Seismic response of MDOF systems (shear-type frames)
It results in a "shear" deform.
Could be generalized to frames with finite stiffness of the beams by to beam-to-column stiffness ratio (and accounting for rotational degrees of freedom of the column/beam nodes
static dofs that may be condensed-out statically)
Basic configuration;
M = diag [m_i]
"shear-type"
Shear-type frames:
rigid floors
q_i dynamic dofs
q_i displac relative to ground
m_i
Eqs. of motion
Rigid body motion vector
r^T = {1 1 1}
often
By shifting to principal coordinates
eigenvalue problem
modal participation factors
Notice that both Λ_i and Γ_i
Analysis with Response Spectra
Sa
Sv
Sd
Max displ. in principal coord.

Seismic response of MDOF systems
It results in a "shear deform."
Could be generalized to frames
with finite stiffness of the beams
by to beam-to-column stiffness
ratio
Shear-type frames
rigid floors
dynamic dofs
Basic configuration
diagonal structure column
$\mathbb{M} = diag[m_i]$
eigenvalue problem
$\mathbb{K}\Phi_i = \omega_i^2 \mathbb{M}\Phi_i$
Eqs of motion
rigid body motion vector
$\mathbb{M}\ddot{q} + \mathbb{C}\dot{q} + \mathbb{K}q = -\mathbb{M}r\ddot{u}_g(t)$
often
$r^T = \{1 \ 1 \ 1\}$
$M_{ij}\ddot{q}_j + C_{ij}\dot{q}_j + K_{ij}q_j = -m_i\ddot{u}_g(t) \quad i = 1, n$
By shifting to principal coordinates $p_i(t)$ $[q = \Phi p]$
$\mathbb{M}\ddot{p} + \mathbb{C}\dot{p} + \mathbb{K}p = -\Phi^T\mathbb{M}r\ddot{u}_g(t)$
$diag[M_i = \Phi_i^T \mathbb{M} \Phi_i]$
$diag[K_i = \Phi_i^T K \Phi_i]$
Λ modal participating masses
$\{\Lambda_i = \Phi_i^T \mathbb{M} r\}$
$= \sum_j \Phi_{ij} m_j$
modal damping ratios
2% - 7% (5%)
Numerical integr.
$\ddot{p} + \mathbb{M}^{-1}\mathbb{C}\dot{p} + \mathbb{M}^{-1}\mathbb{K}p = -\mathbb{M}^{-1}\Lambda\ddot{u}_g(t) \rightarrow p(t)$
Γ modal participation factors
$\{\Gamma_i = \frac{\Lambda_i}{M_i} = \frac{\Phi_i^T \mathbb{M} r}{\Phi_i^T \mathbb{M} \Phi_i}\}$
$[\Gamma_i] = [1]$
SDOF $\ddot{u} + 2\xi\omega_1\dot{u} + \omega_1^2 u = -1 \ \ddot{u}_g(t)$
$p_1(t)$
Notice that both Λ_i and Γ_i
clearly depend on the normalization
taken for mode shape vector Φ_i
Analysis with Response Spectra
S_a
S_v
S_d
S_{a_i}
T_i
T
$[\Lambda_i] = [M]$
Max displ. in principal coord.
$p_i^{max} = |\Gamma_i| S_{d_i}$
$= |\Gamma_i| \frac{S_{a_i}}{\omega_i^2}$
Estimates of max displ. in physical coord.
$q_j^{max} = \sum_i |\Phi_{ij}| p_i^{max} = \sum_i |\Phi_{ij}| |\Gamma_i| \frac{S_{a_i}}{\omega_i^2}$
overconservative
estimate
SRSS
Modal internal actions
modal
base shear
Mode i
$q = \Phi_i$
j-th floor
$\tau_i = \sum_j E_{ij}$
This modal participating mass Λ_i defines the base
shear according to mode i
Max. base shear according to mode i
$\tau_i^{max} = \tau_i p_i^{max}$
$= \omega_i^2 \Lambda_i \Gamma_i S_{a_i}/\omega_i^2$
effective
modal
masses

• Eqs of motion

rigid body motion vector

$M\ddot{q} + C\dot{q} + Kq = -M r \ddot{u}_g(t)$

$r^T = \{1 \; 1 \; \dots \; 1\}$

$M_{ij}\ddot{q}_j + C_{ij}\dot{q}_j + K_{ij}q_j = -m_i \ddot{u}_g(t) \quad i = 1, n$

• By shifting to principal coordinates $p_i(t)$ $[q = \Phi p]$

$q_j = \Phi_{ji} p_i = \phi_{i,j} p_i$

$\mathcal{M}\ddot{p} + \mathcal{C}\dot{p} + \mathcal{K}p = -\underbrace{\Phi^T M r}\,\ddot{u}_g(t)$

Λ modal participating masses $\left\{ \Lambda_i = \phi_i^T M r = \sum_j \phi_{i,j} m_j \right\}$

(5%)

Numerical integr.

$\ddot{p} + \dots p = -\underbrace{\mathcal{M}^{-1}\Lambda}_{\Gamma}\,\ddot{u}_g(t) \rightarrow p(t)$

Γ modal participation factors $\left\{ \Gamma_i = \dfrac{\Lambda_i}{M_i} = \dfrac{\phi_i^T M r}{\phi_i^T M \phi_i} \right\}$ $\sim \dfrac{\phi_i}{\phi_i^2}$

$\rightarrow p_i(t) \quad [\Gamma_i] = [1]$

Notice that both Λ_i and Γ_i clearly depend on the normalization taken for mode shape vector ϕ_i.

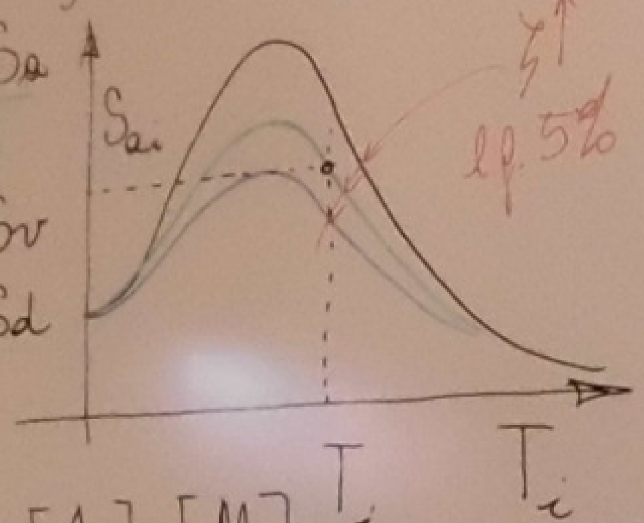
– Analysis with Response Spectra

often S_a

$\dfrac{S_a}{g}$

$\dfrac{S_a}{\omega_i} = S_v$

$\dfrac{S_a}{\omega_i^2} = S_d$



$[\Lambda_i] = [M]$

– Max displ. in principal coord.

$p_i^{max} = |\Gamma_i| S_{d_i} = |\Gamma_i| \dfrac{S_{a_i}}{\omega_i^2}$

– Estimates of max displ. in physical coord.

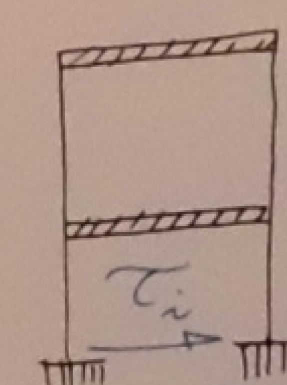
$q_j^{max} = \sum_i |\phi_{ij}| p_i^{max} = \sum_i |\phi_{ij}| |\Gamma_i| \dfrac{S_{a_i}}{\omega_i^2}$ $\quad \phi_i \Gamma_i \sim \dfrac{\phi_i^2}{\phi_i^2}$ indep. of norm. of ϕ_i

overconservative estimate

$SRSS = \sqrt{\sum_i \left(\phi_{ij} \Gamma_i \dfrac{S_{a_i}}{\omega_i^2}\right)^2}$

– Modal internal actions

modal base shear



Mode i

$q = \phi_i$

$\rightarrow \mathbb{E}_i = K\phi_i = \omega_i^2 M \phi_i = \mathbb{F}_{I_i}$

$\rightarrow \mathbb{E}_{i,j}$ j-th floor

$\tau_i = \sum_j \mathbb{E}_{i,j} = \mathbb{E}_i^T r = \omega_i^2 \underbrace{\phi_i^T M}_{\Lambda_i} r = \omega_i^2 \Lambda_i = \tau_i$

$\boxed{\Lambda_i \omega_i^2 = \tau_i}$

Thus modal participating mass Λ_i defines the base shear according to mode i.

– Max. base shear according to mode i

$\tau_i^{max} = \tau_i p_i^{max} = \omega_i^2 |\Lambda_i \Gamma_i| \dfrac{S_{a_i}}{\omega_i^2} = \Lambda_i \dfrac{\Lambda_i}{M_i} S_{a_i} = \dfrac{\Lambda_i^2}{M_i} S_{a_i} = M_{eff_i} S_{a_i}$

$M_{eff_i} = \dfrac{\Lambda_i^2}{M_i} = \dfrac{(\phi_i^T M r)^2}{\phi_i^T M \phi_i}$

effective modal masses

indep. from ϕ_i norm.

- Physical property:

$$\sum_{i \text{ modes}} M_{eff_i} = m_{tot}$$

$$\sum_i \frac{\Lambda_i^2}{M_i} = \sum_i \Lambda_i \underbrace{\frac{\Lambda_i}{M_i}}_{\Gamma_i} = \sum_i \Lambda_i \Gamma_i = \Lambda^T \Gamma$$

$$\mathbb{M}\,\mathbb{r} = \begin{bmatrix} m_1 & & & \\ & \ddots & & \\ & & m_j & \\ & & & \ddots \; m_n \end{bmatrix} \begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix} = \begin{Bmatrix} m_1 \\ \vdots \\ m_j \\ \vdots \\ m_n \end{Bmatrix}$$

$$\mathbb{r}^T \mathbb{M}\,\mathbb{r} = \{1 \dots 1 \dots 1\} \begin{Bmatrix} m_1 \\ \vdots \\ m_j \\ \vdots \\ m_n \end{Bmatrix} = \sum_j m_j$$

$$\mathbb{m} = \Phi^T \mathbb{M}\,\Phi$$

$$\Gamma = \mathbb{m}\,\Lambda$$

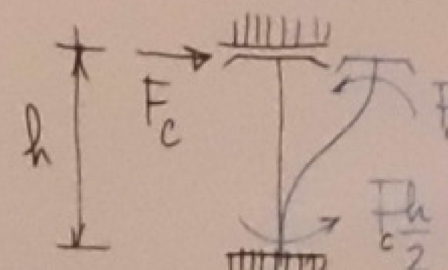
$$\Lambda = \Phi^T \mathbb{M}\,\mathbb{r}$$

$$\Lambda^T \Gamma = \Lambda^T \mathbb{m}\,\Lambda = \mathbb{r}^T \mathbb{M}\,\underbrace{\Phi\,\Phi^{-1}}_{\mathbb{I}}\,\mathbb{m}^{-1}\,\underbrace{\Phi^{-T}\,\Phi^{T}}_{\mathbb{I}}\,\mathbb{M}\,\mathbb{r}$$

$$= \mathbb{r}^T \mathbb{M}\,\mathbb{r} = \sum_j m_j = m_{tot}$$

$M_{eff_i} \rightarrow \%\, m_{tot}$ early for the first one to three modes $\sim 90\%$ of total mass

- Internal actions.



- Starting from the top floor, the total horizontal force is evaluated and divided into the columns by their stiffness ratios

$$F_{c_k} = \frac{\frac{EJ_c}{l_c}}{\frac{EJ}{l}_{tot}} \sum_j \bar{E}_j \Big\}_{\text{floor } k}$$

$k_{tot} = k_1 + k_2$

$$u = \frac{F}{k_{tot}} = \frac{F_1}{k_1} = \frac{F_2}{k_2};\; F_1 = \frac{k_1}{k_{tot}} F$$

- Bending moments at the nodes

- Rotational ... of the floors → axial forces in the columns

$$\Lambda_i = \phi_i^T \mathbb{M}\,\mathbb{r}$$

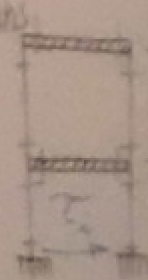
$$\Gamma_i = \frac{\Lambda_i}{M_i} = \frac{\phi_i^T \mathbb{M}\,\mathbb{r}}{\phi_i^T \mathbb{M}\,\phi_i}$$

- Estimates of max internal actions

$$\overset{max}{A}(s) \underset{SRSS}{=} \sqrt{\sum_i \left(\bar{A}_i(s)\, \Gamma_i \frac{S_{a_i}}{\omega_i^2} \right)^2}$$

S_k

modal internal actions $\bar{A}_i(s) \leftarrow S_k$ characteristic actions



$\bar{E}_i = \dots$

$q = \phi_i$

$$\overset{max}{p}_i = |\Gamma_i|\, S_{d_i} = |\Gamma_i| \frac{S_{a_i}}{\omega_i^2}$$

- Max ... according to mode i

- Estimates of max drift in physical ...

$$\overset{max}{q} = \sum_i |\phi_i|\, \overset{max}{p}_i = \sum_i |\phi_i|\, |\Gamma_i| \frac{S_{a_i}}{\omega_i^2}$$

conservative estimate

- Modal internal actions

- Physical property

$$\sum_{i}^{\text{modes}} M_{eff_i} = m_{tot}$$

$$\sum_i \frac{\Lambda_i^2}{M_i} = \sum_i \Lambda_i \underbrace{\frac{\Lambda_i}{M_i}}_{\Gamma_i} = \sum_i \Lambda_i \Gamma_i = \Lambda^T \Gamma$$

$$M r = \begin{bmatrix} m_1 & & \\ & m_j & \\ & & m_n \end{bmatrix} \begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix} = \begin{Bmatrix} m_1 \\ \vdots \\ m_j \\ \vdots \\ m_n \end{Bmatrix}$$

$$r^T M r = \{1 \cdots 1 \cdots 1\} \begin{Bmatrix} m_1 \\ \vdots \\ m_j \\ \vdots \\ m_n \end{Bmatrix} = \sum_j m_j$$

$$\mathcal{M} = \Phi^T M \Phi$$

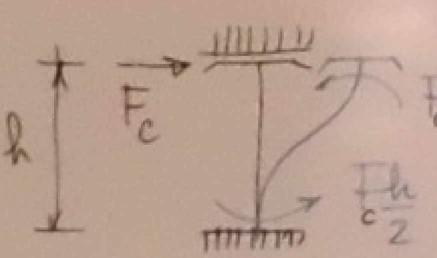
$$\Lambda = \Phi^T M r$$

$$= r^T M \underbrace{\underbrace{\Phi \Phi^{-1}}_{I} M^{-1} \underbrace{\Phi^{-T} \Phi^T}_{I}}_{I} M r$$

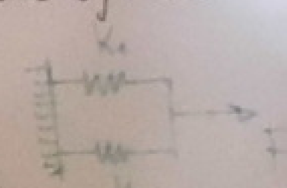
$$= r^T M r = \sum_j m_j = m_{tot}$$

$M_{eff_i} \rightarrow \% \, m_{tot}$ easily for the first one to three modes $\sim 90\%$ of total mass

- Internal actions.



- Starting from the top floor, the total horizontal force is evaluated and divided into the columns by their stiffness ratios



$$F_{c_k} = \frac{\frac{EJ_c}{l_c}}{\frac{EJ}{l}_{tot}} \sum_j \bar{E}_{i_j} \Big|_{\text{floor } k}$$

$K_{tot} = K_1 + K_2$

- Bending moments at the nodes $u = \frac{F}{K_{tot}} = \frac{F_1}{K_1} = \frac{F_2}{K_2}$; $F_1 = \frac{K_1}{K_{tot}} F$

- Estimates of max internal actions

$$\mathcal{A}^{max}_{S_k}(s) \underset{SRSS}{=} \sqrt{\sum_i \left(\bar{\mathcal{A}}_i(s) \Gamma_i \frac{S_{a_i}}{\omega_i^2} \right)^2}$$

$\bar{\mathcal{A}}(s) \leftarrow$ modal internal actions

S_k characteristic sections

$$\Lambda_i = \Phi_i^T M r$$

$$\Gamma_i = \frac{\Lambda_i}{M_i} = \frac{\Phi_i^T M r}{\Phi_i^T M \Phi_i}$$

$$p_i^{max} = |\Gamma_i| S_{d_i} = |\Gamma_i| \frac{S_{a_i}}{\omega_i^2}$$

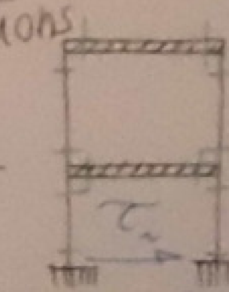
- Estimates of max disp in physical coord

$$q_j^{max} \simeq \sum_i |\phi_{ij}| p_i^{max} = \sum_i |\phi_{ij}| |\Gamma_i| \frac{S_{a_i}}{\omega_i^2}$$

conservative estimate

$$SRSS = \sqrt{\sum_i \left(\phi_{ij} \Gamma_i \frac{S_{a_i}}{\omega_i^2} \right)^2}$$

- Modal internal actions



Mode i

$q = \Phi_i$

j-th floor

$$\bar{E}_i = K \Phi_i = \omega_i^2 M \Phi_i = F_{I_i}$$

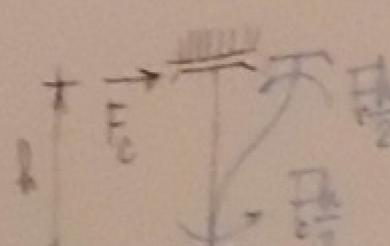
$$\tau_i = \sum_j \bar{E}_{i_j} = \bar{E}_i^T r = \omega_i^2 \underbrace{\Phi_i^T M r}_{\Lambda_i} = \omega_i^2 \Lambda_i = \tau_i$$

Thus modal participating mass Λ_i defines the base shear according to mode i

- Max. base shear according to mode i

$$\tau_i^{max} = \tau_i p_i^{max} = \omega_i^2 \Lambda_i |\Gamma_i| \frac{S_{a_i}}{\omega_i^2} = \Lambda_i \frac{\Lambda_i}{M_i} S_{a_i}$$

– Internal actions.



– Starting from the top floor, the total horizontal force is evaluated and divided into the columns by their stiffness ratios

$$F_{c_k} = \frac{\frac{EI_k}{l_k}}{\frac{EI}{l\,tot}} \sum_j \bar{E}_{ij} \quad k\text{ floor}$$

– Bending moments at the nodes

$$u = \frac{F}{k_{tot}} = \frac{F_1}{k_1} = \frac{F_2}{k_2}; \quad F_i = \frac{k_i}{k_{tot}} F$$

– Estimates of max internal actions:

$$\mathcal{A}^{max}(s) \underset{SRSS}{=} \sqrt{\sum_i \left(\bar{\mathcal{A}}_i(s)\, \Gamma_i \frac{S_{a_i}}{\omega_i^2} \right)^2}$$

S_k

$\bar{\mathcal{A}}_i(s) \leftarrow$ modal internal actions

S_k characteristic sections

$\Lambda_i = \Phi_i^T M r$

$$\Gamma_i = \frac{\Lambda_i}{M_i} = \frac{\Phi_i^T M r}{\Phi_i^T M \Phi_i}$$

$$p_i^{max} = |\Gamma_i| S_{d_i} = |\Gamma_i| \frac{S_{a_i}}{\omega_i^2}$$

– Estimates of max displ. in physical coord.

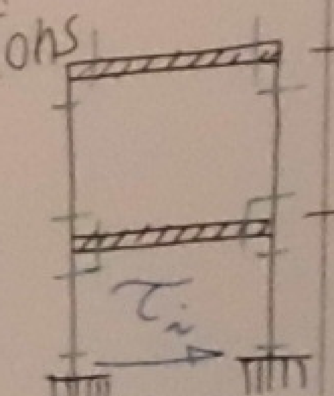
$$q_j^{max} \leq \sum_i |\phi_{ij}| p_i^{max} = \sum_i |\underbrace{\phi_{ij}| |\Gamma_i}_{\sim \phi_i \Gamma_i \sim \frac{\phi_i^2}{\phi_i^2}}| \frac{S_{a_i}}{\omega_i^2}$$

conservative estimate

$$SRSS = \sqrt{\sum_i \left(\phi_{ij} \Gamma_i \frac{S_{a_i}}{\omega_i^2} \right)^2}$$

indep. of norm. of ϕ_i

– Modal internal actions



Mode i

$q = \Phi_i$

$$\bar{\mathbb{E}}_i = K \Phi_i = \omega_i^2 M \Phi_i = \mathbb{F}_{I_i}$$

$\bar{E}_{ij}$ jth floor

$$\tau_i = \sum_j \bar{E}_{ij} = \bar{\mathbb{E}}_i^T r = \omega_i^2 \underbrace{\Phi_i^T M}_{\Lambda_i} r = \boxed{\omega_i^2 \Lambda_i = \tau_i}$$

$$\boxed{\Lambda_i \omega_i^2 = \tau_i}$$

Thus modal participating mass Λ_i defines the base shear according to mode i.

– Max. base shear according to mode i

$$\tau_i^{max} = \tau_i p_i^{max} = \omega_i^2 \Lambda_i \Gamma_i \frac{S_{a_i}}{\omega_i^2} = \Lambda_i \frac{\Lambda_i}{M_i} S_{a_i} = \frac{\Lambda_i^2}{M_i} S_{a_i} = M_{eff_i} S_{a_i}$$

$$M_{eff_i} = \frac{\Lambda_i^2}{M_i} = \frac{(\Phi_i^T M r)^2}{\Phi_i^T M \Phi_i}$$

effective modal masses

indep. from Φ_i norm.