
Elastoplastic bending of beams ("flessione elastoplastica")
- After the introduction on non-linear (plastic) behaviour at the material level, we are now shifting to the structural scale
- Target is the Limit Analysis of frames
(calcolo a rottura dei telai) set to capture plastic collapse mechanisms and loads of frames
(basically acted by bending internal actions)
- Extend the constitutive relation at the material scale to that at the structural scale, namely at the level of the cross section of the beam, with reference to bending (flexural response)
[e.g. reference to steel structures, but also to reinforced concrete structures]
- Material behaviour (tension/compression symmetric)
yield or flow stress
perfectly-plastic behaviour
no hardening
(e.g. metals)
steel
yield strain
0,001 = 10^-3
elastic range
elastic bending stiffness
- Cross section (doubly-symmetric)
reference to rectangular
A = b h
M = M_x
moment of inertia wrt neutral axis
Young's modulus
yield stress
curvature of the axis line of the beam
- Target is to express globally how the whole cross section behaves under bending, given the properties of the (homogeneous material) that composes the beam
- Beam deformation (at increasing M monotonically)
- Hp plane sections remain plane and orthogonal to the deformed axis (also in the plastic range)
- due to double symmetry
rotation of the cross-section
(linear strain field over the cross-section)
- Initially, bending response occurs in the elastic range
in the elastic range
elastic modulus
- Consistently

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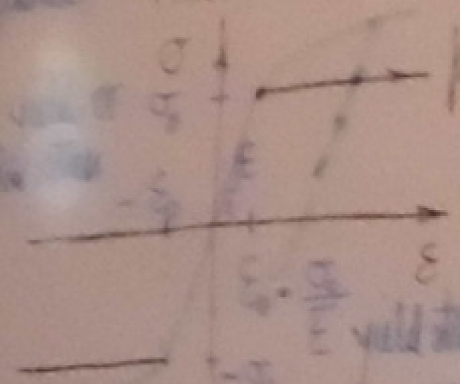
- Material behaviour (tension/compression symmetric)
yield or flow stress
perfectly-plastic behaviour no hardening
(e.g metals) steel
elastic range
$\varepsilon_0 = \frac{\sigma_0}{E}$ yield strain $\simeq \frac{200\ MPa}{200\ GPa}$
$0.001 = 10^{-3}$
$d\varphi_e = \frac{M\,dz}{EJ}$
elastic bending stiffness
- Cross section (doubly symmetric)
=> reference to rectangular
$A = b\,h$
$M = M_x$
$J = J_x = \frac{1}{12} b h^3$
moment of inertia
Young's modulus
$E; \sigma_0$
yield stress
$\varepsilon_{zz}\,dz = d\varphi\,y$
$\varepsilon_{zz}(y) = \frac{d\varphi}{dz}\,y = \chi\,y$
$\chi = \frac{d\varphi}{ds} \simeq \frac{d\varphi}{dz}$
curvature of the medium line of the beam
- Target is to express globally how the whole cross-section behaves under bending, given the properties of the (homogeneous) material that composes the beam
- Beam deformation (at increasing M monotonically)
- Hp plane sections remain plane and orthogonal to the deformed axis (also in the plastic range)
- due to double symmetry
n-n = x-x keeps fixed
rotation of the cross-section
(linear strain field over the cross-section)
$\varepsilon_{zz} = \varepsilon_z$
- Initially, bending response occurs in the elastic range
$\sigma_{zz}(y) = E\,\varepsilon_{zz}(y)$
$M = EJ\chi_e$
$\chi_e = \frac{M}{EJ}$
$= \frac{M}{EJ}\,y$
$M = M_x$
$\sigma_z\,dA$
$M = \int_A \sigma_{zz}\,dA\,y$
static equivalence
$= \int_A E\chi y^2\,dA$
$= E\chi \underbrace{\int_A y^2\,dA}_{J_x}$
$= \frac{M_x}{J_x}\,y$
Navier's formula
$\sigma_{zz} = E\varepsilon_{zz}$
in the elastic range
- Then, according to such a kinematic scheme
$\varepsilon_{zz}^{max} = \chi\,y^{max} = \chi\,\frac{h}{2}$
$\sigma_{zz}^{max} = E\varepsilon^{max} = E\chi\,\frac{h}{2} = \frac{M}{J}\frac{h}{2} = \frac{M}{\frac{J}{h/2}} = \frac{M}{W_e}$
elastic resistance modulus
$W_e = \frac{J_x}{h/2} = \frac{1}{12}\frac{bh^3}{h}\,2 = \frac{1}{6}\,bh^2$
$W_e = \frac{bh^2}{6}$
- Consistently
$M = T \cdot \frac{2}{3}h$
$= \frac{1}{2}\frac{h}{2}\sigma^{max}\,b\,\frac{2}{3}h = \sigma^{max}\frac{bh^2}{6} \Rightarrow \sigma^{max} = \frac{M}{W_e}$
- All points of the cross section keep in the elastic range until
$\varepsilon^{max} \leq \varepsilon_0 \quad , \quad \chi \leq \chi_e = \chi_0$
$\sigma^{max} \leq \sigma_0 \quad , \quad M \leq M_e = W_e\,\sigma_0$
first elastic limit is reached
At first elastic limit $\varepsilon^{max} = \varepsilon_0 \Rightarrow \chi_e\,\frac{h}{2} = \varepsilon_0$

- Material behaviour (tension-compression symmetric)



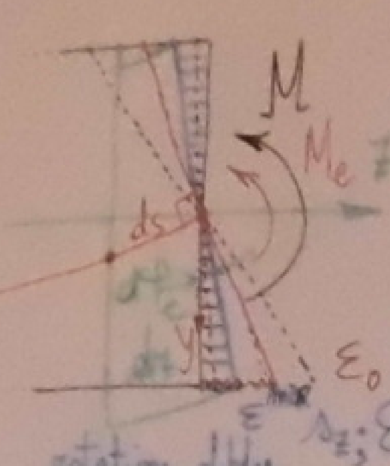
perfectly plastic behaviour no hardening

(e.g. metals, steel)

- Cross section (doubly symmetric)

$\chi = \frac{d\varphi}{ds} \simeq \frac{d\varphi}{dz}$ curvature of the axis line of the beam

- Beam deformation (at increasing M monotonically)



rotation of the cross-section

(linear strain field over the cross-section)

ε_0

- Hp: plane sections remain plane and orthogonal to the deformed axis (also in the plastic range)

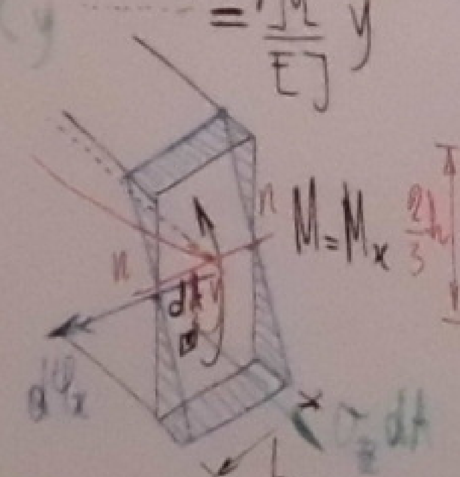
- due to double symmetry n-n = x-x keeps fixed

- Initially, bending response occurs in the elastic range:

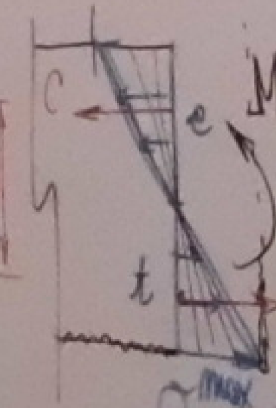
$$\sigma_{zz}(y) = E\, \varepsilon_{zz}(y) = \frac{M}{EJ}\, y$$

$$M = EJ\,\chi_e$$

$$\chi_e = \frac{M}{EJ}$$



$M = M_x$



$$M = \int_A \sigma_{zz}\, dA\, y = \int_A E \chi y^2\, dA = E\chi \underbrace{\int_A y^2\, dA}_{J_x}$$

static equivalence

$\sigma_{zz} = E\,\varepsilon_{zz}$ in the elastic range

Navier's formula $= \frac{M_x}{J_x}\, y$

- Then, according to such a kinematic scheme:

$$\varepsilon_{zz}^{max} = \chi\, y^{max} = \chi\, \frac{h}{2}$$

$$\sigma_{zz}^{max} = E\, \varepsilon^{max} = E\chi\, \frac{h}{2} = \frac{M}{J}\, \frac{h}{2} = \frac{M}{\frac{J}{h/2}} = \frac{M}{W_e}$$

elastic resistance modulus $W_e = \frac{J_x}{h/2} = \frac{1}{12}\, \frac{b h^3}{h}\, 2 = \frac{1}{6}\, b h^2$

$$\boxed{W_e = \frac{b h^2}{6}}$$

- Consistently:

$$M = \underbrace{T}\cdot \frac{2}{3}\, h$$

$$= \frac{1}{2}\, \frac{h}{2}\, \sigma^{max}\, b\, \frac{2}{3}\, h = \sigma^{max}\, \underbrace{\frac{b h^2}{6}}_{W_e} \Rightarrow \sigma^{max} = \frac{M}{W_e}$$

- All points of the cross-section keep in the elastic range until

$$\varepsilon^{max} \leq \varepsilon_0 \,, \quad \chi \leq \chi_e = \chi_0$$

$$\sigma^{max} \leq \sigma_0 \,, \quad M \leq M_e = W_e\, \sigma_0$$

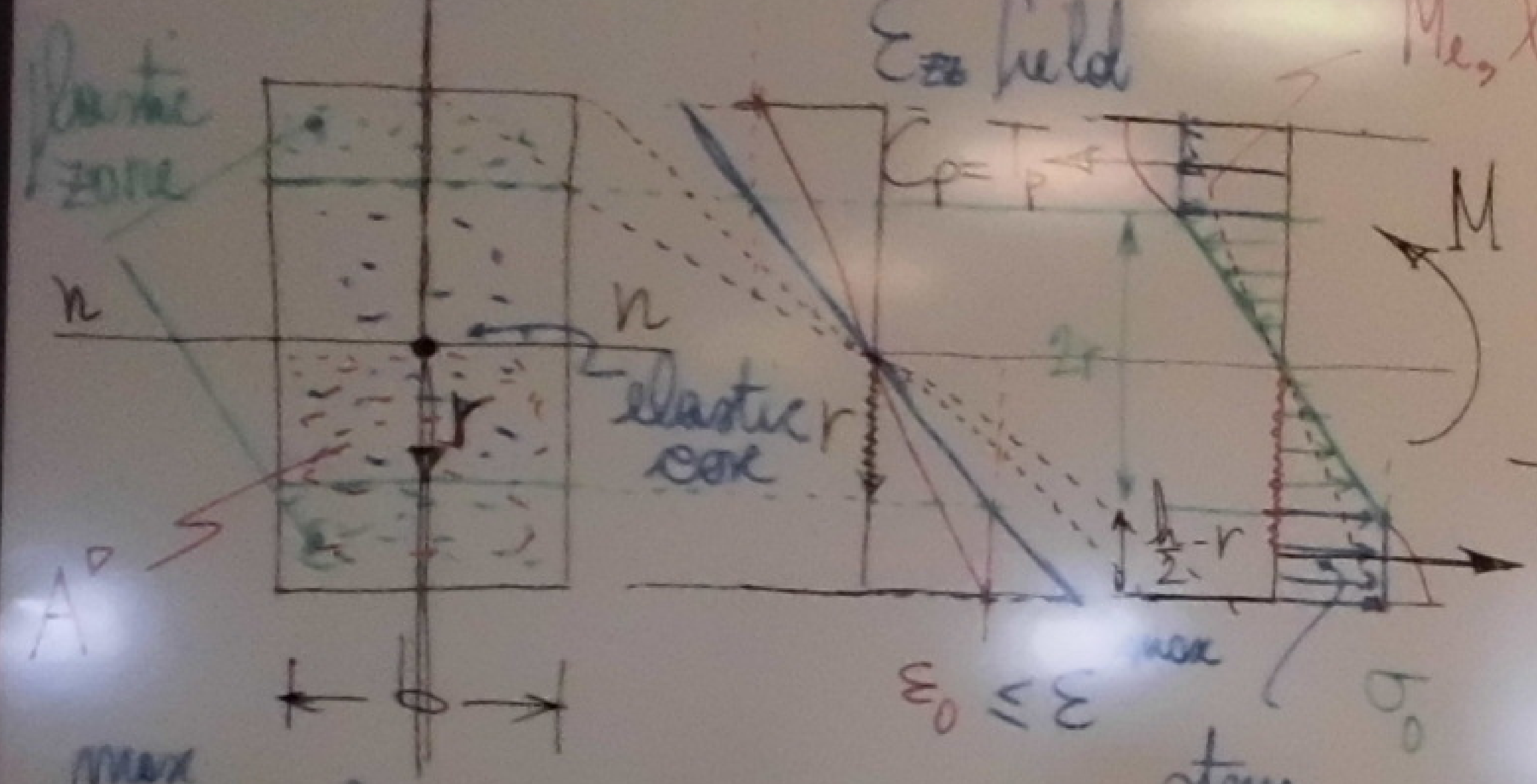
first elastic limit is reached (with =)

At first elastic limit $\varepsilon^{max} = \varepsilon_0 \Rightarrow \boxed{\chi_e\, \frac{h}{2} = \varepsilon_0}\ \boxed{\chi_e = \frac{2\varepsilon_0}{h}}$

- Elastoplastic range (continuous loading)

$\chi \geq \chi_e \quad , \quad \varepsilon^{max} \geq \varepsilon_0$

$M_e, \chi_e = \varepsilon_0 \frac{h}{2}$



$\varepsilon^{max} = \chi \frac{h}{2}$ ← $\varepsilon = \chi y$

$\varepsilon_0 = \chi_e \frac{h}{2}$ — χ_e curvat. at the limit of the elastic range

$\varepsilon_0 = \chi r$ — current curvature in the elastoplastic range

$\varepsilon_0 = \chi r$

$1 = \frac{\chi_e}{\chi}\frac{h}{2r}$

$r = \frac{h}{2}\frac{\chi_e}{\chi}$

$\frac{\chi}{\chi_e} = \frac{\varepsilon^{max}}{\varepsilon_0}$

$$M = \int_A \sigma_{zz}\, y\, dA$$

$$= \sigma_0 \frac{b(2r)^2}{6} + b\left(\frac{h}{2}-r\right)\sigma_0 \frac{1}{2}\left(\frac{h}{2}+r\right) \quad (+50\%!)$$

$2\cdot\frac{1}{2}\left(\frac{h}{2}+r\right)$

$$= \frac{4}{6}\sigma_0 b r^2 + \sigma_0 b\left(\frac{h^2}{4}-r^2\right)$$

$T_p = b\left(\frac{h}{2}-r\right)\sigma_0$

$$= \sigma_0 \frac{bh^2}{4} + \sigma_0 b r^2 \underbrace{\left(\frac{2}{3}-\frac{3}{3}\right)}_{-\frac{1}{3}}$$

global constitutive law at the level of the cross section

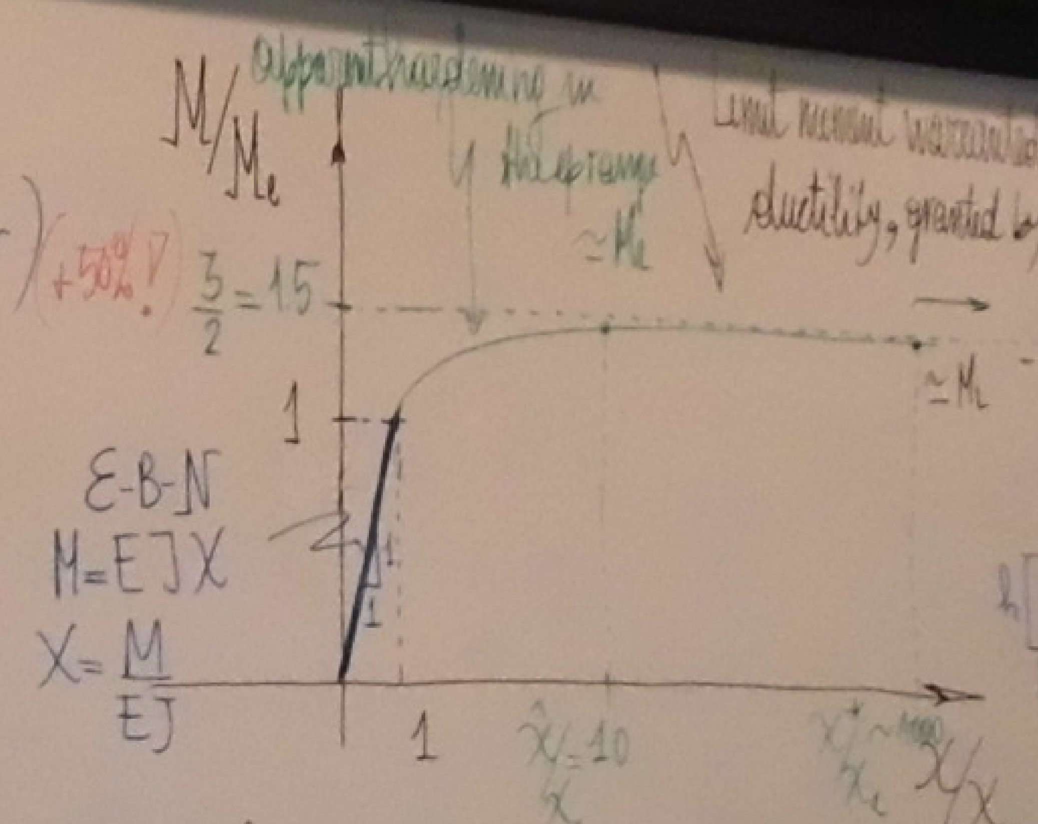
$$= \sigma_0 \frac{bh^2}{4} - \frac{1}{3}\sigma_0 b r^2$$

$$= \sigma_0 \frac{bh^2}{4} - \frac{1}{3}\sigma_0 b \frac{h^2}{4}\left(\frac{\chi_e}{\chi}\right)^2$$

$$M(\chi) = \sigma_0 \frac{bh^2}{4}\left(1-\frac{1}{3}\left(\frac{\chi_e}{\chi}\right)^2\right) = \boxed{M_L\left(1-\frac{1}{3}\left(\frac{\chi_e}{\chi}\right)^2\right)} = M(\chi)$$

non-linear moment/curvature relation in the ep range with hardening

$M_L = \sigma_0 W_{P_L}$ Limit or plastic moment ; where $Z = W_p = \frac{bh^2}{4}$ plastic (resistance) modulus



ε-B-N

$M = EJ\chi$

$\chi = \frac{M}{EJ}$

- Meaning of M_L

$\frac{\chi}{\chi_e} \to \infty \; ; \; M \to M_L$ $\quad \frac{\varepsilon_0}{\varepsilon^{max}}$

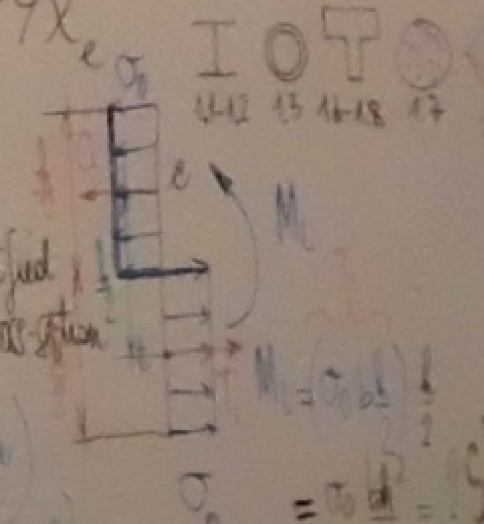
higher bending capacity that may be theoretically achieved (for perfectly plastic behaviour) asymptotically as curvature tends to infinity

- Shape factor

$\alpha = \frac{W_P}{W_e} = \frac{\sigma_0 M_L}{\sigma_0 M_e}$

$= \frac{3}{2} = 1.5$

1.1-1.2 1.5 1.6-1.8 1.7 2.0



$M_L = \sigma_0 \frac{bh^2}{4}$


- Elastoplastic range (continuous loading)
χ > χe , ε^max > ε0
Me, χe = ε0 h/2
ε0 field
M
elastic core
ε0 ≤ ε^max
ε^max = χ h/2
ε = χ y
ε0 = χe h/2
ε0 = χ r
1 = χe/χ h/2r
r = h/2 χe/χ
χ/χe = ε^max/ε0
M = ∫A σzz y dA
= σ0 b(2r)²/6 + b(h/2 − r) σ0 1/2 (h/2 + r)
(+50%!)
= 4/6 σ0 b r² + σ0 b (h²/4 − r²)
Tpl = b(h/2 − r) σ0
= σ0 bh²/4 + σ0 b r² (2/3 − 3/3)
global constitutive law at the level of the cross section
= σ0 bh²/4 − 1/3 σ0 b r²
= σ0 bh²/4 − 1/3 σ0 b h²/4 (χe/χ)²
M(χ) = σ0 bh²/4 (1 − 1/3 (χe/χ)²) = ML (1 − 1/3 (χe/χ)²)
non-linear moment/curvature relation in the ep range with hardening
; where Z = Wp = bh²/4
plastic (resistance) modulus
= M(χ)
M/Me
apparent hardening in the ep range
≃ ML
Limit moment warranted by considerable ductility, granted by the perfect ep beh
3/2 = 1.5
1
E-B-N
M = EJχ
χ = M/EJ
1
10
χ*/χe ~1000
χ/χe
≃ ML
- Shape factor
α = Wp/We = σ0 ML / σ0 Me
h
b
= 3/2 = 1.5
1.1-1.2
1.3
1.6-1.8
1.7
2.0
- Meaning of ML
ε0/ε^max
χ/χe → ∞ ; M → ML
higher bending capacity that may be theoretically achieved (for perfectly-plastic behaviour) asymptotically as curvature tends to infinity (r → 0)
fully plastified cross-section
σ0
ML
ML = (σ0 b h/2) h/2
= σ0 bh²/4
- Limit condition χ → ∞ is just theoretical
ε^max = 1 → χ = χ* = 2/h
dz
ε dz = dz
χ*/χe = 1/ε0 ≃ 1/10^-3 = 1000
M/ML = (1 − 1/3 · 1/10^6) ≃ 1
0.999996
Also, in real structures an admissible level of deformation may be in the order of ε^max = 1% = 0.01 = 10^-2

Limit moment warranted by considerable ductility, granted by the perfect ep beh.
Shape factor
$\alpha = \frac{W_P}{W_e} = \frac{\sigma_0 M_L}{\sigma_0 M_e}$
$= \frac{3}{2} = 1.5$
1.1-1.2
1.3
1.6-1.8
1.7
2.0
Meaning of M_L
$\chi \to \infty$; $M \to M_L$
higher bending capacity that may be theoretically achieved (for perfectly-plastic behaviour)
asymptotically as curvature tends to infinity
fully plastified cross-section
$M_L = \left(\sigma_0 b \frac{h}{2}\right) \frac{h}{2}$
$= \sigma_0 \frac{bh^2}{4} = 2 S_x$
compenetration (not possible)
Limit condition $\chi \to \infty$ is just theoretical
$\varepsilon^{max} = 1 \Rightarrow \chi = \chi^* = \frac{2}{h}$
$\frac{\chi^*}{\chi_e} = \frac{1}{\varepsilon_0} \simeq \frac{1}{10^{-3}} = 1000$
$\frac{M}{M_L} = \left(1 - \frac{1}{3} \frac{1}{10^6}\right) \simeq 1$
0.9999996
Also, in real structures an admissible level of deformation may be in the order of $\varepsilon^{max} = 1\% = 0.01 = 10^{-2}$
$\varepsilon^{max} = 1\% \Rightarrow \chi = \hat{\chi} = \frac{2}{h} 10^{-2}$
$\frac{\hat{\chi}}{\chi_e} = \frac{10^{-2}}{\varepsilon_0} = 10$
$\frac{M}{M_L} = \left(1 - \frac{1}{3} \frac{1}{100}\right) = \frac{299}{300} \simeq 1$
already very near to 1