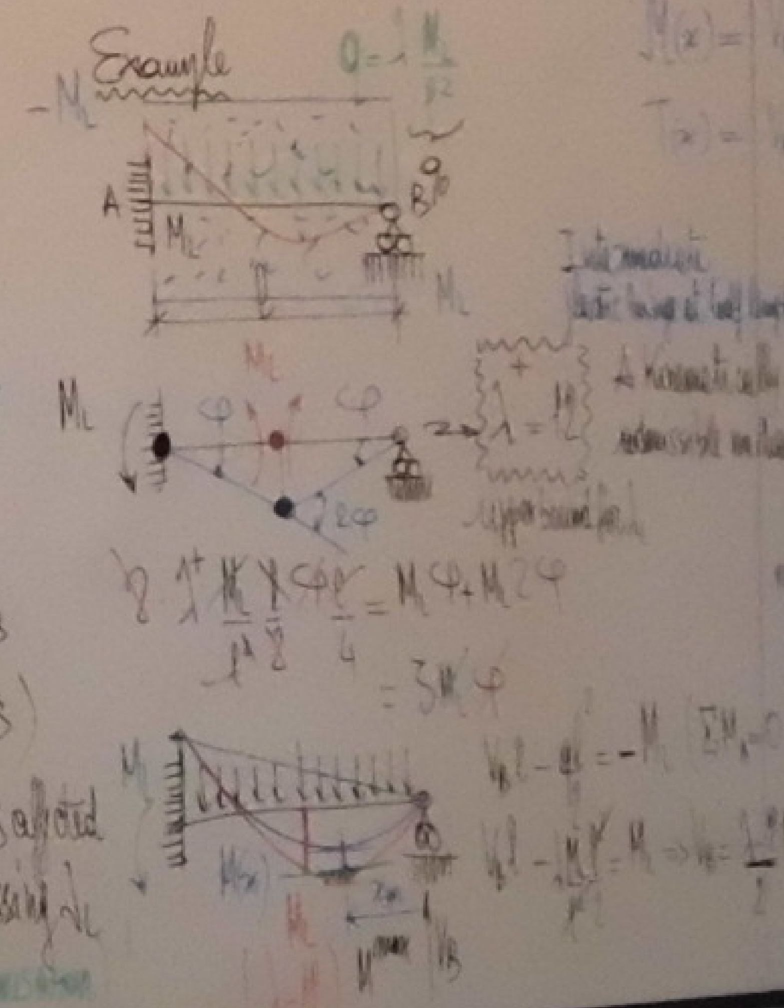
rigid-plastic would provide same λ_L

(though of course they matter in the "real" evolutive elastoplastic response)

- Self-equilibrated imposed co-actions (like temperature variations, settlements) occurring prior to the growing live loads affected by load multiplier λ do not matter in assessing λ_L

- The application of the theorems allows for "manual methods" of calculation of λ_L and collapse mechanism for frame structures. These could be generalized ... (e.g. by the simplex method).

Example



$\lambda^+ \frac{M_L}{l^2} \frac{l}{2} \frac{\varphi l}{4} = M_L \varphi + M_L 2\varphi = 3 M_L \varphi$

PVW
Lip = D
This is the equation that one would use to get λ_L if collapse mode would be known
This is the equation that one would use to get λ+ for the given admissible mechanism
Corollaries:
– Elastic structural properties are irrelevant (for the determination of the collapse cond.)
rigid-plastic would provide same λ_L
– Self-equilibrated impressed co-actions (like temperature variations; settlements) occurring prior to the growing live loads affected by load multiplier λ do not matter in assessing λ_L
– The application of the theorems allows for "manual methods" of calculation of λ_L and attached collapse mechanism for frame structures.
(e.g. by the simplex method).
Example
A
B
λ+ = 12
A kinematically-admissible mechanism
upper bound for λ_L
= 3M_L φ
At λ = 12, M^max = 25/24 M_L
because conformity is violated.
λ+ = 11.6567

$\mathcal{L}_p = 0$

This is the equation that one would use to get λ if collapse mode would be known

$$\Sigma_i \lambda P_i y_i = \Sigma M \varphi_i + M \varphi_i$$

$$\geq 0$$

$\geq 0 \quad > 0 \quad \leq 0 \quad < 0$

$$\Sigma_i \lambda^+ P_{oi} y_i = \Sigma_i M_i^+ \varphi_i + M_i^+ \varphi_i$$

This is the equation that one would use to get λ^+ for the given admissible mechanism

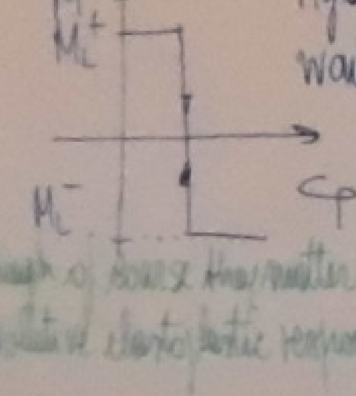
$$\Sigma_i \lambda_L P_{oi} y_i = \Sigma_i M \varphi_i + M \varphi_i$$

$$= \Sigma_i (M_i^+ - M_i)\varphi_i + (M_i^+ - M_i)\varphi_i \geq 0$$

$\geq 0 \quad \geq 0 \quad \leq 0 \quad < 0$

Corollaries:

- Elastic structural properties are irrelevant (for the determination of the collapse cond.)

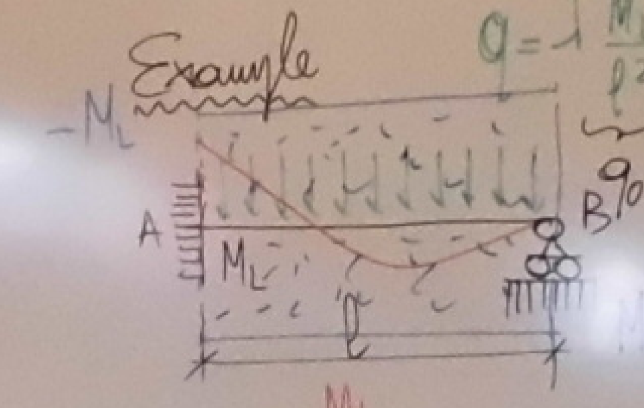


rigid-plastic would provide same λ_L

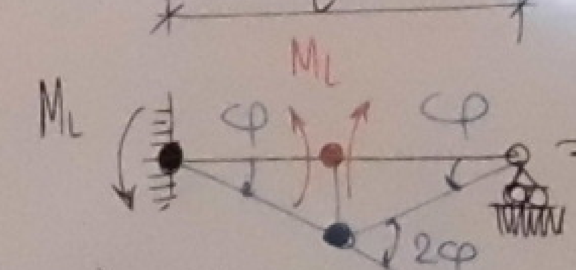
- Self-equilibrated imposed co-actions (like temperature variations, settlements) occurring prior to the growing live loads affected by load multiplier λ do not matter in assessing λ_L

- The application of the theorems allows for "manual methods" of calculation of λ_L and attached collapse mechanism for frame structures. These could be generalized by systematic programming methods and solved computationally (e.g. by the simplex method).

Example $\quad q = \lambda \frac{M_L}{\ell^2}$



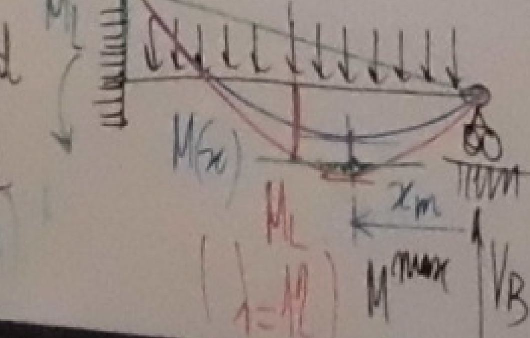
Intermediate plastic hinge at half length



$\Rightarrow \boxed{\lambda^+ = 12}$ A Kinematically-admissible mechanism

upper bound for λ_L

$$2 \cdot \lambda^+ \frac{M_L}{\ell^2} \frac{\ell}{2} \varphi \frac{\ell}{4} = M_L \varphi + M_L 2\varphi = 3 M_L \varphi$$



$$V_B \ell - \frac{q\ell^2}{2} = -M_L \quad (\Sigma M_A = 0)$$

$$V_B \ell - \lambda \frac{M_L}{\ell^2}\frac{\ell^2}{2} = M_L \Rightarrow V_B = \frac{\lambda - 2}{2}\frac{M_L}{\ell}$$

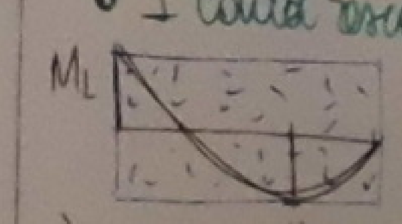
$$M(x) = V_B x - \frac{q x^2}{2} = -\frac{M_L}{\ell} x + \frac{\lambda}{2}\frac{M_L}{\ell^2} x(\ell - x)$$

$$T(x) = V_B - \lambda q_0 x = 0 \Rightarrow x_m = \frac{V_B}{\lambda q_0} = \frac{\lambda - 2}{2\lambda}\ell$$

- At $\lambda = 12$, $M^{max} = \frac{25}{24} M_L = e_m M_L$

Then, this λ^+ is not a λ^-, < 1.0 because conformity is violated.

- I could rescale the loads by $1/e_m$ so that $M = M_L$ at x_m



$$\lambda^- = \frac{\lambda^+}{e_m} = 11.52$$

$$\lambda^- = 11.52 \leq \lambda \leq 12 = \lambda^+$$

rather narrow fork

- Reiterate and set plastic hinge at x_m $\Rightarrow$ new estimate $\lambda^+ = 11.6567$
- New e_m and delimitation $11.656736 \leq \lambda_L \leq 11.656742$