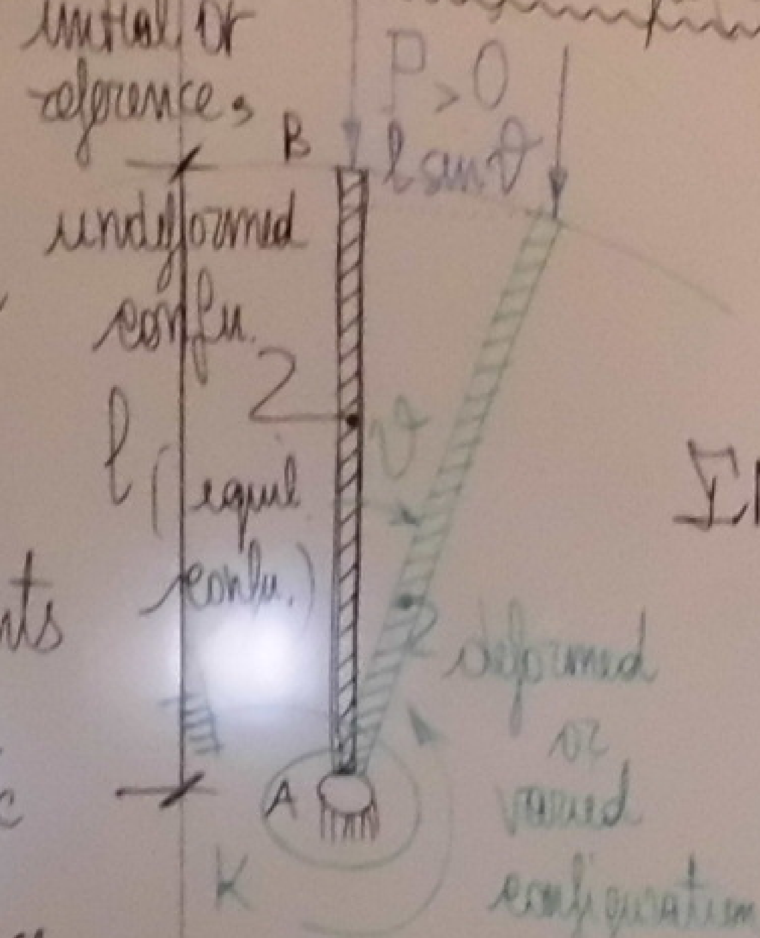
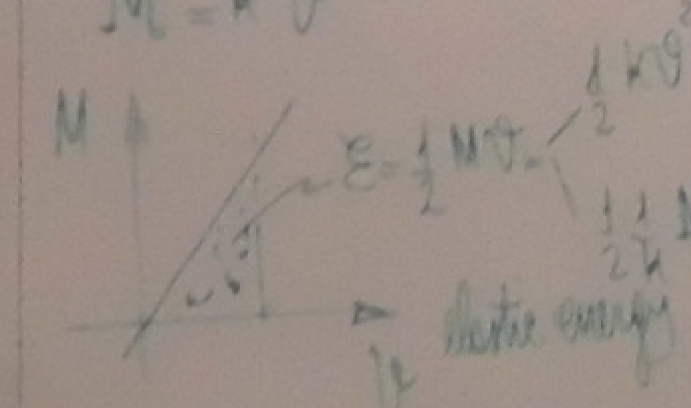
2) Instability of Structures

- Interested in discovering the nature (stable, unstable) of the equilibrium configuration
- Analysis of compressed slender elements by "slowly" increasing loads up to specific critical values (critical load) apt to induce unstable effects.
- Although we have seen that instability criteria should be defined in Liapunov's (dynamical) sense (see introductory class), likely to handle them by a static approach or energy approach

Introductory example (SDOF discrete system)



$M = K\vartheta$



$\mathcal{E} = \frac{1}{2} M\vartheta = \frac{1}{2} K\vartheta^2 = \frac{1}{2}\frac{1}{K} M^2 > 0$

- Static approach looks feasible but it is necessary to write the equilibrium equ. in the deformed configuration

$$\sum M_A^{\vartheta} = 0 \rightarrow \boxed{P\, l \sin\vartheta = K\vartheta}$$

destabilizing effect (overturning moment) linked to compression load P

stabilizing (elastic) effect (stabilizing moment) linked to elastic spring (K)

+ Notice that $\vartheta = 0$ is always an eq. solution
+ Further solutions ruled by non-linear eq.

$$\sin\vartheta = \frac{K}{Pl}\vartheta = \frac{K/l}{P}\vartheta = \frac{1}{p}\vartheta$$

$\left(P = p\frac{K}{l};\ p = \frac{P}{K/l} = \frac{P}{P_{cr}} \right)$ load multiplier $P_{cr} = \frac{K}{l}$

- Geometric non-linearity (P-Δ effect) $p = p(\vartheta) = \frac{\vartheta}{\sin\vartheta}$

- Large displacements meaning that displacements are such to influence (non-linearly) the equilibrium config.
- On the other hand, displacements may keep "geometrically small" (small magnitude, to be possibly linearized) but still such that equilibrium is influenced ...
- 2nd order effects ... approach
- Buckling (...)

Solution trees

1) $p = \frac{P}{K/l} < 1$

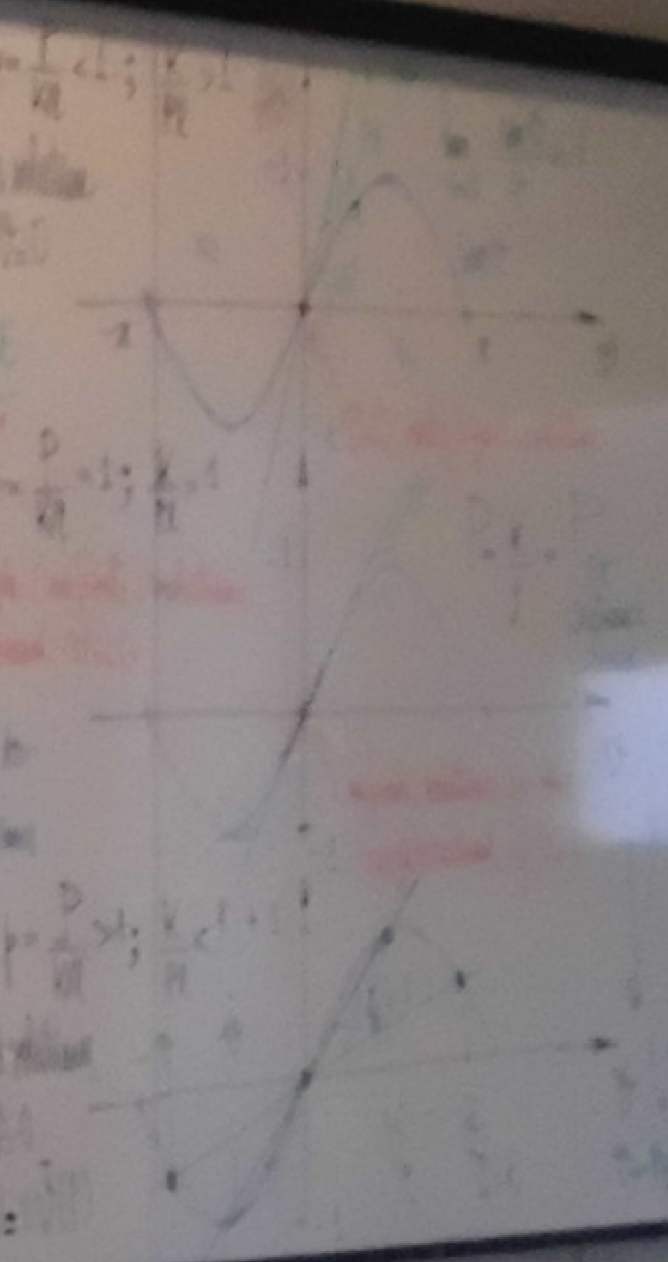
$P < P_{cr}$

2) $p = \frac{P}{K/l} = 1$

3) $p = \frac{P}{K/l} > 1$

$P > P_{cr}$

multiple solutions



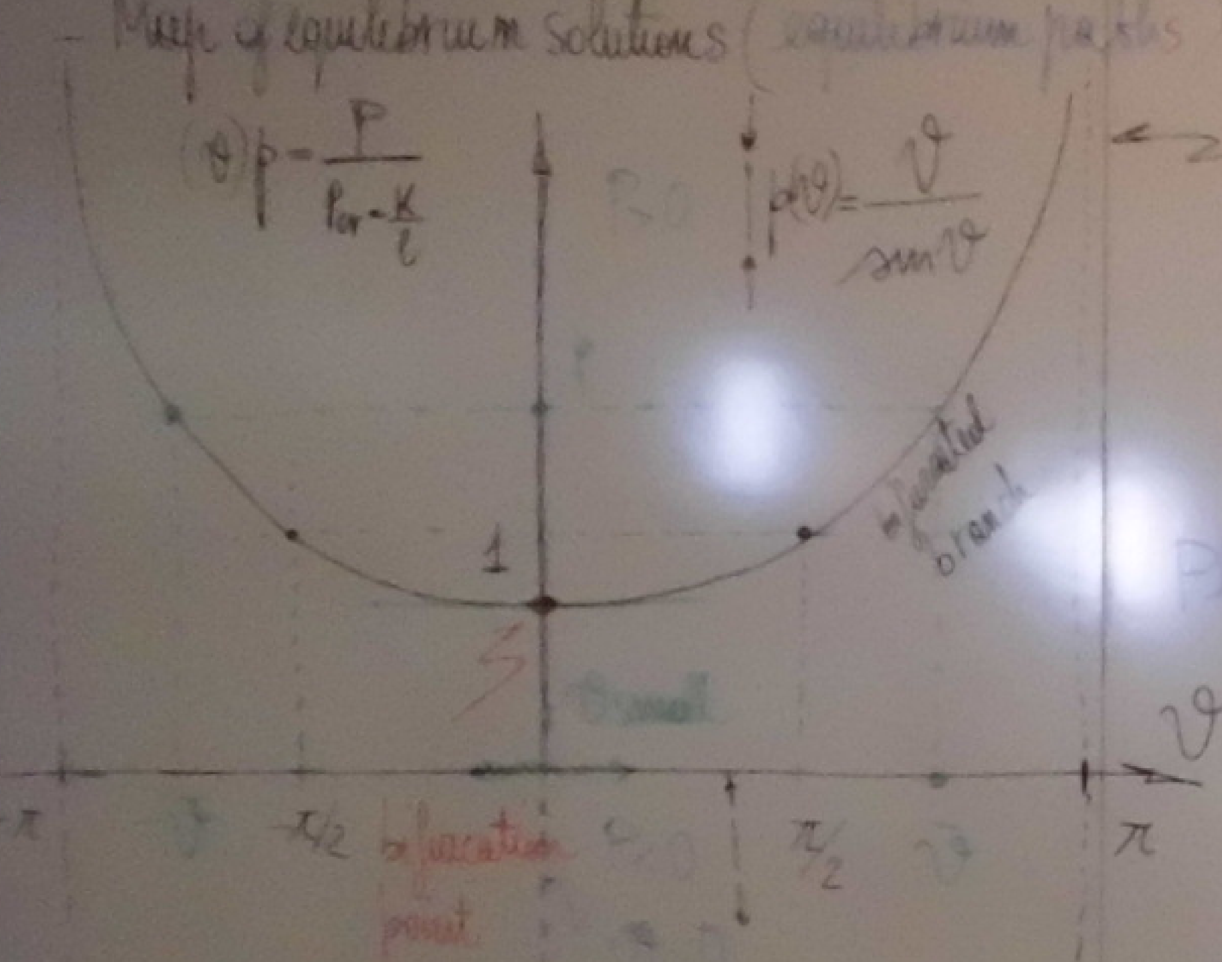

2) Instability of Structures
Introductory example (SDOF discrete system)
- Interested in discovering the nature
of the equilibrium
configuration
- Analysis of compressed slender elements
undeformed
config.
- Static approach looks feasible but it is
necessary to write the equilibrium equ.
in the deformed configuration
ΣM_A = 0 → P l sin ϑ = K ϑ
destabilizing effect
(overturning moment)
linked to compression load P
stabilizing (elastic) effect
(stabilizing moment)
linked to elastic spring (K)
+ Notice that ϑ=0 is always an eq. solution
+ Further solutions ruled by non-linear eq.:
sin ϑ = K/(Pl) ϑ = Kl/P ϑ = 1/p ϑ
- Geometric non-linearity
(P-Δ effect) : p = p(ϑ) = ϑ/sin ϑ
- Large displacements meaning that
displacements are such to influence
(non-linearly) the equilibrium confn.
- on the other hand displacements
"geometrically small"
- 2nd order effects (theory)
approach
- buckling (instability problems)
Solution
cases
1) p = P/(Kl) < 1 ; K/(Pl) > 1
single solution
ϑ=0
P < Pcr
ϑ=0 STABLE
warrants stability
P = Pcr
marks
a transition
(bifurcation)
state from a single to
multiple equil. solutions
2) p = P/(Kl) = 1 ; K/(Pl) = 1
multiple (infinite) solutions
around ϑ=0
P = K/l = Pcr
critical
load
infinite solutions in the
neighborhood of ϑ=0
3) p = P/(Kl) > 1 ; K/(Pl) < 1
P > Pcr
ϑ=0 UNSTABLE
multiple solutions
- ϑ=0
- ϑ = ±ϑ̄(p)
sin ϑ
ϑ=0 only equil. solution
-π
π
ϑ
P


Instability of Structures
Introductory example
Static approach looks feasible but it is necessary to write the equilibrium eq. on the deformed configuration
$\Sigma M_A = 0 \Rightarrow P\, l \sin\vartheta = K\vartheta$
+ Notice that $\vartheta = 0$ is always an eq. solution
+ Further solutions ruled by non-linear eq.:
$\sin\vartheta = \frac{K}{Pl}\vartheta = \frac{K/l}{P}\vartheta = \frac{1}{p}\vartheta$
– Large displacements meaning that displacements are such to influence (non-linearly) the equilibrium confi.
– on the other hand displacements may keep "geometrically small"
– 2nd order effects (theory) ⇒ energy approach
– buckling (instability problems)
Solution cases
1) $p = \frac{P}{K/l} < 1$; $\frac{K}{Pl} > 1$
single solution $\vartheta = 0$
$P < P_{cr}$ $\vartheta = 0$ STABLE
warrants stability
2) $p = \frac{P}{K/l} = 1$; $\frac{K}{Pl} = 1$
multiple (infinite) solutions around $\vartheta = 0$
a transition (bifurcation) state from a single to multiple equil. solutions
3) $p = \frac{P}{K/l} > 1$; $\frac{K}{Pl} < 1$
multiple solutions
– $\vartheta = 0$
– $\vartheta = \pm\bar{\vartheta}(P)$
$P > P_{cr}$
$\vartheta = 0$ UNSTABLE
$f(\vartheta)$
$\lim_{\vartheta \to 0} \frac{\sin\vartheta}{\vartheta} = 1$
$\sin\vartheta$
$\vartheta = 0$ only equil. solution
infinite solutions in the neighbourhood of $\vartheta = 0$
$P = \frac{K}{l} = P_{cr}$ critical load
$\vartheta = \bar{\vartheta}$
$\vartheta = -\bar{\vartheta}$

- Map of equilibrium solutions (equilibrium paths)
(ϑ) p = P / P_cr = K/ℓ
p(ϑ) = ϑ / sin ϑ
P>0
1
ϑ small
ϑ
−π
−π/2
π/2
π
P<0
ϑ = 0
bifurcated branch
non-linear equilibrium path with "locking" at increasing P
Pl sin ϑ = Kϑ
bifurcation point in the equilibrium map
⇝ it marks the critical load
- Inspect now the analysis under "geometrically-small" displacements.
|ϑ| << 1 (ϑ → 0)
linearization of the equil. eqn.
lϑ
sin ϑ ≃ ϑ + ...
Plϑ = Kϑ
linear equil. equat.
(K − Pl) ϑ = 0
K − Pl ≠ 0, ϑ = 0, P ≠ K/l
K − Pl = 0, ϑ arb., P = K/l
K_tot = 0
P = K/l = P_cr
K_tot = K − K_G
elastic stiffn.
geometric stiffness
"ϑ small"
p = P / (K/l)
bifurcation point
bifurcated path
reaching the same critical load
- Then, the analysis under "geometrically-small" displacements provides the same estimate of the critical load (bifurcation point) and
1)
2)
3)

- Map of equilibrium solutions (equilibrium paths)

$p = \dfrac{P}{P_{cr} = \frac{K}{\ell}}$ $\qquad p(\vartheta) = \dfrac{\vartheta}{\sin\vartheta}$



⇐ it marks the critical load

⇐ non-linear equilibrium path with "locking" at increasing P

$P\ell \sin\vartheta = K\vartheta$

- Inspect now the analysis under "geometrically-small" displacements.

$|\vartheta| \ll 1$ $\quad (\vartheta \to 0)$

linearization of the equil. eqn.

$\sin\vartheta \simeq \vartheta + \dots$

$P\ell\vartheta = K\vartheta$

linear equil. equat.

$(K - P\ell)\,\vartheta = 0$

$K_{tot} = K - K_G$

elastic stiffn. geometric stiffness

$K - P\ell \neq 0, \ \vartheta = 0, \ P \neq \frac{K}{\ell}$

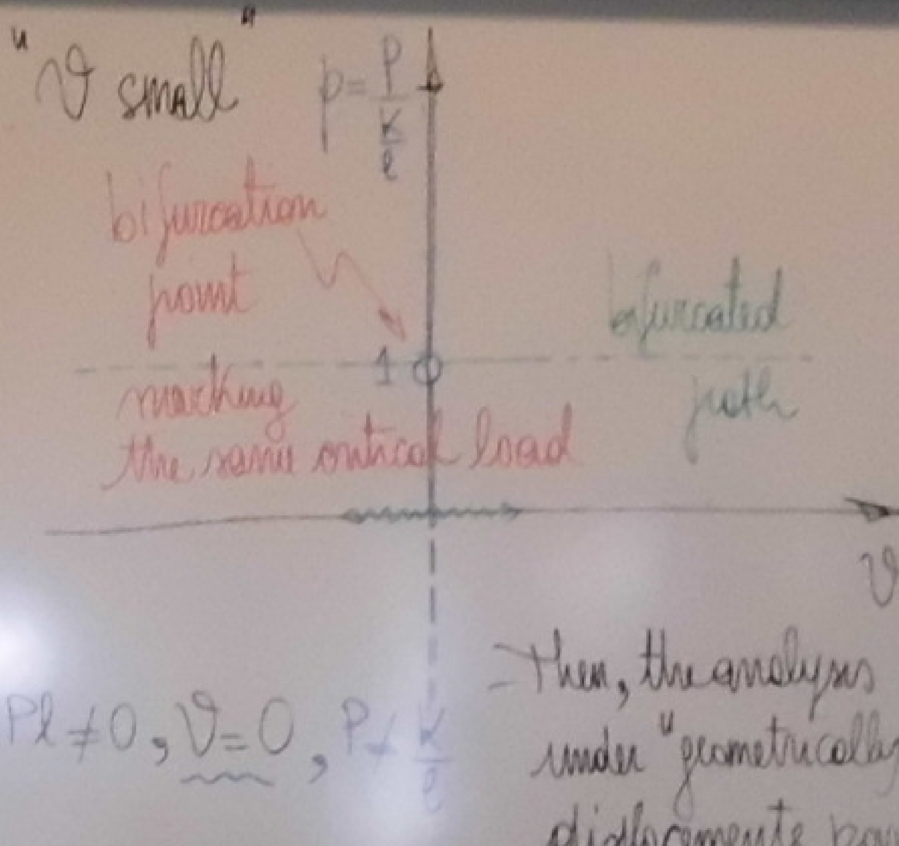
$K - P\ell = 0, \ \vartheta \text{ arb.}, \ P = \frac{K}{\ell}$

$K_{tot} = 0$

$P = \frac{K}{\ell} = P_{cr}$

"ϑ small"

$p = \dfrac{P}{K/\ell}$



- Then, the analysis under "geometrically-small" displacements provides the same estimate of the critical load (bifurcation point) and thus it could be performed instead of the non-linear one with the purpose to get P_{cr}

1) $p = \dfrac{P}{K/\ell} < 1$; $\dfrac{K}{P\ell} > 1$

single solution $\vartheta = 0$

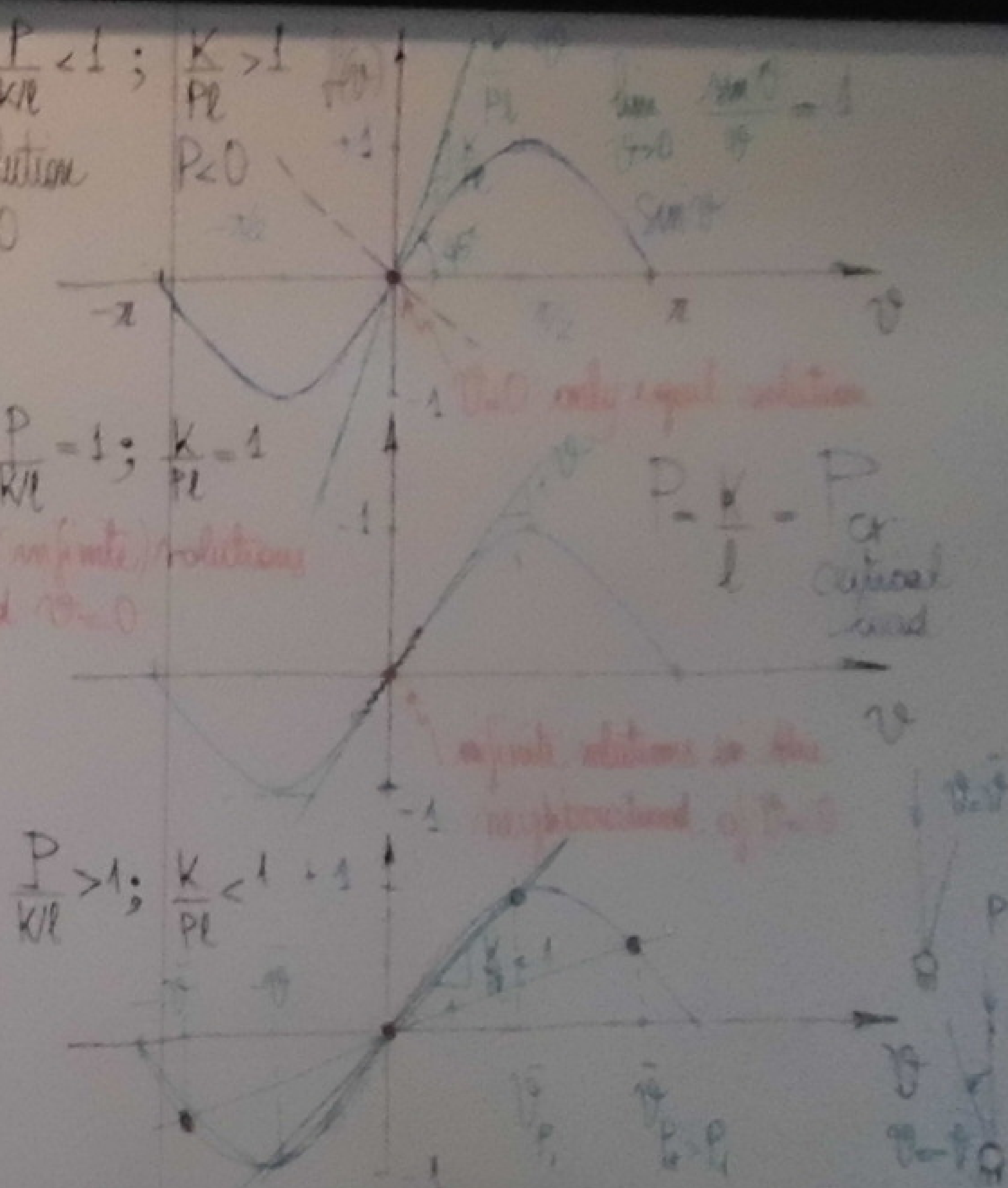
$P < 0$

2) $p = \dfrac{P}{K/\ell} = 1$; $\dfrac{K}{P\ell} = 1$

multiple (infinite) solutions around $\vartheta = 0$

$P = \dfrac{K}{\ell} = P_{cr}$ critical load

3) $p = \dfrac{P}{K/\ell} > 1$; $\dfrac{K}{P\ell} < 1$



- Inspect now the analysis under "geometrically-small" displacements
|ϑ| << 1
linearization of the equil. eqn.
P l ϑ = K ϑ
(K − P l) ϑ = 0
K − P l ≠ 0, ϑ = 0, P ≠ K/l
K − P l = 0, ϑ arb., P = K/l
P = K/l = P_cr
"ϑ small"
bifurcation point
bifurcated path
reaching the same critical load
- Then, the analysis under "geometrically-small" displacements provides the same estimate of the critical load (bifurcation point) and thus it could be performed instead of the non-linear one with the purposes to get P_cr
1) p = P/(K/l) < 1 ; K/Pl > 1
single solution ϑ = 0
lim ϑ→0 sin ϑ / ϑ = 1
sin ϑ
ϑ = 0 only equil. solution
2) p = P/(K/l) = 1 ; K/Pl = 1
multiple (infinite) solutions around ϑ = 0
P = K/l = P_cr
critical load
infinite solutions in the neighbourhood of ϑ = 0
3) p = P/(K/l) > 1 ; K/Pl < 1