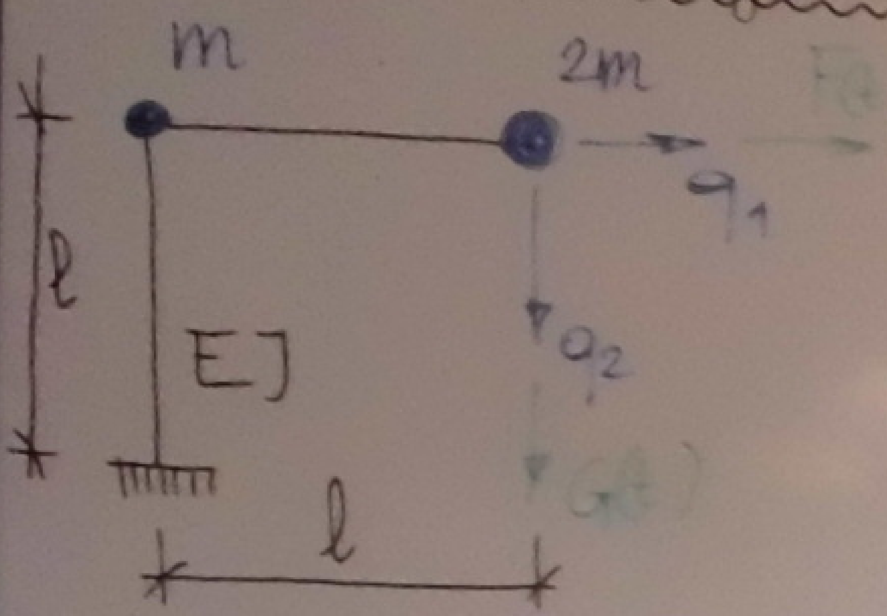
Solution of the eigenvalue problem



$$\mathbb{K}\,\Phi_i = \omega_i^2\,\mathbb{M}\,\Phi_i \Rightarrow \det(\mathbb{K}-\omega_i^2\,\mathbb{M}) = 0$$

$\hookrightarrow \omega_i^2$

$$\frac{6}{7}\frac{EJ}{l^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}\Phi_i = \omega_i^2\, m \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\Phi_i$$

$$6\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}\Phi_i = \omega_i^2 \frac{ml^3}{EJ}\, 7\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\Phi_i$$

$$\lambda_i = \omega_i^2\frac{ml^3}{EJ}\ ;\ \omega_i^2 = \lambda_i\frac{EJ}{ml^3}$$

$$\left(6\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix} - \lambda_i\, 7\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right)\Phi_i = 0$$

$$\omega_i = \sqrt{\lambda_i}\sqrt{\frac{EJ}{ml^3}}$$

Characteristic equ.

$$(6\cdot 8 - 7\cdot 3\lambda_i)(6\cdot 2 - 7\cdot 2\lambda_i) - 6^2\cdot 3^2 = 0$$

$$3(16 - 7\lambda_i)\,2(6 - 7\lambda_i) - 6\cdot 6\cdot 9 = 0$$

$$7\cdot 7\lambda_i^2 - 7\cdot 22\lambda_i + \underbrace{16\cdot 6 - 6\cdot 9}_{7\cdot 6} = 0$$

$$\boxed{7\lambda_i^2 - 22\lambda_i + 6 = 0}$$

Solutions (eigenvalues):

$$\lambda_{1,2} = \frac{11 \mp \sqrt{121-42}}{7} = \frac{11 \mp \sqrt{79}}{7}$$

$\lambda_1 = .3017 \rightarrow \omega_1 = .5493\sqrt{\frac{EJ}{ml^3}}$ first natural freq.

$\lambda_2 = 2.841 \rightarrow \omega_2 = 1.686\sqrt{\frac{EJ}{ml^3}}$ 2nd and last natural freq.

Eigenvectors:

$$\underbrace{\begin{bmatrix} 48-\lambda_i 21 & -18 \\ -18 & 12-14\lambda_i \end{bmatrix}}\begin{Bmatrix} \phi_{i1} \\ \phi_{i2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

singular for $\lambda_{1,2}$

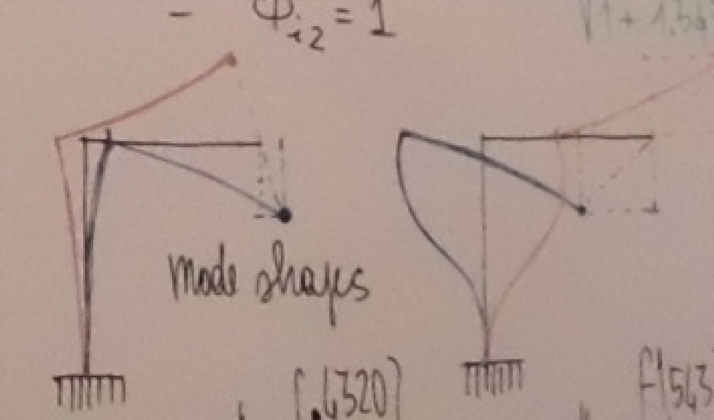
$$\begin{cases} 3(16-7\lambda_i)\phi_{i1} - 36\phi_{i2} = 0 \\ -2\cdot 9\ \phi_{i1} + 2(6-7\lambda_i)\phi_{i2} = 0 \end{cases}$$

$$\frac{\phi_{i1}}{\phi_{i2}} = \frac{6}{16-7\lambda_i} = \frac{6-7\lambda_i}{9}$$

$\lambda_1 \rightarrow .4320$

$\lambda_2 \rightarrow -1.543$

- Normalizations
  - $\phi_{i2} = 1$



mode 1 $\Phi_1 = \begin{Bmatrix} .4320 \\ 1 \end{Bmatrix}$ $\lambda_1 = .3017$

mode 2 $\Phi_2 = \begin{Bmatrix} -1.543 \\ 1 \end{Bmatrix}$ $\lambda_2 = 2.841$

- Unit norm

$$\Phi_i \rightarrow \alpha_i\Phi_i = \frac{\Phi_i}{\sqrt{\Phi_i^T\Phi_i}}$$

$$\alpha_i = \frac{1}{\sqrt{\phi_{i1}^2 + \underbrace{\phi_{i2}^2}_{1}}}$$

$$\alpha_1 = \frac{1}{\sqrt{1+.4320^2}} = .9180 \rightarrow \Phi_1 = \begin{Bmatrix} .3966 \\ .9180 \end{Bmatrix}$$

$$\alpha_2 = \frac{1}{\sqrt{1+1.543^2}} = .5439 \rightarrow \Phi_2 = \begin{Bmatrix} -.8392 \\ .5439 \end{Bmatrix}$$

- Unitary modal masses

$$\Phi_i^T\,\mathbb{M}\,\Phi_i = M_i = \begin{cases} 1, \mathbb{M} = \mathbb{I} \\ m, \mathbb{M} = m\mathbb{I} \end{cases}$$

$$\{\phi_{i1}\ \phi_{i2}\}\begin{Bmatrix} 3\phi_{i1} \\ 2\phi_{i2} \end{Bmatrix} = 1$$

$$3\phi_{i1}^2 + 2\phi_{i2}^2 = 1 \Rightarrow \beta_i = \frac{1}{\sqrt{3\phi_{i1}^2 + 2\phi_{i2}^2}}$$

$$\Phi_1 = \begin{Bmatrix} .2700 \\ .6250 \end{Bmatrix},\ \Phi_2 = \begin{Bmatrix} -.5103 \\ .3307 \end{Bmatrix}$$

- Shift to principal coordinates $q = \Phi p$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} .2700\,p_1 + -.5103\,p_2 \\ .6250\,p_1 + .3307\,p_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} .2700 & -.5103 \\ .6250 & .3307 \end{bmatrix}}_{\Phi}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

Inverse transf.

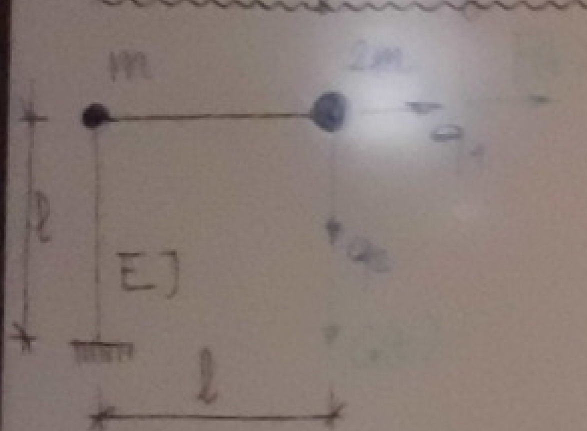
$$p = \Phi^{-1} q\ ;\ \Phi^{-1} = \frac{1}{\det\Phi}\begin{bmatrix} .3307 & .5103 \\ -.6250 & .2700 \end{bmatrix}$$

$$\Phi_i^T\,\mathbb{M}\,q = \underbrace{\Phi_i^T\,\mathbb{M}\,\Phi_i}_{m}\, p_i \rightarrow p_i = \frac{\Phi_i^T\,\mathbb{M}\,q}{m} = \Phi_i^T\begin{Bmatrix} 3q_1 \\ 2q_2 \end{Bmatrix}$$

$$\begin{cases} p_1 = 3\phi_{11}q_1 + 2\phi_{12}q_2 \\ p_2 = 3\phi_{21}q_1 + 2\phi_{22}q_2 \end{cases}$$

$$\begin{cases} p_1 = .8101\,q_1 + 1.250\,q_2 \\ p_2 = -1.531\,q_1 + .6614\,q_2 \end{cases}$$

Solution of the eigenvalue problem



$\mathbb{K}\Phi_i = \omega_i^2 \mathbb{M}\Phi_i \Rightarrow \det(\mathbb{K} - \omega_i^2 \mathbb{M}) = 0$

$\hookrightarrow \omega_i^2$

$\frac{6}{7}\frac{EJ}{\ell^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}\Phi_i = \omega_i^2 m \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\Phi_i$

Characteristic equ.

$(6\cdot 8 - 7\cdot 3\lambda_i)(6\cdot 2 - 7\cdot 2\lambda_i) - 6^2\, 3^2 = 0$

$\cancel{3}(16 - 7\lambda_i)\cancel{2}(6 - 7\lambda_i) - \cancel{6}\,6\,9 = 0$

$7\,7\lambda_i^2 - 7\,22\lambda_i + \underbrace{16\,6 - 6\,9}_{7\,6} = 0$

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$\lambda_{1,2} = \frac{11 \mp \sqrt{121 - 42}}{7} = \frac{11 \mp \sqrt{79}}{7}$

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Eigenvectors:

$\underbrace{\begin{bmatrix} 48 - \lambda_i 21 & -18 \\ -18 & 12 - 14\lambda_i \end{bmatrix}}_{\text{singular for } \lambda_{1,2}}\begin{Bmatrix} \phi_{i1} \\ \phi_{i2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$\begin{cases} 3(16 - 7\lambda_i)\phi_{i1} - 3\,6\phi_{i2} = 0 \\ -2\,9\,\phi_{i1} + 2(6 - 7\lambda_i)\phi_{i2} = 0 \end{cases}$

$\frac{\phi_{i1}}{\phi_{i2}} = \frac{6}{16 - 7\lambda_i} = \frac{6 - 7\lambda_i}{9}$

$\lambda_1 \longrightarrow .4320$

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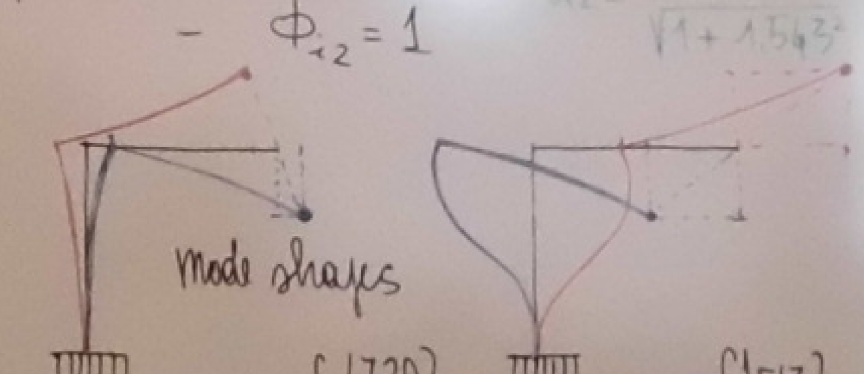
- Normalizations
  - $\phi_{i2} = 1$



mode 1 $\Phi_1 = \begin{Bmatrix} .4320 \\ 1 \end{Bmatrix}$ $\lambda_1 = .3017$

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- Unit norm

$\Phi_i \rightarrow \alpha_i \Phi_i = \frac{\Phi_i}{\sqrt{\Phi_i^T \Phi_i}}$

$\alpha_i = \frac{1}{\sqrt{\phi_{i1}^2 + \underbrace{\phi_{i2}^2}_{1}}}$

$\alpha_1 = \frac{1}{\sqrt{1 + .4320^2}} = .9180 \rightarrow \Phi_1 = \begin{Bmatrix} .3966 \\ .9180 \end{Bmatrix}$

$\alpha_2 = \frac{1}{\sqrt{1 + 1.543^2}} = .5439 \rightarrow \Phi_2 = \begin{Bmatrix} -.8392 \\ .5439 \end{Bmatrix}$

- Unitary modal masses

$\Phi_i^T \mathbb{M} \Phi_i = M_i = \begin{cases} 1, \ \mathbb{M} = \mathbb{I} \\ m, \ \mathbb{M} = m\mathbb{I} \end{cases}$

$\{\phi_{i1}\ \phi_{i2}\}\, m \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} \phi_{i1} \\ \phi_{i2} \end{Bmatrix} = m$

$\{\phi_{i1}\ \phi_{i2}\}\begin{Bmatrix} 3\phi_{i1} \\ 2\phi_{i2} \end{Bmatrix} = 1$

$3\phi_{i1}^2 + 2\phi_{i2}^2 = 1 \Rightarrow \beta_i = \frac{1}{\sqrt{3\phi_{i1}^2 + 2\phi_{i2}^2}}$

$\Phi_1 = \begin{Bmatrix} .2700 \\ .6250 \end{Bmatrix}, \quad \Phi_2 = \begin{Bmatrix} -.5103 \\ .3307 \end{Bmatrix}$

- Shift to principal coordinates $q = \Phi p = \Sigma \Phi_i p_i$

$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} .2700\, p_1 + -.5103\, p_2 \\ .6250\, p_1 + .3307\, p_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} .2700 & -.5103 \\ .6250 & .3307 \end{bmatrix}}_{\Phi}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$

Inverse transf.

$p = \Phi^{-1} q; \quad \Phi^{-1} = \frac{1}{\underbrace{\det \Phi}_{.4087}}\begin{bmatrix} .3307 & .5103 \\ -.6250 & .2700 \end{bmatrix} = \begin{bmatrix} .8101 & 1.250 \\ -1.531 & .6614 \end{bmatrix}$

$\Phi_i^T \mathbb{M} q = \underbrace{\Phi_i^T \mathbb{M} \Phi_i}_{m} p_i \rightarrow p_i = \frac{\Phi_i^T \mathbb{M} q}{m} = \Phi_i^T \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$

$\begin{cases} p_1 = 3\phi_{11} q_1 + 2\phi_{12} q_2 \\ p_2 = 3\phi_{21} q_1 + 2\phi_{22} q_2 \end{cases} = \Phi_i^T \begin{Bmatrix} 3q_1 \\ 2q_2 \end{Bmatrix}$

$\begin{cases} p_1 = .8101\, q_1 + 1.250\, q_2 \\ p_2 = -1.531\, q_1 + .6614\, q_2 \end{cases}$


- Shift to principal coordinates
q = Φ p = Σ Φ_i p_i
q1 = .2700 p1 + -.5103 p2
q2 = .6250 p1 + .3307 p2
.2700 -.5103
.6250 .3307
Inverse transf.
p = Φ^-1 q ; Φ^-1 = 1 / det Φ
.3307 .5103
-.6250 .2700
.8101 1.250
-1.531 .6614
.4082
Φ_i^T M q = Φ_i^T M Φ_i p_i → p_i = Φ_i^T M q / m
m
p1 = 3 Φ11 q1 + 2 Φ12 q2
p2 = 3 Φ21 q1 + 2 Φ22 q2
p1 = .8101 q1 + 1.250 q2
p2 = -1.531 q1 + .6614 q2
- Unit norm
- Unitary modal masses
Normalizations
λ1 → .4320
λ2 → -1.543
Φ1 = .3966 .9180
Φ2 = -.8392 .5439
Mode shapes
mode 1
mode 2
Φ1 = .2700 .6250
Φ2 = -.5103 .3307

- Stiffness matrix in principal coordinates
K = Φ^T K Φ
generalized eigen pb
K_i = ω_i^2 M_i = ω_i^2 m
modal stiffness
diagonal matrix
- Principal coordinates:
- Iterative solution of the eigenvalue problem(s)
Inverse vectorial iteration
G = K^-1 M → G Φ_1 = 1/ω_1^2 Φ_1
H = M^-1 K → H Φ_2 = ω_2^2 Φ_2
standard eigen pb(s)
where
estimate of the eigen
For instance
converges to
Iterative method
Gram-Schmidt

- Stiffness matrix in principal coordinates

$$\mathcal{K} = \Phi^T K \Phi$$

$$K \Phi_i = \omega_i^2 M \Phi_i$$

$$K_i = \omega_i^2 M_i = \omega_i^2 m$$

modal stiffness $= \lambda_i \frac{EJ}{\ell^3} \Rightarrow \mathcal{K} = \frac{EJ}{\ell^3}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

diagonal matrix

$$K = \frac{6}{7}\frac{EJ}{\ell^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}; \quad M = m\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

- Principal coordinates: $M\ddot{q} + Kq = Q(t) =$

$\Rightarrow p_i(t)$

- Iterative solution of the eigenvalue problem (Inverse vectorial iteration)

$$G = K^{-1}M \rightarrow G\Phi_i = \frac{1}{\omega_i^2}\Phi_i$$

$$H = M^{-1}K \rightarrow H\Phi_i = \omega_i^2\Phi_i$$

standard eigv. pb(s)

$$\Phi_i^{(k)} \rightarrow G\Phi_i^{(k)} \simeq \Phi_i^{(k+1)} \xrightarrow{\text{converge to}} \Phi_1, \omega_1$$

$$\Phi_i^{(k)} \rightarrow H\Phi_i^{(k)} \simeq \Phi_i^{(k+1)} \longrightarrow \Phi_n = \Phi_2, \omega_n$$

where

$$G = \frac{7\ell^3}{6EJ}\frac{1}{7}\begin{bmatrix} 2 & 3 \\ 3 & 8 \end{bmatrix} m \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\frac{m\ell^3}{EJ}\underbrace{\begin{bmatrix} 6 & 6 \\ 9 & 16 \end{bmatrix}}_{\tilde{G}}$$

$$H = \frac{1}{m}\begin{bmatrix} \frac{1}{3} & \\ & \frac{1}{2} \end{bmatrix}\frac{6}{7}\frac{EJ}{\ell^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix} = \frac{1}{7}\frac{EJ}{m\ell^3}\underbrace{\begin{bmatrix} 16 & -6 \\ -9 & 6 \end{bmatrix}}_{\tilde{H}}$$

estimate of the eigenv.

$$\omega_i^2 = \frac{\Phi_i^T\Phi_i}{\Phi_i^T G \Phi_i} = \frac{\Phi_i^T H \Phi_i}{\Phi_i^T\Phi_i}$$

$$= \frac{6EJ}{m\ell^3}\frac{\Phi_1^T\Phi_1}{\Phi_1^T\tilde{G}\Phi_1} = \frac{1}{7}\frac{EJ}{m\ell^3}\frac{\Phi_1^T\tilde{H}\Phi_1}{\Phi_1^T\Phi_1}$$

- For instance

- $\Phi_1^{(0)} = \begin{Bmatrix} 0.5 \\ 1 \end{Bmatrix}$
- $\hat{\Phi}_1^{(1)} = \tilde{G}\Phi_1^{(0)} = \begin{bmatrix} 6 & 6 \\ 9 & 16 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 9 \\ 20.5 \end{Bmatrix} \Rightarrow \lambda_1^{(1)} = .3015$
- $\Phi_1^{(1)} = \begin{Bmatrix} .4390 \\ 1 \end{Bmatrix}$
- $\hat{\Phi}_1^{(2)} = \tilde{G}\Phi_1^{(1)} = \begin{Bmatrix} 8.634 \\ 19.95 \end{Bmatrix} \Rightarrow \lambda_1^{(2)} = .3017$
- $\Phi_1^{(3)} = \begin{Bmatrix} .4329 \\ 1 \end{Bmatrix}$

fast convergence in a few iterations

Iterative method

loop over all iterations k

- Trial vector $\Phi_i^{*(k)}$ (e.g. random vector)
- Update $G\Phi_i^{*(k)} = \hat{\Phi}_i^{(k+1)}$
- Normalization $\hat{\Phi}^{(k+1)} \rightarrow \Phi^{(k+1)}$
- Gram-Schmidt normaliz.
- Eigenvalue evaluation $\omega_i^2 =$
- Convergence check $\left|\frac{\omega_i^{2(k+1)} - \omega_i^{2(k)}}{\omega_i^{2(k)}}\right| < tol = 10^{-4} \div 10^{-5}$

Gram-Schmidt normaliz. (ℓ determined eigenvectors)

$$\hat{\Phi}_{\ell+1} = \Phi^* - \sum_{k=1}^{\ell}\alpha_k\Phi_k$$

coefficients to be determined so that: $\hat{\Phi}_{\ell+1} \perp$ to all prev. Φ_k, $k=\ell$

Iterative solution of the eigenvalue problem
(Inverse vectorial iteration)
converge to
For instance
fast convergence in a few iterations
loop over all iterations k
Iterative method
Trial vector
(e.g random vector)
Update
Normalization
Gram-Schmidt normaliz.
Eigenvalue evaluation
by Rayleigh ratio
Convergence check
Gram-Schmidt normaliz.
(l: determined eigenvectors)
coefficients to be determined so that: