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# LOCALIZATION ANALYSIS OF DAMAGED MATERIALS

by

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# LOCALIZATION ANALYSIS OF DAMAGED MATERIALS

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Localization Analysis of Damaged Materials

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Continuous and discontinuous failure indicators at the constitutive level have been extensively developed in recent years in the well-established framework of the theory of plasticity in continuum mechanics. In fact, even if the onset of bifurcation is detected, the medium is always treated as a continuum and concepts of discrete cracking are not considered.

On the other hand, various models describing stiffness degradation, always observed in experiments with loading/unloading/reloading conditions, have been recently proposed in the framework of continuum damage mechanics, even though a unified description of these models is still not established.

Because of the underlying hypothesis of a continuum medium, the localization indicators can still be investigated, when an expression of the tangent operator for the material is given. Thus the extension of the localization analysis to this kind of material models is conceivable when a form for the tangent stiffness is provided. The purpose is to compare these results with the classical outcomes of the theory of plasticity.

Damage models are outlined here as a particular case of the so-called elastic-fracturing formulation. In this theory a constitutive secant law $\boldsymbol{\sigma}$-$\boldsymbol{\epsilon}$ for the material is prescribed, together with a loading function defining the state of damage and a fracturing rule for the evolution of the material stiffness. The model can be associated, in the sense specified in this thesis, and the tangent operator can be fully symmetric.

In this thesis, the localization analysis of the general class of elastic-fracturing materials is outlined. Then, the traditional scalar damage models, where the scalar variable affects all the components of the stiffness in the same way, is considered. The localization diagnosis is performed analytically and geometrically. The expression of the critical hardening modulus and critical localization directions are provided and evaluated for different loading cases.

# DEDICATION

*To my parents, Ezio and Elena*

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# CHAPTER 1

# INTRODUCTION

The apparence of shear bands in metals and soils is well known in the literature and it is due to a state of high deformation localized in a small region of the structure. This phenomenon is usually called strain localization, although different meanings can be attributed to this term. Localization analysis provides a description and possibly a prevision of this phenomenon.

The conditions for the onset of localization have been extensively analyzed in the framework of elastic-plasticity. The results of localization analysis seem to match in general the experimental observations on simple specimens under controlled loading conditions. On the other hand, the need to solve structural problems requires the implementation of these models in a finite element computer code. In this case the problem of mesh dependence strongly affects the computational solution, because of the underlying modification of the governing differential equations.

Regarding this mesh sensitivity, two possible approaches are at hand. On one side it is possible to improve the finite element technology with mesh realignment procedures or enhanced elements with discontinuous interpolation functions. The second possibility is to regularize the description of the material behavior at the constitutive level. A combination of both approaches seems to be most fruitful.

Regarding the second approach, we must say that, before regularizing, it is still necessary to understand at least the behavior of the existing material models with regard to localization. The regularization should be studied later. In general, any material model based on tangential constitutive law, relating the stress and strain rates can be analyzed for localization, once the correspondent localization operator is defined.

Thus it seems appropriate to look at the existing models and examine whether they provide a suitable description of the tangent operators reflecting different localization properties. Thus localization analysis could provide an interesting unifying tool, useful for the comparison of different failure descriptions.

This is in particular the case of recent material models which describe the degradation of the stiffness rather than the strength, as in the theory of plasticity. On the other hand, experiments involving loading/unloading/reloading conditions show elastic/plastic interaction, i.e. the stiffness degrades during plastic loading. Attempts to describe this phenomena gave rise to a number of innovative formulations often combined with plasticity, using the thermodynamic concepts of internal variables. Recent work in continuum damage mechanics did result in numerous constitutive proposals. Thus it is essential to see what localization analysis reveals about the failure prediction of these models.

Before going into localization analysis, the first step is to develop a clear understanding of the models and, if possible, their unified description. To this end, a general expression of the tangent operator is required with clear interpretation of all the variables involved.

Once the tangent operator is available the localization analysis can be performed. Such studies can be done numerically, of course, although the analytical approach is always the most appealing, because it provides general solutions. Recently, analytic solutions were becoming available for a broad class of elastic-plastic material models, thus an attempt to enlarge these solutions to other constitutive descriptions seems reasonable.

The organization of this thesis follows three general themes: review of localization concepts, unified description of stiffness degrading materials and localization analysis of such material models. Thus the thesis is organized in three chapters.

In Chapter 2 the meaning and the derivation of the acoustic tensor will be restated. First the concepts of wave propagation in elastic solids will be addressed and the acoustic tensor will be introduced for the first time within this framework.

The same derivation is then extended to the elastic-plastic case. To start with, the stress and strain-based elastic-plastic formulations will be reviewed and the expressions of the continuum tangent operators will be derived. The interrelations between the stress and strain-based formulations will thereby be clarified.

Localization arises from two different arguments: jump condition of the second order field variables or vanishing elastic-plastic wave speed. The meaning of these two concepts will be extensively discussed. Localization results in the loss of positive definiteness of the localization operator. The mathematical condition can be simplified considering the generalized eigenvalue problem associated with the acoustic tensor. This derivation will be presented in one of the last sections of Chapter 2.

In conclusion, the classical results of $J_2$-plasticity for plane stress will be presented and compared with the outcomes of the general $3D$ analysis. The graphic phase velocity plot representation will be used to illustrate various loading cases.

In Chapter 3 a unified description of elastic-fracturing and damage models will be presented. The attempt is here to provide a general framework that can unify most of the existing models in the literature. In fact, at the end of the chapter, some existing models will be reviewed and presented within the general framework. All models possess the same underlying structure and the expression of the tangent operator can be cast into a common format.

The concepts of associativity at various levels will be examined. Also, the relation between flow, fracturing and damage rules, will be outlined. In fact these different evolution equations represent the same phenomenon: some irreversible process leading to energy dissipation. This energy dissipation results in the degradation of the stiffness. In Chapter 3 only the elastic damage behavior is analyzed. The coupling with plasticity should be subject of further work.

Finally, Chapter 4 contains the localization analysis of the proposed elastic-fracturing and damage models; it constitutes the main part of the thesis, including new developments. The general concept for diffuse and discontinuous failure conditions for elastic-fracturing and damage materials will be outlined. Then, analytical solutions will be provided for the class of scalar damage models in the traditional sense, i.e. the models in which the scalar variable affects all the components of the stiffness in the same way. We are going to show that existing solutions of localization analysis for stress-based elastic-plastic materials apply directly to that case. Also, an extension for strain-based formulations will be presented, at least for the plane stress case.

The geometric interpretation of the localization condition in the Mohr plane will also be examined. The analytical results along this derivation will be compared with the previous ones and various loading cases will be analyzed. The geometric representation will be outlined for the general $3D$ and plane stress cases, within the strain-based framework. A general discussion of the validity of the representation for stress or strain-based elastic-plastic or elastic-fracturing materials will also be addressed.

In Chapter 5 final conclusions and remarks for further work will be made.

# CHAPTER 2

## LOCALIZATION ANALYSIS: BASIC CONCEPTS

The appearance of shear bands in metals subjected to uniaxial tension is well known in the literature under the name of *Lüders bands* (NADAI [50]). These shear bands show a failure condition inside the material, with large deformations being concentrated in the narrow region of the band. This phenomenon is usually attributed to the formation of spatial discontinuities of the strain increment field and is called *localization of the deformation* or *strain localization*.

There were several theoretical attempts to capture the onset of localization and to determine the direction and the amplitude of these shear bands. Following the earlier works of NADAI [50], THOMAS [82], HILL [29], RUDNICKI AND RICE [65], recently many researchers studied the problem in a systematic manner, using the flow theory of plasticity and developed general mathematical conditions governing the failure behavior (e.g. SOBH [75], BORRÉ AND MAIER [11], PERIĆ [58], OTTOSEN AND RUNESSON [56], WILLAM AND DIETSCHE [88], NEILSEN AND SCHREYER [52], for a comprehensive review).

On the computational front, substantial research effort was directed to eliminate the mesh dependence of the solution, by developing standard finite elements formulations which were able to follow the post-bifurcation localization, using realignment procedures or special enhanced finite elements with discontinuous interpolation capabilities.

These finite element capabilities might be combined in conjunction with advanced *regularization theories* at the constitutive level that prevent localization to appear in a discontinuous fashion. The medium is considered to remain a continuum, with high deformation gradients being concentrated into a finite small region of the body, which remains still continuous.

These enriched theories can be based on both *local* and *non-local* approaches (WILLAM AND DIETSCHE [88]). The classic local approach considers the generic response function (e.g. strain) at a certain point of the body as a function of other point variables, evaluated at the same material point (e.g. the local strain measure depends only on the displacement gradient at that point).

In non-local theories gradients of the displacement function are evaluated in the vicinity of the material point, thus a spatial average is taken into account to evaluate the point value (higher grade material theories) (e.g. PIJAUDIER-CABOT AND BAŽANT [59]).

Higher order local theories use some non-local concepts, introducing geometric parameters in terms of a characteristic length, that take into account the non-local effect (*second order gradient formulation*, *micropolar Cosserat theory*).

The failure state inside the body can be detected according to two different concepts, using the following terminology:

- *diffuse* failure
- *discontinuous* failure

The first one is related to material instability at the constitutive level and does not imply bifurcations of field variables with jumps and formation of spatial discontinuities. The second one, termed discontinuous bifurcation, indicates the appearance of a discontinuity surface exhibiting jumps in the value of certain variables (e.g. $\dot{\boldsymbol{\epsilon}}$ for a discontinuity of the $2^{nd}$ order).

The two failure concepts are different and in general the conditions that govern the appearance of the two phenomena are not the same, depending on the particular state of stress/strain and on the constitutive model used.

Localization analysis has been extensively developed in the framework of elastic-plasticity, but it is theoretically possible to extend it to any constitutive model that provides an expression for the tangent operator $\mathbf{E}_t$ relating the rates of stresses to that of strains. This is for example the case for material models with stiffness degradation (*elastic-fracturing, elastic-damage*) that will be considered in the following chapters of this thesis.

Localization features are normally detected via spectral analysis of a $2^{nd}$ order tensor conveniently defined by double contraction of the tangent operator with the normal vector of the possible discontinuity surface. This operator is often called the *characteristic*, *localization* or *acoustic tensor* since its definition originates in wave propagation arguments.

In the following sections of this chapter, the concepts of wave propagation in elastic solids, which lead to the definition of the elastic acoustic tensor and the classic approach of plastic localization, will be reviewed together with some illustrating examples for $J_2$-plasticity.

## 2.1 The linear elastic operators

In linear elasticity, the constitutive law, i.e. the relation between stresses and strains, can be expressed in a finite form by means of the $4^{th}$ order tensor $\mathbf{E}_o$, which possesses two minor symmetries and a major symmetry. Assuming the existence of a strain energy function, the elasticity operator contains in general 21 independent constants.

$$\boldsymbol{\sigma} = \mathbf{E}_o : \boldsymbol{\epsilon} \qquad \Leftrightarrow \qquad \sigma_{ij} = E^o_{ijkl}\, \epsilon_{kl} \tag{2.1}$$

The symbol " : " denotes double tensor contraction, as shown in indicial notation.

In the case of isotropic behavior, the elastic operator is expressed in terms of two independent moduli, the *Lamé's constants* $\lambda$ and $\mu$, which are related to Young's modulus $E$ and to the Poisson's ratio $\nu$:

$$E^o_{ijkl} = \lambda\ \delta_{ij}\delta_{kl} + 2\,\mu\ \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.2}$$

$$\lambda = \frac{\nu\,E}{(1+\nu)\,(1-2\nu)} \quad ; \qquad \mu = \frac{E}{2\,(1+\nu)}$$

The elastic operator is obtained considering that the most general isotropic $4^{th}$ order tensor $\mathbf{T}$ can be formed from all possible combinations of the $2^{nd}$ order Kronecker Delta tensor (here indicated as $\mathbf{I}_2$):

$$T_{ijkl} = \alpha_1\ \delta_{ij}\delta_{kl} + \alpha_2\ \frac{1}{2}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha_3\ \frac{1}{2}\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{2.3}$$

$$\mathbf{T} = \alpha_1\,\mathbf{I}_2 \otimes \mathbf{I}_2 + \alpha_2\,\mathbf{I}_4^s + \alpha_3\,\mathbf{I}_4^{sk} \tag{2.4}$$

In the last expression the compact notation is used and the symbol " $\otimes$ " denotes the outer product of two tensors. $\mathbf{I}_4^s$ and $\mathbf{I}_4^{sk}$ are the $4^{th}$ order symmetric (minor and major symmetries) and skew-symmetric unit tensors respectively. Thereby the $4^{th}$ order unit tensor $\mathbf{I}_4$, which defines a linear transformation of a $2^{nd}$ order tensor into itself (as $A_{ij} = I^4_{ijkl}\,A_{kl}$, i.e. $\mathbf{A} = \mathbf{I}_4 : \mathbf{A}$), must be isotropic

$$\mathbf{I}_4 = \frac{1}{2}\,(\mathbf{I}_4^s + \mathbf{I}_4^{sk}) \qquad \Leftrightarrow \qquad I^4_{ijkl} = \delta_{ik}\,\delta_{jl} \tag{2.5}$$

and satisfies automatically only the major symmetries.

Another interesting expression of the elastic operator, following the notation of NEILSEN AND SCHREYER [53], shows the decomposition of the operator into its volumetric and the deviatoric parts:

$$\mathbf{E}_o = 3K\ \mathbf{P}_v + 2G\ \mathbf{P}_d \tag{2.6}$$

in which $K = (3\lambda + 2G)/3$ and $G = \mu$ denote the bulk and shear moduli, respectively. $\mathbf{P}_v$ and $\mathbf{P}_d$ are two $4^{th}$ order tensors that can be thought as volumetric and deviatoric projection operators, defined as

$$\mathbf{P}_v = \frac{1}{3}\,\mathbf{I}_2 \otimes \mathbf{I}_2 \quad ; \qquad \mathbf{P}_d = \mathbf{I}_4^s - \mathbf{P}_v \tag{2.7}$$

The terminology of projection operators indicates that the application of the operator to a $2^{nd}$ order tensor, for example the stress tensor $\boldsymbol{\sigma}$, resolves into a splitting of the tensor into its volumetric and deviatoric projections

$$\mathbf{P}_v : \boldsymbol{\sigma} = \frac{1}{3}\, tr(\boldsymbol{\sigma})\, \mathbf{I}_2 = \boldsymbol{\sigma}_{sp} \quad ; \qquad \mathbf{P}_d : \boldsymbol{\sigma} = \boldsymbol{\sigma} - \frac{1}{3}\, tr(\boldsymbol{\sigma})\, \mathbf{I}_2 = \boldsymbol{\sigma}_d \tag{2.8}$$

The two projection operators possess some interesting properties, for example:

$$\mathbf{P}_v : \mathbf{P}_d = \mathbf{P}_d : \mathbf{P}_v = \mathbf{0} \quad ; \qquad \mathbf{P}_v : \mathbf{P}_v = \mathbf{P}_v \quad ; \qquad \mathbf{P}_d : \mathbf{P}_d = \mathbf{P}_d \tag{2.9}$$

The *spectral properties* of $\mathbf{E}_o$. i.e. the solution of the eigenvalue problem

$$\mathbf{E}_o : \mathbf{x} = \eta\, \mathbf{x} \tag{2.10}$$

are readily found from the spectral properties of the two projection operators.
The eigenvalue solution of the volumetric projection operator results in:

$$\mathbf{P}_v : \mathbf{x} = \frac{1}{3} tr(\mathbf{x})\, \mathbf{I}_2 = \eta\, \mathbf{x} \tag{2.11}$$

Thus $\mathbf{P}_v$ has only one $\eta = 1$ non-zero eigenvalue, with the corresponding eigentensor $\mathbf{x} = \mathbf{I}_2$. The remaining 5 eigentensors are in the space of deviatoric symmetric tensors orthogonal to $\mathbf{I}_2$ (i.e. such that $\mathbf{x} : \mathbf{I}_2 = 0$).
Regarding the deviatoric projection:

$$\mathbf{P}_d : \mathbf{x} = \mathbf{x} - \frac{1}{3}\, tr(\mathbf{x})\, \mathbf{I}_2 = \eta\, \mathbf{x} \tag{2.12}$$

We obtain 1 eigenvalue $\eta = 1$ with the multiplicity of five, with corresponding deviatoric eigentensors orthogonal to $\mathbf{I}_2$. The $\mathbf{I}_2$ eigentensor is associated with the zero eigenvalue $\eta = 0$.
Finally we conclude that $\mathbf{E}_o$ has one eigenvalue $\eta = 3K$ with multiplicity of one and one eigenvalue $\eta = 2G$ with multiplicity of five.

The inverse strain/stress relationship, expressed in terms of the compliance $4^{th}$ order tensor $\mathbf{C}_o$ can be obtained inverting the elastic stiffness: $\mathbf{C}_o = \mathbf{E}_o^{-1}$. To this end, we might consider the so called *Shermann and Morrison formula* (SHERMANN AND MORRISON [69]) that was originally proposed for the inversion of a square matrix in which only one term has been modified. This is in fact the case of the elastic operator, which appears as a rank-one modification of the term $2\mu\,\mathbf{I}_4^s$; the dyadic modification $\lambda\,\mathbf{I}_2 \otimes \mathbf{I}_2$ defines a rank-one tensor and thus can be written, in the proper eigenbasis, as a matrix with only one term different from zero.

From the original paper by SHERMANN AND MORRISON [69], the inverse of the matrix $A_{RS} = a_{RS} + \Delta a_{RS}$ is given by (no sum on repeated indices):

$$(A^{-1})_{rj} = (a^{-1})_{rj} - \frac{(a^{-1})_{rR}\,(a^{-1})_{Sj}\,\Delta a_{RS}}{1 + (a^{-1})_{SR}\,\Delta a_{RS}} \tag{2.13}$$

and applying it to our case, using $\mathbf{I}_2 : \mathbf{I}_2 = 3$:

$$\mathbf{C}_o = \frac{1}{2\mu}\,\mathbf{I}_4^s - \frac{\lambda}{4\mu^2}\,\frac{\mathbf{I}_4^s : \mathbf{I}_2 \otimes \mathbf{I}_2 : \mathbf{I}_4^s}{1 + \dfrac{\lambda}{2\mu}\,\mathbf{I}_2 : \mathbf{I}_4^s : \mathbf{I}_2} = \frac{1}{2\mu}\mathbf{I}_4^s - \frac{\lambda}{2\,\mu\,(2\mu + 3\lambda)}\mathbf{I}_2 \otimes \mathbf{I}_2 \tag{2.14}$$

The strain/stress constitutive relation may be obtained by double contracting the compliance with the stress tensor:

$$\begin{aligned} \boldsymbol{\epsilon} = \mathbf{C}_o : \boldsymbol{\sigma} &= \frac{1}{2\mu}\,\mathbf{I}_4^s : \boldsymbol{\sigma} - \frac{\lambda}{2\,\mu\,(3\lambda + 2\mu)}\,\mathbf{I}_2 : \boldsymbol{\sigma}\quad \mathbf{I}_2 \\ &= \frac{1}{2\mu}\,\boldsymbol{\sigma} - \frac{\lambda}{2\,\mu\,(3\lambda + 2\mu)}\,tr(\boldsymbol{\sigma})\,\mathbf{I}_2 \end{aligned} \tag{2.15}$$

In conclusion the constitutive law for an elastic isotropic material (*generalized Hooke's law)*, in both stress/strain and strain/stress forms reads

$$\begin{aligned} \sigma_{ij} &= \lambda\,\delta_{ij}\,\epsilon_{kk} + 2\,\mu\,\epsilon_{ij} \\ \epsilon_{ij} &= \frac{-\lambda}{2\,\mu\,(3\lambda + 2\mu)}\,\delta_{ij}\,\sigma_{kk} + \frac{1}{2\,\mu}\,\sigma_{ij} \end{aligned} \tag{2.16}$$

Hence the linear isotropic elasticity material stiffness and compliance, satisfying major and all minor symmetries, can be written in compact notation as

$$\boxed{\begin{aligned} \mathbf{E}_o &= \lambda\quad \mathbf{I}_2 \otimes \mathbf{I}_2 + 2\mu\quad \mathbf{I}_4^s \\ \mathbf{C}_o &= \frac{-\lambda}{2\,\mu\,(3\lambda + 2\mu)}\quad \mathbf{I}_2 \otimes \mathbf{I}_2 + \frac{1}{2\mu}\quad \mathbf{I}_4^s \end{aligned}} \tag{2.17}$$

## 2.2 Wave propagation in elastic solids

Mechanical waves in solids are the results of a *disturbance* that propagates inside the medium (e.g. ACHENBACH [1], GRAFF [26]). The disturbance is created by a local excitation. This cannot be detected at the same time at all positions of the body, since we have to expect a resistance offered by the medium.

The disturbance is transmitted point to point and it involves a propagation of strain. Thus it appears clear that the *stiffness* characteristics of the medium (e.g. $E$ for an elastic isotropic homogeneous material) have to play some role in the transmission of the motion. If the medium is not deformable (i.e. $E \to \infty$) we expect to obtain a simultaneous response in every point of the medium, with an infinite velocity of propagation (i.e. $c \to \infty$).

The other material property that plays an important role is the *inertia*. The more the body tends to remain in the undeformed configuration (i.e. the larger the density $\rho$), the slower the propagation will be. If the inertia is small (i.e. $\rho \to 0$), we expect a large value of the velocity of propagation $c$ (i.e. $c \to \infty$).

In fact the velocity of propagation always assumes the form of a square root of a parameter defining the stiffness divided by the mass density of the material (e.g. $c = \sqrt{E/\rho}$).

For the sake of clarifying the meaning of a few terms, which will be used in the paper, let us consider wave propagation in three dimensions. At every time instant $t$ we can draw a surface that contains all points reached by the same disturbance. This surface is called the *wavefront*. The directions of propagation are always perpendicular to the wavefront and are called *rays*.

For example in a *spherical wave* the wavefront is the surface of a sphere and rays are the radial directions. In the case of a *plane wave* (still in three dimensions) the wavefront is a plane and the propagation direction is identified by the unit normal $\vec{\mathbf{N}}$ to the plane.

The direction of motion is called the *polarization direction* (defined by a unit vector $\vec{\mathbf{M}}$) and in general can be different from $\vec{\mathbf{N}}$.

The mathematical description of the simplest one-dimensional wave motion is given by the *simple wave function* $\phi(x - c\,t)$ (ACHENBACH [1]), where: $s = (x - c\,t)$ (i.e. the argument of the function) is the *phase* of the wave function, $x$ is the actual spatial position, $t$ is the time instant and $c$ is the *phase velocity* of the wave. Thus the wave function represents the amount of disturbance traveling in the $x$ direction with a velocity $c$.

### 2.2.1 Governing equation: Navier's equation of motion

The governing wave propagation equation can be derived using classic concepts of *continuum mechanics*. In this theory the medium is idealized at a macroscopic level as a continuum in which all the field variables (e.g. displacements, strain, stress, density) vary in a continuous manner from point to point inside the body.

If the medium is considered *elastic, isotropic* and *homogeneous* the constitutive law is given by the finite generalized Hooke's law $\boldsymbol{\sigma} = \mathbf{E}_o : \boldsymbol{\epsilon}$.

The magnitude of the displacement gradient ($\nabla = \partial/\partial\mathbf{x}$) is considered small and the infinitesimal strains are given in a linear form as the symmetric part of the displacement gradient tensor, neglecting terms of higher order. Thus the *strain-displacement* relations are:

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) \qquad \Leftrightarrow \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2.18}$$

The *equilibrium* conditions are given by the balance of linear momentum

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho\,\ddot{\mathbf{u}} \qquad \Leftrightarrow \qquad \sigma_{ij,j} + f_i = \rho\,\ddot{u}_i \tag{2.19}$$

where $\mathbf{f}$ denotes the body force and $\rho$ is the density. The balance of angular momentum is satisfied by the symmetry of the stress tensor (i.e. $\sigma_{ij} = \sigma_{ji}$), based on the usual assumption of non-polar continua.

Thus, the basic relations formulate a problem of 15 equations (3 equilibrium equations, 6 strain-displacement relations and 6 constitutive equations) in 15 unknown (3 displacement, 6 strain and 6 stress components). Now it is possible to substitute strain-displacement relations and generalized Hooke's law in the equilibrium statement and to find the three equations (of the second order) in the three unknown displacement components $u_i$.

Substituting the Hooke's law into the equilibrium equations (2.19) and using the expression (2.2) for $\mathbf{E}_o$, we obtain

$$E^o_{ijkl}\, \epsilon_{kl,_j} + f_i = \rho\, \ddot{u}_i \tag{2.20}$$

$$\lambda\, \delta_{ij}\, \delta_{kl}\, \epsilon_{kl,_j} + \mu\, \delta_{ik}\, \delta_{jl}\, \epsilon_{kl,_j} + \mu\, \delta_{il}\, \delta_{jk}\, \epsilon_{kl,_j} + f_i = \rho\, \ddot{u}_i \tag{2.21}$$

Using the properties of the Kronecker delta, $\delta_{ij}$

$$\lambda\, \delta_{ij}\, \epsilon_{hh,_j} + \mu\, \epsilon_{ij,_j} + \mu\, \epsilon_{ij,_j} + f_i = \rho\, \ddot{u}_i \tag{2.22}$$

$$\lambda\, \epsilon_{hh,_i} + 2\,\mu\, \epsilon_{ij,_j} + f_i = \rho\, \ddot{u}_i \tag{2.23}$$

Finally, using the strain-displacement relations:

$$\lambda\, \frac{1}{2}\left(u_{h,_{hi}} + u_{h,_{hi}}\right) + 2\,\mu\, \frac{1}{2}\left(u_{i,_{jj}} + u_{j,_{ij}}\right) + f_i = \rho\, \ddot{u}_i \tag{2.24}$$

$$\lambda\, u_{h,_{hi}} + \mu\, u_{i,_{jj}} + \mu\, u_{j,_{ij}} + f_i = \rho\, \ddot{u}_i \tag{2.25}$$

This is the so called *Navier's equation of motion* and can be rewritten in direct notation, using $\nabla^2 = \nabla \cdot \nabla$

$$\boxed{\begin{array}{c} \mu\, u_{i,_{jj}} + (\lambda + \mu)\ u_{k,_{ki}} + f_i = \rho\, \ddot{u}_i \\[2ex] \mu\, \nabla^2 \mathbf{u} + (\lambda + \mu)\ \nabla\left(\nabla \cdot \mathbf{u}\right) + \mathbf{f} = \rho\, \ddot{\mathbf{u}} \end{array}} \tag{2.26}$$

In the general theory of wave propagation the body force term is dropped. Particular solutions can be found, when these force fields are given.

### 2.2.2 Plane body waves

A simple assumption for a three-dimensional wave propagation in an unbounded medium considers *plane wavefronts*. This seems a reasonable idealization of a spherical wave when the distance between the source of perturbation and the generic point is considerably large.

The expression of a plane wave is given in a general form in eqn. (2.27), in terms of the wave function $\phi$. This form is a solution of the Navier's equation of motion when the wave speed assumes two different values $c_{_L}$ and $c_{_T}$ depending on the relative orientation of the polarization and propagation directions. If the two directions coincide, the wave is called *longitudinal*, if the two directions are mutually orthogonal we speak of a *transverse* wave.

The *harmonic* plane wave is obtained as a particular case in which the wave function is an harmonic function. This represents a basic solution useful to describe any periodic or non-periodic function by means of the superposition principle (Fourier Series or Fourier Transform).

The general wave expression (2.27) is a solution of the Navier's equation if the *Fresnel-Hadamard* propagation condition holds. In this condition the determination of the wave speeds and the polarization directions results in an eigenvalue problem for a conveniently defined tensor, the so-called *elastic acoustic tensor*.

### 2.2.3 Longitudinal and transverse waves

A plane body wave propagating with velocity $c$ in an unbounded medium can be represented in the form (e.g. ACHENBACH [1], GURTIN [27]):

$$\boxed{\mathbf{u}(\mathbf{x}, t) = \phi(\mathbf{x} \cdot \vec{\mathbf{N}} - ct)\,\vec{\mathbf{M}} = \phi(s)\,\vec{\mathbf{M}}} \tag{2.27}$$

where

- $\phi(s)$ is a given $C^2(-\infty, \infty)$ function with $\phi'' = d^2\phi/ds^2 \neq 0$
- $\mathbf{x}$ is the position vector
- $c$ is the velocity of propagation

- $\vec{\mathbf{M}}$ is the direction of polarization which defines the particle motion
- $\vec{\mathbf{N}}$ is the direction of propagation.

Therefore the argument $s$ of the wave function $\phi$ (i.e. the phase) represents the relative distance in the $\vec{\mathbf{N}}$ direction between the generic plane $\mathbf{x} \cdot \vec{\mathbf{N}} = const$ perpendicular to the propagation direction $\vec{\mathbf{N}}$ and the actual position $c\,t$ of the wave front. With the minus sign in (2.27) we obtain a plane wavefront propagating with velocity $c$ in the positive $\vec{\mathbf{N}}$ direction.

If we want to investigate whether expression (2.27) represents a solution of the equation of motion, we substitute eqn. (2.27) into the Navier's equation of motion for zero body force:

$$\mu\,\nabla^2\mathbf{u} + (\lambda+\mu)\;\nabla\,(\nabla\cdot\mathbf{u}) = \rho\,\ddot{\mathbf{u}} \tag{2.28}$$

Differentiating expression (2.27) with respect to spatial coordinates and time by means of the chain rule, we obtain the following relations (ACHENBACH [1], GURTIN [27]), in indicial and direct forms:

$$u_{i,j} = \phi'(s)\,N_j\,M_i \quad\Leftrightarrow\quad \nabla\mathbf{u} = \phi'(s)\,\left(\vec{\mathbf{M}}\otimes\vec{\mathbf{N}}\right) \tag{2.29}$$

$$u_{i,i} = \phi'(s)\,N_i\,M_i \quad\Leftrightarrow\quad \nabla\cdot\mathbf{u} = \phi'(s)\,\left(\vec{\mathbf{N}}\cdot\vec{\mathbf{M}}\right) \tag{2.30}$$

$$u_{i,ij} = \phi''(s)\,N_i\,M_i\,N_j \quad\Leftrightarrow\quad \nabla\,(\nabla\cdot\mathbf{u}) = \phi''(s)\,\left(\vec{\mathbf{N}}\cdot\vec{\mathbf{M}}\right)\vec{\mathbf{N}} \tag{2.31}$$

$$u_{i,jj} = \phi''(s)\,N_j\,M_i\,N_j = \phi''(s)\,M_i \quad\Leftrightarrow\quad \nabla^2\mathbf{u} = \phi''(s)\,\vec{\mathbf{M}} \tag{2.32}$$

$$\dot{u}_i = -c\,\phi'(s)\,M_i \quad\Leftrightarrow\quad \dot{\mathbf{u}} = -c\,\phi'(s)\,\vec{\mathbf{M}} \tag{2.33}$$

$$\ddot{u}_i = c^2\,\phi''(s)\,M_i \quad\Leftrightarrow\quad \ddot{\mathbf{u}} = c^2\,\phi''(s)\,\vec{\mathbf{M}} \tag{2.34}$$

Substituting these expressions into Navier's equation (2.28) results in:

$$\mu\,\phi''(s)\,\vec{\mathbf{M}} + (\lambda+\mu)\,\phi''(s)\,\left(\vec{\mathbf{N}}\cdot\vec{\mathbf{M}}\right) = \rho\,c^2\,\phi''(s)\,\vec{\mathbf{M}} \tag{2.35}$$

Collecting the term $\phi''(s)$, results in

$$\left[(\mu-\rho c^2)\,\vec{\mathbf{M}} + (\lambda+\mu)\,\left(\vec{\mathbf{N}}\cdot\vec{\mathbf{M}}\right)\,\vec{\mathbf{N}}\right]\,\phi''(s) = \mathbf{0} \tag{2.36}$$

Now, since $\vec{\mathbf{N}}$ and $\vec{\mathbf{M}}$ are two different unit vectors, there are two ways to satisfy the previous equation, with $\phi''(s) \neq 0$ (" $\times$ " denotes the vector product):

$$\begin{array}{lccc} \vec{\mathbf{N}} \times \vec{\mathbf{M}} = \mathbf{0} & \text{i.e.} & \vec{\mathbf{N}} \parallel \vec{\mathbf{M}} & (\vec{\mathbf{M}} = \pm\vec{\mathbf{N}}) \\ & \text{or} & & \\ \vec{\mathbf{N}} \cdot \vec{\mathbf{M}} = 0 & \text{i.e.} & \vec{\mathbf{N}} \perp \vec{\mathbf{M}} & \end{array} \tag{2.37}$$

In the first case the propagation direction $\vec{\mathbf{N}}$ and the polarization direction $\vec{\mathbf{M}}$ coincide and the wave is therefore called a **longitudinal wave**. These waves are also known as *irrotational.* In fact the rotation of the displacement field vanishes in this case:

$$curl\ \mathbf{u} = \nabla \times \mathbf{u} = \phi'(s)\,(\vec{\mathbf{N}} \times \vec{\mathbf{M}}) = \mathbf{0} \tag{2.38}$$

In addition these waves are also called *dilatational* (**P-wave**: primary, pressure or punch waves) since the volumetric strain $\theta = u_{i,i} = \nabla \cdot \mathbf{u}$ is not zero.

$$\nabla \cdot \mathbf{u} = \phi''(s)\,(\vec{\mathbf{N}} \cdot \vec{\mathbf{M}}) = \pm\, \phi''(s) \neq 0 \tag{2.39}$$

The expression of the longitudinal wave velocity $c_L$ which satisfies the eqn. (2.36) is given by

$$(\mu - \rho c^2)\,\vec{\mathbf{N}} + (\lambda + \mu)\,\vec{\mathbf{N}} = \mathbf{0} \qquad (\forall\,\vec{\mathbf{N}}) \tag{2.40}$$

and finally

$$\lambda + 2\mu - \rho c^2 = 0 \quad \Rightarrow \quad c = c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{1}{\rho}\,\frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}} \tag{2.41}$$

From the last expression it is possible to see that, in the incompressible case $\nu = 1/2$, $c_L \to \infty$. The longitudinal velocity is often written in another form (BOLT [10]), using the *bulk modulus* $K = (3\lambda + 2\mu)/3$, which gives the ratio $p/\theta$ between mean pressure $p = \sigma_{ii}/3$ and volumetric strain $\theta$:

$$c = c_L = \sqrt{\frac{1}{\rho}\,\left(K + \frac{4}{3}\mu\right)} \tag{2.42}$$

On the other hand, in the second case, the polarization direction $\vec{\mathbf{M}}$, i.e. the direction of the motion, is perpendicular to the propagation direction $\vec{\mathbf{N}}$ and the wave is called a **transverse** or **shear** wave (**S-wave**: secondary, shear wave). Using the same arguments as before, the shear wave can also be called *rotational* or *equivoluminal* wave, because the rotation is not zero (in general $\phi'(s) \neq 0$ with $\phi''(s) \neq 0$) and the dilatational strain $\theta$ vanishes.

The transverse velocity $c_T$ satisfying eqn. (2.36) is

$$(\mu - \rho c^2)\, \vec{\mathbf{N}} = \mathbf{0} \qquad\qquad (\forall\, \vec{\mathbf{N}}) \tag{2.43}$$

which leads to

$$c = c_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{1}{\rho}\,\frac{E}{2\,(1+\nu)}} \tag{2.44}$$

The longitudinal wave velocity is always larger than the transverse wave velocity and the relative ratio is given by

$$\frac{c_L}{c_T} = \sqrt{\frac{\lambda + 2\mu}{\mu}} = \sqrt{\frac{2\,(1-\nu)}{(1-2\nu)}} \qquad\qquad (0 < \nu < 1/2) \tag{2.45}$$

### 2.2.4 The elastic acoustic tensor

The general expression (2.27) for a plane wave is a solution of the Navier's equation of motion if the conditions previously stated hold. It renders the possible wave speeds and polarization directions. Another way for the derivation of the same conditions leads to the statement of the *Fresnel-Hadamard propagation* condition, after definition of the *elastic acoustic tensor*. At this stage the problem reduces to an eigenvalue analysis of the elastic acoustic tensor.

The steps follow the same as before, i.e. to impose balance of linear momentum, use the strain-displacement relations and introduce an elastic constitutive law. Let us rewrite the general expression (2.27) for the wave motion and the relative displacement gradient

$$\mathbf{u} = \phi(\mathbf{x} \cdot \vec{\mathbf{N}} - ct)\, \vec{\mathbf{M}} = \phi(s)\, \vec{\mathbf{M}} \qquad \nabla \mathbf{u} = \phi'(s)\, \vec{\mathbf{M}} \otimes \vec{\mathbf{N}} \tag{2.46}$$

The infinitesimal strain tensor is the symmetric part of $\nabla \mathbf{u}$:

$$\boldsymbol{\epsilon} = \phi'(s)\, \frac{1}{2}\, (\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} + \vec{\mathbf{N}} \otimes \vec{\mathbf{M}}) \tag{2.47}$$

Now, using the generalized Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{E}_o : \boldsymbol{\epsilon} = \phi'(s)\, \frac{1}{2}\, \mathbf{E}_o : (\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} + \vec{\mathbf{N}} \otimes \vec{\mathbf{M}}) \tag{2.48}$$

which in indicial notation is written as

$$\sigma_{ij} = \phi'(s)\, \frac{1}{2}\, (E^o_{ijhk}\, M_k\, N_h + E^o_{ijhk}\, N_h\, M_k) \tag{2.49}$$

The elastic material stiffness $\mathbf{E}_o$ is symmetric, using the first and second minor symmetries ($E^o_{ijhk} = E^o_{jihk}$ and $E^o_{ijhk} = E^o_{ijkh}$), the previous statement becomes

$$\sigma_{ij} = \phi'(s)\, \frac{1}{2}\, 2\, (E^o_{ijhk} N_k M_h) \tag{2.50}$$

We have now to consider the divergence of $\boldsymbol{\sigma}$, for the substitution into the equilibrium equation $\sigma_{ij,j} = \rho\, \ddot{u}_i$ for zero body force. The divergence is given by

$$\sigma_{ij,j} = \phi''(s)\, N_j\, E^o_{ijhk}\, N_k\, M_h = \tag{2.51}$$

$$= \phi''(s)\, (N_j\, E^o_{jihk}\, N_k)\, M_h = \phi''(s)\, Q^{ep}_{ih} M_h \tag{2.52}$$

Where $Q^e_{ih}$ is the *elastic acoustic tensor*, which is a second order tensor derived through two single contractions of $\mathbf{E}_o$ with $\vec{\mathbf{N}}$:

$$\boxed{\mathbf{Q}_e = \vec{\mathbf{N}} \cdot \mathbf{E}_o \cdot \vec{\mathbf{N}} \qquad Q^e_{ih} = N_j E^o_{jihk} N_k} \tag{2.53}$$

The elastic acoustic tensor depends on the elastic material properties and on the orientation of the plane wavefront. $\mathbf{Q}_e$ is symmetric because $\mathbf{E}_o$ is symmetric

(GURTIN [27]).

Finally, using the relation

$$\ddot{\mathbf{u}} = c^2\, \phi''(s)\, \vec{\mathbf{M}} \tag{2.54}$$

we obtain

$$\begin{aligned} \phi''(s)\, Q^e_{ih}\, M_h &= \rho\, c^2\, \phi''(s)\, M_i \\ Q^e_{ih}\, M_h &= \rho\, c^2\, M_i \end{aligned} \tag{2.55}$$

and in direct notation

$$\boxed{\mathbf{Q}_e \cdot \vec{\mathbf{M}} = \rho\, c^2\, \vec{\mathbf{M}}} \tag{2.56}$$

which is the *Fresnel-Hadamard propagation* condition. In other words $(\rho c_i^2)$ is an eigenvalue of $\mathbf{Q}_e$ and $\vec{\mathbf{M}}_i$ is the associated eigenvector (i.e. the polarization direction of the wave with $c_i$ phase velocity).

The relation (2.36) which did result for isotropic linear elasticity in the solutions of longitudinal and transverse waves can be derived from the Fresnel-Hadamard condition. Let us find the expression of the elastic acoustic tensor using eqn. (2.53) and eqn. (2.2) for the elastic stiffness

$$E^o_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\, (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.57}$$

$$Q^e_{jk} = N_i\, E^o_{ijkl}\, N_l = \tag{2.58}$$

$$= \lambda\, \delta_{ij}\delta_{kl}\, N_i\, N_l + \mu\, \delta_{ik}\delta_{jl}\, N_i\, N_l + \mu\, \delta_{il}\delta_{jk}\, N_i\, N_l = \tag{2.59}$$

$$= \lambda\, N_j\, N_k + \mu\, N_k\, N_j + \mu\, N_n\, N_n\, \delta_{jk} \tag{2.60}$$

$$= (\lambda + \mu)\, N_j\, N_k + \mu\, \delta_{jk} \tag{2.61}$$

and in direct notation

$$\mathbf{Q}_e = (\lambda + \mu)\, (\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) + \mu\, \mathbf{I}_2 \tag{2.62}$$

Finally the Fresnel-Hadamard condition $\mathbf{Q}_e \cdot \vec{\mathbf{M}} - \rho\, c^2\, \vec{\mathbf{M}} = \mathbf{0}$ gives:

$$(\lambda + \mu)\; (\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \cdot \vec{\mathbf{M}} + \mu\, \mathbf{I}_2 \cdot \vec{\mathbf{M}} - \rho\, c^2\, \vec{\mathbf{M}} = \mathbf{0} \tag{2.63}$$

Now $(\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \cdot \vec{\mathbf{M}} = (\vec{\mathbf{N}} \cdot \vec{\mathbf{M}})\, \vec{\mathbf{N}}$ and the previous can be rewritten as

$$(\mu - \rho\, c^2)\; \vec{\mathbf{M}} + (\lambda + \mu)\; (\vec{\mathbf{N}} \cdot \vec{\mathbf{M}})\, \vec{\mathbf{N}} = \mathbf{0} \tag{2.64}$$

which is exactly the same as eqn. (2.36).

In summary the Fresnel-Hadamard propagation condition is a necessary and sufficient condition that the wave defined by eqn. (2.27) is a solution of the elastic wave equation (GURTIN [27]). The spectral properties of the elastic acoustic tensor $\mathbf{Q}_e$ provide wave speeds and polarization directions.

The definition of the acoustic tensor should be generalized to non-linear material behavior, which will be presented in the following sections and chapters. The acoustic tensor will be constructed again in the same way but contracting twice the tangent operator that defines the incremental constitutive law of the material. The natural extension considered in the literature at the onset was the flow theory of plasticity which extends the wave arguments of elastic solids to that of elastic-plastic solids.

## 2.3 The elastic-plastic tangent operators

The classic flow theory of plasticity provides incrementally linear $\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\epsilon}}$ or $\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\sigma}}$ constitutive relations in terms of the *elastic-plastic tangent operators*: stiffness $\mathbf{E}_{ep}$ or compliance $\mathbf{C}_{ep}$

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}} \quad ; \qquad \dot{\boldsymbol{\epsilon}} = \mathbf{C}_{ep} : \dot{\boldsymbol{\sigma}} \tag{2.65}$$

The formulation assumes the existence of a loading function $F = 0$ which defines the current limit of the elastic domain. Dual formulations can be developed depending on the loading function which may be expressed either in terms of stresses or strains: this leads to *stress-based* and *strain-based* formulations (CAROL ET AL. [14]).

### 2.3.1 Stress-based formulation

A general form of the yield surface is given by

$$F(\boldsymbol{\sigma}, \mathbf{p}) = 0 \tag{2.66}$$

where $F$ is the numerical value of the yield function, a function of the stress tensor $\boldsymbol{\sigma}$ and of a set of parameters $\mathbf{p}$, which govern the evolution of the surface by means of appropriate hardening rules. For example, in the case of kinematic and isotropic hardening/softening eqn.(2.66) can be written as

$$F(\boldsymbol{\sigma}, \boldsymbol{\epsilon}_p) = f(\boldsymbol{\sigma} - \boldsymbol{\alpha}(\boldsymbol{\epsilon}_p)) - Y(\boldsymbol{\epsilon}_p) = 0 \tag{2.67}$$

where:

- $\boldsymbol{\alpha}$ is a stress tensor which specifies the current position of the center of the yield surface.
- $Y$ is a scalar function that gives the actual limit of the elastic domain.
- $f$ is a scalar function that characterizes the yield criterion.
- $\boldsymbol{\epsilon}_p$ is the plastic strain.

The differential equations expressing stress increment, plastic flow rule and consistency condition can be stated as follows

$$\begin{aligned} \dot{\boldsymbol{\sigma}} &= \mathbf{E}_o : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_p) \\ \dot{\boldsymbol{\epsilon}}_p &= \dot{\lambda}\, \mathbf{m} \\ \dot{F} &= \frac{\partial F}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial \lambda}\, \dot{\lambda} = 0 \end{aligned} \tag{2.68}$$

where:

- $\mathbf{E}_o$ is the elastic material stiffness.
- $\dot{\boldsymbol{\epsilon}}_p$ is the plastic strain rate.
- $\dot{\lambda} \geq 0$ is the plastic multiplier.
- $\mathbf{m}$ is the direction of flow, normally defined via the plastic potential $Q$, where $\mathbf{m} = \dfrac{\partial Q}{\partial \boldsymbol{\sigma}}$.

The differential consistency condition $\dot{F} = 0$ can be simplified to $\dot{F} = \mathbf{n} : \dot{\boldsymbol{\sigma}} - H\dot{\lambda} = 0$, after introducing the gradient of the yield function $\mathbf{n}$ and the hardening/softening modulus $H$ (in this chapter indicated also by $E_p$ as plastic modulus)

$$\mathbf{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}} \; ; \qquad H = \frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial \boldsymbol{\epsilon}_p} : \frac{\partial \boldsymbol{\epsilon}_p}{\partial \lambda} \tag{2.69}$$

The plastic formulation is called *associated* if the two gradient directions $\mathbf{n}$ and $\mathbf{m}$ are coaxial. The hardening parameter $H$ is positive for hardening, zero for perfect elastic-plasticity and negative for softening.

Loading and unloading are determined by the complementarity conditions:

$$\dot{\lambda} \geq 0 \; , \qquad F \leq 0 \; , \qquad F\dot{\lambda} = 0 \tag{2.70}$$

The plastic multiplier can be determined solving the consistency condition in respect to $\dot{\lambda}$. Now two possibilities appear: expressing $\dot{\lambda}$ in terms of $\dot{\boldsymbol{\epsilon}}$ or in terms of $\dot{\boldsymbol{\sigma}}$. The first option leads to the tangent stiffness, the second one to the tangent compliance.

**Tangent stiffness**

In the first case, substituting the flow rule ($\dot{\boldsymbol{\epsilon}}_p = \dot{\lambda}\,\mathbf{m}$) and the generalized Hooke's law into the consistency condition $\dot{F} = 0$,

$$\mathbf{n} : \mathbf{E}_o : (\dot{\boldsymbol{\epsilon}} - \dot{\lambda}\mathbf{m}) - H\dot{\lambda} = 0 \tag{2.71}$$

we obtain

$$\dot{\lambda} = \frac{\mathbf{n} : \mathbf{E}_o : \dot{\boldsymbol{\epsilon}}}{H + \mathbf{n} : \mathbf{E}_o : \mathbf{m}} = \frac{\mathbf{n} : \dot{\boldsymbol{\sigma}}_e}{h} \geq 0 \tag{2.72}$$

where $h > 0$, for $\dot{\lambda} \geq 0$, which assumes positive plastic dissipation resolving in irreversible plastic deformation ($\dot{\lambda} \geq 0$ if ($\mathbf{n} : \dot{\boldsymbol{\sigma}}_e \geq 0$) and $\dot{\lambda} = 0$ if $\mathbf{n} : \dot{\boldsymbol{\sigma}}_e \leq 0$). The limiting value $h = 0$ gives the limiting negative value of $H = -\mathbf{n} : \mathbf{E}_o : \mathbf{m}$, which depends on the actual state of stress in the yield criterion and on the elasticity operator. Fig. 2.1 shows the admissible values of $H = E_p$ (i.e. for $h > 0$) as a function of the principal stress state $-2/\sqrt{3} \leq (\sigma_1/\sigma_Y) \leq 2/\sqrt{3}$ on the yield von Mises surface (plane stress case with $\nu = 0.3$). The expression of $h$ will change for different hardening rules.

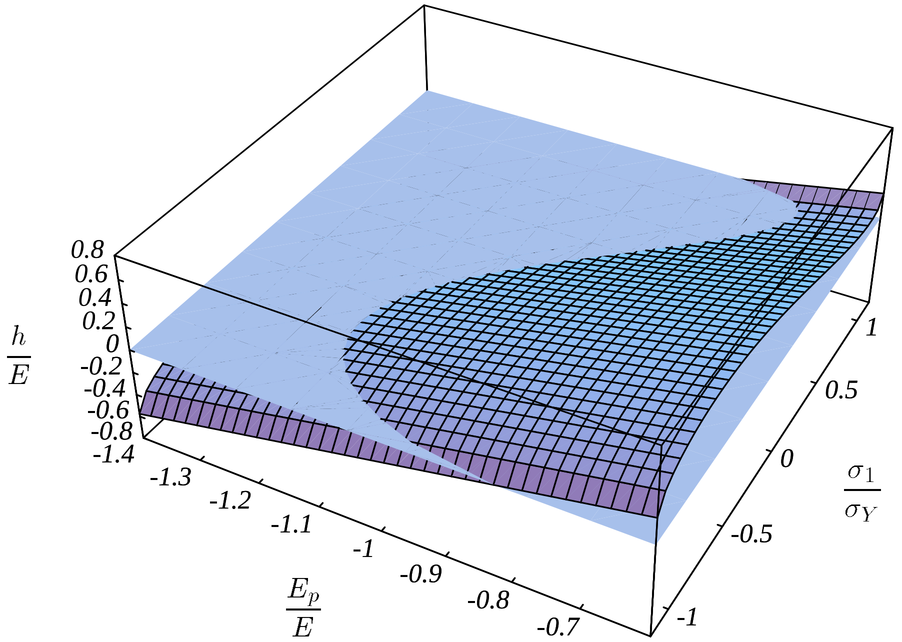



Figure 2.1: Plane stress $J_2$-plasticity ($\nu = 0.3$): admissible values for $E_p$

Finally, generalized Hooke's law leads to the elastic-plastic operator

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_o : \dot{\boldsymbol{\epsilon}} - \mathbf{E}_o : \dot{\boldsymbol{\epsilon}}_p = \mathbf{E}_o : \dot{\boldsymbol{\epsilon}} - \mathbf{E}_o : \mathbf{m}\,\dot{\lambda} = \tag{2.73}$$

$$= \mathbf{E}_o : \dot{\boldsymbol{\epsilon}} - \mathbf{E}_o : \mathbf{m}\,\frac{\mathbf{n} : \mathbf{E}_o : \dot{\boldsymbol{\epsilon}}}{h} = \tag{2.74}$$

$$= \left(\mathbf{E}_o - \frac{\mathbf{E}_o : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}_o}{h}\right) : \dot{\boldsymbol{\epsilon}} = \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}} \tag{2.75}$$

where $\mathbf{E}_{ep}$ denotes the elastic-plastic tangent operator (*tangent material stiffness*) for plastic loading ($\dot{\lambda} > 0$)

$$\boxed{\mathbf{E}_{ep} = \mathbf{E}_o - \frac{\mathbf{E}_o : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}_o}{H + \mathbf{n} : \mathbf{E}_o : \mathbf{m}}} \tag{2.76}$$

In indicial notation:

$$E^{ep}_{ijkl} = E^{o}_{ijkl} - \frac{E^{o}_{ijrs}\, m_{rs}\, n_{pq}\, E^{o}_{pqkl}}{H + n_{pq}\, E^{o}_{pqrs}\, m_{rs}} \tag{2.77}$$

**Tangent compliance**

In the inverse relation, the plastic multiplier is expressed as a function of the stress rate

$$\dot{\lambda} = \frac{\mathbf{n} : \dot{\boldsymbol{\sigma}}}{H} \tag{2.78}$$

Substituting into the elastic strain/stress relation

$$\begin{aligned} \dot{\boldsymbol{\epsilon}} &= \mathbf{C}_o : \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\epsilon}}_p = \mathbf{C}_o : \dot{\boldsymbol{\sigma}} + \frac{\mathbf{n} : \dot{\boldsymbol{\sigma}}}{H}\,\mathbf{m} = & (2.79)\\ &= (\mathbf{C} + \frac{\mathbf{m} \otimes \mathbf{n}}{H}) : \dot{\boldsymbol{\sigma}} & (2.80) \end{aligned}$$

we obtain the expression of the elastic-plastic *tangent material compliance* for plastic loading ($\dot{\lambda} > 0$)

$$\boxed{\mathbf{C}_{ep} = \mathbf{C}_o + \frac{\mathbf{m} \otimes \mathbf{n}}{H}} \tag{2.81}$$

In indicial notation:

$$C^{ep}_{ijkl} = C^{o}_{ijkl} + \frac{m_{ij}\, n_{kl}}{H} \tag{2.82}$$

<u>Remarks</u>

- The expression of the tangent compliance looks simpler than the one for the tangent stiffness, but it exhibits the limitation $H > 0$, if we require $\dot{\lambda} \geq 0$ for $\mathbf{n} : \dot{\boldsymbol{\sigma}} \geq 0$. Thus the perfectly elastic-plastic case can not be considered and only hardening behavior can be described.
- Because of the rank-one modification type the tangent compliance could be derived by inversion of the tangent stiffness $\mathbf{C}_{ep} = \mathbf{E}_{ep}^{-1}$ using the Shermann and Morrison formula (see eqn. (2.13)) and vice versa.
- In the case of associate flow, with $\mathbf{m} = \mathbf{n}$ the elastic-plastic tangent operators preserve all the symmetries of the elastic tensors $\mathbf{E}_o$, $\mathbf{C}_o$.
- In both cases the tangent operators appear to be a rank-one modification of the elastic operators, i.e. the additional term is a rank-one tensor of the kind $\mathbf{a} \otimes \mathbf{b}$ with only one non-zero eigenvalue. However, in general we might expect that two

eigenvalues of $\mathbf{E}_o$ will be modified; one of the five $2G$ associated with the shear modes and the $3K$ eigenvalue related to the bulk mode. In fact, considering the decomposition of both $\mathbf{a}$ and $\mathbf{b}$ in their volumetric and deviatoric parts, the outer product reads

$$\mathbf{a} \otimes \mathbf{b} = (\mathbf{a}_v + \mathbf{a}_d) \otimes (\mathbf{b}_v + \mathbf{b}_d) = \mathbf{a}_v \otimes \mathbf{b}_v + \mathbf{a}_d \otimes \mathbf{b}_d + \mathbf{a}_v \otimes \mathbf{b}_d + \mathbf{a}_d \otimes \mathbf{b}_v \quad (2.83)$$

The last two terms are $4^{th}$ order tensors with zero eigenvalues and will not introduce any modification of the eigenvalues. This point can be readily shown considering the trace of the matrix form $[\mathbf{a}_v \otimes \mathbf{b}_d]$, where $\mathbf{a}_v$ can be expressed as $\mathbf{a}_v = \dfrac{tr(\mathbf{a}_v)}{3}\ \mathbf{I}_2$

$$\begin{aligned} tr([\mathbf{a}_v \otimes \mathbf{b}_d]) &= a^v_{11}b^d_{11} + a^v_{22}b^d_{22} + a^v_{33}b^d_{33} + a^v_{12}b^d_{12} + a^v_{13}b^d_{13} + a^v_{23}b^d_{23} \\ &= \frac{tr(\mathbf{a}_v)}{3}\ (b^d_{11} + b^d_{22} + b^d_{33}) = 0 \end{aligned} \quad (2.84)$$

because $tr(\mathbf{b}_d) = 0$. Thus the only possible non-zero eigenvalue which must be equal to the trace of $\mathbf{a}_v \otimes \mathbf{b}_d$ is zero. The same result is obtained for the other term $\mathbf{a}_d \otimes \mathbf{b}_v$.

For the first term in eqn. (2.83), the eigenvalue problem reads

$$\begin{aligned} (\mathbf{a}_v \otimes \mathbf{b}_v) : \mathbf{x} \quad &= \quad \frac{1}{9}\, tr(\mathbf{a}_v)\, tr(\mathbf{b}_v)(\mathbf{I}_2 \otimes \mathbf{I}_2) : \mathbf{x} \quad = \quad \eta\, \mathbf{x} \\ & \quad\ \ \frac{1}{9}\ tr(\mathbf{a}_v)\ \ tr(\mathbf{b}_v)\ \ tr(\mathbf{x})\ \ \mathbf{I}_2 \quad = \quad \eta\, \mathbf{x} \end{aligned} \quad (2.85)$$

and we obtain

$$\begin{aligned} &\eta = 0 && \forall\ \ \mathbf{x}_d \\ &\eta = \frac{1}{3}\, tr(\mathbf{a}_v)\, tr(\mathbf{b}_v) && \mathbf{x} = \mathbf{I}_2 \end{aligned} \quad (2.86)$$

These eigentensors are the same as those of the operator $\mathbf{P}_v$ in Section (2.1). Thus we can conclude that the eigenvalue $3K$ of the bulk mode of $\mathbf{E}_o$ will be modified as $3K + tr(\mathbf{a}_v)\, tr(\mathbf{b}_v)/3$. Similarly, for the second term in eqn. (2.83), the eigenvalue problem gives

$$\mathbf{a}_d \otimes \mathbf{b}_d : \mathbf{x} = \eta\, \mathbf{x} \quad (2.87)$$

$$\begin{aligned} &\eta = 0 && \mathbf{x} \perp \mathbf{b}_d \quad (\text{i.e. } \mathbf{b}_d : \mathbf{x} = 0) \\ &\eta = \mathbf{b}_d : \mathbf{a}_d && \mathbf{x} \parallel \mathbf{a}_d \end{aligned} \tag{2.88}$$

Recalling again the results of the eigenvalue problem for $\mathbf{P}_d$, in Section (2.1), we conclude that only the shear eigenvalue $2G$ of $\mathbf{E}_o$ will be modified as $2G+\mathbf{b}_d : \mathbf{a}_d$. Therefore, when both $\mathbf{a}$ and $\mathbf{b}$ have a deviatoric as well as a volumetric part the bulk eigenvalue and one of the shear eigenvalues of $\mathbf{E}_o$ will be modified. If one of them possesses just the volumetric/deviatoric part and the other one has both volumetric and deviatoric components or the same component, only the bulk/shear eigenvalue will be modified. Finally no change of the eigenspectrum of $\mathbf{E}_o$ is recovered if one tensor is purely volumetric and the other one is purely deviatoric. In the usual case of deviatoric plasticity with $\mathbf{P}_v : \mathbf{m} = \mathbf{0}$ the eigenvalue related to the bulk modulus is not modified and thus it is possible to exclude complex conjugate eigenvalues in the case of non-symmetric tangent operators (SOBH ET AL. [76]), i.e. the eigenspectrum remains real valued.

### 2.3.2 Strain-based formulation

All the developments described in the previous subsection can be extended in a dual fashion when the yield function is expressed in terms of strains, rather than stress:

$$F(\boldsymbol{\epsilon}, \mathbf{p}) = 0 \tag{2.89}$$

The rate equations include now the expression for the strain increment, plastic stress flow rule and consistency condition

$$\begin{aligned} \dot{\boldsymbol{\epsilon}} &= \mathbf{C}_o : (\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_p) \\ \dot{\boldsymbol{\sigma}}_p &= \dot{\lambda}\, \bar{\mathbf{m}} \\ \dot{F} &= \frac{\partial F}{\partial \boldsymbol{\epsilon}} : \dot{\boldsymbol{\epsilon}} + \frac{\partial F}{\partial \lambda}\, \dot{\lambda} = 0 \end{aligned} \tag{2.90}$$

Note that the plastic stress increment is related to the plastic strain increment considering the identity of the first two rate equations in (2.68) and (2.90)

$$\mathbf{E}_o : \dot{\boldsymbol{\epsilon}} + \dot{\boldsymbol{\sigma}}_p = \dot{\boldsymbol{\sigma}} \quad \Leftrightarrow \quad \dot{\boldsymbol{\sigma}} = \mathbf{E}_o : \dot{\boldsymbol{\epsilon}} - \mathbf{E}_o : \dot{\boldsymbol{\epsilon}}_p \qquad \Rightarrow \qquad \dot{\boldsymbol{\sigma}}_p = -\mathbf{E}_o : \dot{\boldsymbol{\epsilon}}_p \tag{2.91}$$

The consistency condition can be simplified introducing the new gradient $\bar{\mathbf{n}} = \frac{\partial F}{\partial \boldsymbol{\epsilon}}$ and hardening parameter $\bar{H} = -\frac{\partial F}{\partial \lambda}$

$$\dot{F} = \bar{\mathbf{n}} : \dot{\boldsymbol{\epsilon}} - \bar{H}\,\dot{\lambda} = 0 \tag{2.92}$$

Again, solving this equation with respect to $\dot{\lambda}$ and expressing the rate of the plastic multiplier in terms of $\dot{\boldsymbol{\epsilon}}$ or $\dot{\boldsymbol{\sigma}}$ alternatively

$$\dot{\lambda} = \frac{\bar{\mathbf{n}} : \mathbf{C}_o : \dot{\boldsymbol{\sigma}}}{\bar{H} + \bar{\mathbf{n}} : \mathbf{C}_o : \bar{\mathbf{m}}} \quad ; \qquad \dot{\lambda} = \frac{\bar{\mathbf{n}} : \dot{\boldsymbol{\epsilon}}}{\bar{H}} \tag{2.93}$$

we recover the dual formulation of elastic-plastic stiffness and compliance

$$\boxed{\mathbf{C}_{ep} = \mathbf{C}_o - \frac{\mathbf{C}_o : \bar{\mathbf{m}} \otimes \bar{\mathbf{n}} : \mathbf{C}_o}{\bar{H} + \bar{\mathbf{n}} : \mathbf{C}_o : \bar{\mathbf{m}}} \quad ; \qquad \mathbf{E}_{ep} = \mathbf{E}_o + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}}} \tag{2.94}$$

Note that the expression for the stiffness looks simpler than in the stress-based case, but the hardening parameter $\bar{H}$ can now assume only positive values, because in strain space the loading function can only expand.

Interesting relations can be readily established between the gradients $\mathbf{n}$, $\mathbf{m}$ and $\bar{\mathbf{n}}$, $\bar{\mathbf{m}}$ (Runesson and Mroz [66], Carol et al. [14]). From the eqn. (2.91) eliminating $\dot{\lambda}$ we find

$$\bar{\mathbf{m}} = -\mathbf{E}_o : \mathbf{m} \quad ; \qquad \mathbf{m} = -\mathbf{C}_o : \bar{\mathbf{m}} \tag{2.95}$$

For the gradients:

$$\bar{\mathbf{n}} = \left.\frac{\partial F}{\partial \boldsymbol{\sigma}}\right|_{\lambda} : \left.\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\epsilon}}\right|_{\lambda} = \mathbf{n} : \mathbf{E}_o = \mathbf{E}_o : \mathbf{n} \quad ; \qquad \mathbf{n} = \mathbf{C}_o : \bar{\mathbf{n}} \tag{2.96}$$

Finally, using this relation, and equating the expression of the plastic multiplier rates

$$\frac{\mathbf{n} : \mathbf{E}_o : \dot{\boldsymbol{\epsilon}}}{H + \mathbf{n} : \mathbf{E}_o : \mathbf{m}} = \dot{\lambda} = \frac{\bar{\mathbf{n}} : \dot{\boldsymbol{\epsilon}}}{\bar{H}} = \frac{\mathbf{n} : \mathbf{E}_o : \dot{\boldsymbol{\epsilon}}}{\bar{H}} \tag{2.97}$$

we obtain $\bar{H} = H + \mathbf{n} : \mathbf{E}_o : \mathbf{m}$, which also verifies the restriction $\bar{H} > 0$ stated earlier.

## 2.4 Diffuse failure

In the literature, the term *diffuse* or *continuous failure* indicates in general a state of bifurcation inside the material which is associated with a singularity of the tangent material stiffness (*limit points*) or loss of positive definiteness of the second order work density (*loss of stability in the small*). The term continuous bifurcation refers to the fact that the material still remains continuous, i.e. the field variables do not exhibit any spatial discontinuities as in discontinuous bifurcation, that will be investigated in Section 2.6.

According to the first criterion, diffuse failure is associated with the loss of positive definite properties of the elastic-plastic tangent operator $\mathbf{E}_{ep}$:

$$det(\mathbf{E}_{ep}) \leq 0 \tag{2.98}$$

This condition infers material branching, i.e. a singularity of the tangential material stiffness. Considering the eigenvalue problem for $\mathbf{E}_{ep}$

$$\mathbf{E}_{ep} : \mathbf{x} = \eta \, \mathbf{x} \tag{2.99}$$

the critical eigentensor $\dot{\boldsymbol{\epsilon}}_{dif}$ is the one which renders stationary values of the stress rate

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}}_{dif} = \mathbf{0} \qquad for \quad \dot{\boldsymbol{\epsilon}}_{dif} \neq \mathbf{0} \tag{2.100}$$

that is $\mathbf{x} = \dot{\boldsymbol{\epsilon}}_{dif}$ is the eigentensor of $\mathbf{E}_{ep}$ associated with the minimum eigenvalue diminishing to zero.

For both symmetric and non-symmetric material operators the singularity condition is reached when

$$\det(\mathbf{E}_{ep}) = 0 \qquad \Rightarrow \qquad H = E_p = 0 \qquad \text{with} \qquad \mathbf{x} = \alpha \, \mathbf{m} \tag{2.101}$$

i.e. the critical direction is given by the flow rule and the zero eigenvalue is obtained for perfect elastic-plasticity when $H = E_p = 0$ (SOBH ET AL. [76]). This is clearly shown assuming an eigentensor for $\mathbf{E}_{ep}$ of the form $\mathbf{x} = \alpha\, \mathbf{m}$

$$\begin{aligned} \mathbf{E}_{ep} : \mathbf{x} &= \alpha\, (\mathbf{E}_o : \mathbf{m} - \frac{\mathbf{E}_o : \mathbf{m}\, (\mathbf{n} : \mathbf{E}_o : \mathbf{m})}{h}) &= \alpha\, \eta\, \mathbf{m} = \mathbf{0} \\ \frac{H}{H + \mathbf{n} : \mathbf{E}_o : \mathbf{m}}\ (\mathbf{E}_o : \mathbf{m}) &= \mathbf{0} \end{aligned} \tag{2.102}$$

which leads to $H = 0$ (the denominator $H + \mathbf{n} : \mathbf{E}_o : \mathbf{m}$ is always positive).

Runesson and Mroz [66] developed solutions of the generalized eigenvalue problem

$$\mathbf{E}^s_{ep} : \mathbf{x} = \eta^*\, \mathbf{E}_o : \mathbf{x} \qquad \mathbf{E}_{ep} : \mathbf{x} = \eta^*\, \mathbf{E}_o : \mathbf{x} \tag{2.103}$$

where, $\mathbf{E}^s_{ep} = (\mathbf{E}_{ep} + \mathbf{E}^T_{ep})/2$ is the symmetrized operator, in terms of eigenpairs for both associated and non-associated cases. It is interesting to note that the associated flow rule modifies just one normalized eigenvalue, with the corresponding eigentensor being proportional to $\mathbf{m} = \mathbf{n}$:

$$\begin{aligned} \eta^*_1 &= \frac{H}{H + \mathbf{n} : \mathbf{E}_o : \mathbf{m}} \\ \eta^*_k &= 1 \ , \quad k = 2, 3, ..., 6 \end{aligned} \tag{2.104}$$

Comparing this result to the remarks made in Section 2.3.1 we conclude that in this case, even if just one eigenvalue of $\mathbf{E}_o^{-1} : \mathbf{E}_{ep}$ is modified, still two eigenvalues of $\mathbf{E}_{ep}$ are modified in general, depending on the format of $\mathbf{n} = \mathbf{m}$. The generalized eigenvalue problem is useful for the determination of the Limit Points condition but does not provide the eigenvalues and eigentensors of the actual tangent operator $\mathbf{E}_{ep}$.

The Limit Point statement represents a *weak* failure condition. In fact the other criterion of internal material instability, given by non-positive values of the second order work density (according to Hill's *"stability in the small"* condition), reads for plastic loading:

$$d^2W = \frac{1}{2}\, \dot{\boldsymbol{\epsilon}} : \dot{\boldsymbol{\sigma}} = \frac{1}{2}\, \dot{\boldsymbol{\epsilon}} : \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}} \leq 0 \qquad for \quad \dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}_{dif} \neq \mathbf{0} \tag{2.105}$$

This condition is different, in general, if $\mathbf{E}_{ep}$ is non-symmetric in the case of non-associate flow ($\mathbf{n} \neq \mathbf{m}$). In fact, the quadratic form of the energy, because of the symmetry of the strain tensor, mobilizes only the symmetric part of $\mathbf{E}_{ep}$:

$$d^2W = \frac{1}{2}\,\dot{\boldsymbol{\epsilon}} : \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}} = \frac{1}{2}\,\dot{\boldsymbol{\epsilon}} : \mathbf{E}^s_{ep} : \dot{\boldsymbol{\epsilon}} \tag{2.106}$$

Linear algebra of the non-symmetric operators states that the lowest eigenvalue of the symmetrized operator provides a lower bound for the real eigenspectrum of the non-symmetric operator (and vice versa for the highest eigenvalue) (Mirsky [47]). These *"Bromwich"* bounds are also established for the imaginary spectrum, considering the skew-symmetric tensor $\mathbf{E}^{sk}_{ep} = (\mathbf{E}_{ep} - \mathbf{E}^T_{ep})/2$. In other words:

$$\begin{aligned} \eta_{min}[\mathbf{E}^s_{ep}] \;\leq\; \Re\,(\eta[\mathbf{E}_{ep}]) \;\leq\; \eta_{max}[\mathbf{E}^s_{ep}] \\ \eta_{min}[\mathbf{E}^{sk}_{ep}] \;\leq\; \Im\,(\eta[\mathbf{E}_{ep}]) \;\leq\; \eta_{max}[\mathbf{E}^{sk}_{ep}] \end{aligned} \tag{2.107}$$

Clearly it is possible that the lowest $\mathbf{E}^s_{ep}$ eigenvalue becomes zero, while the lowest $\mathbf{E}_{ep}$ real eigenvalue is still positive. In short $det(\mathbf{E}_{ep}) = 0$ is necessary and sufficient for diffuse failure, where $\dot{\boldsymbol{\sigma}} = 0$, while $d^2W = 0$ is only a necessary condition.

In fact Maier and Hueckel [41] developed an explicit expression for the critical hardening modulus $H_1 \geq 0$, that signals the first loss of stability

$$H_1 = \frac{1}{2}\,(\sqrt{\mathbf{n} : \mathbf{E}_o : \mathbf{n}} \quad \sqrt{\mathbf{m} : \mathbf{E}_o : \mathbf{m}} - \mathbf{n} : \mathbf{E}_o : \mathbf{m}) \;\geq\; 0 \tag{2.108}$$

$H_1$ is zero only if the flow rule is associated, otherwise $H_1 > 0$. This result shows again that, for non-associated flow, the loss of stability can take place in the hardening regime, before the singularity of $\mathbf{E}_{ep}$ is reached, when $H = 0$.

Fig. 2.2 shows the variation of the normalized quantity $det(\mathbf{E}_{ep})/det(\mathbf{E}_o)$, which is equal to $\eta^*_1$ in eqn. (2.104), as a function of $H = E_p$ and the plane stress state $\sigma_1$ which varies on the yield surface of the von Mises locus, for the associated case, when $\nu = 0.3$. It is apparent that $det(\mathbf{E}_{ep}) < 0$ for $E_p < 0$ and $det(\mathbf{E}_{ep}) > 0$ for $E_p > 0$.

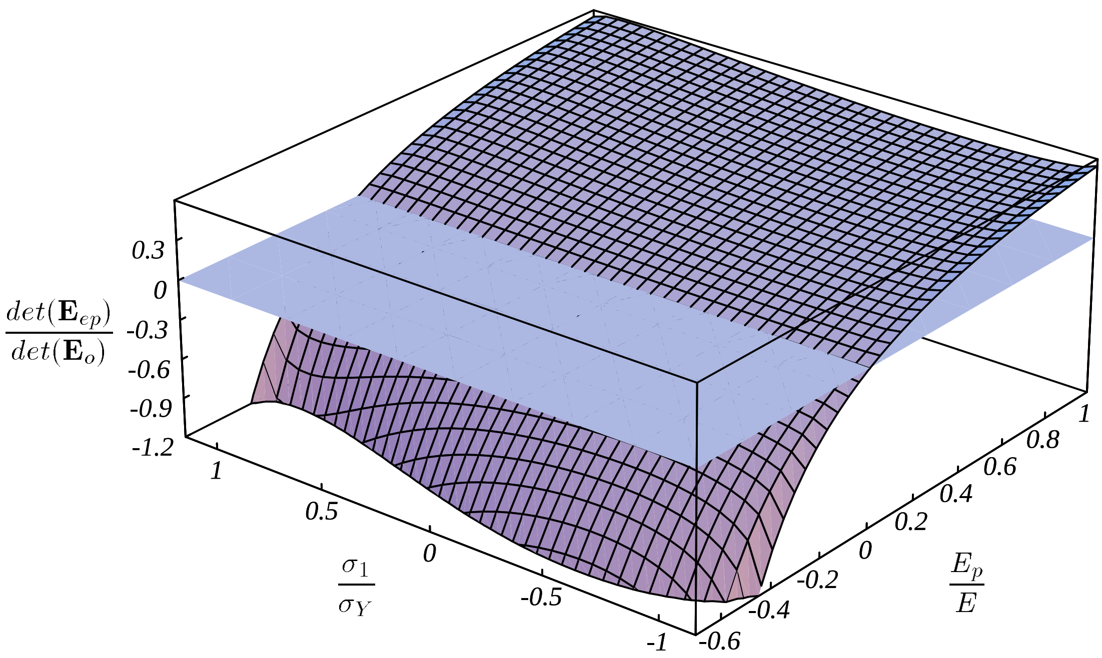



Figure 2.2: Plane stress $J_2$-plasticity ($\nu = 0.3$): singularity of $\mathbf{E}_{ep}$ for $E_p = 0$

## 2.5 The elastic-plastic acoustic tensor

The argument of localization derives neatly from the study of the propagation of *acceleration waves* in solids HILL [29], GURTIN [27], ACHENBACH [1]. A wave represents a disturbance which propagates inside the medium (dispersion analysis in wave mechanics). One can think of having a surface (*wavefront*) moving relatively to the material and across which certain variables are discontinuous. For an acceleration wave the discontinuity is of the $2^{nd}$ order, since it involves variables obtained after two differentiations with respect to time and/or space: the acceleration and the spatial gradient of velocity. Velocity and displacement are still continuous. In other words, denoting with "+" and "−" the two parts of the body divided by the discontinuity surface, the velocity field $\dot{\mathbf{u}}$ is continuous, i.e.

$$[\![\, \dot{\mathbf{u}} \,]\!] = \dot{\mathbf{u}}^+ - \dot{\mathbf{u}}^- = \mathbf{0} \tag{2.109}$$

but the velocity gradient ($\nabla = \partial/\partial \mathbf{x}$) can jump across the wavefront,

$$[\![\, \nabla \dot{\mathbf{u}} \,]\!] = \nabla \dot{\mathbf{u}}^{+} - \nabla \dot{\mathbf{u}}^{-} \neq \mathbf{0} \tag{2.110}$$

This reflects a jump condition for the infinitesimal strain increment $\dot{\boldsymbol{\epsilon}}$, which is the symmetric part of the velocity gradient

$$[\![\, \dot{\boldsymbol{\epsilon}} \,]\!] = \frac{1}{2} [\![\, \nabla \dot{\mathbf{u}} + \nabla \dot{\mathbf{u}}^{T} \,]\!] \neq \mathbf{0} \tag{2.111}$$

The jump of the velocity gradient $[\![\, \nabla \dot{\mathbf{u}} \,]\!]$ is given by *Maxwell's compatibility condition* and the *Fresnel-Hadamard propagation condition* involves the elastic-plastic acoustic tensor. It can be derived according to the classic propagation argument of acceleration waves, or considering the case of plane acceleration wave, in the same way as for the elastic case (Ottosen and Runesson [57]). In the next we will follow the second approach and use the first one to derive the expression of the jump condition.

Considering the plane wave (2.27), where the discontinuity surface is represented by the plane with normal $\vec{\mathbf{N}}$, the propagating displacement reads

$$\mathbf{u} = \phi(\mathbf{x} \cdot \vec{\mathbf{N}} - ct)\, \vec{\mathbf{M}} = \phi(s)\, \vec{\mathbf{M}} \tag{2.112}$$

with the meaning explained in Section 2.2.3. In this case we need one more differentiation with respect to time and thus we require that $\phi(s)$ is $C^3(-\infty, \infty)$.

From the expression (2.112) for the displacement, we can find the spatial gradient:

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j} = \phi' N_j M_i \tag{2.113}$$

In compact notation this can be stated as

$$\nabla \mathbf{u} = \phi'\, \vec{\mathbf{M}} \otimes \vec{\mathbf{N}} \tag{2.114}$$

On the other hand, taking derivatives of $\mathbf{u}$ with respect to time:

$$\dot{\mathbf{u}} = -c\, \phi'(s)\, \vec{\mathbf{M}} \qquad \ddot{\mathbf{u}} = c^2\, \phi''(s)\, \vec{\mathbf{M}} \qquad \dddot{\mathbf{u}} = -c^3\, \phi'''(s)\, \vec{\mathbf{M}} \tag{2.115}$$

and from eqn. (2.114)

$$\nabla\dot{\mathbf{u}} = -c\,\phi''\,\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} \tag{2.116}$$

Thus the jump in the velocity gradient can be expressed as

$$[\![\,\nabla\dot{\mathbf{u}}\,]\!] = -c\,[\![\,\phi''\,]\!]\,\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} = \dot{\gamma}\,\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} \tag{2.117}$$

which is known as one of *Maxwell's compatibility* conditions (GURTIN [27]). This equation can be used to derive the *localization* tensor and the condition for the onset of discontinuous bifurcation.

The expression of the jump condition can be derived independently of the assumption of plane wave (OTTOSEN AND RUNESSON [57]). Consider the vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$, which is assumed to be *constant* on the surface $S$, with zero derivative along the surface

$$\frac{d\mathbf{v}}{ds} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial s} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot \vec{\mathbf{T}} = \mathbf{0} \tag{2.118}$$

where $\vec{\mathbf{T}}$ is an arbitrary vector tangent to the surface $S$. Thus the vector $\partial v_1/\partial\mathbf{x}$ has to be normal to $\vec{\mathbf{T}}$, i.e. $\partial v_1/\partial\mathbf{x} = M_1'\,\vec{\mathbf{N}}$, where $\mathbf{N} \perp \vec{\mathbf{T}}$ and $M_1'$ is a scalar. The same conclusion holds for the other vectors $\partial v_2/\partial\mathbf{x}$, $\partial v_3/\partial\mathbf{x}$, thus finally

$$\frac{\partial v_i}{\partial x_k} = M_i'\,N_k \qquad \Leftrightarrow \qquad \nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \vec{\mathbf{M}}' \otimes \vec{\mathbf{N}} \tag{2.119}$$

Now taking $\mathbf{v} = [\![\,\dot{\mathbf{u}}\,]\!] = const = 0$ along the surface we obtain again the same expression for the jump condition of the velocity gradient

$$\nabla[\![\,\dot{\mathbf{u}}\,]\!] = [\![\,\nabla\dot{\mathbf{u}}\,]\!] = \vec{\mathbf{M}}' \otimes \vec{\mathbf{N}} = \dot{\gamma}\,\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} \tag{2.120}$$

Note that it is always possible to use a particular reference system which simplifies the expression of the jump (NEILSEN AND SCHREYER [52]). In fact, selecting a cartesian system with the axis $x_1$ aligned with the vector $\vec{\mathbf{N}}$ and the axis $x_2$ on the discontinuity surface chosen such that the plane $(x_1, x_2)$ contains the vector $\vec{\mathbf{M}}'$ the two vectors $\vec{\mathbf{N}}$ and $\vec{\mathbf{M}}'$ read

$$\vec{\mathbf{N}}^T = \{\,1 \;\; 0 \;\; 0\,\} \qquad\qquad \vec{\mathbf{M}}'^T = \{\,\alpha \;\; 2\beta \;\; 0\,\} \tag{2.121}$$

where $\alpha^2 + 4\beta^2 = 1$. If $\alpha = 0$ and $\vec{\mathbf{M}}' \perp \vec{\mathbf{N}}$, the shear mode is recovered. The tensile opening mode is obtained when $\beta = 0$, i.e. $\vec{\mathbf{M}}' \parallel \vec{\mathbf{N}}$. Now the outer product expressing the jump condition yields the velocity gradient

$$[\![ \nabla \dot{\mathbf{u}} ]\!] = \vec{\mathbf{M}}' \otimes \vec{\mathbf{N}} = \begin{bmatrix} \alpha & 0 & 0 \\ 2\beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.122}$$

and the corresponding jump of the strain rate follows as

$$[\![ \dot{\boldsymbol{\epsilon}} ]\!] = \frac{1}{2}\,(\vec{\mathbf{M}}' \otimes \vec{\mathbf{N}} + \vec{\mathbf{N}} \otimes \vec{\mathbf{M}}') = \begin{bmatrix} \alpha & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.123}$$

The eigenvalues of the strain rate jump are readily determined as

$$\eta_1 = \frac{\alpha + 1}{2} > 0\ ; \qquad \eta_2 = 0\ ; \qquad \eta_3 = \frac{\alpha - 1}{2} < 0 \tag{2.124}$$

Thus the discontinuous bifurcation mode $[\![ \dot{\boldsymbol{\epsilon}} ]\!]$ exhibits always a zero intermediate eigenvalue between a positive and a negative eigenvalue. This property can be used to detect the onset of localization without analyzing the acoustic tensor (NEILSEN AND SCHREYER [52]).

Now let us go back to the wave propagation analysis and augment it by the *Fresnel-Hadamard propagation* condition (GURTIN [27]), which leads to the definition of the elastic-plastic acoustic tensor and to an analogous argument for localization.

To derive the Fresnel-Hadamard condition let us consider the rate of balance equations for zero body forces

$$\nabla \cdot \dot{\boldsymbol{\sigma}} = \rho\, \ddot{\mathbf{u}} \qquad \Leftrightarrow \qquad \dot{\sigma}_{ij,j} = \rho\, \ddot{u}_i \tag{2.125}$$

In this case, the additional time derivative is necessary since the constitutive law is expressed in rate form. It is possible to eliminate $\dot{\boldsymbol{\sigma}}$ using expression (2.116) for the velocity gradient and the elastic-plastic constitutive law. The rate of deformation is defined by the symmetric part of $\nabla \dot{\mathbf{u}}$:

$$\dot{\boldsymbol{\epsilon}} = -c\,\phi''\,\frac{1}{2}\,(\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} + \vec{\mathbf{N}} \otimes \vec{\mathbf{M}}) \tag{2.126}$$

Using the constitutive law, assuming plastic/plastic bifurcation mode on both sides of the discontinuity surface

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}} = -c\,\phi''\,\frac{1}{2}\,\mathbf{E}_{ep} : (\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} + \vec{\mathbf{N}} \otimes \vec{\mathbf{M}}) \tag{2.127}$$

Taking advantage of the minor symmetries of $\mathbf{E}_{ep}$ gives in indicial notation:

$$\dot{\sigma}_{ij} = -c\,\phi''\,\frac{1}{2}\,2\,(E^{ep}_{ijhk} N_k M_h) \tag{2.128}$$

The divergence of $\dot{\boldsymbol{\sigma}}$ in eqn. (2.125) can be expanded into

$$\begin{aligned} \dot{\sigma}_{ij,j} &= -c\,\phi'''\,N_j\,E^{ep}_{ijhk}\,N_k\,M_h = & (2.129)\\ &= -c\,\phi'''\,(N_j\,E^{ep}_{jihk}\,N_k)\,M_h = -c\,\phi'''\,Q^{ep}_{ih} M_h & (2.130) \end{aligned}$$

Where $Q^{ep}_{ih}$ is the *elastic-plastic acoustic tensor*, which is a second order tensor after two single contractions of $\mathbf{E}_{ep}$ with $\vec{\mathbf{N}}$:

$$\boxed{\mathbf{Q}_{ep} = \vec{\mathbf{N}} \cdot \mathbf{E}_{ep} \cdot \vec{\mathbf{N}} \qquad \Leftrightarrow \qquad Q^{ep}_{ih} = N_j E^{ep}_{jihk} N_k} \tag{2.131}$$

The acoustic tensor depends on the material properties, the actual stress state and on the orientation of the plane wavefront. $\mathbf{Q}_{ep}$ is symmetric if and only if $\mathbf{E}_{ep}$ is symmetric GURTIN [27]).

Finally, using the relation

$$\ddot{\mathbf{u}} = -c^3\,\phi'''(s)\,\vec{\mathbf{M}} \tag{2.132}$$

for the right hand side of equation (2.125) we obtain:

$$\begin{aligned} -c\,\phi'''\,Q^{ep}_{ih}\,M_h &= -\rho\,c^3\,\phi'''\,M_i & (2.133)\\ Q^{ep}_{ih}\,M_h &= \rho\,c^2\,M_i & (2.134) \end{aligned}$$

In direct notation

$$\boxed{\mathbf{Q}_{ep} \cdot \vec{\mathbf{M}} = \rho\,c^2\,\vec{\mathbf{M}}} \tag{2.135}$$

This is the *Fresnel-Hadamard propagation* condition. In other words $(\rho c_i^2)$ is the eigenvalue of $\mathbf{Q}_{ep}$ and $\vec{\mathbf{M}}_i$ is the associated eigenvector which defines the polarization direction of the wave with $c_i$ phase velocity.

Finally, contracting (2.135) with $\vec{\mathbf{M}}$, the wave velocity follows from the quadratic form of $\mathbf{Q}_{ep}$:

$$\vec{\mathbf{M}} \cdot \mathbf{Q}_{ep} \cdot \vec{\mathbf{M}} = \rho\, c^2\, (\vec{\mathbf{M}} \cdot \vec{\mathbf{M}}) = \rho\, c^2 \tag{2.136}$$

Summarizing the two expressions derived here, that will be used in the next section as arguments for the onset of localization (SOBH [75], STEINMANN AND WILLAM [77]),

$$\boxed{\begin{aligned} &[\![\, \nabla \dot{\mathbf{u}} \,]\!] = \dot{\gamma}\, \vec{\mathbf{M}} \otimes \vec{\mathbf{N}} \\ &\rho\, c^2 = \vec{\mathbf{M}} \cdot \mathbf{Q}_{ep} \cdot \vec{\mathbf{M}} \end{aligned}} \tag{2.137}$$

## 2.6 Discontinuous failure

The onset of localization can be derived through two different approaches which lead to analogous results. These approaches are identical if $\mathbf{Q}_{ep}$ is symmetric for associate plastic flow rule.

In the first approach, the condition for the formation of a spatial discontinuity with a jump in the velocity gradient is derived in a quasi-static environment, without the use of wave propagation theory concepts. The singularity comes out from the balance of linear momentum, using Maxwell's compatibility condition.

In the second approach the concepts of wave propagation are considered: the singularity surface is seen as a stationary wavefront, when the phase velocity, i.e. the velocity of the propagating discontinuity, vanishes for a certain polarization direction $\vec{\mathbf{M}}$. In this manner the static case is recovered.

### 2.6.1 Localization as a jump condition

From the static viewpoint, Maxwell's compatibility condition can be thought as an expression which specifies kinematic jump condition of the velocity gradient across the discontinuity surface defined by the normal $\vec{\mathbf{N}}$, $\forall$ variable $\vec{\mathbf{M}}$

$$[\![\, \nabla \dot{\mathbf{u}} \,]\!] = \dot{\gamma}\; \vec{\mathbf{M}} \otimes \vec{\mathbf{N}} \tag{2.138}$$

Here, parameter $\dot{\gamma} \neq 0$ defines the magnitude of the jump.

The balance of linear momentum across the surface states that, according to the lemma of Cauchy:

$$[\![\, \dot{\mathbf{t}} \,]\!] = [\![\, \dot{\boldsymbol{\sigma}} \,]\!] \cdot \vec{\mathbf{N}} = \mathbf{0} \tag{2.139}$$

In the following, the standard form of plastic/plastic bifurcation will be considered: the material is assumed to be loaded in the plastic range on both sides of the discontinuity surface. In fact, for detecting the onset of localization, the elastic/plastic bifurcation can never occur before the plastic/plastic one, as was stated by Ottosen and Runesson [56].

Using Maxwell's compatibility equation and the elastic-plastic material law, in which $\mathbf{E}_{ep}$ depends on the actual stress state, assuming continuous material behavior across the plane of discontinuity (i.e. $[\![\, \dot{\boldsymbol{\sigma}} \,]\!] = \mathbf{E}_{ep} : [\![\, \dot{\boldsymbol{\epsilon}} \,]\!]$), one finds:

$$\dot{\gamma}\; E^{ep}_{ijhk} N_k M_h N_i = 0 \tag{2.140}$$

This leads to the definition of the *localization tensor*

$$Q^{ep}_{jh} = N_i\, E^{ep}_{jihk}\, N_k \tag{2.141}$$

and the previous expression can be rewritten in compact notation as:

$$\boxed{\dot{\gamma}\, Q^{ep}_{jh} M_h = 0 \qquad \Leftrightarrow \qquad \dot{\gamma}\; \mathbf{Q}_{ep} \cdot \vec{\mathbf{M}} = \mathbf{0}} \tag{2.142}$$

Therefore the possibility of a non-zero jump, with $\dot{\gamma} \neq 0$ and $\vec{\mathbf{M}} \neq \mathbf{0}$ provides the condition for the onset of localization in terms of

$$det(\mathbf{Q}_{ep}) = det(\vec{\mathbf{N}} \cdot \mathbf{E}_{ep} \cdot \vec{\mathbf{N}}) = 0 \tag{2.143}$$

This condition is usually called the condition for *"loss of ellipticity"* because of the corresponding change in the elliptic characteristic equations of static equilibrium. This constitutes a weak localization statement as will be explained in the next subsection. The singularity condition of the determinant of $\mathbf{Q}_{ep}$ may be satisfied for a certain critical direction $\vec{\mathbf{N}}$, depending on the actual stress state.

In general this discontinuous failure condition leads to different results when compared with diffuse failure analysis; i.e. for certain stress states the elastic-plastic tangent operator is singular, but the localization tensor is not. For example, the equi-biaxial $\sigma_1 = \sigma_2$ plane stress state for perfect $J_2$-plasticity does not show a singularity of $\mathbf{Q}_{ep}$, while $det(\mathbf{E}_{ep}) = 0$ for $H = 0$. Furthermore the critical direction of the two failure conditions are different

$$\dot{\boldsymbol{\epsilon}}_{dif} \neq [\![ \dot{\boldsymbol{\epsilon}} ]\!]_{dis} \tag{2.144}$$

### 2.6.2 Localization as a stationarity condition

Stationarity conditions of the wavefront are reached when $c \rightarrow 0$ for certain polarization directions. This leads to the zero value criterion of the quadratic form

$$\boxed{\rho c^2 = \vec{\mathbf{M}} \cdot \mathbf{Q}_{ep} \cdot \vec{\mathbf{M}} = 0} \tag{2.145}$$

Using the same reasoning need for analyzing the singularities of the elastic-plastic tangent operator we realize that this statement represents a strong condition for localization than $det(\mathbf{Q}_{ep}) = 0$, thus it is called the criterion for *"loss of strong ellipticity"* of the static equilibrium. On the other hand, one can think of this as the condition for loss of hyperbolicity of the dynamic equation of motion.

The quadratic form (2.145) mobilizes only the symmetric part of $\mathbf{Q}_{ep}$ and the lowest eigenvalue gives an upper bound for the lowest real part of the eigenvalue of $\mathbf{Q}_{ep}$,

$$\Re(\rho\, c^2)_{min}[\mathbf{Q}_{ep}] \;\geq\; (\rho\, c^2)_{min}[\mathbf{Q}^s_{ep}] \tag{2.146}$$

In other words, the condition of vanishing wave speed can be reached before $\mathbf{Q}_{ep}$ is singular, thus the strong statement is more restrictive.

## 2.7 Eigenvalues of the acoustic tensor

The solution of the eigenvalue problem for the acoustic tensor in elastic-plasticity has been extensively studied in the literature. Three very comprehensive papers summarize the analytic results of localization analysis for a broad class of elastic-plastic materials with non-associated flow rules. Ottosen and Runesson [56] provide the solution of the localization condition $det(\mathbf{Q}_{ep})$=0 in the general three-dimensional case, where they give the expressions for the critical hardening modulus and the critical localization direction. In a sequel paper, the general results were applied to the two-dimensional case by Runesson et al. [67], where kinematic or static constraints are applied. Finally, in a third paper, Ottosen and Runesson [57] give the solution of the original eigenvalue problem for $\mathbf{Q}_{ep}$ and discuss the possibility of imaginary wave speeds.

In all three approaches, the localization condition $det(\mathbf{Q}_{ep})$=0 was compacted into a single scalar equation, considering the generalized eigenvalue problem

$$\mathbf{Q}_{ep} \cdot \mathbf{x} = \eta^* \, \mathbf{Q}_e \cdot \mathbf{x} \qquad \Leftrightarrow \qquad (\mathbf{Q}_e^{-1} \cdot \mathbf{Q}_{ep}) \cdot \mathbf{x} = \eta^* \, \mathbf{x} \tag{2.147}$$

where $\eta^*_i$ denote the eigenvalues of the matrix

$$\mathbf{B} = \mathbf{Q}_e^{-1} \cdot \mathbf{Q}_{ep} = \mathbf{I}_2 - \frac{1}{h}\, \mathbf{Q}_e^{-1} \cdot (\mathbf{b} \otimes \mathbf{a}) \tag{2.148}$$

with $\mathbf{a}$=$\mathbf{n} : \mathbf{E}_o \cdot \vec{\mathbf{N}}$, $\mathbf{b}$=$\vec{\mathbf{N}} \cdot \mathbf{E}_o : \mathbf{m}$. The elastic acoustic tensor can be readily inverted using the Shermann and Morrison formula [69] as

$$\mathbf{Q}_e = \mu \mathbf{I}_2 + (\lambda + \mu)\,(\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \quad \Rightarrow \quad \mathbf{Q}_e^{-1} = \frac{1}{\mu}\,\mathbf{I}_2 - \frac{\lambda + \mu}{\mu\,(\lambda + 2\mu)}\,(\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \tag{2.149}$$

Now it is readily shown that the value 1 is an eigenvalue with multiplicity of two (i.e. $\eta_1^*$=$\eta_2^*$=1). In fact substituting $\eta^*$=1 in eqn. (2.147) the eigenvalue problem results into a homogeneous linear system for the matrix

$$\mathbf{B} - \mathbf{I}_2 = -\frac{1}{h}\,(\mathbf{Q}_e^{-1} \cdot \mathbf{b}) \otimes \mathbf{a} = \mathbf{c} \otimes \mathbf{a} \tag{2.150}$$

which is singular with two of the rows always being proportional to the remaining one. The third normalized eigenvalue can be found by considering the invariant condition

$$tr(\mathbf{B}) = 3 - \frac{1}{h}\,\mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} = \eta_1^* + \eta_2^* + \eta_3^* = 2 + \eta_3^* \tag{2.151}$$

and thus

$$\boxed{\eta_3^* = 1 - \frac{1}{h}\,\mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b}} \tag{2.152}$$

The localization condition can thus be reduced to the question, whether $\eta_3^* \to 0$, as

$$\boxed{det(\mathbf{B}) = \frac{det(\mathbf{Q}_{ep})}{det(\mathbf{Q}_e)} = \eta_1^*\,\eta_2^*\,\eta_3^* = \eta_3^* = 0} \tag{2.153}$$

Solving this equation with respect to the hardening modulus $H$ we obtained the critical hardening value, i.e. the value of $H$ that produces a singularity in $\mathbf{Q}_{ep}$ for a certain direction of $\vec{\mathbf{N}}$

$$H_{cr} = -\mathbf{n} : \mathbf{E}_o : \mathbf{m} + \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} \tag{2.154}$$

The critical value of $H_{cr}$ corresponding to the first onset of discontinuous failure $H_{df}$ is the maximum of $H_{cr}$ for a variable $\vec{\mathbf{N}}$ (see Fig. 2.5, 2.6)

$$H_{df} = \max_{\vec{\mathbf{N}}}[H_{cr}]\Big|_{N_1^2+N_2^2+N_3^2=1} \tag{2.155}$$

and the $\vec{\mathbf{N}}$ corresponding to $H_{df}$ gives the localization direction $\vec{\mathbf{N}}_{cr}$.

Finally, from the previous results it is also possible to determine the eigenvalues of the original eigenvalue problem for $\mathbf{Q}_{ep}$ (Ottosen and Runesson [57]). It is shown that one of the three eigenvalues of $\mathbf{Q}_{ep}$ is never modified and equals the elastic shear wave velocity $\eta_1$=$\mu$=$G$. In fact, after introducing $\mathbf{Q}_e$, the eigenvalue problem reduces to

$$\left[(\mu-\eta)\,\mathbf{I}_2+(\lambda+\mu)\,(\vec{\mathbf{N}}\otimes\vec{\mathbf{N}})-\frac{1}{h}\,\mathbf{b}\otimes\mathbf{a}\right]\cdot\mathbf{x}=\mathbf{0} \tag{2.156}$$

With substitution of $\eta$=$\mu$ the coefficient matrix of $\mathbf{x}$ is singular with two rows proportional to the remaining one.

Considering eqn. (2.153), the remaining two eigenvalues are obtained as

$$\begin{aligned} det(\mathbf{Q}_{ep}) &= det(\mathbf{Q}_e)\;\;\eta_1^*\,\eta_2^*\,\eta_3^* \\ G\,\eta_2\,\eta_3 &= G\;G\;(\lambda+2G)\;\;\eta_3^* \end{aligned} \tag{2.157}$$

which gives

$$\eta_2\,\eta_3=G\;(\lambda+2G)\;\eta_3^*=\beta \tag{2.158}$$

Note that $\beta$=0 for $\eta_3^*$=0. Furthermore, considering the invariance of the trace operation

$$\begin{aligned} tr(\mathbf{Q}_{ep})=tr(\mathbf{Q}_e)-\frac{1}{h}\,tr(\mathbf{b}\otimes\mathbf{a}) &= \eta_1+\eta_2+\eta_3 \\ 2G+(\lambda+2G)-\frac{1}{h}\,\mathbf{b}\cdot\mathbf{a} &= G+\eta_2+\eta_3 \end{aligned} \tag{2.159}$$

This determines

$$\eta_2+\eta_3=G+(\lambda+2G)-\frac{1}{h}\,\mathbf{b}\cdot\mathbf{a}=\alpha \tag{2.160}$$

In summary we have two equations

$$\begin{aligned} \eta_2+\eta_3 &= \alpha \\ \eta_2\,\eta_3 &= \beta \end{aligned} \tag{2.161}$$

which result in the second order equation

$$\eta^2-\alpha\,\eta+\beta=0 \tag{2.162}$$

for the unknown eigenvalues $\eta$ of $\mathbf{Q}_{ep}$.

In general $\eta_2 > \eta_3$ and $\eta_3$=0 if $\eta_3^*$=0. Ottosen and Runesson [57] showed that if $\eta_3^*$<0, then $\eta_3$<0, i.e. the eigenvalue $\eta_3$ remains real valued but negative, with a corresponding imaginary wave speed (*"divergence instability"*). At last they also showed that the two eigenvalues $\eta_2$, $\eta_3$ remain real-valued for a broad class of non-associated flow rules, i.e. imaginary eigenvalues are excluded (*"flutter instability"*).

In summary, similarly to the eigenanalysis of the elastic-plastic operator, the generalized eigenvalue problem shows the modification of only one eigenvalue, but the original localization tensor leads to the variation of two eigenvalues (see the phase velocity plots in the next section), depending on the loading state and on the orientation $\vec{\mathbf{N}}$. The generalized eigenvalue problem is useful for establishing analytic results for the onset of localization in the simple form (2.153).

The following figures show the normalized quantity $det(\mathbf{Q}_{ep})/det(\mathbf{Q}_e)$ as a function of the orientation of the possible discontinuity surface $\theta$ in the plane $\sigma_3$=0. The von Mises yield criterion is considered, with loading in uniaxial tension and pure shear.

The numerical results match the analytic solution presented by Ottosen and Runesson [56], which exhibits a dependence on Poisson's ratio that does not appear for the plane stress case. Note that a significative amount of softening is necessary for inducing localization in the uniaxial tension case. In fact the $3D$ case appears to be less susceptible to localization with respect to loading in plane stress, as it will be shown in the next section.

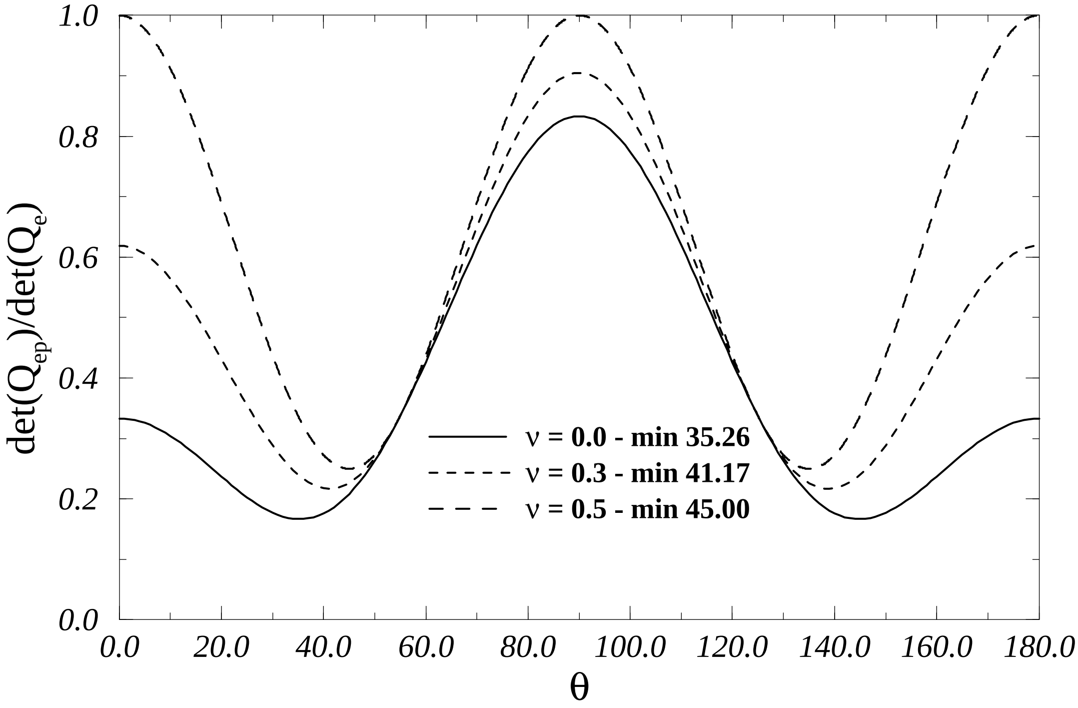



Figure 2.3: $3D$ $J_2$-plasticity ($E_p = 0$): uniaxial tension $\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$

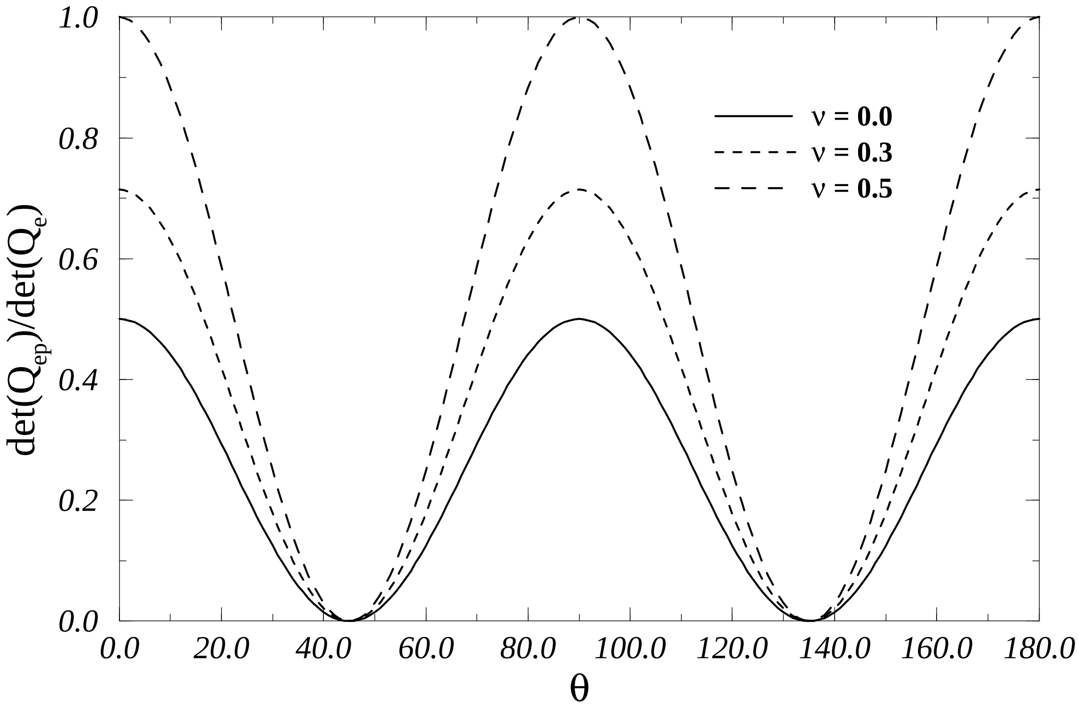



Figure 2.4: $3D$ $J_2$-plasticity ($E_p = 0$): pure shear $\sigma_1 = -\sigma_2 = \sigma$, $\sigma_3 = 0$

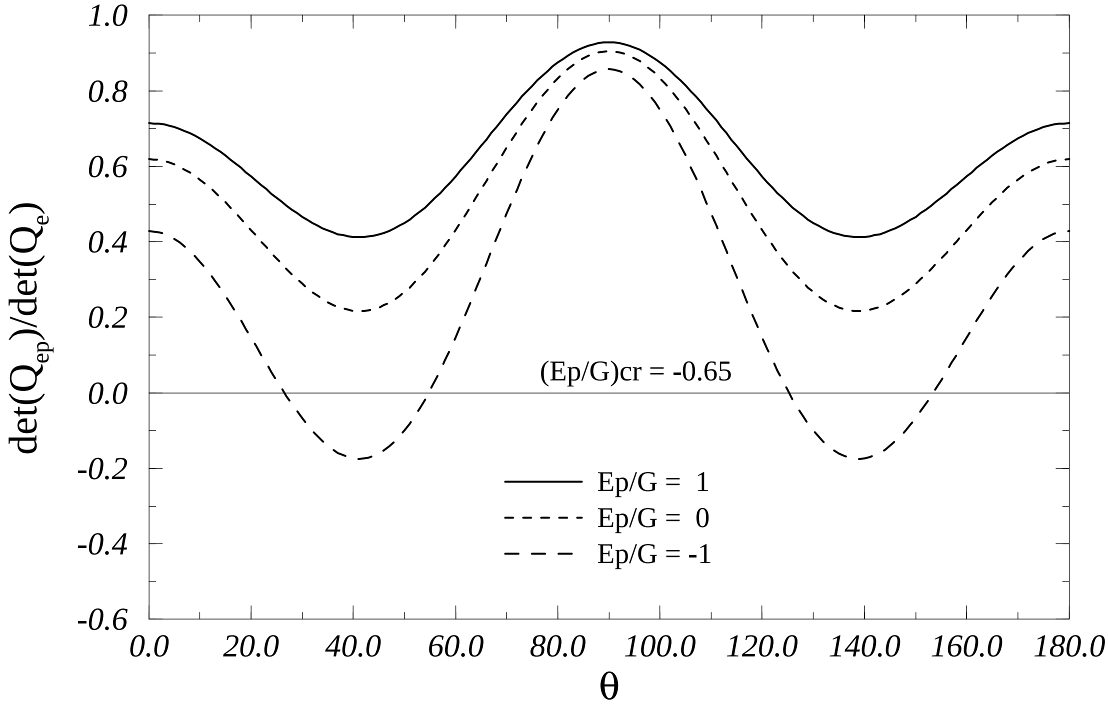



Figure 2.5: $3D$ $J_2$-plasticity ($\nu = 0.3$): uniaxial tension $\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$

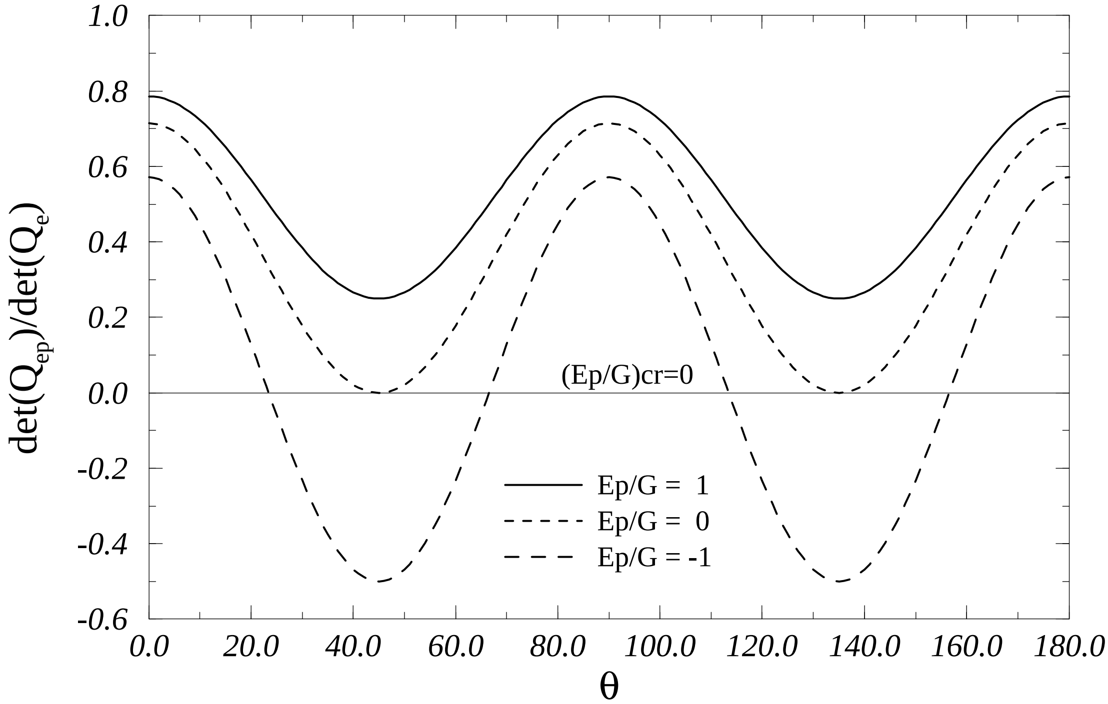



Figure 2.6: $3D$ $J_2$-plasticity ($\nu = 0.3$): pure shear $\sigma_1 = -\sigma_2 = \sigma$, $\sigma_3 = 0$

## 2.8 Plane stress $J_2$-plasticity: some results

Dimensional reduction degenerates the three-dimensional material description to the two-dimensional context. In the general procedure we impose kinematic or static constraints suitable for the planar representation chosen to derive the expression for the tangent operators (Runesson et al. [67], Steinmann and Willam [79], Sabban [68]).

In the case of plane stress it is customary to impose the static constraint $\dot{\sigma}_{33}$=0, $\dot{\sigma}_{13}$=0, $\dot{\sigma}_{23}$=0 into the strain/stress relation in the form of the tangent compliance

$$\dot{\epsilon}_{ij} = C^{ep}_{ij\alpha\beta}\, \dot{\sigma}_{\alpha\beta} \tag{2.163}$$

where the Greek subscripts denote indices varying between 1 and 2. Eliminating the out-of-plane components of $\dot{\boldsymbol{\epsilon}}$

$$\dot{\epsilon}_{\alpha\beta} = C^{ep}_{\alpha\beta\gamma\delta}\, \dot{\sigma}_{\gamma\delta} \;\; ; \qquad C^{ep}_{\alpha\beta\gamma\delta} = C^{o}_{\alpha\beta\gamma\delta} + \frac{m_{\alpha\beta}\, n_{\gamma\delta}}{H} \tag{2.164}$$

The stiffness is obtained by inversion, with the Shermann and Morrison formula [69]

$$E^{ep}_{\alpha\beta\gamma\delta} = E^{o}_{\alpha\beta\gamma\delta} - \frac{E^{o}_{\alpha\beta\zeta\xi}\, m_{\zeta\xi}\, n_{\omega\theta}\, E^{o}_{\omega\theta\gamma\delta}}{H + n_{\eta\theta} E^{o}_{\eta\theta\phi\psi} m_{\phi\psi}} \tag{2.165}$$

Here $E^{o}_{\alpha\beta\gamma\delta}$ is obtained by inversion of $C^{o}_{\alpha\beta\gamma\delta}$ and represents the plane stress operator

$$E^{o}_{\alpha\beta\gamma\delta} = \frac{E}{1-\nu^2}\, [\nu\, \delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1-\nu}{2}\, (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})] \tag{2.166}$$

which reads in matrix form

$$\mathbf{E}^{pss}_{o} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{2.167}$$

Since the mathematical expression of the tangent operator is different when compared with the $3D$ or the plane strain case, we expect different results for the localization condition. This is indeed the case, as shown in what the following figures, confirmed by the analytic results of Runesson et al. [67].

The plane stress case appears to be more susceptible to localization. In fact, the static constraint does not put any limitation on the jump of the out-of-plane strain increment $\dot{\epsilon}_{33}\neq 0$ as is the case for the $3D$ and plane strain descriptions. For example, considering the $J_2$-plasticity case, onset of localization is detected for uniaxial tension with $E_p$=0 (Fig. 2.8), while softening is required for the $3D$ case (Fig. 2.5). The critical orientations $\theta_{cr}$ between $\vec{\mathbf{N}}_{cr}$ and the load axis are also different: the plane stress analysis yields the classic result of $35.26^o$, independently by $\nu$. Instead, the $3D$ analysis of the critical orientation, which depends on $\nu$, yields $41.17^o$ for $\nu$=0.3 (the result $35.26^o$ is recovered only for $\nu$=0).

In the following figures, the spectral properties of $\mathbf{Q}_{ep}$ for plane stress $J_2$-plasticity with $E_p = 0$ (perfect elastic-plasticity) are investigated for various limit stress states on the Von Mises locus (see Fig. 2.7). In the ranges $A - B$, $F - G$ localization is not possible without softening (SOBH [75], SABBAN [68]). Fig. 2.7 also shows the critical orientation for $\vec{\mathbf{N}}$ and the corresponding polarization vector $\vec{\mathbf{M}}$: mode I opening (i.e. $\vec{\mathbf{N}} \parallel \vec{\mathbf{M}}$) is obtained for biaxial tension with $\sigma_1 = 2\sigma_2$, mode II opening (i.e. $\vec{\mathbf{N}} \perp \vec{\mathbf{M}}$) is reached for the pure shear case and a mixed mode of failure appears in uniaxial tension.

The phase velocity plots (Fig. 2.11, 2.13, 2.15, 2.17) show clearly the spectral property of $\mathbf{Q}_{ep}$ (in this case $\mathbf{Q}_{ep}$ can be written as a 2x2 matrix and has only two eigenvalues). In this plot the normalized eigenvalues of $\mathbf{Q}_{ep}$ are represented in a polar diagram: the distance from the origin of the intersection of the straight line with orientation $\vartheta$ with the curve represents the normalized eigenvalue. The normalization is done relatively to the value of the largest eigenvalue of $\mathbf{Q}_e$, which corresponds to the longitudinal wave speed $\lambda + 2\mu$.

Note that in the plane stress case the longitudinal wave speed is modified as compared to that of the $3D$ description, even in the linear elastic case. It can be obtained considering the eigenvalues of the elastic acoustic tensor determined by two single contractions of the elastic stiffness (2.166)

$$Q^e_{\beta\gamma} = N_\alpha\, E^o_{\alpha\beta\gamma\delta}\, N_\delta = \frac{E}{1-\nu^2}\,[\frac{1+\nu}{2} N_\beta N_\gamma + \frac{1-\nu}{2}\,\delta_{\beta\gamma}] \tag{2.168}$$

In matrix form, this leads to

$$\mathbf{Q} = \frac{E}{(1-\nu^2)} \begin{bmatrix} N_1^2 + N_2^2\,(1-\nu)/2 & (1-\nu)/2 + \nu\, N_1\, N_2 \\ (1-\nu)/2 + \nu\, N_1\, N_2 & N_2^2 + N_1^2\,(1-\nu)/2 \end{bmatrix} \tag{2.169}$$

The eigenvalues are given by (with $N_1^2 + N_2^2 = 1$)

$$\begin{aligned} (\rho c_L^2) &= \frac{E}{(1-\nu^2)}\,(N_1^2 + N_2^2) = \frac{4\,\mu}{(\lambda + 2\,\mu)}\,(\lambda+\mu) \\ (\rho c_T^2) &= \frac{E}{2\,(1+\nu)}\,(N_1^2 + N_2^2) = \mu \end{aligned} \tag{2.170}$$

Thus the amplitude of the two circles in the phase velocity plots for $\nu$=0.3 are
$\left(\frac{c_L}{\lambda+2\mu}\right)^2 = \frac{E}{1-\nu^2}\,\frac{1}{\lambda+2\mu} = 0.82$
$\left(\frac{c_T}{\lambda+2\mu}\right)^2 = \mu\,\frac{1}{\lambda+2\mu} = 0.29$

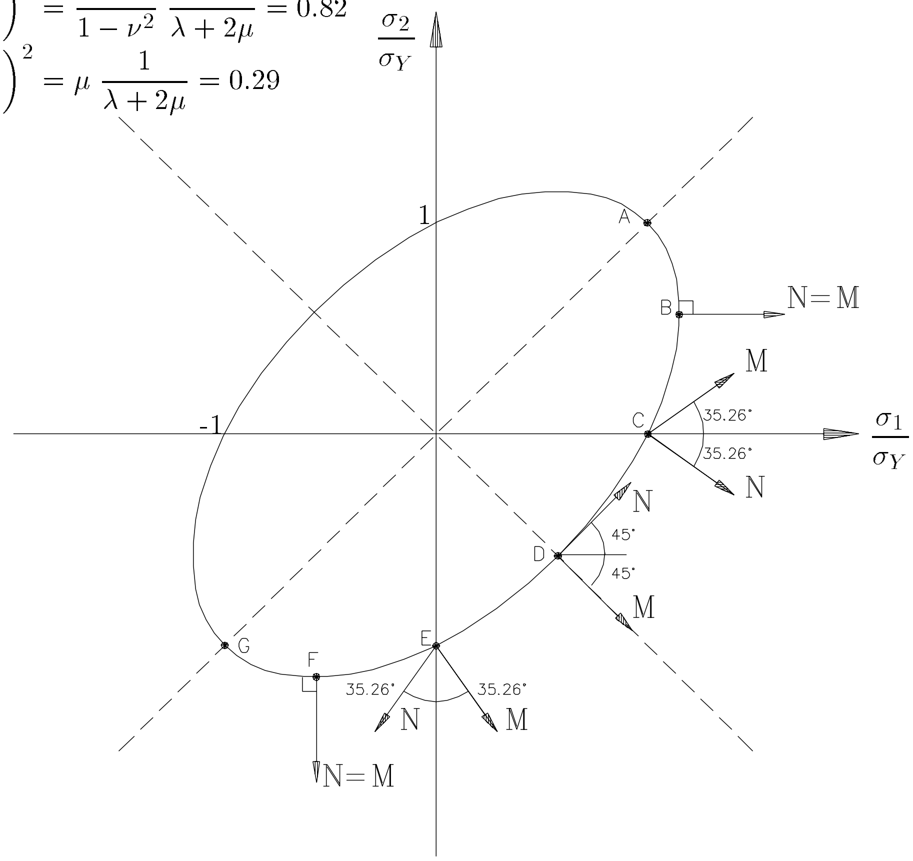



Figure 2.7: Plane stress von Mises locus: critical orientations of **N** and **M**

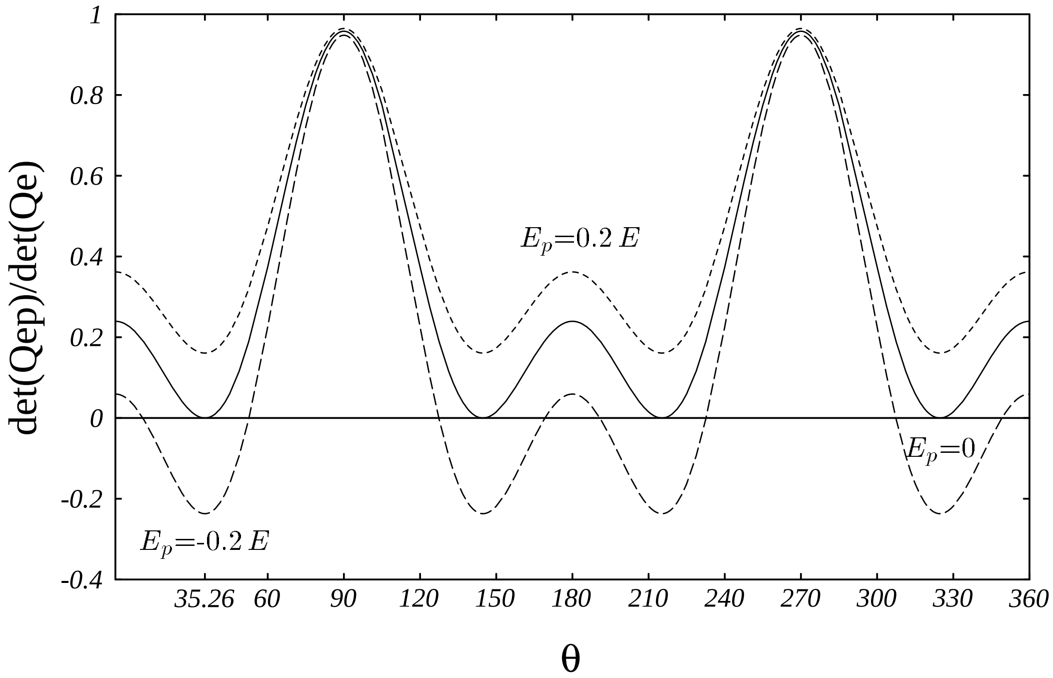



Figure 2.8. (C) Uniaxial tension ($\nu = 0.3$): influence of the hardening parameter $E_p$

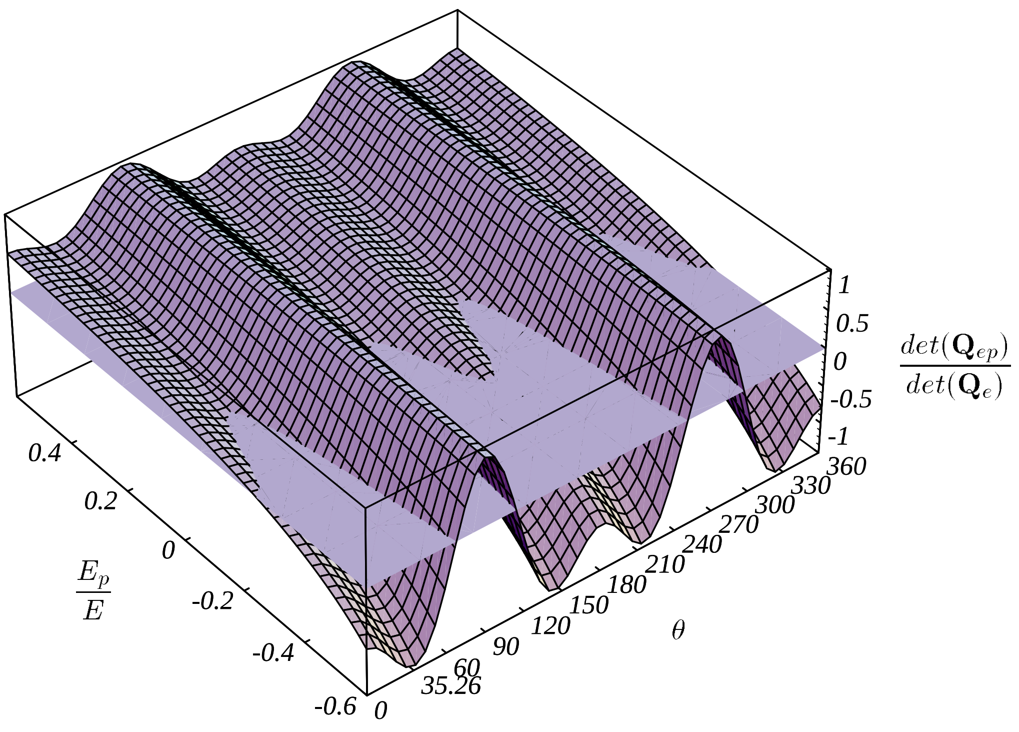



Figure 2.9: (C) Uniaxial tension ($\nu = 0.3$): $det(\mathbf{Q}_{ep})$ as a function of $E_p$, $\theta$

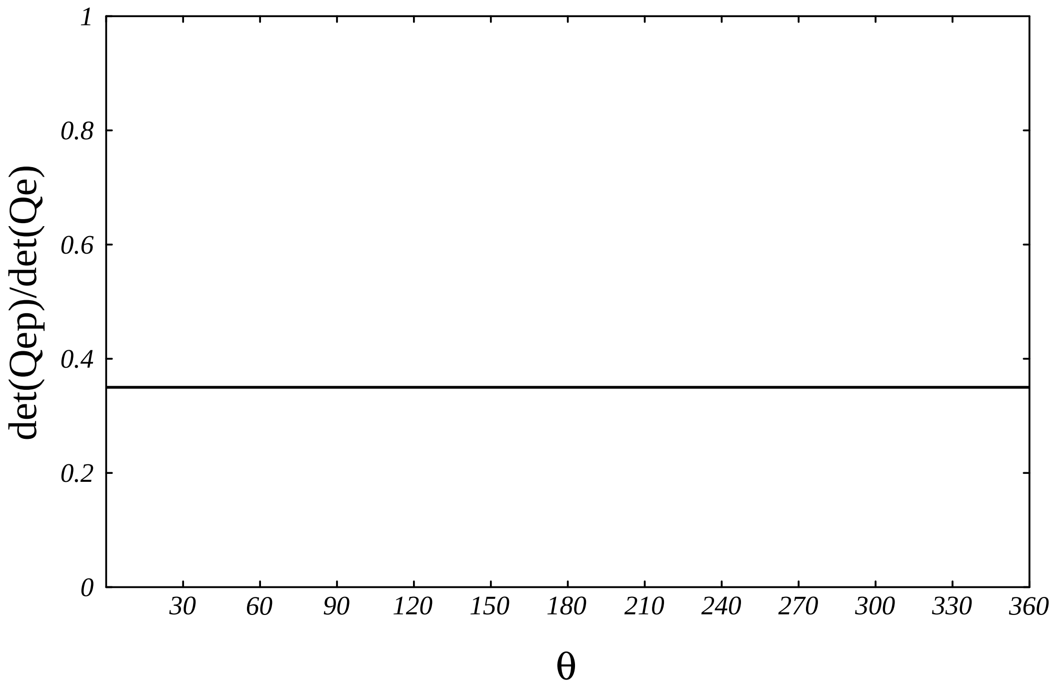



Figure 2.10: (A) Equi-biaxial tension ($E_p = 0$, $\nu = 0.3$): singularity of $\mathbf{Q}_{ep}$

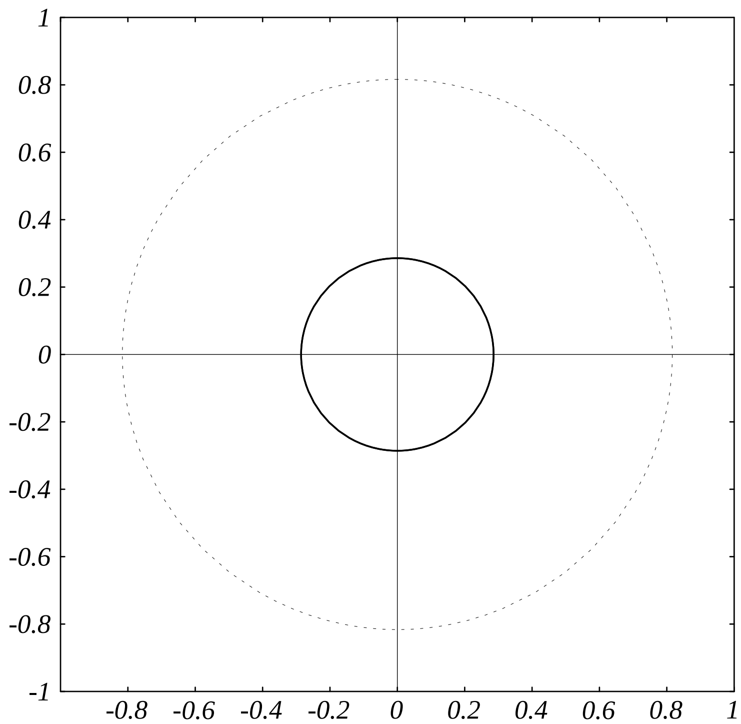



Figure 2.11: (A) Equi-biaxial tension ($E_p = 0$, $\nu = 0.3$): phase velocity plot

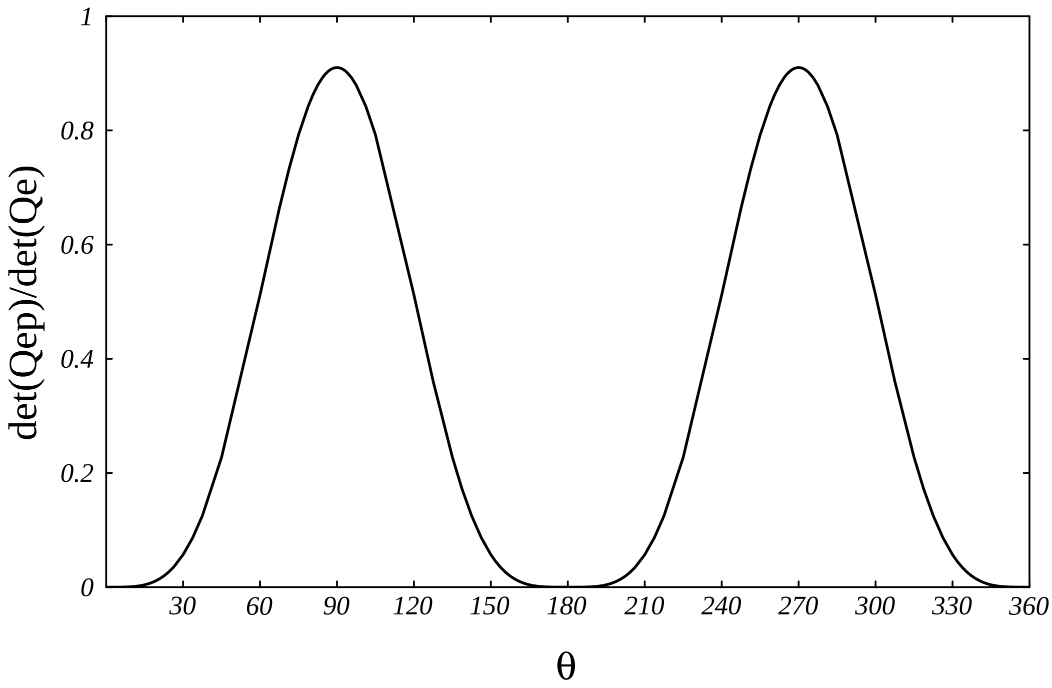



Figure 2.12: (B) $\sigma_1 = 2\ \sigma_2$ ($E_p = 0$, $\nu = 0.3$): singularity of $\mathbf{Q}_{ep}$

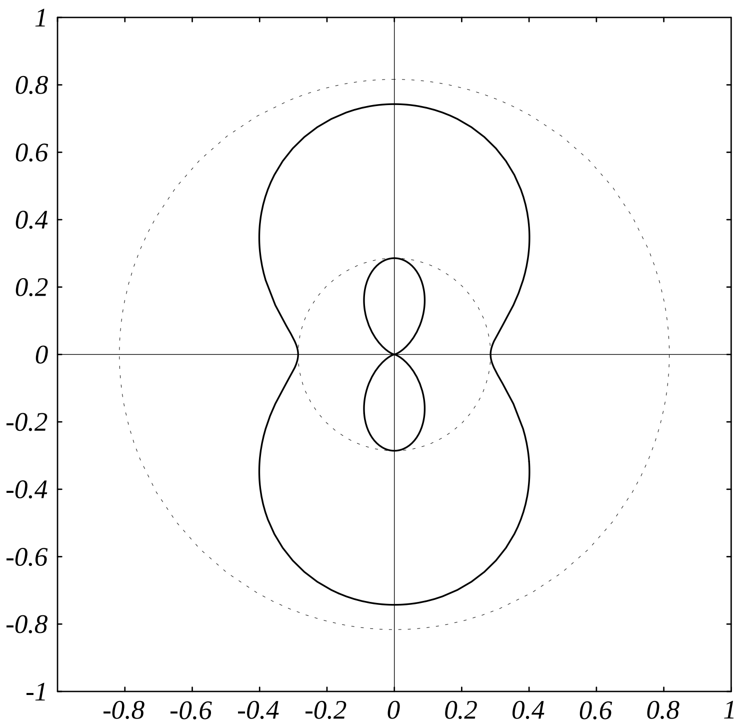



Figure 2.13: (B) $\sigma_1 = 2\ \sigma_2$ ($E_p = 0$, $\nu = 0.3$): phase velocity plot

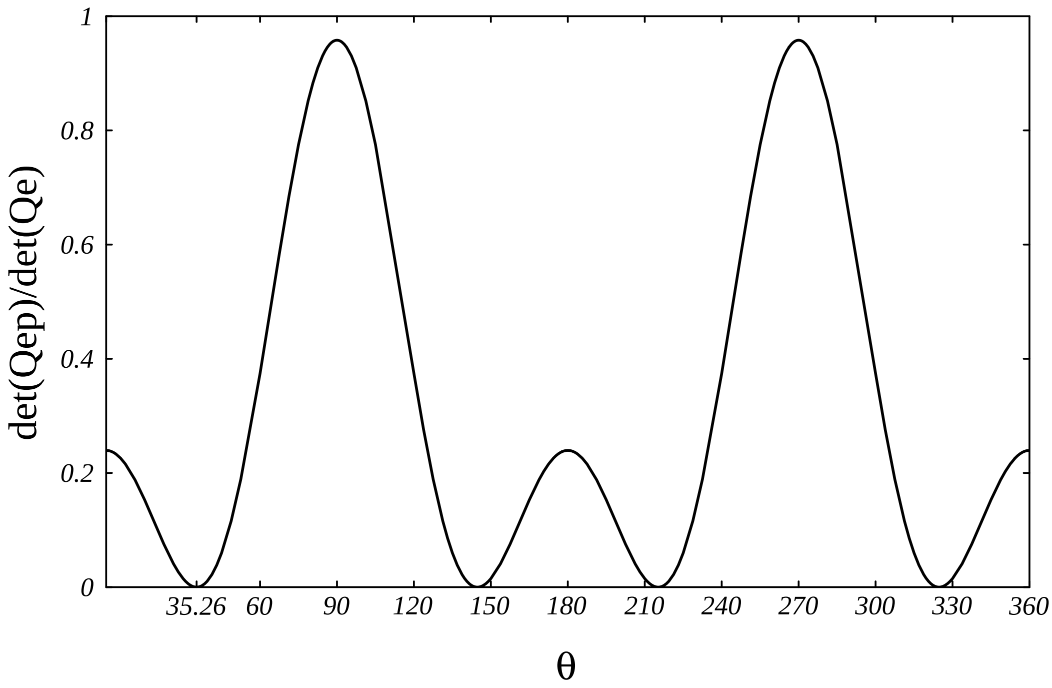



Figure 2.14: (C) Uniaxial tension ($E_p = 0$, $\nu = 0.3$): singularity of $\mathbf{Q}_{ep}$

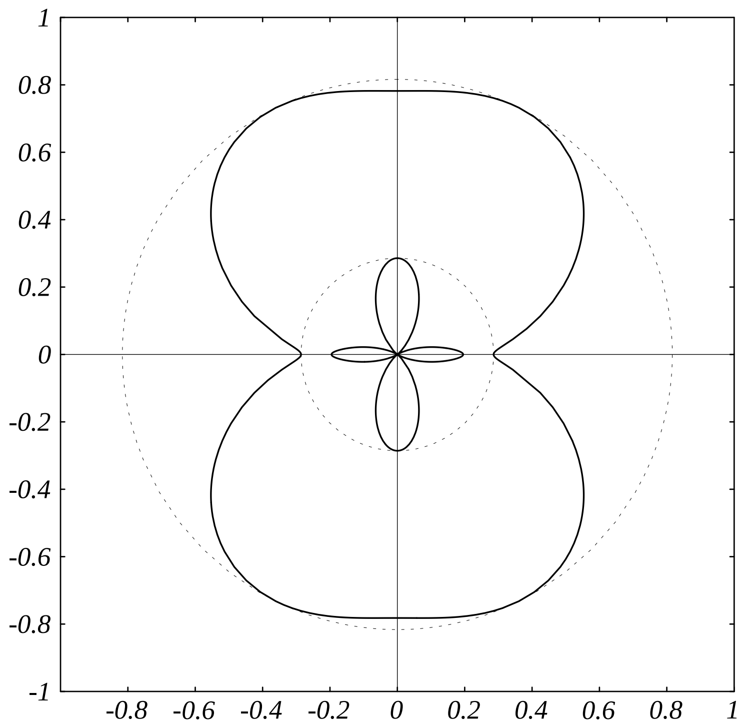



Figure 2.15: (C) Uniaxial tension ($E_p = 0$, $\nu = 0.3$): phase velocity plot

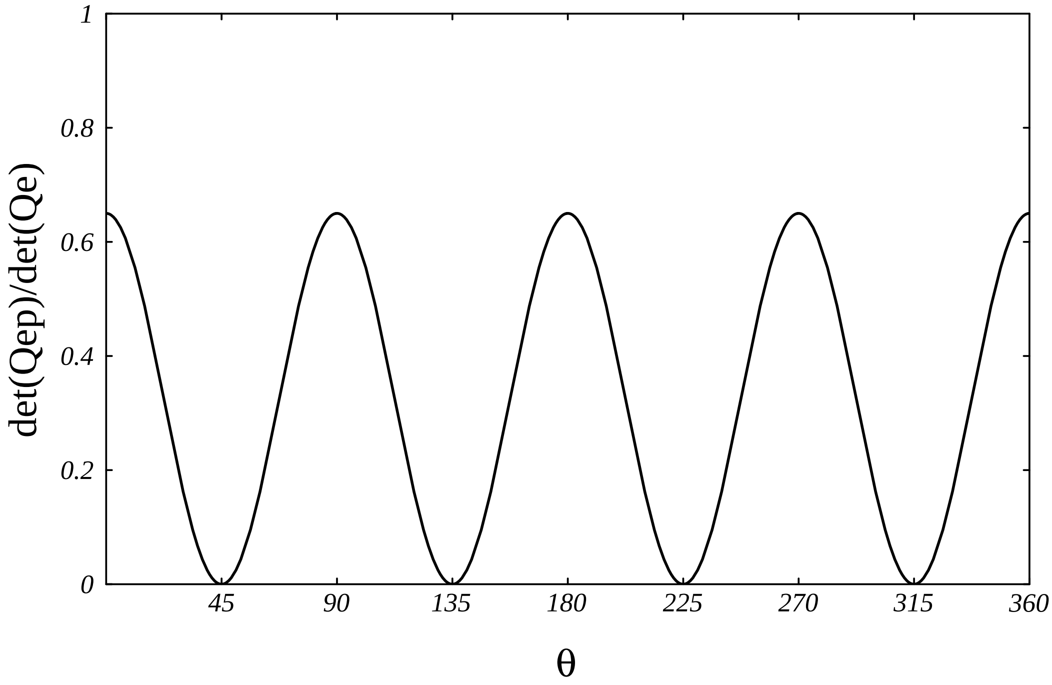



Figure 2.16: (D) Pure shear ($E_p = 0$, $\nu = 0.3$): singularity of $\mathbf{Q}_{ep}$

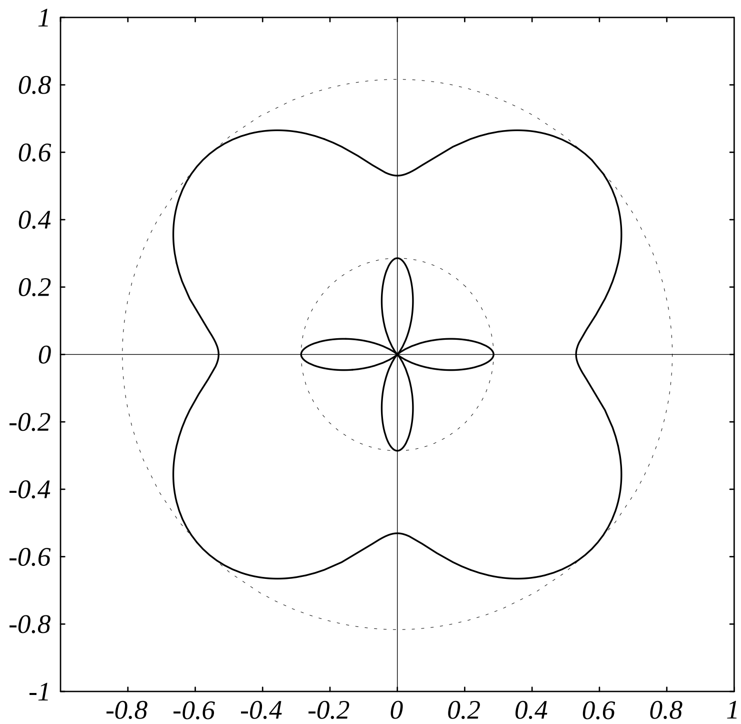



Figure 2.17: (D) Pure shear ($E_p = 0$, $\nu = 0.3$): phase velocity plot

# CHAPTER 3

## CONTINUUM MECHANICS MODELS FOR MATERIALS WITH STIFFNESS DEGRADATION

Experimental observations on load-unload tests primarily in tension, but also in compression, indicate for many non-ferrous materials a degradation of the initial stiffness. This constitutive degradation features features are common to different structural materials, when they are loaded beyond peak: concrete, ceramics, composites (Fig. 3.1).

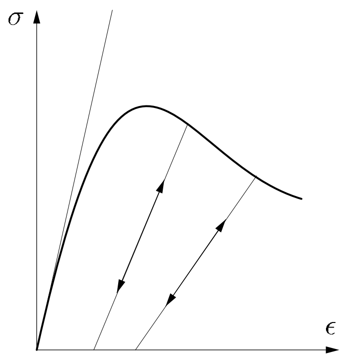
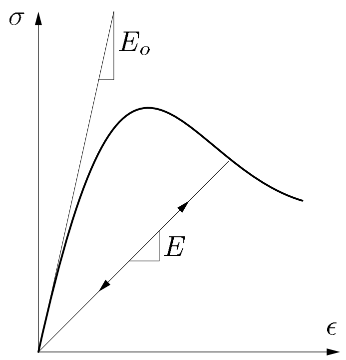



Figure 3.1: Materials with stiffness degradation

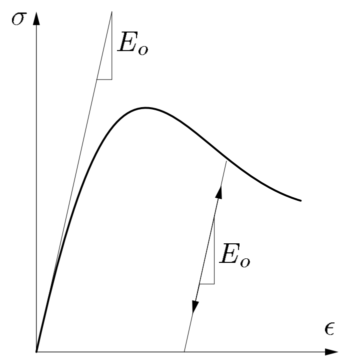
Recently a lot of research did focus on developing constitutive models which are able to capture the stiffness degradation. However, it appears that a unified and widely accepted description of this kind is still not available, especially if one compares the current status in this field with the mature theory of plasticity.

The classic theory of plasticity focuses on the description of material strength, taking into account the non-reversible processes involved with the dissipation of the energy, which results in non-recoverable plastic deformations. However, elastic unloading is traditionally assumed to occur with the initial elastic stiffness and the degradation of elastic stiffness is normally neglected (Fig. 3.2b).

Dougill [21] developed an *elastic-fracturing* model for degrading stiffness, utilizing and extending the concepts of the theory of plasticity, i.e. adopting a loading function which defines the threshold for the fracturing state and a flow rule for inelastic stresses and stiffness. The ideal fracturing material unloads always towards the origin of the stress/strain diagram (Fig. 3.2a), without any permanent strain. The term "fracturing" should not be misleading: in this approach, fracture is not localized and the model does not consider discrete cracks and fracture processes along the line of fracture mechanics concepts. The fracturing process is homogenized throughout the body and the medium is still considered to remain an intact continuum.




a) Elastic-fracturing material




b) Elastic-plastic material

Figure 3.2: Elastic unloading/reloading stiffnesses

Kachanov [33] introduced the concept of damage relating the current stress carrying area to the original cross section. The area reduction effect leads to an increase of the effective stress acting on the material with the stiffness degradation, expressing the relation between nominal stress and nominal strain. This can be consider the point of departure of *continuum damage mechanics* which has been developed in recent years.

It is apparent that the combination of theory of plasticity and elastic-fracturing or damage mechanics concepts results in a satisfactory description of all the inelastic processes involved, exhibiting irreversible strains after unloading and decreasing slopes of the unloading branch due to the progressing damage. In fact, which couple the two effects various models have been proposed (e.g. HUECKEL AND MAIER [30], DRAGON AND MRÓZ [23], BAŽANT AND KIM [2], SIMO AND JU [71]).

However, before using such combined description, it seems more important to establish a unified description of stiffness degradation, which encompasses continuum damage models as a particular case of the elastic-fracturing theory (CAROL ET AL. [14]).

In the following, the theory of elastic-fracturing materials will be briefly reviewed and its relation with some existing damage models will be addressed. In particular, the tangent operators, which are required for localization analysis of the elastic-fracturing/damaging description, will be derived.

## 3.1 Elastic-fracturing materials. Stress-based formulation

As we have seen, elastic-fracturing materials assume elastic unloading always to occur towards the origin. Thus, the constitutive formulation can be stated in "total" terms, as a secant relation

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon} \qquad\qquad \boldsymbol{\epsilon} = \mathbf{C} : \boldsymbol{\sigma} \tag{3.1}$$

where $\mathbf{E}$ and $\mathbf{C}$ denote the secant material stiffness and compliance respectively ($\mathbf{C} = \mathbf{E}^{-1}$ and vice versa). Because the unloading is assumed to be elastic, the secant stiffness changes only during the loading path and retains all the symmetries of the $4^{th}$ order elasticity tensor, i.e, the major symmetries are preserved, in addition to the minor symmetries.

The rate of change of secant stiffness and compliance can be readily determined, considering the identity $\mathbf{C} : \mathbf{E} = \mathbf{E} : \mathbf{C} = \mathbf{I}_4$. Differentiation of this identity

$$\dot{\mathbf{C}} : \mathbf{E} + \mathbf{C} : \dot{\mathbf{E}} = \mathbf{0} \tag{3.2}$$

renders

$$\dot{\mathbf{E}} = -\mathbf{E} : \dot{\mathbf{C}} : \mathbf{E} \qquad\qquad \dot{\mathbf{C}} = -\mathbf{C} : \dot{\mathbf{E}} : \mathbf{C} \tag{3.3}$$

These relations will be used in the following when a change of variable $\dot{\mathbf{C}}/\dot{\mathbf{E}}$ is required.

The other important assumption in the fracturing formulations is the existence of a loading function defining the current fracturing state, similar to the yield function in the classic theory of plasticity. For the stress-based case the surface is defined as a function of the current stress state $\boldsymbol{\sigma}$ and of some internal variables which account for the history $\mathbf{b}$: $F = F(\boldsymbol{\sigma}, \mathbf{b}) = 0$. Progressive fracture is mobilized when the stress state reaches the fracture surface and the strain rate leads to an elastic stress rate directed outside the loading surface. As fracture progresses the stiffness decreases correspondingly.

Irreversible degradation of the elastic stiffness can be related to the existence of an additional inelastic strain component $\boldsymbol{\epsilon}_i$ such that the stress rate $\dot{\boldsymbol{\sigma}}$ is obtained by multiplying the secant stiffness $\mathbf{E}$ with the elastic strain rate $\dot{\boldsymbol{\epsilon}}_e = \dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_i$ (see Fig. 3.3 for the uniaxial case)

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_i) \tag{3.4}$$

Now, even if unloading leads back to the origin without irreversible deformation, the process is dissipative because part of the energy is spent in the degradation of the material and cannot be recovered (see Fig. 3.4 in Section 3.1.2).

Following the procedures of plasticity theory, a flow rule for the inelastic strain

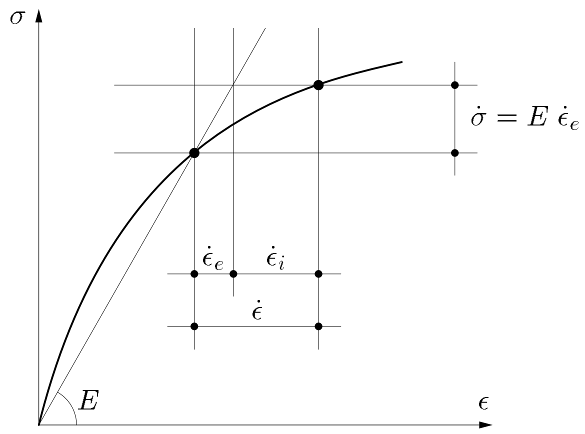



Figure 3.3: Definition of the inelastic strain rate

rate is adopted and the differential consistency condition is imposed

$$\begin{aligned} \dot{\boldsymbol{\epsilon}}_i &= \dot\lambda\ \mathbf{m} \\ \dot F &= \mathbf{n} : \dot{\boldsymbol{\sigma}} - H\dot\lambda = 0 \end{aligned} \tag{3.5}$$

Here $\mathbf{n}$ denotes the gradient of the loading function and $H$ the hardening parameter

$$\mathbf{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}} \qquad\qquad H = -\frac{\partial F}{\partial \lambda} \tag{3.6}$$

As was shown in section 2.3.1 the rate of change of the inelastic multiplier can be expressed either in terms of $\dot{\boldsymbol{\epsilon}}$ or $\dot{\boldsymbol{\sigma}}$, depending on strain/stress control

$$\dot\lambda = \frac{\mathbf{n} : \mathbf{E} : \dot{\boldsymbol{\epsilon}}}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}} = \frac{\mathbf{n} : \dot{\boldsymbol{\sigma}}_e}{h} \geq 0 \qquad \text{or} \qquad \dot\lambda = \frac{\mathbf{n} : \dot{\boldsymbol{\sigma}}}{H} \geq 0 \tag{3.7}$$

This leads to the tangent stiffness and compliance relations respectively, for progressive fracturing

$$\boxed{\mathbf{E}_{ef} = \mathbf{E} - \frac{\mathbf{E} : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}} \qquad\qquad \mathbf{C}_{ef} = \mathbf{C} + \frac{\mathbf{m} \otimes \mathbf{n}}{H}} \tag{3.8}$$

The subscript $_{ef}$ stands for elastic-fracturing, keeping in mind that these are tangent operators of the differential linearization. A more general notation should resort to $\mathbf{E}_t$, $\mathbf{C}_t$, without specification of the underlying model.

There are two important differences in expressions (3.8) with respect to the dual plastic counterparts:

- the operators $\mathbf{E}$, $\mathbf{C}$ represent here the current secant stiffness and compliance and are different from the initial values $\mathbf{E}_o$, $\mathbf{C}_o$ adopted in plasticity.
- $\mathbf{E}$, $\mathbf{C}$ are not constant and evolution laws have to be prescribed.

The rate of change $\dot{\mathbf{E}}$, $\dot{\mathbf{C}}$ are not independently of each other, but they must be related to the assumed flow rule for the inelastic strain. In fact, this inelastic strain contains, at the strain level, the information of the degradation of the stiffness. Considering the secant material relation and differentiating it

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon} \qquad \Rightarrow \qquad \dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\epsilon}} + \dot{\mathbf{E}} : \boldsymbol{\epsilon} \tag{3.9}$$

From the definition of inelastic behavior $\dot{\boldsymbol{\sigma}} = \mathbf{E} : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_i)$, the inelastic strain can be compared

$$-\,\mathbf{E} : \dot{\boldsymbol{\epsilon}}_i = \dot{\mathbf{E}} : \boldsymbol{\epsilon} \qquad \Rightarrow \qquad \dot{\boldsymbol{\epsilon}}_i = -\mathbf{C} : \dot{\mathbf{E}} : \boldsymbol{\epsilon} \tag{3.10}$$

Using the first expression (3.3) $\dot{\mathbf{E}} = -\mathbf{E} : \dot{\mathbf{C}} : \mathbf{E}$

$$\dot{\boldsymbol{\epsilon}}_i = \dot{\mathbf{C}} : \boldsymbol{\sigma} \tag{3.11}$$

It is important to note now that this last equation can be used to determine the inelastic strain rate, once the rate of compliance change is known. In the general $3D$ case, the inverse implication does not apply, i.e., it is not possible to obtain uniquely the $4^{th}$ order tensor $\dot{\mathbf{C}}$ which gives the prescribed $2^{nd}$ tensor $\dot{\boldsymbol{\epsilon}}_i$ from (3.11).

Thus it is appropriate to define a generalized flow rule for the compliance, called the *fracturing rule*, which describes directly the compliance rate at the $4^{th}$ order tensor level (Ortiz [54])

$$\dot{\mathbf{C}} = \dot{\lambda}\, \mathbf{M} \tag{3.12}$$

where $\dot{\lambda}$ is the same inelastic multiplier which appears in the flow rule for the inelastic strains (3.5) and $\mathbf{M}$ is a $4^{th}$ order tensor which prescribes the “direction” of the compliance rate. $\mathbf{M}$ must possess all the symmetries as $\mathbf{C}$ and $\dot{\mathbf{C}}$.

Substituting strain flow rule and fracturing rule in eqn. (3.11)

$$\dot{\lambda}\ \mathbf{m} = \dot{\lambda}\ \mathbf{M} : \boldsymbol{\sigma} \tag{3.13}$$

and eliminating the inelastic multipliers, we obtain the relation between the two evolution rules

$$\boxed{\mathbf{m} = \mathbf{M} : \boldsymbol{\sigma}} \tag{3.14}$$

Thus, when the fracturing rule is assigned, the flow rule directly follows. Again, the inverse is not possible and the definition of the flow rule is not enough for the unique determination of the fracturing rule.

When the fracturing rule is assigned, the rate of change of stiffness can be readily obtained, considering the first eqn. in (3.3)

$$\dot{\mathbf{E}} = -\mathbf{E} : \dot{\mathbf{C}} : \mathbf{E} = -\dot{\lambda}\ \mathbf{E} : \mathbf{M} : \mathbf{E} = \dot{\lambda}\ \bar{\mathbf{M}} \tag{3.15}$$

where $\bar{\mathbf{M}}$ is the $4^{th}$ order tensor defining the fracturing rule for the stiffness.

Using eqn. (3.14) the equations for the stress-based fracturing material can now be summarized as

$$\boxed{\begin{array}{c} \dot{\boldsymbol{\sigma}} = \mathbf{E}_{ef} : \dot{\boldsymbol{\epsilon}} \quad \text{with} \quad \mathbf{E}_{ef} = \mathbf{E} - \dfrac{\mathbf{E} : \mathbf{M} : \boldsymbol{\sigma} \otimes \mathbf{n} : \mathbf{E}}{H + \mathbf{n} : \mathbf{E} : \mathbf{M} : \boldsymbol{\sigma}} \\[2ex] \dot{\mathbf{E}} = -\dfrac{\mathbf{n} : \mathbf{E} : \dot{\boldsymbol{\epsilon}}}{H + \mathbf{n} : \mathbf{E} : \mathbf{M} : \boldsymbol{\sigma}}\ \mathbf{E} : \mathbf{M} : \mathbf{E} \end{array}} \tag{3.16}$$

### 3.1.1 Associativity in strain space

The concept of plastic associativity can be readily extended also for the elastic-fracturing material. The important point is that there are different levels of associativity, as we will show in the next sections.

For now, considering the flow rule for the inelastic strains $\dot{\boldsymbol{\epsilon}}_i = \dot{\lambda}\ \mathbf{m}$ we can define a fracturing model to be *associated in the strain space* if the flow direction is parallel to the gradient of the yield function

$$\textit{Associativity in strain space} \qquad \Rightarrow \qquad \mathbf{n} \parallel \mathbf{m} \tag{3.17}$$

Since the flow rule must be derived from the fracturing rule, this means that the fracturing rule $\mathbf{M}$ must be chosen in the way such that

$$(\mathbf{M} : \boldsymbol{\sigma}) \quad \parallel \quad \mathbf{n} \tag{3.18}$$

where $\mathbf{n}$ is assigned once the loading function is defined. This demonstrates the necessity of defining associativity at the $4^{th}$ order tensor level, from which the associativity at the $2^{nd}$ order level follows.

If the fracturing model is associated the tangent operators are fully symmetric. This property, as we have seen in elastic-plasticity, plays an important rule when the failure indicators of the operators are analyzed.

### 3.1.2 Associativity in compliance space

Before introducing the concept of associativity in the compliance space it is necessary to consider some thermodynamic aspects of the fracturing process, with the determination of the energy dissipation related to the degradation of the stiffness. The *free energy* per unit volume is the amount of energy that can be recovered after unloading. For elastic-fracturing material it is given by the elastic energy $w_e = \boldsymbol{\sigma} : \boldsymbol{\epsilon}/2$, which may be evaluated either with the current secant stiffness or the compliance relation

$$w_e(\boldsymbol{\epsilon}, \mathbf{E}) = \frac{1}{2}\, \boldsymbol{\epsilon} : \mathbf{E} : \boldsymbol{\epsilon} \qquad \text{or} \qquad w_e(\boldsymbol{\sigma}, \mathbf{C}) = \frac{1}{2}\, \boldsymbol{\sigma} : \mathbf{C} : \boldsymbol{\sigma} \tag{3.19}$$

The rate of elastic energy is obtained by differentiation

$$w_e = \frac{1}{2}\, \boldsymbol{\sigma} : \boldsymbol{\epsilon} \qquad \Rightarrow \qquad \dot{w}_e = \frac{1}{2}\, (\dot{\boldsymbol{\sigma}} : \boldsymbol{\epsilon} + \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}) \tag{3.20}$$

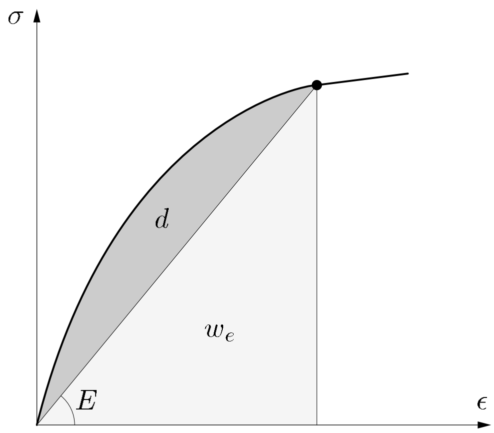



Figure 3.4: Energy dissipation $d$ and free energy $w_e$

Using for example the strain energy description, the rate of change of elastic energy is obtained as

$$\begin{aligned}
\dot{w}_e &= \left.\frac{\partial w_e}{\partial \boldsymbol{\epsilon}}\right|_{\mathbf{E}} : \dot{\boldsymbol{\epsilon}} + \left.\frac{\partial w_e}{\partial \mathbf{E}}\right|_{\boldsymbol{\epsilon}} :: \dot{\mathbf{E}} \\
&= \mathbf{E} : \boldsymbol{\epsilon} : \dot{\boldsymbol{\epsilon}} + \frac{1}{2}\, \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon} :: \dot{\mathbf{E}} \\
&= \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \bar{\mathbf{Y}} :: \dot{\mathbf{E}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \dot{d}
\end{aligned} \tag{3.21}$$

where the scalar quantity $\dot{d}$ denotes the dissipation and the fourth order tensor

$$(-\bar{\mathbf{Y}}) = \left.\frac{\partial w_e}{\partial \mathbf{E}}\right|_{\boldsymbol{\epsilon}} = \frac{1}{2}\, \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon} \tag{3.22}$$

will be termed the thermodynamic force associated to the stiffness.

The *fracturing dissipation* is given by the total amount of energy spent in loading minus the energy recoverable after unloading, as it is shown in Fig. 3.4 (Dougill [21]).

$$d = \int_0^t \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}\, dt - \frac{1}{2}\, \boldsymbol{\sigma} : \boldsymbol{\epsilon} = \int_0^t \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}\, dt - w_e \tag{3.23}$$

Differentiation yields

$$\dot{d} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \dot{w}_e = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \frac{1}{2}\, (\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \dot{\boldsymbol{\sigma}} : \boldsymbol{\epsilon}) = \frac{1}{2}\, (\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\sigma}} : \boldsymbol{\epsilon}) \tag{3.24}$$

Now, recalling that

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon} \quad \text{and} \quad \dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\epsilon}} + \dot{\mathbf{E}} : \boldsymbol{\epsilon} \tag{3.25}$$

Substitution into expression (3.24) and use of the symmetries of the secant stiffness, render

$$\dot{d} = \frac{1}{2}\left(\mathbf{E} : \boldsymbol{\epsilon} : \dot{\boldsymbol{\epsilon}} - \mathbf{E} : \dot{\boldsymbol{\epsilon}} : \boldsymbol{\epsilon} - \dot{\mathbf{E}} : \boldsymbol{\epsilon} : \boldsymbol{\epsilon}\right) = -\frac{1}{2}\,\boldsymbol{\epsilon} : \dot{\mathbf{E}} : \boldsymbol{\epsilon} \tag{3.26}$$

which justifies the introduction of $\dot{d}$ in the last term of eqn. (3.21).

The rate of fracturing dissipation can also be expressed in terms of stress and compliance rate, using eqn. (3.3) or in terms of stress and inelastic strain rate $\dot{\boldsymbol{\epsilon}}_i$ by means of eqn. (3.11)

$$\dot{d} = \frac{1}{2}\,\boldsymbol{\sigma} : \dot{\mathbf{C}} : \boldsymbol{\sigma} = \frac{1}{2}\,\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}_i \tag{3.27}$$

Because the degradation of the stiffness is an irreversible process we might expect the rate of fracturing dissipation being non-negative, as a manifestation of the second principle of thermodynamics: $\dot{d}\geq 0$. Utilizing the fracturing rule for the compliance $\dot{\mathbf{C}} = \dot{\lambda}\,\mathbf{M}$, with $\dot{\lambda}\geq 0$

$$\dot{d} = \frac{1}{2}\,\boldsymbol{\sigma} : \dot{\mathbf{C}} : \boldsymbol{\sigma} = \dot{\lambda}\,\frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{M} : \boldsymbol{\sigma} \geq 0 \tag{3.28}$$

we obtain a second requirement for the fracturing rule, i.e. that $\mathbf{M}$ not only must be symmetric but also positive definite.

It is now necessary to introduce the concept of a *thermodynamic force*, often referred to in the literature, in the contest of thermodynamics. Looking at expression (3.27) we can say that the thermodynamic force associated to $\boldsymbol{\epsilon}_i$ is $\boldsymbol{\sigma}/2$. In general the thermodynamic force associated to its conjugate quantity is such that the inner product with the quantity itself gives the rate of fracturing dissipation. For the subsequent purpose it is convenient to define the thermodynamic force $(-\mathbf{Y})$ associated to the secant compliance $\mathbf{C}$ and conversely the thermodynamic force $\bar{\mathbf{Y}}$ associated to the secant stiffness $\mathbf{E}$ (the minus sign is usually introduced in the literature)

$$\dot{d} = (-\mathbf{Y}) :: \dot{\mathbf{C}} \qquad \text{and} \qquad \dot{d} = \bar{\mathbf{Y}} :: \dot{\mathbf{E}} \tag{3.29}$$

Comparing these expressions with the relations previously obtained for $\dot{d}$ one can readily find

$$-\mathbf{Y} = \frac{1}{2}\,\boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \qquad \text{and} \qquad -\bar{\mathbf{Y}} = \frac{1}{2}\,\boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon} \tag{3.30}$$

The thermodynamic force associated to the compliance is the generalization of the stress tensor (thermodynamic force associated to the strain) at the $4^{th}$ order tensor level. Thus it is natural to define a gradient of the loading surface in the space of thermodynamic forces:

$$\mathbf{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}} \qquad \Rightarrow \qquad \mathbf{N} = \frac{\partial F}{\partial(-\mathbf{Y})} \tag{3.31}$$

Since $\mathbf{Y}$ is fully symmetric, $\mathbf{N}$ is symmetric as well.

Now we are able to define associativity in the compliance space (or at the $4^{th}$ order level) in the case that the two tensors $\mathbf{N}$ and $\mathbf{M}$ are coaxial

$$\textit{Associativity in compliance space} \qquad \Rightarrow \qquad \mathbf{N} \parallel \mathbf{M} \tag{3.32}$$

This associativity at the higher order level implies also associativity at the $2^{nd}$ order level. In fact the gradient of the fracturing function $\mathbf{n}$ can be readily related to the tensor $\mathbf{N}$ as

$$\mathbf{n} = \frac{\partial F}{\partial \boldsymbol{\sigma}} = \frac{\partial F}{\partial(-\mathbf{Y})} :: \frac{\partial(-\mathbf{Y})}{\partial \boldsymbol{\sigma}} \tag{3.33}$$

Expanding the derivatives

$$\begin{aligned} n_{ij} &= N_{pqrs}\,\tfrac{1}{2}\,(\sigma_{pq}\frac{\partial \sigma_{rs}}{\partial \sigma_{ij}} + \sigma_{rs}\frac{\partial \sigma_{pq}}{\partial \sigma_{ij}}) \\ &= N_{pqrs}\,\frac{1}{2}\,(\sigma_{pq}\delta_{ri}\delta_{sj} + \sigma_{rs}\delta_{pi}\delta_{qj}) \\ &= \tfrac{1}{2}\,(N_{pqij}\sigma_{pq} + N_{ijrs}\sigma_{rs}) = N_{ijrs}\sigma_{rs} \end{aligned} \tag{3.34}$$

we finally obtain

$$\boxed{\mathbf{n} = \mathbf{N} : \boldsymbol{\sigma}} \tag{3.35}$$

Because inelastic flow and fracturing rules are related in the same way ($\mathbf{m} = \mathbf{M} : \boldsymbol{\sigma}$) we can state that associativity at the level of $\mathbf{N}$ and $\mathbf{M}$ infers associativity at the level of $\mathbf{n}$ and $\mathbf{m}$

$$\mathbf{N} \parallel \mathbf{M} \qquad \Rightarrow \qquad \mathbf{n} \parallel \mathbf{m} \tag{3.36}$$

### 3.2 Elastic-fracturing materials. Strain-based formulation

As in plasticity, it is possible to develop a dual theory for the case in which the fracturing function is thought to be a function of strain $\boldsymbol{\epsilon}$ and history variables $\mathbf{p}$, such that $F = F(\boldsymbol{\epsilon}, \mathbf{p})$. All the developments are analogous to strain-based plasticity (section 2.3.2).

The counterpart of the inelastic strain is now the inelastic stress that identifies the second contribution of the stress rate obtained by differentiation of the secant relation (3.9) (DOUGILL [21])

$$\dot{\boldsymbol{\sigma}}_i = \dot{\mathbf{E}} : \boldsymbol{\epsilon} \tag{3.37}$$

Alternatively the inelastic stress rate can be defined as the rate of stress to subtract from the total stress rate such that the remaining elastic stress rate gives the strain rate when contracted with the secant compliance

$$\dot{\boldsymbol{\epsilon}} = \mathbf{C} : (\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}_i) = \dot{\boldsymbol{\epsilon}}_e + \dot{\boldsymbol{\epsilon}}_i \tag{3.38}$$

The interrelations between inelastic stress and strain rates are simply

$$\dot{\boldsymbol{\epsilon}}_i = -\mathbf{C} : \dot{\boldsymbol{\sigma}}_i \qquad \text{and} \qquad \dot{\boldsymbol{\sigma}}_i = -\mathbf{E} : \dot{\boldsymbol{\epsilon}}_i \tag{3.39}$$

Following the usual steps, the flow rule for the inelastic stress rate and the subsequent consistency condition read

$$\begin{aligned} \dot{\boldsymbol{\sigma}}_i &= \dot{\lambda}\ \bar{\mathbf{m}} \\ \dot{F} &= \bar{\mathbf{n}} : \dot{\boldsymbol{\epsilon}} - \bar{H}\,\dot{\lambda} = 0 \end{aligned} \tag{3.40}$$

where the gradient of the fracturing function $\bar{\mathbf{n}}$ and the hardening parameter $\bar{H}$ are now defined as

$$\bar{\mathbf{n}} = \frac{\partial F}{\partial \boldsymbol{\epsilon}}\ ; \qquad\qquad \bar{H} = -\frac{\partial F}{\partial \lambda} \tag{3.41}$$

Solving the consistency condition for $\dot{\lambda}$ and expressing the increment of the plastic multiplier in terms of $\dot{\boldsymbol{\epsilon}}$ or $\dot{\boldsymbol{\sigma}}$

$$\dot{\lambda} = \frac{\bar{\mathbf{n}} : \mathbf{C} : \dot{\boldsymbol{\sigma}}}{\bar{H} + \bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}}}\ ; \qquad \dot{\lambda} = \frac{\bar{\mathbf{n}} : \dot{\boldsymbol{\epsilon}}}{\bar{H}} \tag{3.42}$$

we recover the dual formulation of elastic-fracturing compliance and stiffness

$$\boxed{\mathbf{C}_{ef} = \mathbf{C} - \frac{\mathbf{C} : \bar{\mathbf{m}} \otimes \bar{\mathbf{n}} : \mathbf{C}}{\bar{H} + \bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}}} \quad ; \qquad \mathbf{E}_{ef} = \mathbf{E} + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}}} \tag{3.43}$$

The same remarks as in plasticity apply to this rank-one modification of the secant stiffness with respect to the stress-based counterpart: the parameter $\bar{H}$ can now assume only positive values. Again, note that the two operators are completely similar to the plasticity case, but instead of the initial stiffness and compliance their actual secant values appear: evolution laws for $\mathbf{C}$ or $\mathbf{E}$ must complement the tangent formulation.

The two flow rules and the two gradients are related to each other as follows (see section 2.3.2)

$$\begin{aligned} \bar{\mathbf{m}} &= -\mathbf{E} : \mathbf{m} \qquad & \mathbf{m} &= -\mathbf{C} : \bar{\mathbf{m}} \\ \bar{\mathbf{n}} &= \mathbf{E} : \mathbf{n} \qquad & \mathbf{n} &= \mathbf{C} : \bar{\mathbf{n}} \end{aligned} \tag{3.44}$$

while the hardening parameters are interrelated by

$$\bar{H} = H + \mathbf{n} : \mathbf{E} : \mathbf{m} > 0 \tag{3.45}$$

The fracturing rule at the $4^{th}$ order tensor level, which is analogue to the flow rule for the inelastic stress, describes the rate of elastic stiffness in terms of

$$\dot{\mathbf{E}} = \dot{\lambda}\, \bar{\mathbf{M}} \tag{3.46}$$

As before, the flow rule and fracturing rule are not independent, and the first one follows from the second one. In fact, substituting the two rules into the relation $\dot{\boldsymbol{\sigma}}_i = \dot{\mathbf{E}} : \boldsymbol{\epsilon}$

$$\dot{\lambda}\, \bar{\mathbf{m}} = \dot{\lambda}\, \bar{\mathbf{M}} : \boldsymbol{\epsilon} \tag{3.47}$$

Eliminating the inelastic multipliers

$$\boxed{\bar{\mathbf{m}} = \bar{\mathbf{M}} : \boldsymbol{\epsilon}} \tag{3.48}$$

The two fracturing rules for stiffness and compliance are related. In fact, using the relations (3.3), one can develop the interrelation between $\bar{\mathbf{M}}$ and $\mathbf{M}$

$$\bar{\mathbf{M}} = -\mathbf{E} : \mathbf{M} : \mathbf{E} \qquad\qquad \mathbf{M} = -\mathbf{C} : \bar{\mathbf{M}} : \mathbf{C} \tag{3.49}$$

Finally, the governing equations of strain-based fracturing material description in strain control can be summarized as follows

$$\boxed{\begin{array}{c} \dot{\boldsymbol{\sigma}} = \mathbf{E}_{ef} : \dot{\boldsymbol{\epsilon}} \quad \text{with} \quad \mathbf{E}_{ef} = \mathbf{E} - \dfrac{\bar{\mathbf{M}} : \boldsymbol{\epsilon} \otimes \bar{\mathbf{n}}}{\bar{H}} \\[2ex] \dot{\mathbf{E}} = \dfrac{\bar{\mathbf{n}} : \dot{\boldsymbol{\epsilon}}}{\bar{H}} \; \bar{\mathbf{M}} \end{array}} \tag{3.50}$$

### 3.2.1 Associativity in stress and stiffness spaces

Associativity at the stress level or more appropriately, as a generalization, associativity at the $2^{nd}$ order level can be recovered when the tensor $\bar{\mathbf{n}}$ and $\bar{\mathbf{m}}$ are coaxial

$$\textit{Associativity in stress space} \quad \Rightarrow \quad \bar{\mathbf{n}} \parallel \bar{\mathbf{m}} \tag{3.51}$$

and the tangent operators become fully symmetric.

The concept of associativity can be extended to the $4^{th}$ order tensor level, in the stiffness space. The generalized gradient to the fracturing function will be now defined as

$$\bar{\mathbf{N}} = \frac{\partial F}{\partial(-\bar{\mathbf{Y}})} \tag{3.52}$$

where $\bar{\mathbf{Y}} = -\dfrac{1}{2}\, \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}$ is the thermodynamic force associated to the stiffness. The relation with the gradient to the fracturing function can be obtained as

$$\bar{\mathbf{n}} = \frac{\partial F}{\partial(-\bar{\mathbf{Y}})} : \frac{\partial(-\bar{\mathbf{Y}})}{\partial \boldsymbol{\epsilon}} = \bar{\mathbf{N}} : \boldsymbol{\epsilon} \tag{3.53}$$

Associativity at the stiffness level can now be defined for coaxial $\bar{\mathbf{N}}$, $\bar{\mathbf{M}}$

$$\textit{Associativity in stiffness space} \quad \Rightarrow \quad \bar{\mathbf{N}} \parallel \bar{\mathbf{M}} \tag{3.54}$$

Once again, because $\bar{\mathbf{m}}$ and $\bar{\mathbf{n}}$ are related to $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ with the same type of relation (compare (3.53) with (3.48)) we conclude that associativity at the stress level follows from associativity at the stiffness level

$$\bar{\mathbf{N}} \parallel \bar{\mathbf{M}} \qquad \Rightarrow \qquad \bar{\mathbf{n}} \parallel \bar{\mathbf{m}} \tag{3.55}$$

At last we remark that dual relations to (3.49) can be derived for the tensors $\mathbf{N}$ and $\bar{\mathbf{N}}$

$$\bar{\mathbf{N}} = \mathbf{E} : \mathbf{N} : \mathbf{E} \qquad\qquad \mathbf{N} = \mathbf{C} : \bar{\mathbf{N}} : \mathbf{C} \tag{3.56}$$

The equations developed for the elastic-fracturing materials are summarized in Table 3.1.

Table 3.1: Elastic-fracturing materials: summary of equations

| **ELASTIC-FRACTURING MATERIALS** | | |
|---|---|---|
| Secant relation | $\boldsymbol{\epsilon} = \mathbf{C} : \boldsymbol{\sigma}$ | $\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon}$ |
| Formulation | STRESS-BASED $F(\boldsymbol{\sigma}, \mathbf{p})$ | STRAIN-BASED $F(\boldsymbol{\epsilon}, \mathbf{p})$ |
| | *STRAIN SPACE* | *STRESS SPACE* |
| Flow rule | $\dot{\boldsymbol{\epsilon}}_i = \dot{\lambda}\, \mathbf{m}$ | $\dot{\boldsymbol{\sigma}}_i = \dot{\lambda}\, \bar{\mathbf{m}}$ |
| Gradient/Hardening | $\mathbf{n} = \dfrac{\partial F}{\partial \boldsymbol{\sigma}} \qquad H = \dfrac{\partial F}{\partial \lambda}$ | $\bar{\mathbf{n}} = \dfrac{\partial F}{\partial \boldsymbol{\epsilon}} \qquad \bar{H} = \dfrac{\partial F}{\partial \lambda}$ |
| Tangent stiffness | $\mathbf{E}_{ef} = \mathbf{E} - \dfrac{\mathbf{E} : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}}$ | $\mathbf{E}_{ef} = \mathbf{E} + \dfrac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}}$ |
| Tangent compliance | $\mathbf{C}_{ef} = \mathbf{C} + \dfrac{\mathbf{m} \otimes \mathbf{n}}{H}$ | $\mathbf{C}_{ef} = \mathbf{C} - \dfrac{\mathbf{C} : \bar{\mathbf{m}} \otimes \bar{\mathbf{n}} : \mathbf{C}}{\bar{H} + \bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}}}$ |
| Relationships | $\mathbf{m} = -\mathbf{C} : \bar{\mathbf{m}}$<br>$\mathbf{n} = \mathbf{C} : \bar{\mathbf{n}}$<br>$H = \bar{H} + \bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}}$ | $\bar{\mathbf{m}} = -\mathbf{E} : \mathbf{m}$<br>$\bar{\mathbf{n}} = \mathbf{E} : \mathbf{n}$<br>$\bar{H} = H + \mathbf{n} : \mathbf{E} : \mathbf{m}$ |
| Associativity | $\mathbf{n} = \mathbf{m}$ | $\bar{\mathbf{n}} = \bar{\mathbf{m}}$ |
| | *COMPLIANCE SPACE* | *STIFFNESS SPACE* |
| Fracturing rule | $\dot{\mathbf{C}} = \dot{\lambda}\, \mathbf{M}$ | $\dot{\mathbf{E}} = \dot{\lambda}\, \bar{\mathbf{M}}$ |
| Thermodynamic force | $(-\mathbf{Y}) = \left.\dfrac{\partial w_e}{\partial \mathbf{C}}\right\rvert_{\boldsymbol{\sigma}} = \dfrac{1}{2}\, \boldsymbol{\sigma} \otimes \boldsymbol{\sigma}$ | $(-\bar{\mathbf{Y}}) = \left.\dfrac{\partial w_e}{\partial \mathbf{E}}\right\rvert_{\boldsymbol{\epsilon}} = \dfrac{1}{2}\, \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}$ |
| Generalized gradient | $\mathbf{N} = \dfrac{\partial F}{\partial(-\mathbf{Y})}$ | $\bar{\mathbf{N}} = \dfrac{\partial F}{\partial(-\bar{\mathbf{Y}})}$ |
| Flow/Fracturing rules | $\mathbf{m} = \mathbf{M} : \boldsymbol{\sigma}$ | $\bar{\mathbf{m}} = \bar{\mathbf{M}} : \boldsymbol{\epsilon}$ |
| Gradient/Gen. gradient | $\mathbf{n} = \mathbf{N} : \boldsymbol{\sigma}$ | $\bar{\mathbf{n}} = \bar{\mathbf{N}} : \boldsymbol{\epsilon}$ |
| Relationships | $\mathbf{M} = -\mathbf{C} : \bar{\mathbf{M}} : \mathbf{C}$<br>$\mathbf{N} = \mathbf{C} : \bar{\mathbf{N}} : \mathbf{C}$ | $\bar{\mathbf{M}} = -\mathbf{E} : \mathbf{M} : \mathbf{E}$<br>$\bar{\mathbf{N}} = \mathbf{E} : \mathbf{N} : \mathbf{E}$ |
| Associativity | $\mathbf{N} = \mathbf{M}$ | $\bar{\mathbf{N}} = \bar{\mathbf{M}}$ |

### 3.3 Damage models as elastic-fracturing models

Continuum damage mechanics has developed intensively in recent years. In this material description appropriate damage variables are introduced with the main purpose of describing the degrading stiffness. The constitutive framework in which damage mechanics has been introduced is actually quite broad: the undamaged material can behave not only elastically but also in different inelastic manners, e.g. elastic-plastic or visco-plastic.

In the following, only the case of elastic behavior for the virgin material is considered and the description of different damage models is developed in the framework of the elastic-fracturing materials. In this case, even though it is possible to cover just the limited spectrum of the elastic damage theories, a unified view can be outlined: many damage material models can be described as a particular case of the general formulation.

For the elastic-fracturing materials the main assumption consists in the definition of a fracturing rule for the compliance or the stiffness. This means that in general the $4^{th}$ order tensor $\mathbf{M}$ or $\bar{\mathbf{M}}$ must be prescribed, with 21 independent components. Thus it appears normal to look for a reduced set of variables able to describe fully the evolution of stiffness and compliance.

Damage variables are used here in this sense and are indicated with the symbol $\mathcal{D}$ or $\bar{\mathcal{D}}$, denoting in general a tensor of various order: scalar, vector, second order, fourth order or even eight order. The type of the damage variable comes from the typical concepts of the continuum damage mechanics theory that will be reviewed later.

With the introduction of the damage variables we can express the actual compliance and stiffness as a continuous and differentiable function of the initial values and the current state of damage in the general following way

$$\mathbf{C} = \mathbf{C}(\mathbf{C}_o, \mathcal{D}) \qquad\qquad \mathbf{E} = \mathbf{E}(\mathbf{E}_o, \bar{\mathcal{D}}) \tag{3.57}$$

The first expression leads to a compliance-based formulation. The second stiffness-based approach is more popular because damage is directly expressed as degradation of the stiffness, rather than an increase of the compliance. This approach will be analyzed later on, following the procedure developed for elastic-fracturing materials. The overscored bar indicates the dual damage variables in the stiffness space even though in the literature these are really the primary variables.

### 3.3.1 Compliance-based description

The differentiation of the material compliance leads to the general expression

$$\mathbf{C} = \mathbf{C}(\mathbf{C}_o, \mathcal{D}) \qquad \Rightarrow \qquad \dot{\mathbf{C}} = \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \dot{\mathcal{D}} \tag{3.58}$$

where the symbol " $\star$ " indicates the full contraction of all the indices of $\mathcal{D}$. Thus if the increment of damage is prescribed, together with the derivatives of the function $\mathbf{C}$, the rate of compliance can be determined.

It is natural to extend the flow rule to the damage space, by stating that the rate of damage variable is proportional to the tensor $\mathcal{M}$ which defines the direction of the damage flow

$$\textit{Damage rule} \qquad \Rightarrow \qquad \dot{\mathcal{D}} = \dot{\lambda}\, \mathcal{M} \tag{3.59}$$

On the other hand, in elastic-fracturing descriptions, the fracturing rule for $\dot{\mathbf{C}} = \dot{\lambda}\ \mathbf{M}$ is assigned. Thus, equating the two expressions for the compliance rate and using the damage rule (3.58) yields

$$\dot{\lambda}\, \mathbf{M} = \dot{\lambda}\, \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \mathcal{M} \tag{3.60}$$

Elimination of the inelastic multipliers leads to

$$\boxed{\mathbf{M} = \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \mathcal{M}} \tag{3.61}$$

Thus when the damage rule is prescribed the fracturing rule follows directly. Note that, even if the damage tensor $\mathcal{D}$ is of an order lower than 4, the implication is still possible because the information at the $4^{th}$ order tensor level is stored in the assigned function $\mathbf{C}$.

For a given $\mathbf{M}$, the flow rule for the inelastic strain, $\dot{\boldsymbol{\epsilon}}_{in} = \dot{\lambda}\ \mathbf{m}$, follows

$$\mathbf{m} = \mathbf{M} : \boldsymbol{\sigma} = \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \mathcal{M} : \boldsymbol{\sigma} \tag{3.62}$$

The flow rule describes the rate of the damage strain and its direction $\mathbf{m}$ can be substitute in the expressions of the tangent operators (3.8), that we could denote by $\mathbf{E}_{ed}$ and $\mathbf{C}_{ed}$, whereby the subscript $_{ed}$ stands for elastic-damage. Note that this interpretation of the damage model as a particular case of the elastic-fracturing theory implies a loading function which defines the state of damage.

### 3.3.2 Stiffness-based description

In the traditional stiffness approach the damage variables directly describe the degradation of the elastic stiffness. Thus the material stiffness tensor $\mathbf{E}$ is prescribed and can be differentiated with respect to the damage variables $\bar{\mathcal{D}}$

$$\mathbf{E} = \mathbf{E}(\mathbf{E}_o, \bar{\mathcal{D}}) \qquad \Rightarrow \qquad \dot{\mathbf{E}} = \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \dot{\bar{\mathcal{D}}} \tag{3.63}$$

The new set of damage variables $\bar{\mathcal{D}}$ is different from the previous set $\mathcal{D}$, which was used for defining the compliance function, although they might be interrelated.

Following the procedure outlined in the previous section it is convenient to establish a second damage rule

$$\textit{Damage rule} \quad \Rightarrow \quad \dot{\bar{\mathcal{D}}} = \dot{\lambda}\ \bar{\mathcal{M}} \tag{3.64}$$

that can be substituted in (3.63). Using the fracturing rule for stiffness evolution $\dot{\mathbf{E}} = \dot{\lambda}\ \bar{\mathbf{M}}$ and eliminating the inelastic multipliers we obtain the relation between $\bar{\mathbf{M}}$ and $\bar{\mathcal{M}}$

$$\boxed{\bar{\mathbf{M}} = \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \bar{\mathcal{M}}} \tag{3.65}$$

The direction $\bar{\mathbf{m}}$ of the flow rule for the damage stress $\dot{\boldsymbol{\sigma}}_d = \dot{\lambda}\ \bar{\mathbf{m}}$ can now be obtained from the direction $\bar{\mathbf{M}}$ of the fracturing rule

$$\bar{\mathbf{m}} = \bar{\mathbf{M}} : \boldsymbol{\epsilon} = \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \bar{\mathcal{M}} : \boldsymbol{\epsilon} \tag{3.66}$$

It can be substituted into the expression of the tangent operators (3.43) defined in terms of the damage direction $\bar{\mathcal{M}}$.

### 3.3.3 Associativity in damage space

For the definition of associativity in the damage space we need to introduce the generalized gradients of the damage loading condition $\mathcal{N}$ or $\bar{\mathcal{N}}$, which compare with the direction of the damage growth $\mathcal{M}$ or $\bar{\mathcal{M}}$. Thus it is necessary to revise the expression of fracturing dissipation $\dot{d}$ and define the generalized forces $(-\mathcal{Y})$, $\bar{\mathcal{Y}}$ associated with the damage variables $\mathcal{D}$ and $\bar{\mathcal{D}}$.

The fracturing dissipation was expressed in terms of compliance and stiffness rates and the conjugate thermodynamic forces as

$$\dot{d} = (-\mathbf{Y}) :: \dot{\mathbf{C}} \qquad \text{and} \qquad \dot{d} = \bar{\mathbf{Y}} :: \dot{\mathbf{E}} \tag{3.67}$$

Substituting expressions (3.58) and (3.63) for the compliance and stiffness rates into these relations we wind up with

$$\dot{d} = (-\mathbf{Y}) :: \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \dot{\mathcal{D}} \qquad \text{and} \qquad \dot{d} = \bar{\mathbf{Y}} :: \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \dot{\bar{\mathcal{D}}} \tag{3.68}$$

The thermodynamic forces associated to damage can now be define in such a way that the format of damage dissipation is recovered as

$$\dot{d} = (-\mathcal{Y}) \star \dot{\mathcal{D}} \qquad \text{and} \qquad \dot{d} = \bar{\mathcal{Y}} \star \dot{\bar{\mathcal{D}}} \tag{3.69}$$

Comparing these equations with (3.68), the expressions of the thermodynamic forces associated to damage are simply

$$(-\mathcal{Y}) = (-\mathbf{Y}) :: \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \qquad \text{and} \qquad \bar{\mathcal{Y}} = \bar{\mathbf{Y}} :: \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \tag{3.70}$$

We are now able to define the generalized gradients of the damage function with respect to the thermodynamic forces as

$$\mathcal{N} = \frac{\partial F}{\partial(-\mathcal{Y})} \qquad \text{and} \qquad \bar{\mathcal{N}} = \frac{\partial F}{\partial(-\bar{\mathcal{Y}})} \tag{3.71}$$

Associativity at the damage level will be recovered when $\mathcal{N}$ is coaxial with $\mathcal{M}$ or when $\bar{\mathcal{N}}$ is coaxial with $\bar{\mathcal{M}}$

$$\textit{Associativity in damage space} \quad \Rightarrow \quad \mathcal{N} \parallel \mathcal{M} \quad \text{or} \quad \bar{\mathcal{N}} \parallel \bar{\mathcal{M}} \tag{3.72}$$

Associativity in damage space can be related to the associativity at the $4^{th}$ order tensor level. In fact considering the generalized gradients $\mathbf{N}$ and $\bar{\mathbf{N}}$

$$\begin{aligned} \mathbf{N} &= \frac{\partial F}{\partial(-\mathbf{Y})} = \frac{\partial F}{\partial(-\mathcal{Y})} \star \frac{\partial(-\mathcal{Y})}{\partial(-\mathbf{Y})} = \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \mathcal{N} \\ \bar{\mathbf{N}} &= \frac{\partial F}{\partial(-\bar{\mathbf{Y}})} = \frac{\partial F}{\partial(-\bar{\mathcal{Y}})} \star \frac{\partial(-\bar{\mathcal{Y}})}{\partial(-\bar{\mathbf{Y}})} = \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \bar{\mathcal{N}} \end{aligned} \tag{3.73}$$

where the derivatives of $(-\mathcal{Y})$ and $(-\bar{\mathcal{Y}})$ with respect to $(-\mathbf{Y})$ and $(-\bar{\mathbf{Y}})$ are readily found because they are evaluated for $\mathcal{D} = const$ and $\bar{\mathcal{D}} = const$ respectively.

In sum the relations between the generalized gradients read

$$\boxed{\mathbf{N} = \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \mathcal{N} \qquad \text{and} \qquad \bar{\mathbf{N}} = \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \bar{\mathcal{N}}} \tag{3.74}$$

Comparing with the expressions for $\mathbf{M}/\mathcal{M}$ (3.61) and $\bar{\mathbf{M}}/\bar{\mathcal{M}}$ (3.65) we conclude that the associativity in the damage space implies associativity in the compliance and stiffness spaces (and consequently also associativity in the strain and stress space)

$$\begin{aligned} \mathcal{N} \parallel \mathcal{M} \quad &\Rightarrow \quad \mathbf{N} \parallel \mathbf{M} \quad \Rightarrow \quad \mathbf{n} \parallel \mathbf{m} \\ \bar{\mathcal{N}} \parallel \bar{\mathcal{M}} \quad &\Rightarrow \quad \bar{\mathbf{N}} \parallel \bar{\mathbf{M}} \quad \Rightarrow \quad \bar{\mathbf{n}} \parallel \bar{\mathbf{m}} \end{aligned} \tag{3.75}$$

For the sake of completeness we derive the relations between $\mathcal{M}$ and $\bar{\mathcal{M}}$, as it was done for $\mathbf{M}/\bar{\mathbf{M}}$ and $\mathbf{m}/\bar{\mathbf{m}}$ and also the relations $\mathcal{N}/\bar{\mathcal{N}}$, similar to $\mathbf{N}/\bar{\mathbf{N}}$ and $\mathbf{n}/\bar{\mathbf{n}}$.

Let us focus on the expression of $\bar{\mathcal{M}}$ as a function of $\mathcal{M}$. Using the first of (3.49) and substituting (3.61) for $\mathbf{M}$ and (3.65) for $\bar{\mathbf{M}}$, we obtain after some algebra

$$\begin{aligned}
\bar{\mathbf{M}} &= -\mathbf{E} : \mathbf{M} : \mathbf{E} \\
\bar{\mathbf{M}} &= -\mathbf{E} : \frac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \mathcal{M} : \mathbf{E} = -\left(\mathbf{E} : \frac{\partial \mathbf{C}}{\partial \mathcal{D}} : \mathbf{E}\right) \star \mathcal{M} \\
\frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \bar{\mathcal{M}} &= -\left(\mathbf{E} : \frac{\partial \mathbf{C}}{\partial \mathcal{D}} : \mathbf{E}\right) \star \mathcal{M} \\
\left(\frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}}\right) \star \bar{\mathcal{M}} &= -\frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \left(\mathbf{E} : \frac{\partial \mathbf{C}}{\partial \mathcal{D}} : \mathbf{E}\right) \star \mathcal{M}
\end{aligned} \tag{3.76}$$

Solving for $\bar{\mathcal{M}}$

$$\bar{\mathcal{M}} = -\left(\frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}}\right)^{-1} \star \left(\frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \left(\mathbf{E} : \frac{\partial \mathbf{C}}{\partial \mathcal{D}} : \mathbf{E}\right)\right) \star \mathcal{M} \tag{3.77}$$

An analogous expression can be obtain for the inverse relation

$$\mathcal{M} = -\left(\frac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \frac{\partial \mathbf{C}}{\partial \mathcal{D}}\right)^{-1} \star \left(\frac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \left(\mathbf{C} : \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} : \mathbf{C}\right)\right) \star \bar{\mathcal{M}} \tag{3.78}$$

Finally, using the relations (3.56), the correspondent relations for $\mathcal{N}$, $\bar{\mathcal{N}}$ read

$$\bar{\mathcal{N}} = \left(\frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}}\right)^{-1} \star \left(\frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \left(\mathbf{E} : \frac{\partial \mathbf{C}}{\partial \mathcal{D}} : \mathbf{E}\right)\right) \star \mathcal{N} \tag{3.79}$$

$$\mathcal{N} = \left(\frac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \frac{\partial \mathbf{C}}{\partial \mathcal{D}}\right)^{-1} \star \left(\frac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \left(\mathbf{C} : \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} : \mathbf{C}\right)\right) \star \bar{\mathcal{N}} \tag{3.80}$$

The expressions developed for the damage model as a special case of elastic-fracturing materials are summarized in Tables 3.2 and 3.3.

Table 3.2: Damage-EF models, compliance formulation: summary of equations

| **DAMAGE-ELASTIC-FRACTURING MATERIALS** | |
|---|---|
| | *COMPLIANCE DAMAGE SPACE* |
| Formulation | STRESS-BASED $F(\boldsymbol{\sigma}, \mathbf{p})$ |
| Secant relation | $\boldsymbol{\epsilon} = \mathbf{C}(\mathbf{C}_o, \mathcal{D}) : \boldsymbol{\sigma}$ |
| Damage rule | $\dot{\mathcal{D}} = \dot{\lambda}\ \mathcal{M}$ |
| Thermodynamic force | $(-\mathcal{Y}) = \left.\dfrac{\partial w_\epsilon}{\partial \mathcal{D}}\right\vert_{\boldsymbol{\sigma}} = (-\mathbf{Y}) :: \dfrac{\partial \mathbf{C}}{\partial \mathcal{D}}$ |
| Generalized gradient | $\mathcal{N} = \dfrac{\partial F}{\partial(-\mathcal{Y})}$ |
| Fracturing/Damage rules | $\mathbf{M} = \dfrac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \mathcal{M}$ |
| Gen. gradient/Gen. gradient | $\mathbf{N} = \dfrac{\partial \mathbf{C}}{\partial \mathcal{D}} \star \mathcal{N}$ |
| Relationships | $\mathcal{M} = -\left(\dfrac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \dfrac{\partial \mathbf{C}}{\partial \mathcal{D}}\right)^{-1} \star \left(\dfrac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \left(\mathbf{C} : \dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} : \mathbf{C}\right)\right) \star \bar{\mathcal{M}}$ <br> $\mathcal{N} = \left(\dfrac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \dfrac{\partial \mathbf{C}}{\partial \mathcal{D}}\right)^{-1} \star \left(\dfrac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \left(\mathbf{C} : \dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} : \mathbf{C}\right)\right) \star \bar{\mathcal{N}}$ |
| Associativity | $\mathcal{N} = \mathcal{M}$ |

Table 3.3: Damage-EF models, stiffness formulation: summary of equations

| **DAMAGE-ELASTIC-FRACTURING MATERIALS** | |
|---|---|
| | *STIFFNESS DAMAGE SPACE* |
| Formulation | STRAIN-BASED $F(\boldsymbol{\epsilon}, \mathbf{p})$ |
| Secant relation | $\boldsymbol{\sigma} = \mathbf{E}(\mathbf{E}_o, \bar{\mathcal{D}}) : \boldsymbol{\epsilon}$ |
| Damage rule | $\dot{\bar{\mathcal{D}}} = \dot{\lambda}\, \bar{\mathcal{M}}$ |
| Thermodynamic force | $\bar{\mathcal{Y}} = \left.\dfrac{\partial w_e}{\partial \bar{\mathcal{D}}}\right\|_{\boldsymbol{\epsilon}} = \bar{\mathbf{Y}} :: \dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}}$ |
| Generalized gradient | $\bar{\mathcal{N}} = \dfrac{\partial F}{\partial(-\bar{\mathcal{Y}})}$ |
| Fracturing/Damage rules | $\bar{\mathbf{M}} = \dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \bar{\mathcal{M}}$ |
| Gen. gradient/Gen. gradient | $\bar{\mathbf{N}} = \dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} \star \bar{\mathcal{N}}$ |
| Relationships | $\bar{\mathcal{M}} = -\left(\dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}}\right)^{-1} \star \left(\dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \left(\mathbf{E} : \dfrac{\partial \mathbf{C}}{\partial \mathcal{D}} : \mathbf{E}\right)\right) \star \mathcal{M}$ <br> $\bar{\mathcal{N}} = \left(\dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}}\right)^{-1} \star \left(\dfrac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} :: \left(\mathbf{E} : \dfrac{\partial \mathbf{C}}{\partial \mathcal{D}} : \mathbf{E}\right)\right) \star \mathcal{N}$ |
| Associativity | $\bar{\mathcal{N}} = \bar{\mathcal{M}}$ |

### 3.4 Damage variables in continuum damage mechanics

The purpose of continuum damage mechanics is a constitutive setting between nominal stress and strain, i.e. static and kinematic variables that have physical meaning. In our framework this means that the secant relations $\boldsymbol{\sigma} = \mathbf{E}(\mathbf{E}_o, \bar{\mathcal{D}}) : \boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon} = \mathbf{C}(\mathbf{C}_o, \mathcal{D}) : \boldsymbol{\sigma}$ respectively, connect the damage formulation to the elastic-fracturing theory. Regarding this, the following notation remark seems necessary.

In the usual continuum damage mechanics approach, the damage variables are directly related to the degradation of the stiffness and are normally considered as primary variables with respect to the dual expressions which describe the compliance evolution. Thus, in the following, the usual continuum damage mechanics notation will be used, even though this may create some confusion with the previous notation because it might appear to be the opposite. The damage variables $\bar{\mathcal{D}}$ related to the stiffness will be denoted without overscored bar, as $\alpha$, $D$, $\boldsymbol{\phi}$, $\boldsymbol{\alpha}$. The dual compliance variables $\mathcal{D}$ will be indicated by the overscored bar ($\mathcal{D} \to \bar{\alpha}$, $\bar{D}$, $\bar{\boldsymbol{\phi}}$, $\bar{\boldsymbol{\alpha}}$).

Consider a uniaxial tension test of a specimen of original length $l_o$ and cross section $A_o$ subjected to the load $P$, resulting in the elongation $\delta$, then the nominal stress and strain are simply $\sigma = P/A_o$, $\epsilon = \delta/l_o$. On the other hand, we can think that all the imperfections and microcracks, diffused evenly throughout the body, create a distributed state of damage that reduces the amount of compact material that carries the load (see Fig. 3.5 for a symbolic representation). Thus the undamaged part of the body (of area $A$) will be subjected to a larger state of true stress and true strain, that are called *"effective"* stress and strain measures (e.g. SIMO AND JU [71]).

Effective stress and strain are related by the constitutive law for the undamaged material, that in our case will be considered elastic. For the constitutive law in the physical space, relating the nominal stress and nominal strain some definitions and concepts of continuum damage mechanics need to be revisited.

Instead of establishing directly the $\sigma$-$\epsilon$ law, we use the undamaged elastic constitutive law. Thus, additional assumptions are needed for relating nominal stress and strain with their effective counterparts, i.e. we need two more relations. The approach is shown in Fig. 3.6. One of the two relations introduces the damage variable adopted in the theory, which relates $\sigma$ to $\sigma_{eff}$ or $\epsilon$ to $\epsilon_{eff}$. The second equation between nominal and effective quantities is usually established using homogenization concepts of *strain equivalence*, *stress equivalence* or *energy equivalence*, that will be briefly reviewed in the following.

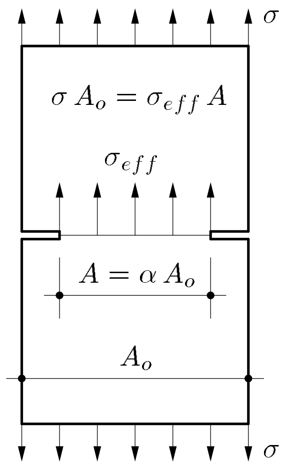



Figure 3.5: Nominal and effective stresses in $1D$

| | **STRESS** | **STRAIN** |
|---|---|---|
| NOMINAL | $\boldsymbol{\sigma}$ | $\boldsymbol{\epsilon}$ |
| EFFECTIVE | $\boldsymbol{\sigma}_{eff}$ | $\boldsymbol{\epsilon}_{eff}$ |

Figure 3.6: Nominal and effective constitutive laws

### 3.4.1 Strain equivalence

With the hypothesis of strain equivalence the material is thought to consist of a parallel arrangement of fibers subjected to the same straining. As damage progresses part of the fibers are lost, thus the total area is reduced, but the strain in each fiber remains the same

$$\epsilon = \epsilon_{eff} \tag{3.81}$$

On the other hand, because of the reduction of the stress-carrying area, the effective stress in increased and is given by the equilibrium condition (see Fig. 3.5)

$$\sigma \; A_o = \sigma_{eff} \; A \qquad \Rightarrow \qquad \sigma = \frac{A}{A_o} \, \sigma_{eff} = \alpha \; \sigma_{eff} \tag{3.82}$$

where $\alpha$ is an integrity variable (VALANIS [84]) varying between 0 and 1: for $\alpha = 0$ the material is completely damaged, for $\alpha = 1$ the material is undamaged. Note, the dual damage variable $D = 1 - \alpha$ in the literature, increases between 0 and 1 as damage progresses.

We have now all the relations that are needed for establishing the nominal constitutive law. Using the elastic law for the undamaged material $\sigma_{eff} = E_o \; \epsilon_{eff}$ and substituting $\epsilon_{eff} = \epsilon$, $\sigma_{eff} = \sigma/\alpha$ we obtain

$$\sigma = \alpha \, E_o \, \epsilon = (1 - D) \, E_o \, \epsilon \tag{3.83}$$

Thus we can think that the entire damage process results in a degradation of the stiffness $E = \alpha \; E_o$.

The $3D$ generalization is straight forward. At this stage the relation between nominal and effective stresses can be expressed using the $4^{th}$ order integrity tensor $\boldsymbol{\alpha}$ or the damage tensor $\mathbf{D}$, which possesses only the minor symmetries

$$\boldsymbol{\sigma} = \boldsymbol{\alpha} : \boldsymbol{\sigma}_{eff} = (\mathbf{I}_4^s - \mathbf{D}) : \boldsymbol{\sigma}_{eff} \tag{3.84}$$

On the other hand, assuming equivalent strains, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{eff}$, the secant stiffness can be derived substituting the elastic law for the undamaged material into the previous relation

$$\boldsymbol{\sigma} = \boldsymbol{\alpha} : \mathbf{E}_o : \boldsymbol{\epsilon}_{eff} = \boldsymbol{\alpha} : \mathbf{E}_o : \boldsymbol{\epsilon} \tag{3.85}$$

Thus the secant stiffness is reduced to

$$\boxed{\mathbf{E} = \boldsymbol{\alpha} : \mathbf{E}_o = (\mathbf{I}_4^s - \mathbf{D}) : \mathbf{E}_o} \tag{3.86}$$

which is in fact a generalization of the uniaxial case.

### 3.4.2 Stress equivalence

The dual approach of stress equivalence assumes that the material consists of a series of layers subjected to the same stress

$$\sigma = \sigma_{eff} \tag{3.87}$$

instead the effective strain is reduced because of the additional strain due to micro-cracks. A new integrity variable $\alpha$ is introduced, since it is not possible to assign it the same physical meaning as before. The relation between effective and nominal strain reads

$$\epsilon_{eff} = \alpha\ \epsilon \tag{3.88}$$

Note that this is the opposite when compared with the previous description for effective stresses. In fact if we want to obtain an expression of the stiffness similar to the one for strain equivalence, it is convenient to define the variable $\alpha$ in the following way.

$$\alpha\ \epsilon = \epsilon_{eff} = C_o\ \sigma_{eff} = C_o\ \sigma \tag{3.89}$$

thus

$$\epsilon = \frac{C_o}{\alpha}\ \sigma = C\ \sigma \tag{3.90}$$

Inverting, the same expression $E = \alpha\ E_o$ of stiffness degradation is obtained.

For the $3D$ case, however, the resulting expression of the secant stiffness is different because of the tensor nature of the operators. The generalized tensor variable

$\boldsymbol{\alpha}$ can now be defined as

$$\boldsymbol{\epsilon}_{eff} = \boldsymbol{\alpha} : \boldsymbol{\epsilon} \tag{3.91}$$

For the stresses,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{eff} = \mathbf{E}_o : \boldsymbol{\epsilon}_{eff} = \mathbf{E}_o : \boldsymbol{\alpha} : \boldsymbol{\epsilon} \tag{3.92}$$

Thus the expression of the stiffness looks slightly different

$$\boxed{\mathbf{E} = \mathbf{E}_o : \boldsymbol{\alpha}} \tag{3.93}$$

It is important to note that both expressions (3.86) and (3.93) for the secant stiffness do not maintain the major symmetries of the tensor $\mathbf{E}$ for a general form of the damage tensor $\boldsymbol{\alpha}$, even if $\boldsymbol{\alpha} = \boldsymbol{\alpha}^T$ possesses major symmetries. In contrast, the energy equivalence approach will maintain symmetry.

### 3.4.3 Energy equivalence

With this approach, once the relation between nominal and effective quantities is established, the other condition is obtained by imposing the same internal virtual work in physical and effective spaces.

If degradation rule for stresses is prescribed, $\sigma = \alpha\ \sigma_{eff}$, with $0 \leq \alpha \leq 1$, the equality of virtual works gives

$$\epsilon\ \delta\sigma = \epsilon_{eff}\ \delta\sigma_{eff} \qquad \Rightarrow \qquad \epsilon_{eff} = \alpha\ \delta\epsilon \tag{3.94}$$

On the other hand, when the relation between nominal and effective strain $\epsilon_{eff} = \alpha\ \epsilon$ is assigned, for a virtual variation of the strain fields the internal work expression read

$$\sigma\ \delta\epsilon = \sigma_{eff}\ \delta\epsilon_{eff} \qquad \Rightarrow \qquad \sigma = \alpha\ \sigma_{eff} \tag{3.95}$$

The energy equivalence approach yields thus an intermediate formulation because both effective stress and strain are not equal to the nominal ones when damage progresses.

The expression of the stiffness is readily found using the elastic law in the effective space and above mentioned relations

$$\sigma = \alpha\, \sigma_{eff} = \alpha\, E_o \epsilon_{eff} = \alpha E_o \alpha\, \epsilon \tag{3.96}$$

$$E = \alpha\, E_o\, \alpha = \alpha^2\, E_o \tag{3.97}$$

In the $3D$ extension two different expressions for the secant stiffness can be found. When stresses are first related by the integrity tensor variable $\boldsymbol{\alpha}$ as $\boldsymbol{\sigma} = \boldsymbol{\alpha} : \boldsymbol{\sigma}_{eff}$ the equality of internal virtual works yields

$$\boldsymbol{\epsilon} : \delta\boldsymbol{\sigma} = \boldsymbol{\epsilon} : \boldsymbol{\alpha} : \delta\boldsymbol{\sigma}_{eff} = \boldsymbol{\epsilon}_{eff} : \delta\boldsymbol{\sigma}_{eff} \qquad \Rightarrow \qquad \boldsymbol{\epsilon}_{eff} = \boldsymbol{\alpha}^{T} : \boldsymbol{\epsilon} \tag{3.98}$$

where the transpose symbol means $\boldsymbol{\alpha}^{T}_{ijkl} = \boldsymbol{\alpha}_{klij}$. The secant stiffness is found in the usual way

$$\boldsymbol{\sigma} = \boldsymbol{\alpha} : \boldsymbol{\sigma}_{eff} = \boldsymbol{\alpha} : \mathbf{E}_o : \boldsymbol{\epsilon}_{eff} = \boldsymbol{\alpha} : \mathbf{E}_o : \boldsymbol{\alpha}^{T} : \boldsymbol{\epsilon} \tag{3.99}$$

$$\boxed{\mathbf{E} = \boldsymbol{\alpha} : \mathbf{E}_o : \boldsymbol{\alpha}^{T}} \tag{3.100}$$

As remarked before, the expression of the stiffness is now fully symmetric, i.e. $\mathbf{E}^{T} = \mathbf{E}$. This is an important requirement, at least in the case when the stiffness represents the secant elastic unloading stiffness which must satisfy the major symmetries.

In the second approach, the strains are first prescribed as $\boldsymbol{\epsilon}_{eff} = \boldsymbol{\alpha} : \boldsymbol{\epsilon}$. The identity of virtual internal works yields

$$\boldsymbol{\sigma} : \delta\boldsymbol{\epsilon} = \boldsymbol{\sigma}_{eff} : \delta\boldsymbol{\epsilon}_{eff} = \boldsymbol{\sigma}_{eff} : \boldsymbol{\alpha} : \delta\boldsymbol{\epsilon} \qquad \Rightarrow \qquad \boldsymbol{\sigma} = \boldsymbol{\alpha}^{T} : \boldsymbol{\sigma}_{eff} \tag{3.101}$$

$$\boldsymbol{\sigma} = \boldsymbol{\alpha}^{T} : \boldsymbol{\sigma}_{eff} = \boldsymbol{\alpha}^{T} : \mathbf{E}_o : \boldsymbol{\epsilon}_{eff} = \boldsymbol{\alpha}^{T} : \mathbf{E}_o : \boldsymbol{\alpha} : \boldsymbol{\epsilon} \tag{3.102}$$

$$\boxed{\mathbf{E} = \boldsymbol{\alpha}^{T} : \mathbf{E}_o : \boldsymbol{\alpha}} \tag{3.103}$$

which is also symmetric. The same stiffness expression (3.100) would be recovered if we start from the assumption $\boldsymbol{\epsilon}_{eff} = \boldsymbol{\alpha}^{T} : \boldsymbol{\epsilon}$, which would give also $\boldsymbol{\sigma} = \boldsymbol{\alpha} : \boldsymbol{\sigma}_{eff}$.

The different stiffness expressions in the various homogenization approaches can be compacted into a general multiplicative format involving a $8^{th}$ order *damage effect tensor* $\bar{\boldsymbol{\Omega}}$ which contracts four times the initial stiffness. The same general format can also be applied to the expression for the secant compliance, using the dual $8^{th}$ order *damage effect tensor* $\boldsymbol{\Omega}$ on the compliance.

$$\mathbf{E} = \boldsymbol{\Omega} :: \mathbf{E}_o \qquad\qquad \mathbf{C} = \bar{\boldsymbol{\Omega}} :: \mathbf{C}_o \tag{3.104}$$

For example, one can verify that the expressions (3.86), (3.93), (3.100) can be recovered assuming an $8^{th}$ order tensor that in indicial notation reads

$$\begin{aligned} \text{Strain equivalence} \quad &\Rightarrow \quad \Omega_{ijklpqrs} = \alpha_{ijpq}\,\delta_{kr}\,\delta_{ls} \\ \text{Stress equivalence} \quad &\Rightarrow \quad \Omega_{ijklpqrs} = \delta_{ip}\,\delta_{jq}\,\alpha_{rskl} \\ \text{Energy equivalence} \quad &\Rightarrow \quad \Omega_{ijklpqrs} = \alpha_{ijpq}\,\alpha_{klrs} \end{aligned} \tag{3.105}$$

The tensor $\boldsymbol{\Omega}$ could also be defined directly as a function of a damage variable $\bar{\mathcal{D}}$ not necessarily related to $\boldsymbol{\alpha}$.

### 3.4.4 Types of damage variables. Scalar damage

Various types of damage variables have been proposed in the literature: scalar, vector, second order tensor, fourth order tensor, eight order tensor. Most of the formulations can be recast in the general framework briefly illustrated in the previous sections. In other cases different concepts are used, the direct expression of the stiffness may be prescribed or obtained without resorting to equivalence or homogenization concepts.

The main purpose of this subsection will be a general format of scalar damage, that will be analyzed in the next chapter with respect to localization properties. Thus only a short overview of the other types of damage formulations will be presented.

**Scalar damage. Traditional formulation**

The simplest and traditional way to express damage using just a single scalar variable assumes an isotropic format of the damage effect tensor $\boldsymbol{\alpha}$ on the stress or on the strain as $\boldsymbol{\alpha} = \alpha \mathbf{I}_4^s$. In this case the relations between nominal and effective quantities in the strain and stress equivalence approaches read

$$\boldsymbol{\sigma} = \alpha\ \boldsymbol{\sigma}_{eff} = (1 - D)\ \boldsymbol{\sigma}_{eff} \qquad \boldsymbol{\epsilon}_{eff} = \alpha\ \boldsymbol{\epsilon} = (1 - D)\ \boldsymbol{\epsilon} \tag{3.106}$$

Consequently, the stress and strain equivalence approaches result in the expression of the stiffness

$$\mathbf{E} = \alpha\ \mathbf{E}_o = (1 - D)\ \mathbf{E}_o \tag{3.107}$$

In contrast, the energy equivalence approach yields

$$\mathbf{E} = \alpha^2\ \mathbf{E}_o = (1 - D)^2\ \mathbf{E}_o \tag{3.108}$$

which can both be written as $\mathbf{E} = \theta\ \mathbf{E}_o$, with $\theta = \alpha$ in the first case and $\theta = \alpha^2$ in the second case. The secant compliance is readily found by inverting $\mathbf{E}$ into $\mathbf{C} = \bar{\theta}\ \mathbf{C}_o$, with $\bar{\theta} = 1/\theta$.

As final remark we note that the scalar damage format is isotropic, because it affects all the stiffness and compliance components in the same way, although this is not the most general isotropic description (see the discussion on $4^{th}$ order damage tensors).

Differentiation of the stiffness yields $\dot{\mathbf{E}} = \dot{\theta}\mathbf{E}_o$ that, when compared with the fracturing rule for the stiffness $\dot{\mathbf{E}} = \dot{\lambda}\ \bar{\mathbf{M}}$, results in a direction of the stiffness rate of the kind

$$\bar{\mathbf{M}} = k\ \mathbf{E}_o \qquad k{=}scalar \tag{3.109}$$

Note that for $k = 1$, $\bar{\mathbf{M}} = \mathbf{E}_o$ and for $k = \theta$, $\bar{\mathbf{M}} = \mathbf{E}$.

On the other hand, similar conclusions can be drawn for the compliance fracturing rule $\dot{\mathbf{C}} = \dot{\lambda}\ \mathbf{M}$. Using the relations in Table 3.1, we obtain

$$\mathbf{M} = -\mathbf{C} : \bar{\mathbf{M}} : \mathbf{C} = -\bar{\theta}^2\ k\ \mathbf{C}_o = \bar{k}\ \mathbf{C}_o \qquad \bar{k}{=}scalar \tag{3.110}$$

These equations for the fracturing rules furnish interesting results when projected onto the $2^{nd}$ order level of the flow rule for damage strains and stresses ($\dot{\boldsymbol{\epsilon}}_d = \dot{\lambda}\ \mathbf{m},\ \ \dot{\boldsymbol{\sigma}}_d = \dot{\lambda}\ \bar{\mathbf{m}}$ respectively)

$$\mathbf{m} = \mathbf{M} : \boldsymbol{\sigma} = \bar{k}\ \mathbf{C}_o : \boldsymbol{\sigma} \qquad\qquad \bar{\mathbf{m}} = \bar{\mathbf{M}} : \boldsymbol{\epsilon} = k\ \mathbf{E}_o : \boldsymbol{\epsilon} \tag{3.111}$$

If $\bar{k} = k = 1$ the two flow directions are given by $\mathbf{m} = \boldsymbol{\epsilon}_o,\ \ \bar{\mathbf{m}} = \boldsymbol{\sigma}_o$ where $\boldsymbol{\epsilon}_o = \mathbf{C}_o : \boldsymbol{\sigma}$ would be the undamaged strain and $\boldsymbol{\sigma}_o = \mathbf{E}_o : \boldsymbol{\epsilon}$ the undamaged stress. If $\bar{k} = \bar{\theta}$ and $k = \theta$ the flow directions are given by the actual stress or strain. Both rules are acceptable and identical because they differs just by a scalar quantity that can be included in the damage multiplier ($\boldsymbol{\sigma} = \theta\, \boldsymbol{\sigma}_o$ and $\boldsymbol{\epsilon} = \bar{\theta}\boldsymbol{\epsilon}_o$).

The interrelation between the fracturing rule and the damage rule is readily established for scalar damage descriptions. In this case the damage rule does not contain any direction and the damage flow direction is represented by a scalar quantity $\mathcal{M} = k$. Using the relationship $\bar{\mathbf{M}}$-$\mathcal{M}$ and considering the secant stiffness expression

$$\mathbf{E} = \mathbf{E}(\mathbf{E}_o, \theta) = \theta\ \mathbf{E}_o \tag{3.112}$$

we obtain $\bar{\mathbf{M}} = \dfrac{\partial \mathbf{E}}{\partial \theta}\ \mathcal{M} = k\ \mathbf{E}_o$ as mentioned above.

Let us investigate whether the scalar damage formulation maintains *associativity.* The loading function $F$ must be selected such that $\mathbf{N} \parallel \mathbf{M}$ or $\bar{\mathbf{N}} \parallel \mathbf{M}$. We recall that

$$\mathbf{N} = \frac{\partial F}{\partial \left( \dfrac{1}{2}\, \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)} \qquad\qquad \bar{\mathbf{N}} = \frac{\partial F}{\partial \left( \dfrac{1}{2}\, \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon} \right)} \tag{3.113}$$

Thus we will study an expression for the damage loading function $F$ of the form

$$F(\boldsymbol{\sigma}, \mathbf{p}) = \frac{1}{2}\, \boldsymbol{\sigma} : \mathbf{M} : \boldsymbol{\sigma} - A(\mathbf{p}) \qquad\qquad F(\boldsymbol{\epsilon}, \mathbf{p}) = \frac{1}{2}\, \boldsymbol{\epsilon} : \bar{\mathbf{M}} : \boldsymbol{\epsilon} - A(\mathbf{p}) \tag{3.114}$$

with $\mathbf{M}$ and $\bar{\mathbf{M}}$ being independent of $\boldsymbol{\sigma}$, gives after differentiation $\mathbf{N} = \mathbf{M}$ and $\bar{\mathbf{N}} = \bar{\mathbf{M}}$. Note that the first contribution in the expressions of the loading function represents the elastic energy which is recoverable during unloading if $\mathbf{M} = \mathbf{C}$ and

$\bar{\mathbf{M}} = \mathbf{E}$, for $k = \theta$ and $\bar{k} = \bar{\theta}$. The previous expressions for $F$ represent the most general associated format for this kind of scalar damage model.

Let us examine the differentiation of $F$. Considering the expression for $F$ in the stress space and recalling that the derivatives are evaluated for $\lambda = \text{const}$, we observe that the second term $A$ does not play any role in the gradient. Using the chain rule

$$\mathbf{N} = \frac{\partial \left( \dfrac{1}{2}\, \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} :: \mathbf{M} \right)}{\partial \left( \dfrac{1}{2}\, \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)} = \frac{\partial \left( \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)}{\partial \left( \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)} :: \mathbf{M} + \left( \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right) :: \frac{\partial \mathbf{M}}{\partial \left( \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)} \tag{3.115}$$

If $\mathbf{M}$ is not dependent on $\boldsymbol{\sigma}$ the second contribution vanishes. The derivative in the first term leads to the definition of an $8^{th}$ order unit tensor $\mathbf{I}_8$ that contracted with the $4^{th}$ order $\mathbf{M}$ gives again $\mathbf{M}$.

$$\frac{\partial \left( \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)}{\partial \left( \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)} = \mathbf{I}_8 \qquad \Rightarrow \qquad \frac{\partial \left( \sigma_{ij} \sigma_{kl} \right)}{\partial \left( \sigma_{pq} \sigma_{rs} \right)} = \delta_{ip}\, \delta_{jq}\, \delta_{kr}\, \delta_{ls} = I^4_{ijpq}\, I^4_{klrs} = I^8_{ijklpqrs} \tag{3.116}$$

And, as a final result,

$$\mathbf{N} = \mathbf{I}_8 :: \mathbf{M} = \mathbf{M} \tag{3.117}$$

**Scalar damage. General formulation**

In the traditional scalar damage formulation all the components of the stiffness are affected in the same way. As was shown by Ju [32], that infers that, for an isotropic $\mathbf{E}_o$, only the elastic modulus $E_o$ is changed as $E = \theta E_o$, while Poisson's ratio $\nu_o$ remains constant. In fact both Lamé's constants are affected by the same factor $\theta$ as $\lambda = \theta\, \lambda_o$ and $\mu = \theta\, \mu_o$. Thus

$$\frac{\lambda}{\mu} = \frac{\lambda_o}{\mu_o} = \frac{1}{\dfrac{1}{2\nu_o} - 1} \qquad \Rightarrow \qquad \nu = \nu_o \tag{3.118}$$

Clearly, other scalar damage models can be proposed which affect in different ways the various components of the elastic stiffness. A general approach for expressing scalar damage defines the secant operator as follow

$$\mathbf{E} = \mathbf{E}_o + \lambda\,\bar{\mathbf{M}}\,; \qquad\qquad \mathbf{C} = \mathbf{C}_o + \lambda\,\mathbf{M} \tag{3.119}$$

where $\bar{\mathbf{M}}$ and $\mathbf{M}$ are constant tensors. Thus the actual value of stiffness and compliance is given by the initial undamaged operator and a variation which is proportional to the current amount of damage.

It is important to note that if $\bar{\mathbf{M}}$ and $\mathbf{M}$ are isotropic tensors the scalar formulation results in an isotropic damage model. Otherwise, for anisotropic $\bar{\mathbf{M}}$, $\mathbf{M}$, the scalar damage representation will result in an anisotropic effect on the secant operators (see also the following discussion on $4^{th}$ order damage tensors).

Differentiation of the expressions (3.119) yields

$$\dot{\mathbf{E}} = \dot{\lambda}\,\bar{\mathbf{M}} \qquad\qquad \dot{\mathbf{C}} = \dot{\lambda}\,\mathbf{M} \tag{3.120}$$

In this case, the tensors appearing in (3.119) are the ones that define the fracturing rule in the theory of elastic-fracturing materials.

Again the formulation leads to an associated model if the format (3.114) for the loading function $F$ is prescribed. Let us explore possible formulations for the tensors $\bar{\mathbf{M}}$ and $\mathbf{M}$.

The traditional formulation can be readily recovered if we assume $\bar{\mathbf{M}} = \mathbf{E}_o$ or $\mathbf{M} = \mathbf{C}_o$. A linear variation of $E$ with $D$ is recovered, while Poisson's ratio $\nu$ remains constant (Fig. 3.7, 3.8)

$$\boxed{\mathbf{M} = \mathbf{C}_o \qquad \Rightarrow \qquad E = (1-D)\,E_o \;\; ; \qquad \nu = \nu_o} \tag{3.121}$$

Let us consider the expression of the compliance modified by $\mathbf{M} = \mathbf{I}_4^s$, which results in an isotropic damage format. The variable $\lambda$ is related to the damage variable $D$ in the usual way, such that $\dfrac{1}{2G} = \dfrac{1}{1-D}\,\dfrac{1}{2G_o}$:

$$\mathbf{C} = \mathbf{C}_o + \lambda\,\mathbf{M} = -\frac{\nu_o}{E_o}\,\mathbf{I}_2 \otimes \mathbf{I}_2 + \frac{1}{1-D}\,\frac{1+\nu_o}{E_o}\,\mathbf{I}_4^s \tag{3.122}$$

Solving the system of two equations

$$\begin{cases} -\dfrac{\nu}{E} = -\dfrac{\nu_o}{E_o} \\ \dfrac{1+\nu}{E} = \dfrac{1}{1-D}\,\dfrac{1+\nu_o}{E_o} \end{cases} \tag{3.123}$$

for $E$ and $\nu$, we obtain

$$\boxed{\mathbf{M} = \mathbf{I}_4 \qquad \Rightarrow \qquad E = \frac{(1-D)\,E_o}{1+\nu_o\,D} \quad ; \qquad \nu = \frac{(1-D)\,\nu_o}{1+\nu_o\,D}} \tag{3.124}$$

Young's modulus and Poisson's ratio are thus both decreasing in the same way with $D$ as shown in Fig. 3.7 and 3.8.

An associated scalar model is recovered with a damage function containing $\mathbf{M} = \mathbf{I}_4^s$, as was proposed by ORTIZ [54]:

$$F = \frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{I}_4 : \boldsymbol{\sigma} - A(\mathbf{p}) = \frac{1}{2}\,\boldsymbol{\sigma} : \boldsymbol{\sigma} - A(\mathbf{p}) = 0 \tag{3.125}$$

Another interesting model is obtained when the deviatoric projection operator $\mathbf{P}_d = \mathbf{I}_4^s - \mathbf{P}_v = \mathbf{I}_4^s - \dfrac{1}{3}\,\mathbf{I}_2 \otimes \mathbf{I}_2$ is assumed for $\mathbf{M}$. The proposal is interesting because in this case only the shear modulus is modified, while the bulk modulus remains constant. Furthermore, an associated formulation of this model results in a damage function of the von Mises type. In fact, with $\mathbf{M} = \mathbf{P}_d$

$$F = \frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{P}_d : \boldsymbol{\sigma} - A(\mathbf{p}) = \frac{1}{2}\,\boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d - A(\mathbf{p}) = J_2 - A(\mathbf{p}) = 0 \tag{3.126}$$

where $\boldsymbol{\sigma}_d$ denotes the deviatoric stress and $J_2$ the second deviatoric invariant. Thus we could label the model as a $J_2$-damage formulation (NEILSEN AND SCHREYER [53]).

The variation of the elastic modulus and Poisson's ratio are readily found. Using the two projection operators the expression of the compliance reads

$$\mathbf{C} = \frac{1}{3K}\,\mathbf{P}_v + \frac{1}{2G}\,\mathbf{P}_d \tag{3.127}$$

where $3K = \dfrac{E}{1-2\nu}$. Introducing the damage variable $D$ in the usual way, we obtain the following system of two equations

$$\begin{cases} \dfrac{1-2\nu}{E} = \dfrac{1-2\nu_o}{E_o} \\ \dfrac{1+\nu}{E} = \dfrac{1}{1-D}\,\dfrac{1+\nu_o}{E_o} \end{cases} \tag{3.128}$$

for the two unknowns $E$ and $\nu$. Solution yields

$$\boxed{\mathbf{M} = \mathbf{P}_d \quad \Rightarrow \quad E = \frac{3\,(1-D)\;E_o}{3-(1-2\nu_o)\,D} \;\; ; \qquad \nu = \frac{3\nu_o + (1-2\nu_o)\,D}{3-(1-2\nu_o)\,D}} \tag{3.129}$$

The resulting non-linear increase of Poisson's ratio is illustrated in Fig. 3.7 and 3.8, together with the non-linear decrease of the stiffness.

Now, other damage models could be proposed, for instance by imposing $\mathbf{M} = \mathbf{P}_v$, which will modify just the bulk modulus, although the physical meaning of the resulting models must be always verified. For example an anisotropic $\mathbf{M}$, will lead to an anisotropic damage formulation.

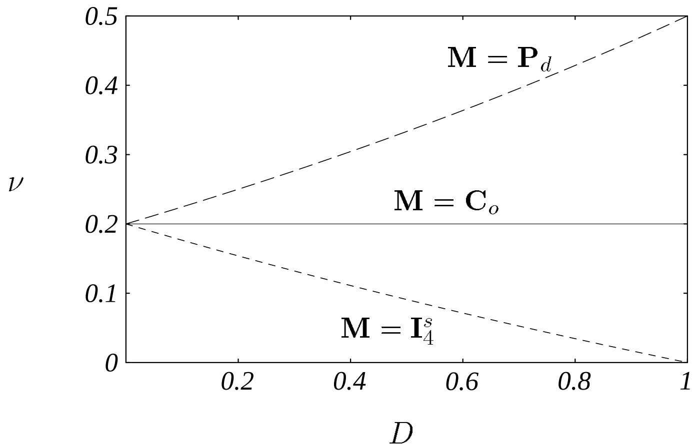



Figure 3.7: Scalar damage models: variation of Poisson's ratio ($\nu_o = 0.2$)

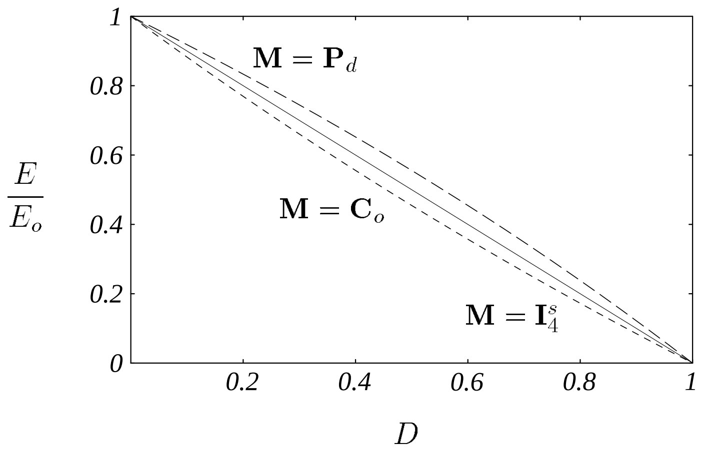



Figure 3.8: Scalar damage models: variation of Young's modulus ($\nu_o = 0.2$)

**Vector damage**

The vectorial damage description is defined by a damage vector $\mathbf{D}$ with the components $D_i$. Now two possible interpretations can be considered. In the original proposal by Krajcinovic and Fonseka [34], the damage vector identifies the density and orientation of a set of parallel microcracks inside the body. We could label this damage vector as a physical damage vector. On the other hand, when the damage is characterized by a set of scalar variables that could be grouped in a vector we could also call that vector a damage vector, even though no physical meaning can be attributed to it (Carol et al. [14]).

An expression of the secant stiffness for the physical damage vector was proposed by Krajcinovic and Fonseka [34] in the form

$$E_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + 2\mu\, \delta_{ik}\delta_{jl} + C_1\, (\delta_{ij} D_k D_l + \delta_{kl} D_i D_j) + C_2\, (\delta_{jk} D_i D_l + \delta_{il} D_j D_k) \quad (3.130)$$

where $C_1$ and $C_2$ are two material constants and $D_i$ are the components of the damage vector related to the void densities in the planes of normal $N_i$. For no damage the void density is zero, all the components of $\mathbf{D}$ remain zero and the initial stiffness $\mathbf{E}_o$ is recovered. When full damage in the direction $\vec{\mathbf{N}}$ is prescribed ($D_N = 1$), then the stiffness $E_N = E_{ijkl}N_iN_jN_kN_l$ in that direction must vanish. This results in a condition for the two constants $C_1$ and $C_2$

$$E_N = \lambda + 2\mu + 2(C_1 + C_2)\, D_N \tag{3.131}$$

For $D_N = 1$ they satisfy

$$C_1 + C_2 = -\frac{\lambda}{2} - \mu \tag{3.132}$$

The proposed form of the secant stiffness appears to be derived independently of all the concepts previously introduced. However the general format (3.104) could be recovered assuming $C_1 = -\lambda/2,\ C_2 = -\mu$, with a $8^{th}$ order tensor formed with combinations of the Kronecker deltas and damage vector components.

The physical damage vector approach obviously enriches the scalar damage description because it introduces the orientation of microcracks. However it seems that one damage vector it is not sufficient to represent a general family of voids which are randomly distributed. A set of three damage vectors representing the different families or three perpendicular damage vectors could give a more appropriate description. The last possibility lends itself to the definition of a second order damage tensor with eigenvectors coincident with the three mutually perpendicular damage vectors.

### Second order damage tensor

The definition of a second order damage tensor appears to be a reasonable approach because of the underlying physical meaning of a second order tensor. For instance, considering the generic plane with orientation $\vec{\mathbf{N}}$, the damage in that plane, that could be expressed in the Kachanov's sense as a reduction of area, is obtained as $D_N = \vec{\mathbf{N}} \cdot \mathbf{D} \cdot \vec{\mathbf{N}}$ once the damage tensor $\mathbf{D}$ is defined.

In the literature the tensor $\mathbf{D}$ is usually defined such that $\mathbf{D} = \mathbf{0}$ for no damage and $\mathbf{D} = \mathbf{I}_2$ for full damage (Murakami and Ohno [48], Sidoroff [70], Cordebois and Sidoroff [17], Murakami [49], Mazars et al. [46]). On the other hand, other authors consider the integrity tensor $\boldsymbol{\phi} = \mathbf{I}_2 - \mathbf{D}$, which varies between $\mathbf{I}_2$ and $\mathbf{0}$ (BETTEN [8], VALANIS [84]).

The damage tensor is usually introduced in order to relate nominal and effective stresses

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{eff} \cdot \boldsymbol{\phi} \qquad \text{or} \qquad \boldsymbol{\sigma} = \boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{eff} \tag{3.133}$$

However this formulation presents the disadvantage that, even for a symmetric $\boldsymbol{\phi}$ or $\mathbf{D}$, it is not possible to maintain in general a symmetric format of $\boldsymbol{\sigma}$, even if we assume a symmetric $\boldsymbol{\sigma}_{eff}$ and vice versa. In fact, according to the normal assumption, the physical effective stress is symmetric (for non-polar continuum) and also the nominal stress should be symmetric.

Two forms of symmetrization were recently proposed (CAROL ET AL. [14]):

$$\begin{aligned} \text{“\textit{product-type}”} \quad &\Rightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\omega} \cdot \boldsymbol{\sigma}_{eff} \cdot \boldsymbol{\omega} \\ \text{“\textit{summation-type}”} \quad &\Rightarrow \quad \boldsymbol{\sigma} = \frac{1}{2}\left(\boldsymbol{\phi} \cdot \boldsymbol{\sigma}_{eff} + \boldsymbol{\sigma}_{eff} \cdot \boldsymbol{\phi}\right) \end{aligned} \tag{3.134}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\omega}$ are symmetric tensors and $\boldsymbol{\omega} = \boldsymbol{\phi}^{1/2}$, i.e. $\boldsymbol{\phi} = \boldsymbol{\omega} \cdot \boldsymbol{\omega}$

The expression of the secant stiffness can be readily obtained after definition of the $4^{th}$ order tensor $\boldsymbol{\alpha}$, which relates nominal and effective stresses as $\boldsymbol{\sigma} = \boldsymbol{\alpha} : \boldsymbol{\sigma}_{eff}$, by means of the stress equivalence or the energy equivalence approach. In addition, the expression of the $8^{th}$ order tensor $\boldsymbol{\Omega}$ can be developed.

For the two types of symmetrization $\boldsymbol{\alpha}$ is built up as outer product of two second order tensors, but in this case compact notation cannot be used because the indices are mixed. In indicial notation the expressions for $\boldsymbol{\alpha}$ read

$$\begin{aligned} \text{“\textit{product-type}”} \quad &\Rightarrow \quad \alpha_{ijkl} = \omega_{ik}\,\omega_{jl} \\ \text{“\textit{summation-type}”} \quad &\Rightarrow \quad \alpha_{ijkl} = \frac{1}{2}\left(\phi_{ik}\delta_{jl} + \delta_{ik}\phi_{lj}\right) \end{aligned} \tag{3.135}$$

Note that $\boldsymbol{\alpha}$ is symmetric, i.e. $\alpha_{ijkl} = \alpha_{klij}$, because of the symmetry of $\boldsymbol{\phi}$ and $\boldsymbol{\omega}$.

An interesting expression of the secant stiffness is obtained from the product-type symmetrization when the energy equivalence approach is used. In this case the secant stiffness is fully symmetric and is given by $\mathbf{E} = \boldsymbol{\alpha} : \mathbf{E}_o : \boldsymbol{\alpha}^T$. Substituting the first expression of (3.135) and using the definition $\omega_{ik}\omega_{kj} = \phi_{ij}$ we obtain, after some algebra

$$E_{ijkl} = \lambda_o \, \phi_{ij}\phi_{kl} + 2\mu_o \frac{1}{2}(\phi_{ik}\phi_{jl} + \phi_{il}\phi_{jk}) \tag{3.136}$$

This degradation model was recently proposed by VALANIS [84] for the secant stiffness. The actual stiffness looks very similar to the initial undamaged one: the components of $\mathbf{I}_2$ are replaced by the components of $\boldsymbol{\phi}$. Also it is easy to check that for $\boldsymbol{\phi} = \mathbf{I}_2$, i.e. for full integrity, the initial stiffness is recovered.

Finally the expression of the $8^{th}$ order tensor $\boldsymbol{\Omega}$ can be developed as

$$\Omega_{ijklpqrs} = \alpha_{ijpq}\,\alpha_{rskl} = \omega_{ip}\,\omega_{jq}\,\omega_{kr}\,\omega_{ls} \tag{3.137}$$

**Fourth order damage tensor**

The $4^{th}$ order damage tensor description appears to be the most comprehensive representation which can be obtained with a reasonable complexity. In fact, the use of an eight order tensor appears to be too complicated and difficult to handle. The $4^{th}$ order tensor description has also the advantage of corresponding $4^{th}$ order descriptions of stiffness and compliance.

The $4^{th}$ order tensor is usually prescribed by (3.84) expressing the relation between effective and nominal stresses or by the expression (3.91) for the strains

$$\boldsymbol{\sigma} = \boldsymbol{\alpha} : \boldsymbol{\sigma}_{eff} = (\mathbf{I}_4^s - \mathbf{D}) : \boldsymbol{\sigma}_{eff} \;; \qquad \boldsymbol{\epsilon}_{eff} = \boldsymbol{\alpha} : \boldsymbol{\epsilon} \tag{3.138}$$

Following the notation of JU [32], the damage effect tensor on stresses $\mathbf{D}$ is introduced for generalizing the scalar damage variable $D$. Note that this definition of $\mathbf{D}$ is normally presented in the strain equivalence approach, where the final expression of the secant stiffness appears to generalize the scalar description $D$, that affects all the components $E_{ijkl}$ in the same way, $\mathbf{E} = (1 - D)\,\mathbf{E}_o$:

$$\mathbf{E} = \boldsymbol{\alpha} : \mathbf{E}_o = (\mathbf{I}_4^s - \mathbf{D}) : \mathbf{E}_o \tag{3.139}$$

Once the tensor $\boldsymbol{\alpha}$ is defined, other expressions for the stiffness can be found using also the stress or energy equivalence approaches. Note again that only the last one provides a fully symmetric secant stiffness tensor.

The main advantage of the use of a $4^{th}$ order tensor is its extension to describe anisotropic damage. According to Ju [32], anisotropic damage is obtained when the damage effect tensor affects the directional properties of the initial stiffness tensor. Conversely an isotropic damage preserves the directional characteristics of $\mathbf{E}_o$. Thus, for instance, if $\mathbf{E}_o$ is isotropic, the use of an isotropic $\mathbf{D}$ leads to an actual secant stiffness $\mathbf{E}$ which is also isotropic. If $\mathbf{E}_o$ is anisotropic, then $\mathbf{E}$ is also anisotropic.

The most general isotropic damage tensor, leading to an isotropic damage description, could be assigned as

$$\mathbf{D} = d_1 \mathbf{I}_2 \otimes \mathbf{I}_2 + d_2 \mathbf{I}_4^s \tag{3.140}$$

where $d_1$ and $d_2$ are two scalar damage variables. Thus the most general form of isotropic damage involves the use of a $4^{th}$ order tensor which depends on two scalar parameters. The isotropic scalar damage description is only a particular case, even though the scalar representation is usually intended as synonym for isotropic damage.

## 3.5 Some examples from the literature

The elastic-fracturing theory presented earlier, with or without the introduction of damage variables, renders a general and unified description that encompasses many of the models with stiffness degradation proposed in the literature. Often these models were originally proposed starting from different assumptions and every paper appeared to be completely independent from the others. In fact, the following discussion is based on an underlying common denominator, even though not all the theories can be included.

In the following subsections, different damage models presented by different authors are analyzed within the present theory of elastic stiffness degradation. The models have been chosen because they provide expressions for the tangent operators and are useful for subsequent localization analysis. The expressions of the tangent operators is recalled using the present theory.

### 3.5.1 Mazars and Lemaitre [1984]

Mazars and Lemaitre [44] proposed an isotropic scalar model characterized by the scalar variable $D$ that relates effective and nominal stresses which affects the stiffness in terms of

$$\boldsymbol{\sigma} = (1 - D)\,\boldsymbol{\sigma}_{eff}\,; \qquad \mathbf{E} = (1 - D)\,\mathbf{E}_o \tag{3.141}$$

Indeed, this is the traditional scalar damage formulation in the sense of stress equivalence mentioned in the previous section.

The model fits the description of elastic-fracturing materials in the strain-based formulation. Its derivation is quite general and let us say classic. For instance, the proposal by Simo and Ju [71], that will be also presented as an example, follows the same ideas.

The model is derived on the basis of thermodynamics and the first step assumes the existence of a free energy potential of the form

$$\Psi = \frac{1}{2}\,(1 - D)\,\boldsymbol{\epsilon} : \mathbf{E}_o : \boldsymbol{\epsilon} \tag{3.142}$$

The stresses and the thermodynamic force $\bar{\mathcal{Y}}$ associated to the damage are derived as equations of state

$$\begin{aligned} \boldsymbol{\sigma} &= \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}} = (1 - D)\,\mathbf{E}_o : \boldsymbol{\epsilon} = (1 - D)\,\boldsymbol{\sigma}_o \\ \bar{\mathcal{Y}} &= \frac{\partial \Psi}{\partial D} = -\frac{1}{2}\,\boldsymbol{\epsilon} : \mathbf{E}_o : \boldsymbol{\epsilon} \end{aligned} \tag{3.143}$$

where $\boldsymbol{\sigma}_o$ denotes the undamaged stress. The thermodynamic force matches our definition in the general theory. In fact, using the expression in Table 3.3 we obtain, with $\bar{\mathcal{D}} = D$

$$\bar{\mathcal{Y}} = \bar{\mathbf{Y}} :: \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} = \frac{1}{2}\, \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon} :: (-\mathbf{E}_o) = -\frac{1}{2}\, \boldsymbol{\epsilon} : \mathbf{E}_o : \boldsymbol{\epsilon} \tag{3.144}$$

A partition of the stress tensor is then introduced from differentiation of the first expression in (3.143)

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}_e + \dot{\boldsymbol{\sigma}}_d = (1 - D)\; \mathbf{E}_o : \dot{\boldsymbol{\epsilon}} + (-\dot{D}\; \mathbf{E}_o : \boldsymbol{\epsilon}) \tag{3.145}$$

The important assumption is the existence of the damage condition as

$$F(\bar{\epsilon}, D) = \bar{\epsilon} - r(D) = 0 \tag{3.146}$$

where the equivalent strain $\bar{\epsilon}$ denotes the second invariant of the strain tensor

$$\boxed{\bar{\epsilon} = \sqrt{\boldsymbol{\epsilon} : \boldsymbol{\epsilon}}} \tag{3.147}$$

This definition of $\bar{\epsilon}$ constitutes the main difference with the model by Simo and Ju [71]. It will lead to a non-associated damage model. In fact in this case the damage function $F$ does not contain the energy type term, necessary to obtain associativity, as remarked in the previous section.

The rate of equivalent strain is then obtained by differentiation

$$\dot{\bar{\epsilon}} = \frac{1}{\bar{\epsilon}}\, \boldsymbol{\epsilon} : \dot{\boldsymbol{\epsilon}} \tag{3.148}$$

The evolution of damage is assumed in the form

$$\dot{D} = \dot{\bar{\epsilon}}\; \bar{\mathcal{M}}(\bar{\epsilon}) = \dot{\lambda}\; \bar{\mathcal{M}}(\bar{\epsilon}) \tag{3.149}$$

where $\dot{\bar{\epsilon}} = \dot{r} = \dot{\lambda}$ because of the consistency condition $\dot{F} = 0$.

The tangent operator can now be readily found, although the paper does not provide an explicit expression. Simo and Ju [71] give this expression in equation (18). The rate of damaged stress can be expressed in terms of the total strain rate $\dot{\boldsymbol{\epsilon}}$ as

$$\dot{\boldsymbol{\sigma}}_d = \boldsymbol{\sigma}_o\; \dot{D} = \boldsymbol{\sigma}_o\; \bar{\mathcal{M}}\; \dot{\bar{\epsilon}} = \bar{\mathcal{M}}\; \boldsymbol{\sigma}_o\; \frac{1}{\bar{\epsilon}}\; \boldsymbol{\epsilon} : \dot{\boldsymbol{\epsilon}} = \frac{\bar{\mathcal{M}}}{\bar{\epsilon}}\; (\boldsymbol{\sigma}_o \otimes \boldsymbol{\epsilon}) : \dot{\boldsymbol{\epsilon}} \tag{3.150}$$

Coming back to the stress decomposition, after collection of $\dot{\boldsymbol{\epsilon}}$, we obtain $\dot{\boldsymbol{\sigma}} = \mathbf{E}_{ed} : \dot{\boldsymbol{\epsilon}}$, with

$$\boxed{\mathbf{E}_{ed} = (1-D)\,\mathbf{E}_o - \frac{\bar{\mathcal{M}}}{\bar{\epsilon}}\,\boldsymbol{\sigma}_o \otimes \boldsymbol{\epsilon}} \tag{3.151}$$

The elastic-damage tangent operator is not symmetric as was expected. The same expression can be found from our theory, once all the contributions in

$$\mathbf{E}_{ef} = \mathbf{E} + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} \tag{3.152}$$

are known. They are obtained as follows (see Tables 3.1 and 3.3)

$$\begin{aligned}
\bar{\mathbf{n}} &= \frac{\partial F}{\partial \boldsymbol{\epsilon}} = \frac{\partial F}{\partial \bar{\epsilon}}\,\frac{\partial \bar{\epsilon}}{\partial \boldsymbol{\epsilon}} = \frac{\boldsymbol{\epsilon}}{\bar{\epsilon}} \\
\bar{\mathbf{m}} &= \bar{\mathcal{M}} \star \frac{\partial \mathbf{E}}{\partial \bar{D}} : \boldsymbol{\epsilon} = -\bar{\mathcal{M}}\,\mathbf{E}_o : \boldsymbol{\epsilon} = -\bar{\mathcal{M}}\,\boldsymbol{\sigma}_o \\
\bar{H} &= -\frac{\partial F}{\partial \lambda} = -\frac{\partial F}{\partial r}\,\frac{\partial r}{\partial \lambda} = -(-1)\,1 = 1 \\
\mathbf{E} &= (1-D)\,\mathbf{E}_o
\end{aligned} \tag{3.153}$$

After substitution in the expression for $\mathbf{E}_{ef}$, the same tangent stiffness is recovered as in (3.151).

### 3.5.2 Ortiz [1985]

Ortiz [54] proposed a comprehensive model for the inelastic behavior of concrete. The formulation considers three fundamental aspects: the behavior of the mortar, the behavior of the aggregate and the interaction between mortar and aggregate. In the current study we will focus only on the part of the model that describe the constitutive law for the mortar as an elastic fracturing material, although in his paper this terminology is never used and Ortiz refers to it as a damage model.

Furthermore, the formulation considers the closure of microcracks, by introducing projection operators which define the positive and negative parts of the stress and strain tensors. Without loss of generality and considering that, for our purposes, we want recast the model into the framework of the present theory, these additional aspects will not be considered.

The main assumption of Ortiz's damage description is the existence of a damage function and a damage rule (fracturing rule in our terminology) for the compliance, that can be written, in the present notation, as

$$F = \bar{F} - A = \frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{M} : \boldsymbol{\sigma} - \frac{1}{2}\,t(\lambda)^2 = 0\;; \qquad \dot{\mathbf{C}} = \dot{\lambda}\,\mathbf{M} \tag{3.154}$$

The author defines the damage rule to be associated if

$$\mathbf{M} = \frac{\partial^2 \bar{F}}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} \qquad \Rightarrow \qquad \bar{F} = \frac{1}{2}\,\boldsymbol{\sigma} : \frac{\partial^2 \bar{F}}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} \tag{3.155}$$

which requires that $\bar{F}$ is a homogeneous function of degree two. In fact, a general homogeneous function $f(\boldsymbol{\sigma})$ of degree $n$ satisfies

$$\begin{aligned} f(c\,\boldsymbol{\sigma}) &= c^n\; f(\boldsymbol{\sigma}) \\ \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} &= n\; f(\boldsymbol{\sigma}) \end{aligned} \tag{3.156}$$

where the second proposition represents Euler's theorem of homogeneous functions. Differentiating the last expression in (3.156), after some manipulations, the following equations can be obtained

$$\begin{aligned} \frac{\partial^2 f}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} &= (n-1)\,\frac{\partial f}{\partial \boldsymbol{\sigma}} \\ \boldsymbol{\sigma} : \frac{\partial^2 f}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} &= n\,(n-1)\,f \end{aligned} \tag{3.157}$$

For $n = 2$, we recover equation (3.155) substituting $f = \bar{F}$.

Now it is clear that the model is termed associated because of the resulting expression of the inelastic strain rate

$$\dot{\boldsymbol{\epsilon}}_i = \dot{\mathbf{C}} : \boldsymbol{\sigma} = \dot{\lambda}\,\mathbf{M} : \boldsymbol{\sigma} = \dot{\lambda}\,\frac{\partial^2 \bar{F}}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} = \dot{\lambda}\,\frac{\partial \bar{F}}{\partial \boldsymbol{\sigma}} \qquad \Rightarrow \qquad \mathbf{m} = \mathbf{n} \tag{3.158}$$

where the first condition in (3.157) was used.

In the following, Ortiz proposes that the damage process must be driven by the thermodynamic force $\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}/2$ associated to the compliance and assumes that the damage direction depends on the state of stress through the thermodynamic forces. A form of $\mathbf{M}$ of this kind, which is consistent with the associativity assumption is

$$\mathbf{M}(\boldsymbol{\sigma}) = \frac{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}}{\boldsymbol{\sigma} : \boldsymbol{\sigma}} \tag{3.159}$$

In fact, in this case, the resulting expression for $\bar{F}$ reduces to

$$\bar{F} = \frac{1}{2}\, \boldsymbol{\sigma} : \frac{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}}{\boldsymbol{\sigma} : \boldsymbol{\sigma}} : \boldsymbol{\sigma} = \frac{1}{2}\, \boldsymbol{\sigma} : \boldsymbol{\sigma}\, \frac{\boldsymbol{\sigma} : \boldsymbol{\sigma}}{\boldsymbol{\sigma} : \boldsymbol{\sigma}} = \frac{1}{2}\, \boldsymbol{\sigma} : \boldsymbol{\sigma} \tag{3.160}$$

Differentiation yields

$$\frac{\partial \bar{F}}{\partial \boldsymbol{\sigma}} = \boldsymbol{\sigma} \qquad \frac{\partial^2 \bar{F}}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} = \mathbf{I}_4 \tag{3.161}$$

This proves that the resulting $\bar{F}$ is homogeneous of degree two, even though $\mathbf{M} \neq \dfrac{\partial^2 \bar{F}}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}} = \mathbf{I}_4$. Thus there is some inconsistency in this formulation of associated damage.

Associativity can be readily checked using our theory. The direction of flow for the inelastic strain reads

$$\mathbf{m} = \mathbf{M} : \boldsymbol{\sigma} = \frac{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}}{\boldsymbol{\sigma} : \boldsymbol{\sigma}} : \boldsymbol{\sigma} = \boldsymbol{\sigma} \tag{3.162}$$

which equals the gradient of the damage function $\mathbf{n} = \boldsymbol{\sigma}$. Thus associativity at the strain level is recovered. However the model is not associated at the compliance level and this reflects the inconsistency earlier mentioned. The generalized gradient $\mathbf{N}$

$$\mathbf{N} = \frac{\partial \bar{F}}{\partial \left( \dfrac{1}{2}\, \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)} = \frac{\partial \left( \boldsymbol{\sigma} : \boldsymbol{\sigma} \right)}{\partial \left( \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)} = \mathbf{I}_4 \tag{3.163}$$

differs from $\mathbf{M}$. This is an example of a constitutive model that is associated in strain space but not in compliance space. Full associativity would be recovered if $\mathbf{M} = \mathbf{I}_4$.

Finally, the tangent stiffness can now be evaluated using the general expression in Table 3.1

$$\mathbf{E}_{ef} = \mathbf{E} - \frac{\mathbf{E} : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}} \tag{3.164}$$

when all the ingredients are known:

$$\begin{aligned} &\mathbf{n} = \mathbf{m} = \boldsymbol{\sigma} \\ &H = -\frac{\partial F}{\partial \lambda} = -\frac{\partial F}{\partial t}\,\frac{\partial t}{\partial \lambda} = t\,t' \end{aligned} \tag{3.165}$$

The final expression, which compares with equation (3.66) in the paper by Ortiz, is thus recovered for $\mathbf{s} = \boldsymbol{\sigma}$

$$\boxed{\mathbf{E}_{ef} = \mathbf{E} - \frac{\mathbf{E} : \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} : \mathbf{E}}{t\,t' + \boldsymbol{\sigma} : \mathbf{E} : \boldsymbol{\sigma}}} \tag{3.166}$$

### 3.5.3 Simo and Ju [1987]

Simo and Ju [71] formulated an elastic-plastic continuum damage model, using dual strain- or stress-based formulations. They used thermodynamics and the effective stress/strain concept in conjunction with strain/stress equivalence.

The damage formulation is a scalar isotropic damage model in the traditional sense. Later, the authors developed also an anisotropic damage model, that introduces positive projection operators for tensile damage. In the following, only the scalar isotropic damage model will be considered, which is limited here to elastic-damage.

#### Strain-based scalar isotropic damage model

The starting point is the existence of a free energy potential $\Psi$, that, neglecting the plastic part reads

$$\Psi(\boldsymbol{\epsilon}, D) = (1 - D)\ \Psi_o(\boldsymbol{\epsilon}) \tag{3.167}$$

where $\Psi_o$ denotes the initial elastic stored energy of the undamaged material. For the linear elastic case $\Psi_o = \frac{1}{2}\,\boldsymbol{\epsilon} : \mathbf{E}_o : \boldsymbol{\epsilon}$. The stress is obtained by differentiation as

$$\boldsymbol{\sigma} = \left.\frac{\partial \Psi}{\partial \boldsymbol{\epsilon}}\right|_D = (1 - D)\,\frac{\partial \Psi_o}{\partial \boldsymbol{\epsilon}} \tag{3.168}$$

For the linear case, with $\frac{\partial \Psi_o}{\partial \boldsymbol{\epsilon}} = \mathbf{E}_o : \boldsymbol{\epsilon} = \boldsymbol{\sigma}_o$,

$$\boldsymbol{\sigma} = (1 - D)\,\mathbf{E}_o : \boldsymbol{\epsilon} = \mathbf{E} : \boldsymbol{\epsilon} \tag{3.169}$$

the secant stiffness $(1 - D)\,\mathbf{E}_o$ is introduced. This format corresponds to the traditional scalar damage model developed in the previous section. Note that the initial stiffness is recovered as $\mathbf{E}_o = \frac{\partial^2 \Psi_o}{\partial \boldsymbol{\epsilon} \otimes \partial \boldsymbol{\epsilon}}$.

The following step assumes a damage function of the kind

$$F(\boldsymbol{\epsilon}, \mathbf{p}) = F(\bar{\tau}, \lambda) = \bar{\tau} - r(\lambda) \tag{3.170}$$

where $\bar{\tau}$ is called the equivalent strain, following the terminology introduced by Mazars and Lemaitre [44]. Actually it represents an undamaged energy norm of the strain tensor

$$\boxed{\bar{\tau} = \sqrt{2\,\Psi_o(\boldsymbol{\epsilon})} = \sqrt{\boldsymbol{\epsilon} : \mathbf{E}_o : \boldsymbol{\epsilon}}} \tag{3.171}$$

The following damage rule is adopted

$$\begin{aligned} \dot{D} &= \dot{\lambda}\,\bar{\mathcal{M}}(\bar{\tau}, D) \\ \dot{r} &= \dot{\lambda} \end{aligned} \tag{3.172}$$

Note that all the terms, and $\bar{\mathcal{M}}$ in particular $\bar{\mathcal{M}}$, are scalars.

We can now evaluate all the ingredients for constructing the tangent operator of the elastic-fracturing strain-based formulation. The gradient of the damage function reads

$$\bar{\mathbf{n}} = \left.\frac{\partial F}{\partial \boldsymbol{\epsilon}}\right|_D = \frac{\partial F}{\partial \bar{\tau}}\,\frac{\partial \bar{\tau}}{\partial \boldsymbol{\epsilon}} = 1\,\frac{1}{\bar{\tau}}\,\frac{\partial \Psi_o}{\partial \boldsymbol{\epsilon}} = \frac{\boldsymbol{\sigma}_o}{\bar{\tau}} \tag{3.173}$$

The hardening parameter is obtained as

$$\bar{H} = -\frac{\partial F}{\partial \lambda} = -\frac{\partial F}{\partial r}\,\frac{\partial r}{\partial \lambda} = -(-1)\;1 = 1 \tag{3.174}$$

As a generalization, the hardening parameter could be redefined as $\bar{h} > 0$ if the different hypothesis $\dot{r} = \bar{h}\;\dot{\lambda}$ is assumed. $\bar{h}$ will appear directly in the expression of the tangent operator.

The flow direction can be derived from the damage rule as

$$\bar{\mathbf{m}} = \bar{\mathcal{M}} \star \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} : \boldsymbol{\epsilon} = \bar{\mathcal{M}}\;\frac{\partial \mathbf{E}}{\partial D} : \boldsymbol{\epsilon} = -\bar{\mathcal{M}}\;\mathbf{E}_o : \boldsymbol{\epsilon} = -\bar{\mathcal{M}}\;\boldsymbol{\sigma}_o \tag{3.175}$$

Finally the tangent stiffness, comparable with the expression (17) in the paper, reads

$$\boxed{\mathbf{E}_{ef} = \mathbf{E} + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} = (1-D)\,\mathbf{E}_o - \frac{\bar{\mathcal{M}}}{\bar{\tau}}\;\frac{\boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o}{\bar{h}}} \tag{3.176}$$

Note that the damage model results in our associated formulation because $\bar{\mathbf{n}} \parallel \bar{\mathbf{m}} \parallel \boldsymbol{\sigma}_o$. Following the remarks on the traditional scalar damage formulation, we can say that associativity is due to the assumed energy norm $\bar{\tau}$ in the characterization of $F$.

**Stress-based scalar isotropic model**

In the dual stress-based formulation the starting assumption is the existence of a complementary free energy potential, which reads for the elastic-damage case

$$\Lambda(\boldsymbol{\sigma}, D) = \bar{D}\;\Lambda_o(\boldsymbol{\sigma}) = \frac{1}{(1-D)}\;\Lambda_o(\boldsymbol{\sigma}) \tag{3.177}$$

where $\Lambda_o(\boldsymbol{\sigma})$ is the initial elastic complementary energy stored in the undamaged material. For the linear case, $\Lambda_o(\boldsymbol{\sigma}) = \frac{1}{2}\;\boldsymbol{\sigma} : \mathbf{C}_o : \boldsymbol{\sigma}$. The strains are now derived by differentiation of the complementary energy potential

$$\boldsymbol{\epsilon} = \left.\frac{\partial \Lambda}{\partial \boldsymbol{\sigma}}\right|_{\bar{D}} = \bar{D}\;\frac{\partial \Lambda_0(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \bar{D}\;\mathbf{C}_o : \boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\sigma} = \bar{D}\;\boldsymbol{\epsilon}_o \tag{3.178}$$

The secant compliance thus follows the usual format $\mathbf{C} = \bar{D}\ \mathbf{C}_o$ of the traditional scalar damage formulation. The initial elastic compliance is also recovered as $\mathbf{C}_o = \dfrac{\partial^2 \Lambda_o}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}}$.

The damage function in the stress space is expressed in the same way as before

$$F(\boldsymbol{\sigma}, \mathbf{p}) = F(\bar{\tau}, \lambda) = \bar{\tau} - r(\lambda) = 0 \tag{3.179}$$

where the equivalent strain $\bar{\tau}$ is the undamaged complementary energy norm of the stress tensor

$$\boxed{\bar{\tau} = \sqrt{2\,\Lambda_o(\boldsymbol{\sigma})} = \sqrt{\boldsymbol{\sigma} : \mathbf{C}_o : \boldsymbol{\sigma}}} \tag{3.180}$$

The equivalent damage rule for $\bar{D}$ is assumed as

$$\begin{aligned} \dot{\bar{D}} &= \dot{\lambda}\ \mathcal{M}(\bar{\tau}, \bar{D}) \\ \dot{r} &= \dot{\lambda} \end{aligned} \tag{3.181}$$

Gradient to the damage function, hardening parameter and inelastic stress flow direction are readily obtained as follows

$$\begin{aligned} \mathbf{n} &= \left.\frac{\partial F}{\partial \boldsymbol{\sigma}}\right|_{\bar{D}} = \frac{\partial F}{\partial \bar{\tau}}\ \frac{\partial \bar{\tau}}{\partial \boldsymbol{\sigma}} = 1\ \frac{1}{\bar{\tau}}\ \frac{\partial \Lambda_o}{\partial \boldsymbol{\sigma}} = \frac{\boldsymbol{\epsilon}_o}{\bar{\tau}} \\ H &= -\frac{\partial F}{\partial \lambda} = -\frac{\partial F}{\partial r}\ \frac{\partial r}{\partial \lambda} = -(-1)\ 1 = 1 \\ \mathbf{m} &= \mathcal{M} \star \frac{\partial \mathbf{C}}{\partial \mathcal{D}} : \boldsymbol{\sigma} = \mathcal{M}\ \frac{\partial \mathbf{C}}{\partial \bar{D}} : \boldsymbol{\sigma} = \mathcal{M}\ \mathbf{C}_o : \boldsymbol{\sigma} = \mathcal{M}\ \boldsymbol{\epsilon}_o \end{aligned} \tag{3.182}$$

The same remark as before applies to the choice of the hardening parameter: $H = h$ could be explicitly assumed with an evolution law of the initial elastic limit at the onset of damage, $\dot{r} = h\dot{\lambda}$.

The expression of the tangent compliance, matches equation (67) in the paper by Simo and Ju and is finally recovered as

$$\boxed{\mathbf{C}_{ef} = \mathbf{C} + \frac{\mathbf{m} \otimes \mathbf{n}}{H} = \bar{D}\,\mathbf{C}_o + \frac{\mathcal{M}}{\bar{\tau}}\ \frac{\boldsymbol{\epsilon}_o \otimes \boldsymbol{\epsilon}_o}{h}} \tag{3.183}$$

As noted before the model is associated because of the underlying expression of the elastic energy type in $F$.

The tangent stiffness could also be obtained by inversion of the compliance relation, or with the direct use of the general expression

$$\mathbf{E}_{ef} = \mathbf{E} - \frac{\mathbf{E} : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}}$$
$$\boxed{\mathbf{E}_{ef} = (1-D)\,\mathbf{E}_o - \frac{(1-D)^2\,\mathcal{M}}{\bar{\tau}} \; \frac{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}}{h + (1-D)\,\mathcal{M}\,\bar{\tau}}} \tag{3.184}$$

As an example, let us consider the relation between $\mathcal{M}$ and $\bar{\mathcal{M}}$ in Table 3.2, which really looks quite complicated, but simplifies in the scalar case to

$$\begin{aligned}
\mathcal{M} &= -\left(\frac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \frac{\partial \mathbf{C}}{\partial \mathcal{D}}\right)^{-1} \star \left(\frac{\partial \mathbf{C}}{\partial \mathcal{D}} :: \left(\mathbf{C} : \frac{\partial \mathbf{E}}{\partial \bar{\mathcal{D}}} : \mathbf{C}\right)\right) \star \bar{\mathcal{M}} \\
&= -(\mathbf{C}_o :: \mathbf{C}_o)^{-1}\,(\mathbf{C}_o :: (\bar{D}\,\mathbf{C}_o : (-\mathbf{E}_o) : \bar{D}\,\mathbf{C}_o))\,\bar{\mathcal{M}} \\
&= \bar{D}^2\,(\mathbf{C}_o :: \mathbf{C}_o)^{-1}\,(\mathbf{C}_o :: \mathbf{C}_o)\,\bar{\mathcal{M}} = \bar{D}^2\,\bar{\mathcal{M}}
\end{aligned} \tag{3.185}$$

To check the result it suffices to consider that

$$\bar{D} = \frac{1}{(1-D)} \qquad \Rightarrow \qquad \dot{\bar{D}} = \frac{1}{(1-D)^2}\,\dot{D} = \bar{D}^2\,\dot{D} \tag{3.186}$$

Recalling the two damage rules, and after elimination of $\dot{\lambda}$, we recover

$$\dot{\lambda}\,\mathcal{M} = \dot{\lambda}\,\bar{D}^2\,\bar{\mathcal{M}} \qquad \Rightarrow \qquad \mathcal{M} = \bar{D}^2\,\bar{\mathcal{M}} \tag{3.187}$$

### 3.5.4 Ju [1989]

In a recent paper JU [31] revisited the previous work and proposed an energy based coupled elastic-plastic damage model. The main assumption is the split of the strain tensor into an elastic-damage and a plastic-damage part as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{ed} + \boldsymbol{\epsilon}_{pd} \tag{3.188}$$

The free energy function must contain the information of both damage and plasticity and is defined as

$$\Psi(\boldsymbol{\epsilon}_{ed}, \mathbf{q}, D) = (1 - D)\ \Psi_o(\boldsymbol{\epsilon}_{ed}, \mathbf{q}) \tag{3.189}$$

where $\mathbf{q}$ denotes a set of plastic variables and $\Psi_o(\boldsymbol{\epsilon}_{ed}, \mathbf{q})$ the potential energy of the undamaged material, in contrast with the previous work, in which $\Psi_o(\boldsymbol{\epsilon})$.

The stress is again derived by differentiation of the free energy function with respect to the elastic-damage strain $\boldsymbol{\epsilon}_{ed}$

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}_{ed}} = (1 - D)\ \frac{\partial \Psi_o}{\partial \boldsymbol{\epsilon}_{ed}} \tag{3.190}$$

Differentiating once more with respect to $\boldsymbol{\epsilon}_{ed}$ the secant stiffness is obtained as

$$\mathbf{E} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\epsilon}_{ed}} = (1 - D)\ \frac{\partial^2 \Psi_o}{\partial \boldsymbol{\epsilon}_{ed} \otimes \partial \boldsymbol{\epsilon}_{ed}} = (1 - D)\ \mathbf{E}_o \tag{3.191}$$

For the effective stress, defined as $\boldsymbol{\sigma}_{eff} = \dfrac{\boldsymbol{\sigma}}{(1 - D)}$, we obtain

$$\boldsymbol{\sigma}_{eff} = \frac{\partial \Psi_o}{\partial \boldsymbol{\epsilon}_{ed}} \tag{3.192}$$

The thermodynamic force associated to the damage variable $D$ is given in a dual way

$$\bar{\mathcal{Y}} = \frac{\partial \Psi(\boldsymbol{\epsilon}_{ed}, \mathbf{q}, D)}{\partial D} = -\Psi_o(\boldsymbol{\epsilon}_{ed}, \mathbf{q}) \tag{3.193}$$

In difference to the formulation by MAZARS AND LEMAITRE [44], this is not only specified by the elastic part of the damage energy.

The thermodynamic force is important for the definition of the damage process and the damage function is now defined as

$$\boxed{F(\Psi_o, r) = \Psi_o(\boldsymbol{\epsilon}_{ed}, \mathbf{q}) - r = 0} \tag{3.194}$$

For our purposes, because we want to capture the elastic-damage behavior, the new assumption for the loading function does not introduce any difference with respect to the previous case. In fact, in the absence of plastic flow $\dot{\boldsymbol{\epsilon}}_{pd} = \dot{\mathbf{q}} = \mathbf{0}$, $\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}_{ed}$ and by differentiation

$$\dot{\Psi}_o = \frac{\partial \Psi_o}{\partial \boldsymbol{\epsilon}_{ed}} : \dot{\boldsymbol{\epsilon}}_{ed} = \boldsymbol{\sigma}_{eff} : \dot{\boldsymbol{\epsilon}} \tag{3.195}$$

The damage rule is assigned as before:

$$\begin{aligned} \dot{D} &= \dot{\lambda}\, \bar{\mathcal{M}} \\ \dot{r} &= \dot{\lambda} \end{aligned} \tag{3.196}$$

Taking the derivative with respect to time, equation (3.190) reduces to

$$\dot{\boldsymbol{\sigma}} = (1 - D)\, \frac{\partial^2 \Psi_o}{\partial \boldsymbol{\epsilon}_{ed} \otimes \partial \boldsymbol{\epsilon}_{ed}} : \dot{\boldsymbol{\epsilon}} - \dot{D}\, \boldsymbol{\sigma}_{eff} \tag{3.197}$$

From the damage rule and the consistency condition, $\dot{D} = \bar{\mathcal{M}}\, \dot{r} = \bar{\mathcal{M}}\, \dot{\Psi}_o = \bar{\mathcal{M}}\, \boldsymbol{\sigma}_{eff} : \dot{\boldsymbol{\epsilon}}$. Thus the last term in the stress rate expression above reads

$$\boldsymbol{\sigma}_{eff}\, \dot{D} = \boldsymbol{\sigma}_{eff}\, \bar{\mathcal{M}}\, \boldsymbol{\sigma}_{eff} : \dot{\boldsymbol{\epsilon}} = \bar{\mathcal{M}}\, (\boldsymbol{\sigma}_{eff} \otimes \boldsymbol{\sigma}_{eff}) : \dot{\boldsymbol{\epsilon}} \tag{3.198}$$

Collecting $\dot{\boldsymbol{\epsilon}}$ terms, we obtain the expression of the tangent operator as

$$\boxed{\mathbf{E}_{ed} = (1 - D)\, \frac{\partial^2 \Psi_o}{\partial \boldsymbol{\epsilon}_{ed} \otimes \partial \boldsymbol{\epsilon}_{ed}} - \bar{\mathcal{M}}\;\; \boldsymbol{\sigma}_{eff} \otimes \boldsymbol{\sigma}_{eff}} \tag{3.199}$$

The same expression can be derived directly in the framework of the elastic-fracturing theory after definition of the variables in the tangent operator for the strain-based formulation (3.152).

$$\begin{aligned}
\bar{\mathbf{n}} &= \frac{\partial F}{\partial \boldsymbol{\epsilon}} = \frac{\partial F}{\partial \Psi_o} \frac{\partial \Psi_o}{\partial \boldsymbol{\epsilon}_{ed}} = \boldsymbol{\sigma}_{eff} \\
\bar{\mathbf{m}} &= \bar{\mathcal{M}} \star \frac{\partial \mathbf{E}}{\partial \bar{D}} : \boldsymbol{\epsilon} = -\bar{\mathcal{M}} \, \mathbf{E}_o : \boldsymbol{\epsilon} = -\bar{\mathcal{M}} \, \boldsymbol{\sigma}_{eff} \\
\bar{H} &= -\frac{\partial F}{\partial \lambda} = -\frac{\partial F}{\partial r} \frac{\partial r}{\partial \lambda} = -(-1)\, 1 = 1 \\
\mathbf{E} &= (1-D) \, \frac{\partial^2 \Psi_o}{\partial \boldsymbol{\epsilon}_{ed} \otimes \partial \boldsymbol{\epsilon}_{ed}}
\end{aligned} \tag{3.200}$$

And after substitution

$$\mathbf{E}_{ef} = \mathbf{E} + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} = (1-D) \, \frac{\partial^2 \Psi_o}{\partial \boldsymbol{\epsilon}_{ed} \otimes \partial \boldsymbol{\epsilon}_{ed}} - \bar{\mathcal{M}} \;\; \boldsymbol{\sigma}_{eff} \otimes \boldsymbol{\sigma}_{eff} \tag{3.201}$$

As expected the tangent stiffness is fully symmetric, because the damage function is defined by a term representing the elastic free energy.

### 3.5.5 Benallal, Billardon and Geymonat [1989]

Benallal, Billardon and Geymonat [5] presented a general constitutive description for rate independent materials with stiffness degradation for both elastic and plastic-damage. The reversible behavior is described by means of a free energy potential $\Psi(\boldsymbol{\epsilon}, \bar{\boldsymbol{\alpha}})$ function of the strain tensor and of a set of internal variables $\bar{\boldsymbol{\alpha}}$ of various tensorial nature.

The elastic-damage behavior of the model can be readily recast into the framework of the strain-based description for elastic-fracturing materials. In this case the internal variables $\bar{\boldsymbol{\alpha}}$ are the damage variables $\bar{\mathcal{D}}$.

The thermodynamic forces associated with the strains and the internal variables are defined as

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}} \qquad\qquad \bar{\mathcal{A}} = -\frac{\partial \Psi}{\partial \bar{\boldsymbol{\alpha}}} \tag{3.202}$$

Note that according to our definition, the thermodynamic force is associated to the damage variable $\bar{\mathcal{D}}$, when $\bar{\boldsymbol{\alpha}} = \bar{\mathcal{D}}$, $\bar{\mathcal{A}} = -\bar{\mathcal{Y}}$. The second differentiation for $\bar{\boldsymbol{\alpha}} = const$, leads to the secant stiffness as

$$\mathbf{E} = \left.\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\epsilon}}\right|_{\bar{\boldsymbol{\alpha}}} = \frac{\partial^2 \Psi}{\partial \boldsymbol{\epsilon} \otimes \partial \boldsymbol{\epsilon}} \tag{3.203}$$

The domain of reversibility is now defined in the space of the two conjugate variables $\bar{\boldsymbol{\alpha}}$, $\bar{\mathcal{A}}$, and the damage condition is given as

$$F(\boldsymbol{\epsilon}, \bar{\boldsymbol{\alpha}}) = f(\bar{\mathcal{A}}, \bar{\boldsymbol{\alpha}}) = 0 \tag{3.204}$$

The evolution of the internal variables is determined by a potential $g(\bar{\mathcal{A}}, \bar{\boldsymbol{\alpha}})$ such that

$$\dot{\bar{\boldsymbol{\alpha}}} = \dot{\lambda}\, \frac{\partial g}{\partial \bar{\mathcal{A}}} \tag{3.205}$$

In our terminology, for elastic-damage materials, this infers $\dot{\bar{\boldsymbol{\alpha}}} = \dot{\bar{\mathcal{D}}}$ and $\bar{\mathcal{M}} = \dfrac{\partial g}{\partial \bar{\mathcal{A}}}$.

The expression of the tangent stiffness can be derived directly from the relation

$$\mathbf{E}_{ef} = \mathbf{E} + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} \tag{3.206}$$

Let us consider first the expression of the hardening parameter $\bar{H}$. Using the chain rule for differentiating $f(\bar{\mathcal{A}}(\boldsymbol{\epsilon}, \bar{\boldsymbol{\alpha}}), \bar{\boldsymbol{\alpha}})$

$$\bar{H} = -\frac{\partial F}{\partial \lambda} = -\frac{\partial f}{\partial \bar{\mathcal{A}}} \star \frac{\partial \bar{\mathcal{A}}}{\partial \bar{\boldsymbol{\alpha}}} \star \frac{\partial \bar{\boldsymbol{\alpha}}}{\partial \lambda} - \frac{\partial f}{\partial \bar{\boldsymbol{\alpha}}} \star \frac{\partial \bar{\boldsymbol{\alpha}}}{\partial \lambda} \tag{3.207}$$

Now, following the notation in the paper [5], the tensor variable $\Pi$ is defined as

$$\Pi = \frac{\partial^2 \Psi}{\partial \bar{\boldsymbol{\alpha}} \otimes \partial \bar{\boldsymbol{\alpha}}} = -\frac{\partial \bar{\mathcal{A}}}{\partial \bar{\boldsymbol{\alpha}}} \tag{3.208}$$

where the same definition of $\bar{\mathcal{A}}$ has been used. Furthermore, from the assumed damage rule (3.205), we can obtain the identity

$$\frac{\partial \bar{\boldsymbol{\alpha}}}{\partial \lambda} = \frac{\partial g}{\partial \bar{\mathcal{A}}} \tag{3.209}$$

Thus the expression of the hardening parameter $\bar{H} = \bar{h} > 0$ reads

$$\bar{h} = \frac{\partial f}{\partial \bar{\mathcal{A}}} \star \Pi \star \frac{\partial g}{\partial \bar{\mathcal{A}}} - \frac{\partial f}{\partial \bar{\boldsymbol{\alpha}}} \star \frac{\partial g}{\partial \bar{\mathcal{A}}} \tag{3.210}$$

The gradient to the damage function is found as

$$\bar{\mathbf{n}} = \frac{\partial F}{\partial \boldsymbol{\epsilon}} = \frac{\partial f}{\partial \bar{\mathcal{A}}} \star \frac{\partial \bar{\mathcal{A}}}{\partial \boldsymbol{\epsilon}} = \frac{\partial f}{\partial \bar{\mathcal{A}}} \star \Lambda \tag{3.211}$$

where the second tensor quantity $\Lambda = \dfrac{\partial \bar{\mathcal{A}}}{\partial \boldsymbol{\epsilon}} = -\dfrac{\partial^2 \Psi}{\partial \bar{\boldsymbol{\alpha}} \otimes \partial \boldsymbol{\epsilon}}$ has been introduced.

The direction of the inelastic stress rate is obtained considering the time derivative of stress in (3.202)

$$\dot{\boldsymbol{\sigma}} = \frac{\partial^2 \Psi}{\partial \boldsymbol{\epsilon} \otimes \partial \boldsymbol{\epsilon}} : \boldsymbol{\epsilon} + \frac{\partial^2 \Psi}{\partial \boldsymbol{\epsilon} \otimes \partial \bar{\boldsymbol{\alpha}}} \star \dot{\bar{\boldsymbol{\alpha}}} = \mathbf{E} : \dot{\boldsymbol{\epsilon}} - \Lambda^{T} \star \frac{\partial g}{\partial \bar{\mathcal{A}}}\, \dot{\lambda} \tag{3.212}$$

Now identifying the second contribution as the inelastic stress rate $\boldsymbol{\sigma}_i = \dot{\lambda}\; \bar{\mathbf{m}}$, we obtain

$$\bar{\mathbf{m}} = -\Lambda^{T} \star \frac{\partial g}{\partial \bar{\mathcal{A}}} \tag{3.213}$$

Finally the tangent stiffness reads

$$\boxed{\mathbf{E}_{ef} = \mathbf{E} + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} = \mathbf{E} - \frac{\left(\Lambda^{T} \star \dfrac{\partial g}{\partial \bar{\mathcal{A}}}\right) \otimes \left(\dfrac{\partial f}{\partial \bar{\mathcal{A}}} \star \Lambda\right)}{\bar{h}}} \tag{3.214}$$

Note that the model is in general non-associated if $f \neq g$. In that case the tangent operator is no longer symmetric. If $f = g$ the model is associated at the damage level and consequently at the $4^{th}$ order and $2^{nd}$ order levels.

### Scalar elastic damage

The simple example of a scalar damage variable is considered in the paper [5] as an example. The free energy is prescribed in the usual form

$$\Psi(\boldsymbol{\epsilon}, D) = \frac{1}{2}\,(1 - D)\,\boldsymbol{\epsilon} : \mathbf{E}_0 : \boldsymbol{\epsilon} \tag{3.215}$$

and an associated model is considered with the damage condition

$$f(\bar{\mathcal{Y}}, D) = g(\bar{\mathcal{Y}}, D) = (-\bar{\mathcal{Y}}) - r(D) = 0 \tag{3.216}$$

with

$$\begin{cases} (-\bar{\mathcal{Y}}) = -\dfrac{\partial \Psi}{\partial D} = \dfrac{1}{2}\, \boldsymbol{\epsilon} : \mathbf{E}_0 : \boldsymbol{\epsilon} \\ \\ r(D) = (-\bar{\mathcal{Y}}_o) + \bar{h}\, D \end{cases} \tag{3.217}$$

Let us now evaluate all the quantities necessary for the definition of the tangent stiffness (3.214). The hardening parameter is obtained as follows

$$\bar{H} = -\frac{\partial F}{\partial \lambda} = -\frac{\partial f}{\partial r}\,\frac{\partial r}{\partial D}\,\frac{\partial D}{\partial \lambda} = -\,(-1)\;\bar{h}\;1 = \bar{h} \tag{3.218}$$

The generalized gradient $\dfrac{\partial f}{\partial \bar{\mathcal{A}}}$ and the damage direction $\dfrac{\partial g}{\partial \bar{\mathcal{A}}}$ are identical and are $\dfrac{\partial f}{\partial(-\bar{\mathcal{Y}})} = 1$. The remaining tensor variable $\Lambda$ is symmetric and assumes the meaning of undamaged stress, as

$$\Lambda = -\frac{\partial^2 f}{\partial \bar{\boldsymbol{\alpha}} \otimes \partial \boldsymbol{\epsilon}} = -\frac{\partial^2 f}{\partial D \otimes \partial \boldsymbol{\epsilon}} = \mathbf{E}_o : \boldsymbol{\epsilon} = \boldsymbol{\sigma}_o = \boldsymbol{\sigma}_o^T = \Lambda^T \tag{3.219}$$

The secant stiffness is recovered in the usual way as

$$\mathbf{E} = \frac{\partial^2 \Psi}{\partial \boldsymbol{\epsilon} \otimes \partial \boldsymbol{\epsilon}} = (1 - D)\; \mathbf{E}_o \tag{3.220}$$

Finally, the familiar expression of a symmetric tangent stiffness is obtained in the form

$$\boxed{\mathbf{E}_{ed} = (1 - D)\; \mathbf{E}_o - \frac{\boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o}{\bar{h}}} \tag{3.221}$$

### 3.5.6 Neilsen and Schreyer [1992]

As the last example Neilsen and Schreyer [53] proposed recently a general elastic damage model that can be described as a special case of the stress-based elastic-fracturing theory. In this work thermodynamics concepts are not introduced and the starting point is the secant relation typical for elastic-fracturing and elastic-damage formulations: $\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon}$.

The main assumption is the existence of a damage threshold surface in the $\boldsymbol{\sigma}$-space defined as

$$\boxed{F(\boldsymbol{\sigma}, \mathbf{E}) = \frac{1}{2}\, \boldsymbol{\sigma} : \mathbf{P} : \boldsymbol{\sigma} - r(\mathbf{E}) = 0} \tag{3.222}$$

where the tensor $\mathbf{P}$ is a general symmetric, constant, positive-definite, $4^{th}$ order tensor. Note that for the particular choice $\mathbf{P} = \mathbf{M}$, where $\mathbf{M}$ defines the compliance fracturing rule in our terminology, the model is associated.

The second assumption is the evolution equation for the stiffness, defined by a $4^{th}$ order tensor $\mathbf{R}$, that is proportional to the tensor $\bar{\mathbf{M}}$ in our notation:

$$\dot{\mathbf{E}} = \dot{\lambda}\,(-\mathbf{R}) \qquad \Rightarrow \qquad \bar{\mathbf{M}} = -\mathbf{R} \tag{3.223}$$

This specifies the fracturing rule for the compliance. In fact, using the relations in Table 3.1, one can obtain

$$\mathbf{M} = -\mathbf{C} : \bar{\mathbf{M}} : \mathbf{C} = \mathbf{C} : \mathbf{R} : \mathbf{C} \tag{3.224}$$

The expression of the tangent stiffness operator is readily found. Let us evaluate all the necessary ingredients.

The hardening parameter $H$ is determined as

$$H = -\frac{\partial F}{\partial \lambda} = -\frac{\partial F}{\partial \mathbf{E}} :: \frac{\partial \mathbf{E}}{\partial \lambda} = \frac{\partial F}{\partial \mathbf{E}} :: \mathbf{R} \tag{3.225}$$

The gradient $\mathbf{n}$ and the inelastic strain flow direction $\mathbf{m}$ are

$$\begin{aligned} \mathbf{n} &= \frac{\partial F}{\partial \boldsymbol{\sigma}} = \mathbf{P} : \boldsymbol{\sigma} \\ \mathbf{m} &= \mathbf{M} : \boldsymbol{\sigma} = \mathbf{C} : \mathbf{R} : \mathbf{C} : \boldsymbol{\sigma} = \mathbf{C} : \mathbf{R} : \boldsymbol{\epsilon} \end{aligned} \tag{3.226}$$

The contraction of $\mathbf{E}$ with $\mathbf{m}$ gives $\mathbf{E} : \mathbf{m} = \mathbf{R} : \boldsymbol{\epsilon}$ and the tangent stiffness is obtained as follows

$$\boxed{\mathbf{E}_{ef} = \mathbf{E} - \frac{\mathbf{E} : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}} = \mathbf{E} - \frac{\mathbf{R} : \boldsymbol{\epsilon} \otimes \mathbf{n} : \mathbf{E}}{H + \mathbf{n} : \mathbf{R} : \boldsymbol{\epsilon}}} \tag{3.227}$$

which compares with equation (25) in the paper [53].

The authors note that the operator is symmetric when $\mathbf{R}$ is chosen such that $\mathbf{R} : \boldsymbol{\epsilon}$ is parallel to $\mathbf{n} : \mathbf{E}$ and they propose

$$\mathbf{R} = \mathbf{E} : \mathbf{P} : \mathbf{E} \tag{3.228}$$

which verifies the previous requirement of associativity. In fact, solving the previous relation with respect to $\mathbf{P}$

$$\mathbf{P} = \mathbf{C} : \mathbf{R} : \mathbf{C} = \mathbf{M} \tag{3.229}$$

This means that the damage model is associated at the $4^{th}$ order tensor level because

$$\mathbf{N} = \frac{\partial F}{\partial \left( \dfrac{1}{2}\, \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \right)} = \mathbf{P} = \mathbf{M} \tag{3.230}$$

Of course this implies also associativity at the $2^{nd}$ order level. In fact

$$\mathbf{m} = \mathbf{M} : \boldsymbol{\sigma} = \mathbf{P} : \boldsymbol{\sigma} = \mathbf{n} \tag{3.231}$$

**Scalar elastic damage**

A simple scalar damage version of the model is considered as an example. Assuming for the secant stiffness and the tensor operator $\mathbf{P}$

$$\mathbf{E} = (1 - D)\, \mathbf{E}_o \,; \qquad\qquad \mathbf{P} = \mathbf{C}_o \tag{3.232}$$

then the model is associated because

$$\bar{\mathbf{M}} = \mathbf{R} = \mathbf{E}_o \qquad \Rightarrow \qquad \mathbf{M} = \mathbf{C} : \mathbf{E}_o : \mathbf{C} = \frac{\mathbf{C}_o}{(1-D)^2} \quad \| \quad \mathbf{N} = \mathbf{P} = \mathbf{C}_o \tag{3.233}$$

The damage condition takes the form

$$F = \frac{1}{2}\, \boldsymbol{\sigma} : \mathbf{C}_o : \boldsymbol{\sigma} - (r_o + H\ D) = 0 \tag{3.234}$$

Hence

$$\begin{aligned} \mathbf{R} : \boldsymbol{\epsilon} &= \mathbf{E}_o : \boldsymbol{\epsilon} = \frac{\mathbf{E} : \boldsymbol{\epsilon}}{(1-D)} = \frac{\boldsymbol{\sigma}}{(1-D)} \\ \mathbf{n} : \mathbf{E} &= \boldsymbol{\sigma} : \mathbf{C}_o : (1-D)\, \mathbf{E}_o = (1-D)\, \boldsymbol{\sigma} \end{aligned} \tag{3.235}$$

such that the numerator of the rank-one modification of the secant stiffness reads

$$\mathbf{R} : \boldsymbol{\epsilon} \otimes \mathbf{n} : \mathbf{E} = \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \tag{3.236}$$

The triple product in the denominator yields $\mathbf{n} : \mathbf{R} : \boldsymbol{\epsilon} = \boldsymbol{\sigma} : \mathbf{C}_o : \mathbf{E}_o : \boldsymbol{\epsilon} = \boldsymbol{\sigma} : \boldsymbol{\epsilon}$ and the final expression of the tangent operator reduces to the simple format

$$\boxed{\mathbf{E}_{ed} = (1-D)\, \mathbf{E}_o - \frac{\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}}{H + \boldsymbol{\epsilon} : \boldsymbol{\sigma}}} \tag{3.237}$$

which corresponds to equation (35) in the paper [53].

**Scalar von Mises damage**

An interesting second proposal, that we already discussed briefly in the previous general discussion of scalar damage formulations, is recovered when the tensor $\mathbf{P}$ is the deviatoric projection operator $\mathbf{P}_d = \mathbf{I}_4^s - \mathbf{P}_v = \mathbf{I}_4^s - \frac{1}{3}\, \mathbf{I}_2 \otimes \mathbf{I}_2$. The loading function is now of the von Mises type and only the shear modulus is modified during damage, i.e. $G = (1-D)\, G_o$.

The model is associated if

$$\mathbf{R} = \mathbf{E} : \mathbf{P}_d : \mathbf{E} = 4\, G^2\, \mathbf{P}_d \tag{3.238}$$

where the last identity is obtained after recalling that $\mathbf{E}$ can be rewritten as

$$\mathbf{E} = 3\, K_o\, \mathbf{P}_v + 2\, G\, \mathbf{P}_d \tag{3.239}$$

and using the properties $\mathbf{P}_v : \mathbf{P}_d = \mathbf{0}$ and $\mathbf{P}_d : \mathbf{P}_d = \mathbf{P}_d$.

The gradient to the damage function in the stress space is now the deviatoric stress, which is the usual case for von Mises type yield functions: $\mathbf{n} = \mathbf{P}_d : \boldsymbol{\sigma} = \boldsymbol{\sigma}_d$.

Recalling that in this case the volumetric and deviatoric responses completely decouple, the deviatoric strain response reduces to the "Henchy"-format of the deformation theory of plasticity

$$\boldsymbol{\epsilon}_d = \frac{\boldsymbol{\sigma}_d}{2\, G} \tag{3.240}$$

The corresponding tangent operator can be readily computed using the following terms

$$\begin{aligned}
\mathbf{R} : \boldsymbol{\epsilon} \otimes \mathbf{n} : \mathbf{E} &= 4\,G^2\,\mathbf{P}_d : \boldsymbol{\epsilon} \otimes \boldsymbol{\sigma}_d : \mathbf{E} \\
&= 4\,G^2\,\frac{\boldsymbol{\sigma}_d}{2\,G} \otimes 2\,G\,\boldsymbol{\sigma}_d \\
&= 4\,G^2\;\;\boldsymbol{\sigma}_d \otimes \boldsymbol{\sigma}_d \\
\mathbf{n} : \mathbf{R} : \boldsymbol{\epsilon} &= \boldsymbol{\sigma}_d : 4\,G^2\,\mathbf{P}_d : \boldsymbol{\epsilon} \\
&= 4\,G^2\;\;\boldsymbol{\sigma}_d : \frac{\boldsymbol{\sigma}_d}{2\,G} \\
&= 2\,G\;\;\boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d
\end{aligned} \tag{3.241}$$

The tangential material stiffness is finally recovered as follows

$$\boxed{\mathbf{E}_{ed} = \mathbf{E} - 4\,G^2\;\frac{\boldsymbol{\sigma}_d \otimes \boldsymbol{\sigma}_d}{H + 2\,G\;\boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d}} \tag{3.242}$$

which corresponds to the tangent formulation of the von Mises flow type of elastic-plasticity.

### 3.5.7 Summary of the scalar damage formulations

The expressions of the tangent stiffness operator for the reviewed scalar damage models are summarized in the following Table 3.4.

It appears that all the associated formulations lead to the same structure of the rank-one modification, involving a dyadic product of the stress tensor or of the undamaged stress tensor $\boldsymbol{\sigma}_o = \mathbf{E}_o : \boldsymbol{\epsilon}$. The second format is typical of the strain-based formulations, where $\boldsymbol{\epsilon}$ is the governing variable, and the dyadic product really means $\mathbf{E}_o : \boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon} : \mathbf{E}_o$.

Considering that the two stress tensors are related as $\boldsymbol{\sigma} = (1-D) : \boldsymbol{\sigma}_o$ it appears possible to think to a unique structure of the operator. On the other hand, if one wants to keep the original variable of the description, the following distinction can be maintained.

$$\begin{aligned} &\textit{Strain-based} \quad \Rightarrow \quad \mathbf{E}_{ed} = (1-D)\,\mathbf{E}_o - \beta(\boldsymbol{\epsilon}, D)\;\;\boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o \\ &\textit{Stress-based} \quad \Rightarrow \quad \mathbf{E}_{ed} = (1-D)\,\mathbf{E}_o - \gamma(\boldsymbol{\sigma}, D)\;\;\boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \end{aligned} \tag{3.243}$$

where $\boldsymbol{\sigma}_o$ is intended as above.

Table 3.4: Tangent operator for traditional scalar damage models: $\alpha = (1 - D)$

| **TRADITIONAL SCALAR DAMAGE FORMULATIONS** | | |
|---|---|---|
| *STRAIN-BASED FORMULATIONS* | | |
| Author | Tangent Operator | Variables |
| MAZARS AND LEMAITRE | $\mathbf{E}_{ed} = \alpha\,\mathbf{E}_o - \dfrac{\bar{\mathcal{M}}}{\bar{\epsilon}}\;\boldsymbol{\sigma}_o \otimes \boldsymbol{\epsilon}$ | $\bar{\epsilon} = \sqrt{\boldsymbol{\epsilon} : \boldsymbol{\epsilon}}$ <br> $\bar{\mathcal{M}} = \bar{\mathcal{M}}(\bar{\epsilon})$ |
| SIMO AND JU | $\mathbf{E}_{ed} = \alpha\,\mathbf{E}_o - \dfrac{\bar{\mathcal{M}}}{\bar{\tau}}\;\boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o$ | $\bar{\tau} = \sqrt{\boldsymbol{\epsilon} : \mathbf{E}_o : \boldsymbol{\epsilon}}$ <br> $\bar{\mathcal{M}} = \bar{\mathcal{M}}(\bar{\tau}, D)$ |
| JU | $\mathbf{E}_{ed} = \alpha\,\mathbf{E}_o - \bar{\mathcal{M}}\;\boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o$ | $\bar{\tau} = \dfrac{1}{2}\,\boldsymbol{\epsilon} : \mathbf{E}_o : \boldsymbol{\epsilon}$ <br> $\bar{\mathcal{M}} = \bar{\mathcal{M}}(\bar{\tau}, D)$ |
| BENALLAL ET AL. | $\mathbf{E}_{ed} = \alpha\,\mathbf{E}_o - \dfrac{1}{\bar{h}}\;\boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o$ | $\boldsymbol{\sigma}_o = \mathbf{E}_o : \boldsymbol{\epsilon}$ |
| *STRESS-BASED FORMULATIONS* | | |
| Author | Tangent Operator | Variables |
| SIMO AND JU | $\mathbf{E}_{ed} = \alpha\,\mathbf{E}_o - \dfrac{\alpha^2\,\mathcal{M}}{\bar{\tau}\;(h + \alpha\,\mathcal{M}\,\bar{\tau})}\;\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}$ | $\bar{\tau} = \sqrt{\boldsymbol{\sigma} : \mathbf{C}_o : \boldsymbol{\sigma}}$ <br> $\mathcal{M} = \mathcal{M}(\bar{\tau}, D)$ |
| NEILSEN AND SCHREYER | $\mathbf{E}_{ed} = \alpha\,\mathbf{E}_o - \dfrac{1}{H + \boldsymbol{\epsilon} : \boldsymbol{\sigma}}\;\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}$ | |

# CHAPTER 4

# FAILURE INDICATORS FOR ELASTIC - FRACTURING AND DAMAGE MODELS

The main object of this thesis is to extend the classical localization analysis for elastic-plastic solids to materials with stiffness degradation. However, for localization analysis of degrading materials an expression of the tangent operators has to be established.

In the previous chapter this general expression of the tangent stiffness was developed in the framework of elastic-fracturing materials. These material descriptions are characterized by a secant constitutive relation, a fracturing surface and a fracturing rule defining the rate of change of the stiffness of compliance.

Different damage models can be interpreted as fracturing models when a reduced set of variables is provided for the definition of the actual secant stiffness or compliance evolution. A number of traditional scalar damage models was reviewed at the end of the previous chapter and a general format of the tangent material stiffness was presented.

The next step is to investigate the failure indicators connected with these operators. The diffuse failure condition, discussed in chapter 2 is associated with loss of positive definiteness of the tangent operator and indicates bifurcation without formation of spatial discontinuities. Discontinuous failure is checked with the analysis of the spectral properties of the acoustic elastic-fracturing or elastic-damage tensor.

In the following sections the diffuse failure condition is analyzed for the general formulation of elastic-fracturing models, for which the framework of the discontinuous bifurcation is also considered. Furthermore, a geometric interpretation of the localization condition, in terms of Mohr's circle coordinates, which appeals because of its engineering meaning, is considered.

Finally, scalar damage models in the traditional sense are analyzed and analytical solutions for the critical localization directions are provided for the strain and stress based formulation, in the $3D$ and $2D$ cases.

## 4.1 Diffuse failure for elastic-fracturing materials

The limit point condition for the elastic-fracturing tangential stiffness is straight forward, once the results from elastic-plasticity are known. In fact, the structure of the tangent operator is the same as in elastic-plasticity, with the only difference that the elastic secant stiffness $\mathbf{E}$ appears, instead of the initial elastic stiffness $\mathbf{E}_o$.

For a stress-based elastic-fracturing formulation the tangent material stiffness reads

$$\mathbf{E}_{ef} = \mathbf{E} - \frac{\mathbf{E} : \mathbf{m} \otimes \mathbf{n} : \mathbf{E}}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}} \tag{4.1}$$

The limit point condition is achieved for the perfect elastic-fracturing case with a critical eigentensor which is coaxial with the direction of flow of the inelastic strains. Let us suppose an eigentensor for $\mathbf{E}_{ef}$ of the form $\mathbf{x} = \alpha \mathbf{m}$, with $\alpha = const$, then substitution into the eigenvalue problem for $\mathbf{E}_{ef}$ gives

$$\mathbf{E}_{ef} : \mathbf{x} = \alpha \left( \mathbf{E} : \mathbf{m} - \frac{\mathbf{E} : \mathbf{m} \, (\mathbf{n} : \mathbf{E} : \mathbf{m})}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}} \right) = \eta \, \alpha \, \mathbf{m} \tag{4.2}$$

Imposing the condition that the relative eigenvalue $\eta$ vanishes, we obtain

$$\frac{H + \mathbf{n} : \mathbf{E} : \mathbf{m} - \mathbf{n} : \mathbf{E} : \mathbf{m}}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}} \; (\mathbf{E} : \mathbf{m}) = \frac{H}{H + \mathbf{n} : \mathbf{E} : \mathbf{m}} \; (\mathbf{E} : \mathbf{m}) = \mathbf{0} \tag{4.3}$$

This means that $H = 0$ as long as the denominator $H + \mathbf{n} : \mathbf{E} : \mathbf{m}$ remains positive and $\mathbf{E}$ remains non singular, i.e. $\mathbf{E} \neq \mathbf{0}$.

We conclude that diffuse failure for associated as well as non-associated elastic-fracturing stress-based models occurs if

$$\boxed{det(\mathbf{E}_{ef}) = 0 \quad \Rightarrow \quad H = 0 \quad \text{with} \quad \dot{\boldsymbol{\epsilon}}_{cr} = \alpha\, \mathbf{m}} \tag{4.4}$$

In the dual strain-based formulation similar results can be obtained. In this case the tangent stiffness is expressed by the simpler form

$$\mathbf{E}_{ef} = \mathbf{E} + \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}}}{\bar{H}} \tag{4.5}$$

Recalling the relation between the direction of the inelastic strain rate tensor $\mathbf{m}$ and the stress rate $\bar{\mathbf{m}}$ and the relation between the two hardening parameters

$$\mathbf{m} = -\mathbf{C} : \bar{\mathbf{m}} \qquad\qquad H = \bar{H} + \bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}} \tag{4.6}$$

we find a critical eigentensor of $\mathbf{E}_{ef}$ of the type $\mathbf{x} = \alpha\, \mathbf{C} : \bar{\mathbf{m}}$ and a critical hardening parameter $\bar{H} = -\bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}}$. In fact, substitution of the eigentensor into the eigenvalue problem for $\mathbf{E}_{ef}$

$$\mathbf{E}_{ef} : \mathbf{x} = \alpha \left( \mathbf{E} : \mathbf{C} : \bar{\mathbf{m}} + \frac{\bar{\mathbf{m}}\ (\bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}})}{\bar{H}} \right) = \eta\, \mathbf{x} \tag{4.7}$$

gives, for $\eta = 0$

$$\frac{\bar{H} + \bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}}}{\bar{H}}\ \mathbf{m} = \mathbf{0} \tag{4.8}$$

which recovers the critical value of hardening. In conclusion, the diffuse failure condition for strain-based elastic-fracturing models can be summarized as

$$\boxed{det(\mathbf{E}_{ef}) = 0 \quad \Rightarrow \quad \bar{H} = -\bar{\mathbf{n}} : \mathbf{C} : \bar{\mathbf{m}} \quad \text{with} \quad \dot{\boldsymbol{\epsilon}}_{cr} = \alpha\, \mathbf{C} : \bar{\mathbf{m}}} \tag{4.9}$$

### 4.1.1 Traditional scalar damage model. Plane stress

The general result for the elastic-fracturing theory can be extended to the case of damage models. For example let us consider the scalar damage model proposed by NEILSEN AND SCHREYER [53]. The scalar damage variable $D$ is introduced in the traditional way and affects all the components of the initial stiffness in the same manner

$$\mathbf{E} = (1 - D)\ \mathbf{E}_o \tag{4.10}$$

The damage model is considered associated and the damage surface is defined in terms of the undamaged elastic free energy, such that

$$F = \frac{1}{2}\ \boldsymbol{\sigma} : \mathbf{C}_o : \boldsymbol{\sigma} - r = 0 \tag{4.11}$$

where $r$ defines the limit of the elastic domain. The expression of the operator, for the general $3D$ case, reads

$$\mathbf{E}_{ed} = (1 - D)\ \mathbf{E}_o - \frac{1}{H + \boldsymbol{\sigma} : \mathbf{C} : \boldsymbol{\sigma}}\ \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \tag{4.12}$$

For explicitness, let us analyze the case *plane stress*. The tangent operator can be written in the $3 \times 3$ matrix format, that is useful for implementation in a FE computer code:

$$\mathbf{E}_{ed} = \frac{(1-D)E_o}{1-\nu_o^2} \begin{bmatrix} 1 & \nu_o & 0 \\ \nu_o & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu_o}{2} \end{bmatrix} - \frac{1}{H + \dfrac{2\,r}{1-D}} \begin{bmatrix} \sigma_{11}\sigma_{11} & \sigma_{11}\sigma_{22} & \sigma_{11}\sigma_{12} \\ \sigma_{22}\sigma_{11} & \sigma_{22}\sigma_{22} & \sigma_{22}\sigma_{12} \\ \sigma_{12}\sigma_{11} & \sigma_{12}\sigma_{22} & \sigma_{12}\sigma_{12} \end{bmatrix} \tag{4.13}$$

where $\boldsymbol{\sigma}^T = \{\sigma_{11}\ \ \sigma_{22}\ \ \sigma_{12}\}$ is a state of stress on the loading surface such that $\boldsymbol{\sigma}^T \cdot \mathbf{C} \cdot \boldsymbol{\sigma} = 2r$ (in matrix notation). In this case Poisson's ratio remains constant and equals its initial value $\nu_o$; Young's modulus however decreases linearly with damage growth.

The normalized determinant of the elastic-damage tangent stiffness reads

$$\frac{det(\mathbf{E}_{ed})}{det(\mathbf{E}_o)} = \frac{(1-D)^4\ H}{(1-D)\ H + 2\ r} \tag{4.14}$$

and it is plotted in fig. 4.1 as a function of the hardening parameter $H$ and the damage variable D. There are two singularities, one for $H = 0$ and one for $D = 1$, where the secant stiffness vanishes entirely.

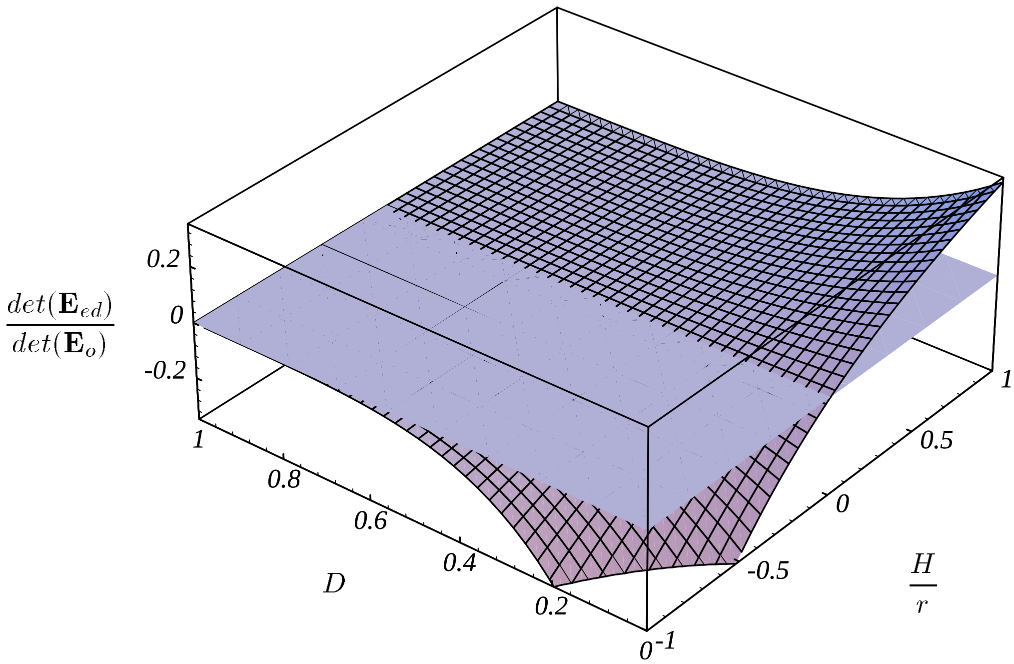



Figure 4.1: Associated scalar damage model: singularity of $\mathbf{E}_{ed}$ for $H = 0$

## 4.2 Discontinuous failure for elastic-fracturing materials

With the expression for the tangent elastic-fracturing material stiffness it is possible to apply localization analysis to this kind of material description. The acoustic tensor can be formed via contraction of the tangent operator with the orientation $\vec{\mathbf{N}}$ of the discontinuity surface

$$\mathbf{Q}_{ef} = \vec{\mathbf{N}} \cdot \mathbf{E}_{ef} \cdot \vec{\mathbf{N}} \tag{4.15}$$

Using the expressions of the operators for the stress-based and strain-based versions of the elastic-fracturing theory, the following general format of the acoustic tensor can be obtained

$$\mathbf{Q}_{ef} = \mathbf{Q}_e - \frac{\mathbf{b} \otimes \mathbf{a}}{h} \tag{4.16}$$

where $\mathbf{Q}_e = \vec{\mathbf{N}} \cdot \mathbf{E} \cdot \vec{\mathbf{N}}$ is the elastic acoustic tensor formed with the current secant stiffness and:

$$Stress\text{-}based \begin{cases} h = H + \mathbf{n} : \mathbf{E} : \mathbf{m} \\ \mathbf{a} = \mathbf{n} : \mathbf{E} \cdot \vec{\mathbf{N}} \\ \mathbf{b} = \vec{\mathbf{N}} \cdot \mathbf{E} : \mathbf{m} \end{cases} \qquad Strain\text{-}based \begin{cases} h = \bar{H} \\ \mathbf{a} = \bar{\mathbf{n}} \cdot \vec{\mathbf{N}} \\ \mathbf{b} = -\vec{\mathbf{N}} \cdot \bar{\mathbf{m}} \end{cases} \tag{4.17}$$

The onset of the fracturing/fracturing bifurcation is obtained when the lowest eigenvalues of $\mathbf{Q}_{ef}$ diminishes to zero and the localization operator becomes singular

$$det(\mathbf{Q}_{ef}) = 0 \tag{4.18}$$

This condition can be evaluated considering the generalized eigenvalue problem for $\mathbf{Q}_{ef}$, similarly to what has been done for the elastic-plastic case

$$\mathbf{Q}_{ef} \cdot \mathbf{x} = \eta^* \, \mathbf{Q}_e \cdot \mathbf{x} \quad \Leftrightarrow \quad (\mathbf{Q}_e^{-1} \cdot \mathbf{Q}_{ef}) \cdot \mathbf{x} = \eta^* \, \mathbf{x} \tag{4.19}$$

where $\eta^*$ are the eigenvalues of the matrix

$$\mathbf{B} = \mathbf{Q}_e^{-1} \cdot \mathbf{Q}_{ef} = \mathbf{I}_2 - \frac{1}{h} \, \mathbf{Q}_e^{-1} \cdot (\mathbf{b} \otimes \mathbf{a}) \tag{4.20}$$

Two eigenvalues of $\mathbf{B}$ coincide and equal 1, i.e. $\eta_1^* = \eta_2^* = 1$. This can be checked substituting $\eta^* = 1$ into the eigenvalue problem for $\mathbf{B}$, that result in a homogeneous linear system for the singular matrix

$$\mathbf{B} - \mathbf{I}_2 = -\frac{1}{h} \, (\mathbf{Q}_e^{-1} \cdot \mathbf{b}) \otimes \mathbf{a} = \mathbf{c} \otimes \mathbf{a} \tag{4.21}$$

The remaining eigenvalue can be extracted from the trace operation of $\mathbf{B}$

$$tr(\mathbf{B}) = 3 - \frac{1}{h} \, \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} = \eta_1^* + \eta_2^* + \eta_3^* = 2 + \eta_3^* \tag{4.22}$$

as

$$\boxed{\eta_3^* = 1 - \frac{1}{h} \, \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b}} \tag{4.23}$$

The determinant of $\mathbf{B}$ represents the normalized determinant of the elastic-fracturing acoustic tensor and the localization condition is recovered when the third eigenvalue of $\mathbf{B}$ vanishes

$$\boxed{det(\mathbf{B}) = \frac{det(\mathbf{Q}_{ef})}{det(\mathbf{Q}_e)} = \eta_1^* \, \eta_2^* \, \eta_3^* = \eta_3^* = 0} \tag{4.24}$$

The localization condition $\eta_3^* = 0$ can be solved with respect to $h$, giving the critical value of $h$ leading to localization for a certain direction $\vec{\mathbf{N}}$, and for a certain stress/strain fracturing state

$$h_{cr} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} \tag{4.25}$$

This can be explicitly expressed with respect to the hardening parameters $H$ or $\bar{H}$, depending on the type of fracturing formulation

$$\begin{aligned} &\textit{Stress-based} \quad \Rightarrow \quad H_{cr} = -\mathbf{n} : \mathbf{E} : \mathbf{m} + \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} \\ &\textit{Strain-based} \quad \Rightarrow \quad \bar{H}_{cr} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} \end{aligned} \tag{4.26}$$

The first onset of localization can be detected for the maximum value of $h_{cr}$ recovered varying $\vec{\mathbf{N}}$ in all the possible directions. The direction corresponding to the maximum value of $h_{cr}$ is the critical localization direction for that stress/strain fracturing state

$$H_{df} = \max_{\vec{\mathbf{N}}}[H_{cr}] \Bigg|_{N_1^2+N_2^2+N_3^2=1} \qquad \bar{H}_{df} = \max_{\vec{\mathbf{N}}}[\bar{H}_{cr}] \Bigg|_{N_1^2+N_2^2+N_3^2=1} \tag{4.27}$$

The maximization problem (4.27) was solved analytically by Ottosen and Runesson [56] for a broad class of non-associated stress-based elastic-plastic isotropic models. This solution readily extends to the strain-based formulation. In fact the maximization problem on $\vec{\mathbf{N}}$ is not affected by the term $-\mathbf{n} : \mathbf{E} : \mathbf{m}$, which is independent of $\vec{\mathbf{N}}$. The critical localization direction will not be affected but only the critical hardening parameter.

In the fracturing rule, the hypothesis for stiffness or compliance evolutions must be inserted. If the fracturing model is interpreted as a damage model, the damage variable is introduced, relating the actual secant stiffness to the initial one.

In the following sections we will show how the elastic-plastic solution can be extended to the case of traditional scalar isotropic damage models, because of the particular structure of the secant stiffness $\mathbf{E}$, which retains the form of $\mathbf{E}_o$.

### 4.2.1 Geometric representation of the localization condition

An interesting representation of the localization condition in the strain or stress coordinates of Mohr, was recently introduced by BENALLAL [6], PIJAUDIER-CABOT AND BENALLAL [60], for a specific stress-based elastic-plastic model and a scalar damage strain-based model.

The localization condition $det(\mathbf{Q}) = 0$ is represented by an ellipse in the Mohr's plane, although the validity of the representation for a complete general case has not been fully investigated.

The idea is to express the localization condition $\eta_3^* = 0$ in terms of the Mohr's components of stress or strain. The traction vector in the generic plane of orientation $\vec{\mathbf{N}}$ is obtained with the single contraction of $\vec{\mathbf{N}}$ with the stress or strain tensor

$$\mathbf{t}_\sigma = \vec{\mathbf{N}} \cdot \boldsymbol{\sigma} \qquad\qquad \mathbf{t}_\epsilon = \vec{\mathbf{N}} \cdot \boldsymbol{\epsilon} \tag{4.28}$$

and the components normal to the plane and tangential to the plane are

$$\begin{aligned} \sigma &= \vec{\mathbf{N}} \cdot \boldsymbol{\sigma} \cdot \vec{\mathbf{N}} \qquad\qquad & \epsilon &= \vec{\mathbf{N}} \cdot \boldsymbol{\epsilon} \cdot \vec{\mathbf{N}} \\ \tau &= \vec{\mathbf{N}} \cdot \boldsymbol{\sigma} \cdot \vec{\mathbf{T}} \qquad\qquad & \gamma &= \vec{\mathbf{N}} \cdot \boldsymbol{\epsilon} \cdot \vec{\mathbf{T}} \end{aligned} \tag{4.29}$$

where $\vec{\mathbf{T}} \perp \vec{\mathbf{N}}$ is a vector in the plane orthogonal to $\vec{\mathbf{N}}$. Now let us consider the general expression of the tangent operator

$$\mathbf{E}_t = \mathbf{E} - \beta \, \mathbf{b}' \otimes \mathbf{a}' \tag{4.30}$$

where $\mathbf{E}$ designates the initial stiffness for the elastic-plastic case and represents the secant stiffness for the elastic-fracturing or elastic-damage case. The scalar quantity $\beta$ might be a function of the hardening parameters, the damage variables and the stress or strain state. The two second order tensors $\mathbf{a}'$, $\mathbf{b}'$ are function of the stress or the strain tensor. For example for the strain-based case

$$\mathbf{a}' = \bar{\mathbf{n}} \qquad\qquad \mathbf{b}' = \bar{\mathbf{m}} \tag{4.31}$$

Now, if isotropy is assumed, the loading function can be expressed in terms of the three invariants of the strain tensor and the gradient reads

$$\begin{aligned} \bar{\mathbf{n}} = \frac{\partial F(I_1, I_2, I_3)}{\partial \boldsymbol{\epsilon}} &= \frac{\partial F}{\partial I_1}\,\frac{\partial I_1}{\partial \boldsymbol{\epsilon}} + \frac{\partial F}{\partial I_2}\,\frac{\partial I_2}{\partial \boldsymbol{\epsilon}} + \frac{\partial F}{\partial I_3}\,\frac{\partial I_3}{\partial \boldsymbol{\epsilon}} \\ &= F_1(\boldsymbol{\epsilon})\,\mathbf{I}_2 + F_2(\boldsymbol{\epsilon})\,\boldsymbol{\epsilon} + F_3(\boldsymbol{\epsilon})\,\boldsymbol{\epsilon}^2 \end{aligned} \tag{4.32}$$

If a potential function for the direction of flow $\bar{\mathbf{m}}$ is also assumed, the same kind of expansion can be derived for $\bar{\mathbf{m}}$. As a result, the two tensors $\mathbf{a}'$ and $\mathbf{b}'$ can be represented in general as

$$\begin{aligned} &\mathbf{a}' = a_1(\boldsymbol{\epsilon})\,\mathbf{I}_2 + a_2(\boldsymbol{\epsilon})\,\boldsymbol{\epsilon} + a_3(\boldsymbol{\epsilon})\,\boldsymbol{\epsilon}^2 \quad \textit{or} \quad \mathbf{a}' = a_1(\boldsymbol{\sigma})\,\mathbf{I}_2 + a_2(\boldsymbol{\sigma})\,\boldsymbol{\sigma} + a_3(\boldsymbol{\sigma})\,\boldsymbol{\sigma}^2 \\ &\mathbf{b}' = b_1(\boldsymbol{\epsilon})\,\mathbf{I}_2 + b_2(\boldsymbol{\epsilon})\,\boldsymbol{\epsilon} + b_3(\boldsymbol{\epsilon})\,\boldsymbol{\epsilon}^2 \quad \textit{or} \quad \mathbf{b}' = b_1(\boldsymbol{\sigma})\,\mathbf{I}_2 + b_2(\boldsymbol{\sigma})\,\boldsymbol{\sigma} + b_3(\boldsymbol{\sigma})\,\boldsymbol{\sigma}^2 \end{aligned} \tag{4.33}$$

Now let us consider the localization conditions (4.23), (4.24) and suppose that $\mathbf{E}$ maintains the same form of the initial elastic isotropic stiffness (e.g. for traditional scalar damage models): the elastic acoustic tensor and its inverse are computed as

$$\mathbf{Q}_e = \mu\mathbf{I}_2 + (\lambda + \mu)\,(\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \quad \Rightarrow \quad \mathbf{Q}_e^{-1} = \frac{1}{\mu}\,\mathbf{I}_2 - \frac{\lambda + \mu}{\mu\,(\lambda + 2\mu)}\,(\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \tag{4.34}$$

where $\lambda$ and $\mu$ are the current values of the Lamé's constants. Considering for example the strain-based formulation, the localization condition can be expressed as

$$\frac{1}{\beta} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} \tag{4.35}$$

where

$$\begin{aligned} \mathbf{a} = \mathbf{a}' \cdot \vec{\mathbf{N}} = a_1(\boldsymbol{\epsilon})\,\vec{\mathbf{N}} + a_2(\boldsymbol{\epsilon})\,\mathbf{t}_\epsilon + a_3(\boldsymbol{\epsilon})\,\mathbf{t}_\epsilon \cdot \boldsymbol{\epsilon} \\ \mathbf{b} = \mathbf{b}' \cdot \vec{\mathbf{N}} = b_1(\boldsymbol{\epsilon})\,\vec{\mathbf{N}} + b_2(\boldsymbol{\epsilon})\,\mathbf{t}_\epsilon + b_3(\boldsymbol{\epsilon})\,\mathbf{t}_\epsilon \cdot \boldsymbol{\epsilon} \end{aligned} \tag{4.36}$$

Thus (4.35) contains in general all the following terms

$$
\begin{aligned}
\vec{\mathbf{N}} \cdot \mathbf{I}_2 \cdot \vec{\mathbf{N}} &= 1 &&\Rightarrow 1 \\
\vec{\mathbf{N}} \cdot \mathbf{I}_2 \cdot \mathbf{t}_\epsilon &= \epsilon &&\Rightarrow \epsilon \\
\vec{\mathbf{N}} \cdot \mathbf{I}_2 \cdot \boldsymbol{\epsilon} \cdot \mathbf{t}_\epsilon &= \mathbf{t}_\epsilon \cdot \mathbf{t}_\epsilon &&\Rightarrow \epsilon^2 + \gamma^2 \\
\mathbf{t}_\epsilon \cdot \mathbf{I}_2 \cdot \mathbf{t}_\epsilon &= \mathbf{t}_\epsilon \cdot \mathbf{t}_\epsilon &&\Rightarrow \epsilon^2 + \gamma^2 \\
\mathbf{t}_\epsilon \cdot \mathbf{I}_2 \cdot \boldsymbol{\epsilon} \cdot \mathbf{t}_\epsilon &= \vec{\mathbf{N}} \cdot \boldsymbol{\epsilon}^3 \cdot \vec{\mathbf{N}} &&\Rightarrow 1\,,\ \epsilon\,,\ \epsilon^2 + \gamma^2 \\
\mathbf{t}_\epsilon \cdot \boldsymbol{\epsilon} \cdot \mathbf{I}_2 \cdot \boldsymbol{\epsilon} \cdot \mathbf{t}_\epsilon &= \vec{\mathbf{N}} \cdot \boldsymbol{\epsilon}^4 \cdot \vec{\mathbf{N}} &&\Rightarrow 1\,,\ \epsilon\,,\ \epsilon^2 + \gamma^2 \\
\\
\vec{\mathbf{N}} \cdot \vec{\mathbf{N}} \otimes \vec{\mathbf{N}} \cdot \vec{\mathbf{N}} &= 1 &&\Rightarrow 1 \\
\vec{\mathbf{N}} \cdot \vec{\mathbf{N}} \otimes \vec{\mathbf{N}} \cdot \mathbf{t}_\epsilon &= \epsilon &&\Rightarrow \epsilon \\
\vec{\mathbf{N}} \cdot \vec{\mathbf{N}} \otimes \vec{\mathbf{N}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{t}_\epsilon &= \mathbf{t}_\epsilon \cdot \mathbf{t}_\epsilon &&\Rightarrow \epsilon^2 + \gamma^2 \\
\mathbf{t}_\epsilon \cdot \vec{\mathbf{N}} \otimes \vec{\mathbf{N}} \cdot \mathbf{t}_\epsilon &= \epsilon^2 &&\Rightarrow \epsilon^2 \\
\mathbf{t}_\epsilon \cdot \vec{\mathbf{N}} \otimes \vec{\mathbf{N}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{t}_\epsilon &= \epsilon\ \mathbf{t}_\epsilon \cdot \mathbf{t}_\epsilon &&\Rightarrow \epsilon\ (\epsilon^2 + \gamma^2) \\
\mathbf{t}_\epsilon \cdot \boldsymbol{\epsilon} \cdot \vec{\mathbf{N}} \otimes \vec{\mathbf{N}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{t}_\epsilon &= (\mathbf{t}_\epsilon \cdot \mathbf{t}_\epsilon)^2 &&\Rightarrow (\epsilon^2 + \gamma^2)^2
\end{aligned}
\tag{4.37}
$$

where the Cayley-Hamilton theorem $\boldsymbol{\epsilon}^3 = I_1\ \boldsymbol{\epsilon}^2 + I_2\ \boldsymbol{\epsilon} + I_3\ \mathbf{I}_2$ has been used to reduce $\boldsymbol{\epsilon}^3$ and $\boldsymbol{\epsilon}^4$.

We conclude that in general the localization condition plots in the Mohr's stress or strain plane in the form of a $4^{th}$ order curve. If the potential functions do not depend on the third invariant, a $2^{nd}$ order curve is obtained. This is for example the case of $J_2$-plasticity, where the localization condition is represented by an ellipse in the Mohr's stress plane (Benallal [6]). As we will see in the next sections, the localization condition for a traditional scalar isotropic damage strain-based model, results also in an ellipse in the Mohr's strain plane (Pijaudier-Cabot and Benallal [60]).

## 4.3 Traditional scalar stress-based associated damage

The general format of a stress-based associated scalar damage model was investigated in the previous chapter. All the components of the initial stiffness are affected by the same coefficient $(1 - D)$ and a loading function based on elastic energy expansion leads to an associated model.

Thus in these models the gradient to the damage function is proportional to the undamaged strain $\boldsymbol{\epsilon}_o = \mathbf{C}_o : \boldsymbol{\sigma}$ and the general format of the tangent operator reads

$$\mathbf{E}_{ed} = (1 - D)\ \mathbf{E}_o - \gamma(\boldsymbol{\sigma}, D)\ \ \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \tag{4.38}$$

where the scalar function $\gamma$ contains the hardening parameter $H$.

This scalar damage case is a particular form of the general elastic-fracturing formulation and the localization condition leads to the critical value of the hardening parameter (4.26). Considering that

$$\mathbf{E} = (1 - D)\ \mathbf{E}_o \qquad \text{and} \qquad \mathbf{Q}_e^{-1} = \frac{\mathbf{Q}_o^{-1}}{1 - D} \tag{4.39}$$

the expression of the critical hardening parameter can be recast as

$$\frac{H_{cr}}{1 - D} = -\mathbf{n} : \mathbf{E}_o : \mathbf{m} + \mathbf{a} \cdot \mathbf{Q}_o^{-1} \cdot \mathbf{b} \tag{4.40}$$

with

$$\mathbf{a} = \mathbf{n} : \mathbf{E}_o \cdot \vec{\mathbf{N}} \qquad\qquad \mathbf{b} = \vec{\mathbf{N}} \cdot \mathbf{E}_o : \mathbf{m} \tag{4.41}$$

which is the same format proposed in elastic-plasticity by Ottosen and Runesson [56] for the general $3D$ case, except for the scalar factor $(1 - D)$. Thus we conclude that all the analytical results provided in that work extend to this class of scalar damage models, even for the non-associated case.

The critical localization directions can be evaluated following the derivation in that paper. However, the critical hardening parameter might depend on the particular formulation of the model, which results in a different expression of the above mentioned function $\gamma$.

In the following, the analytical derivation will be revisited for the two loading conditions of uniaxial extension and uniaxial tension in the $3D$/plane strain and plane stress cases.

### 4.3.1 3D/plane strain case

Let us follow the elastic-plastic solution proposed by OTTOSEN AND RUNESSON [56]. In our case the model is associated such that $\mathbf{n} = \mathbf{m} = \boldsymbol{\epsilon}_o$. The principal values of the gradient are ordered as

$$n_1 \geq n_2 \geq n_3 \tag{4.42}$$

The first step is the evaluation of the scalar quantity

$$c_{31} = n_3^d + (1 - 2\psi)\; n_1^d + r \qquad \text{with} \qquad \begin{cases} \psi = \dfrac{1}{2\,(1-\nu)} \\ r = 2\,\phi\, tr(\mathbf{n}) \\ \phi = \dfrac{1+\nu}{6\,(1-\nu)} \end{cases} \tag{4.43}$$

Note that the Poisson's ratio is constant with the damage evolution, i.e. $\nu = \nu_o$, because the damage variable $D$ affects bulk and shear moduli in the same way. Now let us consider first the uniaxial extension case.

#### Uniaxial extension

Loading in uniaxial extension indicates a strain state in which on the loading axis a strain $\epsilon_1$ larger then zero is imposed, the out-of-plane strain $\epsilon_2$ is zero and stress $\sigma_3$ is also zero:

$$Uniaxial\ extension \qquad \Rightarrow \qquad \begin{cases} \epsilon_1 > 0 \\ \epsilon_2 = 0 \\ \sigma_3 = 0 \end{cases} \tag{4.44}$$

In the stress-based formulation we need to find the expression of the undamaged strain in terms of the stress components. We can use the relation $\boldsymbol{\epsilon}_o = \mathbf{C}_o : \boldsymbol{\sigma}$, which in matrix format in the principal reference reads

$$\begin{aligned}
\epsilon_1^o &= \frac{1}{E_o}\,\sigma_1 - \frac{\nu}{E_o}\,\sigma_2 - \frac{\nu}{E_o}\,\sigma_3 \\
\epsilon_2^o &= -\frac{\nu}{E_o}\,\sigma_1 + \frac{1}{E_o}\,\sigma_2 - \frac{\nu}{E_o}\,\sigma_3 \\
\epsilon_3^o &= -\frac{\nu}{E_o}\,\sigma_1 - \frac{\nu}{E_o}\,\sigma_2 + \frac{1}{E_o}\,\sigma_3
\end{aligned} \tag{4.45}$$

Imposing the condition $\epsilon_2^o = 0$ and solving it with respect to $\sigma_2$, with $\sigma_3 = 0$ we obtain $\sigma_2 = \nu\ \sigma_1$. After elimination of $\sigma_2$, the two non-zero principal strain components read

$$\left.\begin{aligned}
\epsilon_1^o &= \frac{1-\nu^2}{E_o}\,\sigma_1 \\
\epsilon_3^o &= -\frac{\nu\,(1+\nu)}{E_o}\,\sigma_1
\end{aligned}\right\} \quad \Rightarrow \quad \frac{\epsilon_1^o}{\epsilon_3^o} = 1 - \frac{1}{\nu} \tag{4.46}$$

and the volumetric strain is

$$\epsilon_v^o = \frac{(1+\nu)(1-2\nu)}{E_o}\,\sigma_1 \geq 0 \qquad \text{for} \qquad 0 \leq \nu \leq 0.5 \tag{4.47}$$

Now the quantity $c_{31}$ can be evaluated after some algebra

$$c_{31} = n_3^d + (1-2\psi)\ n_1^d + r = \frac{1}{1-\nu}\ \epsilon_3^o + \frac{\nu}{1-\nu}\ \epsilon_2^o = \frac{1}{1-\nu}\ \epsilon_3^o \leq 0 \tag{4.48}$$

From table 1 in the paper [56], for $n_1^d > n_2^d > n_3^d$

$$N_3^2 = -\frac{c_{31}}{2\psi\,(n_1^d - n_3^d)} = \frac{1}{1 - \dfrac{\epsilon_1^o}{\epsilon_3^o}}\ ; \qquad\qquad N_2 = 0 \tag{4.49}$$

Finally, using the expression for the ratio $\epsilon_1^o/\epsilon_3^o$, we obtain

$$N_3^2 = \nu \qquad\qquad N_1^2 = 1 - \nu \tag{4.50}$$

Indicating with $\theta_{cr}$ the angle between the loading axis $x_1$ and the localization direction $\vec{\mathbf{N}}$

$$\boxed{\tan^2\theta_{cr} = \left(\frac{N_3}{N_1}\right)^2 = \frac{\nu}{1-\nu}} \tag{4.51}$$

The orientation of the shear band thus depends on the value of the Poisson's ratio. For $\nu = 0.3$ a critical orientation of $\theta_{cr} = 33.21^o$ is obtained. This result matches the numerical solution by NEILSEN AND SCHREYER [53], where they predicted a shear band located at approximately $57^o$ from the loading axis. In the next section we will derive the same result for a strain-based formulation, using also the geometric representation of an ellipse in Mohr's coordinates.

**Uniaxial tension**

For the case of uniaxial tension with $\sigma_1 > 0,\;\; \sigma_2 = \sigma_3 = 0$ the $3D$ analysis gives the same result obtained before for the uniaxial extension case. In fact now

$$\left.\begin{aligned} n_1 &= \epsilon_1^o = \frac{\sigma_1}{E_o} \\ n_2 &= \epsilon_2^o = -\nu\frac{\sigma_1}{E_o} \\ n_3 &= \epsilon_3^o = -\nu\frac{\sigma_1}{E_o} \end{aligned}\right\} \qquad \epsilon_v = (1-2\nu)\,\epsilon_1^o \geq 0 \qquad \left\{\begin{aligned} n_1^d &= \frac{2(1+\nu)}{3}\,\epsilon_1^o \\ n_2^d &= -\frac{(1+\nu)}{3}\,\epsilon_1^o \\ n_3^d &= -\frac{(1+\nu)}{3}\,\epsilon_1^o \end{aligned}\right. \tag{4.52}$$

Using the expression (4.48) for $c_{31}$ we obtain

$$c_{31} = n_3^d + (1-2\psi)\,n_1^d + r = \frac{1}{1-\nu}\,\epsilon_3^o + \frac{\nu}{1-\nu}\,\epsilon_2^o = -\frac{\nu\,(1+\nu)}{1-\nu}\,\epsilon_1^o \leq 0 \tag{4.53}$$

Finally, for $n_1^d > n_2^d = n_3^d$, table 1 in [56] gives

$$N_2^2 + N_3^2 = -\frac{c_{31}}{2\psi(n_1^d - n_3^d)} = \nu\;; \qquad N_1^2 = 1-\nu \tag{4.54}$$

which yields a cone of critical directions. For example, for a $\vec{\mathbf{N}}$ in the plane $(1-3)$, with $N_2 = 0$, the previous result is obtained

$$\tan^2\theta_{cr} = \left(\frac{N_3}{N_1}\right)^2 = \frac{\nu}{1-\nu} \tag{4.55}$$

As we will show in the following subsection, this contrasts the plane stress solution for the same uniaxial tension case. Once more, as it was shown in elastic-plasticity, the localization analysis for a general $3D$ case and for a plane stress condition yield different localization results, because of the different underlying structure of the tangent operators.

### 4.3.2 Plane stress case

For the plane stress case, the considerations previously presented still apply. The nature of the scalar damage variable affects all the components of $\mathbf{E}_o$ in the same way, thus the elastic-plastic plane stress solution can be utilized. Runesson et al. [67] developed the solution of the $2D$ cases of plane stress and plane strain for a broad class of non-associated elastic-plastic stress-based models.

In our case the scalar damage model is associated and we can use (49) in their paper

$$\tan^2 \theta_{cr} = -\frac{n_2}{n_1} \tag{4.56}$$

where $\theta_{cr}$ is the critical orientation of $\vec{\mathbf{N}}$ with respect to the $x_1$ axis.

#### Uniaxial tension

In the uniaxial tension case $\sigma_1 > 0, \;\; \sigma_2 = \sigma_3 = 0$, the two components of the gradient $\mathbf{n}$ are given by the two components of the undamaged strain $\boldsymbol{\epsilon}_o$ as

$$n_1 = \epsilon_1^o \qquad\qquad n_2 = \epsilon_2^o = -\nu \, \epsilon_1^o \tag{4.57}$$

and the critical direction is given as

$$\boxed{\tan^2 \theta_{cr} = \nu \qquad \Rightarrow \qquad \theta_{cr} = \sqrt{\nu}} \tag{4.58}$$

We notice that, in comparison to the classical $J_2$-plasticity plane stress case, a dependence on Poisson's ratio is obtained. The following values of $\theta_{cr}$ can be recovered for different values of Poisson's ratios

$$\text{Plane stress uniaxial tension} \quad \Rightarrow \quad \begin{cases} \nu = 0.0 & \Rightarrow \quad \theta_{cr} = \quad 0.00^o \\ \nu = 0.3 & \Rightarrow \quad \theta_{cr} = \quad 28.71^o \\ \nu = 0.5 & \Rightarrow \quad \theta_{cr} = \quad 35.26^o \end{cases} \tag{4.59}$$

For $\nu = 0.5$ the traditional result of $J_2$-plasticity for uniaxial tension in-plane stress is recovered, where $\theta_{cr} = 35.26^o$. For $\nu = 0$ a Rankine type failure is obtained, where $\theta_{cr} = 0$, i.e the shear band is orthogonal to the loading axis. As we have already remarked, the plane stress results are different with respect to the full $3D$ analysis. For $\nu = 0.3$ the two angles compare as $33.21^o$ and $28.71^o$.

These considerations for the critical localization direction are common to all scalar damage models. The expression of the critical hardening parameter will depend on the type of scalar formulation.

For example, considering the stress-based scalar damage model by SIMO AND JU [71] the critical hardening parameter $h$ is equal to zero, i.e. the critical condition is reached for the perfect elastic-damage case, as it was for $J_2$-plasticity. Fig. 4.2, 4.3 show the effect of the Poisson's ratio on the non-dimensional quantity $det(\mathbf{Q}_{ed})/det(\mathbf{Q}_e)$.

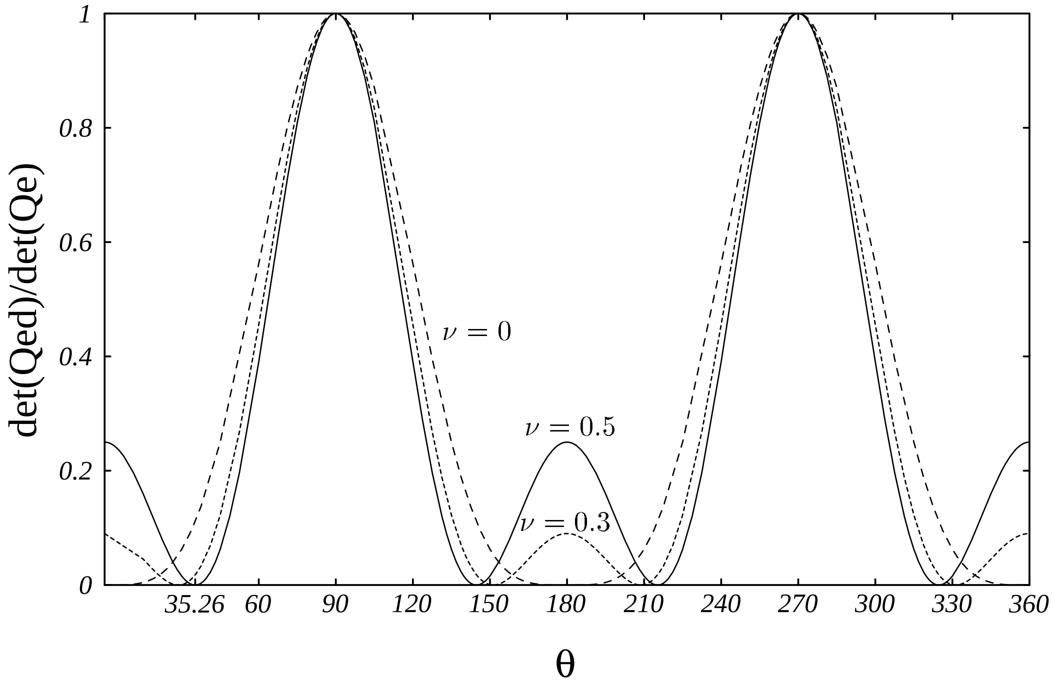



Figure 4.2: Plane stress scalar damage: influence of $\nu$ on $\theta_{cr}$

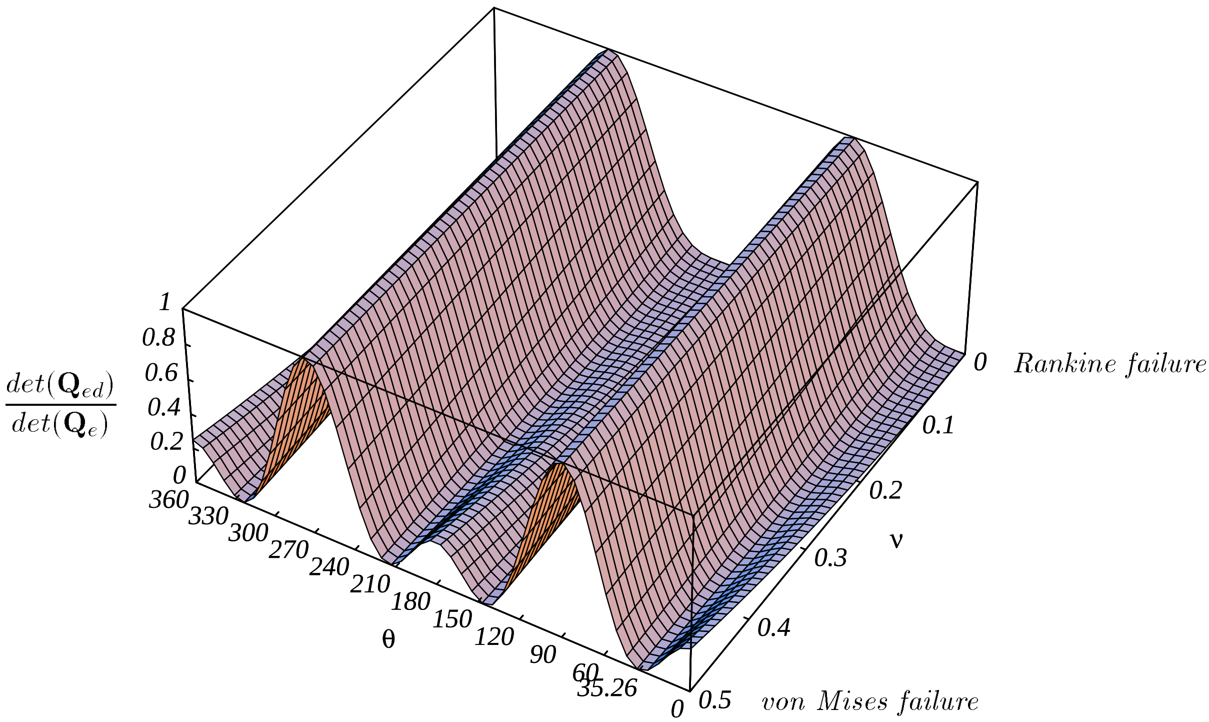



Figure 4.3: Plane stress scalar damage: $det(\mathbf{Q}_{ed})$ as a function of $\theta$, $\nu$

## 4.4 Traditional scalar strain-based associated damage

The strain-based version of traditional scalar damage models was discussed in the previous chapter. The main structure is the same as for the stress-based formulation. The scalar variable $D$ introduces isotropic damage of the initial stiffness $\mathbf{E}_o$ and the model is associated if the loading function contains an energy type term.

In a dual way to the stress-based formulation, the gradient of the damage surface is now expressed by the undamaged stress $\boldsymbol{\sigma}_o = \mathbf{C}_o : \boldsymbol{\epsilon}$ and the tangent operator is given in the general form

$$\mathbf{E}_{ed} = (1 - D)\ \mathbf{E}_o - \beta(\boldsymbol{\epsilon}, D)\ \ \boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o \tag{4.60}$$

where the scalar function $\beta$ contains the hardening parameter $\bar{H}$.

Looking at this scalar damage model as a particular case of an elastic-fracturing strain-based model, we can directly express the localization condition in terms of the critical hardening parameter $\bar{H}$

$$\bar{H}_{cr}\ (1 - D) = \mathbf{a} \cdot \mathbf{Q}_o^{-1} \cdot \mathbf{b} \tag{4.61}$$

with

$$\mathbf{a} = \bar{\mathbf{n}} \cdot \vec{\mathbf{N}} \qquad\qquad \mathbf{b} = -\vec{\mathbf{N}} \cdot \bar{\mathbf{m}} \tag{4.62}$$

Except for the factor (1-D) this will be the same format for the critical hardening parameter of an elastic-plastic strain-based model. Although the solution in this case has not been developed previously, it is still possible to use the general analytical solution by Ottosen and Runesson [56] for a stress-based model. In fact the difference consists in a scalar quantity, independent on $\vec{\mathbf{N}}$ that will not affect the maximization process on $\vec{\mathbf{N}}$.

The fact that the stress-based solution of localization can be used is not surprising because it is enough to find out the corresponding directions $\mathbf{n}$ and $\mathbf{m}$, once the direction $\bar{\mathbf{n}}$ and $\bar{\mathbf{m}}$ in the strain-based format are assigned. In our case, using for the gradients $\bar{\mathbf{n}} = -\bar{\mathbf{m}} = \boldsymbol{\sigma}_o$, corresponds to the stress maximization with

$\mathbf{n} = \mathbf{m} = \mathbf{C}_o : \bar{\mathbf{n}} = \boldsymbol{\epsilon}$ which is proportional to $\boldsymbol{\epsilon}_o$. Thus the same results of the localization condition are inevitably to be expected.

In the following we are going to use the analytical solution developed in [56], showing that the result for a stress-based formulation are recovered for the case of uniaxial tension and extension. Furthermore, the plane stress strain-based solution of the maximization problem will be derived with a proof that the same results still apply.

Finally, the geometric representation of an ellipse in the Mohr's coordinates will be considered. It shows that the critical localization condition leads to the same conclusions.

### 4.4.1 3D/plane strain case: analytical solution

Let us follow the stress-based derivation by Ottosen and Runesson [56] for the *uniaxial extension* case, where the two gradients are given by $\boldsymbol{\sigma}_o$. Using the previous relations

$$\sigma_2^o = \nu\ \sigma_1^o \qquad\qquad \epsilon_1 = \frac{1-\nu^2}{E_o}\ \sigma_1^o \tag{4.63}$$

we obtain

$$\left.\begin{array}{l} \bar{n}_1 = \sigma_1^o = \dfrac{E_o}{1-\nu^2}\ \epsilon_1 \\[2ex] \bar{n}_2 = \sigma_1^o = \nu\ \sigma_1^o \\[2ex] \bar{n}_3 = \sigma_3^o = 0 \end{array}\right\} \qquad \sigma_v^o = (1+\nu)\ \sigma_1^o \geq 0 \qquad \left\{\begin{array}{l} \bar{n}_1^d = \dfrac{2-\nu}{3}\ \sigma_1^o \\[2ex] \bar{n}_2^d = \dfrac{2\nu-1}{3}\ \sigma_1^o \\[2ex] \bar{n}_2^d = -\dfrac{1+\nu}{3}\ \sigma_1^o \end{array}\right. \tag{4.64}$$

The quantity $c_{31}$ reads, after some algebra

$$c_{31} = \bar{n}_3^d + (1-2\psi)\ \bar{n}_1^d + r = -\frac{\nu}{1-\nu}\ \sigma_1^o \leq 0 \tag{4.65}$$

Finally, using table 1 in the paper, for $r \geq 0$ and $c_{31} \leq 0$, with $\bar{n}_1^d > \bar{n}_2^d > \bar{n}_3^d$, the critical direction is given as

$$N_2 = 0\ ; \qquad N_3^2 = -\frac{c_{31}}{2\psi\ (\bar{n}_1^d - \bar{n}_3^d)} = \nu\ ; \qquad N_1^2 = 1-\nu \tag{4.66}$$

and the same critical orientation is recovered:

$$\boxed{\tan^2\theta_{cr} = \left(\frac{N_3}{N_1}\right)^2 = \frac{\nu}{1-\nu}} \tag{4.67}$$

The following numerical value of $\theta_{cr}$ are obtained for various $\nu$

$$\mathit{Uniaxial\ extension} \quad \Rightarrow \quad \begin{cases} \nu = 0.0 & \Rightarrow \quad \theta_{cr} = \quad 0.00^o \\ \nu = 0.3 & \Rightarrow \quad \theta_{cr} = \quad 33.21^o \\ \nu = 0.5 & \Rightarrow \quad \theta_{cr} = \quad 45.00^o \end{cases} \tag{4.68}$$

Thus we are shifting from a Rankine type failure, for $\nu = 0$, to a shear type failure for the incompressible case $\nu = 0.5$. The critical localization directions are the same for all scalar damage models of the strain-based type. The type of formulation will affect the expression of the critical hardening parameter.

Considering the strain-based scalar damage model by Simo and Ju [71] and assuming $\bar{\mathcal{M}} = \bar{\tau}$, the expression of the tangent operator reads

$$\mathbf{E}_{ed} = (1-D)\,\mathbf{E}_o - \frac{\boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o}{\bar{H}} \tag{4.69}$$

The critical hardening parameter is obtained from the localization condition

$$\bar{H}_{cr} = \vec{\mathbf{N}} \cdot \boldsymbol{\sigma}_o \cdot \mathbf{Q}_e^{-1} \cdot \boldsymbol{\sigma}_o \cdot \vec{\mathbf{N}} \tag{4.70}$$

Fig. 4.4, 4.5 show the effect of the Poisson's ratio on the non-dimensional quantity $\bar{H}_{cr}\ E/(\sigma_1^o)^2$. The maximum value of $\bar{H}_{cr}$ is thus depending on the Poisson's ratio and it is obtained for the critical values of $\theta$ previously derived.

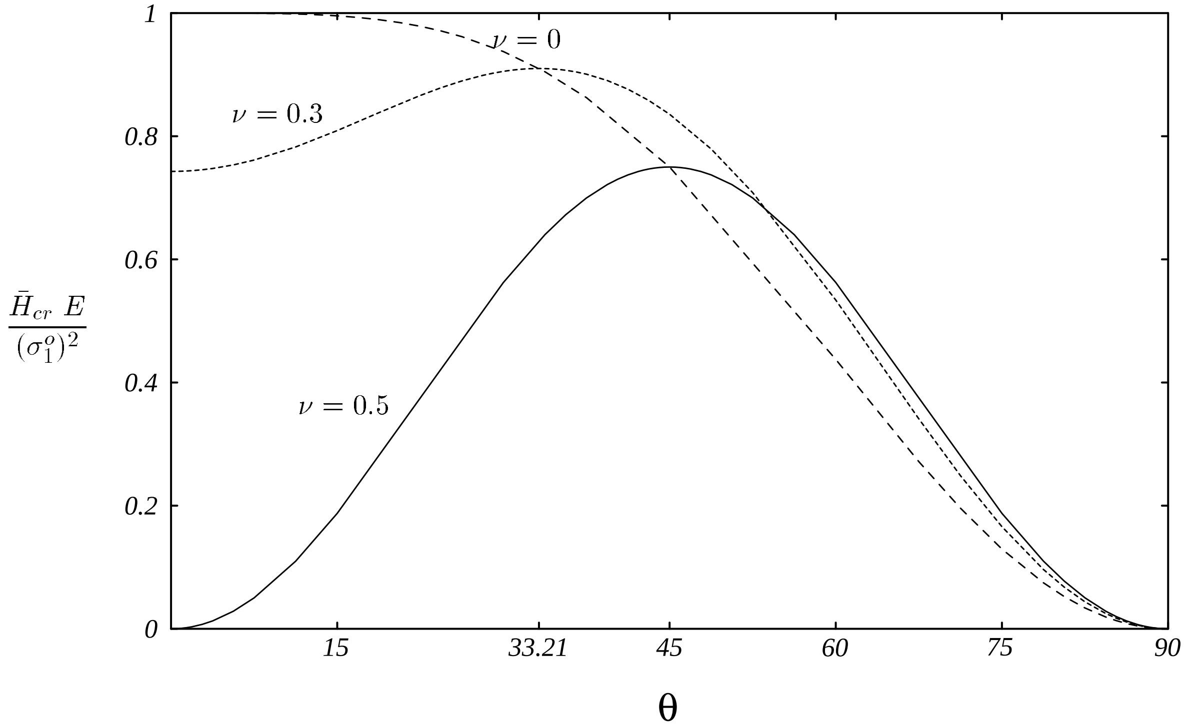



Figure 4.4: Scalar damage - uniaxial extension: influence of $\nu$ on $\bar{H}_{df}$

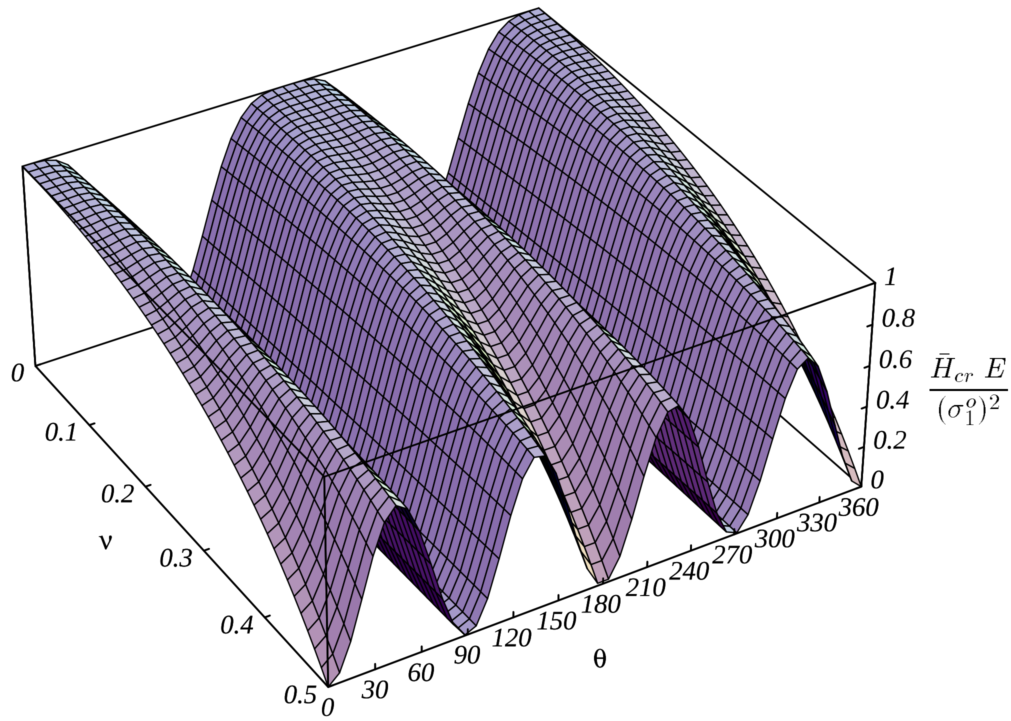



Figure 4.5: Scalar damage - uniaxial extension: $\bar{H}_{cr}$ as a function of $\theta,\ \nu$

### 4.4.2 3D/plane strain case: geometric solution

The same results can be derived using the geometric representation proposed by Pijaudier-Cabot and Benallal [60]. We have seen that the localization condition for the family of scalar strain-based damage models can be expressed by equation (4.35)

$$\frac{1}{\beta} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} \tag{4.71}$$

For the associated case, $\mathbf{a} = \mathbf{b} = \vec{\mathbf{N}} \cdot \boldsymbol{\sigma}_o = \vec{\mathbf{N}} \cdot \mathbf{E}_o : \boldsymbol{\epsilon}$ and

$$\mathbf{Q}_e^{-1} = \frac{\mathbf{Q}_o^{-1}}{1-D} = \frac{1}{1-D} \left[ \frac{1}{\mu_o} \mathbf{I}_2 - \frac{\lambda_o + \mu_o}{\mu_o \, (\lambda_o + 2\mu_o)} \, (\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \right] \tag{4.72}$$

Considering the expression of the initial elastic stiffness, the vector $\mathbf{a}$ is readily found as

$$\begin{aligned} \mathbf{a} &= \vec{\mathbf{N}} \cdot \mathbf{E}_o : \boldsymbol{\epsilon} \\ &= \vec{\mathbf{N}} \cdot (\lambda_0 \, \mathbf{I}_2 \otimes \mathbf{I}_2 + 2\mu_o \, \mathbf{I}_4^s) : \boldsymbol{\epsilon} \\ &= \vec{\mathbf{N}} \cdot (\lambda_o \, tr(\boldsymbol{\epsilon}) \, \mathbf{I}_2 + 2\mu_o \, \boldsymbol{\epsilon}) \\ &= \lambda_o \, tr(\boldsymbol{\epsilon}) \, \vec{\mathbf{N}} + 2\mu_o \, \mathbf{t}_\epsilon \end{aligned} \tag{4.73}$$

Substituting the expression of $\mathbf{Q}_e^{-1}$ in (4.71), the localization condition reads

$$\frac{1-D}{\beta} = \frac{1}{\mu_o} \, \mathbf{a} \cdot \mathbf{a} - \frac{\lambda_o + \mu_o}{\mu_o \, (\lambda_o + 2\mu_o)} \, (\mathbf{a} \cdot \vec{\mathbf{N}})^2 \tag{4.74}$$

Now, recalling the relations (4.37), we can expand the two scalar products

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= (\lambda_o \, tr(\boldsymbol{\epsilon}))^2 + 4 \, \lambda_o \, \mu_o \, tr(\boldsymbol{\epsilon}) \, \epsilon + 4 \, \mu_o^2 \, (\epsilon^2 + \gamma^2) \\ &= 4 \, \mu_o^2 \, \gamma^2 + [\lambda_o \, tr(\boldsymbol{\epsilon}) + 2\mu_o \, \epsilon]^2 \\ \mathbf{a} \cdot \vec{\mathbf{N}} &= \lambda_o \, tr(\boldsymbol{\epsilon}) + 2\mu_o \, \epsilon \end{aligned} \tag{4.75}$$

The localization condition reads, after some algebra

$$\boxed{\frac{1-D}{\beta} = 4 \, \mu_o \, \gamma^2 + \frac{1}{\lambda_o + 2\mu_o} \, [\lambda_o \, tr(\boldsymbol{\epsilon}) + 2\mu_o \, \epsilon]^2} \tag{4.76}$$

which is the equation of a set of ellipses in the Mohr's strain plane $(\epsilon - \gamma)$. The amplitude of the ellipse depends on the actual value of the function $\beta$, i.e., on the current damage and strain states and on the hardening parameter.

Once a strain state is assigned, in terms of the principal components $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$, the strain state is represented by its three Mohr's circles. As long as the ellipse encloses the largest Mohr's circle the localization condition is not activated. i.e. $det(\mathbf{Q}_{ed}) > 0$. The critical value of the hardening parameter for that state of strain is obtained when the ellipse is tangent to the largest Mohr's circle. The localization condition is thus activated for the first time and the critical localization direction results from the Mohr's analysis. If the ellipse crosses the largest circle in two points a cone of critical directions in which $det(\mathbf{Q}_{ed}) = 0$ is obtained (see fig. 4.6).

The critical orientation can now be determined imposing that the ellipse is tangent to the largest Mohr's circle. We can recast the equations of the ellipse and the largest Mohr's circle in the following system

$$\left\{ \begin{array}{l} \gamma_e^2 = \dfrac{1-D}{4\mu_o \beta} - \dfrac{\mu_o}{\lambda_o + 2\mu_o} \left( \epsilon + \dfrac{\lambda_o}{2\mu_o}\, tr(\boldsymbol{\epsilon}) \right)^2 \\ \gamma_c^2 = \left( \dfrac{\epsilon_1 - \epsilon_3}{2} \right)^2 - \left( \epsilon - \dfrac{\epsilon_1 + \epsilon_3}{2} \right)^2 \end{array} \right. \tag{4.77}$$

The condition $\gamma_e^2 = \gamma_c^2$ results in the second order equation $a\,\epsilon^2 + b\,\epsilon + c = 0$, with

$$\begin{aligned} a &= -\frac{\lambda_o + \mu_o}{\lambda_o + 2\mu_o} \\ b &= (\epsilon_1 + \epsilon_3) + \frac{\lambda_o}{\lambda_o + 2\mu_o}\, tr(\boldsymbol{\epsilon}) \\ c &= \left( \frac{\epsilon_1 - \epsilon_3}{2} \right)^2 - \left( \frac{\epsilon_1 + \epsilon_3}{2} \right)^2 + \frac{\lambda_o^2}{4\mu_o\,(\lambda_o + 2\mu_o)}\, tr^2(\boldsymbol{\epsilon}) - \frac{1-D}{4\mu_o\beta} \end{aligned} \tag{4.78}$$

The tangent condition is recovered when the discriminant $\Delta = b^2 - 4ac$ is zero, with two coincident solutions. This equation will lead to the value of the hardening parameter necessary for discontinuous failure, i.e. the maximum value of $\bar{H}_{cr}$

$$\boxed{b^2 - 4ac = 0 \qquad \Rightarrow \qquad \bar{H}_{df}} \tag{4.79}$$

When $\Delta = 0$ the solution of the system gives

$$\begin{aligned} \epsilon &= \frac{\lambda_o + 2\mu_o}{2\,(\lambda_o + \mu_o)} \left[ (\epsilon_1 + \epsilon_3) + \frac{\lambda_o}{\lambda_o + 2\mu_o}\, tr(\boldsymbol{\epsilon}) \right] \\ &= (\epsilon_1 + \epsilon_3) + \frac{\lambda_o}{2(\lambda_o + \mu_o)}\, \epsilon_2 \end{aligned} \tag{4.80}$$

such that

$$\left( \epsilon - \frac{\epsilon_1 + \epsilon_3}{2} \right) = \frac{\epsilon_1 + \epsilon_3}{2} + \frac{\lambda_o}{2(\lambda_o + \mu_o)}\, \epsilon_2 \tag{4.81}$$

Finally, the critical orientation is obtained via Mohr's analysis as

$$\tan^2(2\theta_{cr}) = \frac{\gamma_c^2}{\left( \epsilon - \dfrac{\epsilon_1 + \epsilon_3}{2} \right)^2} = \frac{\left( \dfrac{\epsilon_1 - \epsilon_3}{2} \right)^2 - \left( \dfrac{\epsilon_1 + \epsilon_3}{2} + \dfrac{\lambda_o}{2(\lambda_o + \mu_o)}\, \epsilon_2 \right)^2}{\left( \dfrac{\epsilon_1 + \epsilon_3}{2} + \dfrac{\lambda_o}{2(\lambda_o + \mu_o)}\, \epsilon_2 \right)^2} \tag{4.82}$$

Noticing that $\dfrac{\lambda_o}{\lambda_o + \mu_o} = 2\nu_o = 2\nu$

$$\boxed{\tan^2(2\theta_{cr}) = \frac{(\epsilon_1 - \epsilon_3)^2 - (\epsilon_1 + \epsilon_3 + 2\nu\, \epsilon_2)^2}{(\epsilon_1 + \epsilon_3 + 2\nu\, \epsilon_2)^2}} \tag{4.83}$$

and using the trigonometric relation

$$\tan\theta = \frac{\tan 2\theta}{1 + \dfrac{1}{\cos 2\theta}} \tag{4.84}$$

the localization condition can be also expressed in terms of the critical angle $\theta_{cr}$ as

$$\boxed{\tan^2(\theta_{cr}) = \frac{(\epsilon_1 - \epsilon_3)^2 - (\epsilon_1 + \epsilon_3 + 2\nu\, \epsilon_2)^2}{(2\epsilon_1 + 2\nu\, \epsilon_2)^2}} \tag{4.85}$$

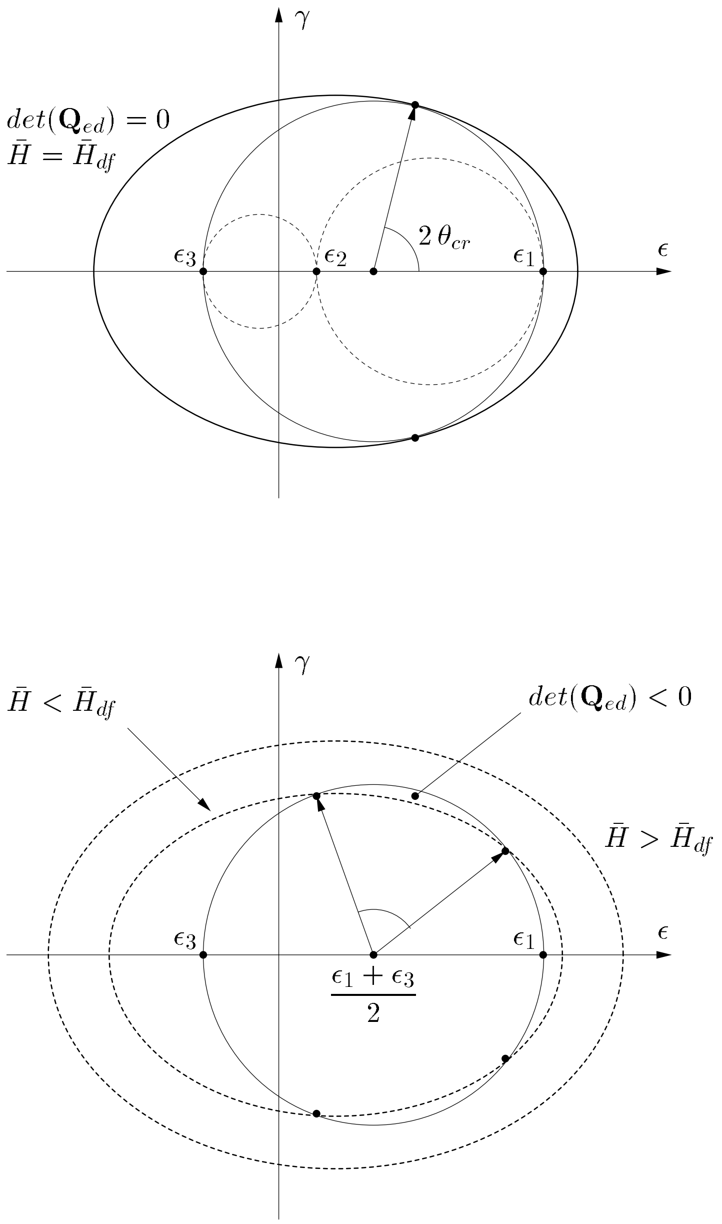



Figure 4.6: Geometric representation of localization in the Mohr's strain plane

Many interesting conclusions can be drawn analyzing the previous expression. For the plane strain case in which the intermediate strain is zero, as for the *uniaxial extension* and *pure shear* strain,

$$\tan^2(\theta_{cr}) = -\frac{\epsilon_3}{\epsilon_1} \quad \Rightarrow \quad \begin{cases} \dfrac{\nu}{1-\nu} & \textit{Uniaxial extension} \\ 1 & \textit{Pure shear} \end{cases} \tag{4.86}$$

The same result is obtained as before for the uniaxial extension. For the pure shear case the critical orientation $\theta_{cr} = 45^o$ remains independent of Poisson's ratio.

If the plane strain state imposes two in-plane strain components, with the same sign, the localization condition is never reached, according to Stören and Rice [81], Pijaudier-Cabot and Benallal [60]. In fact, in this case $\epsilon_1 > 0, \;\; \epsilon_2 > 0$, $\epsilon_3 = 0$ and

$$\tan^2(\theta_{cr}) = \frac{\epsilon_1^2 - (\epsilon_1 + 2\nu\, \epsilon_2)^2}{(2\nu\, \epsilon_2)^2} < 0 \tag{4.87}$$

### 4.4.3 Plane stress case: analytical solution

The localization analysis of stress-based elastic-plastic models in-plane stress was carried out by Runesson et al. [67]. This solution can be generalized to the case of strain-based elastic-plastic or scalar damage formulations. In the following we will consider the derivation for the scalar damage model.

We have seen that the localization condition results in an expression for the critical hardening parameter

$$\bar{H}_{cr} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} \tag{4.88}$$

with $\mathbf{a} = \bar{\mathbf{n}} \cdot \vec{\mathbf{N}}, \;\; \mathbf{b} = -\vec{\mathbf{N}} \cdot \bar{\mathbf{m}} = \vec{\mathbf{N}} \cdot \tilde{\mathbf{m}}$. The indices of any tensorial quantity vary between 1 and 2 and will be designated with Greek letters. The elastic acoustic tensor in-plane stress is represented now by a $2 \times 2$ matrix. Its inverse is given as

$$\mathbf{Q}_e^{-1} = \frac{1}{G} \left[ \mathbf{I}_2 - \frac{1+\nu}{2} \, (\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \right] \tag{4.89}$$

where $G$ is the actual value of the shear modulus. After substitution of $\mathbf{Q}_e^{-1}$, the critical hardening parameter reads

$$\bar{H}_{cr} = \frac{1}{G}\left[-\frac{1+\nu}{2}\,(\mathbf{a}\cdot\vec{\mathbf{N}})\,(\vec{\mathbf{N}}\cdot\mathbf{b}) + \mathbf{a}\cdot\mathbf{b}\right] \tag{4.90}$$

and, in indicial notation

$$2G\,\bar{H}_{cr} = -(1+\nu)\,(N_\alpha\,\bar{n}_{\alpha\beta}\,N_\beta)\,(N_\gamma\,\tilde{m}_{\gamma\delta}\,N_\delta) + 2\,N_\alpha\,\bar{n}_{\alpha\beta}\,\tilde{m}_{\beta\gamma}\,N_\gamma \tag{4.91}$$

Now let us make the assumption that $\bar{\mathbf{n}}$ and $\tilde{\mathbf{m}}$ have the same principal directions and two of them define the plane. The in-plane components will be assumed such that $\bar{n}_1 \geq \bar{n}_2$ and $\tilde{m}_1 \geq \tilde{m}_2$. Using the principal reference system, the previous expression simplifies to

$$\begin{aligned} 2G\,\bar{H}_{cr} = \;& -(1+\nu)\,(\bar{n}_1\,N_1^2 + \bar{n}_2\,N_2^2)\,(\tilde{m}_1\,N_1^2 + \tilde{m}_2\,N_2^2) + \\ & +2\,(\bar{n}_1\,\tilde{m}_1\,N_1^2 + \bar{n}_2\,\tilde{m}_2\,N_2^2) \end{aligned} \tag{4.92}$$

Imposing the constraint $N_2^2 = 1 - N_1^2$

$$\begin{aligned} 2G\,\bar{H}_{cr} = \;& -(1+\nu)\,[(\bar{n}_1-\bar{n}_2)\,N_1^2 + \bar{n}_2]\,[(\tilde{m}_1-\tilde{m}_2)\,N_1^2 + \tilde{m}_2] + \\ & +2\,[(\bar{n}_1\tilde{m}_1 - \bar{n}_2\tilde{m}_2)\,N_1^2 + \bar{n}_2\tilde{m}_2] \end{aligned} \tag{4.93}$$

The hardening parameter necessary for the first onset of discontinuous failure is the maximum value of $\bar{H}_{cr}$ for all possible $\vec{\mathbf{N}}$. In this $2D$ problem the maximization can be done directly taking the derivative of the function $\bar{H}_{cr}(\vec{\mathbf{N}})$ with respect to $N_1^2$ and looking for stationary conditions. The first and second derivatives are obtained as follows

$$\begin{aligned} \frac{d\,(2G\,\bar{H}_{cr})}{d\,(N_1^2)} = \;& -(1+\nu)\,(\bar{n}_1-\bar{n}_2)\,[(\tilde{m}_1-\tilde{m}_2)\,N_1^2 + \tilde{m}_2] + \\ & -(1+\nu)\,(\tilde{m}_1-\tilde{m}_2)\,[(\bar{n}_1-\bar{n}_2)\,N_1^2 + \bar{n}_2] + \\ & +2\,(\bar{n}_1\tilde{m}_1 - \bar{n}_2\tilde{m}_2) \end{aligned} \tag{4.94}$$

$$\frac{d^2\,(2G\,\bar{H}_{cr})}{d\,(N_1^2)^2} = -2\,(1+\nu)\,(\bar{n}_1-\bar{n}_2)\,(\tilde{m}_1-\tilde{m}_2) \;\leq\; 0 \tag{4.95}$$

The inequality in the last one assures that stationary point is really a local maximum. Equating the first derivatives to zero we obtain the stationary condition

$$-2\,(1+\nu)\,(\bar{n}_1-\bar{n}_2)\,(\tilde{m}_1-\tilde{m}_2)\,N_1^2-(1+\nu)\,[(\bar{n}_1-\bar{n}_2)\,\tilde{m}_2+(\tilde{m}_1-\tilde{m}_2)\,\bar{n}_2]+$$
$$+2\,(\bar{n}_1\tilde{m}_1-\bar{n}_2\tilde{m}_2)=0 \tag{4.96}$$

that can be rewritten as

$$-\,(1+\nu)\,(c_2-c_1)\,N_1^2+\frac{1+\nu}{2}\,(c_2-c_1)+(1-\nu)\,c_3=0 \tag{4.97}$$

where

$$\begin{cases} c_1=\bar{n}_1\,(\tilde{m}_1-\tilde{m}_2)+\tilde{m}_1\,(\bar{n}_1-\bar{n}_2) \\ c_2=\bar{n}_2\,(\tilde{m}_1-\tilde{m}_2)+\tilde{m}_2\,(\bar{n}_1-\bar{n}_2) \\ c_3=\bar{n}_1\tilde{m}_1-\bar{n}_2\tilde{m}_2 \end{cases} \tag{4.98}$$

Solving for $N_1^2$, $N_2^2=1-N_1^2$

$$N_1^2=\frac{1}{2}+\frac{1-\nu}{1+\nu}\,\frac{c_3}{c_2-c_1}\,;\qquad N_2^2=\frac{1}{2}-\frac{1-\nu}{1+\nu}\,\frac{c_3}{c_2-c_1} \tag{4.99}$$

The critical orientation of the normal to the discontinuity surface with respect to the axis $x_1$ reads

$$\boxed{\tan^2\theta_{cr}=\frac{(1-\nu)\;(c_2-c_1)-2\,(1-\nu)\,c_3}{(1-\nu)\;(c_2-c_1)+2\,(1-\nu)\,c_3}} \tag{4.100}$$

For the associated case $\bar{\mathbf{n}}=\tilde{\mathbf{m}}$, the three quantities $c_1$, $c_2$ and $c_3$ reduce to

$$\begin{cases} c_1=2\,\bar{n}_1\,(\bar{n}_1-\bar{n}_2) \\ c_2=2\,\bar{n}_2\,(\bar{n}_1-\bar{n}_2) \\ c_3=(\bar{n}_1+\bar{n}_2)\,(\bar{n}_1-\bar{n}_2) \end{cases} \tag{4.101}$$

with the critical direction

$$\boxed{\tan^2\theta_{cr}=\frac{\nu\,\bar{n}_1-\bar{n}_2}{\bar{n}_1-\nu\bar{n}_2}} \tag{4.102}$$

**Uniaxial tension and pure shear**

For the uniaxial tension case the in-plane components of the gradient $\bar{n} = \boldsymbol{\sigma}_o$ reduce to

$$\bar{n}_1 = \sigma_1^o \qquad \bar{n}_2 = 0 \tag{4.103}$$

and the critical orientation gives the result

$$\tan^2 \theta_{cr} = \nu \tag{4.104}$$

For the pure shear case $\bar{n}_1 = \sigma_o$, $\bar{n}_2 = -\sigma_o$ the localization direction does not depend on Poisson's ratio

$$\tan^2 \theta_{cr} = 1 \qquad \Rightarrow \qquad \theta_{cr} = 45^o \tag{4.105}$$

### 4.4.4 Plane stress case: geometric solution

The previous derivation for the geometric localization criterion in the general $3D$ case can be readily extended to the plane stress case. The main steps of the formulation are the same, with the difference being that the initial elastic stiffness and elastic acoustic tensor have the structure of the plane stress case. In the following, the range of the indices of any tensorial quantity, indicated with Greek letters, is between 1 and 2.

The localization condition for the strain-based scalar damage models can be expressed in the usual way

$$\frac{1}{\beta} = \mathbf{a} \cdot \mathbf{Q}_e^{-1} \cdot \mathbf{b} = a_\alpha \, (Q_e^{-1})_{\alpha\beta} \, b_\beta \tag{4.106}$$

where for the associated case $\mathbf{a} = \mathbf{b} = \vec{\mathbf{N}} \cdot \boldsymbol{\sigma}_o = \vec{\mathbf{N}} \cdot \mathbf{E}_o : \boldsymbol{\epsilon}$. The inverse of the elastic acoustic tensor reads

$$\mathbf{Q}_e^{-1} = \frac{\mathbf{Q}_o^{-1}}{1-D} = \frac{1}{1-D} \left[ \frac{1}{G_o} \, \mathbf{I}_2 - \frac{1+\nu}{2\,G_o} \, (\vec{\mathbf{N}} \otimes \vec{\mathbf{N}}) \right] \tag{4.107}$$

and the initial elastic stiffness in-plane stress is given as

$$\begin{aligned}\mathbf{E}_o &= \frac{\nu\, E_o}{1-\nu^2}\ \mathbf{I}_2 \otimes \mathbf{I}_2 + 2\, G_o\, \mathbf{I}_4^s \\ E^o_{\alpha\beta\gamma\delta} &= \frac{\nu\, E_o}{1-\nu^2}\ \delta_{\alpha\beta}\delta_{\gamma\delta} + 2\, G_o\, \frac{1}{2}\, (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\end{aligned} \tag{4.108}$$

The expansion of the vector $\mathbf{a}$ can thus be obtained as before

$$\begin{aligned}\mathbf{a} &= \vec{\mathbf{N}} \cdot \mathbf{E}_o : \boldsymbol{\epsilon} \\ &= \vec{\mathbf{N}} \cdot \left( \frac{\nu\, E_o}{1-\nu^2}\ \mathbf{I}_2 \otimes \mathbf{I}_2 + 2\, G_o\, \mathbf{I}_4^s \right) : \boldsymbol{\epsilon} \\ &= \vec{\mathbf{N}} \cdot \left( \frac{\nu\, E_o}{1-\nu^2}\ tr(\boldsymbol{\epsilon})\, \mathbf{I}_2 + 2\, G_o\, \boldsymbol{\epsilon} \right) \\ &= \frac{\nu\, E_o}{1-\nu^2}\ tr(\boldsymbol{\epsilon})\, \vec{\mathbf{N}} + 2\, G_o\, \mathbf{t}_\epsilon\end{aligned} \tag{4.109}$$

where it is important to note that, in this case, $tr(\boldsymbol{\epsilon}) = \epsilon_{\alpha\alpha}$. After substitution of the inverse elastic acoustic tensor, the localization condition gives

$$\frac{1-D}{\beta} = \frac{1}{G_o}\, \mathbf{a} \cdot \mathbf{a} - \frac{1+\nu}{2\, G_o}\, (\mathbf{a} \cdot \vec{\mathbf{N}})^2 \tag{4.110}$$

The expansion of the two scalar products in the previous equation renders

$$\begin{aligned}\mathbf{a} \cdot \mathbf{a} &= (2\, G_o)^2\, [(\frac{\nu}{1-\nu}\, tr(\boldsymbol{\epsilon}))^2 + 2\, \frac{\nu}{1-\nu}\, tr(\boldsymbol{\epsilon})\, \epsilon + \ (\epsilon^2 + \gamma^2)] \\ &= (2\, G_o)^2\, [\gamma^2 + (\epsilon + \frac{\nu}{1-\nu}\, tr(\boldsymbol{\epsilon}))^2] \\ \mathbf{a} \cdot \vec{\mathbf{N}} &= 2\, G_o\, [\epsilon + \frac{\nu}{1-\nu}\, tr(\boldsymbol{\epsilon})]\end{aligned} \tag{4.111}$$

and, the localization condition results in

$$\boxed{\frac{1-D}{\beta} = 4\, G_o\, \gamma^2 + 2\, G_o\, (1-\nu)\, [\epsilon + \frac{\nu}{1-\nu}\, \epsilon_{\alpha\alpha}]^2} \tag{4.112}$$

which can be also written in terms of the Lamé's constants. Recalling that

$$1-\nu = \frac{\lambda_o + 2\mu_o}{2(\lambda_o + \mu_o)} \qquad \frac{\nu}{1-\nu} = \frac{\lambda_o}{\lambda_o + 2\mu_o} \tag{4.113}$$

we obtain

$$\boxed{\frac{1-D}{\beta} = 4\, \mu_o\, \gamma^2 + \mu_o\, \frac{\lambda_o + 2\mu_o}{\lambda_o + \mu_o}\, [\epsilon + \frac{\lambda_o}{\lambda_o + 2\mu_o}\, \epsilon_{\alpha\alpha}]^2} \tag{4.114}$$

This result can be compared with the analogous expression for the $3D$ case. The localization condition is again represented by an ellipse in Mohr's strain coordinates and the previous remarks do extend to this case. For Mohr's circles representing the state of strain it is necessary to evaluate the out-of-plane strain component and order the principal strains as usual $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$. From the condition

$$\sigma_{33} = (\lambda + 2\mu)\, \epsilon_{33} + \lambda\, (\epsilon_{11} + \epsilon_{22}) = o \tag{4.115}$$

we obtain

$$\epsilon_{33} = -\frac{\lambda}{\lambda + 2\mu}\, (\epsilon_{11} + \epsilon_{22}) = -\frac{\nu}{1-\nu}\, (\epsilon_{11} + \epsilon_{22}) \tag{4.116}$$

The intersections between the largest Mohr's circle and the ellipse are given by the condition $\gamma_e^2 = \gamma_c^2$

$$\frac{1-D}{4\,G_o\,\beta} - \frac{1-\nu}{2}\,[\epsilon + \frac{\nu}{1-\nu}\,\epsilon_{\alpha\alpha}]^2 = \left(\frac{\epsilon_1 - \epsilon_3}{2}\right)^2 - \left(\epsilon - \frac{\epsilon_1 + \epsilon_3}{2}\right)^2 \tag{4.117}$$

which results in the second order equation $\;a\,\epsilon^2 + b\,\epsilon + c = 0$, with

$$\begin{aligned} a &= -\frac{1+\nu}{2} \\ b &= (\epsilon_1 + \epsilon_3) + \nu\,\epsilon_{\alpha\alpha} \\ c &= \left(\frac{\epsilon_1 - \epsilon_3}{2}\right)^2 - \left(\frac{\epsilon_1 + \epsilon_3}{2}\right)^2 + \frac{\nu^2}{2\,(1-\nu)}\,\epsilon_{\alpha\alpha}^2 - \frac{1-D}{4\,G_o\beta} \end{aligned} \tag{4.118}$$

The tangent condition reached for $\bar{H}_{df}$ is obtain when the discriminant of the second order equation vanishes

$$\boxed{b^2 - 4ac = 0 \qquad \Rightarrow \qquad \bar{H}_{df}} \tag{4.119}$$

and in that case the solution is

$$\epsilon = \frac{(\epsilon_1 + \epsilon_3) + \nu\,\epsilon_{\alpha\alpha}}{1+\nu} \tag{4.120}$$

such that

$$\epsilon - \frac{\epsilon_1 + \epsilon_3}{2} = \frac{1-\nu}{2\,(1-\nu)}\;(\epsilon_1 + \epsilon_3) + \frac{\nu}{1+\nu}\;\epsilon_{\alpha\alpha} \tag{4.121}$$

The critical orientation is obtained via Mohr's analysis as

$$\tan^2(2\theta_{cr}) = \frac{\left(\dfrac{\epsilon_1 - \epsilon_3}{2}\right)^2 - \left(\dfrac{1-\nu}{2\,(1+\nu)}\;(\epsilon_1 + \epsilon_3) + \dfrac{\nu}{1+\nu}\;\epsilon_{\alpha\alpha}\right)^2}{\left(\dfrac{1-\nu}{2\,(1+\nu)}\;(\epsilon_1 + \epsilon_3) + \dfrac{\nu}{1+\nu}\;\epsilon_{\alpha\alpha}\right)^2} \tag{4.122}$$

After some algebra,

$$\boxed{\tan^2(2\theta_{cr}) = \frac{[(1+\nu)\;(\epsilon_1 - \epsilon_3)]^2 - [(1-\nu)\;(\epsilon_1 + \epsilon_3) + 2\nu\,\epsilon_{\alpha\alpha}]^2}{[(1-\nu)\;(\epsilon_1 + \epsilon_3) + 2\nu\,\epsilon_{\alpha\alpha}]^2}} \tag{4.123}$$

or

$$\boxed{\tan^2(\theta_{cr}) = \frac{[(1+\nu)\;(\epsilon_1 - \epsilon_3)]^2 - [(1-\nu)\;(\epsilon_1 + \epsilon_3) + 2\nu\,\epsilon_{\alpha\alpha}]^2}{[(1+\nu)\;(\epsilon_1 - \epsilon_3) + (1-\nu)\;(\epsilon_1 + \epsilon_3) + 2\nu\;\epsilon_{\alpha\alpha}]^2}} \tag{4.124}$$

### Uniaxial tension

Under uniaxial tension the only non-zero stress component is applied along the loading axis $x_1$. The resulting strains are

$$\begin{cases} \sigma_{11} = \sigma \\ \sigma_{22} = 0 \\ \sigma_{33} = 0 \end{cases} \Rightarrow \begin{cases} \epsilon_{11} = \sigma_{11}/E \\ \epsilon_{22} = -\nu\;\epsilon_{11} \\ \epsilon_{33} = -\nu\;\epsilon_{11} \end{cases} \Rightarrow \begin{cases} \epsilon_1 = \epsilon_{11} \\ \epsilon_2 = -\nu\;\epsilon_{11} \\ \epsilon_3 = -\nu\;\epsilon_{11} \end{cases} \tag{4.125}$$

and $\epsilon_{\alpha\alpha} = (1-\nu)\;\epsilon_{11}$. The critical localization direction is obtained as before

$$\tan^2(\theta_{cr}) = \frac{(1+\nu)^4 - (1-\nu)^2\;(1+\nu)^2}{[(1+\nu)^2 + (1-\nu)\;(1+\nu)]^2} = \frac{(1+\nu)^2 - (1-\nu)^2}{4} = \nu \tag{4.126}$$

**Pure shear**

Under pure shear, the critical localization direction does not depend on $\nu$ and it is given by the classical result of $\theta_{cr} = 45^o$. In fact

$$\begin{cases} \sigma_{11} = \sigma \\ \sigma_{22} = -\sigma \\ \sigma_{33} = 0 \end{cases} \Rightarrow \begin{cases} \epsilon_{11} = (1+\nu)\,\sigma/E \\ \epsilon_{22} = -\epsilon_{11} \\ \epsilon_{33} = 0 \end{cases} \Rightarrow \begin{cases} \epsilon_1 = \epsilon_{11} \\ \epsilon_2 = 0 \\ \epsilon_3 = -\epsilon_{11} \end{cases} \tag{4.127}$$

and $\epsilon_{\alpha\alpha} = 0$. The critical direction is in this case

$$\tan^2(\theta_{cr}) = \frac{[(1+\nu)\,\epsilon_{11}]^2}{[(1+\nu)\,\epsilon_{11}]^2} = 1 \qquad \Rightarrow \qquad \theta_{cr} = 45^o \tag{4.128}$$

Let us derive the expression of the critical hardening parameter. The condition $\Delta = 0$ results in this case in the following relation

$$-\,4\,\frac{1+\nu}{2}\left[-\left(\frac{2\,\epsilon_{11}}{2}\right)^2 + \frac{1-D}{4\,\mu_o\,\beta}\right] = 0 \tag{4.129}$$

Solving with respect to $\beta$, yields

$$\beta_{df} = \frac{1-D}{4\,\mu_o\,\epsilon_{11}^2} \tag{4.130}$$

Solving for $\bar{H}_{df}$ a particular model needs to be introduced, i.e. the function $\beta$ must be specified. Considering again the strain-based scalar damage model by Simo and Ju [71], with $\bar{\mathcal{M}} = \bar{\tau}$, the expression of the tangent operator reads

$$\mathbf{E}_{ed} = (1-D)\,\mathbf{E}_o - \frac{\boldsymbol{\sigma}_o \otimes \boldsymbol{\sigma}_o}{\bar{H}} \tag{4.131}$$

thus $\beta = \dfrac{1}{\bar{H}}$. Solving (4.130) with respect to $\bar{H}_{df}$, we obtain

$$\boxed{\bar{H}_{df} = \frac{(\sigma_1^o)^2}{G}} \tag{4.132}$$

which is again independently of $\nu$. Thus the first discontinuous bifurcation is reached for the critical hardening parameter, with $D = 0$, as shown in fig. 4.7. In this case the normalized determinant of the acoustic elastic-damage tensor reads

$$\frac{det(\mathbf{Q}_{ed})}{det(\mathbf{Q}_e)} = (1 - D)\,\frac{1+\nu}{2}\;\cos^2(2\theta) \tag{4.133}$$

which depends on $\nu$, although the critical localization direction does not depend on Poisson's ratio.

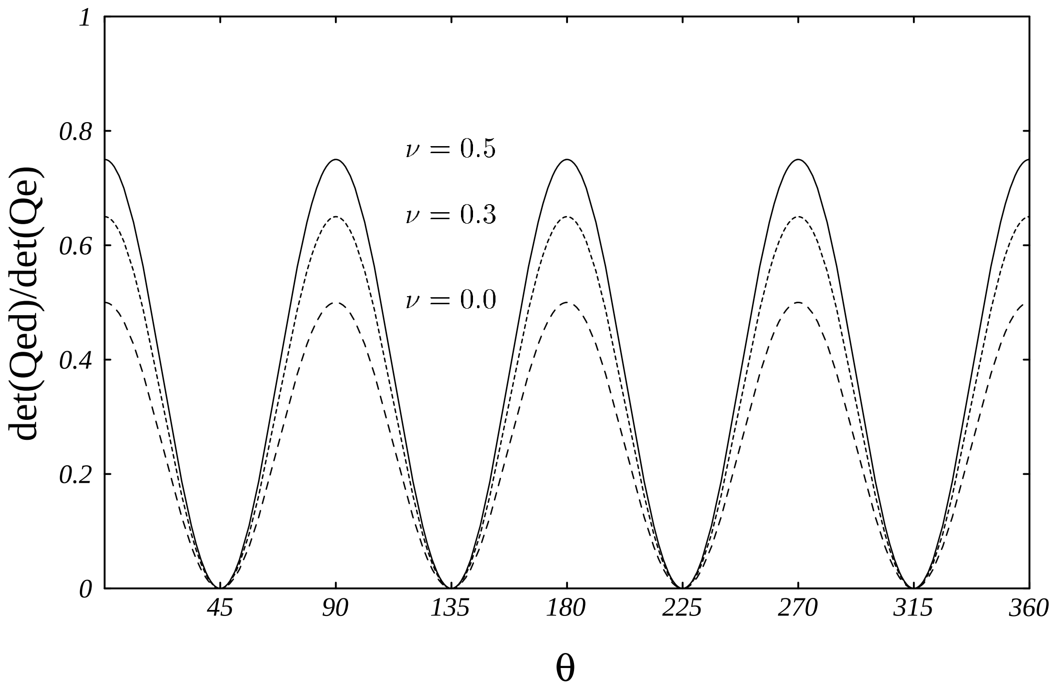



Figure 4.7: Plane stress pure shear loading: no dependence on $\nu$ for $\theta_{cr}$

# CHAPTER 5

# CONCLUSIONS

The main purpose of this work has been the development of analytical solutions for the localization condition in material models with stiffness degradation. This goal has been achieved for a general class of scalar damage models.

Localization analysis has been extensively developed in the last few years in the framework of elastic-plasticity. It provides an important tool for detecting formation of spatial discontinuities at the constitutive level when a load increment is prescribed. The localization criterion for discontinuous bifurcation extends to any material formulation which can be cast into a tangential constitutive format. The extension of localization analysis to different continuum descriptions of material degradation is very important because their comparison will lead to a clear understanding of the validity of the model for realistic failure descriptions.

The concepts of localization are well established and they have been reviewed in Chapter 2 of this thesis. The purpose of that description was to clarify the aspects of wave propagation and discontinuous bifurcation and to show how all the fundamental equations are derived. Also some effort has directed to combine and relate the work of different researchers and present the theory in a unified manner.

Special interesting conclusions emerged during that study. For example, the rank-one modification of the elastic operator, leading to the tangent elastic-plastic operator, was more thoroughly investigated. The conclusion that just one eigenvalue is modified, for deviatoric plasticity, was verified, although two eigenvalues are modified in the case of pressure-sensitive yield conditions and, or flow rules. The relationships between stress and strain-based elastic-plastic formulations were contrasted and clarified.

The description of localization as a jump condition or alternatively as a stationary condition of wave propagation was considered, with the purpose of understanding the full meaning of the acoustic tensor. A clearer insight of the spectral properties of the acoustic tensor was then reached. The way to transform the localization condition into a simple scalar equation was also reviewed. The differences between $3D$ analysis versus $2D$-plane stress computations was highlighted with the perplexing results of $J_2$-plasticity in uniaxial tension.

In Chapter 3 a unified description of stiffness degradation was developed. The main task was to present explicit expressions of the tangent operator which can be used for the construction of the associated localization operator. The derivation of elastic-fracturing materials was pursued, in a dual way to the elastic-plastic case, for both stress-based and strain-based environments. Continuum damage mechanics was then introduced as a particular case of the general elastic-fracturing description: the secant stiffness or the compliance evolution was described by a reduced set of variables, that in the simplest case reduces to a scalar quantity.

The thermodynamic concepts typical of most of the damage formulations were reduced to a minimum, although some of these arguments were necessary for introducing the concept of associativity. Associativity has been defined at two levels: $2^{nd}$ order tensor level for inelastic strain or stress level, $4^{th}$ order tensor level for compliance or stiffness rate or for damage. Finally, the various relations between damage, fracturing and flow rules were developed and presented in tabular form.

A general description of scalar damage models was examined and the differences between isotropic and anisotropic damage were highlighted. At the end of Chapter 3 a review of different models from the recent literature were reviewed as particular cases of the general formulation. The common features of all these models, leading to the same structure of the tangent operator, were summarized at the end.

Chapter 4 actually contains the core of this thesis, with new developments in the field of localization. The elastic-fracturing and continuum damage models were analyzed with respect to their capability to capture failure. Both diffuse and discontinuous failure conditions were investigated in detail. For the diffuse failure condition, essentially the same results as for elastic-plasticity were recovered: the limit points conditions are reached for the case of elastic perfectly fracturing material. Again, the relations between the dual stress and strain-based descriptions were clarified.

The general framework for localization analysis of elastic-fracturing and damage models was presented and explicit expressions of the critical hardening parameters were developed. The localization analysis, which defines the onset of discontinuous bifurcation of damage models, reduces to a natural extension of the elastic-plastic case. For the traditional scalar damage case we also demonstrated that the analytical solutions available in the literature for elastic-plasticity can be directly used.

Some loading cases were considered and expressions of the critical localization direction and critical value of the hardening parameter for onset of localization were derived. Solutions for the strain-based case, usually not considered in elastic-plasticity, were also provided. The derivation for the plane stress strain-based scalar damage or elastic-plasticity was considered. Again, the perplexing differences between full three-dimensional and plane stress analyses were shown. Also, the differences with the elastic-plastic formulation were highlighted. For example, the classical results of plane stress $J_2$-plasticity were compared with the case of plane stress scalar damage. In the second case, Poisson's ratio influences the critical localization direction, although in elastic-plasticity the Poisson's ratio has no effect.

The attractive geometric representation of the localization condition in Mohr's stress or strain plane was then considered. The analytical derivation of the critical directions and hardening parameters were outlined, showing that they match other analytical results, previously derived. Also, an attempt to understand the validity of the geometric representation for a general case was considered. In the most general isotropic case a $4^{th}$ order curve in the Mohr plane does represent the localization condition. The curve is of second order if the loading function does not depend on the third invariant.

Further work could be done on the analysis of non-traditional scalar damage models, non-traditional in the sense mentioned in Chapter 3. For example the von-Mises damage type seems particularly interesting and directly comparable with the traditional $J_2$-plasticity formulation. Also other forms of scalar anisotropic damage should be analyzed in detail.

The more general case of tensorial damage variables of higher order are topics for further work, even though the derivation for the localization condition does not appear to be as straight forward as it was for the simple scalar damage description.

Other possible developments are the extension of the fracturing and damage concepts to other continuum formulations such as the micropolar Cosserat theory, which should lead to regularization effects. The coupling with elastic-plastic and, or visco-plastic models could also be a topic of further research.

Another obvious task would be the implementation of the fracture models into a finite element code for checking localization indicators at the structural level. Mesh dependence problems are still expected if no regularization is introduced. Alignment techniques or alternative enrichment of the underlying element approximations, to capture discontinuities within the element domain, are then appropriate.

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