Principal coordinates (modal dynamic analysis)

- $\mathbb{M}\ddot{q} + \mathbb{K}q = \mathbb{Q}(t)$ [n MDOF system] (1)

  q: generalized coordinates (Lagrange's) representing the physical dofs

- After solution of the eigenvalue problem

$$\mathbb{K}\Phi = \mathbb{M}\Phi\Omega \quad (2)$$

  eigenvector matrix $[\phi_i]$ — eigenvalue matrix diag$[\omega_i^2]$

- Change of variables:

$$q = \Phi p = \sum_{i=1}^{n} \phi_i p_i \quad \text{principal coordinates}$$

$$p = \Phi^{-1} q \quad (\text{if } \Phi \text{ is invertible},$$

granted for linearly indep. eigenvectors, like for orthonormal eigenvectors associated to distinct eigenvalues)

To get explicit inverse relation

$$\phi_r^T \mathbb{M} q = \sum_i \underbrace{\phi_r^T \mathbb{M} \phi_i}_{M_r \delta_{ri}} p_i = M_r p_r$$

$$p_r = \frac{\phi_r^T \mathbb{M} q}{\phi_r^T \mathbb{M} \phi_r} \quad (\mathbb{K})$$

(depends on the normaliz. of ϕ_r)

$p_r \leftrightarrow \phi$

or:

$$\Phi^T \mathbb{M} q = \underbrace{\Phi^T \mathbb{M} \Phi}_{\mathcal{M}} p$$

$$\mathcal{M} = \text{diag}[M_i = \phi_i^T \mathbb{M} \phi_i]$$

$$p = \mathcal{M}^{-1} \Phi^T \mathbb{M} q$$

$$= \mathcal{K}^{-1} \Phi^T \mathbb{K} q \qquad \mathcal{K} = \text{diag}[K_i = \phi_i^T \mathbb{K} \phi_i]$$

- Decoupling of the eqns. of motion in principal coordinates

$$\mathcal{M}\ddot{p} + \mathcal{K}p = \Phi^T \mathbb{Q} = \mathbb{P}(t) \Rightarrow n \text{ eqns} \quad (3)$$

$$\ddot{p}_i + \frac{K_i}{M_i} p_i = \frac{\phi_i^T \mathbb{Q}}{M_i} = \frac{P_i(t)}{M_i} \rightarrow p_i(t) \rightarrow q(t)$$

ω_i^2

$$\ddot{u} + \frac{k}{m} u = \frac{F(t)}{m} \quad \text{SDOF}$$

Response to pure initial conditions (free vibrations) $\mathbb{Q} = 0$

i.es. $\begin{cases} q(t=0) = q_0 \\ \dot{q}(t=0) = \dot{q}_0 \end{cases}$

$$q = \sum_{1}^{n} \phi_i (A_i \sin\omega_i t + B_i \cos\omega_i t)$$

A_i, B_i amplitude coefficients linked to ϕ_i

linear combination of harmonic motions with different frequencies ω_i

Set the i.es

$$\begin{cases} q(0) = \sum_i \phi_i B_i = \Phi B = q_0 \Rightarrow B = \Phi^{-1} q_0 = p_0 \\ \dot{q}(0) = \sum_i \phi_i \omega_i A_i = \Phi\Omega A = \dot{q}_0 \Rightarrow A = (\Phi\Omega)^{-1}\dot{q}_0 = \Omega^{-1}\Phi^{-1}\dot{q}_0 = \Omega^{-1}\dot{p}_0 \end{cases}$$

$[B_i]$, $[A_i]$

Notice that if $q_0 \approx \phi_r \rightarrow B_r = 1$, $B_{i \neq r} = 0$ $(\dot{q}_0 = 0)$

vibration according to mode r only

Procedure of modal analysis

- Set the eqns of motion (1) with $\mathbb{M}$ and $\mathbb{K}$
- Solve the eigenvalue problem (2) numerical iteration method
- Shift to principal coordinates and solve the (decoupled) eqns of motion in principal coord. (3) [direct integration ... or numerical evaluation of Duhamel's integral]
- Get final response in terms of physical coordinates $q(t) = \Phi p(t)$

Principal coordinates

- $\mathbb{M}\ddot{q} + \mathbb{K}q = Q(t)$ [n MDOF system] (1)

q generalized coordinates (Lagrange's) representing the physical dofs

- Change of variables

$q = \Phi p = \sum_i \Phi_i p_i$ principal coordinates

or $\Phi^T \mathbb{M} q = \underbrace{\Phi^T \mathbb{M} \Phi}_{\mathcal{M}} p$

$\mathcal{M} = \mathrm{diag}[M_i = \Phi_i^T \mathbb{M} \Phi_i]$

- Decoupling of the eqns. of motion in principal coordinates

$\mathcal{M}\ddot{p} + \mathcal{K}p = \Phi^T Q = P(t) \Rightarrow$ n eqns (3) $\ddot{p}_i + \frac{K_i}{M_i} p_i = \frac{\Phi_i^T Q}{M_i} = \frac{P_i(t)}{M_i} \rightarrow p_i(t)$

$\ddot{u} + \frac{k}{m} u = \frac{F(t)}{m}$ SDOF

granted for linearly indep. eigenvectors, like (orthonormal) eigenvectors associated to distinct eigenvalues

Response to pure initial conditions (free vibrations)

$Q = 0$

i.c.s $\begin{cases} q(t=0) = q_0 \\ \dot{q}(t=0) = \dot{q}_0 \end{cases}$

$q = \sum_{1}^{n} \Phi_i (A_i \sin\omega_i t + B_i \cos\omega_i t)$ A_i, B_i amplitude coefficients (linked to Φ_i)

linear combination of harmonic motions with different frequencies ω_i

Set the i.c.s

$\begin{cases} q(0) = \sum_i \Phi_i B_i = \Phi B = q_0 \Rightarrow B = \Phi^{-1} q_0 = p_0 \\ \dot{q}(0) = \sum_i \Phi_i \omega_i A_i = \Phi \Omega A = \dot{q}_0 \Rightarrow A = (\Phi\Omega)^{-1} \dot{q}_0 = \Omega^{-1}\Phi^{-1}\dot{q}_0 = \Omega^{-1}\dot{p}_0 \end{cases}$

Notice that if $q_0 \approx \Phi_r \rightarrow B_r = 1$

$B_{i \neq r} = 0$ ($\dot{q}_0 = 0$)

vibration according to mode r only

Procedure of modal analysis

- Set the eqns of motion (1)
  (write $\mathbb{M}$ and $\mathbb{K}$)
- Solve the eigenvalue problem (2)
  (numerical iterative method)
- Shift to principal coordinates and solve the (decoupled) eqns of motion in principal coord. (3)
  [direct step-by-step integration e.g. Newmark or numerical evaluation of Duhamel's integral]
- Get final response in terms of physical coordinates
  $q(t) = \Phi p(t)$

Principal coordinates

- $\mathbb{M}\ddot{q} + \mathbb{K}q = Q(t)$ [a MDOF system]

or $\Phi^T \mathbb{M}\, q = \underbrace{\Phi^T \mathbb{M} \Phi}\, p$

$\mathbb{M} = \text{diag}[M_i = \phi_i^T \mathbb{M} \phi_i]$

- Decoupling of the eqns of motion in principal coordinates

$\mathbb{M}\ddot{p} + \mathcal{K}p = \Phi^T Q = P(t) \Rightarrow n$ eqns $M_i \ddot{p}_i + \frac{K_i}{M_i} p_i = \frac{\phi_i^T Q}{M_i} = \frac{P_i(t)}{M_i} \rightarrow p_i(t) \rightarrow q(t)$

$m\ddot{u} + \frac{k}{m} u = \frac{F(t)}{m}$ SDOF

Response to pure initial conditions (free vibrations) $Q = 0$

i.cs. $\begin{cases} q(t=0) = q_0 \\ \dot{q}(t=0) = \dot{q}_0 \end{cases}$

$q = \sum_{1}^{n} \phi_i (A_i \sin\omega_i t + B_i \cos\omega_i t)$ A_i, B_i amplitude coefficients (linked to ϕ_i)

linear combination of harmonic motions with different frequencies ω_i

Set the i.cs

$\begin{cases} q(0) = \sum_i \phi_i B_i = \Phi \mathbb{B} = q_0 \Rightarrow \mathbb{B} = \Phi^{-1} q_0 = \mathbb{p}_0 \\ \dot{q}(0) = \sum_i \phi_i \omega_i A_i = \Phi\Omega \mathbb{A} = \dot{q}_0 \Rightarrow \mathbb{A} = (\Phi\Omega)^{-1} \dot{q}_0 = \Omega^{-1}\Phi^{-1}\dot{q}_0 = \Omega^{-1}\dot{\mathbb{p}}_0 \end{cases}$

Notice that if $q_0 \approx \phi_r \rightarrow B_r = 1$, $B_{i \neq r} = 0$ ($\dot{q}_0 = 0$)

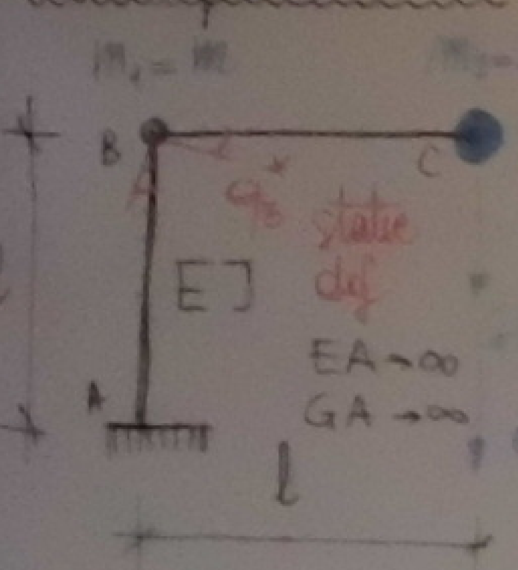
vibration according to mode r only

Procedure of modal analysis:

- Set the eqns of motion (1)
  (write $\mathbb{M}$ and $\mathbb{K}$)
- Solve the eigenvalue problem (2)
  (numerical iterative methods)
- Shift to principal coordinates and solve the (decoupled) eqns of motion in principal coord. (3)
  [direct step-by-step integration, e.g. Newmark or numerical evaluation of Duhamel's integral]
- Get final response in terms of physical coordinates
  $q(t) = \Phi\, \mathbb{p}(t)$

Example (2-DOF)
m1 = m
m2 = 2m
EJ
EA → ∞
GA → ∞
static dof
dynamical dofs
Kinetic energy
Partition of dofs
Explicitely:
static condensation of qs
Determination of stiffness coefficients (displacement method)
Influence coeff. (force method)
compliance
condensed stiffness matrix

Example (2-DOF)



$$\mathbb{M} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

- Partition of dofs (and attached of $\mathbb{E}$ and $\mathbb{K}$)

$$q = \begin{Bmatrix} q_d \\ q_s \end{Bmatrix}$$

$$\mathbb{E} = \begin{Bmatrix} \mathbb{E}_d \\ \mathbb{E}_s \end{Bmatrix} = \underbrace{\begin{bmatrix} \mathbb{K}_{dd} & \mathbb{K}_{ds} \\ \mathbb{K}_{sd} & \mathbb{K}_{ss} \end{bmatrix}}_{\substack{\mathbb{K} \\ 3\times 3}} \begin{Bmatrix} q_d \\ q_s \end{Bmatrix}$$

$$\underset{3\times 3}{\mathbb{K}} = \frac{EJ}{l^3} \begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3^* \end{array} \\ \begin{bmatrix} 12 & 0 & -6l \\ 0 & 3 & -3l \\ -6l & -3l & 7 \end{bmatrix} \end{array}$$

$$\mathbb{K} = \eta^{-1}$$

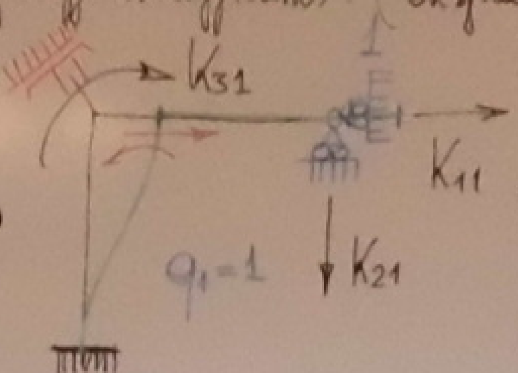
Explicitely

$$\begin{cases} \mathbb{E}_d = \mathbb{K}_{dd}\, q_d + \mathbb{K}_{ds}\, q_s \\ \mathbb{E}_s = \mathbb{K}_{sd}\, q_d + \mathbb{K}_{ss}\, q_s \Rightarrow \boxed{q_s(t) = -\mathbb{K}_{ss}^{-1}\mathbb{K}_{sd}\, q_d(t) + \mathbb{K}_{ss}^{-1}\mathbb{E}_s} \end{cases}$$

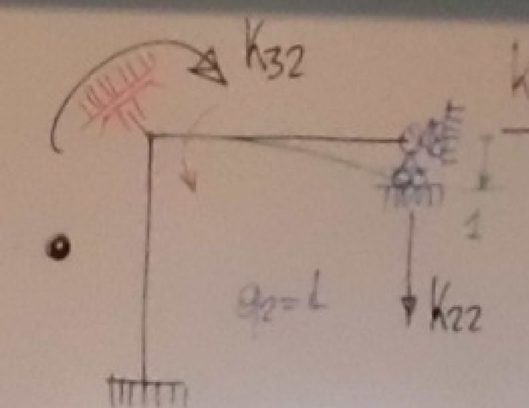
$$\boxed{\mathbb{K}_{dd}^* = \mathbb{K}_{dd} - \mathbb{K}_{ds}\mathbb{K}_{ss}^{-1}\mathbb{K}_{sd}}$$

condensed stiffness matrix linked to dynamic dofs only

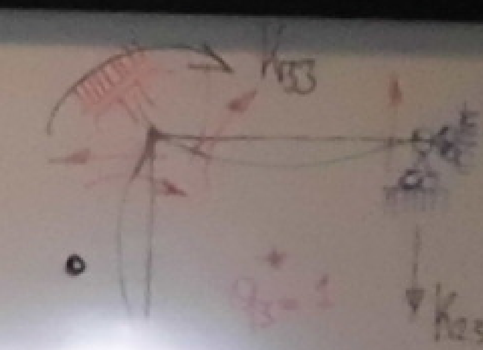
- Determination of stiffness coefficients (displacement method)



$$\begin{cases} K_{11} = 12\dfrac{EJ}{l^3} \\ K_{21} = 0 \\ K_{31} = -\dfrac{6EJ}{l^2} \end{cases}$$



$$\begin{cases} K_{12} = 0 \\ K_{22} = \dfrac{3EJ}{l^3} \\ K_{32} = \end{cases}$$



$$\begin{cases} K_{13} = -\dfrac{6EJ}{l^2} \\ K_{23} = -3\dfrac{EJ}{l^2} \\ K_{33} = \dfrac{4EJ}{l_c} + \dfrac{3EJ}{l_b} \\ \quad = 7\dfrac{EJ}{l} \end{cases}$$

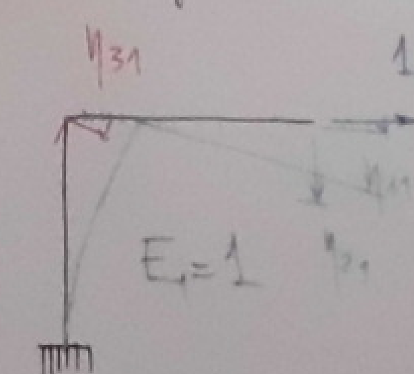
$$\mathbb{K}_{dd}^* = \frac{EJ}{l^3}\left(\begin{bmatrix} 12 & 0 \\ 0 & 3 \end{bmatrix} - \begin{Bmatrix} -6l \\ -3l \end{Bmatrix} \frac{1}{7} \{-6l \;\; -3l\} \right)$$

$$\frac{1}{7}\begin{bmatrix} 36 & 18 \\ 18 & 9 \end{bmatrix}$$

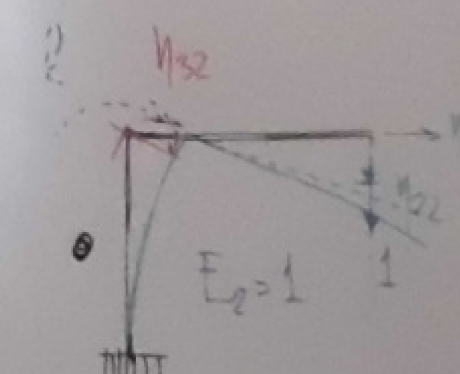
$$\eta = \frac{1}{6}\frac{l^3}{EJ} \begin{bmatrix} 2 & 3 & \frac{3}{l} \\ 3 & 8 & \frac{6}{l} \\ \frac{3}{l} & \frac{6}{l} & \frac{6}{l^2} \end{bmatrix}$$

(Compliance)

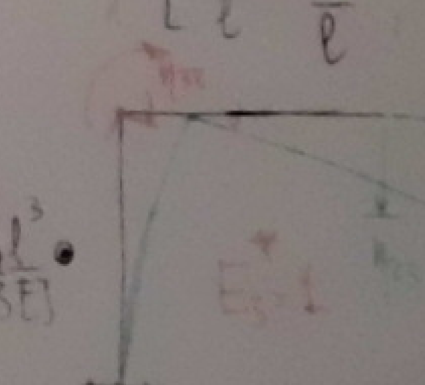
- Influence coeff. (force method)



$$\begin{cases} \eta_{11} = \dfrac{1}{3}\dfrac{l^3}{EJ} \\ \eta_{21} = \eta_{31} l = \dfrac{1}{2}\dfrac{l^3}{EJ} \\ \eta_{31} = \dfrac{1}{2}\dfrac{l^2}{EJ} \end{cases}$$



$$\begin{cases} \eta_{12} = \dfrac{1}{2}\dfrac{l^3}{EJ} \\ \eta_{22} = \eta_{32} l + \dfrac{1}{3}\dfrac{l^3}{EJ_b} = \dfrac{4}{3}\dfrac{l^3}{EJ} \\ \eta_{32} = \dfrac{l^2}{EJ} \end{cases}$$



$$\begin{cases} \eta_{13} = \dfrac{1}{2}\dfrac{l^2}{EJ} \\ \eta_{23} = \dfrac{l^2}{EJ} \\ \eta_{33} = \dfrac{l}{EJ} \end{cases}$$


(displacement method)
K11 = 12EJ/l^3
K21 = 0
K31 = -6EJ/l^2
K12 = 0
K22 = 3EJ/l^3
K32 = -3EJ/l^2
K13 = -6EJ/l^2
K23 = -3EJ/l^2
K33 = 4EJc/lc + 3EJb/lb
= 7 EJ/l
(Compliance)
Influence coeff. (force method)
η11 = 1/3 l^3/EJ
η21 = η31 l = 1/2 l^3/EJ
η31 = 1/2 l^2/EJ
η12 = 1/2 l^3/EJ
η22 = η32 l + 1/3 l^3/EJb = 4/3 l^3/EJ
η32 = l^2/EJ
η13 = 1/2 l^2/EJ
η23 = l^2/EJ
η33 = l/EJ
E1 = 1
E2 = 1
E3 = 1