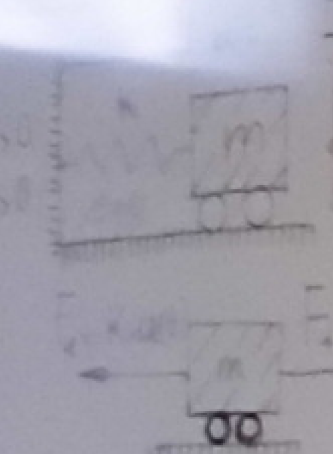

(Undamped) free vibrations
Initial conditions
m > 0
K > 0
u(0) = u0
u̇(0) = u̇0
dynamic equilibrium (according to d'Alembert)
K u = - m ü
natural angular frequency
harmonic motion equation
(2nd-order differential eqn.)
Seek the solution in the form
By substituting into eq. of motion
characteristic equation associated to differential eqn.
algebraic
2 complex conjugate roots purely imaginary
ARGAND plane
General integral
real response
B̃ = Ā (complex)
complex conjugate
angular velocity
Euler representation of complex number
A, B real constants

Undamped free vibrations
Initial conditions
$u(0) = u_0$
$\dot{u}(0) = \dot{u}_0$
$m > 0$
$k > 0$
$F_e = k\,u(t)$
$F_i = -m\,\ddot{u}(t)$
dynamic equilibrium (according to d'Alembert)
$k\,u = -m\,\ddot{u}$
$m\ddot{u} + k u = 0 = F(t)$
natural angular frequency
$\ddot{u} + \omega_1^2 u = 0$
$\omega_1 = \sqrt{\frac{k}{m}}$
$[\omega_1] = \frac{1}{[t]}$
Seek the solution in the form
$u(t) = e^{\lambda t}$
$\dot{u} = \lambda e^{\lambda t}$
$\ddot{u} = \lambda^2 e^{\lambda t} = \lambda^2 u$
By substituting into eq of motion
$\lambda^2 u + \omega_1^2 u = 0$
$(\lambda^2 + \omega_1^2)\, e^{\lambda t} = 0 \quad \forall t$
$\lambda^2 + \omega_1^2 = 0$
$\lambda_{1,2} = \pm i\omega_1$
2 complex conjugate roots
General integral
$u(t) = \tilde{A} e^{i\omega_1 t} + \tilde{B} e^{-i\omega_1 t}$
$\tilde{B} = \overline{A}$ (complex)
real response
$e^{i\vartheta} - e^{-i\vartheta} = 2 i \sin\vartheta$
$e^{i\vartheta} + e^{-i\vartheta} = 2\cos\vartheta$
$\sin\vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$
$\cos\vartheta = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2}$
general integral
$\frac{\dot{u}_0}{\omega_1}\sin\omega_1 t + u_0 \cos\omega_1 t = u(t)$
A, B real constants
$\omega_1 t \Rightarrow$ angle ; $\omega_1 \Rightarrow \frac{\text{angle}}{\text{time}}$ angular velocity
complex conjugate
$e^{i\omega_1 t} = e^{-i\omega_1 t}$
$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$; $e^{-i\vartheta} = \cos\vartheta - i\sin\vartheta = \overline{e^{i\vartheta}}$
ARGAND plane
Euler representation of complex number
Im
Re
$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
By imposing the i.c.s
$u(0) = B = u_0$
$\dot{u}(0) = \omega_1 A = \dot{u}_0$
$A = \frac{\dot{u}_0}{\omega_1}$
$-\omega_1^2 u = -\omega_1^2 A \sin\omega_1 t - \omega_1^2 B\cos\omega_1 t = \ddot{u}(t)$
HARMONIC MOTION
$\sin\omega_1 t = \sin[\omega_1(t + T_1)]$
$\omega_1 T_1 = 2\pi$
$T_1 = \frac{2\pi}{\omega_1}$
$f_1 = \frac{1}{T_1} = \frac{\omega_1}{2\pi}$
$\omega_1 = 2\pi f_1$
angular frequency
natural cyclic frequency
$u(t)$
$R \sin\omega_1 t$
$+R$
$-R$
u_0
$2R$
natural period
$T_1 = \frac{2\pi}{\omega_1}$
Representations in terms of amplitude and phase angles
$= R\cos(\omega_1 t - \varphi)$
complementary angles
$= R\sin(\omega_1 t + \psi)$
amplitude
phase
$= R\sin\omega_1 t\cos\psi + R\cos\omega_1 t\sin\psi$
$s = Hz$
$\frac{u_0}{\dot{u}_0/\omega_1} = \tan\varphi = \frac{A}{B} = \frac{1}{\tan\psi}$
$R = \sqrt{A^2 + B^2}$

Initial conditions
$u(0) = u_0$
$\dot u(0) = \dot u_0$



$F_I = -m\ddot u$

dynamic equilibrium according to d'Alembert

$m\ddot u + ku = 0$

Seek the solution in the form

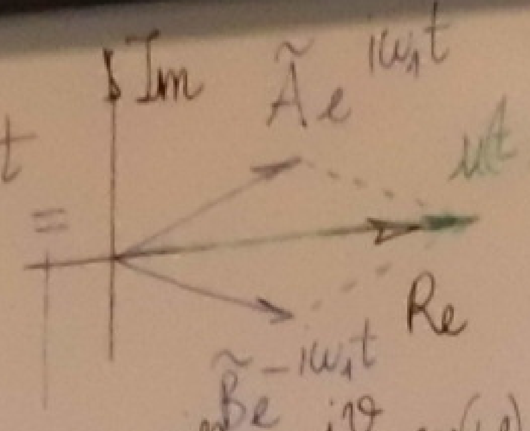
By substituting into eq. of motion

$\ddot u + \omega_1^2 u = 0$

General integral

$u(t) = \tilde{A} e^{i\omega_1 t} + \tilde{B} e^{-i\omega_1 t}$

$\tilde{B} = \bar{\tilde{A}}$ (complex)



$e^{i\vartheta} - e^{-i\vartheta} = 2i\sin\vartheta$

$e^{i\vartheta} + e^{-i\vartheta} = 2\cos\vartheta$ $\Rightarrow$ $\sin\vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$; $\cos\vartheta = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2}$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

general integral

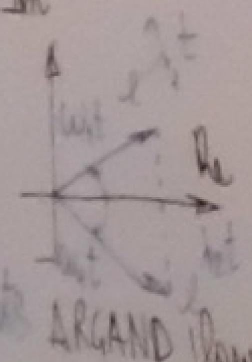
$\boxed{\underset{\frac{\dot u_0}{\omega_1}}{A}\sin\omega_1 t + \underset{u_0}{B}\cos\omega_1 t = u(t)}$

A, B real constants

$\omega_1 A\cos\omega_1 t - \omega_1 B\sin\omega_1 t = \dot u(t)$

$\omega_1 t \Rightarrow$ angle : $\omega_1 \Rightarrow \frac{\text{angle}}{\text{time}}$ angular velocity

$-\omega_1^2 u = -\omega_1^2 A\sin\omega_1 t - \omega_1^2 B\cos\omega_1 t = \ddot u(t)$



ARGAND plane

$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$; $e^{-i\vartheta} = \cos\vartheta - i\sin\vartheta = \overline{e^{i\vartheta}}$

Euler representation of complex number

- By imposing the i.c.s

$u(0) = \boxed{B = u_0}$

$\dot u(0) = \omega_1 A = \dot u_0$

$\boxed{A = \frac{\dot u_0}{\omega_1}}$

HARMONIC MOTION

$\sin\omega_1 t = \sin[\omega_1(t+T_1)]$

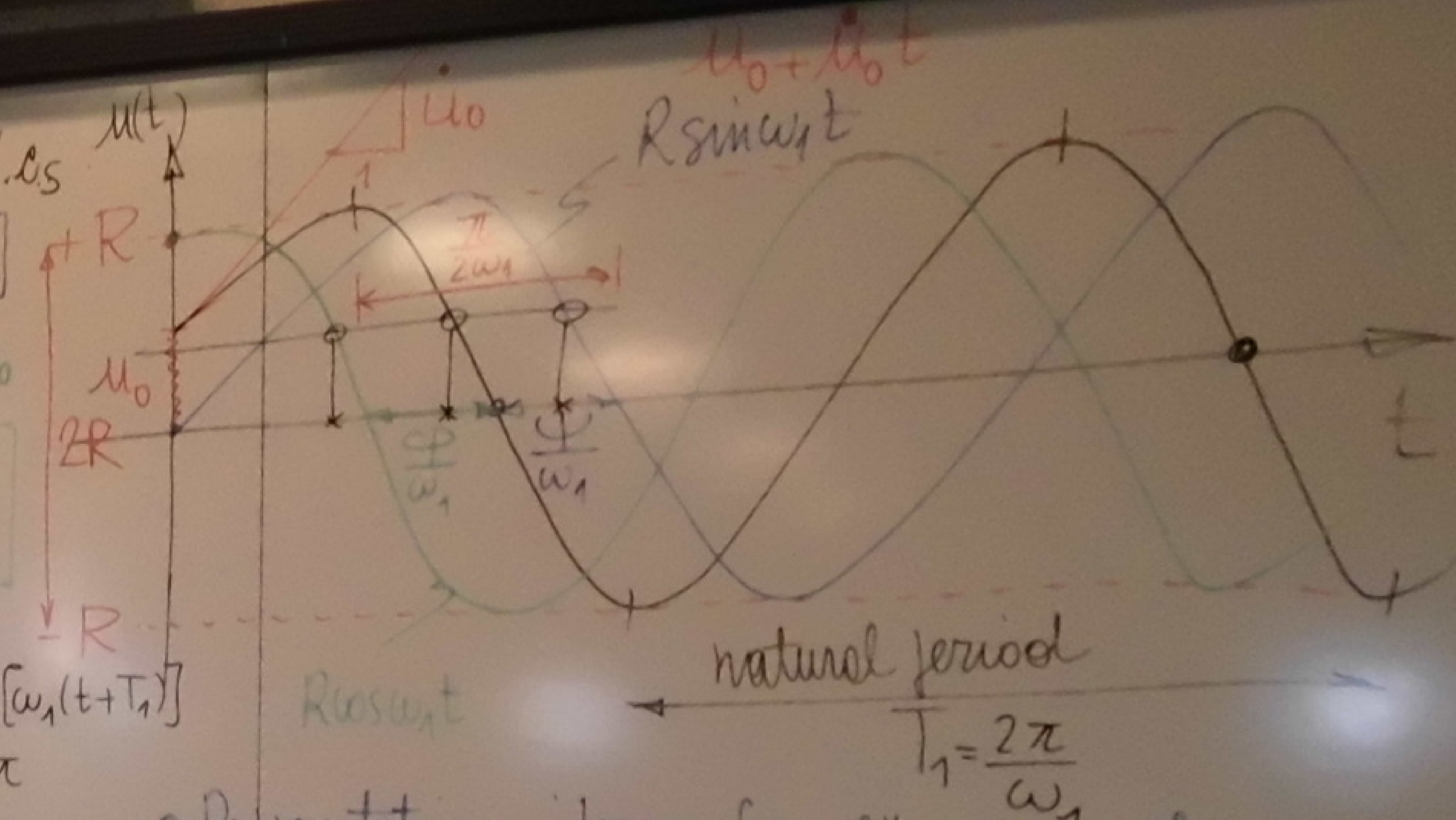
$\omega_1 T_1 = 2\pi$

$T_1 = \frac{2\pi}{\omega_1}$

$f_1 = \frac{1}{T_1} = \frac{\omega_1}{2\pi}$

$\boxed{\omega_1 = 2\pi f_1}$

angular frequency rad/s ; natural cyclic frequency s^{-1} = Hz



- Representations in terms of amplitude and phase angles

$u(t) = R\cos(\omega_1 t - \varphi)$ delay shift wrt pure cosine

complementary angles

$= R\sin(\omega_1 t + \psi)$ (anticipation) wrt pure sine advance

↑amplitude phases

$u = R\cos\omega_1 t\cos\varphi + R\sin\omega_1 t\sin\varphi$

$= R\sin\omega_1 t\cos\psi + R\cos\omega_1 t\sin\psi$

$A = R\sin\varphi = R\cos\psi$

$B = R\cos\varphi = R\sin\psi$

$\frac{\dot u_0}{\omega_1 u_0} = \tan\varphi = \frac{A}{B} = \frac{1}{\tan\psi}$; $R = \sqrt{A^2+B^2} = \sqrt{\frac{\dot u_0^2}{\omega_1^2} + u_0^2}$

Alternative representations of the harmonic

$$u(t) = \underbrace{A}_{\frac{\dot{u}_0}{\omega_1}} \sin\omega_1 t + \underbrace{B}_{u_0}\cos\omega_1 t$$

$$= R \cos(\omega_1 t - \varphi)$$

amplitude of displ.

$$= R \sin(\omega_1 t + \psi)$$

$$\begin{cases} R = \sqrt{A^2 + B^2} = \sqrt{\dfrac{\dot{u}_0^2}{\omega_1^2} + u_0^2} \\ \tan\varphi = \dfrac{1}{\tan\psi} = \dfrac{A}{B} = \dfrac{\dot{u}_0}{\omega_1 u_0} \end{cases}$$

$$\begin{cases} A = R\sin\varphi = R\cos\psi \\ B = R\cos\varphi = R\sin\psi \end{cases}$$

$$\rightarrow \dot{u} = \boxed{\omega_1 R} \sin(\omega_1 t - \varphi)$$

amplitude of velocity

amplitude of acceleration

$$\ddot{u} = -\boxed{\omega_1^2 R} \underbrace{\cos(\omega_1 t - \varphi)}_{u(t)}$$

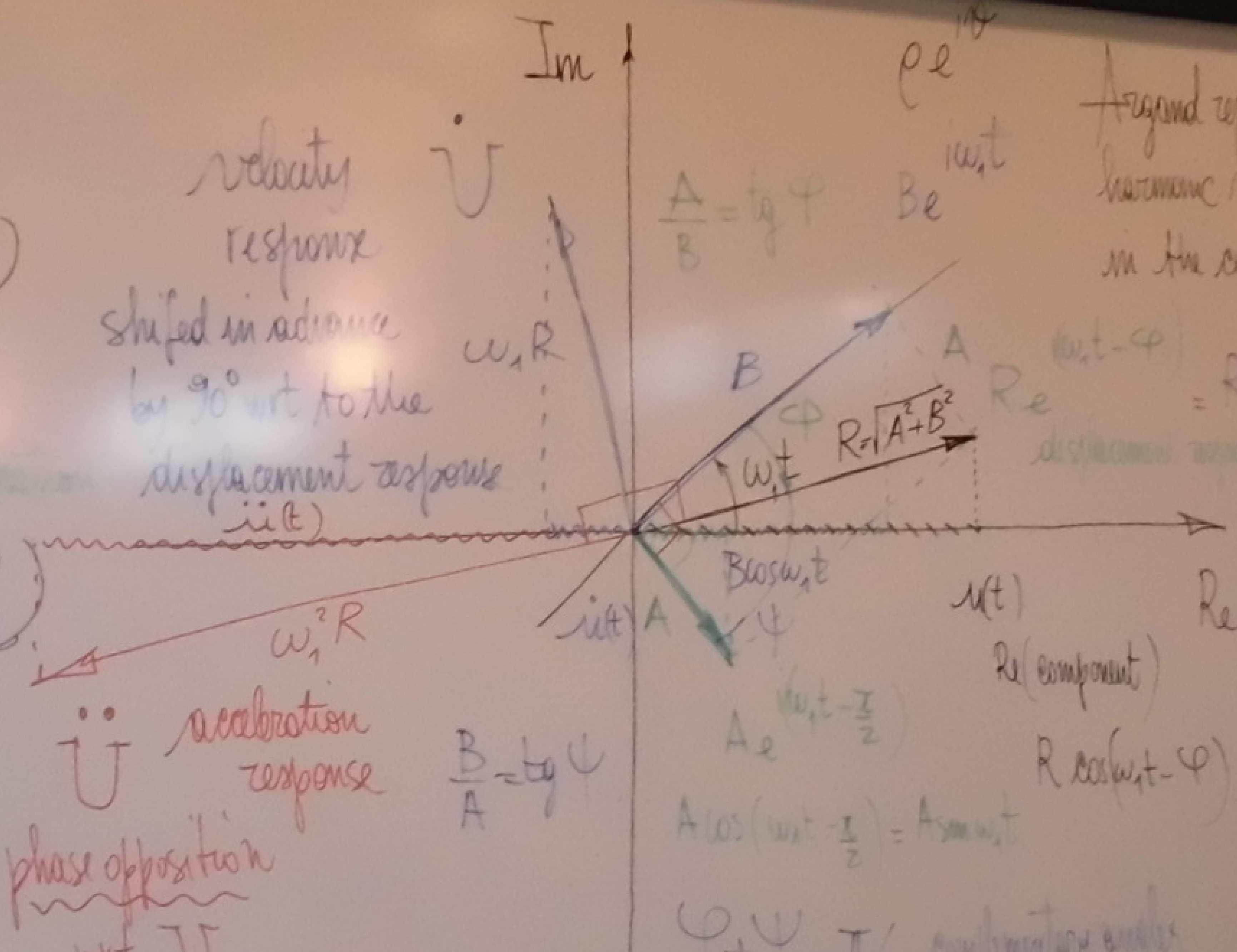
$$= -\omega_1^2 u(t)$$

velocity response $\dot{U}$

shifted in advance by 90° wrt to the displacement response

acceleration response $\ddot{U}$

phase opposition wrt U



$$\frac{B}{A} = tg\,\psi$$

$$\frac{A}{B} = tg\,\varphi$$

$$\rho e^{i\theta}$$

$$Be^{i\omega_1 t}$$

$$Ae^{i(\omega_1 t - \frac{\pi}{2})}$$

$$A\cos(\omega_1 t - \tfrac{\pi}{2}) = A\sin\omega_1 t$$

$$\varphi + \psi = \pi/2 \quad \text{complementary angles}$$

Re(component)

$$R\cos(\omega_1 t - \varphi)$$

Argand representation of harmonic motion by rotating vectors in the complex plane

$$R_e^{i(\omega_1 t - \varphi)} = R_e^{i(\omega_1 t - \frac{\pi}{2} + \psi)} = U$$

displacement response

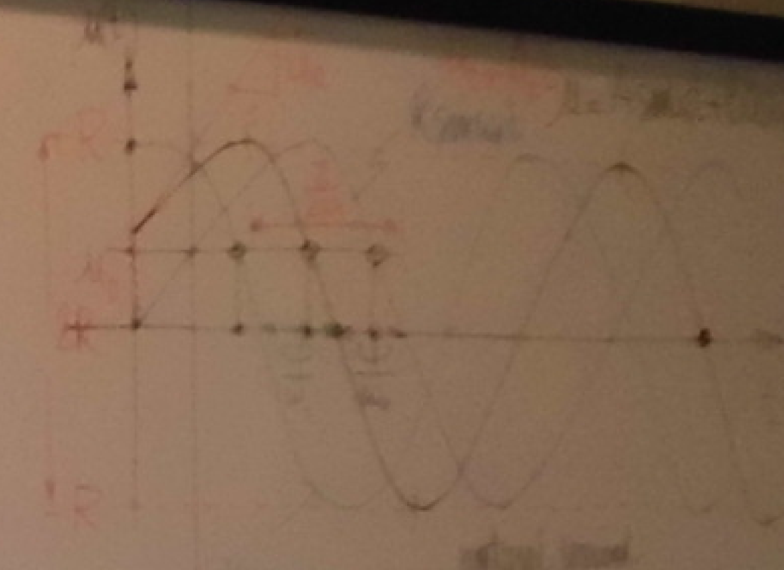
response

$$u(t) = R_e(U(t))$$

$$\dot{U} = R\, i\omega_1 e^{i(\omega_1 t - \varphi)}$$

$$= \omega_1 R e^{i\frac{\pi}{2}} e^{i(\omega_1 t - \varphi)}$$

$$= \omega_1 R e^{i(\omega_1 t - \varphi + \frac{\pi}{2})}$$



• Representation in terms of amplitude and phase

Alternative representations of the harmonic
Argand representation of harmonic motion by rotating vectors in the complex plane
velocity response
shifted in advance by 90° wrt to the displacement response
acceleration response
phase opposition wrt U
complementary angles
Im
Re
Re(component)
response
natural period
Representations in terms of amplitude and phase angles
amplitude
phase


Argand representation of harmonic motion by rotating vectors in the complex plane
$u = A \sin\omega_1 t + B \cos\omega_1 t$
$R \sin\omega_1 t$
Im
Re
response
$u(t) = Re(U(t))$
Re(component)
$R \cos(\omega_1 t - \varphi)$
natural period
$T_1 = \frac{2\pi}{\omega_1}$
Representations in terms of amplitude and phase angles
$u(t) = R \cos(\omega_1 t - \varphi)$
complementary angles
$= R \sin(\omega_1 t + \psi)$
amplitude
phases
(anticipation) wrt pure sine
advance