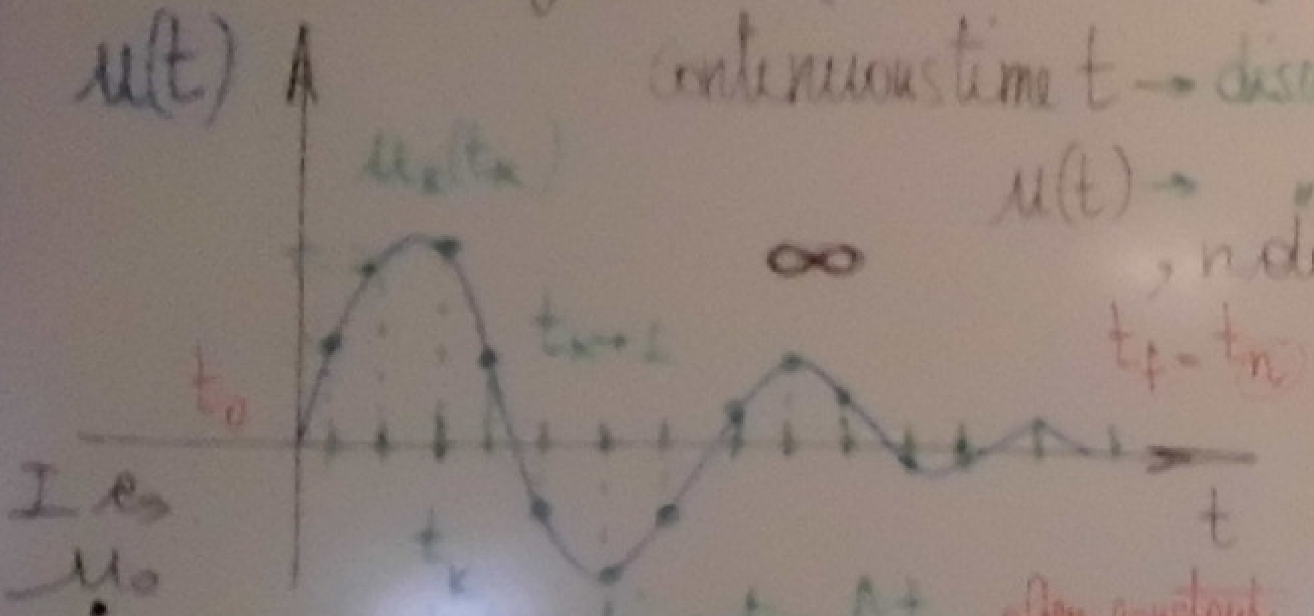
Time integration (direct, step-by-step)



continuous time $t \to$ discrete time t_k

$u(t) \to u_k$

∞ , n discrete

$t_f = t_n$ n of time steps

I.C. u_0, $\dot{u}_0$

time step Δt_k often constant

$\Delta t_k = \Delta t = \frac{t_f - t_0}{n}$

$m\ddot{u} + c\dot{u} + Ku = F(t)$

2^{nd}-order ODE with constant coeffs (time invariant sys.)

Known the solution at $t = t_k$

$\Delta(\ ) = (\ )_{k+1} - (\ )_k$

$m\ddot{u}_k + c\dot{u}_k + Ku_k = F_k(t_k)$

Advance to the solution in $t_k + \Delta t_k = t_{k+1}$

$m\ddot{u}_{k+1} + c\dot{u}_{k+1} + Ku_{k+1} = F_{k+1}(t_{k+1})$

$m\Delta\ddot{u}_k + c\Delta\dot{u}_k + K\Delta u_k = \Delta F_k$ incremental form $(*)$

Finite differences

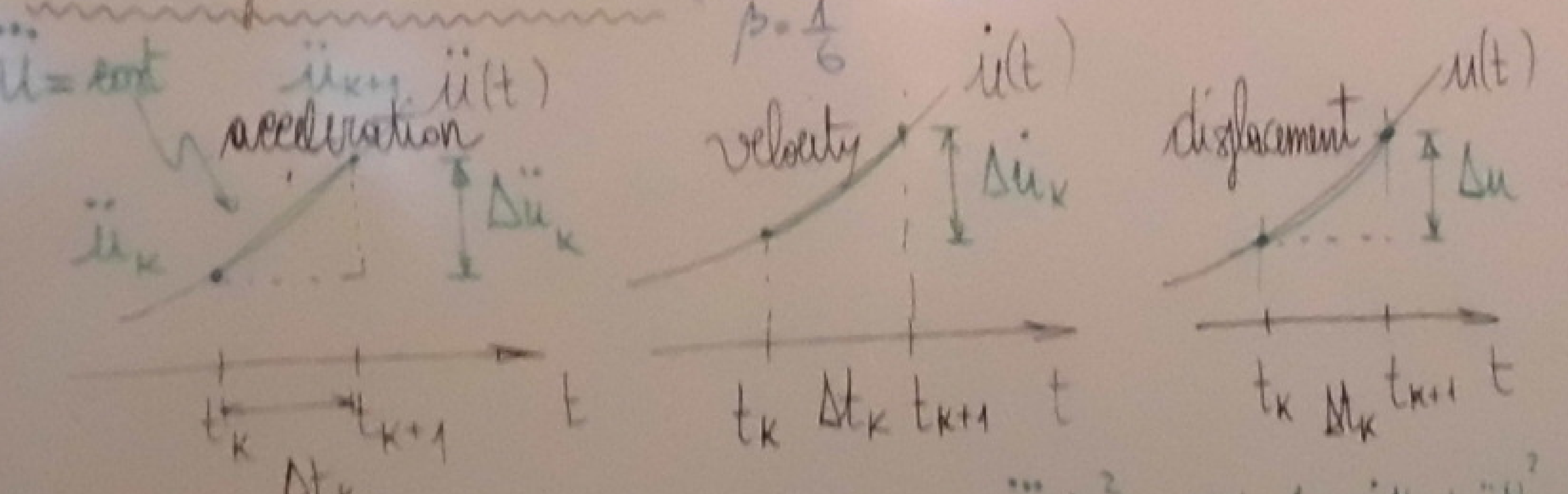
$$\frac{df(x)}{dx} \simeq \frac{\Delta f}{\Delta x}$$

Taylor expansion

$$\Delta f_x = f(x+\Delta x) - f(x) = f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 + \frac{1}{6}f'''(x)\Delta x^3 + \dots$$

Method of linear acceleration $\left(2\beta = \frac{1}{3};\ \gamma = \frac{1}{2}\right)$ $\beta = \frac{1}{6}$

$\dddot{u} = \text{const}$



$\Delta\ddot{u} = \dddot{u}\,\Delta t$ linear in Δt over the time step

$\dddot{u} = \frac{\Delta\ddot{u}}{\Delta t}$

$\Delta\dot{u} = \ddot{u}\Delta t + \frac{1}{2}\dddot{u}\Delta t^2$ quadratic

$= \ddot{u}\Delta t + \frac{1}{2}\frac{\Delta\ddot{u}}{\Delta t}\Delta t^2$

$= \Delta t\left(\ddot{u} + \frac{1}{2}\Delta\ddot{u}\right)$

$\Delta u = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \frac{1}{6}\dddot{u}\Delta t^3$ cubic

$= \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \frac{1}{6}\frac{\Delta\ddot{u}}{\Delta t}\Delta t^3$

$= \dot{u}\Delta t + \frac{1}{2}\Delta t^2\left(\ddot{u} + \frac{1}{3}\Delta\ddot{u}\right)$

Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing lin. acc.

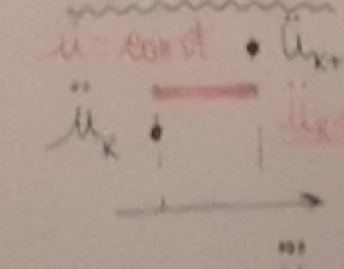
$$\begin{cases} \Delta u = \dot{u}\Delta t + \frac{1}{2}\Delta t^2\left(\ddot{u} + 2\beta\Delta\ddot{u}\right) = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \beta\Delta\ddot{u}\Delta t^2 & (1) \\ \Delta\dot{u} = \Delta t\left(\ddot{u} + \gamma\Delta\ddot{u}\right) = \ddot{u}\Delta t + \gamma\Delta\ddot{u}\Delta t & (2) \end{cases}$$

$(1-2\beta)\ddot{u}_k + 2\beta\ddot{u}_{k+1}$

$(1-\gamma)\ddot{u}_k + \gamma\ddot{u}_{k+1}$ weighted average $(2\beta, \gamma)$

Mean acceleration $\left(2\beta = \frac{1}{2};\ \gamma = \frac{1}{2}\right)$ $\beta = \frac{1}{4}$

$\ddot{u} = \text{const}$



$\frac{\ddot{u}_k + \ddot{u}_{k+1}}{2} = \ddot{u}_k + \frac{1}{2}\Delta\ddot{u}_k$

$\dddot{u} = 0$

$\Delta\dot{u} = \ddot{u}\Delta t = \left(\ddot{u} + \frac{1}{2}\Delta\ddot{u}\right)\Delta t$

$\Delta u = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 = \dot{u}\Delta t + \frac{1}{2}\left(\ddot{u} + \frac{1}{2}\Delta\ddot{u}\right)\Delta t^2$

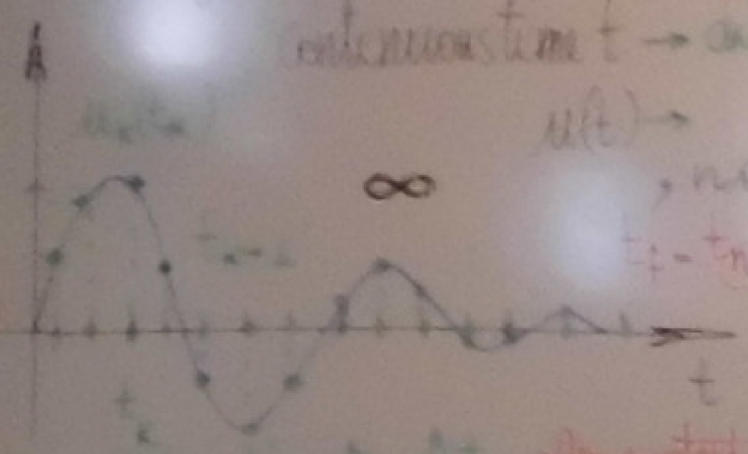
By substituting (3) and (4) into incremental eqn of m. $(*)$

Time integration (direct, step-by-step)

continuous time $t \rightarrow$ discrete time t_k

$u(t) \rightarrow u_k$

∞, n discrete



$t_f - t_0$ n. of time steps

I.C.s u_0, $\dot{u}_0$

time step Δt_k often constant

$\Delta t_k = \Delta t = \frac{t_f - t_0}{n}$

$m\ddot{u} + c\dot{u} + ku = F(t)$

2nd-order ODE with constant coeffs (time invariant sys)

Known the solution at $t = t_k$

$m\ddot{u}_k + c\dot{u}_k + ku_k = F_k(t_k)$

Known is the solution at $t_k + \Delta t_k = t_{k+1}$

$m\ddot{u}_{k+1} + c\dot{u}_{k+1} + ku_{k+1} = F_{k+1}(t_{k+1})$

$m\Delta\ddot{u}_k + c\Delta\dot{u}_k + k\Delta u_k = \Delta F_k$ incremental form (*)

Finite differences

$$\frac{df(x)}{dx} \simeq \frac{\Delta f}{\Delta x}$$

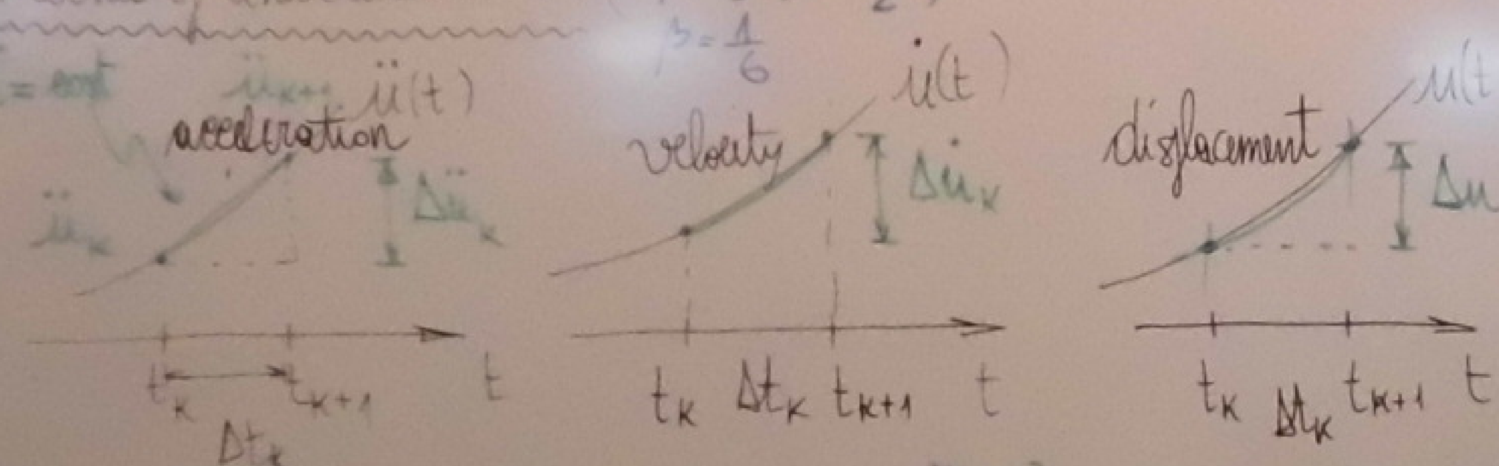
Taylor expansion

$$\Delta f_x = f(x+\Delta x) - f(x) = \dot{f}(x)\Delta x + \frac{1}{2}\ddot{f}(x)\Delta x^2 + \frac{1}{6}\dddot{f}(x)\Delta x^3 + \dots$$

Method of linear acceleration ($2\beta = \frac{1}{3}$; $\gamma = \frac{1}{2}$)

$\beta = \frac{1}{6}$

$\dddot{u} = \text{const}$



$\Delta\ddot{u} = \dddot{u}\,\Delta t$ linear in Δt

$\dddot{u} = \frac{\Delta\ddot{u}}{\Delta t}$

$\Delta\dot{u} = \ddot{u}\Delta t + \frac{1}{2}\dddot{u}\Delta t^2$ quadratic

$= \ddot{u}\Delta t + \frac{1}{2}\frac{\Delta\ddot{u}}{\Delta t}\Delta t^2$

$= \Delta t\left(\ddot{u} + \frac{1}{2}\Delta\ddot{u}\right)$

$\Delta u = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \frac{1}{6}\dddot{u}\Delta t^3$ cubic

$= \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \frac{1}{6}\frac{\Delta\ddot{u}}{\Delta t}\Delta t^3$

$= \dot{u}\Delta t + \frac{1}{2}\Delta t^2\left(\ddot{u} + \frac{1}{3}\Delta\ddot{u}\right)$

Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing lin. acc.

$$\begin{cases} \Delta u = \dot{u}\Delta t + \frac{1}{2}\Delta t^2\left(\ddot{u} + 2\beta\Delta\ddot{u}\right) = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \beta\Delta\ddot{u}\Delta t^2 & (1) \\ \Delta\dot{u} = \Delta t\left(\ddot{u} + \gamma\Delta\ddot{u}\right) = \ddot{u}\Delta t + \gamma\Delta\ddot{u}\Delta t & (2) \end{cases}$$

$(1-2\beta)\ddot{u}_k + 2\beta\ddot{u}_{k+1}$

$(1-\gamma)\ddot{u}_k + \gamma\ddot{u}_{k+1}$ weighted average $(2\beta, \gamma)$

Mean acceleration ($2\beta = \frac{1}{2}$; $\gamma = \frac{1}{2}$)

$\ddot{u} = \text{const}$, $\ddot{u}_k$, $\ddot{u}_{k+1}$, $\beta = \frac{1}{4}$

$\frac{\ddot{u}_k + \ddot{u}_{k+1}}{2} = \ddot{u}_k + \frac{1}{2}\Delta\ddot{u}_k$

$\dddot{u} = 0$

$\Delta\dot{u} = \ddot{u}\Delta t = \left(\ddot{u} + \frac{1}{2}\Delta\ddot{u}\right)\Delta t$

$\Delta u = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2$

$= \dot{u}\Delta t + \frac{1}{2}\left(\ddot{u} + \frac{1}{2}\Delta\ddot{u}\right)\Delta t^2$

Solve (1) wrt $\Delta\ddot{u}$

$$\Delta\ddot{u} = \frac{\Delta u}{\beta\Delta t^2} - \frac{\dot{u}\Delta t}{\beta\Delta t^2} - \frac{1}{2}\frac{\ddot{u}\Delta t^2}{\beta\Delta t^2}$$

$$\boxed{\Delta\ddot{u}(\Delta u) = \frac{\Delta u}{\beta\Delta t^2} - \frac{\dot{u}}{\beta\Delta t} - \frac{\ddot{u}}{2\beta}} \quad (3)$$

Substitute (3) into (2)

$$\Delta\dot{u} = \Delta t\,\ddot{u} + \gamma\Delta t\left(\frac{\Delta u}{\beta\Delta t^2} - \frac{\dot{u}}{\beta\Delta t} - \frac{\ddot{u}}{2\beta}\right)$$

$$= \Delta t\,\ddot{u} + \frac{\gamma}{\beta}\frac{\Delta u}{\Delta t} - \frac{\gamma}{\beta}\dot{u} - \frac{\gamma}{2\beta}\ddot{u}\Delta t$$

$$\boxed{\Delta\dot{u}(\Delta u) = \frac{\gamma}{\beta}\frac{\Delta u}{\Delta t} - \frac{\gamma}{\beta}\dot{u} + \left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}\Delta t} \quad (4)$$

By substituting (3) and (4) into incremental eqn of m. (*)

$$m\left(\frac{\Delta u}{\beta\Delta t^2} - \frac{\dot{u}}{\beta\Delta t} - \frac{\ddot{u}}{2\beta}\right) + c\left(\frac{\gamma}{\beta}\frac{\Delta u}{\Delta t} - \frac{\gamma}{\beta}\dot{u} + \left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}\Delta t\right) + k\Delta u = \Delta F$$

$$\left(\frac{m}{\beta\Delta t^2} + \frac{\gamma}{\beta}\frac{c}{\Delta t} + k\right)\Delta u = \Delta F + \frac{m}{\beta}\left(\frac{\dot{u}}{\Delta t} + \frac{\ddot{u}}{2}\right) + c\left(\frac{\gamma}{\beta}\dot{u} - \left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}\Delta t\right) \Rightarrow \boxed{\tilde{K}\Delta u = \Delta\tilde{F}}$$

effective stiffness

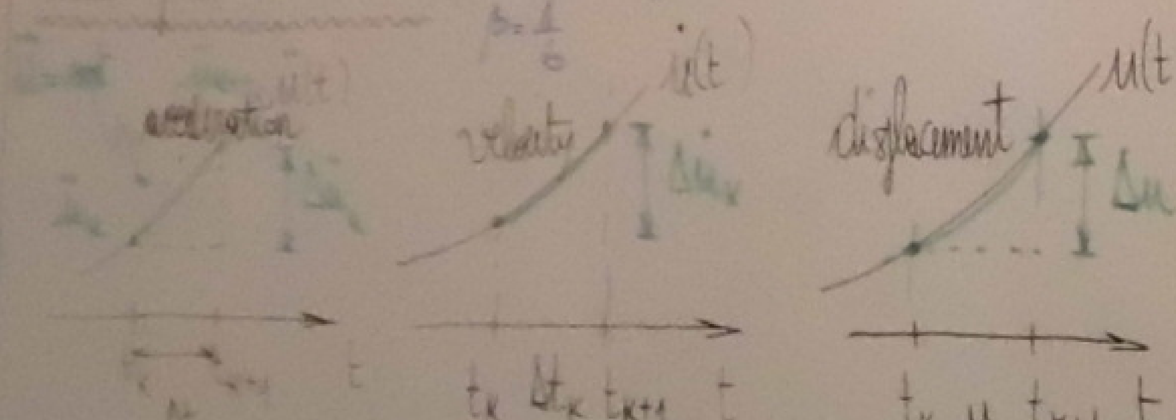
$\Rightarrow \Delta u = \tilde{K}^{-1}\cdot\Delta\tilde{F}$

Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing lin. acc.

$$\begin{cases} \Delta u = \dot{u}\Delta t + \frac{1}{2}\Delta t^2 \underbrace{\left(\ddot{u} + 2\beta\Delta\ddot{u}\right)}_{(1-2\beta)\ddot{u}_k + 2\beta\ddot{u}_{k+1}} = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \beta\Delta\ddot{u}\Delta t^2 & (1) \\ \Delta\dot{u} = \Delta t \underbrace{\left(\ddot{u} + \gamma\Delta\ddot{u}\right)}_{(1-\gamma)\ddot{u}_k + \gamma\ddot{u}_{k+1}} = \ddot{u}\Delta t + \gamma\Delta\ddot{u}\Delta t & (2) \end{cases}$$

weighted average $(2\beta, \gamma)$



Mean acceleration $\left(2\beta = \frac{1}{2};\ \gamma = \frac{1}{2}\right)$ $\beta = \frac{1}{4}$

$\ddot{u} = \text{const}$

$\ddot{u}_k$ $\ddot{u}_{k+1}$

$\frac{\ddot{u}_k + \ddot{u}_{k+1}}{2} = \ddot{u}_k + \frac{1}{2}\Delta\ddot{u}_k$

$\dddot{u} = 0$

$\Delta\dot{u} = \ddot{u}\Delta t = \left(\ddot{u} + \frac{1}{2}\Delta\ddot{u}\right)\Delta t$

$\Delta u = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2$

$= \dot{u}\Delta t + \frac{1}{2}\left(\ddot{u} + \frac{1}{2}\Delta\ddot{u}\right)\Delta t^2$

Solve (1) wrt $\Delta\ddot{u}$:

$$\Delta\ddot{u} = \frac{\Delta u}{\beta\Delta t^2} - \frac{\dot{u}\cancel{\Delta t}}{\beta\Delta t^{\cancel{2}}} - \frac{1}{2}\frac{\ddot{u}\cancel{\Delta t^2}}{\beta\cancel{\Delta t^2}}$$

$$\boxed{\Delta\ddot{u}(\Delta u) = \frac{\Delta u}{\beta\Delta t^2} - \frac{\dot{u}}{\beta\Delta t} - \frac{\ddot{u}}{2\beta}} \quad (3)$$

Substitute (3) into (2):

$$\Delta\dot{u} = \Delta t\,\ddot{u} + \gamma\Delta t\left(\frac{\Delta u}{\beta\Delta t^2} - \frac{\dot{u}}{\beta\Delta t} - \frac{\ddot{u}}{2\beta}\right)$$

$$= \Delta t\,\ddot{u} + \frac{\gamma}{\beta}\frac{\Delta u}{\Delta t} - \frac{\gamma}{\beta}\dot{u} - \frac{\gamma}{2\beta}\ddot{u}\Delta t$$

$$\boxed{\Delta\dot{u}(\Delta u) = \frac{\gamma}{\beta}\frac{\Delta u}{\Delta t} - \frac{\gamma}{\beta}\dot{u} + \left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}\Delta t} \quad (4)$$

By substituting (3) and (4) into incremental equ. of m. (*)

$$m\left(\frac{\Delta u}{\beta\Delta t^2} - \frac{\dot{u}}{\beta\Delta t} - \frac{\ddot{u}}{2\beta}\right) + c\left(\frac{\gamma}{\beta}\frac{\Delta u}{\Delta t} - \frac{\gamma}{\beta}\dot{u} + \left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}\Delta t\right) + k\Delta u = \Delta F$$

$$\underbrace{\left(\frac{m}{\beta\Delta t^2} + \frac{\gamma}{\beta}\frac{c}{\Delta t} + k\right)}_{\text{effective stiffness}}\Delta u = \underbrace{\Delta F + \frac{m}{\beta}\left(\frac{\dot{u}}{\Delta t} + \frac{\ddot{u}}{2}\right) + c\left(\frac{\gamma}{\beta}\dot{u} - \left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}\Delta t\right)}_{\text{effective incremental force}} \Rightarrow \boxed{\tilde{K}\Delta u = \widetilde{\Delta F}}$$

linear

$$\Rightarrow \Delta u = \tilde{K}^{-1} \cdot \widetilde{\Delta F}$$

Iterative loop

(Start: $u_0, \dot{u}_0$; $\ddot{u}_0$ from eq. of motion)

① Initial state $u_k, \dot{u}_k, \ddot{u}_k, F_k \leftrightarrow m\ddot{u}_k + c\dot{u}_k + K u_k = F_k$

② Set constants β, γ and evaluate (at given $\Delta t = \text{const}$)

$$\tilde{K} = \frac{m}{\beta \Delta t^2} + \frac{\gamma}{\beta}\frac{c}{\Delta t} + K \quad \text{once for all } (\Delta t_k = \Delta t)$$

$$\Delta\tilde{F} = \Delta F + \frac{m}{\beta}\left(\frac{\dot{u}_k}{\Delta t} + \frac{\ddot{u}_k}{2}\right) + c\left(\frac{\gamma}{\beta}\dot{u}_k - \left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_k \Delta t\right)$$

③ Solve for increment Δu

$$\Delta u = \tilde{K}^{-1} \cdot \Delta\tilde{F}$$

④ Update variables:

$$u_{k+1} = u_k + \Delta u_k$$

$$\dot{u}_{k+1} = \dot{u}_k + \Delta\dot{u}_k \leftarrow (4)$$

$$\ddot{u}_{k+1} = \ddot{u}_k + \Delta\ddot{u}_k \leftarrow (3)$$

loop over the time steps $k = 1, \dots, n-1$

Monitoring of the goodness of the numerical solution

Numerical considerations

- In general N.M. is implicit (meaning that the solution at time t_{k+1} is not automatically known from values at t_k (one has to calculate $\Delta u = \tilde{K}^{-1} \cdot \Delta\tilde{F}$)
- Explicit iff $\beta = 0$ (forward), consistently with estimate of Δu from $\tilde{K}\Delta u = \Delta\tilde{F}$ for $\beta \to 0$
- Accuracy of the 2^{nd}-order iff $\boxed{\gamma = 1/2}$ $\tilde{\Delta u} \sim \Delta t^2$ $\left(\Delta t \simeq \frac{T_1}{10}\right)$
- Numerical stability (solution shall be bound in time at variable Δt)
  - unconditionally stable if $2\beta \geq \gamma \geq 1/2$ (any Δt fine)
  - conditionally stable if $2\beta < \gamma \geq 1/2$

$$\Delta t \leq \Delta t_{cr} = \frac{\Omega_{cr}}{\omega_1} = \frac{\Omega_{cr}}{2\pi} T_1 \quad \text{where } \Omega_{cr} = \frac{\xi\left(\gamma - \frac{1}{2}\right) + \sqrt{\xi^2\left(\gamma - \frac{1}{2}\right)^2 + \frac{\gamma}{2} - \beta}}{\frac{\gamma}{2} - \beta} = \frac{1}{\sqrt{\frac{\gamma}{4} - \beta}}$$

ξ damping ratio

common estimate $\xi = 0$ or $\boxed{\gamma = \frac{1}{2}}$

Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing [illegible]

$$\begin{cases} \Delta u = \dot{u}\Delta t + \frac{1}{2}\Delta t^2\left(\ddot{u} + 2\beta\Delta\ddot{u}\right) = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \beta\Delta\ddot{u}\Delta t^2 & (1) \\ \Delta\dot{u} = \Delta t\left(\ddot{u} + \gamma\Delta\ddot{u}\right) = \ddot{u}\Delta t + \gamma\Delta\ddot{u}\Delta t & (2) \end{cases}$$

Method	β	γ	Type	Stability
Mean acceleration	$\frac{1}{4}$	$\frac{1}{2}$	implicit	[illegible] stable
Linear acceleration	$\frac{1}{6}$	$\frac{1}{2}$	implicit	[illegible] $\Omega_{cr} = \frac{\sqrt{3}}{\pi}$
Central difference	0	$\frac{1}{2}$	explicit	$\Omega_{cr} = \frac{T}{\pi}$

Iterative loop
1 Initial state $u_k, \dot{u}_k, \ddot{u}_k$ ← $m\ddot{u}_k + c\dot{u}_k + Ku_k = F_k$
2 Set constants β, γ and evaluate (at given Δt = const)
$\tilde{K} = \frac{m}{\beta \Delta t^2} + \frac{\gamma}{\beta}\frac{c}{\Delta t} + K$
for all ($\Delta t_k = \Delta t$)
3 Solve for increment Δu
$\Delta u = \tilde{K}^{-1} \Delta\tilde{F}$
4 Update variables
$u_{k+1} = u_k + \Delta u_k$
$\dot{u}_{k+1} = \dot{u}_k + \Delta\dot{u}_k$ ← (4)
$\ddot{u}_{k+1} = \ddot{u}_k + \Delta\ddot{u}_k$ ← (5)
Numerical considerations
- In general N.M. is implicit (meaning that the solution at time t_{k+1} is not automatically known from values at t_k
calculate $\Delta u = \tilde{K}^{-1} \cdot \Delta\tilde{F}$
- Explicit iff β=0 (forward), consistently with estimate of Δu from $\tilde{K}\Delta u = \Delta\tilde{F}$ for β→0
- Accuracy of the 2nd-order iff $\gamma = 1/2$ $\widetilde{\Delta u} \sim \Delta t^2$ ($\Delta t \simeq \frac{T_1}{10}$)
- Numerical stability (solution shall be bound in time at variable Δt)
- unconditionally stable if $2\beta \geq \gamma \geq 1/2$ (any Δt fine)
- conditionally stable if $2\beta < \gamma \geq 1/2$
damping ratio
$\Delta t < \Delta t_{cr} = \frac{\Omega_{cr}}{\omega_1} = \frac{\Omega_{cr}}{2\pi} T_1$ where $\Omega_{cr} = \frac{\xi(\gamma - \frac{1}{2}) + \sqrt{\xi^2(\gamma-\frac{1}{2})^2 + \frac{\gamma}{2} - \beta}}{\frac{\gamma}{2} - \beta}$
Newmark method (~1959) (γ, β method)
Family of integration methods, generalizing lin. acc
$\Delta u = \dot{u}\Delta t + \frac{1}{2}\Delta t^2(\ddot{u} + 2\beta\Delta\ddot{u}) = \dot{u}\Delta t + \frac{1}{2}\ddot{u}\Delta t^2 + \beta\Delta\ddot{u}\Delta t^2$ (1)
$\Delta\dot{u} = \Delta t(\ddot{u} + \gamma\Delta\ddot{u}) = \ddot{u}\Delta t + \gamma\Delta\ddot{u}\Delta t$ (2)
Method | β | γ | Type | Stability
Mean acceleration | 1/4 | 1/2 | implicit | Uncond. stable
Linear acceleration | 1/6 | 1/2 | implicit | $\Delta t_{cr} = \frac{\sqrt{3}}{\pi} T_1$
Central differences | 0 | 1/2 | explicit | $\Delta t_{cr} = \frac{T_1}{\pi}$

(3) Solve for increment Δu

(4) Update variables

Numerical considerations

- In general N.M. is implicit (meaning that the solution at time t_{k+1} is not automatically known from values at t_k; one has to calculate $\Delta u = \tilde{K}^{-1} \cdot \Delta \tilde{F}$
- Explicit iff $\beta = 0$ (forward), consistently with estimate of Δu from $\tilde{K}\,\Delta u = \Delta \tilde{F}$ for $\beta \to 0$
- Accuracy of the 2^{nd}-order iff $\boxed{\gamma = 1/2}$ $\quad \widetilde{\Delta u} \sim \Delta t^2$ $\quad \boxed{\Delta t \simeq \frac{T_1}{10}}$
- Numerical stability (solution shall be bound in time at variable Δt)
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  - conditionally stable if $2\beta < \gamma$, $\gamma \geq 1/2$

$$\Delta t \leq \Delta t_{cr} = \frac{\Omega_{cr}}{\omega_1} = \frac{\Omega_{cr}}{2\pi} T_1 \quad \text{where} \quad \Omega_{cr} = \frac{\xi\left(\gamma - \frac{1}{2}\right) + \sqrt{\xi^2\left(\gamma - \frac{1}{2}\right)^2 + \frac{\gamma}{2} - \beta}}{\frac{\gamma}{2} - \beta} = \frac{1}{\sqrt{\frac{\gamma}{2} - \beta}}$$

ξ damping ratio

Conserv. estimate: $\xi = 0$ or $\boxed{\gamma = \frac{1}{2}}$

Newmark method (~1959) (γ, β method)

Family of integration methods, generalizing lin. acc.

$$\begin{cases} \Delta u = \dot{u}\,\Delta t + \frac{1}{2}\Delta t^2\left(\ddot{u} + 2\beta\,\Delta\ddot{u}\right) = \dot{u}\,\Delta t + \frac{1}{2}\ddot{u}\,\Delta t^2 + \beta\,\Delta\ddot{u}\,\Delta t^2 & (1) \\ \Delta\dot{u} = \Delta t\left(\ddot{u} + \gamma\,\Delta\ddot{u}\right) = \ddot{u}\,\Delta t + \gamma\,\Delta\ddot{u}\,\Delta t & (2) \end{cases}$$

Method	β	γ	Type	Stability
Mean acceleration	$\frac{1}{4}$	$\frac{1}{2}$	implicit	Uncond. stable
Linear acceleration	$\frac{1}{6}$	$\frac{1}{2}$	implicit	Conditionally stable $\Delta t_{cr} = \frac{\sqrt{3}}{\pi} T_1$
Central differences	0	$\frac{1}{2}$	explicit	$\boxed{\Delta t_{cr} = \frac{T_1}{\pi}}$