Università degli studi di Bergamo
Scuola di Ingegneria (Dalmine)
CCS Ingegneria Edile
LM-24 Ingegneria delle Costruzioni Edili

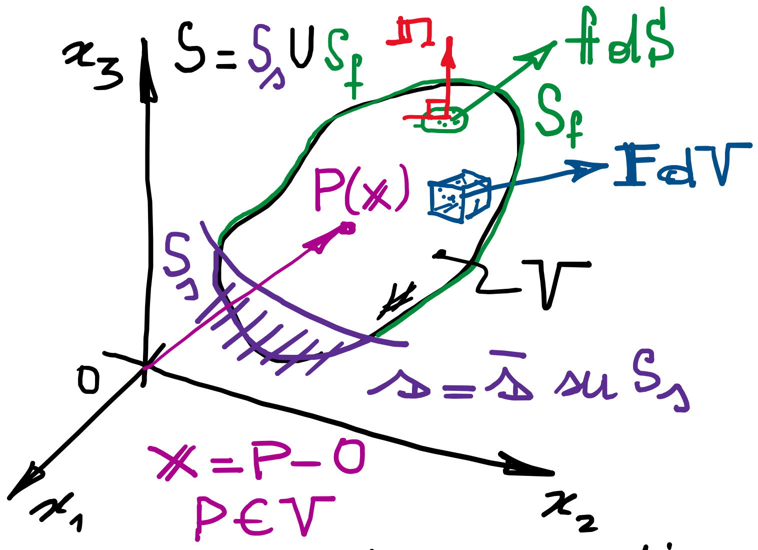
Complementi di Scienza delle Costruzioni

(ICAR/08 - SdC; 6 CFU)

prof. Egidio RIZZI
egidio.rizzi@unibg.it

LEZIONE 14

Legame costitutivo (elastico) [nella formulazione del "problema elastico" (lineare)]



Risposta tenso-deformativa: (da determinare)

tensore sforzo di Cauchy $\boldsymbol{\sigma}(\mathbf{x}) \leftrightarrow \sigma_{ij}(x_k)$: 6

tensore deformazione (piccole deformazioni) $\boldsymbol{\varepsilon}(\mathbf{x}) \leftrightarrow \varepsilon_{ij}(x_k)$: 6

vettore spostamento $\mathbf{s}(\mathbf{x}) \leftrightarrow s_i(x_k)$: 3

n° componenti incognite 15 (12)

Equazioni governanti:

statica dei continui

equilibrio: $\operatorname{div}\boldsymbol{\sigma} + \mathbf{F} = \mathbf{0} \leftrightarrow \sigma_{ij,i} + F_j = 0_j$ in V $(\forall\, \mathbf{x} \in V)$: 3

[con c.c. $\mathbf{t}_n = \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{f} \leftrightarrow n_i\, \sigma_{ij} = f_j$ su S_f]

cinematica dei continui

congruenza: $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{s} + \nabla \mathbf{s}^T) \leftrightarrow \varepsilon_{ij} = \frac{1}{2}(s_{i,j} + s_{j,i})$ in V : 6

[con c.c. $\mathbf{s} = \bar{\mathbf{s}} \leftrightarrow s_i = \bar{s}_i$ su S_s]

$\mathbf{S} = \mathbf{0} \leftrightarrow \varepsilon_{ij,kl} + \varepsilon_{kl,ij} \overset{81 \to 6}{=} \varepsilon_{ik,jl} + \varepsilon_{jl,ik}$ $(S_{ij,i} = 0)$: 3 indipendenti

legame costitutivo

legame sforzi/deform.

: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ o $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \leftrightarrow \sigma_{ij} = \sigma_{ij}(\varepsilon_{kl})$ o $\varepsilon_{ij} = \varepsilon_{ij}(\sigma_{kl})$: 6

(comportamento meccanico del materiale)

n° equazioni 15 (12)

Legame costitutivo iperelastico (in generale non lineare)

legame olonomo (legge intera), perfettam. reversibile

Trasformata di Legendre di $\omega(\boldsymbol{\varepsilon})$

$$\boldsymbol{\sigma}:\boldsymbol{\varepsilon} = \omega(\boldsymbol{\varepsilon}) + \gamma(\boldsymbol{\sigma})$$

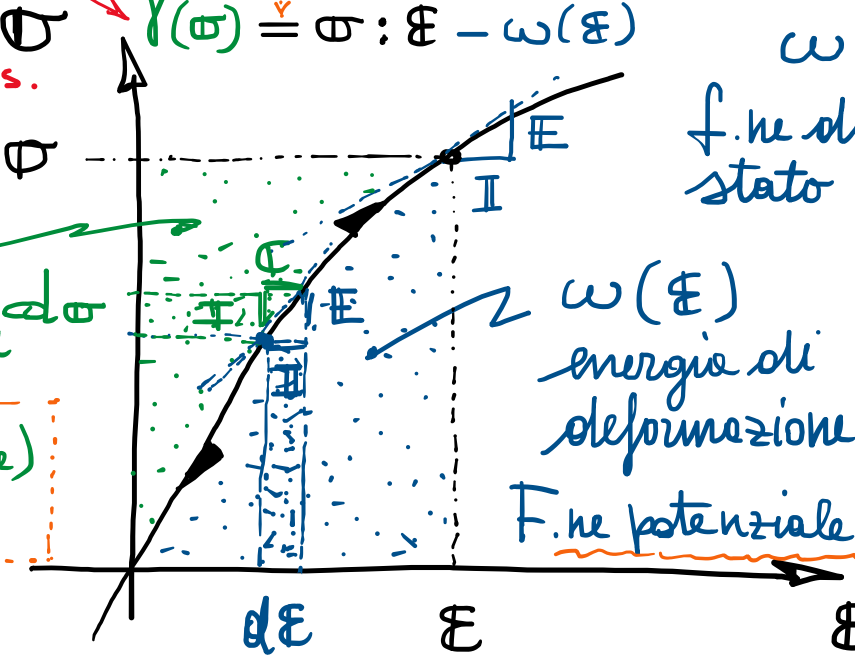
$$\gamma(\boldsymbol{\sigma}) = \boldsymbol{\sigma}:\boldsymbol{\varepsilon} - \omega(\boldsymbol{\varepsilon})$$

$$\gamma(\boldsymbol{\sigma}) = \int_0^{\boldsymbol{\sigma}} \underbrace{\boldsymbol{\varepsilon}:d\boldsymbol{\sigma}}_{d\gamma} > 0 \quad \text{def. pos.} \quad \forall \boldsymbol{\sigma} \neq \mathbf{0}$$

energia complementare $\gamma(\boldsymbol{\sigma})$

F.ne potenziale di deformazione

$$\boldsymbol{\varepsilon} = \frac{\partial \gamma(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \leftrightarrow \varepsilon_{ij} = \frac{\partial \gamma(\sigma_{kl})}{\partial \sigma_{ij}}$$



$$\omega(\boldsymbol{\varepsilon}) = \int_0^{\boldsymbol{\varepsilon}} \underbrace{\boldsymbol{\sigma}:d\boldsymbol{\varepsilon}}_{d\omega} > 0 \quad \forall \boldsymbol{\varepsilon} \neq \mathbf{0}$$

$\sigma_{ij}\, d\varepsilon_{ij}$

definita positiva

f.ne di stato

$d\omega$ differenziale esatto (della f.ne $\omega(\boldsymbol{\varepsilon})$)

CNS $\dfrac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \dfrac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}}$ (Th. Schwarz)

sse $\boldsymbol{\sigma} = \dfrac{\partial \omega(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \leftrightarrow \sigma_{ij} = \dfrac{\partial \omega(\varepsilon_{kl})}{\partial \varepsilon_{ij}}$

$\omega(\boldsymbol{\varepsilon})$ energia di deformazione

F.ne potenziale di sforzo

(George GREEN ~ 1839)

$$\mathbb{C}(\boldsymbol{\sigma}) = \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\sigma}} = \frac{\partial^2 \gamma(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}\, \partial \boldsymbol{\sigma}} > 0$$

$$C_{ijkl} = \frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial^2 \gamma(\sigma_{rs})}{\partial \sigma_{ij}\, \partial \sigma_{kl}}$$

tensore di cedevolezza (tangente, $\mathbb{C}_t$)

[idem] $C_{ijkl} = C_{jikl} = C_{ijlk}$

$C_{ijkl} = C_{klij}$

$$\mathbb{C} = \mathbb{E}^{-1}$$

$$\mathbb{C}^{-1} = \mathbb{E}$$

(invertibilità data def. pos.)

$$\mathbb{E}(\boldsymbol{\varepsilon}) = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \frac{\partial^2 \omega(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}\, \partial \boldsymbol{\varepsilon}} > 0$$

$$E_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 \omega(\varepsilon_{rs})}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}}$$

$3^4 = 81$ compon.

tensore di rigidezza (tangente, $\mathbb{E}_t$) 4° ord.

36 ← • simmetrie minori $E_{ijkl} = E_{jikl} = E_{ijlk}$ ($\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}^T = \boldsymbol{\varepsilon}$)

21 ← • simmetrie maggiori $E_{ijkl} = E_{klij}$ (Th. di Schwarz)

• **Legame iperelastico lineare**

$\forall \sigma \neq 0$

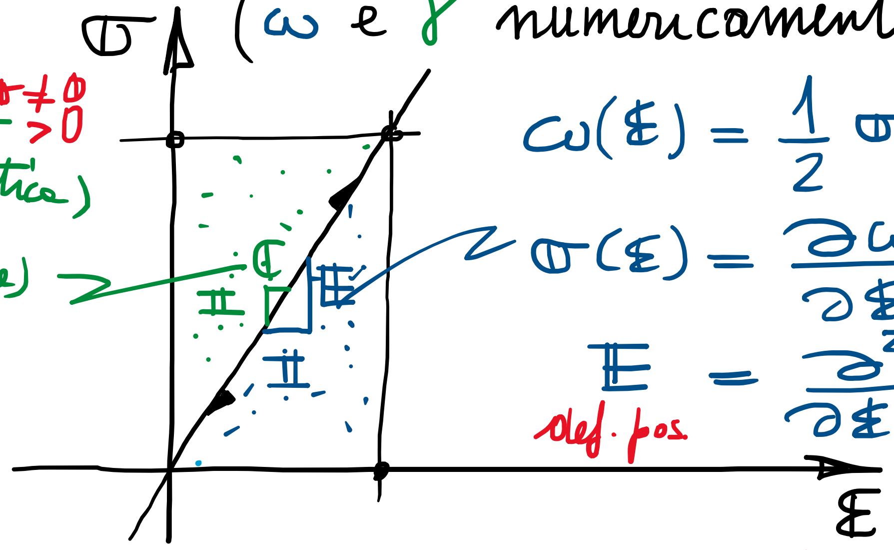
$$\gamma(\sigma) = \frac{1}{2}\,\varepsilon : \sigma = \frac{1}{2}\,\sigma : \mathbb{C} : \sigma > 0 \quad \text{(quadratica)}$$

$$\varepsilon(\sigma) = \frac{\partial \gamma}{\partial \sigma} = \mathbb{C} : \sigma \quad \text{(lineare)}$$

$$\mathbb{C} = \frac{\partial^2 \gamma}{\partial \sigma \partial \sigma} \quad \text{(costante, 21 coeff.)}$$

def. pos.

(ω e γ numericamente uguali)



$\forall \varepsilon \neq 0$

$$\omega(\varepsilon) = \frac{1}{2}\,\sigma : \varepsilon = \frac{1}{2}\,\varepsilon : \mathbb{E} : \varepsilon > 0 \quad \text{(quadratica)}$$

$$\sigma(\varepsilon) = \frac{\partial \omega}{\partial \varepsilon} = \mathbb{E} : \varepsilon \quad \text{(lineare)}$$

$$\mathbb{E} = \frac{\partial^2 \omega}{\partial \varepsilon \partial \varepsilon} \quad \text{(costante, 21 coefficienti)}$$

def. pos.

[generalm. anisotropo]

• **Legame iperelastico lineare isotropo** (comp. meccanico indipendente dalle direzioni)

21 → 2 parametri indipendenti (es. E, ν)

Costanti di Lamé (prima e seconda) λ μ

legame inverso $\varepsilon(\sigma)$

$$\varepsilon = -\frac{\nu}{E}\,\mathrm{tr}\,\sigma\,\mathbb{I} + \boxed{\frac{1+\nu}{E}}\,\sigma$$

$$\frac{1}{2G}\,; \quad G = \mu = \frac{E}{2(1+\nu)} > 0$$

legame diretto $\sigma(\varepsilon)$

$$\sigma = \lambda\,\mathrm{tr}\,\varepsilon\,\mathbb{I} + 2\mu\,\varepsilon$$

$\mu = G$ modulo di taglio > 0

$0 < E$: modulo di Young

$G \to \infty$, $\nu \to -1$

$\varepsilon_{11} \neq 0$; $\varepsilon_{22} = \varepsilon_{33} = 0$ $\quad \lambda = \dfrac{\sigma_{22} = \sigma_{33}}{\varepsilon_{11}}$

ν : coeff. di contrazione trasversale o di Poisson

$$\nu = -\frac{\varepsilon_{22} = \varepsilon_{33}}{\varepsilon_{11}}$$

$$-1 < \nu < \frac{1}{2}$$

per def. pos. di $\mathbb{C}$

$$\mathrm{tr}\,\varepsilon = -\frac{\nu}{E}\,\mathrm{tr}\,\sigma\,3 + \frac{1+\nu}{E}\,\mathrm{tr}\,\sigma \longleftrightarrow \mathrm{tr}\,\sigma = \lambda\,\mathrm{tr}\,\varepsilon\,3 + 2\mu\,\mathrm{tr}\,\varepsilon$$

deformaz. volumetrica $\quad \vartheta = \dfrac{1-2\nu}{E}\,\underbrace{\dfrac{\mathrm{tr}\,\sigma}{3}}_{p}\,3$

$$3\,\underbrace{\frac{\mathrm{tr}\,\sigma}{3}}_{p} = \underbrace{(3\lambda + 2\mu)}_{3K}\,\underbrace{\mathrm{tr}\,\varepsilon}_{\vartheta} \Rightarrow p = K\,\vartheta$$

$$\boxed{K = \lambda + \frac{2}{3}\,\mu}$$

p tensione media

$$\vartheta = \frac{p}{K}\,; \quad p = K\,\vartheta \text{ con } K = \frac{E}{3(1-2\nu)} > 0 \text{ modulo di volume}$$

$K \to \infty$, $\nu \to 1/2$ (incompr.)

$$\lambda = K - \frac{2}{3}\,G$$

Rappresentazione matriciale di $\varepsilon = \mathbb{C}:\sigma$ e di $\sigma = \mathbb{E}:\varepsilon$

$$\underbrace{\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} = 2\varepsilon_{12} \\ \gamma_{23} = 2\varepsilon_{23} \\ \gamma_{13} = 2\varepsilon_{13} \end{Bmatrix}}_{\{\varepsilon\}\ 6\times 1} = \underbrace{\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & & & \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & & 0 & \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & & & \\ & & & \frac{1}{G} & & \\ & 0 & & & \frac{1}{G} & \\ & & & & & \frac{1}{G} \end{bmatrix}}_{\substack{\text{matrice di cedevolezza} \\ [\mathbb{C}]\ 6\times 6}} \cdot \underbrace{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}}_{\{\sigma\}\ 6\times 1} \leftrightarrow \{\sigma\} = \underbrace{\begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & 2\mu & & \\ & 0 & & & 2\mu & \\ & & & & & 2\mu \end{bmatrix}}_{\substack{\text{matrice di rigidezza} \\ [\mathbb{E}]\ 6\times 6}} \cdot \{\varepsilon\}$$

del materiale

γ_{ij}
ε_{ij}

$\tau_{ij} = G\gamma_{ij}$
μ
$= G\,2\varepsilon_{ij}$

- Materiale isotropo (infiniti piani di simmetria materiale, ogni direzione è di simmetria): 2 parametri elastici indipendenti, es. E, ν ; E, G ; $\boxed{K, G}$; $\lambda, \mu = G$; ecc.

Vedi leggi volumetrica e deviatorica disaccoppiate $\quad p = K v$; $v = \frac{p}{K}$

$s = 2G\,e$, $e = \frac{s}{2G}$

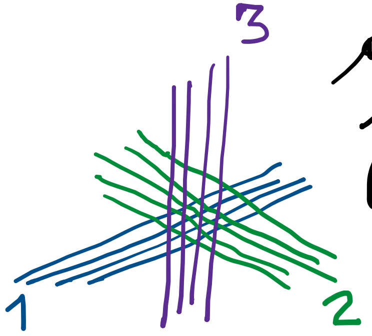
deviatore di deformaz.

deviatore di sforzo

$$\lambda = K - \frac{2}{3}G = \frac{E}{3(1-2\nu)} - \frac{\cancel{2}}{3}\,\frac{E}{\cancel{2}(1+\nu)} = \frac{E}{\cancel{3}(1-2\nu)(1+\nu)}\underbrace{(\cancel{1}+\nu-\cancel{1}+2\nu)}_{\cancel{3}\nu} = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\lambda \to \infty \begin{cases} \nu \to -1 \\ \nu \to 1/2 \end{cases}$$

- **Materiale ortotropo** (simmetria materiale rispetto a tre piani mut. $\perp$)



direzioni di ortotropia (mutuamente $\perp$)
(es. legno, composito, roccia)

3: E_1, E_2, E_3

3: $\nu_{12}, \nu_{13}, \nu_{23}$

3: G_{12}, G_{23}, G_{13}

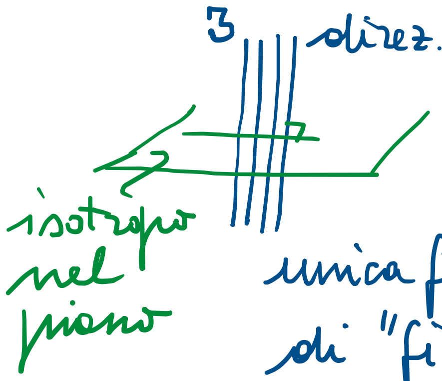
9 parametri indipendenti

$$\frac{\nu_{ji}}{E_i} = \frac{\nu_{ij}}{E_j} \quad \text{per simm. } \mathbb{C}^T = \mathbb{C}$$

$$[\mathbb{C}] = \left[\begin{array}{ccc|ccc} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & & & \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & & 0 & \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{32}}{E_2} & \frac{1}{E_3} & & & \\ \hline & & & \frac{1}{G_{12}} & & \\ & 0 & & & \frac{1}{G_{23}} & \\ & & & & & \frac{1}{G_{13}} \end{array}\right]$$

rif. ort.

- **Materiale trasversalmente isotropo** (asse di simmetria del materiale)



direz. trasversa al piano

unica famiglia di "fibre"

2: $E_1 = E_2 = E$, $E_T = E_3$

2: $\nu_{12} = \nu$, $\nu_T = \nu_{31} = \nu_{32}$

1: $\cancel{G_{12} = G = \frac{E}{2(1+\nu)}}$, $G_T = G_{23} = G_{13}$

5 parametri indipendenti

$$[\mathbb{C}] = \left[\begin{array}{ccc|ccc} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu_T}{E_T} & & & \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu_T}{E_T} & & 0 & \\ -\frac{\nu_T}{E_T} & -\frac{\nu_T}{E_T} & \frac{1}{E_T} & & & \\ \hline & & & \frac{2(1+\nu)}{E} & & \\ & 0 & & & \frac{1}{G_T} & \\ & & & & & \frac{1}{G_T} \end{array}\right]$$