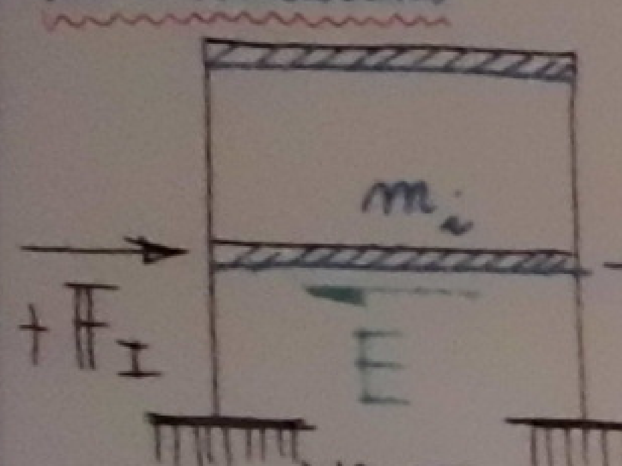
Principal modes of vibration

time invariant — mass matrix — stiffness matrix



MDOF

$Q(t)$ $\quad \underbrace{\mathbb{M}\ddot{q}}_{-\mathbb{F}_I} + \underbrace{\mathbb{K}q}_{E \leftrightarrow \mathcal{E}} = Q(t)$

n $\quad -F_{I_i} = M_{ij}\ddot{q}_j \qquad q = \{q_i\}$

M_{ij} : inertia force on i due to unit acceleration on j

- in general $\mathbb{M}$ is a non-diagonal matrix
- is linked to the kinetic energy $T = \frac{1}{2}\dot{q}^T\mathbb{M}\dot{q} = \frac{1}{2}\dot{q}_i M_{ij}\dot{q}_j > 0 \ \forall \dot{q} \neq 0$

quadratic form associated to mass matrix $\mathbb{M}$ — symmetric $M_{ji} = M_{ij}$ — Positive Definite

power $P = (-\mathbb{F}_F)^T\dot{q} = \dot{q}^T(-\mathbb{F}_I) = \dot{q}^T\mathbb{M}\ddot{q}$

$\underset{\mathbb{M}=\text{const}}{=} \frac{d}{dt}\left(\frac{1}{2}\dot{q}^T\mathbb{M}\dot{q}\right) = \frac{dT}{dt}$

Lagrange's eqns.

$\mathcal{L} = T - V$

$= T - \mathcal{E} = \frac{1}{2}\dot{q}^T\mathbb{M}\dot{q} - \frac{1}{2}q^T\mathbb{K}q$

$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = Q = \frac{\delta L(\delta q)}{\delta q}$

$\mathbb{M}\ddot{q} + \mathbb{K}q = Q$

- Mechanical systems:

the definition of T and $V = \mathcal{E}$ allows for the derivation of the n eqns. of motion

Free vibrations (undamped)

$\mathbb{M}\ddot{q} + \mathbb{K}q = 0$

$+ \ q = \phi_i \sin(\omega_i t + \psi_i)$ harmonic motion with amplitude vector ϕ_i = const and angular frequency ω_i

principal mode of vibration $\quad \cos(\omega t - \varphi)$

$(A \sin\omega t + B\cos\omega t)$

$e^{i\omega t}$

period $T_i = \frac{2\pi}{\omega_i} = \frac{1}{f_i}$

$+ \ \dot{q} = \omega\phi\cos(\omega t + \psi)$

$+ \ \ddot{q} = -\omega^2\phi\sin(\omega t + \psi) = -\omega^2 q$

By substituting:

$(-\mathbb{M}\omega^2\phi + \mathbb{K}\phi)\sin(\omega t + \psi) = 0 \ \forall t$

$$\boxed{\begin{gathered}\underset{n\times n}{\mathbb{K}}\phi_i = \omega_i^2\underset{n\times n}{\mathbb{M}}\phi_i \\ (\mathbb{K} - \omega_i^2\mathbb{M})\phi_i = 0\end{gathered}}$$

$n \times n$ generalized eigenvalue problem associated to matrices $\mathbb{M}$ and $\mathbb{K}$

standard eigenv. pb. $A x = \lambda x$

generalized eigenv. pb. $A x = \lambda B x$

Characteristics

- ω_i^2 eigenvalues

n eigenvalues

real, positive ($\mathbb{K}$ and $\mathbb{M}$ are symmetric and PD matrices)

generally distinct for real systems

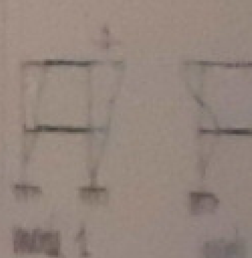
- ϕ_i eigenvectors → mode shapes

are known except for a scalar factor

ϕ_i → also $\lambda\phi_i$ is an eigenvector

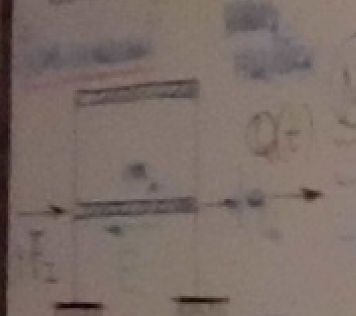
Normalized in different ways

- $\phi_i^T\phi_i = 1$
- set a single component $\phi_{ij} = 1$
- either $\phi_i^T\mathbb{M}\phi_i = 1$



Principal modes of vibration
mass matrix
stiffness matrix
$Q(t)$
$\mathbb{M}\ddot{q} + \mathbb{K}q = Q(t)$
$-F_I$
$E \leftrightarrow \mathcal{E}$
MDOF
$-F_{I_i} = M_{ij}\ddot{q}_j$
M_{ij} inertia force on i due to unit acceleration on j
- in general $\mathbb{M}$ is a non-diagonal matrix
- is linked to the kinetic energy $T = \frac{1}{2}\dot{q}^T\mathbb{M}\dot{q} = \frac{1}{2}\dot{q}_i M_{ij}\dot{q}_j > 0 \; \forall \dot{q} \neq 0$
power $P = (-F_I)^T\dot{q} = \dot{q}^T(-F_I) = \dot{q}^T\mathbb{M}\ddot{q}$
$= \frac{d}{dt}\left(\frac{1}{2}\dot{q}^T\mathbb{M}\dot{q}\right) = \frac{dT}{dt}$
- symmetric $M_{ij} = M_{ji}$
- Positive Definite
Lagrange's eqns
$\mathcal{L} = T - V$
$= T - \mathcal{E} = \frac{1}{2}\dot{q}^T\mathbb{M}\dot{q} - \frac{1}{2}q^T\mathbb{K}q$
$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = Q = \frac{\delta L(\delta q)}{\delta q}$
$\mathbb{M}\ddot{q} + \mathbb{K}q = Q$
- Mechanical systems:
the definition of T and $V = \mathcal{E}$ allows for the derivation of the n eqns. of motion
Free vibrations (undamped)
$\mathbb{M}\ddot{q} + \mathbb{K}q = 0$
$+ \; q = \phi_i \sin(\omega t + \psi_i)$
$\cos(\omega t - \varphi)$
principal mode of vibration
$(A \sin\omega t + B\cos\omega t)$
$e^{i\omega t}$
harmonic motion with amplitude vector ϕ_i = const and angular frequency ω_i
period $T_i = \frac{2\pi}{\omega_i} = \frac{1}{f_i}$
$+ \; \dot{q} = \omega\phi\cos(\omega t + \psi)$
$+ \; \ddot{q} = -\omega^2\phi\sin(\omega t + \psi) = -\omega^2 q$
By substituting:
$(-\mathbb{M}\omega^2\phi + \mathbb{K}\phi)\sin(\omega t + \psi) = 0 \; \forall t$
$\mathbb{K}_{n\times n}\phi = \omega^2\mathbb{M}_{n\times n}\phi$
$(\mathbb{K} - \omega^2\mathbb{M})\phi = 0$
n×n generalized eigenvalue problem associated to matrices $\mathbb{M}$ and $\mathbb{K}$
standard eigenv. pb.
$Ax = \lambda x$
generalized eigenv. pb.
$Ax = \lambda Bx$
Characteristics
fundamental frequency (first mode)
- ω_i^2 eigenvalues
n eigenvalues natural angular frequencies
last mode
real, positive ($\mathbb{K}$ and $\mathbb{M}$ are symmetric and PD matrices) $\Rightarrow$ ω_i are real numbers
generally distinct for real systems
Roots of the characteristic eqn
$\det(\mathbb{K} - \omega_i^2\mathbb{M}) = 0$
- ϕ_i eigenvectors $\Rightarrow$ mode shapes
are known except for a scalar factor
$\phi_i \Rightarrow$ also $\lambda\phi_i$ is an eigenvector
Normalized in different ways:
- $\phi_i^T\phi_i = 1$ unitary Euclidean norm
- set a single component $\phi_{ij} = 1$ (e.g. last floor, or first floor)
- either $\phi_i^T\mathbb{M}\phi_i = 1$ unitary modal mass or $\phi_i^T\mathbb{K}\phi_i = 1$ unitary modal stiffness
mode 1
mode 2
For $\omega_r^2 \neq \omega_s^2$
two distinct eigenvalues
$\Rightarrow$ the associated eigenvectors are orthonormal
$\mathbb{K}\phi_r = \omega_r^2\mathbb{M}\phi_r$
$\mathbb{K}\phi_s = \omega_s^2\mathbb{M}\phi_s$
then
$\phi_r^T\mathbb{M}\phi_s = 0$
$\phi_r^T\mathbb{K}\phi_s = 0$
$s \neq r$

Principal modes of vibration



$Q(t)$ $M\ddot{q} - Kq = Q(t)$

$E \leftrightarrow \mathcal{E}$

$F_2 = M_{..}\ddot{q}$

- in general M is a non-diagonal matrix

$T = \frac{1}{2}\dot{q}^T M \dot{q} > 0$

$L = T - V = T - \mathcal{E} = \frac{1}{2}\dot{q}^T M \dot{q} - \frac{1}{2} q^T K q$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q = \frac{\delta L(\delta q)}{\delta q}$

$M\ddot{q} + Kq = Q$

- the definition of T and $V = \mathcal{E}$ allows for the derivation of the eq. of motion

Free vibrations (undamped)

$M\ddot{q} + Kq = 0$

+ $q = \phi_i \sin(\omega_i t + \psi_i)$ harmonic motion with amplitude vector ϕ_i = const and angular frequency ω_i

principal mode of vibration

$(A \sin\omega t + B\cos\omega t)$

$e^{i\omega t}$

period $T_i = \frac{2\pi}{\omega_i} = \frac{1}{f_i}$

+ $\dot{q} = \omega\phi\cos(\omega t + \psi)$

+ $\ddot{q} = -\omega^2\phi\sin(\omega t + \psi) = -\omega^2 q$

By substituting:

$(-M\omega^2\phi + K\phi)\sin(\omega t + \psi) = 0 \quad \forall t$

$$\boxed{\begin{gathered} \underset{n\times n}{K}\phi = \omega^2 \underset{n\times n}{M}\phi \\ (K - \omega^2 M)\phi = 0 \end{gathered}}$$

$n \times n$ generalized eigenvalue problem associated to matrices M and K

standard eigenv. pb.: $Ax = \lambda x$

generalized eigenv. pb.: $Ax = \lambda Bx$

Characteristics

- ω_i^2 eigenvalues $\Rightarrow$ $\omega_1 \le \omega_2 \le \dots \omega_i \le \dots \le \omega_n$ (fundamental frequency (first mode) … last mode)

n eigenvalues natural angular frequencies

real, positive (K and M are symmetric and PD matrices) $\Rightarrow$ ω_i are real numbers

generally distinct for real systems

- ϕ_i eigenvectors $\Rightarrow$ mode shapes

are known except for a scalar factor

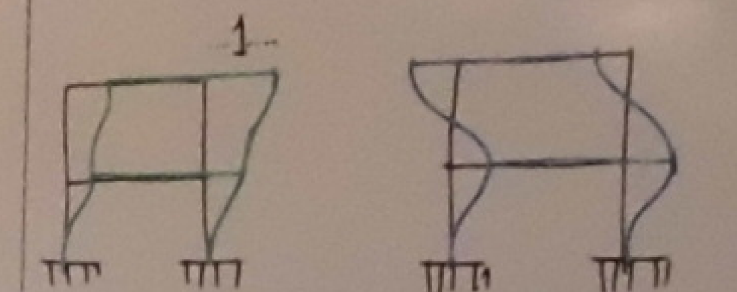
$\phi_i \Rightarrow$ also $\lambda\phi_i$ is an eigenvector

Normalized in different ways:

- $\phi_i^T\phi_i = 1$ unitary Euclidean norm $\quad \phi_i \to \frac{\phi_i}{\|\phi_i\|}$
- set a single component $\phi_{1j} = 1$ (e.g. last floor, or first floor)
- either $\phi_i^T M \phi_i = 1$ unitary mode mass or $\phi_i^T K \phi_i = 1$ unitary mode stiffness

Roots of the characteristic eqn

$\det(K - \omega_i^2 M) = 0$



mode 1   mode 2

For $\omega_r^2 \neq \omega_s^2$

two distinct eigenvalues

$\Rightarrow$ the associated eigvectors are orthonormal (wrt M, K)

$\phi_s^T\,(K\phi_r = \omega_r^2 M\phi_r)$

$\phi_r^T\,(K\phi_s = \omega_s^2 M\phi_s)$ $\quad -$

$\phi_s^T K\phi_r - \phi_s^T K^T\phi_r = \omega_r^2\phi_s^T M\phi_r + $

$K^T = K \quad -\omega_s^2\phi_s^T M^T\phi_r$

$0 = \underbrace{(\omega_r^2 - \omega_s^2)}_{\neq 0}\phi_s^T M\phi_r \quad M^T = M$

Then $\boxed{\begin{gathered}\phi_s^T M\phi_r = 0 \\ \phi_s^T K\phi_r = 0\end{gathered}}$ $s \neq r$

- Given the orthonormal property:

- $\phi_r^T \mathbb{M} \phi_s = M_r \delta_{rs}$ — $M_r = \phi_r^T \mathbb{M} \phi_r$ if $r=s$ modal mass (>0); 0 if $r \neq s$ — dependent on eigenvector normalization
- $\phi_r^T \mathbb{K} \phi_s = K_r \delta_{rs}$ — $K_r = \phi_r^T \mathbb{K} \phi_r$ if $r=s$ modal stiffness (>0); 0 if $r \neq s$

- Then, since

$$\phi_i^T (\mathbb{K} \phi_i - \omega_i^2 \mathbb{M} \phi_i) \Rightarrow \omega_i^2 = \frac{\frac{1}{2}\phi_i^T \mathbb{K} \phi_i}{\frac{1}{2}\phi_i^T \mathbb{M} \phi_i} = \frac{K_i}{M_i} = \frac{\mathcal{E}_i^+}{T_i^+} > 0$$ Rayleigh ratio

(recall SDOF $\omega_1^2 = \frac{K}{m}$)

- independent from the eigenvector normalization
- estimate of approx. frequency $\tilde{\omega}_i$ from approx of ϕ_i

- By introducing:

$\underset{n \times n}{\Phi} = [\phi_1 \vdots \phi_2 \vdots \cdots \vdots \phi_i \vdots \cdots \vdots \phi_n]$ eigenvector matrix (non singular if eigenvalues are distinct, as typical of real systems)

$\Omega^2 = \mathrm{diag}[\omega_i^2]$ eigenvalue matrix

one may write the eigv. problem as $\mathbb{K}\Phi = \mathbb{M}\Phi\Omega^2$

$$\boxed{\begin{array}{c} \underset{n\times n}{\mathbb{K}} \phi_i = \omega_i^2 \underset{n \times n}{\mathbb{M}} \phi_i \\ (\mathbb{K} - \omega_i^2 \mathbb{M}) \phi_i = 0 \end{array}}$$

By multiplying from the left by Φ^T

$$\underbrace{\Phi^T \mathbb{K} \Phi}_{\mathcal{K}} = \underbrace{\Phi^T \mathbb{M} \Phi}_{\mathcal{M}} \Omega^2$$

modal stiffness matrix $\mathcal{K} = \mathrm{diag}[K_i]$

modal mass matrix $\mathcal{M} = \mathrm{diag}[M_i]$

- Then

$$\Omega^2 = \mathcal{M}^{-1} \mathcal{K} = \mathrm{diag}\left[\omega_i^2 = \frac{K_i}{M_i}\right]$$

clearly generalizing the SDOF concept

- Generalized eigenvalue problem $\Rightarrow$ standard form

$$\overbrace{\mathbb{K}^{-1} \mathbb{M}}^{G} \phi_i = \frac{1}{\omega_i^2} \phi_i$$

$$\underbrace{\mathbb{M}^{-1} \mathbb{K}}_{H} \phi_i = \omega_i^2 \phi_i$$

numerical solution of the eig. pb.

iterative methods $\rightarrow$ first mode; $\rightarrow$ last mode

- Shift to principal coordinates

lagrangian $q^{(t)} = \sum_{i=1}^{n} \phi_i p_i = \phi_1 p_1 + \phi_2 p_2 + \cdots + \phi_i p_i + \cdots + \phi_n p_n$ (physical meaning)

$= \Phi p^{(t)}$

Inverse relation if Φ is non-singular $p = \Phi^{-1} q$

- Tacking back the eqs. of motion and shifting to principal coordinates

$$\mathbb{M}\ddot{q} + \mathbb{K} q = Q$$

$$\Phi^T \quad \mathbb{M}\Phi\ddot{p} + \mathbb{K}\Phi p = Q$$


Given the orthonormal property:
$\Phi_r^T \mathbb{M} \Phi_s = M_r \delta_{rs}$
$\Phi_r^T \mathbb{K} \Phi_s = K_r \delta_{rs}$
if r = s modal mass
if r ≠ s
dependent on eigenvector normalization
if r = s modal stiffness
recall SDOF $\omega_i^2 = \frac{K}{m}$
Then, since
> 0 Rayleigh ratio
By introducing
independent from the eigenvector normalization
$\Phi = [\phi_1 \, \phi_2 \, \dots \, \phi_i \, \dots \, \phi_n]$ eigenvector matrix
n×n
$\Omega = \mathrm{diag}[\omega_i^2]$
one may write the eigen. problem as $\mathbb{K}\Phi = \mathbb{M}\Phi\Omega^2$
By multiplying from the left by Φ^T
$\Phi^T \mathbb{K} \Phi = \Phi^T \mathbb{M} \Phi \, \Omega^2$
modal stiffness matrix
= diag [K_i]
= diag [M_i]
modal mass matrix
Then
$\Omega^2 = \mathcal{M}^{-1} \mathcal{K}$
$= \mathrm{diag}\left[\omega_i^2 = \frac{K_i}{M_i}\right]$
clearly generalizing the SDOF concept
$(\mathbb{K} - \omega_i^2 \mathbb{M})\phi_i = 0$
$\mathbb{K}^{-1}\mathbb{M}\,\phi_i = \frac{1}{\omega_i^2}\phi_i$
$\mathbb{M}^{-1}\mathbb{K}\,\phi_i = \omega_i^2\phi_i$
numerical
solution of the eign pb.
iterative methods
first mode
last mode
Shift to principal coordinates
Lagrangian
(physical meaning)
Inverse relation of Φ is non-singular
Tacking back the eqns of motion and shifting to principal coordinates
$\mathbb{M}\ddot{q} + \mathbb{K}q = Q$
recall SDOF
n decoupled eqns of motion in principal coordinates p_i

– Shift to principal coordinates
lagrangian q(t) = Σ Φ_i p_i = Φ_1 p_1 + Φ_2 p_2 + ... + Φ_i p_i + ... + Φ_n p_n
(physical meaning)
= Φ p(t)
p = {p_i} mathematical meaning
coeff. linear combination of Φ_i
Inverse relation if Φ is non-singular : p = Φ^-1 q
– Tacking back the eqns. of motion and shifting to principal coordinates
M q̈ + K q = Q
Φ^T (M Φ p̈ + K Φ p = Q
(Φ^T M Φ) p̈ + (Φ^T K Φ) p = Φ^T Q = P
M diagonal K
p(t)
recall SDOF
M_i p̈_i + K_i p_i = P_i = Φ_i^T Q
n decoupled eqns. of motion in principal coordinates p_i
Φ^T K Φ = Φ^T M Φ Ω^2
modal mass matrix
modal stiffness matrix
= diag [K_i]
= diag [M_i]
– Then
Ω^2 = M^-1 K
= diag [ω_i^2 = K_i / M_i]
clearly generalizing the SDOF concept
Rayleigh ratio
iterative methods
→ first mode
→ last mode