# Harmonic force - complex variables

$F \sin\omega t \Rightarrow \ddot{u} + 2\zeta\omega_1 \dot{u} + \omega_1^2 u = \omega_1^2 u_{st} \sin\omega t$

$F \cos\omega t \Rightarrow \ddot{v} + 2\zeta\omega_1 \dot{v} + \omega_1^2 v = \omega_1^2 u_{st} \cos\omega t$

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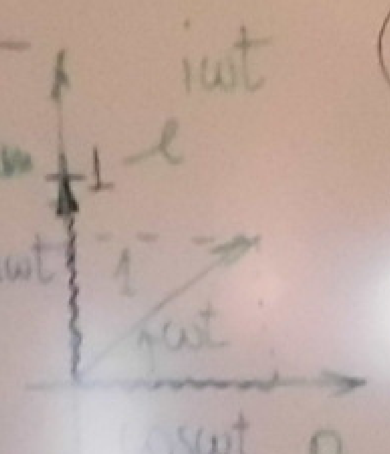
$F e^{i\omega t} \Rightarrow \ddot{z} + 2\zeta\omega_1 \dot{z} + \omega_1^2 z = \omega_1^2 u_{st} e^{i\omega t}$

where $z = v + i u$ ; $e^{i\omega t} = \cos\omega t + i \sin\omega t$

$Re(z)$ resp. to $F\cos\omega t$ $Im(z)$ response to $F\sin\omega t$




Seek for particular solution:

$z_p(t) = \underset{\text{complex}}{Z} e^{i\omega t} = \underset{\text{real}}{Z^*} e^{i(\omega t - \xi)}$

dynamic amplification factor $\quad u_{st} = \frac{F}{k}$

expected $Z^* = |Z| = N u_{st}$

$\dot{z}_p(t) = i\omega Z e^{i\omega t} = \underbrace{Z^* e^{-i\xi}} e^{i\omega t}$ $\quad \xi$ phase angle

$= \omega Z e^{i(\omega t + \pi/2)}$

$\ddot{z}_p(t) = -\omega^2 \underbrace{Z e^{i\omega t}}_{z_p}$

$(*)\ Z = Z^* e^{-i\xi} \Rightarrow |Z| = Z^* \cdot 1$

By substituting:

$\cdot i \quad (\omega_1^2 - \omega^2) Z e^{i\omega t} + 2\zeta\omega_1 i\omega Z e^{i\omega t} = \omega_1^2 u_{st} e^{i\omega t}, \forall t$

$+$

$\beta = \frac{\omega}{\omega_1}$ frequency ratio

$(1-\beta^2) + i\, 2\zeta\beta = \frac{u_{st}}{Z}$

$Z = \frac{u_{st}}{1-\beta^2 + i\, 2\zeta\beta} \cdot \frac{1-\beta^2 - i\, 2\zeta\beta}{1-\beta^2 - i\, 2\zeta\beta}$

$= \underbrace{\frac{1-\beta^2}{D} u_{st}} - i \underbrace{\frac{2\zeta\beta}{D} u_{st}}$

$i = e^{i\pi/2}$

$-i = e^{-i\pi/2}$

where $D = (1-\beta^2)^2 + (2\zeta\beta)^2$

$= Z_1 - i Z_2$

$\cdot\ \frac{Z_1}{u_{st}} = \frac{1-\beta^2}{D}$

$\cdot\ \frac{Z_2}{u_{st}} = \frac{2\zeta\beta}{D}$

dynamic ampl. factor

$\frac{Z^*}{u_{st}} = \frac{|Z|}{u_{st}} = \sqrt{\frac{Z_1^2}{u_{st}^2} + \frac{Z_2^2}{u_{st}^2}} = \frac{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}{D} = \frac{\sqrt{D}}{D} = \boxed{\frac{1}{\sqrt{D}} = N(\beta, \zeta)}$

$(*)\ Z = Z_1 - i Z_2$

$= N u_{st} (\cos\xi - i \sin\xi)$

$\begin{cases} Z_1 = N u_{st} \cos\xi \\ Z_2 = N u_{st} \sin\xi \end{cases}$

$\cos\xi = \frac{Z_1}{N u_{st}} = \frac{1-\beta^2}{D}\sqrt{D} = \frac{1-\beta^2}{\sqrt{D}}$

$\sin\xi = \frac{Z_2}{N u_{st}} = \frac{2\zeta\beta}{D}\sqrt{D} = \frac{2\zeta\beta}{\sqrt{D}}$

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$\frac{Z_2}{Z_1} = \tan\xi = \frac{2\zeta\beta}{1-\beta^2}$

phase shift $\underset{(\beta,\zeta)}{\xi} = \operatorname{atan} \frac{2\zeta\beta}{1-\beta^2}$

Finally: $z_p(t) = (Z_1 - i Z_2) e^{i\omega t}$

$= Z_1 e^{i\omega t} - i Z_2 e^{i\omega t}$

$= Z_1 e^{i\omega t} + Z_2 e^{i(\omega t - \pi/2)}$

$e^{i\omega t} = \cos\omega t + i \sin\omega t$

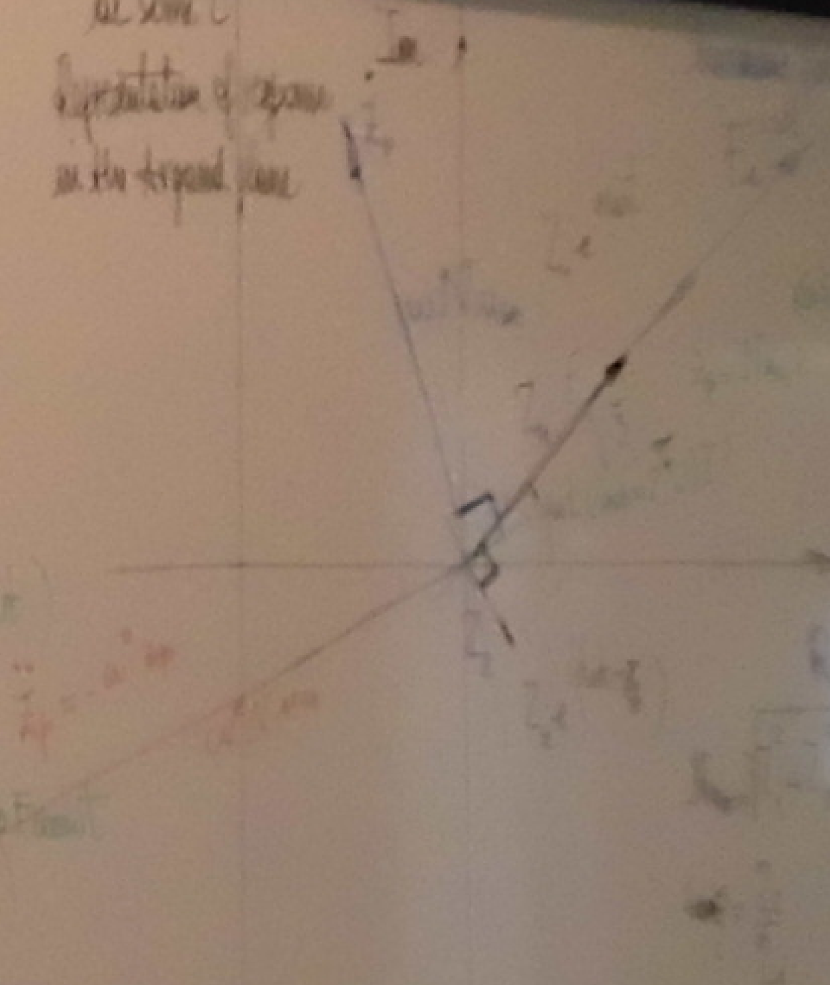
$= (Z_1 - i Z_2)(\cos\omega t + i \sin\omega t)$

$= \underbrace{Z_1 \cos\omega t + Z_2 \sin\omega t}_{Re(z_p(t)) \text{ response to } F\cos\omega t}$

$+ i \underbrace{(Z_1 \sin\omega t - Z_2 \cos\omega t)}_{Im(z_p(t)) \text{ response to } F\sin\omega t}$

at some t

Representation of response in the complex plane



Harmonic force - complex variables
Seek for particular solution
dynamic amplification factor
By substituting
β = ω/ω1 frequency ratio
(1−β²) + i 2ζβ = ust/Z
Z = ust/(1−β²+i2ζβ)
= Z1 − i Z2
Z* = |Z| = √(Z1²+Z2²) = √((1−β²)²+(2ζβ)²)/D = √D/D = 1/√D = N(β,ζ)
dynamic ampl. factor
(*) Z = Z1 − iZ2
= N ust (cos ξ − i sin ξ)
Z1 = N ust cos ξ   cos ξ = Z1/(N ust) = (1−β²)/D √D = (1−β²)/√D
Z2 = N ust sin ξ   sin ξ = Z2/(N ust) = 2ζβ/D √D = 2ζβ/√D
Z2/Z1 = tan ξ = 2ζβ/(1−β²)
phase shift ξ = atan 2ζβ/(1−β²) (β, ζ)
Finally: zp(t) = (Z1 − iZ2) e^{iωt}
= Z1 e^{iωt} − iZ2 e^{iωt}
= Z1 e^{iωt} + Z2 e^{i(ωt−π/2)}
= (Z1 − iZ2)(cos ωt + i sin ωt)
= Z1 cos ωt + Z2 sin ωt
+ i(Z1 sin ωt − Z2 cos ωt)
at some t
Representation of response in the Argand plane
Im
Re
Z1 e^{iωt}
Z2 e^{i(ωt−π/2)}
ω N ust
N ust = √(Z1²+Z2²)
tan ξ = Z2/Z1
Harmonic force

Finally: $z_p(t) = (Z_1 - iZ_2)\, e^{i\omega t}$
$= Z_1 e^{i\omega t} - iZ_2 e^{i\omega t}$
$= Z_1 e^{i\omega t} + Z_2 e^{i(\omega t - \pi/2)}$
frequency ratio
$Z = Z_1 - iZ_2$
$Z_1 = N u_{st} \cos\xi$
$Z_2 = N u_{st} \sin\xi$
$\frac{Z_2}{Z_1} = \tan\xi = \frac{2\zeta\beta}{1-\beta^2}$
phase shift $\xi = \mathrm{atan}\frac{2\zeta\beta}{1-\beta^2}$
$= (Z_1 - iZ_2)(\cos\omega t + i\sin\omega t)$
$= Z_1\cos\omega t + Z_2\sin\omega t$
$+ i(Z_1\sin\omega t - Z_2\cos\omega t)$
$\ddot{z}_p = -\omega^2 z_p$
at some t
Representation of response in the Argand plane
Im
Re
$\ddot{z}_p$
$\omega N u_{st}$
$Z_1 e^{i\omega t}$
$Z_2 e^{i(\omega t - \frac{\pi}{2})}$
$F_e^{i\omega t}$
harmonic force
$z_p = N u_{st}\, e^{i(\omega t - \xi)}$
$N u_{st} = \sqrt{Z_1^2 + Z_2^2}$
$\tan\xi = \frac{Z_2}{Z_1}$

Force diagram
in the Argand plane
F sin ξ = 2ζβNF
F cos ξ = (1-β²)NF
Im
Re
along z_p response direction: equil.
F_d; F sin ξ
β²NF
F sin ξ
F cos ξ = (1-β²) NF
F_i
2ζβNF
NF
F_e
along z_p response direction: dyn. equil. F_e, F_i; F cos ξ
harmonic force
F e^{iωt}
|F_e| = k|z_p| = k N F_st/k = NF
|F_d| = c|ż_p| = 2ζ√(km) ωN F_st/k = 2ζβNF
|F_i| = m|z̈_p| = m ω² N F_st/k = β²NF
D_n = (1-β_n²)² + (2ζβ_n)²
Z_1/u_st = (1-β²)/D
Z_2/u_st = 2ζβ/D
Z* = |Z|/u_st = 1/√D = N(β, ζ)
Response to periodic force
F(t) = F f(t)
mean value
period
t
- real cases
- analytical solution (series)
Fourier series
discrete frequencies
coefficients
FILTER

Force diagram
in the Argand plane
Im
Re
Response to periodic force
Fourier series
discrete frequencies
mean value
coefficients
period
- real cases
artificial periodic force
- analytical solution (series)
FILTER
SYSTEM
F(t) generally continuous


Response to periodic force
Fourier Series
F(t) = A0 + Σ An cos ωn t + Bn sin ωn t = Σ Cn e^(iωn t)
Cn = 1/2 (An − i Bn)
discrete frequencies ωn = n ω, ω = 2π/T fund freq
cos ωn t = (e^(iωn t) + e^(−iωn t))/2
sin ωn t = (e^(iωn t) − e^(−iωn t))/2i
βn = ωn/ω1 = n ω/ω1 = nβ
F(t) = F f(t)
mean value
A0 = 1/T ∫0T F(t) dt
An = 2/T ∫0T F(t) cos ωn t dt
Bn = 2/T ∫0T F(t) sin ωn t dt
coefficients
period
up(t) ≅ A0/K + Σ An/K (Z1n/ust cos ωn t + Z2n/ust sin ωn t)
FILTER
Bn/K (Z1n/ust sin ωn t − Z2n/ust cos ωn t)
periodic
SYSTEM ω1
response
β̄ = √(1 − 2ζ²) ≈ 1
βn = nβ = n ω/ω1
sin ωn t sin ωm t dt: T/2 if m = n
cos nωt cos mωt dt: 0 if m ≠ n
sin ωn t cos ωm t dt = 0
F(t) generally continuous
∫0T |F(t)| dt ≠ ∞
real cases
"artificial" periodic force
0 ≤ t ≤ t1, T = t1
analytical solution (series)
Fourier transform
Im
Re
Fel
Fd
Fin