Università degli studi di Bergamo

Scuola di Ingegneria (Dolmine)

CCS Ingegneria Edile

LM-24 Ingegneria delle Costruzioni Edili

Dinamica, Instabilità e Anelasticità delle Strutture

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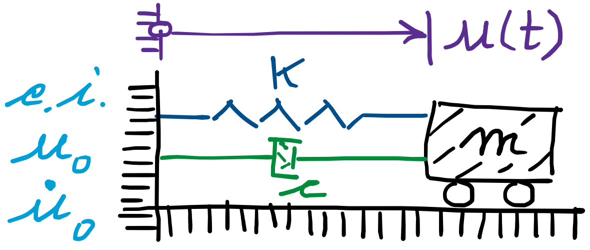
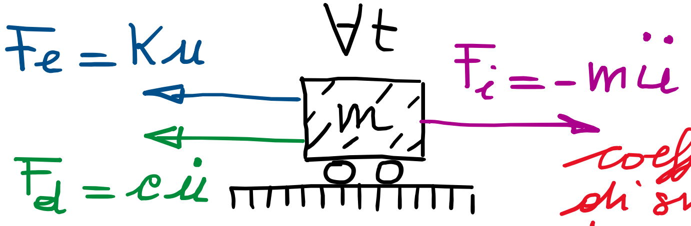
LEZIONE 03

Oscillazioni libere smorzate



m, c, K cost (>0)

sistemi tempo-invarianti



Eq. del moto (equilibrio dinamico):

$$\cancel{m}\,\ddot u(t) + \underbrace{\frac{c}{m}}_{2\zeta\omega_1}\dot u(t) + \underbrace{\frac{K}{m}}_{\omega_1^2} u(t) = \frac{0}{m} \qquad (\zeta, \omega_1)$$

coefficiente di smorzamento

$\frac{c}{m} = 2\zeta\omega_1$ definizione

$\omega_1^2 \Rightarrow \omega_1 = \sqrt{\frac{K}{m}}$ pulsazione naturale del sistema non smorzato

fattore di smorzamento (relativo al critico)

$$\zeta = \frac{c}{2m\omega_1} = \frac{c}{2m\sqrt{\frac{K}{m}}} = \frac{c}{\underbrace{2\sqrt{Km}}_{c_{cr}}} = \frac{c}{c_{cr}} \qquad (c = c_{cr},\ \zeta = 1)$$

$[\zeta] = [1]$

Per strutture civili:

$\zeta \simeq 1\%$ $(2\% - 7\%)$

tipic. $5\% = 0.05$

$\zeta \ll 1 \simeq 0.01$

eq. differenziale del 2° ordine e coeff. cost.

$$\ddot u + 2\zeta\omega_1\dot u + \omega_1^2 u = 0 \qquad (*)$$

Si cercano soluz. nella forma:

$u(t) = e^{\lambda t}$

$\dot u(t) = \lambda e^{\lambda t}$

$\ddot u(t) = \lambda^2 e^{\lambda t}$

Sostituendo in (*)

$$\underbrace{(\lambda^2 + 2\zeta\omega_1\lambda + \omega_1^2)}_{=0}\,\cancel{e^{\lambda t}} = 0 \quad \forall t$$

eq. caratteristica (associata all'eq.ne differenziale di partenza) algebrica, di 2° grado

$$\lambda^2 + 2\zeta\omega_1\lambda + \omega_1^2\lambda^0 = 0$$

$\Rightarrow$ radici (due) $\lambda_{1,2}$ dell'eq.ne caratteristica (poli)

$$\lambda^2 + 2\zeta\omega_1\lambda + \omega_1^2 = 0 \Rightarrow \lambda_{1,2} = -\zeta\omega_1 \pm \sqrt{\zeta^2\omega_1^2 - \omega_1^2}$$

$$= -\zeta\omega_1 \pm \omega_1\sqrt{\zeta^2-1} = \omega_1\left(-\zeta \pm \sqrt{\zeta^2-1}\right)$$

$$\zeta^2 - 1 = -(1-\zeta^2) = i^2(1-\zeta^2)$$

Casistica radici:

smorzamento

$\zeta < 1$ subcritico $\quad \lambda_{1,2} = -\zeta\omega_1 \pm i\,\omega_1\sqrt{1-\zeta^2}$ due radici complesse coniugate

$\zeta = 1$ critico $\quad \lambda_{1,2} = -\omega_1$ due radici reali coincidenti (<0)

$\zeta > 1$ supercritico $\quad \lambda_{1,2} = -\zeta\omega_1 \pm \omega_1\sqrt{\zeta^2-1}$ due radici reali distinte (<0)

$c < c_{cr}$

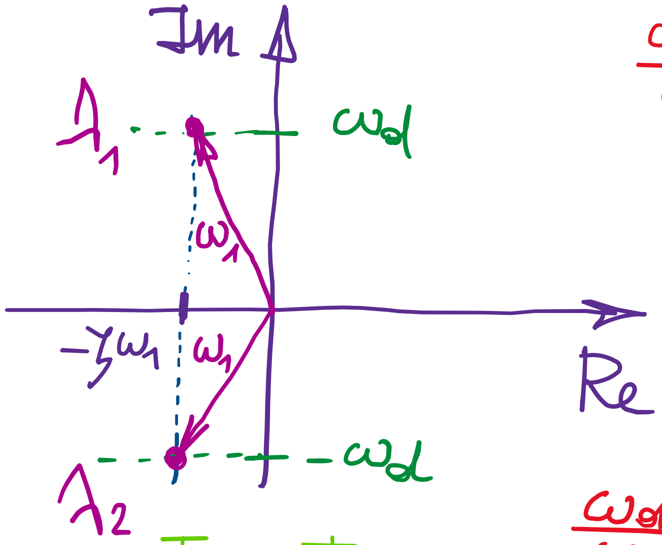
• Caso subcritico ($\zeta < 1$):

$$\lambda_{1,2} = \underbrace{-\zeta\omega_1}_{\substack{Re[\lambda_{1,2}] \\ <0}} \pm \underbrace{i\,\omega_1\sqrt{1-\zeta^2}}_{Im[\lambda_{1,2}]}$$

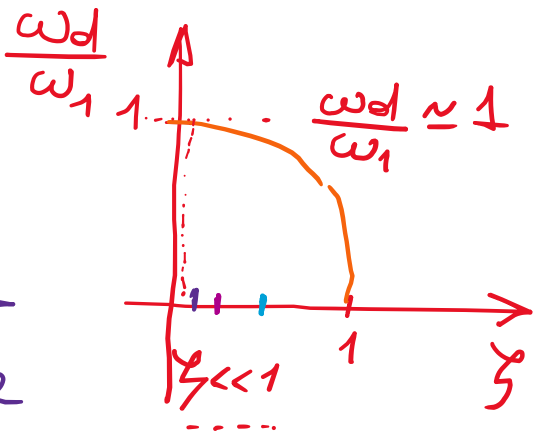
$$|\lambda_{1,2}| = \omega_1$$

$$\omega_d = \omega_1\sqrt{1-\zeta^2} \le \omega_1 \simeq \omega_1 \ (\zeta \ll 1)$$

pulsazione naturale del sistema smorzato



$T_d \ge T_1$
$\simeq T_1$



$$\frac{\omega_d}{\omega_1} = \sqrt{1-\zeta^2}; \quad \left(\frac{\omega_d}{\omega_1}\right)^2 = 1-\zeta^2$$

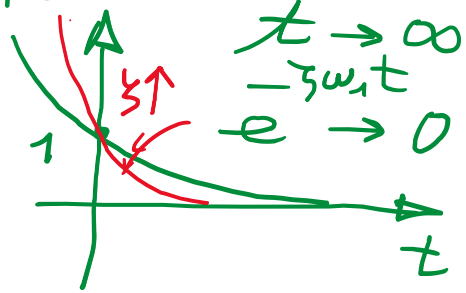
$$\left(\frac{\omega_d}{\omega_1}\right)^2 + \zeta^2 = 1$$

Pertanto: $\quad (\omega_d = \omega_1\sqrt{1-\zeta^2})$

$$\lambda_{1,2} = -\zeta\omega_1 \pm i\,\omega_d$$

comp.te di moto armonico di pulsazione ω_d

$$\text{Integrale } e^{\lambda_{1,2}t} = e^{(-\zeta\omega_1 \pm i\omega_d)t} = \underbrace{e^{-\zeta\omega_1 t}}_{\text{ampiezza esponenzialmente decadente in } t} \cdot \overbrace{e^{\pm i\omega_d t}}^{\sin\omega_d t,\ \cos\omega_d t}$$



$t \to \infty \quad e^{-\zeta\omega_1 t} \to 0$

Integrale generale:

$$u(t) = e^{-\zeta\omega_1 t}\cdot\left(\overbrace{A}^{\frac{\dot u_0 + \zeta\omega_1 u_0}{\omega_d}} \sin\omega_d t + \overbrace{B}^{u_0}\cos\omega_d t\right)$$

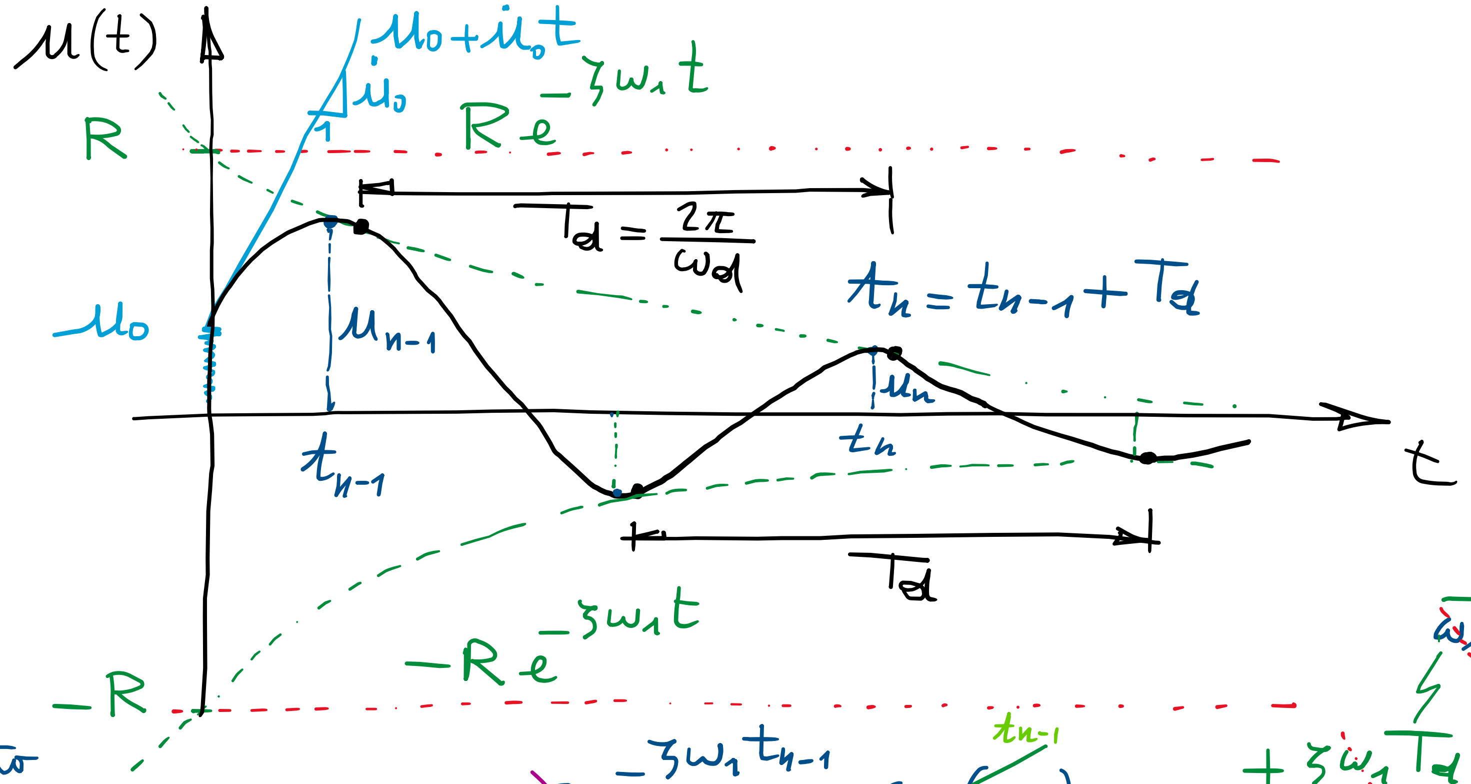
$$= R\,e^{-\zeta\omega_1 t}\cos(\omega_d t - \varphi) = R\,e^{-\zeta\omega_1 t}\sin(\omega_d t + \psi)$$

$$\Rightarrow \dot u(t) = -\zeta\omega_1 e^{-\zeta\omega_1 t}(A\sin\omega_d t + B\cos\omega_d t) + e^{-\zeta\omega_1 t}\,\omega_d\,(A\cos\omega_d t - B\sin\omega_d t)$$

Moto oscillatorio effetto di un moto armonico di periodo $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_1\sqrt{1-\zeta^2}} = \frac{T_1}{\sqrt{1-\zeta^2}}$
con ampiezza decadente in t (esponenzialmente) e rapidità legata a ζ.

Dalle c.i.:

$$\begin{cases} u_0 = u(0) = B \Rightarrow \boxed{B = u_0} \\ \dot u_0 = \dot u(0) = -\zeta\omega_1 B + \omega_d A \Rightarrow \boxed{A = \dfrac{\dot u_0 + \zeta\omega_1 B}{\omega_d} = \dfrac{\dot u_0 + \zeta\omega_1 u_0}{\omega_d}} \end{cases}$$



moto oscillatorio aperiodico con ampiezza decrescente e valori max/min che si riducono, ogni T_d, dello stesso rapporto $r=e^{\delta}$

rapporto tra ampiezze max/min successive

$$r=r_n=\frac{u_{n-1}}{u_n}=\frac{\cancel{R}\,e^{-\zeta\omega_1 t_{n-1}}\,\cancel{\cos(\overset{t_{n-1}}{\;})}}{\cancel{R}\,e^{-\zeta\omega_1(t_{n-1}+T_d)}\,\cancel{\cos(\underset{t_n}{\overset{t_{n-1}+T_d}{\;}})}}=e^{+\zeta\omega_1 T_d}=\underline{\text{cost}}=e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

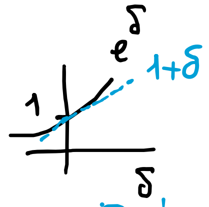
$$T_d=\frac{2\pi}{\cancel{\omega_1}\sqrt{1-\zeta^2}}$$

$$\delta=\ln r=\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\underset{\zeta\ll1}{\simeq}2\pi\zeta\;\Rightarrow\;\boxed{\zeta=\frac{\delta}{2\pi}}$$

decremento logaritmico

$\delta\ll1$

$$\hookrightarrow r=e^{\delta}\simeq1+\delta\;\Rightarrow\;\delta=r-1=\frac{u_{n-1}}{u_n}-1=\frac{u_{n-1}-u_n}{u_n}=\frac{\Delta u_n}{u_n}$$



approx per δ piccolo

stima pratica di δ per $\zeta\ll1$ (e anche $\delta\ll1$)

$$\delta^2(1-\zeta^2)=4\pi^2\zeta^2$$
$$\delta^2-\zeta^2\delta^2=4\pi^2\zeta^2$$
$$\delta^2=(4\pi^2+\delta^2)\zeta^2$$

stima di ζ (moto sperim. δ)

$$\boxed{\zeta=\frac{\delta}{\sqrt{4\pi^2+\delta^2}}=\frac{\delta}{2\pi\sqrt{\underbrace{1+(\delta/2\pi)^2}_{\simeq1}}}}$$

• Caso critico ($\zeta = 1$) $c = c_{cr}$

$\lambda_{1,2} = -\omega_1 \Rightarrow e^{-\omega_1 t}$; $t\, e^{-\omega_1 t}$ (verific.) ← ulteriore integrale

NB.: Il caso critico discrimina tra risposta oscillatoria e non

$$u(t) = e^{-\omega_1 t} \cdot (A + Bt)$$

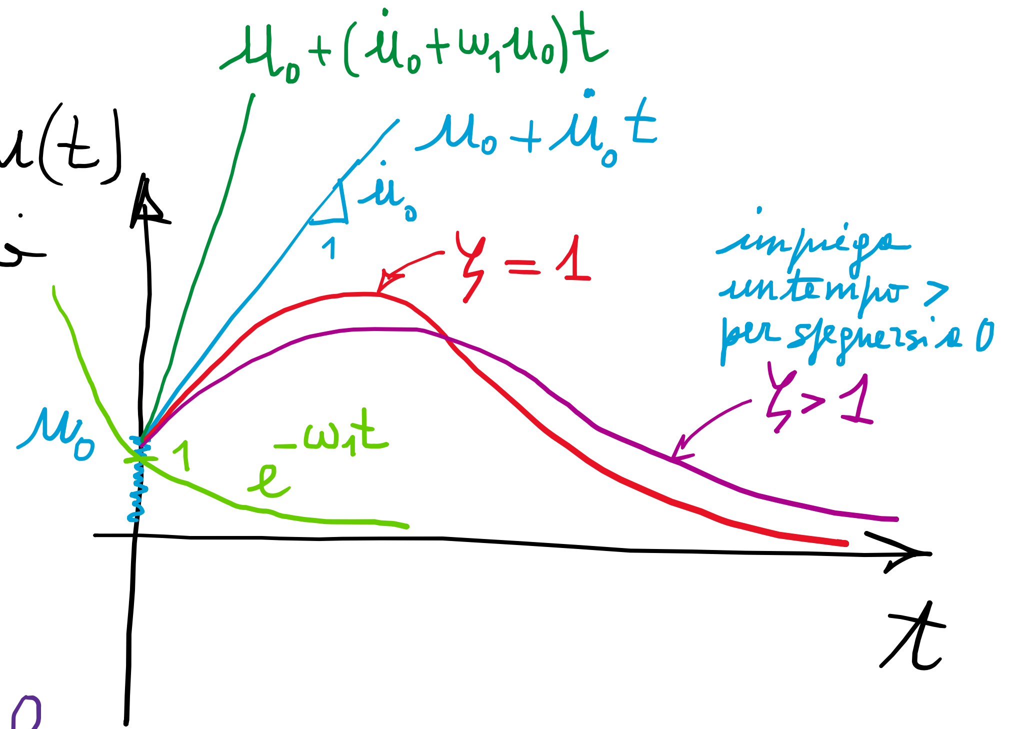
moto non oscillatorio decadente in t

$\boxed{u_0 = A}$

$\dot u_0 = -\omega_1 A + B \Rightarrow \boxed{B = \dot u_0 + \omega_1 u_0}$

$$= e^{-\omega_1 t} \cdot \left(u_0 + (\dot u_0 + \omega_1 u_0) t\right)$$

(risposta "resta della parte delle c.i.")



• Caso supercritico ($\zeta > 1$) $c > c_{cr}$

$\lambda_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_1 < 0$

$\lambda_1 - \lambda_2 = 2\sqrt{\zeta^2 - 1}\, \omega_1 > 0$

$$u(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$

$$\dot u(t) = \lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}$$

$\begin{aligned} u_0 &= A + B \\ \dot u_0 &= \lambda_1 A + \lambda_2 B \end{aligned} \Rightarrow \begin{aligned} \lambda_2 u_0 - \dot u_0 &= -(\lambda_1 - \lambda_2) A \\ \lambda_1 u_0 - \dot u_0 &= (\lambda_1 - \lambda_2) B \end{aligned} \Rightarrow \begin{aligned} A &= -\frac{\lambda_2 u_0 - \dot u_0}{\lambda_1 - \lambda_2} \\ B &= \frac{\lambda_1 u_0 - \dot u_0}{\lambda_1 - \lambda_2} \end{aligned}$

finale:

$$u(t) = -\frac{\lambda_2 u_0 - \dot u_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{\lambda_1 u_0 - \dot u_0}{\lambda_1 - \lambda_2} e^{\lambda_2 t}$$

resta moto non oscillatorio con tendenza asintotica

Concetti fondamentali:

- Oscillazioni libere smorzate:

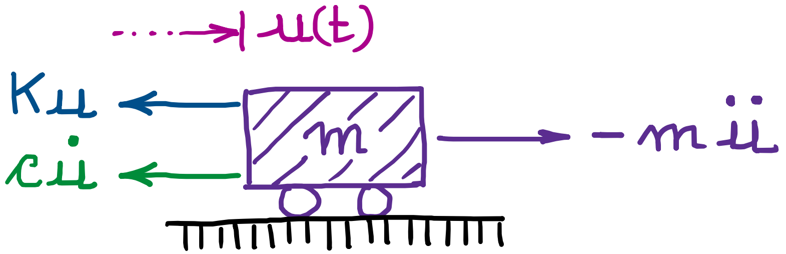
$\omega_d = \omega_1\sqrt{1-\zeta^2}$ pulsaz. naturale sistema smorzato ← (tipicamente subcritico, $\zeta<1$ e $\ll 1$, $\zeta \sim 1\% = 0.01$)

$$\zeta = \frac{c}{2\sqrt{Km}} = \frac{c}{c_{cr}}$$

fattore di smorzamento

$m, c, K = \text{cost}$

sistema tempo-invariante



con c.i. $\begin{cases} u_0 \\ \dot u_0 \end{cases}$ @ $t=t_0$

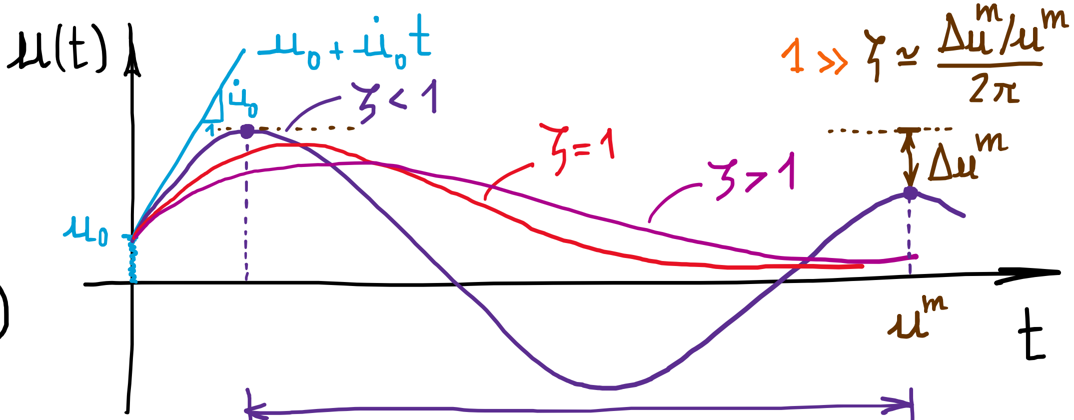
$$m\ddot u(t) + c\dot u(t) + Ku(t) = 0 \Rightarrow \ddot u(t) + \overbrace{2\zeta\omega_1}^{\frac{c}{m}}\dot u(t) + \overbrace{\omega_1^2}^{\frac{K}{m}} u(t) = 0 \qquad \omega_1 = \sqrt{\frac{K}{m}}$$

- Integrale generale (soluzione) ⇒ moto oscillatorio smorzato ad ampiezza variabile per $\zeta<1$, moto non oscillatorio smorzato per $\zeta \geq 1$.

$\zeta<1$ $\quad u(t) = e^{-\zeta\omega_1 t}(A\sin\omega_d t + B\cos\omega_d t)$

$\zeta=1$ $\quad u(t) = e^{-\omega_1 t}(A + Bt)$

$\zeta>1$ $\quad u(t) = e^{-\zeta\omega_1 t}\left(A e^{\omega_1\sqrt{\zeta^2-1}\,t} + B e^{-\omega_1\sqrt{\zeta^2-1}\,t}\right)$



$$1 \gg \zeta \simeq \frac{\Delta u^m / u^m}{2\pi}$$

$$T_d = \frac{T_1}{\sqrt{1-\zeta^2}} \quad \left(T_1 = \frac{2\pi}{\omega_1}\right)$$

SOMMARIO (Lez. 03)

- Oscillazioni libere smorzate (in risposta alle sole c.i.).
- Fattore di smorzamento (~ 1% per strutture civili).
- Radici dell'eq.ne caratteristica: poli.
- Casistica: subcritico → moto oscillatorio con ampiezza decadente.
  critico → moto aperiodico non oscillatorio.
  supercritico → idem, con ampiezza iniziale e picco inferiore.
- Decremento logaritmico e stima del fattore di smorzamento.
- Integrale generale e imposizione delle c.i.
- Next step: visto l'integrale generale dell'eq.ne omogenea con termine noto nullo. Da sovrapporsi ad integrale particolare dipendente dalla forzante → risposta forzata.