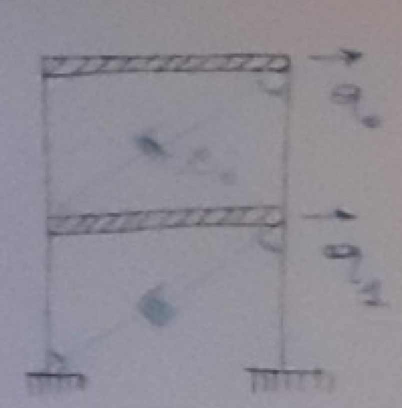

Structural damping (MDOF systems)
n of dofs
- either damping may be set by specific damper elements (viscous dampers) => mechanical systems
- or damping may be smeared over the whole structure (inherent structural damping).
- to be described by viscous linear damping, at a first (good) approximation
- eqns. of motion change to:
by Lagrange equations
damping coefficient
Total dissipation function (Rayleigh)
damping matrix
Example:

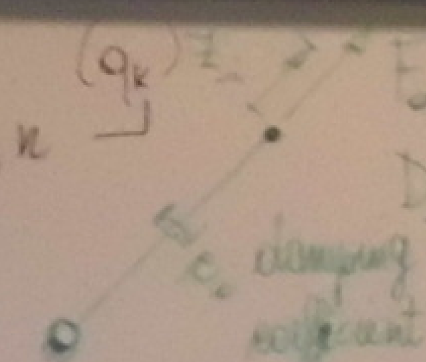
Structural damping (MDOF systems, n of dofs)



- either damping may be set by specific damper elements (viscous dampers) ⇒ mechanical systems
- or damping may be smeared over the whole structure (inherent structural damping).
- to be described by viscous linear damping, at a first (and) approximation
- eqns of motion change to:

$$M\ddot{q} + \underbrace{C\dot{q}}_{F_d = -Q_d} + Kq = Q(t) \longleftrightarrow \text{by Lagrange equations} \quad Q_d = -\frac{\partial D}{\partial \dot{q}} = -F_d$$

(q_k) $1, n$



$$F_{d_i} = c_i \dot{z}_i = \dot{z}_i c_i$$

$$D_i = \frac{1}{2} \underbrace{F_{d_i} \dot{z}_i}_{P_i} = \frac{1}{2} \dot{z}_i c_i \dot{z}_i$$

$i = 1, n_d$ dampers

$$z_i = z_i(q) \Rightarrow \dot{z}_i = \frac{\partial z_i}{\partial q}^T \dot{q} = \dot{q}^T \frac{\partial z_i}{\partial q}$$

$$\underset{n_d \times 1}{z} = \{z_i\} \; ; \; \underset{n_d \times 1}{F_d^z} = \{F_{d_i}\}$$

$$F_d^z = C_z \dot{z} \; ; \; \underset{n_d \times n_d}{C_z} = \text{diag}[c_i]$$

$$\dot{z} = \frac{\partial z}{\partial q}^T \dot{q} = \dot{q}^T \frac{\partial z}{\partial q}$$

Total dissipation function (Rayleigh)

$$D = \frac{1}{2} F_d^{zT} \dot{z}$$

$$= \frac{1}{2} \dot{z}^T C_z \dot{z} = \frac{1}{2} \dot{q}^T \underbrace{\frac{\partial z}{\partial q} C_z \frac{\partial z}{\partial q}^T}_{C} \dot{q}$$

damping matrix, symmetric, positive semidefinite

$$= \sum_i D_i$$

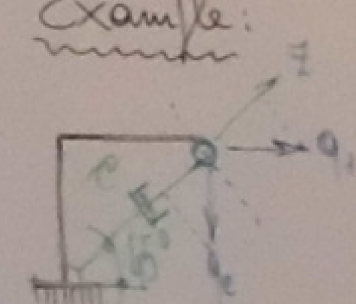
$$= \sum_i \frac{1}{2} \dot{z}_i c_i \dot{z}_i$$

$$= \sum_i \frac{1}{2} \dot{q}^T \underbrace{\frac{\partial z_i}{\partial q} c_i \frac{\partial z_i}{\partial q}^T}_{C_i} \dot{q} \qquad C_{ij} = \frac{\partial^2 D}{\partial \dot{q}_i \partial \dot{q}_j}$$

$$= \frac{1}{2} \dot{q}^T \underbrace{\sum_{i=1}^{n_d} \frac{\partial z_i}{\partial q} c_i \frac{\partial z_i}{\partial q}^T}_{} \dot{q} = \frac{1}{2} \dot{q}^T \underbrace{C\dot{q}}_{F_d} = \frac{1}{2} P_d$$

$$C = \sum_i C_i$$

Example:



$$\dot{z} = \frac{\dot{q}_1}{\sqrt{2}} - \frac{\dot{q}_2}{\sqrt{2}}$$

$$\frac{\partial z}{\partial q_i} = \pm \frac{1}{\sqrt{2}}$$

$$C = \frac{\partial z}{\partial q} c \frac{\partial z}{\partial q}^T = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} c \frac{1}{\sqrt{2}} \{1 \; -1\} = \frac{c}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

singular

Otherwise

$$D = \frac{1}{2} c \dot{z}^2 = \frac{1}{2} c \frac{1}{2} (\dot{q}_1 - \dot{q}_2)^2 = \frac{1}{2} c \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2 - 2\dot{q}_1\dot{q}_2)$$

$$= \frac{1}{2} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \frac{1}{2} c \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = \frac{1}{2} \dot{q}^T C \dot{q}$$

- By changing to principal coordinates $q = \Phi p$

$$\underbrace{(\Phi^T M \Phi)}_{} \ddot{p} + (\Phi^T C \Phi) \dot{p} + \underbrace{(\Phi^T K \Phi)}_{} p(t) = P(t) = \Phi^T Q(t)$$

$\mathcal{M} = \text{diag}[M_i]$ $\quad \mathcal{C}$ non diag $\quad \mathcal{K} = \text{diag}[K_i]$

matrices in principal coordinates

- A sufficient condition to get $\mathcal{C} = \text{diag}[C_i]$ [inherent damping]

$C = \alpha M + \beta K$ linear comb. ("classical" or Rayleigh damping)

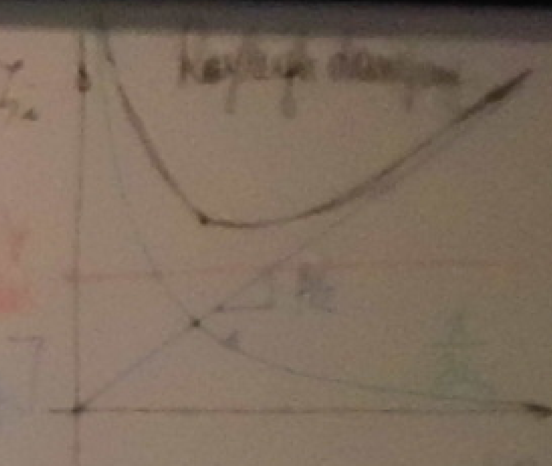
$\mathcal{C} = \alpha \mathcal{M} + \beta \mathcal{K} \Rightarrow C_i = \alpha M_i + \beta K_i$

- Assume that $\mathcal{C}$ is a diagonal matrix, prescribed in terms of modal damping ratios ζ_i so that $\frac{C_i}{M_i} = 2\zeta_i \omega_i \Rightarrow C_i = 2\zeta_i M_i \sqrt{\frac{K_i}{M_i}} = 2\zeta_i \sqrt{K_i M_i}$

$$C = \Phi^{-T} \mathcal{C} \Phi^{-1}$$

Meaning $\; 2\zeta_i M_i \omega_i = \alpha \frac{M_i}{M_i \omega_i} + \beta \frac{K_i}{M_i \omega_i} \Rightarrow \zeta_i = \frac{1}{2} \left(\frac{\alpha}{\omega_i} + \beta \omega_i \right)$

modal damping ratios



α, β so that ζ_1, ζ_2 are set

Total dissipation function (Rayleigh)

$D = \frac{1}{2} F_d^T \dot{z}$

$= \frac{1}{2} \dot{z}^T C_z \dot{z}$

$= \sum_i D_i$

$= \sum_i \frac{1}{2} \dot{z}_i c_i \dot{z}_i$

$= \sum_i \frac{1}{2} \dot{q}^T \frac{\partial z_i}{\partial q} c_i \frac{\partial z_i}{\partial q}^T \dot{q}$ $\qquad C_{ij} = \frac{\partial^2 D}{\partial \dot{q}_i \partial \dot{q}_j}$

$= \frac{1}{2} \dot{q}^T \underbrace{\sum_i \frac{\partial z_i}{\partial q} c_i \frac{\partial z_i}{\partial q}^T}_{\mathbb{C}} \dot{q} = \frac{1}{2} \dot{q}^T \underbrace{\mathbb{C} \dot{q}}_{F_d}$

damping matrix

symmetric, positive semidefinite

Example

$\mathbb{C} = \sum_i \mathbb{C}_i$ $\qquad = \frac{1}{2} P_d$

$\dot{z} = \frac{\dot{q}_1}{\sqrt{2}} - \frac{\dot{q}_2}{\sqrt{2}}$ $\qquad \frac{\partial z}{\partial q_1} = +\frac{1}{\sqrt{2}}$

$\mathbb{C} = \frac{\partial z}{\partial q} c \frac{\partial z}{\partial q}^T = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} c \frac{1}{\sqrt{2}} \{1 \ \ -1\} = \frac{c}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

singular

- By changing to principal coordinates $q = \Phi p$

$\underbrace{(\Phi^T \mathbb{M} \Phi)}_{\mathcal{M} = \mathrm{diag}[M_i]} \ddot{p} + \Phi^T \mathbb{C} \Phi \dot{p} + \underbrace{(\Phi^T \mathbb{K} \Phi)}_{\mathcal{K} = \mathrm{diag}[K_i]} p(t) = \mathbb{P}(t) = \Phi^T \mathbb{Q}(t)$

$\mathcal{C}$ non diag

matrices in principal coordinates

- A sufficient condition to get $\mathcal{C} = \mathrm{diag}[C_i]$ [inherent damping]

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$\mathbb{C} = \Phi^{-T} \mathcal{C} \Phi^{-1}$

Meaning $2\zeta_i \cancel{M_i \omega_i} = \alpha \frac{M_i}{M_i \omega_i} + \beta \frac{K_i}{M_i \omega_i}$ $\quad \omega_i^2$

$\Rightarrow \zeta_i = \frac{1}{2} \left(\frac{\alpha}{\omega_i} + \beta \omega_i \right)$

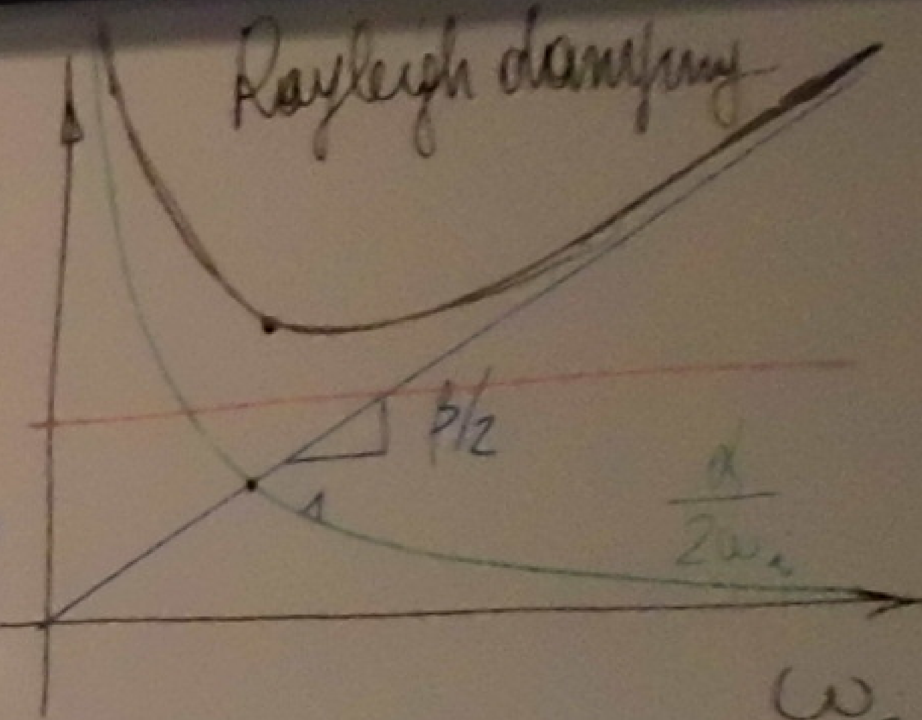
modal damping ratios

Otherwise

$D = \frac{1}{2} c \dot{z}^2 = \frac{1}{2} c \frac{1}{2} (\dot{q}_1 - \dot{q}_2)^2 = \frac{1}{2} c \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2 - 2\dot{q}_1 \dot{q}_2)$

$= \frac{1}{2} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}^T \frac{1}{2} c \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = \frac{1}{2} \dot{q}^T \mathbb{C} \dot{q}$

ζ_i

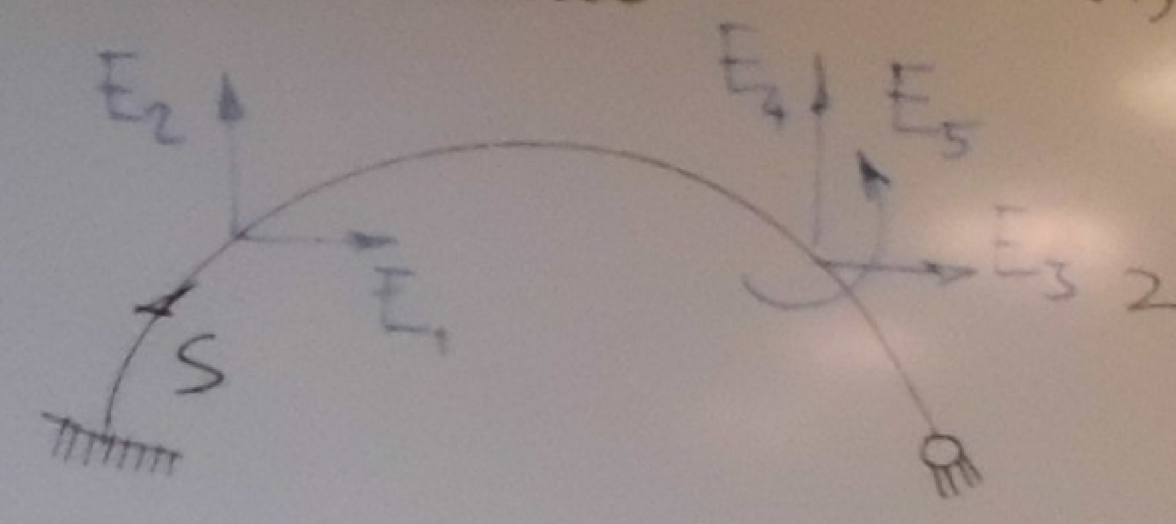


α, β so that ζ_1, ζ_2 are set

$\zeta_i \simeq 2\% - 7\%$

Internal Actions

$\mathcal{A} \Rightarrow N, T, M$



$\Rightarrow$ Known the structural response $q(t)$ o $p(t)$

$$\mathbb{E}(t) = \mathbb{K}\, q(t) = \mathbb{Q}(t) - \mathbb{M}\ddot{q}(t) - \mathbb{C}\dot{q}(t)$$

- At any time instant t, the distribution of elastic forces on the structure allows to evaluate $\mathcal{A}(s,t)$ as in statics.
- By superposition of effects
  - $q_j = 1 \Rightarrow \mathbb{E}_j \leftrightarrow$ column j of stiffness matrix $\mathbb{K} \leftrightarrow \mathcal{A}_j(s) \rightarrow \mathcal{A}(s)$
  - $p_k = 1 \Rightarrow \bar{\mathbb{E}}_k = \mathbb{K}\,\phi_k = \omega_k^2 \mathbb{M}\,\phi_k = \mathbb{E}_{Ik} \leftrightarrow \bar{\mathcal{A}}_k(s) \rightarrow \bar{\mathcal{A}}(s)$ modal internal actions

$q = \Phi\, p \leftrightarrow q_j = \Phi_{jk}\, p_k = \phi_{kj}\, p_k$ $\quad \Phi = [\phi_1 \cdots \phi_k \cdots \phi_n]$

$p = \Phi\, q$

- $\mathcal{A}(s,t) = \sum_j \mathcal{A}_j(s)\, q_j(t) = \mathcal{A}^T(s)\, q(t) = \mathcal{A}^T(s)\, \Phi\, p(t)$
  $\phantom{\mathcal{A}(s,t)} = \sum_k \bar{\mathcal{A}}_k(s)\, p_k(t) = \bar{\mathcal{A}}^T(s)\, p(t) = \bar{\mathcal{A}}(s)\, \Phi\, q(t)$

$$\begin{cases} \bar{\mathcal{A}}^T = \mathcal{A}^T \Phi \Rightarrow \bar{\mathcal{A}} = \Phi^T \mathcal{A} \\ \mathcal{A}^T = \bar{\mathcal{A}}\, \Phi^{-1} \Rightarrow \mathcal{A} = \Phi^{-T} \bar{\mathcal{A}} \end{cases}$$

- Envelope estimates aim at determining maximum values of structural response, in terms of displacements and internal actions

• $\max_t q_j(t) = q_j^{max} \underset{\text{estimate}}{=} \sum_k |\phi_{kj}|\, p_k^{max}$ too conservative

Square Root of Sum of Squares SRSS $= \sqrt{\sum_k (\phi_{kj}\, p_k^{max})^2}$ shown to provide consistent estimates

• $\max_t \mathcal{A}(s,t) = \mathcal{A}^{max}(s) = \sqrt{\sum_k (\bar{\mathcal{A}}_k(s)\, p_k^{max})^2} = \sqrt{\sum_j (\mathcal{A}_j(s)\, q_j^{max})^2}$

Example

$$\mathbb{K} = \frac{6}{7}\frac{EJ}{\ell^3} \begin{array}{cc} q_1 & q_2 \end{array} \begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}$$

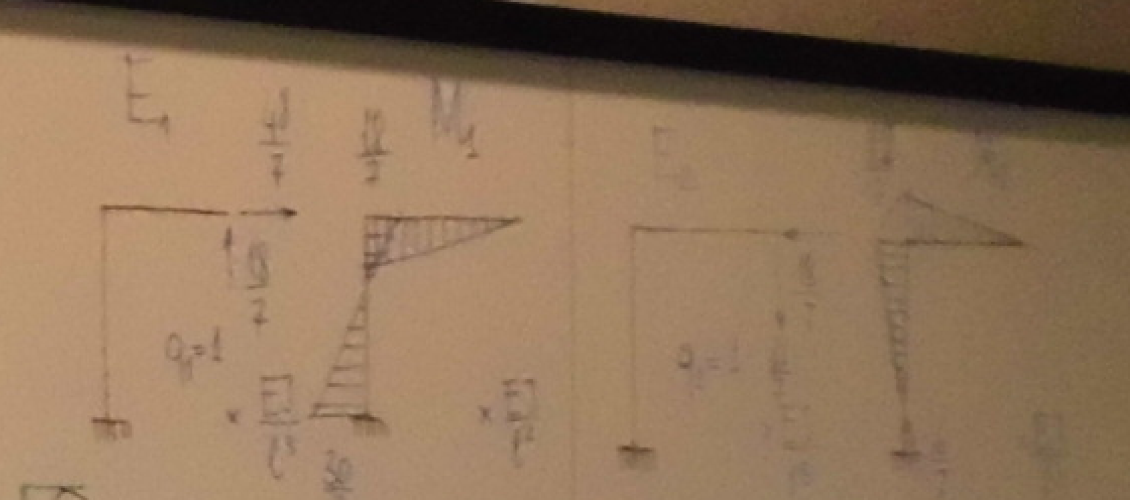
$$\mathbb{M} = m \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

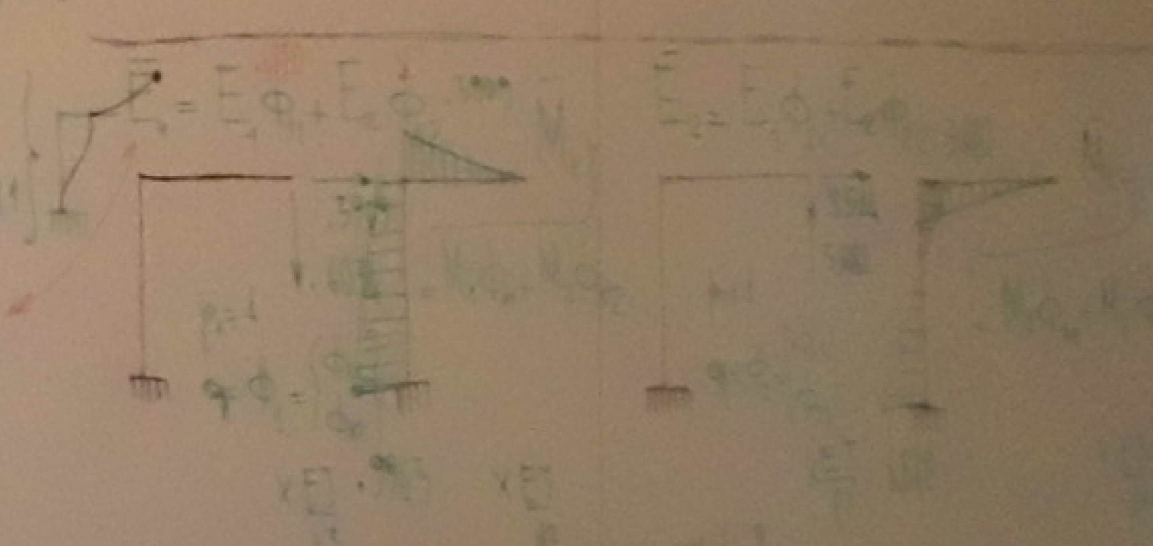
$\left(\omega_1^2 = \lambda_1 \frac{EJ}{m\ell^3}\right)$

1st mode $\lambda_1 = .3017$; $\phi_1 = \begin{Bmatrix} .4320 \\ 1 \end{Bmatrix}$

2nd mode $\lambda_2 = 2.841$; $\phi_2 = \begin{Bmatrix} 1 \\ -.4681 \end{Bmatrix}$

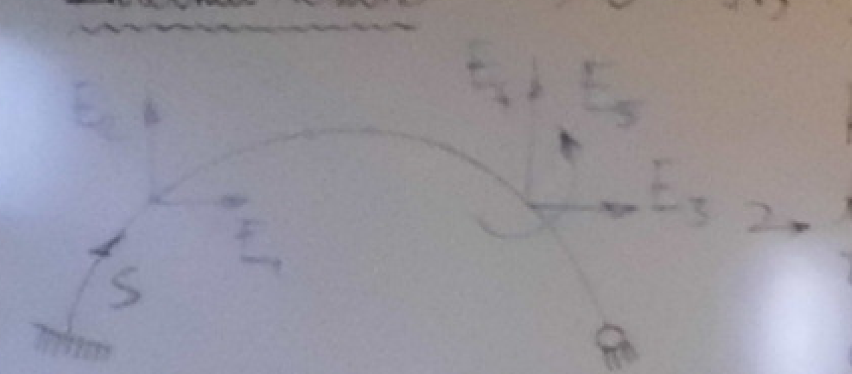
$\bar{\mathbb{E}}_k = \mathbb{E}_j\, \Phi_{jk} = \mathbb{E}_j\, \phi_{kj}$





modal forces: $\bar{\mathbb{E}}_1 = \mathbb{K}\phi_1 = \omega_1^2 \mathbb{M}\phi_1 = \mathbb{E}_{I1}$

Internal Actions $\mathcal{A} \to N, T, M$



Known the structural response $q(t)$ o $p(t)$

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- By superposition of effects

$q = \Phi p \leftrightarrow q_j = \Phi_{jk} p_k = \phi_{kj} p_k$ $\Phi = [\phi_1 \ \phi_k \cdots \phi_n]$

$p = \Phi^{-1} q$

- $\mathcal{A}(s,t) = \sum_j \mathcal{A}_j(s) q_j(t) = \mathcal{A}(s) q(t) = \mathcal{A}(s) \Phi p(t)$
  $= \sum_k \bar{\mathcal{A}}_k(s) p_k(t) = \bar{\mathcal{A}}(s) p(t) = \bar{\mathcal{A}}(s) \Phi^{-1} q(t)$

$$\begin{cases} \bar{\mathcal{A}}^T = \mathcal{A}^T \Phi \Rightarrow \bar{\mathcal{A}} = \Phi^T \mathcal{A} \\ \mathcal{A}^T = \bar{\mathcal{A}}^T \Phi^{-1} \Rightarrow \mathcal{A} = \Phi^{-T} \bar{\mathcal{A}} \end{cases}$$

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$\bar{\mathbb{E}}_k = \mathbb{E}_j \Phi_{jk} = \mathbb{E}_j \phi_{kj}$

Example

$$\mathbb{K} = \frac{6}{7}\frac{EJ}{\ell^3} \begin{array}{c} \begin{matrix} q_1 & q_2 \end{matrix} \\ \begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix} \end{array}$$

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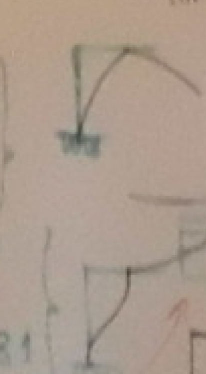

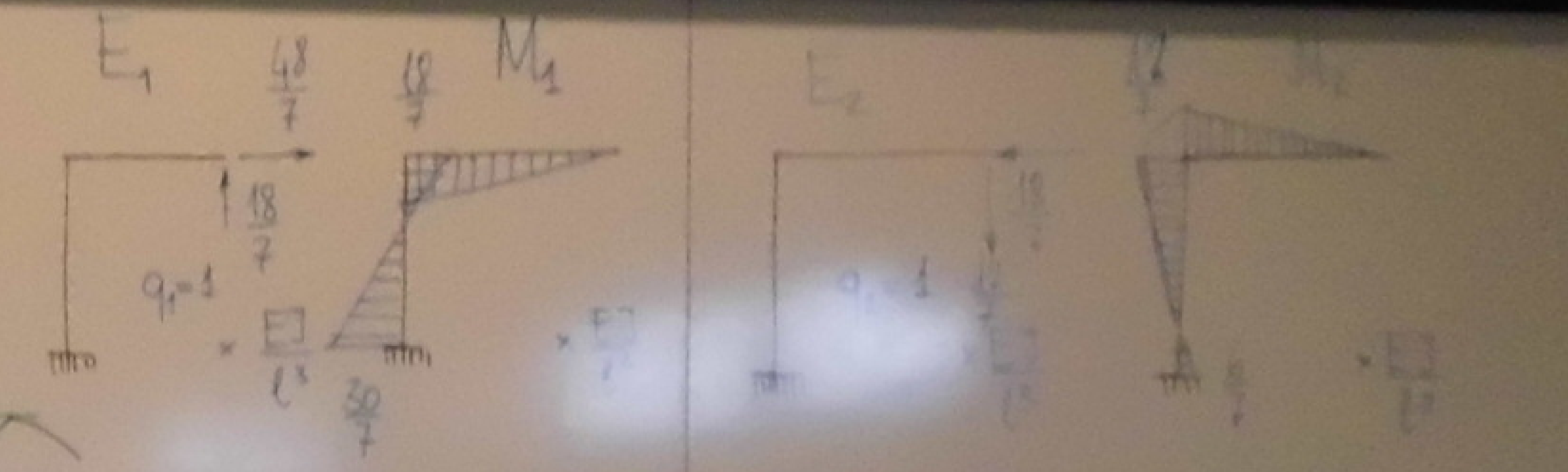

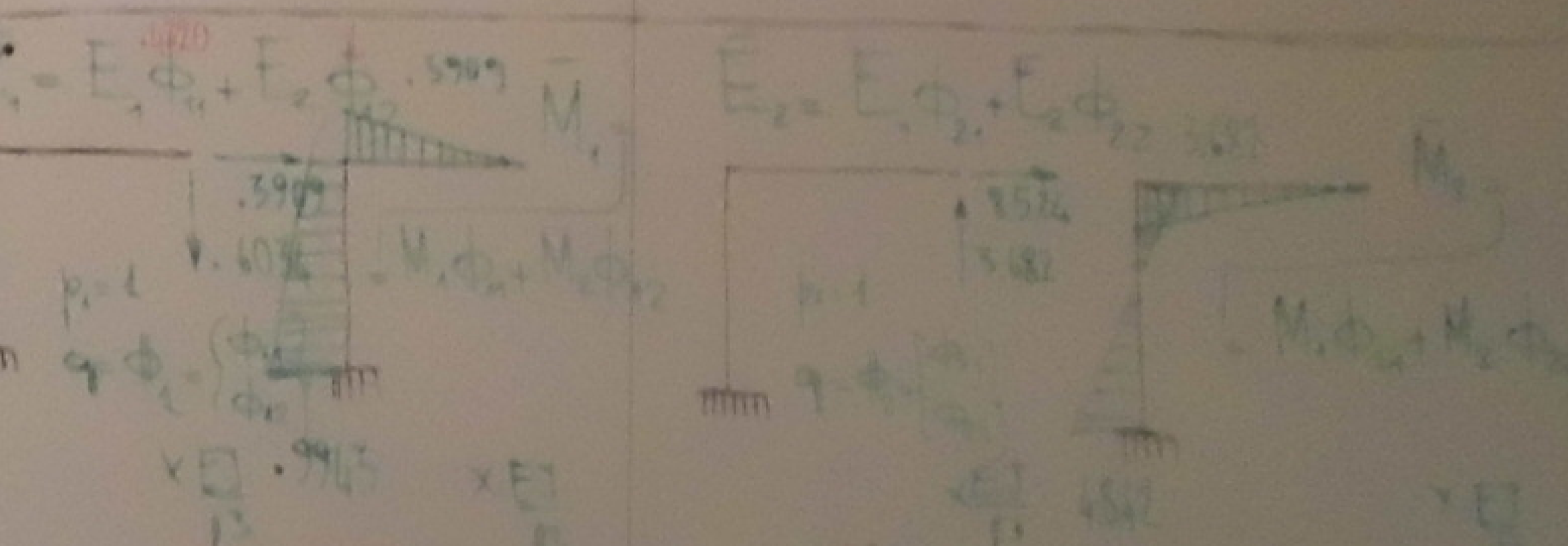

modal forces: $\mathbb{E}_i = \mathbb{K}\phi_i = \omega_i^2 \mathbb{M}\phi_i = \mathbb{F}_{Ii} = \lambda_i \frac{EJ}{m\ell^3} m \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \phi_{i1} \\ \phi_{i2} \end{Bmatrix}$

Example
E1
M1
E2
M2
48/7
18/7
30/7
12/7
6/7
q1 = 1
q2 = 1
× EJ/l³
× EJ/l²
Envelope estimates aim at obtaining maximum values
of structural response, in terms of displacements and internal actions
too conservative
modal forces