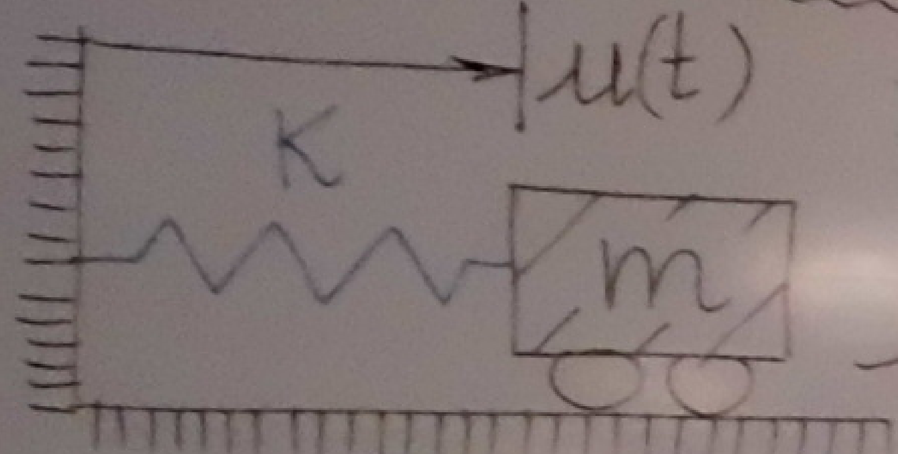
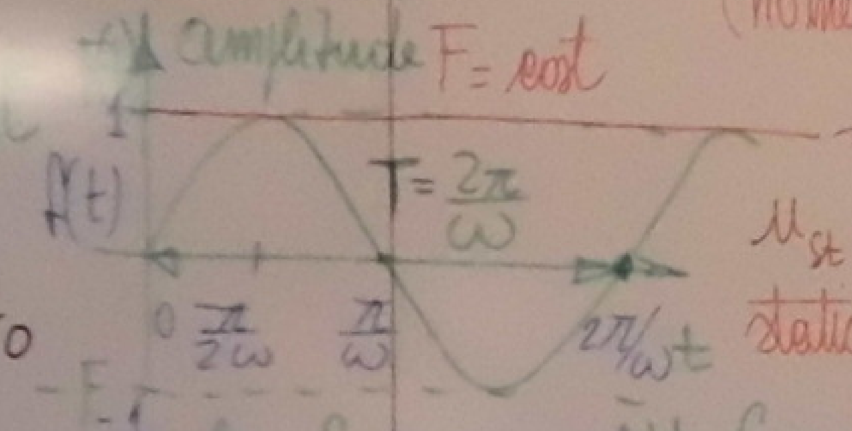
Forced vibrations (undamped, harmonic force) $\zeta = 0$



$u_0, \dot{u}_0$ i.cs.

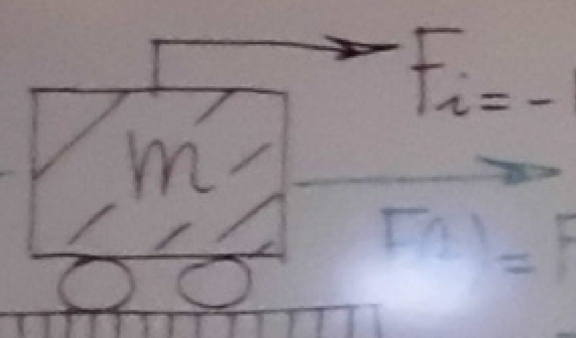


(no inertia effects)

$u_{st} = \frac{F}{k}$ static displac. by $\ddot{F} = 0$

ω angular frequency of the force

$\omega_1 = \sqrt{\frac{K}{m}}$, $F_e = Ku$



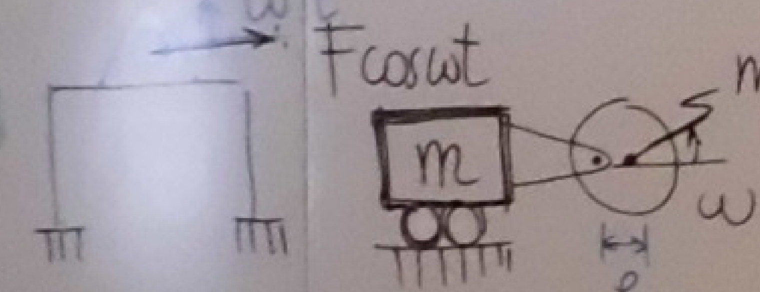
$F_i = -m\ddot{u}$

$F(t) = F\sin\omega t$

$= F f(t)$

magnitude ┘ [1]

$F = m_r e \omega^2$



Harmonic force:

- many cases are induced by harmonic force
- sample of periodic force; allows to represent general periodic actions by Fourier series
- concepts: resonance, dynamic amplification factor, phase shift
- convenient analytical treatment and attached solution

Eq of motion: u_{st}

$$m\ddot{u} + Ku = F\sin\omega t \quad \Big/ m, \ \frac{K}{m} = \omega_1^2, \ \frac{F}{m} = \omega_1^2 \frac{F}{K}$$

$$\boxed{\ddot{u} + \omega_1^2 u = \omega_1^2 u_{st} \sin\omega t}$$

$$u(t) = u_{gh}(t) + u_p(t)$$

i.cs.

general integral of the associated homogeneous eqn $= A\sin\omega_1 t + B\cos\omega_1 t$

harmonic motion with frequency ω_1 + ?

particular integral linked to $F(t)$

Seek particular integral in the form:

$u_p(t) = U\sin\omega t$ $U = ?$

$\frac{d}{dt}$ $\begin{cases} \dot{u}_p(t) = \omega U \cos\omega t \\ \ddot{u}_p(t) = -\omega^2 U \sin\omega t \end{cases}$

By substituting

$$-\omega^2 U \sin\omega t + \omega_1^2 U \sin\omega t = \omega_1^2 u_{st} \sin\omega t$$

$$(\omega_1^2 - \omega^2) U = \omega_1^2 u_{st}$$

$$U = \frac{\omega_1^2}{\omega_1^2 - \omega^2} u_{st} = \frac{1}{1 - \left(\frac{\omega}{\omega_1}\right)^2} u_{st}$$

$\beta = \frac{\omega}{\omega_1}$ frequency ratio [1]

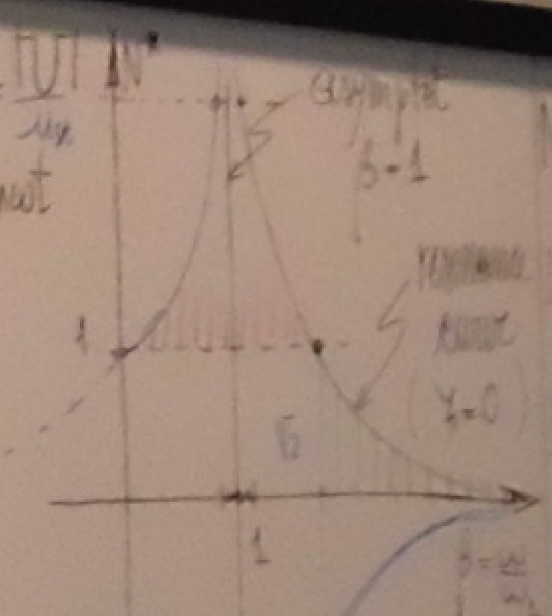
$U = \pm N u_{st}$

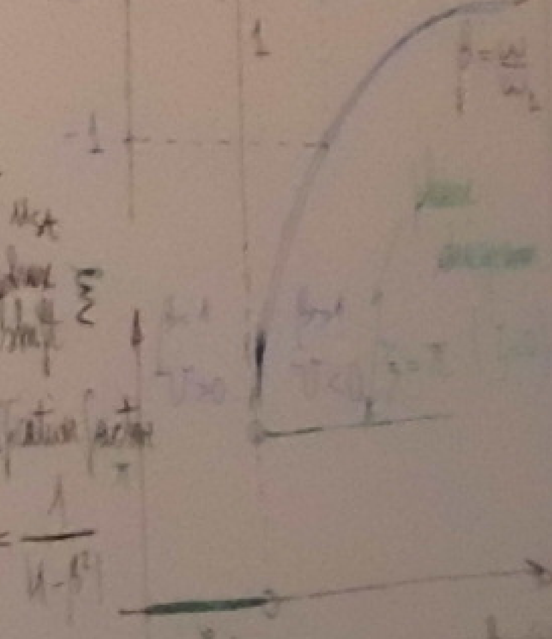
$(\beta)\ U = \frac{1}{1-\beta^2} u_{st}$, $N = \frac{|U|}{u_{st}} = \frac{1}{|1-\beta^2|}$ dynamic amplification factor

$\omega < \omega_1$, $\beta < 1$, $U > 0$ $\xi = 0$ in phase

$\omega > \omega_1$, $\beta > 1$, $U < 0$ $\xi = \pi$ in phase opposition

with the $\beta \to 1$ $U \to \infty$ force resonance

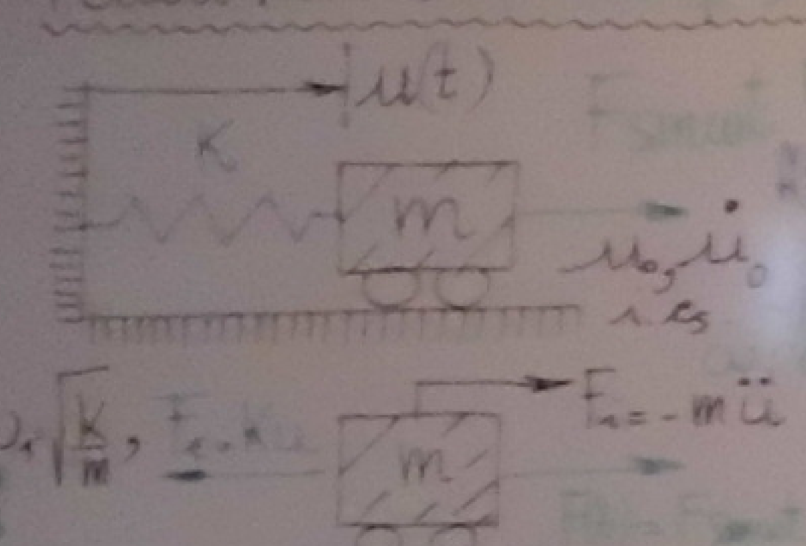




Forced vibrations

$\zeta = 0$ undamped, harmonic force

(no inertia effects)



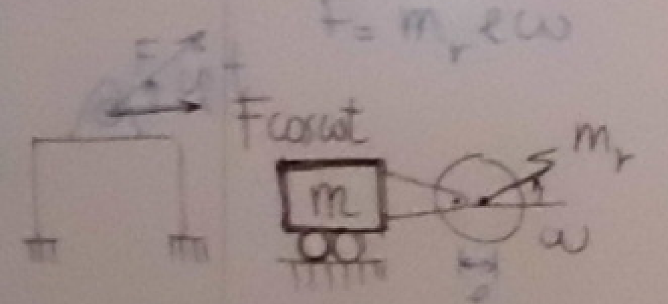
$F = \omega t$

$T = \frac{2\pi}{\omega}$

angular frequency of the force

$u_{st} = \frac{F}{K}$ static displac. by $\dot{F} = 0$

$F = m_r e \omega^2$



Harmonic force

- many cases are excited by harmonic force
- concepts: resonance, dynamic amplification factor, phase shift
- convenient analytical treatment and attached solution

Eq of motion: u_{st}

$$m\ddot{u} + \frac{Ku}{m} = \frac{F\sin\omega t}{m}\frac{K}{K}$$

$$\boxed{\ddot{u} + \omega_1^2 u = \omega_1^2 u_{st} \sin\omega t}$$

$$u(t) = u_h(t) + u_p(t)$$

i.cs

general integral of the associated homogeneous eqn $= A\sin\omega_1 t + B\cos\omega_1 t$ harmonic motion with frequency ω_1 + ?

particular integral linked to $F(t)$

Seek particular integral in the form:

$u_p(t) = U\sin\omega t$ $\quad U = ?$

$\frac{d}{dt}$: $\dot{u}_p(t) = \omega U\cos\omega t$, $\ddot{u}_p(t) = -\omega^2 U\sin\omega t$

By substituting:

$$-\omega^2 U\sin\omega t + \omega_1^2 U\sin\omega t = \omega_1^2 u_{st}\sin\omega t$$

$$(\omega_1^2 - \omega^2)U = \omega_1^2 u_{st}$$

$$U = \frac{\omega_1^2}{\omega_1^2 - \omega^2} u_{st} = \frac{1}{1 - \left(\frac{\omega}{\omega_1}\right)^2} u_{st}$$

$\beta = \frac{\omega}{\omega_1}$ frequency ratio [1]

$(\beta)\ U = \frac{1}{1-\beta^2} u_{st}$, $N = \frac{|U|}{u_{st}} = \frac{1}{|1-\beta^2|}$ dynamic amplification factor

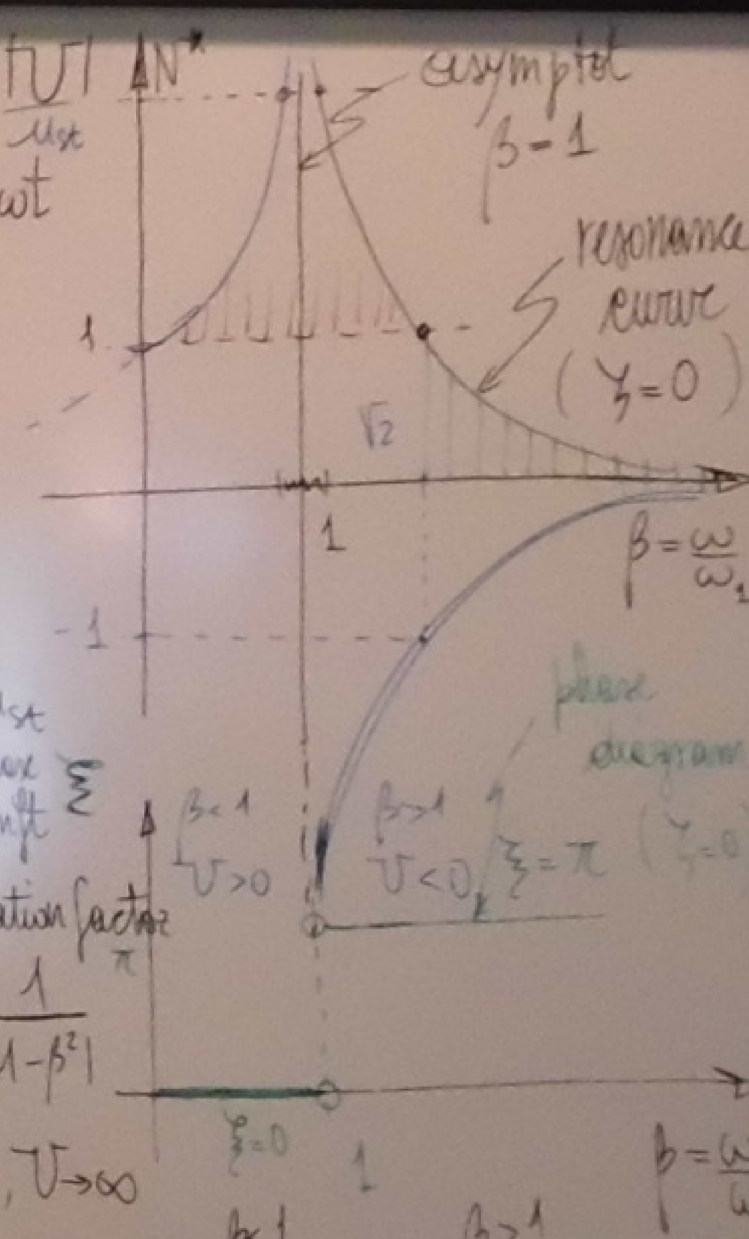
$\forall t\ N = \frac{|U|}{u_{st}}$

$U = \pm N u_{st}$ $\quad \beta < 1$, $\beta > 1$ phase shift ξ

$\omega < \omega_1$, $\beta < 1$, $U > 0$, $\xi = 0$ in phase

$\omega > \omega_1$, $\beta > 1$, $U < 0$, $\xi = \pi$ in phase opposition

with the force $\beta \to 1$, $U \to \infty$ resonance



$N \simeq 10 \Rightarrow$ inelastic response (leaving out from the hypothesis of validity of the present linear equation)

$N \geq 1$, $0 < \beta \leq \sqrt{2}$

$N \leq 1$, $\beta \geq \sqrt{2}$

$$u(t) = N u_{st}\sin(\omega t - \xi) \ \text{vs}\ = U\sin\omega t$$

$$N u_{st}(\sin\omega t\cos\xi - \cos\omega t\sin\xi) = U\sin\omega t$$

$\cos\xi = \pm 1$ $\leftarrow$ $\sin\xi = 0$: $\xi = 0$ in phase, $\xi = \pi$

$$= \frac{1}{1-\beta^2} u_{st}\sin\omega t \quad \text{"in phase"} \quad \text{phase opposition}$$

$$= \frac{1}{\sqrt{(1-\beta^2)^2}} u_{st}\sin(\omega t - \xi) = N u_{st} \sin\omega t$$

$\beta > 1$ "phase opposition"

Eq of motion
By substituting
$\forall t\ N = \frac{|U|}{u_{st}}$
$-\omega^2 U \sin\omega t + \omega_1^2 U \sin\omega t = \omega_1^2 u_{st} \sin\omega t$
$(\omega_1^2 - \omega^2) U = \omega_1^2 u_{st}$
$U = \frac{\omega_1^2}{\omega_1^2 - \omega^2} u_{st}$
particular integral linked to F(t)
$\frac{1}{1-\left(\frac{\omega}{\omega_1}\right)^2} u_{st}$
$U = \pm N u_{st}$
phase shift
frequency ratio [1]
dynamic amplification factor
$N = \frac{|U|}{u_{st}} = \frac{1}{|1-\beta^2|}$
in phase
in phase opposition
with the force
$\beta \to 1,\ U \to \infty$
resonance
asymptot
$\beta = 1$
resonance curve
$(\zeta = 0)$
$\beta = \frac{\omega}{\omega_1}$
phase diagram
$\xi = \pi$
$N \simeq 10 \Rightarrow$ inelastic response (leaving out from the hypothesis of validity of the present linear equation)
$N \geq 1,\quad 0 \leq \beta \leq \sqrt{2}$
$N \leq 1,\quad \beta \geq \sqrt{2}$
$u(t) = N u_{st} \sin(\omega t - \xi) = U \sin\omega t$
$N u_{st} (\sin\omega t \cos\xi - \cos\omega t \sin\xi) = U \sin\omega t$
$\cos\xi = \pm 1$
$\sin\xi = 0$
$\xi = 0$ in phase
$\xi = \pi$
$= \frac{1}{1-\beta^2} u_{st} \sin\omega t$
"in phase"
phase opposition
$= \frac{1}{\sqrt{(1-\beta^2)^2}} u_{st} \sin(\omega t - \xi) = \pm N u_{st} \sin\omega t$
"phase opposition"

Final solution (general integral): $[\beta \neq 1]$

$$u(t) = A\sin\omega_1 t + B\cos\omega_1 t + \frac{1}{1-\beta^2} u_{st} \sin\omega t =$$

$$\dot{u}(t) = \omega_1 A\cos\omega_1 t - \omega_1 B\sin\omega_1 t + \frac{u_{st}}{1-\beta^2}\,\omega\cos\omega t$$

Set the i.c.s:

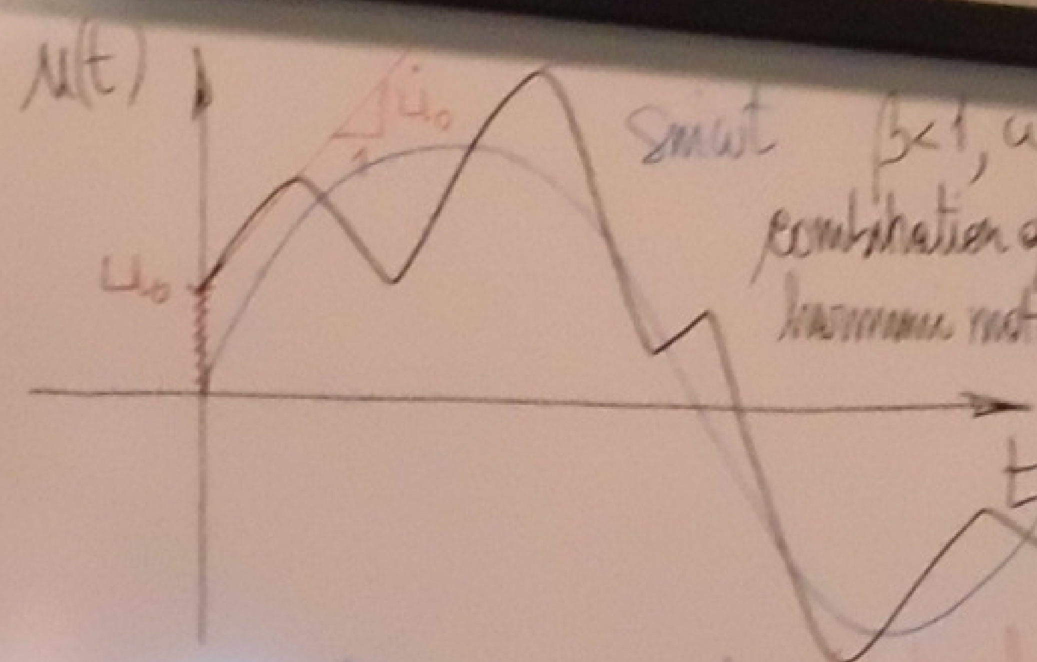
$$\begin{cases} u(0) = \boxed{B = u_0} \\ \dot{u}(0) = \omega_1 A + \dfrac{\omega}{1-\beta^2} u_{st} = \dot{u}_0 \end{cases}$$

$$\boxed{A = \frac{\dot{u}_0}{\omega_1} - \frac{\beta}{1-\beta^2} u_{st}}$$

$$= \underbrace{\left(\frac{\dot{u}_0}{\omega_1} - \frac{\beta}{1-\beta^2} u_{st}\right)\sin\omega_1 t + u_0\cos\omega_1 t}_{\text{"transient response" } \omega_1\ (\xi \neq 0)} + \underbrace{\frac{1}{1-\beta^2} u_{st} \sin\omega t}_{\text{"steady-state" response } \omega}$$

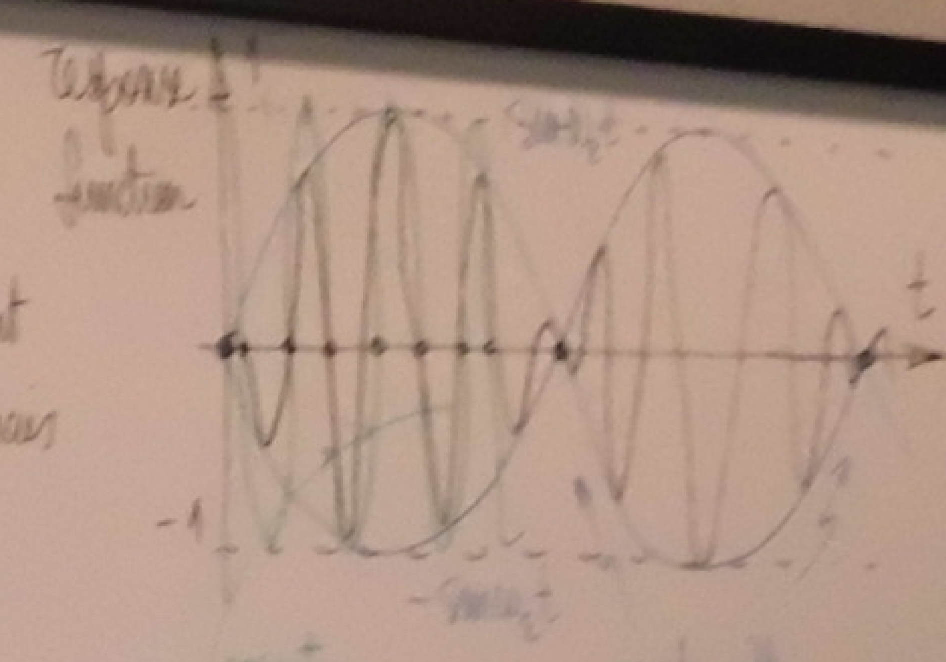
$$= \underbrace{\frac{\dot{u}_0}{\omega_1}\sin\omega_1 t + u_0\cos\omega_1 t}_{\text{response to given i.c.s (non-homogeneous)}} + \underbrace{\frac{u_{st}}{1-\beta^2}\left(\sin\omega t - \beta\sin\omega_1 t\right)}_{\text{response to harmonic force for zero i.c.s (system initially at rest)}}$$

(if $u_0 = 0$, $\dot{u}_0 = 0$ ⇒ vanishes)



$\beta < 1,\ \omega < \omega_1$
combination of two harmonic motions of different frequencies

(not a harmonic motion)



Beating $\beta \simeq 1,\ \omega \simeq \omega_1$

$\simeq 1$

$\sin\omega t - \beta\sin\omega_1 t \simeq \sin\omega t - \sin\omega_1 t$

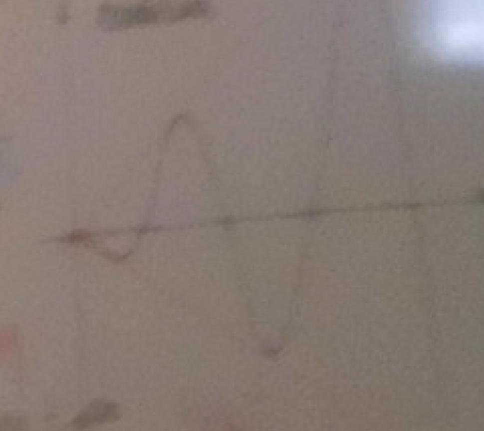
small

$\sin p - \sin q = 2\sin\frac{p-q}{2}\cos\frac{p+q}{2}$

Prostaphaeresis formulas

Amplitude of motion ... increasing in time from zero to high values

long period $T = \frac{2\pi}{\omega_\varepsilon}$



Final solution (general integral) $[\beta \neq 1]$

$$u(t) = A\sin\omega_1 t + B\cos\omega_1 t + \frac{1}{1-\beta^2} u_{st} \sin\omega t$$

$$\dot{u}(t) = \omega_1 A\cos\omega_1 t - \omega_1 B\sin\omega_1 t + \frac{u_{st}}{1-\beta^2}\omega\cos\omega t$$

Set the i.c.s

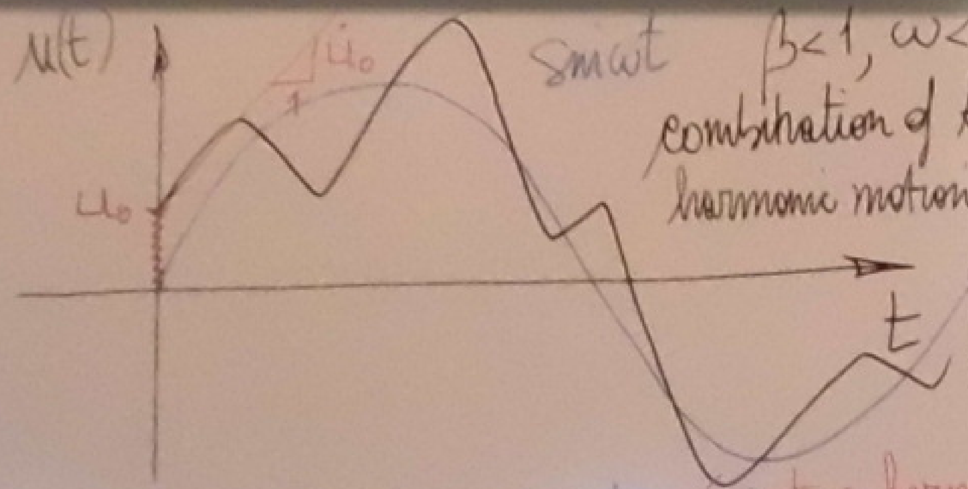
$$\begin{cases} u(0) = \boxed{B = u_0} \\ \dot{u}(0) = \omega_1 A + \frac{\omega}{1-\beta^2} u_{st} = \dot{u}_0 \end{cases}$$

$$\boxed{A = \frac{\dot{u}_0}{\omega_1} - \frac{\beta}{1-\beta^2} u_{st}}$$

$$= \underbrace{\left(\frac{\dot{u}_0}{\omega_1} - \frac{\beta}{1-\beta^2} u_{st}\right)\sin\omega_1 t + u_0\cos\omega_1 t}_{\text{"transient response" } \omega_1 \ (\xi \neq 0)} + \underbrace{\frac{1}{1-\beta^2} u_{st} \sin\omega t}_{\text{"steady-state" response } \omega}$$

$$= \underbrace{\frac{\dot{u}_0}{\omega_1}\sin\omega_1 t + u_0\cos\omega_1 t}_{\text{response to pure i.c.s (non-homogeneous)}} + \underbrace{\frac{u_{st}}{1-\beta^2}(\sin\omega t - \beta\sin\omega_1 t)}_{\text{response to harmonic force for zero i.c.s (system initially at rest)}}$$

(if $u_0 = 0$, $\dot{u}_0 = 0 \Rightarrow$ vanishes)

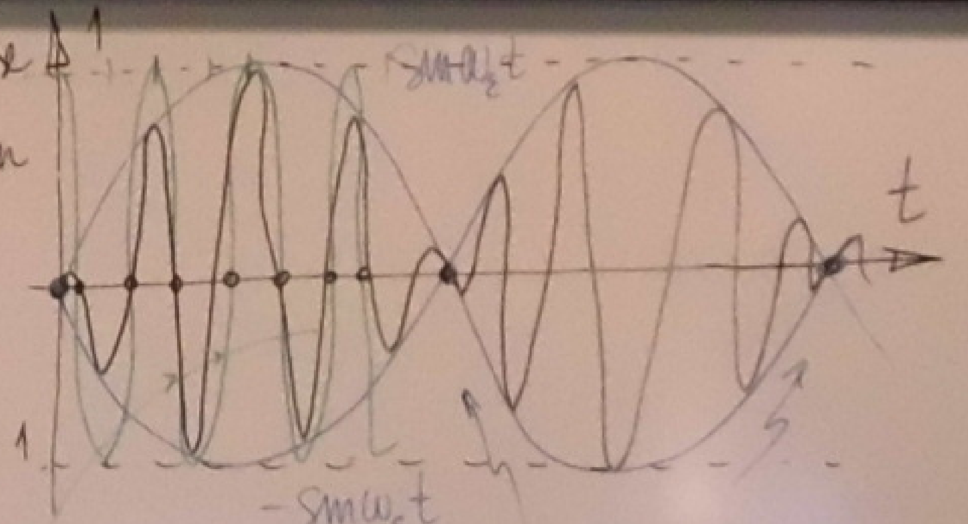


$\sin\omega t$ $\beta < 1, \omega < \omega_1$

combination of two harmonic motions of different frequencies

(not a harmonic motion)

response function



nodes with zero amplitude repeated after each T^E

Beating $\beta \simeq 1, \omega \simeq \omega_1$

$$\sin\omega t - \overset{\simeq 1}{\beta}\sin\omega_1 t \simeq \sin\omega t - \sin\omega_1 t$$

$$\sin p - \sin q = \frac{1}{2}\sin\frac{p-q}{2}\cos\frac{p+q}{2}$$

Prostaferesi's formulas

$$\simeq \frac{1}{2}\sin\underbrace{\frac{\omega-\omega_1}{2}}_{\text{small } \omega_\varepsilon}t \cdot \cos\underbrace{\frac{\omega+\omega_1}{2}}_{\sim\omega}t$$

high period $T^E = \frac{2\pi}{\omega_\varepsilon}$

Amplitude of motion dramatically varying in time from zero to high values ($N \to \infty$, $\beta \to 1$)

(to be avoided in practical applications)

Prelude to resonance

Resonance condition $\beta = 1$, $\omega = \omega_1$

$u_p(t) = U\sin\omega_1 t$ no longer works

$U \to \infty$

$\sin\omega_1 t$ is no longer a particular integral because is also included in u_{gen}

$$u_p(t) = \boxed{U}\, t\cos\omega t$$

$$\dot{u}_p(t) = U(\cos\omega t - t\omega\sin\omega t)$$

$$\ddot{u}_p(t) = U(\underbrace{-\omega\sin\omega t - \omega\sin\omega t}_{-2\omega\sin\omega t} - \omega^2 t\cos\omega t)$$

$$U(-2\omega\sin\omega t - \cancel{\omega^2 t\cos\omega t} + \cancel{\omega_1^2 t\cos\omega t}) = \omega_1 u_{st}\sin\omega t$$

$$-2\omega U = \omega^2 u_{st}$$

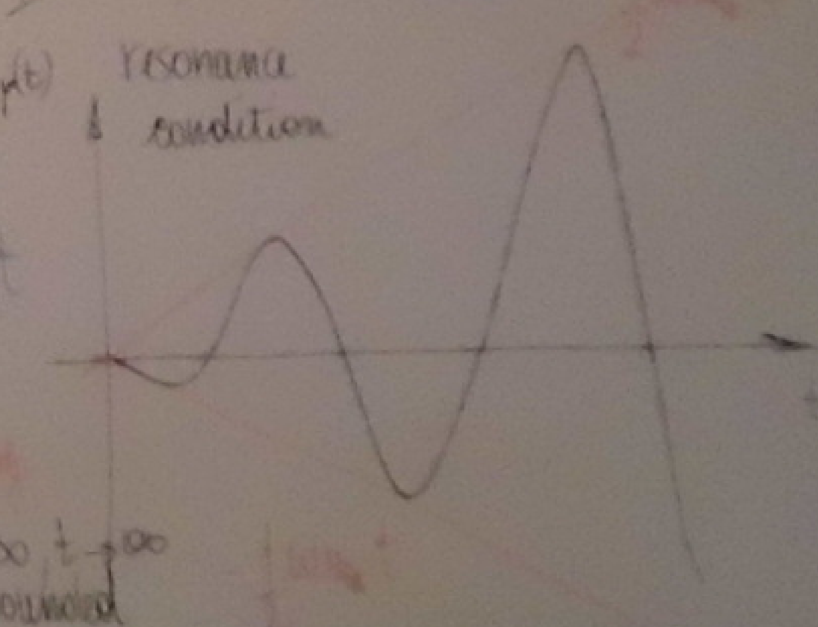
$$U = -\frac{1}{2}\omega u_{st}$$

$$u_p(t) = -\frac{1}{2}\omega u_{st}\, t\cos\omega t$$

amplitude linearly increases in time $\to \infty$ $t \to \infty$

unbounded

$u_p(t)$ resonance condition



Resonance condition β=1, ω=ω1
V→∞
u_p(t) = V sin ω1 t no longer works
sin ω1 t is no longer a particular integral because is represented in u_ge
u_p(t) = V t cos ωt
u̇_p(t) = V(cos ωt − t ω sin ωt)
ü_p(t) = V(−ω sin ωt − ω sin ωt − ω² t cos ωt)
−2ω sin ωt
V(−2ω sin ωt − ω² t cos ωt + ω1² t cos ωt) = ω1² u_st sin ωt
−2ω V = ω² u_st
V = −1/2 ω u_st
u_p(t) = −1/2 ω u_st t cos ωt
amplitude linearly increasing in time →∞, t→∞
unbounded
u_p(t)
resonance condition
1/2 ω u_st t
−1/2 ω u_st t
t
Beating β≃1, ω≃ω1
sin ωt − β sin ω1 t ≃ sin ωt − sin ω1 t
small ω_E
≃ω
long period T^E = 2π/ω_E
Amplitude of motion dramatically varying in time from zero to high values (N→∞, β→1)
(to be avoided in practical applications)
Prelude to resonance
nodes with zero amplitude repeated after each T^E
response function
sin ω_E t
−sin ω_E t
cos ωt
t
β<1, ω<ω1
combination of two harmonic motions of different frequencies
sin ωt
not a harmonic motion
"steady-state" response