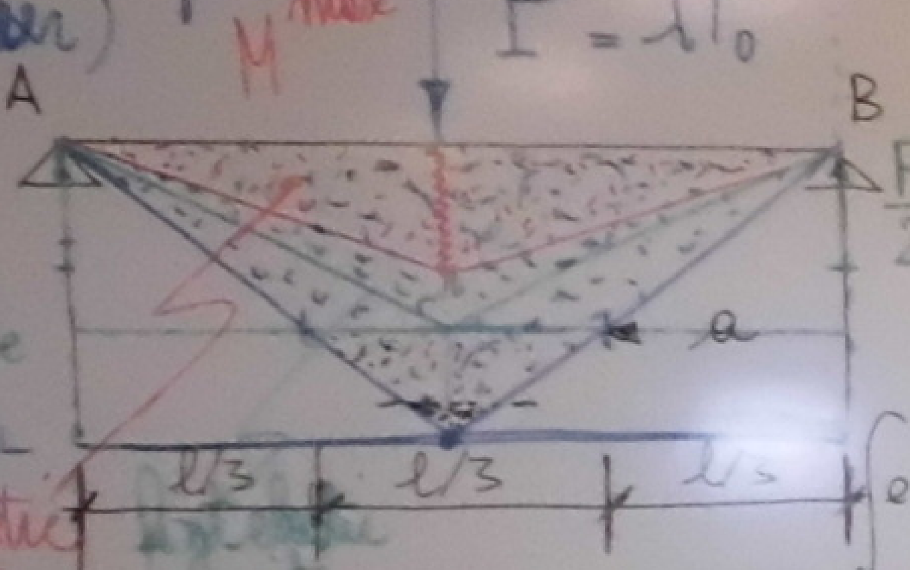
Plastic ~~hinge~~ concept and structural plastic collapse

λ load multiplier (pure number)

$P = \lambda P_0$

$P_0 = \frac{M_L}{l}$ base load



rectangular cross-section

$\alpha = \frac{W_P}{W_e} = \frac{M_L}{M_e} = \frac{3}{2}$ shape factor

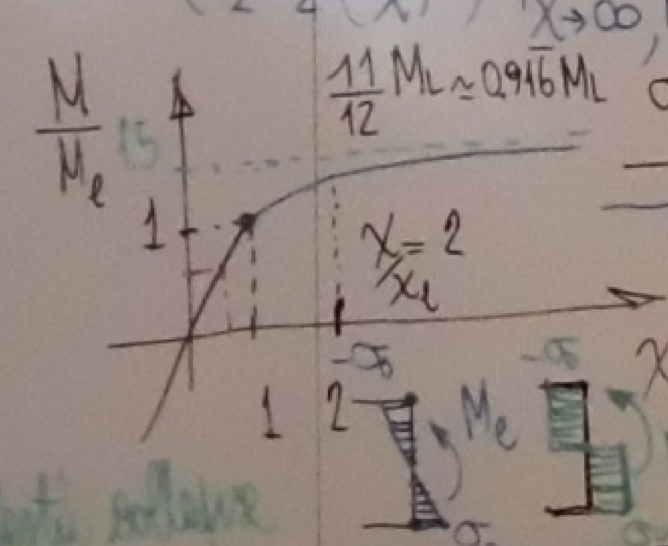
$$\begin{cases} e.\ M = EJ\chi & \text{if } \chi \le \chi_e \\ ep.\ M = M_L\left(1 - \frac{1}{3}\left(\frac{\chi_e}{\chi}\right)^2\right) = M_e\left(\frac{3}{2} - \frac{1}{2}\left(\frac{\chi_e}{\chi}\right)^2\right) & \text{if } \chi_e \le \chi < +\infty \end{cases}$$

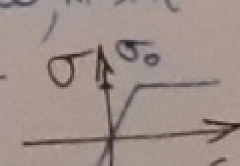
$\chi \to \infty,\ M \to M_L$

- Let us refer to a statically-determined structure where the bending moment distribution is set just by equilibrium. Specifically, the maximum moment is obtained as:

$$M^{max} = \frac{P}{2}\frac{l}{2} = \frac{Pl}{4} = \lambda \frac{M_L}{\cancel{l}}\frac{\cancel{l}}{4} = \frac{\lambda}{4} M_L$$

- One considers the amount of $P_L = \lambda_L \frac{M_L}{l}$ leading to plastic collapse





material behav. p.p.

- First yield is achieved when:

$$M^{max} = M_e \Rightarrow \frac{\lambda}{4} M_L = M_e \Rightarrow \lambda = 4 \overbrace{\frac{M_e}{M_L}}^{2/3} = \frac{8}{3}$$

Then, linear elastic range applies to $0 \le \lambda \le \frac{8}{3}$

- Going beyond that, plastification applies, to a central region that has a max extension of $\frac{l}{3}$ at $M^{max} = M_L \Rightarrow \frac{\lambda}{4}\cancel{M_L} = \cancel{M_L} \Rightarrow \lambda = 4$

$$M_a = \frac{P}{2} a = M_e \Rightarrow a = \frac{2M_e}{P} = \frac{2M_e}{\lambda \frac{M_L}{l}}$$

$$\frac{a}{l} = \frac{2}{\lambda}\frac{M_e}{M_L} = \frac{4}{3}\frac{1}{\lambda}$$

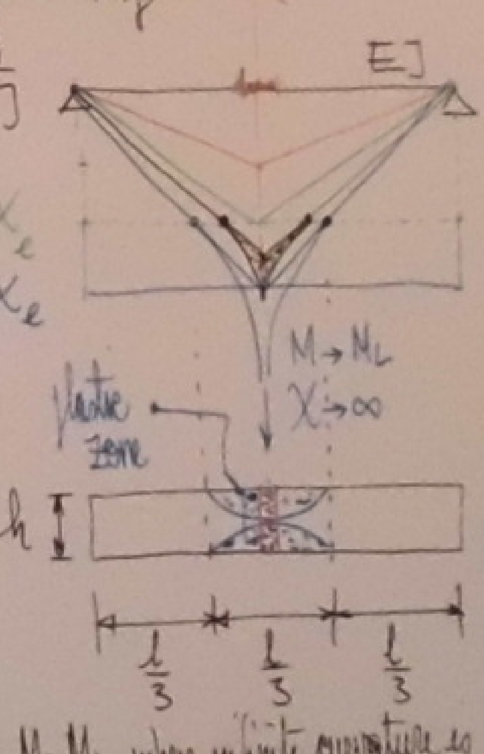
$\lambda = \frac{8}{3};\ \frac{a}{l} = \frac{1}{2}$

$\lambda = 4;\ \frac{a}{l} = \frac{1}{3}$

- Elastoplastic range applies when $\frac{8}{3} \le \lambda \le 4$ and $\lambda = \lambda_L = 4$ obviously marks plastic collapse, in the sense that infinite local curvature may be reached in the cross-section at half length

- Curvature plot in the elastic range

$\chi = \frac{M}{EJ}$

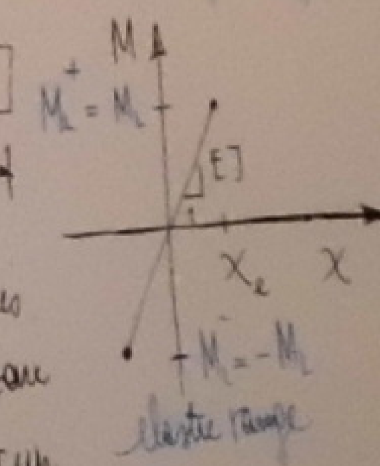


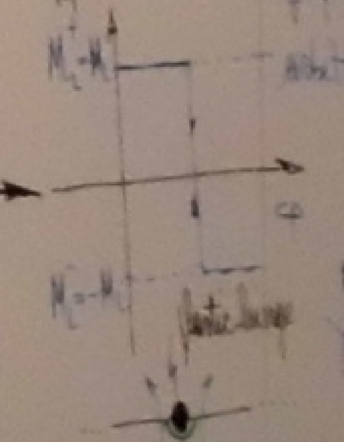
- At $M = M_L$, where infinite curvature is reached at half length, the plastic zone has diffused over $l/3$, around it, so up to a quite wide zone wrt the beam length, but high values of curvature are localized to a very narrow region around half length.

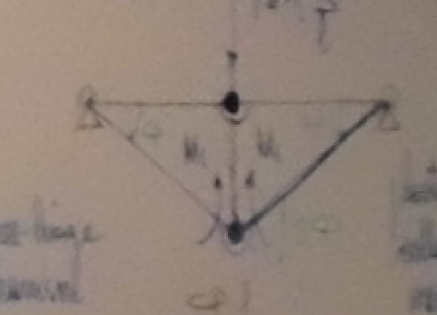
- This leads to the concept of plastic hinge, where a localized rotation may occur in the plastic range at constant $M = M_L$

$$\int \chi\, dx = \int \frac{d\varphi}{dx} dx = \Delta\varphi$$ finite rotation

- Assumption of structural behaviour







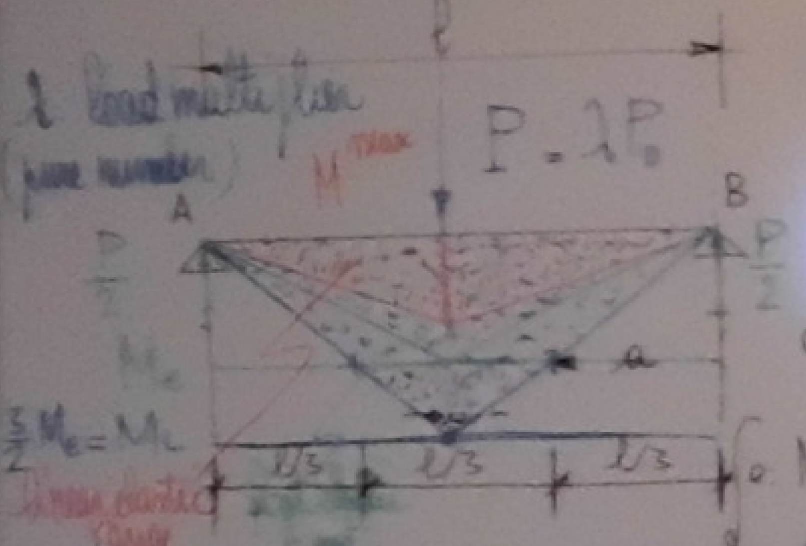
- Since $\lambda = \lambda_L = 4$ is obviously the collapse load multiplier ...

$\Rightarrow \lambda_L = 4$

Plastic hinge concept and structural plastic collapse

λ load multiplier (pure number)

$P = \lambda P_0$



$P_0 = \frac{M_L}{\ell}$ base load

rectangular cross-section

$\alpha = \frac{W_P}{W_e} = \frac{M_L}{M_e} = \frac{3}{2}$ shape factor

$\frac{3}{2} M_e = M_L$

linear elastic range

$M = EJ\chi$ if $\chi \leq \chi_e$

or $M = M_L\left(1 - \frac{1}{3}\left(\frac{\chi_e}{\chi}\right)^2\right) = M_e\left(\frac{3}{2} - \frac{1}{2}\left(\frac{\chi_e}{\chi}\right)^2\right)$ if $\chi_e \leq \chi < +\infty$

$\chi \to \infty$, $M \to M_L$

- Let us refer to a statically determined structure where the bending moment distribution is set just by equilibrium. Specifically, the maximum moment is attained as:

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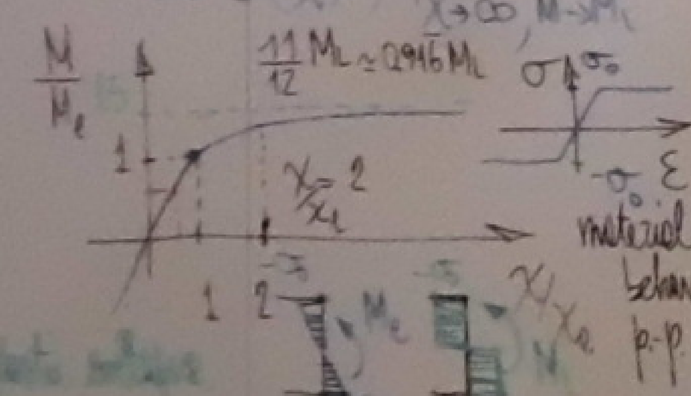

$\frac{11}{12} M_L \simeq 0.916 M_L$

$\frac{\chi}{\chi_e} = 2$

material behav. p-p

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$$M_a = \frac{P}{2}a = M_e \Rightarrow a = \frac{2M_e}{P} = \frac{2M_e}{\lambda \frac{M_L}{\ell}}$$

$$\frac{a}{\ell} = \frac{2}{\lambda}\frac{M_e}{M_L} = \frac{4}{3}\frac{1}{\lambda}$$

$\lambda = \frac{8}{3}$; $\frac{a}{\ell} = \frac{1}{2}$

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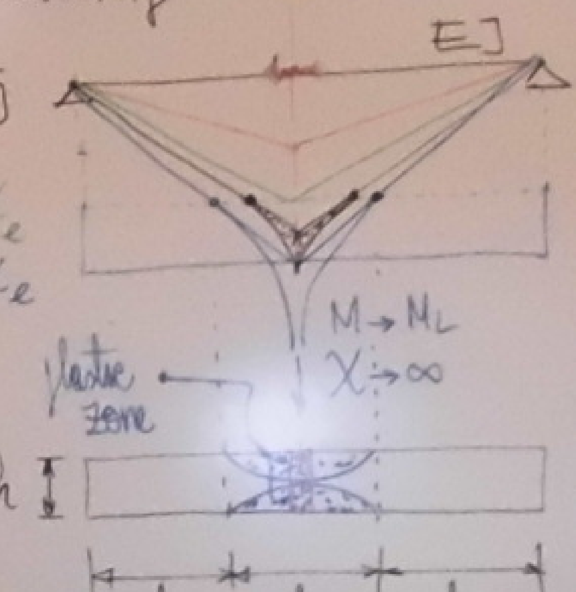
- Curvature plot

in the elastic range $\chi = \frac{M}{EJ}$

$\chi = \chi_e$

$1.5\chi_e$



$M \to M_L$

$\chi \to \infty$

plastic zone

- At $M = M_L$, where infinite curvature is reached at half length, the plastic zone has diffused over $\ell/3$, around it, so up to a quite wide zone wrt the beam length, but high values of curvature are localized to a very narrow region around half length

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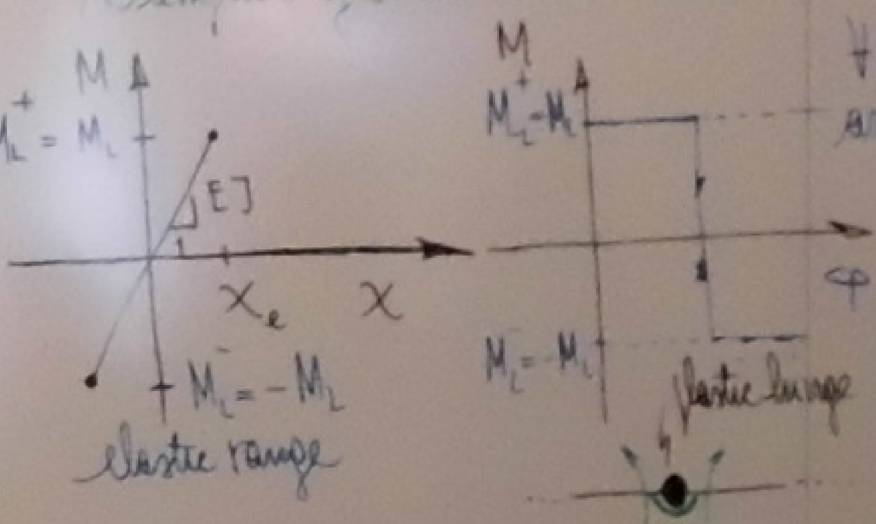
$$\int_{high\ curv.} \chi\, dx = \int \frac{d\varphi}{dx}\, dx = \Delta\varphi \quad \text{finite rotation}$$

- Assumption of structural behaviour



$M_L^+ = M_L$

$M_L^- = -M_L$

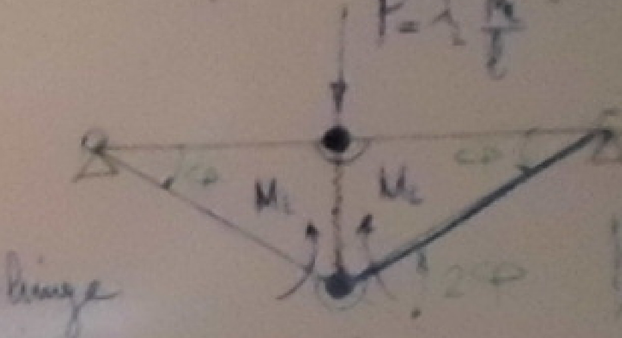
elastic range

plastic hinge

- Since $\lambda = \lambda_L = 4$ is obviously the collapse load multiplier in this case, one wonders if that value of λ_L can be evaluated with the plastic hinge concept

- Energy balance (to get λ_L)

$P = \lambda_L \frac{M_L}{\ell}$



three-hinge mechanism

plastic collapse mechanism

- $\forall\varphi$ arbitrary $\mathcal{L}_e = \lambda_L \frac{M_L}{\ell}\varphi\frac{\ell}{2} - M_L 2\varphi = 0$

$\boxed{\lambda_L = 4}$ collapse multiplier

- Otherwise $\mathcal{L}_e = \mathcal{L}_i = D \Rightarrow \lambda_L = 4$

internal plastic dissipation

work produced in the plastic hinge at constant $M = M_L$ on considered rotation in the plastic hinge

$\Delta\varphi_p = 2\varphi$

$\mathcal{L}_i = \mathcal{L}_p = D = 2M_L\varphi$

Plastic hinge concept and structural plastic collapse

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$M_e = \frac{P}{2} a = M_e \Rightarrow a = \frac{2 M_e}{P} = \frac{2 M_e}{\lambda M_L / l}$

$\frac{a}{l} = \frac{2}{\lambda} \frac{M_e}{M_L} = \frac{4}{3} \frac{1}{\lambda}$

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in the elastic range

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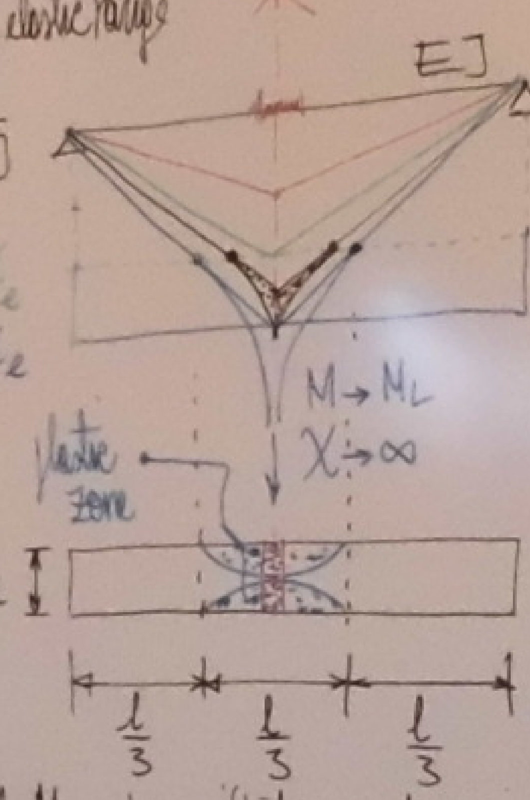

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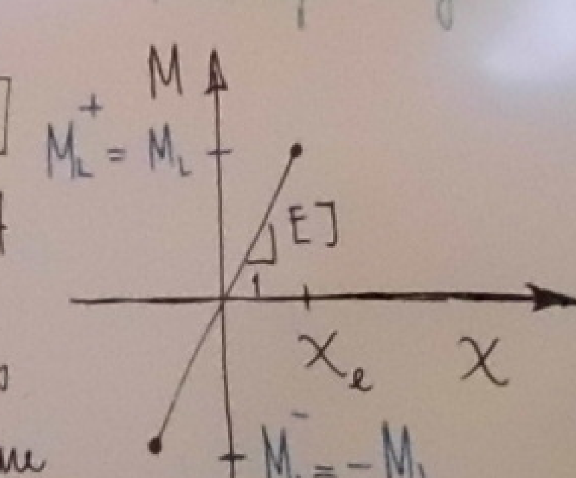
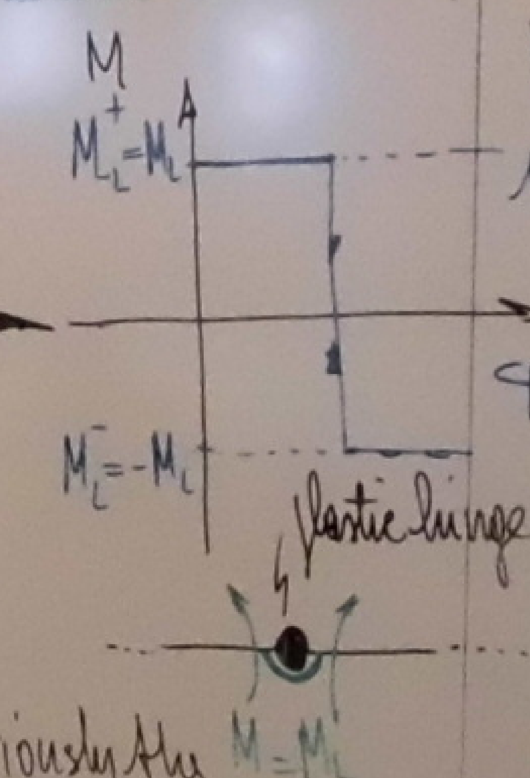
- Assumption of structural behaviour:





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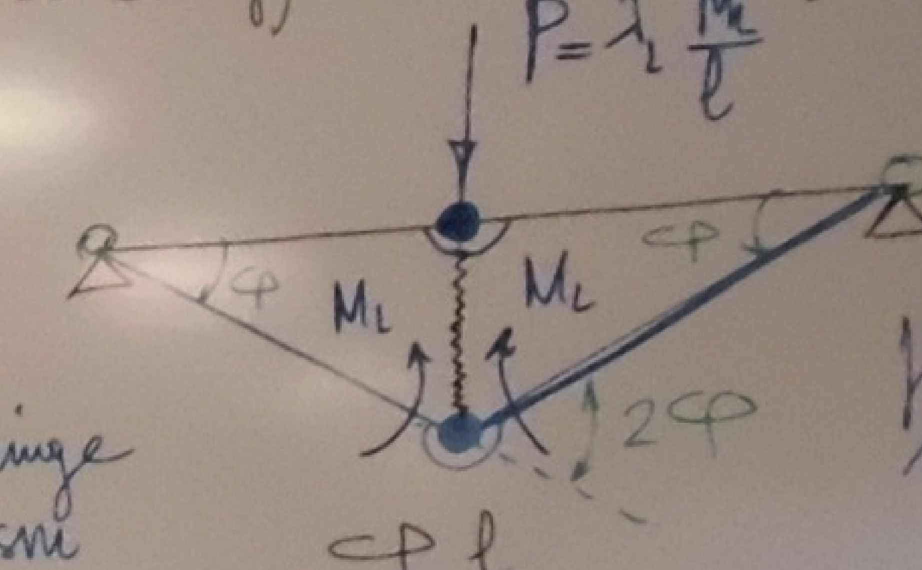

three-hinge mechanism

plastic collapse mechanism

$\forall \varphi$ arbitrary • $\mathcal{L}_e = \lambda_L \frac{M_L}{l} \cdot \varphi \frac{l}{2} - M_L 2\varphi = 0$

internal plastic dissipation

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$\mathcal{L}_e = \mathcal{L}_i = D \Rightarrow \lambda_L = 4$

M_L

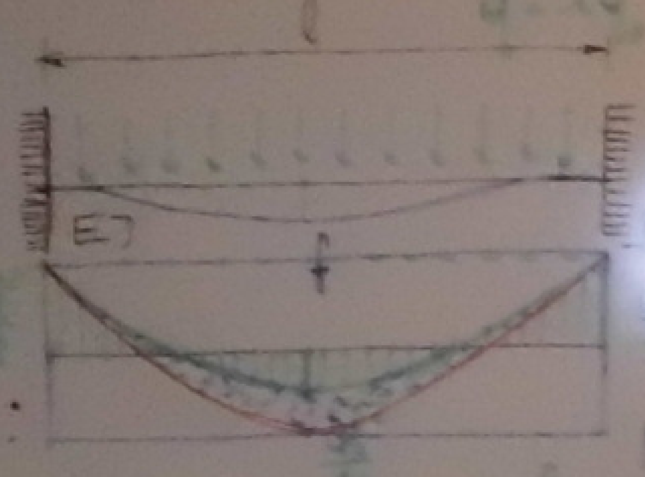
$\Delta\varphi_p = 2\varphi$

$\mathcal{L}_i = \mathcal{L}_p = D = 2 M_L \varphi$

work produced in the plastic hinge at constant $M = M_L$ for undefined (unbounded) plastic rotation in the plastic hinge

Elastoplastic evolutive analysis (in the plastic hinge concept)
EJ
elastic solution :
- First yield :
elastic range: 0 ≤ λ ≤ λe = 8
- In the plastic hinge hyp., first plastic hinges appear as: 0 ≤ λ ≤ λE = 12
- Further load is tolerated by the beam:
statically-undeterminated structure (hyperstatic).

Elasto-plastic evolutive analysis (on the plastic hinge concept)



• $f = \frac{1}{384}\frac{ql^4}{EJ} = \frac{1}{96\cdot 4}\frac{\lambda M_L}{l^2}\frac{l^4}{EJ} \Rightarrow \bar{f} = f\Big/\frac{M_L l^2}{EJ} = \frac{\lambda}{96\cdot 4}$

statically-undetermined structure (hyperstatic).

– First yield : elastic range: $0 \le \lambda \le \lambda_e = 8$

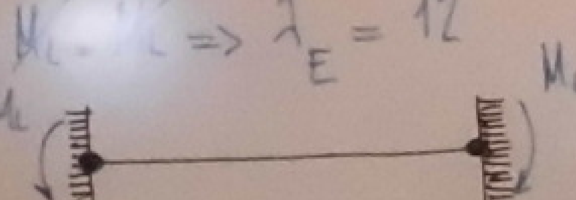
$$M^{max} = \frac{ql^2}{12} = M_e \Rightarrow \frac{\lambda}{12} M_L = M_e \Rightarrow \lambda_e = 12\frac{M_e}{M_L} = 12\,\frac{2}{3} = 8$$

$$\bar{f} = \frac{8}{96\cdot 4} = \frac{2}{96}$$

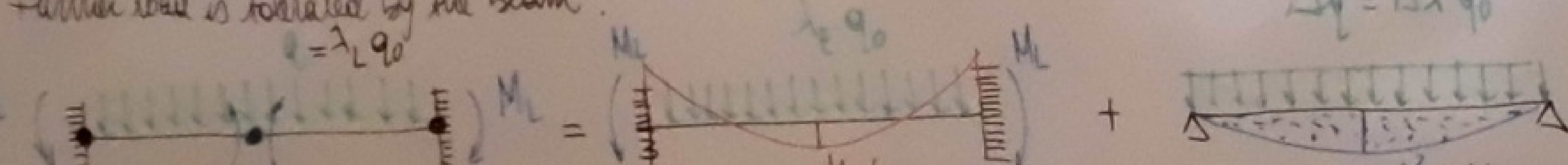
– In the plastic hinge hyp., first plastic hinges appear as: $0 \le \lambda \le \lambda_E = 12$

$$M^{max} = \frac{ql^2}{12} = M_L \Rightarrow \frac{\lambda}{12 M_L} M_L = M_L \Rightarrow \lambda_E = 12$$

$$\bar{f}_E = \frac{12}{96\cdot 4} = \frac{3}{96} = \frac{1}{32}$$



– Further load is tolerated by the beam:



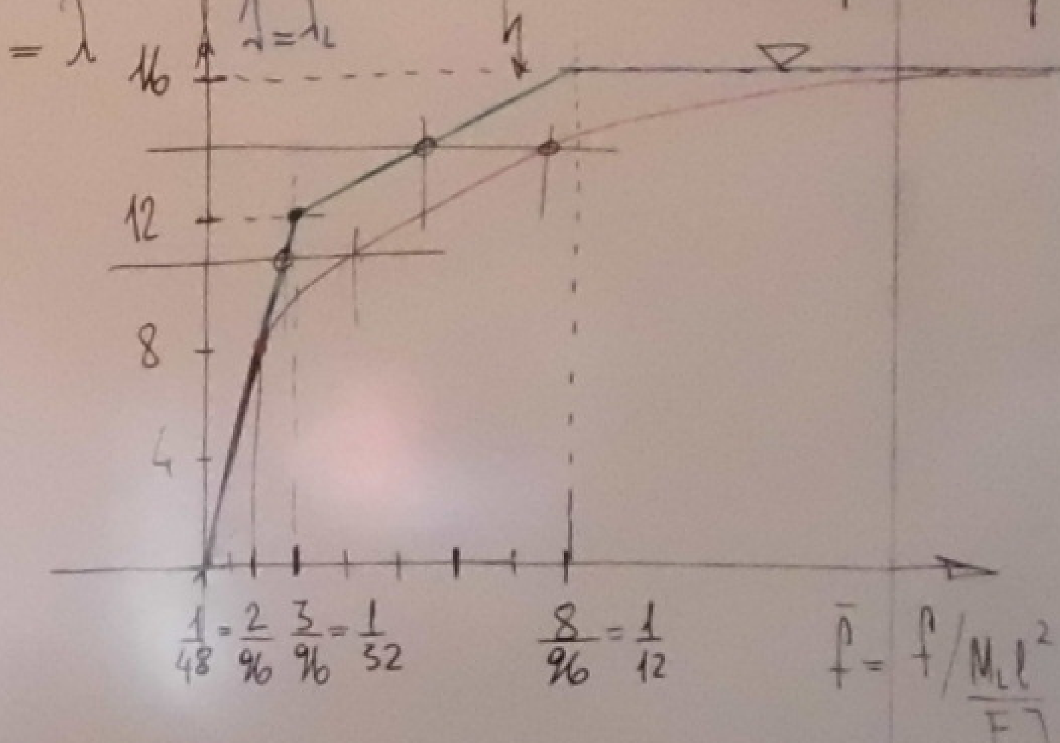
$\lambda = \lambda_L$

$$\frac{\Delta q l^2}{8} = \frac{M_L}{2}$$

$$\Delta\lambda \frac{M_L}{l^2}\frac{l^2}{8} = \frac{M_L}{2} \Rightarrow \Delta\lambda = 4$$

$$\Delta f = \frac{5}{384}\frac{\Delta q l^4}{EJ} = \frac{5}{96\cdot 4}\,4\,\frac{M_L}{l^2}\frac{l^4}{EJ} = \frac{5}{96}\frac{M_L l^2}{EJ} \Leftrightarrow \frac{8}{96}$$

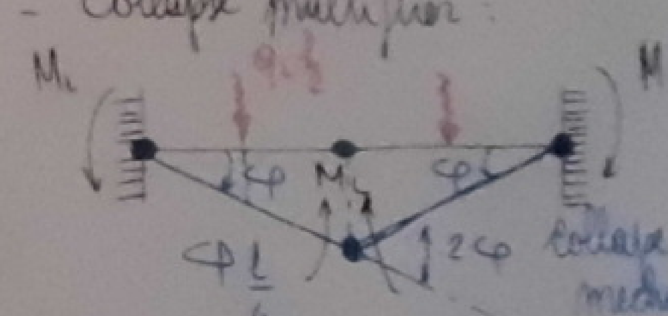
$$\frac{q}{M_L/l^2} = \lambda$$



evolutive elasto-plastic response with sequence of plastic hinge activations

piece-wise linear load deflection curve that, though underestimating the deflections, targets the final collapse load multiplier

– Collapse multiplier:



$\forall\varphi \quad \mathcal{L}_e = \lambda\, \frac{M_L}{l^2}\, \frac{l}{2}\, \varphi \frac{l}{4} = M_L\, 4\varphi = D$

$\lambda = \lambda_L = 16$ collapse load multiplier


First yield :
elastic range : 0 ≤ λ ≤ λ_e = 8
In the plastic hinge hyp., first plastic hinges appear at : 0 ≤ λ ≤ λ_E = 12
evolutive elasto-plastic response with sequence of plastic hinge activations
piece-wise linear load deflection curve
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Collapse multiplier :
collapse mechanism
λ = λ_L = 16
collapse load multiplier