Response to generic loading (Duhamel convolution integral)
F(t)
ΔF_i   F_{i+1} = F_i + ΔF_i
ΔI_i = F_i Δτ_i
time step
INPUT
t_k − τ_i
τ_{i+1} = τ_i + Δτ_i
t_k
F(t_i) ≃ Σ_{i=0}^{k} F_i Δτ_i δ(t_k, τ_i)
ΔI_i
Dirac delta function
(unit pulse loading)
superposition of discrete pulses ΔI_i
Δτ_i → 0   infinite   dI
F(t) = ∫_0^t F(τ) dτ δ(t, τ)
t − τ
h(t_k, τ_i)
OUTPUT (Response)
response to unit pulse
Alternatively
u_0(t_k) = Σ_{i=0}^{k} F_i Δτ_i h(t_k, τ_i)
where h(t) = 1/(m ω_d) e^{−ζ ω_1 t} sin ω_d t
ω_d = ω_1 √(1 − ζ²)
u_0(t) = ∫_0^t F(τ) dτ h(t, τ)
= ∫_0^t F(τ) h(t − τ) dτ
F(t_k) = F_0 + Σ_{i=1}^{k} ΔF_i
A(t) = 1/k (1 − e^{−ζ ω_1 t} (ζ ω_1/ω_d sin ω_d t + cos ω_d t))
A_0 = A(0) = 0
response to unit step
u_0(t) = F_0 A(t) + Σ_{i=1}^{k} ΔF_i/Δτ_i A(t, τ_i) Δτ_i
lim Δτ_i → 0
= F_0 A(t) + ∫_0^t Ḟ(τ) A(t − τ) dτ
by parts
= F_0 A(t) + [F(τ) A(t − τ)]_0^t + ∫_0^t F(τ) Ȧ(t − τ) dτ
− rarely generic F(t) represented by an analytic function
− most of the times recorded at selected discrete time instants τ_i = t_i
⇒ vectors t_i , F_i
due to properties of δ
F(t) = ∫_0^t F(τ) δ(t, τ) dτ
t − τ
Convolution integral
Duhamel integral
u_0(t) = 1/(m ω_d) ∫_0^t F(τ) e^{−ζ ω_1 (t−τ)} sin ω_d (t − τ) dτ
sin ω_d t cos ω_d τ − cos ω_d t sin ω_d τ
D.I : − rarely could be performed analytically, unless for "simple" F(t)
− in practice performed by numerical integration

Response to generic loading
OUTPUT (Response)
INPUT
Dirac delta function
(unit pulse loading)
superposition of discrete pulses ΔI_i
where $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_1 t} \sin\omega_d t$
$\omega_d = \omega_1\sqrt{1-\zeta^2}$
$F(t) = \int_0^t F(\tau)\delta(t,\tau)d\tau$
Convolution integral
Duhamel integral
$u(t) = \frac{1}{m\omega_d}\int_0^t F(\tau) e^{-\zeta\omega_1(t-\tau)} \sin\omega_d(t-\tau)d\tau + u_0(t)$
D.I : - rarely could be performed analytically, unless for "simple" F(t)
Alternatively
$F(t_k) = F_0 + \sum_{i=1}^{k} \Delta F_i$
$A(t) = \frac{1}{k}\left(1 - e^{-\zeta\omega_1 t}\left(\zeta\frac{\omega_1}{\omega_d}\sin\omega_d t + \cos\omega_d t\right)\right)$
$A_0 = A(0) = 0$
response to unit step
$u_0(t) = F_0 A(t) + \sum_{i=1}^{k} \frac{\Delta F_i}{\Delta\tau_i} A(t,\tau_i)\Delta\tau_i$
$= F_0 A(t) + \int_0^t \dot{F}(\tau) A(t-\tau)d\tau$
by parts
$= F_0 A(t) + [F(\tau)A(t-\tau)]_0^t + \int_0^t F(\tau)\dot{A}(t-\tau)d\tau$
D.I.
- Rewritings of Duhamel integral
- if $F(t) = F f(t)$
$u_0(t) = u_{st} N(t)$
dynamic ampl. factor
pseudo velocity
time-varying amplitudes
Rectangular rule
Parabolic rule - SIMPSON

OUTPUT (Response) response to unit pulse

$h(t_k, \tau_i)$

where $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_1 t} \sin\omega_d t = \dot{A}(t)$

$\omega_d = \omega_1\sqrt{1-\zeta^2}$

$h(t, \tau)$

Convolution integral

Duhamel integral

$$u(t) = \frac{1}{m\omega_d}\int_0^t F(\tau) e^{-\zeta\omega_1(t-\tau)} \sin\omega_d(t-\tau)\, d\tau + u_0(t)$$

Alternatively

$$F(t_k) = F_0 + \sum_{i=1}^{k} \Delta F_i$$

$$\bullet\ A(t) = \frac{1}{k}\left(1 - e^{-\zeta\omega_1 t}\left(\zeta\frac{\omega_1}{\omega_d}\sin\omega_d t + \cos\omega_d t\right)\right)$$

response to unit step

$A_0 = A(0) = 0$

$$u_0(t) = F_0 A(t) + \sum_{i=1}^{k} \frac{\Delta F_i}{\Delta\tau_i} A(t_k, \tau_i)\Delta\tau_i$$

$\lim_{\Delta\tau_i \to 0}$

$$= F_0 A(t) + \int_0^t \dot{F}(\tau) A(t-\tau)\, d\tau$$

by parts

$$= F_0 A(t) + \left[F(\tau) A(t-\tau)\right]_0^t + \underbrace{\int_0^t F(\tau) \overbrace{\dot{A}(t-\tau)}^{h(t-\tau)}\, d\tau}_{\text{D.I.}}$$

- Rewritings of Duhamel integral

- if $F(t) = F f(t)$

$u_0(t) = u_{st} N(t)$

with $u_{st} = \frac{F}{k}$

$$N(t) = \frac{\omega_1}{\sqrt{1-\zeta^2}}\int_0^t f(\tau) e^{-\zeta\omega_1(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

dynamic ampl. factor (t)

$$- u_0(t) = \frac{1}{\omega_1} V(t)$$

$$V(t) = \frac{1}{m\sqrt{1-\zeta^2}}\int_0^t F(\tau) e^{-\zeta\omega_1(t-\tau)} \sin\omega_d(t-\tau)\, d\tau \simeq V_\zeta(t)$$

pseudo velocity $\zeta \ll 1$, $\omega_d \simeq \omega_1$

$$- u_0(t) = A(t)\sin\omega_d t - B(t)\cos\omega_d t$$

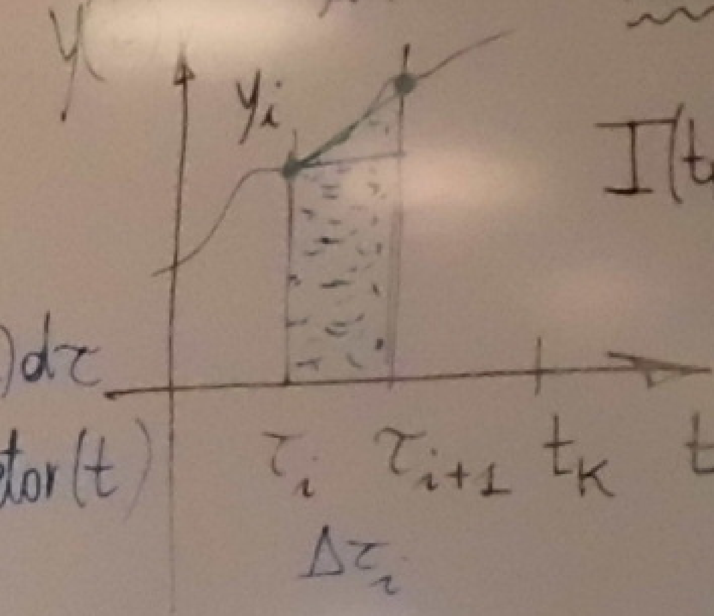
$$A(t) = \frac{e^{-\zeta\omega_1 t}}{m\omega_d}\int_0^t F(\tau) e^{\zeta\omega_1\tau} \cos\omega_d\tau\, d\tau$$

time-varying amplitudes

$$B(t) = \frac{e^{-\zeta\omega_1 t}}{m\omega_d}\int_0^t \underbrace{F(\tau) e^{\zeta\omega_1\tau} \sin\omega_d\tau}_{y(\tau)}\, d\tau$$

- Numerical evaluation

$y_{i+1} = y_i + \Delta y_i$ Rectangular rule



$$I(t_k) \simeq \sum_{i=0}^{k-1} y_i \Delta\tau_i$$

if $\Delta\tau_i = \text{const} = \Delta\tau$ time step $= \frac{t_f - t_0}{n}$

$$= \Delta\tau\left(y_0 + y_1 + \cdots + y_{k-1}\right)$$

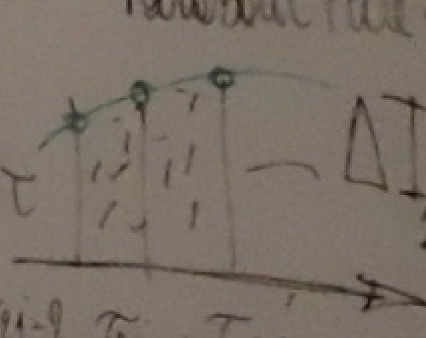
Trapezoidal rule

$$I(t_k) \simeq \sum_{i=0}^{k-1} \frac{y_i + y_{i+1}}{2}\Delta\tau_i$$

$$= \Delta\tau\frac{1}{2}\left(y_0 + 2y_1 + \cdots + 2y_{k-2} + y_{k-1}\right)$$

$2\Delta\tau_i$

Parabolic rule $\to$ SIMPSON



$$\Delta I_{2i-1} = \left(y_{2i-2} + 4y_{2i-1} + y_{2i}\right)\frac{\Delta\tau}{3}$$

even number of $\Delta\tau_i$

Motion of the reference system
relative displ. to the support
apparent force
Typical feature of EQ
PGA ~ 0.1 - 3g
accelerometer
Response
g ≈ 9.81 m/s²
10s - 60s
pseudo-velocity
Response spectra (BIOT, 1932)
amplification
stiff
EQ₁
EQ₂
design
— Occurs frequently, e.g. mechanical systems
A typical civil engineering context
— structure has an extension that is small wrt length of seismic waves
— soil-structure interaction is limited
SSI effects
total


Motion of the reference system
m
apparent force
$\ddot{u}_t(t) = F(t)$
$g = 9.81 \frac{m}{s^2}$
Typical feature of EQ
PGA
$\Delta t = \frac{1}{50} - \frac{1}{200}$ s
accelerometer
10s–60s
Response spectra (BIOT, 1932)
$S_d = \max_t \{u(t); T_1, \xi\}$
$S_v = \max_t \{\dot{u}(t)\} \approx PS_v$
$S_a = \max_t \{\ddot{u}_{tot}(t)\} \approx PS_a$
SSI effects
$\frac{S_a}{g}$
EQ2
EQ1
amplification
stiff
soft (flexible)
seismic isolation
design stage
(Standards, norms)
S_a^d
0.1 s
1 s
natural period
Pseudo-response spectra


Typical feature of EQ
PGA = 0.1-3g
accelerometer
10s-60s
Response spectra (BIOT, 1932)
Envelope of response spectra
EQ2
EQ1
PGA
amplification
stiff
soft (flexible)
seismic isolation
design stage
(Standards, norms)
~ 1÷2 PGA
natural period
0.1 s
1 s
Pseudo-response spectra
motivation
Shear force at the base (base shear)